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Abstract We consider situations where players are part of a network and belong to coalitions in a given coalition structure. We propose the concept of contractual stability to predict the networks that are going to emerge at equilibrium when the consent of coalition partners is needed for adding or deleting links. Two different decision rules for consent are analyzed: simple majority and unanimity. We characterize the coalition structures that make the strongly efficient network contractually stable under the unanimity decision rule and the coalition structures that sustain some critical network as contractually stable under the simple majority decision rule and under any decision rule requiring the consent of any proportion of coalition partners. Requiring the consent of coalition members may help to reconcile stability and efficiency in some classical models of network formation.

Keywords Networks · Coalition structures · Contractual stability · Strong efficiency

JEL Classification A14 · C70

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1 Introduction

The organization of players into networks and coalitions has an important role in the determination of the outcome of many social and economic interactions.¹ There are many situations where players are part of a network and belong to coalitions. In this paper, we consider situations where existing coalitions only constrain a player's ability to add or delete links in the network. A first situation has to do with the network of marriages between some key families during the fifteenth century in Florence. The Medici family (with Cosimo de Medici playing the key role) rose in power and largely consolidated control of the business and politics in Florence. Several strategies helped the Medici gain and hold power. One was the shrewd use of marriage. At first, members of the family arranged marriages for their children to seal economic and political alliances with other Florentine families. Later, the family's horizons were widened, marrying into the Roman nobility, and from then on the Medici gained status through marriages with noble families throughout Europe (see [Padgett and Ansell 1993](#)). A second situation has to do with communication networks (roads, railway tracks or waterways). Basque Y is the name given to the Spanish high-speed rail network being built since 2006 between the three cities of the Basque Country Autonomous Community (Bilbao, Vitoria and San Sebastian).² Since the Basque Y will connect Spain with the European high-speed network, the decision of linking the three cities and of the Y-shaped layout required the consent of the Basque Parliament and the Spanish authorities.³ A third situation has to do with criminal networks. A good example of criminal networks is provided by outlaw motorcycle gangs operating predominantly in the United States and Canada. The most famous are the Hells Angels which gradually evolved into criminal organizations, controlling prostitution and engaging in drug trafficking. There is a well-established hierarchical structure and mode of function in which each individual has a role. For the Hells Angels, the structure is composed of Friends, Hang Arounds, Prospects and Full-Patched members. All individuals who are part of this organization are sponsored by an official member and have to gain the approval of 100 percent of members in order to climb the hierarchy.⁴ Another feature of criminal networks is that connections among different criminal networks became a major feature of the organized crime during the 1990s.⁵ Criminal networks often

¹ See [Jackson \(2008\)](#) or [Goyal \(2007\)](#) for a comprehensive introduction to the theory of social and economic networks.

² See <http://www.euskalyvasca.com/en/home.html>.

³ Another example is the Eurasian gas supply network formed by the former Soviet Union Republics, who produce and deliver gas to the European market. To supply gas, each producer needs to cooperate with transmitters that deliver gas to the market using their transport capacities. In this respect, each producer and its transmitters form a binding supply coalition.

⁴ [Morselli \(2009\)](#) has found that the Hells Angels network of communications among its members during the investigations that led up to the March 2001 crackdown did mirror the hierarchical structure. Most of the higher-level gang members in this network were indirectly involved as suggested by their higher brokerage capital (betweenness centrality and flow betweenness centrality).

⁵ Colombian-Sicilian networks brought together Colombian cocaine suppliers with Sicilian groups possessing local knowledge, well-established heroin distribution networks, extensive bribery and corruption networks, and a full-fledged capability for money laundering. Italian and Russian criminal networks have also forged cooperative relationships. See [Williams \(2001\)](#).

develop contract relationships for the provision of certain kinds of services, such as transportation, security, contract killing, and money laundering.⁶ Thus, the existence of links among players belonging to different coalitions and the fact of requiring the consent of coalition partners for adding or deleting links are commonly observed in many situations.

The aim of this paper is to study the stability of situations where players are part of a network and belong to coalitions in a given coalition structure. We propose the concept of contractual stability to predict the networks that are going to emerge at equilibrium in situations where the deviating coalition can only modify the network structure but not the coalition structure. The idea of contractual stability is that adding or deleting a link needs the consent of coalition partners. For instance, in the context of criminal networks, the signing of a bilateral agreement between two criminals of two different groups may need the consent of those partners within the two different groups. As in [Drèze and Greenberg \(1980\)](#) the word “contractual” is used to reflect the notion that coalitions are contracts binding all members and subject to revision only with consent of coalition partners. Two different decision rules for consent are analyzed: simple majority or unanimity.

We show that the strongly efficient network with the grand coalition is always contractually stable under the unanimity decision rule. In addition, we characterize the coalition structures that make the strongly efficient network contractually stable under the unanimity decision rule for all component additive value functions sustaining the strongly efficient network. We also provide some condition that allows us to characterize the coalition structures that sustain some critical network as contractually stable under the simple majority decision rule and under any decision rule requiring the consent of some proportion of coalition partners. Finally, we look at some classical models (co-author model, symmetric connections model, model of buyer-seller networks) and we observe that requiring the consent of coalition members under the simple majority may help to reconcile stability and efficiency.

Our paper is related to [Myerson \(1977\)](#) who has elaborated communication restrictions by introducing the notion of communication network. In a communication network, a link between two players indicates that they can communicate and negotiate bilaterally the possible gains from cooperation. There are situations in which communication is possible in conferences that can consist of an arbitrary number of players. Hence, [Myerson \(1980\)](#) has modeled the communication possibilities of the players by means of hypergraphs. Each element of an hypergraph is called a conference. Communication and negotiation between players can only take place within a conference if all players of the conference participate. Since a conference can consist of several players, an hypergraph is a generalization of a network, which has bilateral communication channels only.⁷ In our paper, coalitions do not restrict how players can communicate to each other. Coalitions are contracts and each coalition member is entitled to one's say when coalition partners add or delete links to the network.

⁶ For instance, Turkish drug traffickers in Belgium can buy services from Georgian car thieves to meet their transportation needs. See [Williams \(2001\)](#).

⁷ Since each player can belong to several conferences, hypergraphs can also be interpreted as a generalization of coalition structures.

Page et al. (2005) provide a new framework that extends the standard notion of a directed network and also introduces the notion of a supernetwork. A supernetwork specifies how the different directed networks are connected via coalitional moves and coalitional preferences, and thus provides a network representation of agent preferences and the rules governing network formation.⁸ In our paper, we assume that a coalition can move from an undirected network to another undirected network according to the rules of network formation proposed by Jackson and van den Nouweland (2005) (i.e., any new links that are added can only be between players in the deviating coalition, and at least one player of any deleted link must be in the deviating coalition). Moreover, players belong to coalitions in a given coalition structure, and coalitions are contracts. Hence, the contractually stable networks are those networks from which every possible coalitional deviation is deterred because either some deviating player or certain proportion of members of the coalitions of the deviating players in the given coalition structure block the deviation. In other words, the contractually stable networks are those networks which are stable against changes in links by any coalition of individuals that have the consent of the required proportion of members of the coalitions of the deviating players in the given coalition structure.⁹

The paper is organized as follows. In Sect. 2 we introduce networks with coalition structures and we define the concept of contractual stability. In Sect. 3 we provide general results about stability and efficiency and we reconsider classical models of network formation. In Sect. 4 we conclude.

2 Networks with coalition structures

2.1 Notations

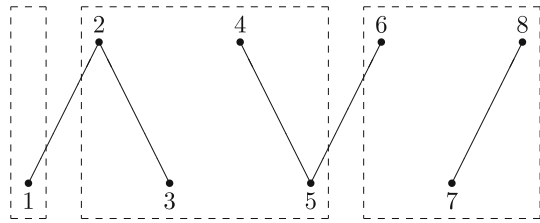
Let $N = \{1, \dots, n\}$ be the finite set of players who are connected in some network relationship. A network g is simply a list of which pairs of players are linked to each other with $ij \in g$ indicating that i and j are linked under the network g . Let g^S be the set of all subsets of $S \subseteq N$ of size 2.¹⁰ So, g^N is the complete network. The set of all possible networks on N is $G = \{g \mid g \subseteq g^N\}$. For any network g , let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g . A path in a network $g \in G$ between i and j is a sequence of players $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g

⁸ Page and Wooders (2009) have introduced a model of network formation whose primitives consist of a feasible set of networks, player preferences, rules of network formation, and a dominance relation on feasible networks. Rules may range from noncooperative, where players may only act unilaterally, to cooperative, where coalitions of players may act in concert. Modeling club structures as bipartite directed networks, Page and Wooders (2010) have formulated the problem of club formation with multiple memberships as a noncooperative game of network formation. See also Bloch and Dutta (2011) for a discussion of some recent literature on the endogenous formation of coalitions and networks.

⁹ In the network formation framework studied by Jackson and van den Nouweland (2005), players do not belong to coalitions. Then, the strongly stable networks are those networks which are stable against changes in links by any coalition of individuals.

¹⁰ Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

Fig. 1 A network with a coalition structure



is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j . A non-empty network $h \subseteq g$ is a component of g , if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j , and for any $i \in N(h)$ and $j \in N(g) \setminus N(h)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$. Knowing the components of a network, we can partition the players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by the network g . That is, $S \in \Pi(g)$ if and only if either there exists $h \in C(g)$ such that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.

Beside having some bilateral links, each player also belongs to some coalition in a given coalition structure. A coalition structure $p = \{S_1, S_2, \dots, S_m\}$ is simply a partition of the player set N , $S_k \cap S_l = \emptyset$ for $k \neq l$, $\cup_{k=1}^m S_k = N$ and $S_k \neq \emptyset$ for $k = 1, \dots, m$. Let $S(i) \in p$ be the coalition to which player i belongs. Let P denote the finite set of coalition structures. A network g with a coalition structure p is denoted by (g, p) . The network $\{12, 23, 45, 56, 78\}$ with the coalition structure $\{\{1\}, \{2, 3, 4, 5\}, \{6, 7, 8\}\}$ is depicted in Fig. 1. There is a link between players 1 and 2, a link between players 2 and 3, a link between players 4 and 5, a link between players 5 and 6, and a link between players 7 and 8, and players 2, 3, 4 and 5 are in the same coalition while players 6, 7 and 8 are in another coalition, and player 1 is alone.

The total societal value only depends on the network structure. A value function is a function $v : G \rightarrow \mathbb{R}$. It keeps track of how the total societal value varies across different networks. Let $\mathcal{V}$ be the set of all possible value functions. An allocation rule is a function $Y : G \times \mathcal{V} \rightarrow \mathbb{R}^N$ that keeps track of how the value is allocated among the players forming a network. It satisfies $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g . A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in G$. Component additive value functions are the ones for which the value of a network is the sum of the value of its components. A network g is *strongly efficient* relative to a value function v if $v(g) \geq v(g')$ for all $g' \in G$.

2.2 Contractual stability

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that no coalition benefits from altering the network. What about possible deviations? A network g' is obtainable from g via S , $S \subseteq N$, if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \notin g'$ and $ij \in g$ implies $\{i, j\} \cap S \neq \emptyset$. Conditions (i) and (ii) have been introduced first by

Jackson and van den Nouweland (2005) to define the networks obtainable from a given network by a coalition S . Condition (i) asks that any new links that are added can only be between players inside S . Condition (ii) requires that there must be at least one player belonging to S for the deletion of a link. This definition identifies possible changes in a network that can be made by a coalition S .¹¹ Hence, it simply gives the possible resulting networks once coalition S has deviated from the existing network (by adding or deleting some links to the existing network).

Once identified all possible coalitional deviations from an existing network, different stability concepts could be studied. In this paper, we propose the concept of contractual stability to predict the networks that are going to emerge at equilibrium in situations where the deviating coalition can only modify the network structure but not the coalition structure. Given the coalition structure, we study the networks that are contractually stable. As in Drèze and Greenberg (1980), we assume that coalitions are contracts binding all members and that adding or deleting a link requires the consent of coalition partners.¹² Two different decision rules for consent are analyzed: simple majority or unanimity.

Definition 1 A network g with a coalition structure p , denoted (g, p) , is contractually stable under the unanimity decision rule with respect to value function v and allocation rule Y if for any $S \subseteq N$, g' obtainable from g via S and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $k \in S(j)$ with $S(j) \in p$ and $j \in S$ such that $Y_k(g', v) \leq Y_k(g, v)$.

Under the unanimity decision rule, the move from a network g to any obtainable network g' needs the consent of every deviating player as well as the consent of every member of the coalitions of the deviating players in the given coalition structure. Then, a network g with a coalition structure p is contractually stable if some deviating player or some member of some coalition of a deviating player in the given coalition structure is not better off from the deviation to any obtainable network g' . This definition of contractual stability would revert to Dutta and Mutuswami (1997) definition of strong stability if players do not belong to coalitions. A network is strongly stable if there is no profitable deviation for any group of players. The definition of strong stability of Dutta and Mutuswami considers a deviation to be valid only if all members of a deviating coalition are strictly better off, while the definition of Jackson and van den Nouweland (2005) is slightly stronger by allowing for a deviation to be valid if some members are strictly better off and others are weakly better off.¹³ The weaker

¹¹ Notice that both types of deviations are allowed. That is, the deviating coalition S can be formed by players belonging to different coalitions of p (i.e., $S \subseteq S_1 \cup \dots \cup S_l$, with $S_k \in p$, for $k = 1, \dots, l$), or by players belonging to a given coalition of p (i.e., $S \subseteq S_i$ with $S_i \in p$).

¹² We want to study the notion of contractual stability in a general framework in which the addition or deletion of some links by some players could affect the payoff obtained by the deviating players and by the members of the coalitions of the deviating players in the given coalition structure. Thus, it seems natural to assume that adding or deleting a link requires not only the consent of the deviating players but also the consent of members of the coalitions of the deviating players in the given coalition structure. An alternative assumption that could be reasonable in some specific situations could be, for instance, to require the consent only for adding links.

¹³ Notice that Jackson and van den Nouweland (2005) version of strongly stability implies pairwise stability from Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one

definition has sense when transfers among players are not possible. In the paper, we adopt the weaker definition in order to restrict more the possible deviations. By doing so, we are able to derive positive results of existence of contractually stable networks when coalitions adopt a decision rule different than the unanimity decision rule.

Definition 2 A network g with a coalition structure p , denoted (g, p) , is contractually stable under the simple majority decision rule with respect to value function v and allocation rule Y if for any $S \subseteq N$, g' obtainable from g via S and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists

- (i) $l \in S$ such that $Y_l(g', v) \leq Y_l(g, v)$, or
- (ii) $\widehat{S} \subseteq S(j)$ with $S(j) \in p$ and $j \in S$ such that $Y_k(g', v) \leq Y_k(g, v)$ for all $k \in \widehat{S}$ and $\#\widehat{S} \geq \#S(j)/2$.

Under the simple majority decision rule, the move from a network g to any obtainable network g' needs the consent of every deviating player as well as the consent of more than half members of each coalition of the deviating players in the given coalition structure. Then, a network g with a coalition structure p is contractually stable if some deviating player or half members of some coalition of a deviating player are not better off from the deviation to any obtainable network g' . Obviously, a network that is contractually stable under the simple majority decision rule is contractually stable under the unanimity decision rule. In fact each decision rule requires the consent of coalition partners above some proportion for a deviation not to be blocked. Let q denote the proportion of coalition partners whose consent is needed for a deviation not to be blocked, $0 \leq q \leq 1$. For instance, the simple majority decision rule reverts to a proportion $q > \frac{1}{2}$ while the unanimity decision rule reverts to a proportion $q = 1$.¹⁴

To illustrate both the framework of networks with coalition structures and the concept of contractual stability we reconsider Jackson and Wolinsky (1996) co-author model.

Example 1 The co-author model (Jackson and Wolinsky 1996). Each player is a researcher who spends time writing papers. If two players are linked, then they are working on a paper together. The amount of time researcher i spends on a given project is inversely related to the number of projects n_i that he is involved in. Formally, player i 's payoff is given by

$$Y_i(g) = \sum_{j:ij \in g} \left(\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right), n_i > 0.$$

Footnote 13 continued

of his links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. However, Dutta and Mutuswami (1997) version of strongly stability only implies the strict version of pairwise stability when no two players strictly benefit from adding a link between them.

¹⁴ The relationship between contractual stability under any decision rule embodied by a proportion q is obvious: a proportion $q' < q$ refines stability. That is, given p , the set of contractually stable networks under q' is (weakly) included in the set of contractually stable networks under q . Indeed, the probability to block a deviation is greater the higher the proportion q . When the proportion approaches zero ($q \rightarrow 0$), coalitional membership has no matter in terms of consent.

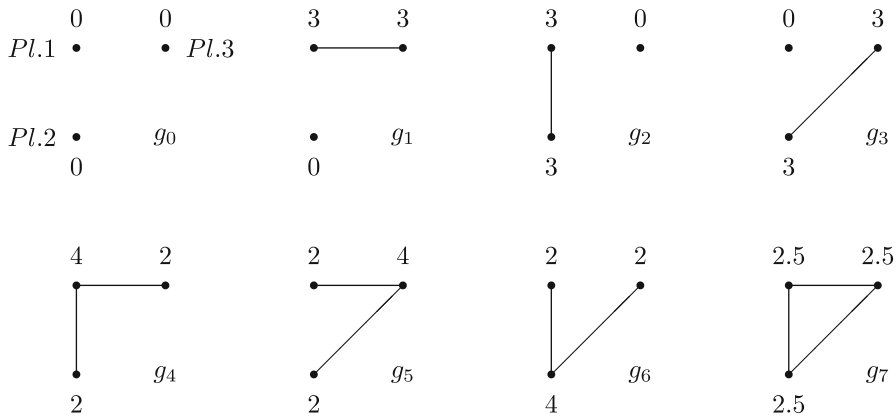


Fig. 2 The co-author model with three players

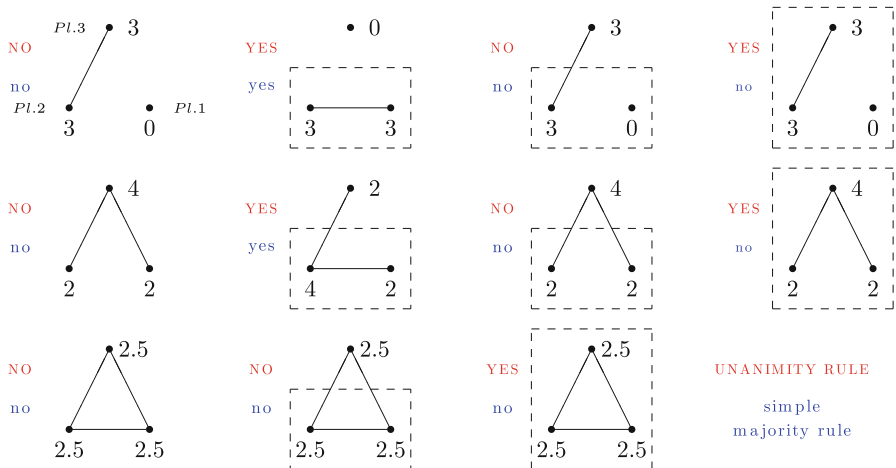


Fig. 3 Contractually stable networks in the co-author model

In Fig. 2 we have depicted the 3-player case. It is easily verified that the complete network g_7 is the unique pairwise stable network but there is no strongly stable network. For instance, players 1 and 2 have incentives to move from g_7 to g_2 by deleting their links with player 3.

We now suppose that each player belongs to a research lab and forms links with other players who do not necessarily belong to the same lab. Inside each lab, the consent of lab members is needed in order to modify the network of collaborations. For instance, consider the network $\{12\}$ with the coalition structure $\{\{1\}, \{2, 3\}\}$. It means that players 1 and 2 are collaborating on a project and that players 2 and 3 belong to the same lab while player 1 belongs to a different lab. If players 1 and 3 want to work together on a new project by building the link 13, they will need the consent of player 2 since players 2 and 3 belong to the same lab. The possible networks with coalition structures and their associated payoffs in the co-author model with three players are depicted in Fig. 3.

Once authors may need consent of coalition (or lab) partners, many forms of collaborations among authors become stable. Indeed, $(g, (\{1\}, \{2\}, \{3\}))$ is not contractually stable; but $(\{12\}, (\{1, 2\}, \{3\}))$ and $(\{12, 23\}, (\{1, 2\}, \{3\}))$ are contractually stable under the simple majority rule. For instance, $(\{12, 23\}, (\{1, 2\}, \{3\}))$ is contractually stable under the simple majority rule because players 1 and 3 would like to add the link 13 but this addition will be blocked by player 2 since player 1 needs the consent of player 2 (player 2 belongs to the same coalition as player 1 while player 3 is alone). Moreover, $\{12, 23\}$ is a strongly efficient network. The decision rule for consent matters. For instance, $(\{13, 23\}, (\{1, 2, 3\}))$ is not contractually stable under the simple majority rule (players 1 and 2 have incentives to add the link 12 and they have the majority of votes within the single lab) but is contractually stable under the unanimity decision rule (player 3 can now veto the formation of link 12).

3 Stability and efficiency

3.1 General results

We first show that, under the unanimity decision rule, there always exists a (g, p) which is contractually stable when g is the strongly efficient network.¹⁵

Proposition 1 *Under the unanimity decision rule, (g, p) is contractually stable if g is strongly efficient and p is the grand coalition $\{N\}$.*

Proof Take some strongly efficient network g^* . Then, $(g^*, \{N\})$ is contractually stable. Indeed, for all g' obtainable from g^* via S there is some player $i \in N$ such that $Y_i(g', v) \leq Y_i(g^*, v)$ and who will block the deviation from g^* to g' . $\square$

Notice that $(g^*, \{N\})$ is not the unique contractually stable network with the grand coalition. Take any network g where some player i receives the maximum he can get for all $g' \in G$. Then, $(g, \{N\})$ is contractually stable since all moves to any g' obtainable from g via S are blocked by player i .

Proposition 2 *Let $\widehat{V} \subseteq \mathcal{V}$ denote the set of component additive value functions under which network g^* is strongly efficient. Then, $\{N\}$ and $\Pi(g^*)$ are the unique coalition structures that make the strongly efficient network g^* contractually stable under the unanimity decision rule for all component additive value function $\widehat{v} \in \widehat{V}$.*

Proof Take any component additive value function $\widehat{v}$ and some strongly efficient network g^* . First, from Proposition 1, we have that $(g^*, \{N\})$ is contractually stable under the unanimity decision rule. Second, we have to show that $(g^*, \Pi(g^*))$ is contractually stable under the unanimity decision rule with respect to $\widehat{v}$, in case $\Pi(g^*) \neq \{N\}$.

¹⁵ All the results of the paper have been obtained when assuming that the rule of consent is exogenously given (either the unanimity or the simple majority) and that it affects both the addition and the deletion of links of any deviating coalition (since, in general, both the addition and the deletion of links could affect the payoff of members of the coalitions of the deviating players in the given coalition structure). In case the consent only affects the addition but not the deletion of links (as suggested by one of the referees), the results would not hold anymore. The study of some specific situation where the rule of consent and its application inside a coalition are determined endogenously could be an interesting future research topic.

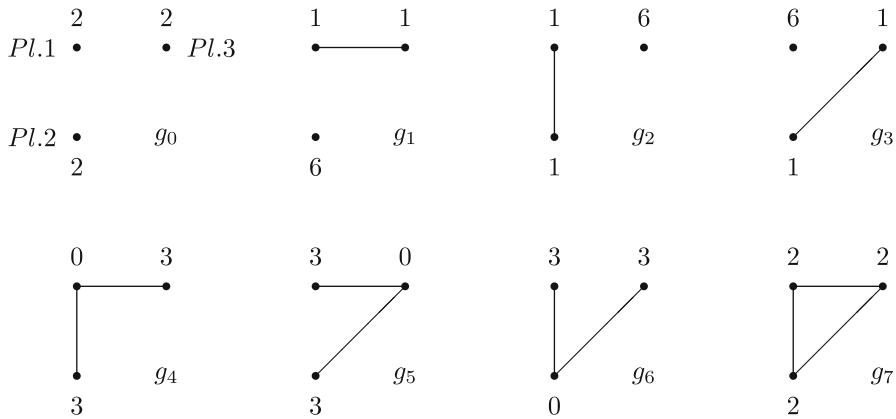


Fig. 4 A value function that is not component additive

Consider a deviation of coalition S from g^* to any obtainable network g' . Two cases have to be considered. (1) There exists $j \in S(i) \setminus S$ for some $i \in S$, $S(i) \in \Pi(g^*)$ such that $Y_j(g', \hat{v}) \leq Y_j(g^*, \hat{v})$. Then, player j will block the deviation from g^* to g' . (2) There does not exist some $j \in S(i) \setminus S$ for some $i \in S$, $S(i) \in \Pi(g^*)$ such that $Y_j(g', \hat{v}) \leq Y_j(g^*, \hat{v})$. Then, g' cannot be a profitable deviation for S , otherwise g^* would not be strongly efficient.

Now, we have to show that (g^*, p) with $p \neq \{N\}$, $\Pi(g^*)$, is not always contractually stable under the unanimity decision rule with respect to some component additive value function $\hat{v}$ under which g^* is strongly efficient. For instance, in the co-author model with three players, $g^* = \{12, 23\}$ is strongly efficient. However, $(\{12, 23\}, \{\{1, 3\}, \{2\}\})$ is not contractually stable under the unanimity decision rule.

Finally, we have to show that $(g^*, \Pi(g^*))$ is not always contractually stable under the unanimity decision rule with respect to a value function v that is not component additive. In Fig. 4 we provide a three-player case where the value function is not component additive. The strongly efficient networks are $\{13\}$, $\{12\}$, $\{23\}$, but nor $(\{13\}, \{\{13\}, \{2\}\})$ nor $(\{12\}, \{\{12\}, \{3\}\})$ nor $(\{23\}, \{\{1\}, \{23\}\})$ are contractually stable under the unanimity decision rule. $\square$

Notice that some (g^*, p) could be contractually stable under the unanimity decision rule for some $\hat{v} \in \hat{V}$ with $p \neq \{N\}$, $\Pi(g^*)$. For instance, in the co-author model $(\{12, 23\}, \{\{1, 2\}, \{3\}\})$ is contractually stable under the unanimity decision rule. However, it is not difficult to find other component additive value function $\hat{v}' \in \hat{V}$ under which $g^* = \{12, 23\}$ is strongly efficient and the deviation of player 2 to $g = \{12\}$ is not deterred. We only need to make players 1 and 2 winning less than the payoff lost by player 3.

What about the existence of contractually stable networks under the simple majority decision rule? Let $D(g)$ be the set of networks that are obtainable from g via profitable deviations. That is, given g , the network g' belongs to $D(g)$ if (i) there exists $S \subseteq N$ such that g' is obtainable from g via S and (ii) $Y_i(g', v) > Y_i(g, v)$ for all $i \in S$. Given the simple majority decision rule, we say that a network g is *critical* with

respect to value function v and allocation rule Y if there exists $T \subseteq N$ such that (i) $\#T \geq \min\{\#D(g), \#N/2\}$ and (ii) $Y_i(g', v) \leq Y_i(g, v)$ for all $i \in T$, for all $g' \in D(g)$. That is, g is critical if any profitable deviation to some obtainable network from g does not benefit a given coalition T containing as many players as the minimum between the number of profitable deviations and the simple majority of players.

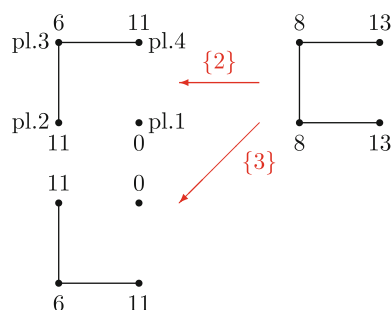
Let us explain better the notion of critical network. Given a coalition structure p and under the simple majority decision rule, a profitable coalitional deviation could only be blocked if at least half members of some coalition of a deviating player in p are not better off from the deviation. Hence, the only possible networks that could be contractually stable under the simple majority decision rule, are networks such that there are some players that do not participate in any profitable coalitional deviation and that do not benefit from any profitable deviation to some obtainable network that other coalitions of players can undertake. Let T be the coalition containing all players that do not benefit by deviating or when other coalitions of players profitably deviate from the network g . Then, if $\#T \geq \min\{\#D(g), \#N/2\}$, the network g is critical. When this is the case, Proposition 3 shows that there always exist some coalition structure p such that (g, p) is contractually stable under the simple majority decision rule. Indeed, if T contains at least half of the total number of players ($\#T \geq \#N/2$), then for $p = \{N\}$ we have that (g, p) is contractually stable under the simple majority decision rule because T can block all profitable deviations from g . If T contains less than half of the total number of players but at least as much players as the number of profitable coalitional deviations from g (i.e., $\#D(g) \leq \#T \leq \#N/2$), then by matching in p a member of each coalition of deviators with some member of coalition T , we have that (g, p) is contractually stable under the simple majority decision rule because T can block all profitable deviations from g . Otherwise, if $\#T < \min\{\#D(g), \#N/2\}$, the network g is not critical and there is no p such that (g, p) is contractually stable.

Proposition 3 *If, given the simple majority decision rule, g is critical with respect to value function v and allocation rule Y , then the coalition structures $p \in P$ such that (g, p) is contractually stable under the simple majority decision rule are of the form $p = \{S_1, \dots, S_k\}$ and satisfy the following condition: for all $g' \in D(g)$, g' obtainable from g via S , there exists some $S_j \in p$ such that $S_j \subseteq S \cup T$, $S_j \cap S \neq \emptyset$ and $\#(S_j \cap S) \leq \#(S_j \cap T)$.*

Proof Suppose that g is critical. Then, there exists a coalition $T \subseteq N$ with $\#T \geq \min\{\#D(g), \#N/2\}$ and such that all its members do not prefer the allocation they could obtain in any network g' obtainable from g via S , $S \cap T = \emptyset$ with $g' \in D(g)$. First, we show the existence of p such that (g, p) is contractually stable under the simple majority decision rule. Two cases have to be considered. (1) Assume that $\min\{\#D(g), \#N/2\} = \#D(g)$. We simply need to match in p a member of each coalition S of deviators to networks $g' \in D(g)$ with some member of coalition T . (2) Assume that $\min\{\#D(g), \#N/2\} = \#N/2$ which implies that $\#T \geq \#N/2$. Then, $(g, \{N\})$ is contractually stable under the simple majority rule since the members of coalition T will block any possible deviation from g .

Second, it is immediate that if (g, p) satisfies the condition (for all $g' \in D(g)$, g' obtainable from g via S , there exists some $S_j \in p$ such that $S_j \subseteq S \cup T$, $S_j \cap S \neq \emptyset$ and $\#(S_j \cap S) \leq \#(S_j \cap T)$), then (g, p) is contractually stable under the simple

Fig. 5 Exchange networks (all payoffs are in 96-th's)



majority decision rule because coalition T can block any profitable deviation of some S to any $g' \in D(g)$. But, if (g, p) does not satisfy this condition, then (g, p) is not contractually stable. $\square$

To illustrate this result we reconsider Jackson and Watts exchange networks model where four players get value from trading goods with each other.

Example 2 *Exchange networks* (Jackson and Watts 2002). Players benefit from trading with other players with whom they are linked, and trade can only flow along links. Players form first a network and then receive random endowments and trade along paths of the network. Trade flows without friction along any path and each connected component trades to a Walrasian equilibrium. The expected utility of a player in a network is calculated by expecting over the Walrasian equilibria that result in the player's connected component as a function of realized endowments.

There are two goods. Players have the same utility function for the two goods, $u(x, y) = x \cdot y$. Players have a random endowment which is independently and identically distributed: $(1, 0)$ with probability $\frac{1}{2}$ and $(0, 1)$ with probability $\frac{1}{2}$. Each network component trades to a Walrasian equilibrium. Thus, $\{12, 23\}$ and $\{12, 23, 13\}$ lead to the same expected trades, but lead to different costs of links. Let $c = 5/96$ be the cost of maintaining a link.¹⁶ There is no pairwise nor strongly stable network in Jackson and Watts exchange networks model with four players. Suppose that players can belong to trade associations. Inside each trade association, the consent of association members is required in order to modify the exchange network. Under the unanimity decision rule, $(\{12, 23, 34\}, \{N\})$ is contractually stable. Notice that, given the simple majority decision rule, the network $\{12, 23, 34\}$ is critical since there is $T = \{1, 4\}$ such that (i) $\#T \geq \min\{\#D(\{12, 23, 34\}), 2\} = 2$ and (ii) $Y_i(g') \leq Y_i(\{12, 23, 34\})$ for all $i \in T$, for all g' such that $g' \in D(\{12, 23, 34\})$. Hence, $(\{12, 23, 34\}, \{\{1, 2\}, \{3, 4\}\})$ is contractually stable under the simple majority decision rule. Indeed, player 1 in coalition $\{1, 2\}$ and player 4 in coalition $\{3, 4\}$ will block the only two profitable deviations from $\{12, 23, 34\}$ of either player 2 or player 3 as shown in Fig. 5.

¹⁶ Ignoring the costs of links, the player's expected utility is increasing and strictly concave in the number of other players that he is connected to: (i) the utility of being alone is 0; (ii) the expected utility of being connected to one player is $1/8$; (iii) the expected utility of being connected to two players is $1/6$; (iv) the expected utility of being connected to three players is $3/16$.

Notice that Proposition 3 could be generalized to show the existence of contractually stable networks with coalition structures under any decision rule requiring the consent of a proportion $q > 0$ of coalition partners. Given a decision rule requiring the consent of a proportion q of coalition partners, we say that a network g is *critical* with respect to value function v and allocation rule Y if there exists $T \subseteq N$ such that (i) $\#T \geq \min\{\#D(g), (1 - q)\#N\}$ and (ii) $Y_i(g', v) \leq Y_i(g, v)$ for all $i \in T$, for all $g' \in G$ such that $g' \in D(g)$.

Proposition 4 *If, given a decision rule requiring the consent of a proportion q of coalition partners, g is critical with respect to value function v and allocation rule Y , then the coalition structures $p \in P$ such that (g, p) is contractually stable under the decision rule requiring the consent of a proportion q of coalition partners are of the form $p = \{S_1, \dots, S_k\}$ and satisfy the following condition: for all $g' \in D(g)$, g' obtainable from g via S , there exists some $S_j \in p$ such that $S_j \subseteq S \cup T$, $S_j \cap S \neq \emptyset$ and $(1 - q)\#S_j \leq \#(S_j \cap T)$.*

Suppose, for instance, that $(1 - q)\#N = \min\{\#D(g), (1 - q)\#N\}$. Then, if g is critical for some decision rule requiring the consent of a proportion q of coalition partners, we have that $(g, \{N\})$ is contractually stable under such decision rule since there are at least $(1 - q)\#N$ coalition partners that will block any profitable deviation from g .¹⁷ Notice that the probability to block a deviation is greater the higher the proportion q . Hence, the higher q , the easier the existence of contractually stable networks with coalition structures under any decision rule requiring the consent of coalition partners above q . For $q = 1$, not only the existence of contractually stable networks is guaranteed but also the contractual stability of the strongly efficient network. However, under the simple majority decision rule, the contractual stability of the strongly efficient network is not guaranteed.

3.2 Some classical models

It is well-known that the relationship between stability and efficiency of networks is context dependent. Contractual stability requires the consent of coalition members in order to modify the network. Does it reconcile individual or group incentives to form links and efficiency? Let us look at some classical examples to see whether requiring the consent of coalition members under the simple majority may help or not to reconcile stability and efficiency.

Example 1 The co-author model (continued). We analyze the co-author model in case $\#N$ is even because, in that case, the strongly efficient network is not pairwise (nor strongly) stable. We show that, under the simple majority decision rule, the strongly efficient networks are always contractually stable.¹⁸

¹⁷ The formal proof is similar to the proof of Proposition 3 and then it is omitted.

¹⁸ The case of $\#N$ odd is not analyzed because, in that case, no characterization of strongly efficient networks exists. Moreover, for $\#N$ odd no strongly stable network exists.

Proposition 5 *Take the co-author model. For $\#N$ even, (g, p) is contractually stable under the simple majority rule if g is a strongly efficient network consisting of $\#N/2$ separate pairs and p is a coalition structure consisting of $\#N/2$ coalitions such that $j \in S(i)$ if and only if $ij \in g$.*

Proof For $\#N$ even, Jackson and Wolinsky (1996, p. 57) have shown that a network g consisting of $\#N/2$ separate pairs is strongly efficient. Thus, for any g' obtainable from g via deviations of S we have that at least one player j , with $S \cap S(j) \neq \emptyset$, is worse off at g' compared to g . We show now that any of such deviations will be blocked. Two cases have to be considered. First, if $S \cap S(j) \neq \emptyset$ and $j \in S$ then j will block the deviation since this deviation is making j worse off. Second, if $S \cap S(j) \neq \emptyset$ and $j \notin S$ then j will block the deviation since j 's agreement is needed given that j 's partner in $S(j)$ belongs to the deviating coalition S . $\square$

Example 3 *The symmetric connections model* (Jackson and Wolinsky 1996). Each player forms links with other players in order to exchange information and each player bears his own costs of maintaining direct links. Player i 's payoff from a network g is given by

$$Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \#\{j : ij \in g\} \cdot c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j), $\delta \in (0, 1)$ and c is the cost of maintaining a direct link. Jackson and Wolinsky have shown that for intermediate costs a conflict between stability and efficiency will arise. For $\delta < c < \delta + ((n-2)/2)\delta^2$, a star network encompassing all players is the unique strongly efficient network but is not pairwise (nor strongly) stable.¹⁹ A star network is simply a network in which all players are linked to one central player and there are no other links. However, once we allow players to belong to coalitions and that deviations need the consent of more than half of the members of the initial coalitions of the deviating players, then the conflict between stability and efficiency may be resolved.

Proposition 6 *Take the symmetric connections model. For $\delta < c < \delta + ((n-2)/2)\delta^2$, (g, p) is contractually stable under the simple majority rule if g is a strongly efficient star network encompassing all players and $\#S(i^*) \geq 2$, $S(i^*) \in p$ with i^* being the center of the star network.*

Proof For $\delta < c < \delta + ((n-2)/2)\delta^2$, Jackson and Wolinsky have shown that a star network encompassing all players is strongly efficient. Since $c > \delta$, a star network encompassing all players where i^* is the center of the star network gives to each player $i \neq i^*$ his highest possible payoff he can obtain in any $g \in G$. The only player who has incentives to modify the star network encompassing all players is i^* who obtains a negative payoff when $c > \delta$. Thus, if player i^* is in a coalition $S(i^*)$ with other players (that is, $\#S(i^*) \geq 2$), those players will block any deviation from the star network encompassing all players. $\square$

¹⁹ For other parameter values, the strongly efficient networks are pairwise stable.

Example 4 A model of buyer-seller networks (Kranton and Minehart 2001). There is one seller who has an indivisible object for sale and n potential buyers who have utilities for the object, denoted u_i , which are uniformly and independently distributed on $[0, 1]$. The object to sell has no value to the seller. Each buyer knows his own valuation, but only the distribution over the buyers' valuations. The seller also knows only the distribution of buyers' valuations. The object is sold by means of a standard second-price auction. Only the buyers who are linked to the seller participate to the auction. Let k be the number of buyers linked to the seller. For a cost per link of c_s to the seller and c_b to the buyer, the allocation rule for any network g with $k \geq 1$ links between the buyers and the seller is

$$Y_i(g) = \begin{cases} \frac{1}{k(k+1)} - c_b & \text{if } i \text{ is a linked buyer} \\ \frac{k-1}{k+1} - kc_s & \text{if } i \text{ is the seller} \\ 0 & \text{if } i \text{ is a buyer without any links.} \end{cases},$$

The value function is

$$v(g) = \frac{k}{k+1} - k(c_s + c_b),$$

which is simply the expected value of the object to the highest valued buyer less the cost of links. Let k^* be the number of links that maximizes $\frac{k}{k+1} - k(c_s + c_b)$.

Proposition 7 Take the buyer-seller network model. (g, p) is contractually stable under the simple majority rule if g is a strongly efficient network with k^* links and p is a coalition structure such that $S(i^*) \in p$ with i^* being the seller and buyer $j \in S(i^*)$ if and only if $i^*j \in g$.

Proof The efficient network is one with k^* links where

$$k^* = \begin{cases} k \text{ such that } (k(k+1))^{-1} \geq c_s + c_b \geq ((k+1)(k+2))^{-1} & \text{if } k \leq n \\ \#N & \text{otherwise} \end{cases}.$$

Take $k^* \neq 0$. Since $(k^*(k^*+1))^{-1} - c_b \geq ((k^*+1)(k^*+2))^{-1} - c_b$, the buyers already linked to the seller will block the addition of new links to the efficient network. Moreover, the buyers linked to the seller have no incentives to cut their links. Since $(k^*-1)(k^*+1)^{-1} - k^*c_s \geq (k^*-2)(k^*)^{-1} - (k^*-1)c_s$ which reverts $2(k^*(k^*+1))^{-1} \geq c_s$, the seller does not want to cut links to the efficient network. Take now $k^* = 0$. Since the empty network is the efficient one, if the seller wants to link to a buyer, then this buyer does not want, or vice versa. $\square$

In the original model of buyer seller networks, a conflict between stability and efficiency is likely to occur when $c_s > 0$. However, once the seller may need the consent of the buyers linked to him, the efficient network becomes stable. While the seller and the buyers with no link have incentives to add links, the decision for adding new links will be turned down by the buyers who are already linked to the seller.

Thus, we observe that contractual stability may help to stabilize the strongly efficient networks in some classical examples. However, contractual stability may also stabilize inefficient networks that were not stable before requiring the consent of group members. For instance, in the co-author model with three players, $(\{12\}, \{\{1, 2\}, \{3\}\})$ is contractually stable under the simple majority rule but is not pairwise nor strongly stable.

4 Conclusion

We have considered situations where players are part of a network and belong to coalitions in a given coalition structure. We have proposed the concept of contractual stability to predict the networks that are going to emerge at equilibrium when the consent of coalition partners is needed for adding or deleting links. Two different decision rules for consent have been studied: simple majority and unanimity. We have shown that the strongly efficient network with the grand coalition is always contractually stable under the unanimity decision rule. We have also characterized the coalition structures that make the strongly efficient network contractually stable under the unanimity decision rule for all component additive value functions sustaining the strongly efficient network. Finally, we have provided some condition that allows us to characterize the coalition structures that sustain some critical network as contractually stable under the simple majority decision rule and under any decision rule requiring the consent of a proportion q of coalition partners. By means of examples, we have shown that requiring the consent of coalition members may help to reconcile stability and efficiency.

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