An unexpected property of tracer transport in a channel flow

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Consider an infinite channel whose cross-sectional area, $S(x)$, depends on the along-channel coordinate *x*. The velocity $U(x)$ is such that the volumetric flow rate,

$$Q = SU , \tag{1}$$

is a constant that can be assumed to be positive without any loss of generality. The diffusivity, *K*(*x*), is positive. The concentration $C(t,x)$ of a passive tracer obeys the equation

$$\frac{\partial(SC)}{\partial t} = -\frac{\partial}{\partial x}\left(QC - SK\frac{\partial C}{\partial x}\right), \tag{2}$$

where *t* denotes the time. At *t*=0, the tracer concentration is zero everywhere, except at *x*=0, where a mass *M* of tracer is released abruptly. Hence, the initial tracer concentration reads

$$C(0,x) = \frac{M\,\delta(x-0)}{\rho S(0)} , \tag{3}$$

where the constant ρ is the fluid density, while δ denotes the Dirac function. As the velocity was assumed to be positive, no tracer particle can reach the upstream "end" of the channel:

$$C(t,-\infty) = 0 . \tag{4}$$

At the opposite "end" of the channel, the concentration must remain finite, implying that its gradient must be zero, i.e.

$$\left[\frac{\partial C}{\partial x}\right]_{x=\infty} = 0 . \tag{5}$$

If the diffusivity is not constant, the partial differential problem (2)-(5) probably admits no analytical solution. However, this does not prevent one from deriving some properties of the tracer transport processes under study. For doing so, it is first necessary to introduce the following variable

$$\xi(x) = \int_0^\infty C(t,x)\,dt . \tag{6}$$

No matter the relative importance of diffusion, the tracer particles eventually are flushed downstream by advection, implying that the following limit must hold true:

$$C(\infty,x) = 0 . \tag{7}$$

Then, by integrating the governing equation over the time and taking into account (2) and (6), one obtains

$$\frac{d}{dx}\left(Q\xi - SK\frac{d\xi}{dx}\right) = \frac{M\,S(x)\,\delta(x-0)}{\rho S(0)} = \frac{M\,\delta(x-0)}{\rho} . \tag{8}$$

The "boundary" conditions necessary to derive the solution of the equation above are obtained by integrating over the time (4) and (5):

$$\xi(-\infty) = 0 , \quad \left[\frac{d\xi}{dx}\right]_{x=\infty} = 0 . \tag{9}$$

The solution of the ordinary differential problem (8)-(9) is

$$\xi(x) = \begin{cases} \dfrac{M}{\rho Q}, & x > 0 \\ \dfrac{M}{\rho Q}\exp\left(-\int_x^0 \dfrac{U(x')\,dx'}{K(x')}\right), & x < 0 \end{cases} \tag{10}$$

As will be seen, this result is conducive to a somewhat unexpected physical interpretation. The amount of tracer that, in the time interval $[0,\infty[$, crosses the section of the channel located at distance x from the point where the tracer was released is

$$\phi(x) = \int_0^\infty \left(\rho QC - \rho SK\frac{\partial C}{\partial x}\right) dt \ . \tag{11}$$

Then, combining (6) and (11) immediately yields

$$\phi(x) = \rho Q\xi - \rho SK\frac{d\xi}{dx} \ . \tag{12}$$

In this relation, the contribution of advective transport, $\phi_a(x)$, and that of diffusion, $\phi_d(x)$, are easily identified, i.e.

$$\phi_a = \rho Q\xi \tag{13}$$

and

$$\phi_d = -\rho SK\frac{d\xi}{dx} \ , \tag{14}$$

with $\phi(x) = \phi_a(x) + \phi_d(x)$. Substituting (10) into (12)-(14) yields the results that are gathered in the table below:

	upstream of the release point: $x<0$	downstream of the release point: $x>0$
advective contribution: $\phi_a(x)$	$\exp\left(-\int_x^0 \frac{U(x')\,dx'}{K(x')}\right)M$	M
advective contribution $\phi_d(x)$	$-\exp\left(-\int_x^0 \frac{U(x')\,dx'}{K(x')}\right)M$	0
total integrated flux: $\phi(x)$	0	M

Consider a channel section located upstream of the release point (x<0). Any tracer particle crossing this section while moving in the upstream direction will at a later time cross it again moving downstream. This is why the total integrated tracer flux is zero for x<0. Downstream of the realease point, the total integrated flux obviously is equal to the mass of tracer released into the channel, M. However, that the diffusive contribution is zero, whatever the diffusivity profile, certainly comes as a surprise.