

Chapter 5

Joint Detection

5.1 Introduction

This chapter is dedicated to the derivation of practical joint detection structures. The matched filter receiver is firstly described in Section 5.2. In Section 5.3, the effect of the selected modulation scheme on the multiuser capacity region is investigated. The matched filter bound (MFB) is derived as an alternative upper bound on the optimal performance that can be expected from real detectors. Infinite linear and decision-feedback (DF) minimum mean square error (MMSE) joint detectors are presented in Section 5.4. Their performance is computed and compared to the multiuser capacity. Their FIR counterparts are derived in Section 5.5, and the computational complexity associated with the coefficient computation is analyzed in detail. The scenario of CAP-CDMA with long codes is analyzed separately in Section 5.6, and reduced-complexity algorithms for the computation of MMSE joint detectors suited to long-code CDMA are proposed. Finally, cyclic-prefixed equalization structures are analyzed in Section 5.7 and performance is compared for different FBs.

5.2 The matched filter receiver

Let us consider a baseband FB-based multiple access system with K users and N signature codes. The continuous-time signal received at the output of the multiple access channel is

$$r(t) = \sum_{m=-\infty}^{+\infty} \sum_{k=1}^K \sum_{p \in \mathcal{C}_k} I_p(m) h_{pk}(t - mT) + n(t) \quad (5.1)$$

where the $h_{pk}(t)$'s denote the equivalent channels including the p -th signature code $s_p(n)$, the pulse shaping filter $p(t)$, the k -th user channel $c_k(t)$, and the receiver filter $f(t)$ (see Chapter 4).

A sufficient statistic of the received signal is obtained by the use of a matched filter bank at the receiver. The outputs of the N matched filters $h_{pk}(-t)$ are sampled at the symbol rate $1/T$, which provides the N sequences :

$$y_p(m) = \int_{-\infty}^{+\infty} r(t) h_{pk}(t - mT). \quad (5.2)$$

This operation can be done by first filtering the received signal with $c_k(-t)$ in the analog domain and sampling the result at the chip rate $1/T_c$:

$$q_k(n) = [r(t) \otimes c_k(-t)](nT_c) \quad (5.3)$$

$$= \sum_{k'=1}^K \sum_{n=-\infty}^{+\infty} v_{k'}(n) g_{kk'}((m-n)T_c) + \nu_k(n) \quad (5.4)$$

where $g_{kk'}(t) = \int_{-\infty}^{+\infty} c_{k'}(\tau) c_k(\tau-t) d\tau$ are the user channel cross-correlation functions and $\nu_k(t) = n(t) \otimes c_k(-t)$ are the filtered noises with covariance $\Gamma_{\nu_k \nu_{k'}}(\tau) = \frac{N_0}{2} g_{kk'}(\tau)$. The N polyphase components of the K filter outputs $q_{\rho k}(n) = q_k(nN + \rho)$ are expressed, with z transforms, as :

$$\begin{pmatrix} \underline{Q}_1(z) \\ \vdots \\ \underline{Q}_K(z) \end{pmatrix} = \begin{pmatrix} \underline{G}_{11}(z) & \cdots & \underline{G}_{1K}(z) \\ \vdots & \ddots & \vdots \\ \underline{G}_{1K}^H(1/z^*) & \cdots & \underline{G}_{KK}(z) \end{pmatrix} \begin{pmatrix} \underline{V}_1(z) \\ \vdots \\ \underline{V}_K(z) \end{pmatrix} + \begin{pmatrix} \underline{U}_1(z) \\ \vdots \\ \underline{U}_K(z) \end{pmatrix} \quad (5.5)$$

The channel correlation polyphase matrices $\underline{G}_{kk'}(z)$ are Toeplitz of size $N \times N$ with element (i, j) given by the z transform of $g(nT + (i-j)T_c)$. They are related to the channel polyphase matrices $\underline{C}_k(z)$ as follows :

$$\underline{G}_{kk}(z) = \frac{T}{M} \underline{C}_k^H(1/z^*) \underline{C}_k(z). \quad (5.6)$$

As $g_{kk}(t)$ are even and real autocorrelation functions, we have the property:

$$\underline{G}_{kk}(z) = \underline{G}_{kk}^H(1/z^*) \quad \forall k \quad (5.7)$$

The noise vector cross-spectra are $\gamma_{\nu_k, \nu_{k'}}(z) = \frac{N_0}{2} \underline{G}_{kk'}(z)$.

The matched filter outputs $y_p(m)$ are finally computed by passing the K sequences $q_k(n)$ through a bank of analysis filters matched to the synthesis bank :

$$y_p(m) = \sum_{n=-\infty}^{+\infty} q_k(n - mN) s_p(n), \quad p \in \mathcal{C}_k \quad (5.8)$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{k=1}^K \sum_{p' \in \mathcal{C}_k} x_{pp'}(m) I_p'(n - m) + w_p(n). \quad (5.9)$$

The N^2 sequences of interference coefficients $x_{pp'}(n)$ are given by

$$x_{pp'}(n) = \int_{-\infty}^{+\infty} h_{pk}(t) h_{p'k'}(t - nT) dt, \quad p \in \mathcal{C}_k, p' \in \mathcal{C}_{k'}. \quad (5.10)$$

The global IIR observation model including the matched filters is given by

$$\begin{pmatrix} \underline{Y}_1(z) \\ \vdots \\ \underline{Y}_K(z) \end{pmatrix} = \begin{pmatrix} \underline{X}_{11}(z) & \cdots & \underline{X}_{1K}(z) \\ \vdots & \ddots & \vdots \\ \underline{X}_{1K}^H(1/z^*) & \cdots & \underline{X}_{KK}(z) \end{pmatrix} \begin{pmatrix} \underline{I}_1(z) \\ \vdots \\ \underline{I}_K(z) \end{pmatrix} + \begin{pmatrix} \underline{W}_1(z) \\ \vdots \\ \underline{W}_K(z) \end{pmatrix} \quad (5.11)$$

or, more compactly,

$$\underline{Y}(z) = \underline{X}(z) \underline{I}(z) + \underline{W}(z). \quad (5.12)$$

The size $N_k \times N_{k'}$ interference matrices $\underline{X}_{kk'}(z)$ are given by

$$\underline{X}_{kk'}(z) = \underline{S}_k^H(1/z^*) \underline{G}_{kk'}(z) \underline{S}_{k'}(z). \quad (5.13)$$

The noise vector cross-spectrum is $\underline{\gamma}_{w,w}(z) = \frac{N_0}{2} \underline{X}(z)$.

Let us now derive a first bound on the joint detector performance, called the *matched filter bound*. This bound is based on the principle that the joint detector cannot do better than removing ISI, ICI and MAI without enhancing the additive noise. These ideal symbol estimates are given by

$$\hat{I}_p(n) = x_{pp}(0) I_p(n) + w_p(n). \quad (5.14)$$

The corresponding signal to noise ratio is

$$\rho_p = \frac{x_{pp}^2(0) \sigma_I^2}{\sigma_{w_p}^2} = \frac{x_{pp}^2(0) \sigma_I^2}{x_{pp}(0) N_0/2} = \frac{2\mathcal{E}_{pk}}{N_0}. \quad (5.15)$$

5.3 Capacity of FB-based multiple access

In this Section, we compute the multiuser capacity of the FB-based multiple access system. This capacity region is smaller than the true capacity region computed in Chapter 3, as additional constraints are introduced : the signals $x_k(t)$ produced by the transmitters are now the outputs of a finite-size FB modulated by mutually independent sequences of information symbols. The power spectral density of the transmitted signals is not arbitrary any more, and the available degrees of freedom depend on both the nature and the size of the selected FB.

5.3.1 Mutual information and capacity region

In order to derive a practical expression of the multiuser capacity region, we firstly propose a slightly modified version of the FB-based multiple-access observation model. The modifications are :

- All K users are assumed to transmit symbols on all N signatures simultaneously. The symbol variance for the k -th user and the n -th signature becomes $\lambda_{pk}^2 \sigma_I^2$. The allocation of a given signature n to a single user k translates into the constraint : $\lambda_{pk} = 1$ and $\lambda_{k'n} = 0$, $\forall k' \neq k$.
- A limited number of N_b symbols per signature are transmitted by the users. This block transmission model actually converges to the initial continuous transmission model for $N_b \rightarrow \infty$.
- The information symbols $I_{pk}(n)$ are assumed to have a Gaussian distribution. This distribution is known to maximize the mutual information between power-constrained transmitted and received signals [52].

Let $\mathcal{S}$ be an arbitrary subset of K_1 users. We denote by $\bar{\mathcal{S}}$ the complementary subset, including $K_2 = K - K_1$ users. The observation model is now

$$\underline{\mathbf{r}}^{N_b} = \sum_{k \in \mathcal{S}} \underline{\mathcal{H}}_k^{N_b} \underline{\mathbf{I}}_k^{N_b} + \sum_{k' \in \bar{\mathcal{S}}} \underline{\mathcal{H}}_{k'}^{N_b} \underline{\mathbf{I}}_{k'}^{N_b} + \underline{\mathbf{n}}^{N_b} \quad (5.16)$$

$$= \underline{\mathcal{H}}_{\mathcal{S}}^{N_b} \underline{\mathbf{I}}_{\mathcal{S}}^{N_b} + \underline{\mathcal{H}}_{\bar{\mathcal{S}}}^{N_b} \underline{\mathbf{I}}_{\bar{\mathcal{S}}}^{N_b} + \underline{\mathbf{n}}^{N_b} \quad (5.17)$$

$$= \underline{\mathbf{r}}_1^{N_b} + \underline{\mathbf{r}}_2^{N_b} + \underline{\mathbf{n}}^{N_b} \quad (5.18)$$

where $\underline{\mathbf{I}}_k^{N_b}$ is the vector of $N_b N$ symbols transmitted by user k , $\underline{\mathbf{r}}^{N_b}$ and $\underline{\mathbf{n}}^{N_b}$ are of size $(N_b + L)M$, and $\underline{\mathbf{H}}_k^{N_b}$ is a size $(N_b + L)M \times N_b N$ block-Toeplitz matrix. The continuous transmission model, obtained by letting N_b go to infinity, is given by

$$\underline{\mathbf{R}}(z) = \underline{\mathbf{H}}_{\mathcal{S}}(z) \underline{\mathbf{I}}_{\mathcal{S}}(z) + \underline{\mathbf{H}}_{\overline{\mathcal{S}}}(z) \underline{\mathbf{I}}_{\overline{\mathcal{S}}}(z) + \underline{\mathbf{N}}(z) \quad (5.19)$$

where $\underline{\mathbf{H}}_{\mathcal{S}}(z)$ and $\underline{\mathbf{H}}_{\overline{\mathcal{S}}}(z)$ are the $M \times K_1 N$ and $M \times K_2 N$ polyphase matrices associated with the composite channels $h_{pk}(t)$.

For a fixed block size N_b and a fixed power allocation defined by the set of coefficients $\{\lambda_{pk}^2\}$, the set of achievable data rates is known to be :

$$\mathcal{C}_{N_b}(\lambda_{pk}^2) = \left\{ \{R_k\} : 0 \leq \sum_{k \in \mathcal{S}} R_k \leq \frac{1}{N_b} \mathcal{I}(\underline{\mathbf{r}}^{N_b}; \underline{\mathbf{I}}_{\mathcal{S}}^{N_b} | \underline{\mathbf{I}}_{\overline{\mathcal{S}}}^{N_b}) \quad \forall \mathcal{S} \subset \{1, \dots, K\} \right\} \quad (5.20)$$

where $\mathcal{I}(\underline{\mathbf{x}}; \underline{\mathbf{y}} | \underline{\mathbf{z}})$ denotes the conditional mutual information between the vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$, for a given vector $\underline{\mathbf{z}}$. This region is a K -dimensional polyhedron defined by $2^K - 1$ constraints, and including $K!$ vertices in the positive quadrant. The global capacity region is then obtained by

$$\mathcal{C} = \text{closure} \left(\lim_{N_b \rightarrow \infty} \left[\bigcup_{\lambda_{pk}^2} \mathcal{C}_{N_b}(\lambda_{pk}^2) \right] \right). \quad (5.21)$$

The union is taken over all admissible input-power distributions, i.e. all admissible sets of coefficients λ_{pk}^2 .

Let us firstly derive the expression of the mutual information $\mathcal{I}(\underline{\mathbf{r}}^{N_b}; \underline{\mathbf{I}}_{\mathcal{S}}^{N_b} | \underline{\mathbf{I}}_{\overline{\mathcal{S}}}^{N_b})$ from the observation model given by (5.16) :

$$\mathcal{I}(\underline{\mathbf{r}}^{N_b}; \underline{\mathbf{I}}_{\mathcal{S}}^{N_b} | \underline{\mathbf{I}}_{\overline{\mathcal{S}}}^{N_b}) = \mathbf{h}(\underline{\mathbf{r}}^{N_b} | \underline{\mathbf{I}}_{\mathcal{S}}^{N_b}) - \mathbf{h}(\underline{\mathbf{r}}^{N_b} | \underline{\mathbf{I}}_{\mathcal{S}}^{N_b}, \underline{\mathbf{I}}_{\overline{\mathcal{S}}}^{N_b}) \quad (5.22)$$

$$= \mathbf{h}(\underline{\mathbf{r}}_1^{N_b} + \underline{\mathbf{n}}^{N_b}) - \mathbf{h}(\underline{\mathbf{n}}^{N_b}) \quad (5.23)$$

where $\mathbf{h}(\underline{\mathbf{x}})$ denotes the differential entropy of the continuous random vector $\underline{\mathbf{x}}$ and $\mathbf{h}(\underline{\mathbf{x}} | \underline{\mathbf{y}})$ denotes the average conditional entropy of $\underline{\mathbf{x}}$ given $\underline{\mathbf{y}}$. The differential entropy of a size- N zero-mean Gaussian vector $\underline{\mathbf{x}} \sim \mathcal{N}(\underline{\mathbf{0}}, \underline{\mathbf{R}}_{xx})$ is known to be

$$\mathbf{h}(\underline{\mathbf{x}}) = \log \left[(\pi e)^N \det(\underline{\mathbf{R}}_{xx}) \right]. \quad (5.24)$$

Introducing this result into (5.22), we get

$$\mathcal{I}(\mathbf{r}; \mathbf{I}_S | \mathbf{I}_{\bar{S}}) = \log \left[\frac{\det(\underline{\mathbf{R}}_{r_1 r_1} + \underline{\mathbf{R}}_{nn})}{\det(\underline{\mathbf{R}}_{nn})} \right] \quad (5.25)$$

$$\begin{aligned} &= \log \left[\det \left(\underline{\mathbf{E}}_{M(N_b+L)} + \frac{\sigma_I^2}{\sigma_n^2} \underline{\mathbf{H}}_S^{N_b} \left(\underline{\Lambda}_S^{N_b} \right)^2 (\underline{\mathbf{H}}_S^{N_b})^T \right) \right] \\ &= \log \left[\det \left(\underline{\mathbf{E}}_{N_b K_1 N} + \frac{\sigma_I^2}{\sigma_n^2} \underline{\Lambda}_S^{N_b} (\underline{\mathbf{H}}_S^{N_b})^T \underline{\mathbf{H}}_S^{N_b} \underline{\Lambda}_S^{N_b} \right) \right] \end{aligned} \quad (5.26)$$

where the diagonal matrix $\underline{\Lambda}_S^{N_b}$ is defined by

$$E \left\{ \underline{\mathbf{I}}_S^{N_b} (\underline{\mathbf{I}}_S^{N_b})^T \right\} = \sigma_I^2 \left(\underline{\Lambda}_S^{N_b} \right)^2. \quad (5.27)$$

Using the asymptotic properties of Toeplitz matrices, it can be demonstrated [57] that the average mutual information in (5.26) becomes

$$\bar{\mathcal{I}}_S = \lim_{N_b \rightarrow \infty} \frac{1}{N_b} \mathcal{I}(\mathbf{r}; \mathbf{I}_S | \mathbf{I}_{\bar{S}}) \quad (5.28)$$

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\mathbf{E}}_{K_1 N} + \frac{\sigma_I^2}{\sigma_n^2} \underline{\Lambda}_S \underline{\mathbf{H}}_S^H (e^{j\Omega}) \underline{\mathbf{H}}_S (e^{j\Omega}) \underline{\Lambda}_S \right) \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\mathbf{E}}_{K_1 N} + \frac{\sigma_I^2}{N_0/2} \underline{\Lambda}_S \underline{\mathbf{X}}_S (e^{j\Omega}) \underline{\Lambda}_S \right) \right] d\Omega \end{aligned} \quad (5.29)$$

when the burst size goes to infinity. The demonstration is not given here, but can be found in [55]. It is based on the observation that the determinant of the matrix in (5.26) is equal to the product of the $N_b K_1 N$ eigenvalues $(1 + \frac{\sigma_I^2}{\sigma_n^2} \rho_i)$ of this matrix. When N_b goes to infinity, the eigenvalues ρ_i converge to those of an equivalent circulant matrix.

The result in (5.28) can be further developed to underline the role of the signature allocation coefficients λ_{pk}^2 :

$$\begin{aligned} \bar{\mathcal{I}}_S &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\mathbf{E}}_M + \frac{\sigma_I^2}{\sigma_n^2} \underline{\mathbf{H}}_S (e^{j\Omega}) \underline{\Lambda}_S^2 \underline{\mathbf{H}}_S^H (e^{j\Omega}) \right) \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\mathbf{E}}_M + \frac{1}{\sigma_n^2} \sum_{k \in \mathcal{S}} \underline{\mathbf{C}}_k (e^{j\Omega}) \gamma_{v_k v_k} (e^{j\Omega}) \underline{\mathbf{C}}_k^H (e^{j\Omega}) \right) \right] d\Omega \end{aligned} \quad (5.30)$$

where the polyphase spectrum $\gamma_{\underline{v}_k \underline{v}_k}(z)$ associated with the chip-rate sequence $v_k(n)$ produced by user k is

$$\begin{aligned} \gamma_{\underline{v}_k \underline{v}_k}(e^{j\Omega}) &= \sigma_I^2 \underline{S}(e^{j\Omega}) \underline{\Lambda}_k^2 \underline{S}^H(e^{j\Omega}) \\ &= \sigma_I^2 \sum_{p=1}^N \lambda_{pk}^2 \underline{S}_p(e^{j\Omega}) \underline{S}_p^H(e^{j\Omega}). \end{aligned} \quad (5.31)$$

For a fixed signature allocation $\{\lambda_{pk}^2\}$, the capacity region is a polyhedron defined by the $2^K - 1$ constraints $\bar{\mathcal{I}}_{\mathcal{S}}$. With K users and N signatures, there are K^N possible polyhedrons, corresponding to the K^N distinct signature allocations. Considering the restrictive two-user scenario, the polyhedron is a pentagon defined by

$$\mathcal{C}_{\infty}(\lambda_{pk}^2) = (R_1, R_2) : \begin{cases} 0 \leq R_1 \leq \bar{\mathcal{I}}_{\{1\}} \\ 0 \leq R_2 \leq \bar{\mathcal{I}}_{\{2\}} \\ 0 \leq R_1 + R_2 \leq \bar{\mathcal{I}}_{\{1,2\}} \end{cases} \quad (5.32)$$

Figure 5.1 illustrates the $2^4 = 16$ regions obtained with $N = 4$ Hadamard signature codes of length 4. Only 14 of these regions are really pentagons. The two remaining ones are degenerated to vertical or horizontal segments : they correspond to the allocation of all signatures to the same user. As given by equation (5.21), the global capacity region is obtained by taking the convex closure of the union of the pentagons. The boundary of the resulting capacity region is given by the dashed line in Figure 5.1. A given rate pair (R_1, R_2) of the boundary is generally achieved by timesharing between two distinct signature allocations.

5.3.2 Optimal signature allocation

The computation of an optimal signature allocation scheme in terms of achievable data rates belongs to the class of integer programming problems. The problem can be considerably simplified by considering its continuous relaxation. A similar strategy was adopted in Chapter 3 to compute the FDMA capacity of frequency selective multiuser channels. The idea, here, is to allow all users to use all signatures simultaneously. The constraint on the power allocation coefficients λ_{pk}^2 becomes

$$0 \leq \lambda_{pk}^2 \leq 1 \quad \sum_{k=1}^K \lambda_{pk}^2 = 1. \quad (5.33)$$

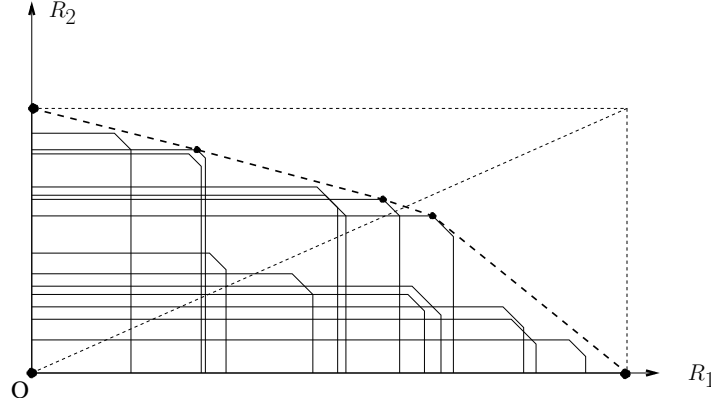


Figure 5.1: Two-user capacity region for a size-4 FB-based transmitter (Hadamard signature codes)

With the relaxed constraints, the new capacity region is generated by the union of an infinite number of polyhedrons. Moreover, as every convex combination of feasible signature allocations is also a feasible signature allocation, *the convex closure operation is not needed any more* to generate the global capacity region. This implies that every point of the global capacity region can be achieved with a fixed signature allocation.

The constraints in (5.33) implicitly assume that the same total power is transmitted on each signature code. The consequence for the polyphase spectra $\gamma_{\underline{v}_k \underline{v}_k}(z)$ is

$$\sum_{k=1}^K \gamma_{\underline{v}_k \underline{v}_k}(z) = \sigma_I^2 \underline{\mathbf{S}}(z) \left(\sum_{k=1}^K \underline{\mathbf{A}}_k \right) \underline{\mathbf{S}}^H(1/z^*) \quad (5.34)$$

$$\leq \sigma_I^2 \underline{\mathbf{S}}(z) \underline{\mathbf{S}}^H(1/z^*) = \sigma_I^2 \underline{\mathbf{E}}_N \quad (5.35)$$

where the inequality of matrices in (5.34) means that the matrix difference is non-negative definite. This is equivalent to the *PSD-sum constraint* introduced in Chapter 3. Alternative constraints could be investigated like the power sum constraint (obtained with $\sigma_I^2 \sum_k \sum_p \lambda_{pk}^2 \leq \bar{P}$) or the individual power constraint (obtained with $\sigma_I^2 \sum_p \lambda_{pk}^2 \leq \bar{P}_k \forall k$), but are beyond the scope of this work.

Like for the true capacity region, the boundary of the FB-based capacity region can be traced out by maximizing the aggregate rate $R_\alpha = \sum_k \alpha_k R_k$

for a given set of relative priorities $\{\alpha_k\}$. The maximum aggregate rate is known to be obtained with successive decoding at the receiver, and a decoding order in accordance with the user relative priorities. Assuming that the users are ordered in ascending order of the relative priorities, the individual data rates are given by

$$R_k = \bar{\mathcal{I}}_{\{k, \dots, K\}} - \bar{\mathcal{I}}_{\{k+1, \dots, K\}} \quad \forall k < K \quad (5.36)$$

$$R_K = \bar{\mathcal{I}}_{\{K\}}. \quad (5.37)$$

The aggregate rate is thus

$$R_\alpha = \sum_{k=1}^K \alpha_k R_k = \sum_{k=1}^K (\alpha_k - \alpha_{k-1}) \bar{\mathcal{I}}_{\{k, \dots, K\}}. \quad (5.38)$$

The quantities $\bar{\mathcal{I}}_{\mathcal{S}}$ in last equation are given by equation (5.30) and depend on the signature allocation. The computation of the optimal signature allocation scheme λ_{pk}^2 is then formulated as follows :

$$\{\lambda_{pk}^2\} = \arg \max_{\lambda_{pk}^2} [R_\alpha(\{\lambda_{pk}^2\})]. \quad (5.39)$$

It can be demonstrated that the objective function (5.38) is a concave function in the parameters λ_{pk}^2 . This property, associated with the linear constraints in (5.33), guarantees that the optimization problem in (5.39) is a standard convex programming problem : a local maximum is also a global maximum, and standard numerical search algorithms can be used to find the solution efficiently.

5.3.3 Maximum balanced rates

One may be interested in a specific point of the capacity region, like the balanced rate solution defined in Chapter 3. In that case, the optimization problem in (5.39) has to be solved iteratively for different sets of relative priorities $\{\alpha_k^{(i)}\}$, until the solution satisfies the balanced rate constraint. Actually, two major difficulties have to be taken into account in the computation of the maximum balanced rates associated with a FB-based system.

Firstly, there is no closed form solution to the problem of optimal signature allocation for a given set of relative priorities. At each iteration i , a new convex programming problem has to be solved.

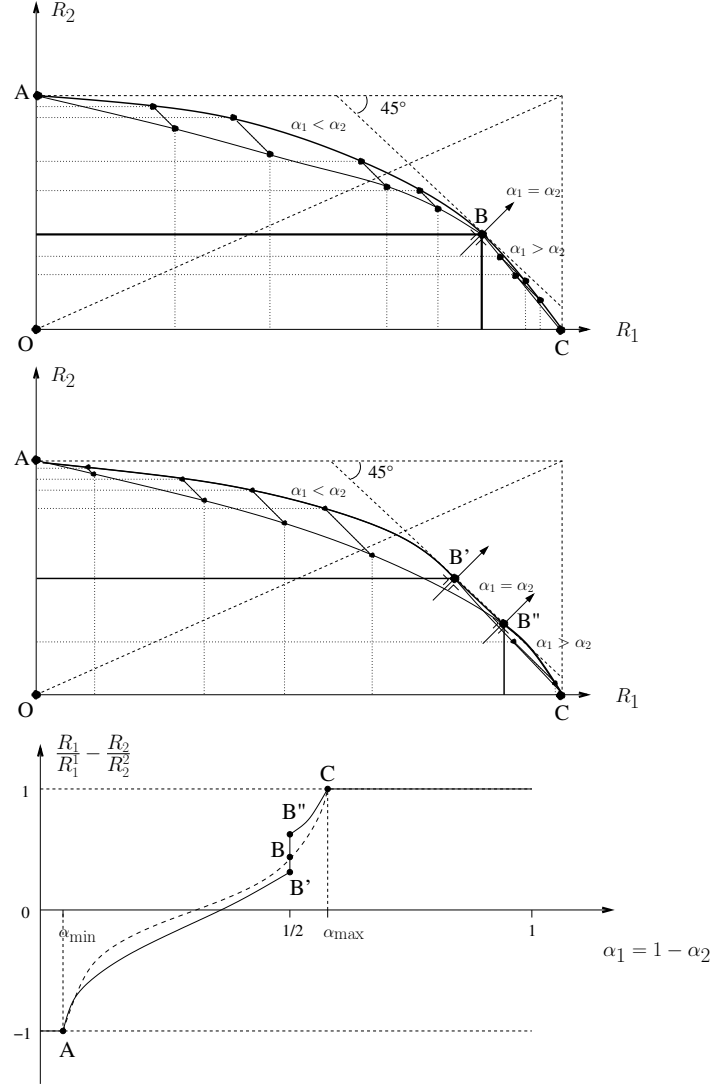


Figure 5.2: Generation of the true capacity region and the FB-constrained capacity region by the union of pentagons

Secondly, the solution proposed in (5.36) is restricted to be a vertex of the K -dimensional polyhedron corresponding to the selected signature allocation. The vertex is chosen in accordance with the ordering of the relative priorities α_k . The boundary of the true capacity region is effectively generated by the union of such vertices. In the case of a FB-constrained capacity region, however, some parts of the boundary are not covered by the union of these vertices. Figure 5.2 illustrates this difference in a two-user case. The first graph gives the true capacity region (as considered in Chapter 3). It is a convex region generated by the union of pentagons. Each pentagon has a lower and an upper vertex. The boundary of the capacity region is the curve ABC. The segment AB of the curve is generated by upper vertices, which means that the decoding order is (1, 2). All points of this segment can be obtained by maximizing the aggregate rate R_α with $\alpha_1 < \alpha_2$. The segment BC of the curve is generated by lower vertices, which means that the decoding order is (2, 1). All points of this segment can be obtained by maximizing the aggregate rate R_α with $\alpha_1 > \alpha_2$. Finally, point B corresponds to the maximum sum rate : it is obtained by maximizing the aggregate rate R_α with $\alpha_1 = \alpha_2$. As the maximum sum rate solution is of the FDMA type, the associated pentagon is degenerated into a rectangle. Indeed, the power is allocated to the users in separate frequency bands, and successive decoding is not needed any more. The second graph gives the corresponding FB-based capacity region. It is also a convex region generated by the union of pentagons. But now the maximum sum rate solution ($\alpha_1 = \alpha_2$) is also a pentagon, with distinct vertices B' and B". Indeed, under the FB constraint, an FDMA power allocation is not feasible any more (because of the spectral overlapping among the different signatures). As a consequence, the whole segment BB' belongs to the boundary of the capacity region, but only points B' and B" are pentagon vertices. The intermediate solutions are obtained by timesharing between the two possible decoding orders, and the corresponding rate pairs (R_1, R_2) are not given by the general solution in (5.36). The third graph gives the weighted rate imbalance $R_1/R_1^1 - R_2/R_2^1$ as a function of $\alpha_1 = 1 - \alpha_2$. The dashed line corresponds to the true capacity region, while the continuous line corresponds to the FB-based capacity region. The maximum balanced rates solution corresponds to the root of this function. If the balanced rate solution is located in one of the intervals AB' or B"C, then the root can be found by searching through the intervals $\alpha_{\min} < \alpha_1 < 1/2$ and $1/2 < \alpha_1 < \alpha_{\max}$, respectively. This case corresponds to the displayed figure. It could happen, however, that the balanced rate solution be located in the interval

B'B" : then the search algorithm would fail to find the root of the function. This problem can be handled easily in the restrictive two-user case, but it is not tractable in the general K -user scenario.

5.3.4 Influence of the FB

We firstly investigate the effect of the FB choice on the SU capacity. In a SU scenario, all signatures can be fully allocated to the same user, i.e. $\lambda_{1p}^2 = 1 \ \forall p$. With a paraunitary FB, this implies that the transmitted polyphase spectrum $\gamma_{\underline{v}_1 \underline{v}_1}(z)$ is

$$\gamma_{\underline{v}_1 \underline{v}_1}(z) = \sigma_I^2 \underline{S}(z) \underline{A}_1^2 \underline{S}^H(z) = \sigma_I^2 \underline{E}_N. \quad (5.40)$$

The SU capacity is then given by

$$\begin{aligned} \bar{\mathcal{I}}_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{E}_M + \frac{\sigma_I^2}{\sigma_n^2} \underline{C}_1(e^{j\Omega}) \underline{C}_1^H(e^{j\Omega}) \right) \right] d\Omega \\ &= \int_{-\infty}^{\infty} \log \left(1 + \frac{\sigma_I^2}{N_0/2} |C_1(f)|^2 \right) df, \end{aligned} \quad (5.41)$$

irrespectively of the selected FB. In particular, the SU FB-based capacity is identical to that of the single carrier modulation obtained by letting $\underline{S}(z) = \underline{E}_N$. As a consequence, it is also independent of N , as the SU single carrier modulation can obviously be represented with a unit-matrix FB of arbitrary size. The single-user FB-based capacity is thus independent of the *choice of signatures* and of the *number of signatures*, provided that the associated FB is paraunitary. The fundamental reason for this is that any paraunitary FB of any size produces the same flat power spectral density at its output when it is modulated by independent sequences of equal variance.

The boundary of the TDMA capacity region, i.e. the hyperplane joining the SU rate solutions, is then also independent of the FB. Consequently, the relative performance of different FBs can be compared adequately through the *multiuser gain* defined in Chapter 3 as the ratio of the maximum balanced rates to the TDMA balanced rates. The multiuser gain is different for every FB. Actually, it depends on the ability of the FB to model the power spectral density of the transmitted signals with as much flexibility as possible. This flexibility is limited by the N degrees of freedom corresponding to the N different signatures.

An upper bound on the FB-constrained capacity region is the (PSD-sum constrained) true capacity region presented in Chapter 3. In the two-user

case, its boundary is represented by the line AEB in Figure 3.6. This upper bound can be approached with a frequency-selective FB of size $N \gg 1$, i.e. with a FB able to shape the transmission PSD of each user with an arbitrary flexibility.

On the opposite, a lower bound on the FB-constrained capacity region is obtained by forcing, for each user, the allocation of a uniform power to all signatures :

$$\lambda_{pk}^2 = \lambda_k^2 \quad \forall p \quad \sum_{k=1}^K \lambda_k^2 = 1. \quad (5.42)$$

With this signature allocation, all users are forced to transmit a flat-PSD signal. The corresponding capacity region is independent of the FB as (5.30) becomes

$$\begin{aligned} \bar{\mathcal{I}}_{\mathcal{S}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\underline{\mathbf{E}}}_M + \frac{\sigma_I^2}{\sigma_n^2} \sum_{k \in \mathcal{S}} \lambda_k^2 \underline{\underline{\mathbf{C}}}_k(e^{j\Omega}) \underline{\underline{\mathbf{C}}}_k^H(e^{j\Omega}) \right) \right] d\Omega \\ &= \int_{-\infty}^{\infty} \log \left(1 + \frac{\sigma_I^2}{N_0/2} \sum_{k \in \mathcal{S}} \lambda_k^2 |C_k(f)|^2 \right) df. \end{aligned} \quad (5.43)$$

This is also the capacity region corresponding to a degenerated FB with $N = 1$ signature, i.e. a FB with the lowest flexibility as possible. In the two-user case, the boundary of this region is represented by the line ADB in Figure 3.6. The same capacity region can be expected from the long-code multiple access system introduced in Section 4.5, as the variability of the signature codes used by each user makes the PSD of the transmitted signals almost uniform.

5.3.5 Capacity vs. MFB

In this section, we show how the SINR bound provided by the MFB can be used to provide a simple upper bound on the multiuser capacity. Equation

(5.28) can be rewritten as

$$\begin{aligned}
\bar{\mathcal{I}}_{\mathcal{S}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\mathbf{E}}_{K_1 N} + \frac{\sigma_I^2}{N_0/2} \underline{\mathbf{\Lambda}}_{\mathcal{S}} \underline{\mathbf{X}}_{\mathcal{S}} (e^{j\Omega}) \underline{\mathbf{\Lambda}}_{\mathcal{S}} \right) \right] d\Omega \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\prod_{i=1}^{K_1 N} \left(\underline{\mathbf{E}}_{K_1 N} + \frac{\sigma_I^2}{N_0/2} \underline{\mathbf{\Lambda}}_{\mathcal{S}} \underline{\mathbf{X}}_{\mathcal{S}} (e^{j\Omega}) \underline{\mathbf{\Lambda}}_{\mathcal{S}} \right)_{ii} \right] d\Omega \\
&= \frac{1}{2\pi} \sum_{k \in \mathcal{S}} \sum_{p=1}^N \int_{-\pi}^{\pi} \log \left[1 + \frac{\sigma_I^2}{N_0/2} \lambda_{pk}^2 \left(\underline{\mathbf{X}}_k \right)_{pp} \right] d\Omega \\
&\leq \sum_{k \in \mathcal{S}} \sum_{p=1}^N \log \left[1 + \frac{\sigma_I^2}{N_0/2} \lambda_{pk}^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\underline{\mathbf{X}}_k \right)_{pp} d\Omega \right] \\
&= \sum_{k \in \mathcal{S}} \sum_{p=1}^N \log \left[1 + \frac{2\mathcal{E}_{pk}}{N_0} \right]. \tag{5.44}
\end{aligned}$$

The first inequality follows from the property that the determinant of a symmetric positive definite matrix is always smaller than the product of its diagonal values. For the second one, it is known that for any positive function $f(x)$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log [f(x)] dx \leq \log \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \right] \tag{5.45}$$

with equality if and only if $f(x)$ is constant. This result shows that, under some very restrictive conditions, the multiuser capacity is simply the sum of the single channel capacities associated with every signature p and every user k , which only depend on the ratios $\mathcal{E}_{pk}/N_0$. Actually, the proposed bound is really valuable if all the following conditions are satisfied :

- Every signature is allocated to a single user.
- The inter-signature orthogonality is not destroyed by the multiuser channel selectivity. This should be valid for any multiuser channel. This condition implies that all signatures should be located in separate frequency bands. A frequency selective FB with non-overlapping subbands should be used for that purpose.
- Each subchannel is sufficiently flat, so that the SINR in every subchannel is roughly equal to the channel SNR at the corresponding frequency. As a consequence, the size N of the frequency-selective FB must be very large.

- The SNR gap Γ is equal to 1, i.e. perfect coding is assumed in every separate subchannel.

A simple signature allocation algorithm can be implemented on the basis of the MFB. The MFB-based aggregate rate $\bar{R}_\alpha$ can be written as

$$\bar{R}_\alpha = \sum_{p=1}^N \sum_{k=1}^K \lambda_{pk}^2 (\alpha_k \bar{R}_{pk}), \quad \text{with } \bar{R}_{pk} \triangleq \mathcal{C} \left(\frac{2\mathcal{E}_{pk}}{N_0} \right) \quad (5.46)$$

with a single $\lambda_{pk}^2 = 1$ and the others equal to 0, for every p . Let us define the following continuous relaxation of the signature allocation constraint :

$$0 \leq \lambda_{pk}^2 \leq 1, \quad \sum_{k=1}^K \lambda_{pk}^2 = 1. \quad (5.47)$$

For equation (5.46) to remain valid, the coefficients λ_{pk}^2 must be re-interpreted as time-sharing coefficients, and not as power allocation coefficients as in the previous Sections. The maximization of the aggregate rate R_α in (5.46) under the relaxed constraints (5.47) is identical to the FDMA subchannel allocation problem analyzed in Section 3.6.1. The solution is given by Algorithm 3.5. The computation of maximum balanced rates, that requires the iterative computation of adequate relative priorities α_k , is obtained by Algorithm 3.6. Linear optimization theory tells us that at least one solution exists that comprises at most $N + (K - 1)$ non-zero components λ_{pk}^2 . This implies that a maximum of $K - 1$ signatures will be shared between couples of users, or alternatively a single signature could be shared between all of the K users, all other signatures being allocated to a single user. Assuming that $N \gg K$, we can see that time-sharing only concerns a small fraction of the available signatures. If time-sharing is not allowed, we have to approximate the continuous solution given by Algorithm 3.6 and force the allocation of each signature to a single user. It results that the balanced rates constraints are not exactly satisfied any more. A measure of the deviation from these conditions is given through the 'user imbalance index' defined as follows, together with a practical upper bound :

$$\Delta_{\text{users}} = \frac{\sigma_k \left(\frac{R_k}{\beta_k} \right)}{\bar{m}_k \left(\frac{R_k}{\beta_k} \right)} \leq \frac{K}{2N} \sqrt{\frac{1}{K^3} \sum_k \left(\frac{1}{\alpha_k} \right)^2} \quad (5.48)$$

where $\sigma_k(\cdot)$ and $\bar{m}_k(\cdot)$ stand for standard deviation and mean across the users respectively. Equation (5.48) shows that the rounding effect associated with a discrete signature allocation is negligible if the ratio N/K

is large. With a large number of signatures, indeed, the boundary of the capacity region can be covered with a better granularity.

The MFB-based signature allocation is suboptimal in the sense that the inter-signature interference is ignored. However, it provides a useful initial allocation for more advanced allocation algorithms as it ensures the best matching as possible between the set of available signatures and the user channel attenuations and noises. The method can be generalized to all situations where a K -user channel can be modeled as a parallel transmission of information in N separate and independent subchannels.

5.4 IIR joint detectors

In the case of a frequency-selective channel, the symbol estimates obtained at the output of the matched filter bank are severely corrupted by interference. A popular solution consists in the use of a MIMO equalizer that intends to compute more reliable estimates of the symbols on the basis of a minimum mean square error criterion (MMSE).

In this section we investigate the structure of IIR linear or decision-feedback (DF) joint detectors designed for the MMSE criterion. These structures are the best possible linear and DF joint detectors, and their corresponding performance provides an upper bound of the performance which can be achieved by their FIR counterparts. This is the main motivation to investigate these IIR detectors. Their FIR counterparts will be investigated in the following section.

For all continuous transmission schemes presented in Chapter 4, a generic IIR observation model was defined as follows (See (4.42), (4.67) and (4.79)) :

$$\underline{\mathbf{R}}(z) = \underline{\mathbf{H}}(z) \underline{\mathbf{I}}(z) + \underline{\mathbf{N}}(z). \quad (5.49)$$

The IIR joint detectors derived in this Section rely on this generic observation model. The baseband FB transmission scheme is assumed in the next derivations, but the application to CAP-FB transmission or CAP-CDMA with long codes is straightforward.

5.4.1 MMSE linear joint detection

A fractionally spaced linear joint detector (JD) builds estimates of the N transmitted symbol sequences $I_p(n)$ as follows :

$$\hat{\underline{\mathbf{I}}}(z) = \underline{\underline{\mathbf{W}}}(z) \underline{\mathbf{R}}(z) \quad (5.50)$$

where $\underline{\underline{W}}(z)$ is a bank of $N \times M$ infinite length filters. The vector of z -transformed estimation errors is defined by :

$$\underline{\underline{\epsilon}}(z) = \underline{\underline{I}}(z) - \hat{\underline{\underline{I}}}(z). \quad (5.51)$$

This IIR linear JD can be designed for an MMSE criterion : the coefficients of the filters $\underline{\underline{W}}(z)$ are then computed in order to minimize the variance of the estimation errors. The orthogonality principle can be used to derive the optimal estimator. It states that the error associated with the optimal estimator is orthogonal to the data samples used to build the estimate. This property can be written as follows :

$$\underline{\underline{\gamma}}_{\underline{\underline{\epsilon}}\underline{\underline{R}}}(z) = \underline{\underline{0}}_{K \times M} \quad (5.52)$$

where $\underline{\underline{\gamma}}_{\underline{\underline{x}}\underline{\underline{y}}}$ stands for the cross-spectrum between $\underline{\underline{x}}(z)$ and $\underline{\underline{y}}(z)$. Using the orthogonality principle (5.52), the observation model (5.49), the estimation model (5.50) and the error definition (5.51), we get

$$\underline{\underline{W}}(z) = \underline{\underline{\gamma}}_{\underline{\underline{I}}\underline{\underline{R}}}(z) \underline{\underline{\gamma}}_{\underline{\underline{R}}\underline{\underline{R}}}^{-1}(z) \quad (5.53)$$

$$= \sigma_I^2 \underline{\underline{H}}^H(1/z^*) \left[\sigma_n^2 \underline{\underline{E}}_M + \sigma_I^2 \underline{\underline{H}}(z) \underline{\underline{H}}^H(1/z^*) \right]^{-1} \quad (5.54)$$

$$= \left[\sigma_I^{-2} \underline{\underline{E}}_N + \sigma_n^{-2} \underline{\underline{H}}^H(1/z^*) \underline{\underline{H}}(z) \right]^{-1} \underline{\underline{H}}^H(1/z^*) \sigma_n^{-2}. \quad (5.55)$$

The last expression is obtained thanks to the matrix inversion lemma.¹ Equation (5.55) shows that the first operation performed by the equalizer is to apply a bank of matched filters whose outputs are down-sampled to the symbol rate $1/T$. This operation is achieved by $\underline{\underline{H}}^H(1/z^*)$. Then the symbol MIMO (multi-input/multi-output) equalization is applied. It has a MIMO transfer function given by the inverse of the key power spectrum:

$$\underline{\underline{S}}_c(z) = \left[\sigma_I^{-2} \underline{\underline{E}}_N + \sigma_n^{-2} \underline{\underline{H}}^H(1/z^*) \underline{\underline{H}}(z) \right]. \quad (5.56)$$

¹Let $\underline{\underline{A}}$ and $\underline{\underline{B}}$ be square positive definite matrices of size $n_1 \times n_1$, $\underline{\underline{D}}$ be a square positive definite matrix of size $n_2 \times n_2$ and $\underline{\underline{C}}, \underline{\underline{F}}$ be arbitrary rectangular matrices of size $n_1 \times n_2$, with

$$\underline{\underline{A}} = \underline{\underline{B}}^{-1} + \underline{\underline{C}} \underline{\underline{D}}^{-1} \underline{\underline{F}}^H.$$

The matrix inversion lemma states that

$$\underline{\underline{A}}^{-1} = \underline{\underline{B}} - \underline{\underline{B}} \underline{\underline{C}} (\underline{\underline{D}} + \underline{\underline{F}}^H \underline{\underline{B}} \underline{\underline{C}})^{-1} \underline{\underline{F}}^H \underline{\underline{B}}$$

With the detector obtained in (5.55), the symbol estimates (5.50) become

$$\hat{\underline{\mathbf{I}}}(z) = \left(\underline{\mathbf{E}}_N - \underline{\mathbf{S}}_c^{-1}(z) \sigma_I^{-2} \right) \underline{\mathbf{I}}(z) + \left(\underline{\mathbf{S}}_c^{-1}(z) \sigma_n^{-2} \underline{\mathbf{H}}^H(1/z^*) \right) \underline{\mathbf{N}}(z). \quad (5.57)$$

For the estimation errors after optimal linear joint detection we have

$$\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(z) = \underline{\mathbf{S}}_c^{-1}(z) = \sigma^2 \underline{\mathbf{E}}_N - \underline{\mathbf{W}}(z) \underline{\gamma}_{\underline{\mathbf{R}}\underline{\mathbf{R}}}(z) \underline{\mathbf{W}}^H\left(\frac{1}{z^*}\right). \quad (5.58)$$

The mean square estimation errors are provided by the diagonal elements of the order 0 matrix in the expansion of $\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(z)$ as a z polynomial :

$$\underline{\Gamma}_{\underline{\epsilon}\underline{\epsilon}}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(e^{j\Omega}) d\Omega. \quad (5.59)$$

5.4.2 MMSE decision-feedback joint detection

The DF joint detector is made of a $(N \times M)$ bank of forward filters, denoted by $\underline{\mathbf{W}}(z)$, and of a causal feedback bank of $N \times N$ filters, denoted by $\underline{\mathbf{B}}(z) - \underline{\mathbf{E}}_N$, and builds an estimate $\hat{\underline{\mathbf{I}}}(z)$ in the following way:

$$\hat{\underline{\mathbf{I}}}(z) = \underline{\mathbf{W}}(z) \underline{\mathbf{R}}(z) - \left[\underline{\mathbf{B}}(z) - \underline{\mathbf{E}}_N \right] \tilde{\underline{\mathbf{I}}}(z). \quad (5.60)$$

The feedback section processes previous decisions $\tilde{\underline{\mathbf{I}}}(z)$. However we assume that they are correct and hence use the original symbols $\underline{\mathbf{I}}(z)$ in the next derivations. The filter bank $\underline{\mathbf{B}}(z)$ is stable, causal and monic. It means that

$$\underline{\mathbf{B}}(z) = \sum_{n \geq 0} \underline{\mathbf{B}}(n) z^{-n} \quad (5.61)$$

where $\underline{\mathbf{B}}(0) - \underline{\mathbf{E}}_N$ is a lower triangular matrix with zeros on the main diagonal. The triangular nature of the order 0 feedback matrix means that each time a new decision is available, it is used for the next estimate to be computed.

If this IIR JD with a causal IIR feedback section is designed for an MMSE criterion, we can again use the orthogonality principle (5.52). With the observation model (5.49), the estimation model (5.60) and the error definition (5.51), it gives

$$\underline{\mathbf{W}}(z) = \underline{\mathbf{B}}(z) \underline{\gamma}_{\underline{\mathbf{I}}\underline{\mathbf{R}}}(z) \underline{\gamma}_{\underline{\mathbf{R}}\underline{\mathbf{R}}}^{-1}(z) \quad (5.62)$$

$$= \underline{\mathbf{B}}(z) \sigma_I^2 \underline{\mathbf{H}}^H(1/z^*) \left[\sigma_n^2 \underline{\mathbf{E}}_M + \sigma_I^2 \underline{\mathbf{H}}(z) \underline{\mathbf{H}}^H(1/z^*) \right]^{-1} \quad (5.63)$$

$$= \underline{\mathbf{B}}(z) \underline{\mathbf{S}}_c^{-1}(z) \underline{\mathbf{H}}^H(1/z^*) \sigma_n^{-2} \quad (5.64)$$

where the last equality comes from the matrix inversion lemma. The power spectrum of the estimation error is now

$$\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(z) = \underline{\mathbf{B}}(z) \underline{\mathbf{S}}_c^{-1}(z) \underline{\mathbf{B}}^H(1/z^*). \quad (5.65)$$

Let us now compute the optimal feedback filters $\underline{\mathbf{B}}(z)$. The key power spectrum can be factored in the following way [58]:

$$\underline{\mathbf{S}}_c(z) = \underline{\mathbf{D}}^H(1/z^*) \underline{\mathbf{A}} \underline{\mathbf{D}}(z), \quad (5.66)$$

where matrix $\underline{\mathbf{A}}$ is diagonal and $\underline{\mathbf{D}}(z)$ is a causal, monic and stable matrix transfer function with causal and stable inverse. Hence the power spectrum of the estimation error (5.65) becomes

$$\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(z) = \underline{\mathbf{B}}(z) \underline{\mathbf{D}}^{-1}(z) \underline{\mathbf{A}}^{-1} \underline{\mathbf{D}}^{-H}(1/z^*) \underline{\mathbf{B}}^H(1/z^*). \quad (5.67)$$

The geometrical mean of the error variances is given by the product of the diagonal elements of the estimation error covariance matrix given by Equation (5.59). For the positive definite error covariance matrix, it is known that

$$\prod_{p=1}^N \left[\underline{\Gamma}_{\underline{\epsilon}\underline{\epsilon}}(0) \right]_{pp} \geq \det \left[\underline{\Gamma}_{\underline{\epsilon}\underline{\epsilon}}(0) \right] \quad (5.68)$$

where equality is only obtained if the error covariance matrix is diagonal. Furthermore, we know from [59] that

$$\begin{aligned} \log \left[\det \left(\underline{\Gamma}_{\underline{\epsilon}\underline{\epsilon}}(0) \right) \right] &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(e^{j\Omega}) \right) \right] d\Omega \\ &= \log (\underline{\mathbf{A}}) \end{aligned} \quad (5.69) \quad (5.70)$$

where the last equality results from the Szegö formula [58]. If the feedback section $\underline{\mathbf{B}}(z)$ is chosen as the spectral factor $\underline{\mathbf{D}}(z)$ defined in (5.66), the power spectrum of the estimation error (5.67) and the associated covariance matrix (5.59) reduce to

$$\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(z) = \underline{\Gamma}_{\underline{\epsilon}\underline{\epsilon}}(0) = \underline{\mathbf{A}}^{-1}. \quad (5.71)$$

As the error power spectrum $\underline{\gamma}_{\underline{\epsilon}\underline{\epsilon}}(e^{j\Omega})$ does no longer depend on Ω , (5.69) is an equality and the determinant of the error covariance matrix is minimized. Furthermore, as the error covariance matrix is a diagonal matrix, (5.68)

is also an equality and the product of the estimation error variances is minimized. The selected feedback matrix $\underline{\underline{B}}(z)$ is thus optimal regarding the MMSE criterion.

The optimal feedback and feedforward filters are finally :

$$\begin{aligned}\underline{\underline{B}}(z) &= \underline{\underline{D}}(z) \\ \underline{\underline{W}}(z) &= \underline{\underline{\Lambda}}^{-1} [\underline{\underline{D}}^H(1/z^*)]^{-1} \underline{\underline{H}}^H(1/z^*) \sigma_n^{-2}.\end{aligned}\quad (5.72)$$

The feedforward filter appears to be made first of a bank of matched filters. Then it is cascaded with an anticausal bank.

The obtained estimation error is white as shown by (5.71). The product of error variances can also be computed by using the Szegö formula [58] as follows :

$$\log [\det (\underline{\underline{\Lambda}}^{-1})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\sigma_I^{-2} \underline{\underline{E}}_N + \sigma_n^{-2} \underline{\underline{H}}^H(1/z^*) \underline{\underline{H}}(z) \right)^{-1} \right] d\Omega. \quad (5.73)$$

Regarding the symbol detection order, the product of the error variances is constant.

5.4.3 Influence of the FB

Invariance properties relative to the choice of the paraunitary FB can be put forward in the single-user configuration ($K = 1$). In that configuration, the global channel matrix can be written as $\underline{\underline{H}}(z) = \underline{\underline{C}}_1(z) \underline{\underline{S}}(z)$ where $\underline{\underline{S}}(z)$ is the full $N \times N$ FB polyphase matrix.

With a paraunitary FB, the key matrix $\underline{\underline{S}}_c(z)$ defined in (5.56) can be written as

$$\underline{\underline{S}}_c(z) = \left[\sigma_I^{-2} \underline{\underline{E}}_N + \frac{2}{N_0} \underline{\underline{S}}^H(1/z^*) \underline{\underline{G}}_{11}(z) \underline{\underline{S}}(z) \right] \quad (5.74)$$

$$= \underline{\underline{S}}^H(1/z^*) \underline{\underline{S}}_{c_0}(z) \underline{\underline{S}}(z) \quad (5.75)$$

where $\underline{\underline{S}}_{c_0}(z) = \sigma_I^{-2} \underline{\underline{E}}_N + \frac{2}{N_0} \underline{\underline{G}}_{11}(z)$ and the paraunitary property was used.

We first investigate the performance of the linear MMSE JD. From (5.58) and (5.74), the arithmetic mean of the error variances is given by :

$$\frac{1}{N} \sum_{p=1}^N \sigma_{\epsilon_p}^2 = \frac{1}{2\pi N} \int_{-\pi}^{\pi} \text{tr} \left[\underline{\underline{S}}^H(e^{j\Omega}) \underline{\underline{S}}_{c_0}^{-1}(e^{j\Omega}) \underline{\underline{S}}(e^{j\Omega}) \right] d\Omega \quad (5.76)$$

An important conclusion may be drawn from the last expression: *at the output of a paraunitary FB transmission scheme with linear MMSE receiver, the arithmetic mean of the error variances does not depend on the filter bank.* In particular, the size of the FB has no impact on this arithmetic mean, which is equal to the unique error variance obtained with a single carrier system. The proof relies on the following matrix properties :

1. Two matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are similar (i.e. they have the same eigenvalues) if there exists an invertible matrix $\underline{\underline{C}}$ such that $\underline{\underline{A}} = \underline{\underline{C}}^{-1}\underline{\underline{B}}\underline{\underline{C}}$.
2. The trace of a matrix is equal to the eigenvalue sum.

The first property can be applied to (5.76) thanks to the paraunitary property of the FB polyphase matrix $\underline{\underline{S}}(z)$.

The arithmetic mean of the error variances is not preserved in the case of a DF-JD, because of the (non-paraunitary) feedback matrix in (5.65). It can be shown, however, that *the geometrical mean of the error variances is independent of the selected FB :*

$$\sqrt[N]{\prod_{p=1}^N \sigma_{\epsilon_p}^2} = \sqrt[N]{\det(\underline{\underline{\Lambda}})^{-1}}. \quad (5.77)$$

The determinant in (5.77) can be evaluated by means of Equation (5.73) as follows :

$$\log [\det (\underline{\underline{\Lambda}}^{-1})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\det \left(\underline{\underline{S}}^H(1/z^*) \underline{\underline{S}}_{c_0}(z) \underline{\underline{S}}(z) \right)^{-1} \right] d\Omega. \quad (5.78)$$

The result does not depend on the FB as long as it satisfies the paraunitary property.

These invariance properties are no longer satisfied in a multiuser configuration. The distribution of error variances depends on the selected signature allocation.

5.4.4 Multiuser capacity and DF joint detection

Decisions about the transmitted symbols can be taken separately in the N subchannels, on the basis of the MMSE estimates $\hat{I}_p(n)$ obtained at the output of the DF-JD. In every subchannel, the MMSE estimate can be

written, according to (4.51), as

$$\hat{I}_p(n) = x_{pp}(0) I_p(n) + J_p(n) + \nu_p(n) \quad (5.79)$$

$$= I_p(n) + (x_{pp}(0) - 1) I_p(n) + J_p(n) + \nu_p(n) \quad (5.80)$$

$$= I_p(n) - \epsilon_p(n) \quad (5.81)$$

where $J_p(n)$ represents the total interference (ISI-ICI-MAI). The fundamental role of the IIR DF joint detector is to transform the initial frequency-selective multiuser (and multi-signature) channel into a set of N parallel and independent AWGN subchannels. The capacity associated with every subchannel, assuming Gaussian-distributed symbols, is given by

$$R_p = \log(1 + \rho_p) \quad \text{with } \rho_p = \frac{x_{pp}^2(0)\sigma_I^2}{\sigma_{J_p}^2 + \sigma_{\nu_p}^2}. \quad (5.82)$$

Error-free transmission is possible at a rate R_p in every subchannel, assuming the use of an ideal AWGN decoder in each separate receiver. The MMSE estimator is known to be biased, i.e. $x_{pp}(0) < 1$. From the MMSE criterion, it can be shown that

$$\sigma_I^2 (1 - x_{pp}(0)) = \sigma_{\epsilon_p}^2. \quad (5.83)$$

for any type of MMSE estimator. The bias of the MMSE estimator is directly proportional to the estimation error variance. As a consequence, the SINR associated with an MMSE estimate can be shown to be

$$\rho_p = \frac{\sigma_I^2}{\sigma_{\epsilon_p}^2} - 1. \quad (5.84)$$

Combining the MMSE-SINR definition (5.84) with the Szegö formula (5.73), the error covariance definition (5.71) and the mutual information definition (5.28), we get

$$\bar{\mathcal{I}}_{\{1, \dots, K\}} = \log [\det (\sigma_I^2 \underline{\Lambda})] \quad (5.85)$$

$$= \log \left[\prod_{p=1}^N \left(\frac{\sigma_I^2}{\sigma_{\epsilon_p}^2} \right) \right] \quad (5.86)$$

$$= \sum_{p=1}^N \log (1 + \rho_p) = \sum_{p=1}^N R_p. \quad (5.87)$$

The very important conclusion of last equation is that *the sum-capacity of the frequency-selective multiuser channel can be achieved by means of an IIR DF-JD followed by ideal AWGN decoders.*

All points of the FB-constrained multiuser capacity region cannot be achieved with the IIR DF-JD, however. Indeed, the boundary of the capacity region is generated by the union of polyhedron vertices as given by Equation (5.36) and illustrated in Figure 5.2. Successive decoding is needed to obtain the set of rates R_k associated with these vertices. With the IIR DF-JD, the only degrees of freedom are the ordering of the decisions for the current block of symbols. The best we can do for a given user with the highest priority α_k is to suppress the causal interference from the $K - 1$ other users, including the contribution of the symbols in the current block. But the anticausal interference cannot be suppressed. Figure 5.3 illustrates this in a two-user scenario. The capacity region for a given signature allocation is a pentagon, with useful vertices A and D. All points of the segment AD are not achievable with the IIR DF-JD. Point B represents the rate pair (R_1, R_2) obtained by ordering the decisions as follows : firstly all subchannels allocated to user 1 (in any order), secondly all subchannels allocated to user 2 (in any order). Point C represents the rate pair obtained with the opposite ordering. A few intermediate points between B and C can be obtained by interleaving the subchannels of both users in the decision ordering. Finally, the whole segment BC is covered by using time-sharing between two or more decision orderings. Even though the available sum rate $R_1 + R_2$ is the same as for the initial pentagon, the useful vertices are now B and C rather than A and D. As a consequence, the multiuser FB-constrained capacity region associated with the IIR DF-JD is strictly smaller than the general FB-constrained capacity region defined in Section 5.2. Only the maximum sum-rate and the single-user rates are equivalent.

While the computation of the sum-capacity in (5.85) is made easy by the Szegö formula (5.73), the derivation of individual data rates R_k requires the computation of the diagonal matrix $\underline{\Lambda}$, which requires in turn the effective spectral factorization of (5.66).

5.5 FIR joint detectors

In the previous section we have obtained the structure and the associated MMSE performance of IIR joint detectors. With continuous transmission, the number of coefficients associated with these optimal detectors is generally infinite. Practically, a finite-length JD is implemented in the receiver.

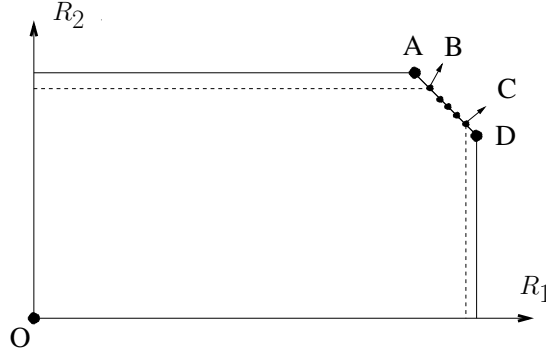


Figure 5.3: *Reduced pentagon associated with the IIR DF-JD*

The received signal is observed within a finite-length window and an estimation of the transmitted symbols is built on the basis of the observed segment. The size and the position of the observation window must be chosen carefully. A feedback section of appropriate length can also be implemented to subtract the causal interference. Both the linear and DF FIR JDs can be designed with an MMSE criterion.

The following parameters are defined :

- $L + 1$ is the *length of the composite channel impulse responses* $h_{pk}(t)$ expressed as a multiple of the symbol duration T . It depends on the length of both the channel impulse responses $c_k(t)$ and the signature codes $s_p(n)$, as given by (4.44).
- $N_f \geq 1$ is the *length of the observation window*, expressed as a multiple of the symbol duration T . This parameter determines the performance of the FIR JD. It should be chosen large enough to provide a performance comparable to that of the optimal (IIR) JD, but not too large in order to limit the computational complexity to a reasonable level.
- Δ is the *decision delay*, expressed as a multiple of the symbol duration T . This parameter determines the position of the observation window, which corresponds to the time interval $[(n - N_f + \Delta)T, (n + \Delta)T]$ for the estimation of the n -th block of symbols. For a given N_f , there is an optimal decision delay that can be found by an exhaustive search. For causal impulse responses of length $L + 1$, the decision delay is

restricted to the interval : $0 \leq \Delta \leq N_f + L - 1$. Outside this interval, the received segment would contain no contribution from the symbols that have to be estimated. For short observation windows, the optimal decision delay is generally the one that allows the capture of the maximum energy from the useful symbols. When a decision-feedback section is implemented, the interference from past symbols is assumed to be perfectly removed : in that case, the decision delay Δ is restricted to the (shorter) interval $N_f - 1 \leq \Delta \leq N_f + L - 1$.

- N_b is the *length of the feedback section*. It gives the number of past blocks of symbols whose contribution is subtracted from the output of the feedforward section to build the DF estimates, in addition to the decisions already taken in the current block. For a given pair (N_f, Δ) , the number of past blocks affecting the output of the feedforward section is $\bar{N}_b = N_f + L - 1 - \Delta$. The feedback parameter should be selected in the interval $0 \leq N_b \leq \bar{N}_b$. In the sequel, a maximum-length feedback section is assumed, i.e. $N_b = \bar{N}_b$.

The following generic FIR observation model is used for the derivation of the MMSE FIR JD coefficients :

$$\underline{\mathbf{r}}_{N_b-L}^\Delta(n) = \left[\underline{\mathcal{H}}_{N_f} \right] \underline{\mathbf{I}}_{N_b}^\Delta(n) + \underline{\mathbf{n}}_{N_b-L}^\Delta(n). \quad (5.88)$$

This model combines the initial FIR observation model (4.49), developed in the context of baseband FB transmission, with the newly defined parameters N_f , Δ and N_b . The extension to CAP-FB transmission is immediate. The case of CAP-CDMA with long codes deserves a specific treatment and is discussed in detail in Section 5.5.

5.5.1 MMSE linear joint detection

The FIR linear JD computes estimates of the transmitted symbols as follows :

$$\hat{\underline{\mathbf{I}}}(n) = \underline{\mathbf{W}} \cdot \underline{\mathbf{r}}_{N_b-L}^\Delta(n) \quad (5.89)$$

where $\underline{\mathbf{W}}$ is a matrix of size $N \times N_f M$. The estimation error is given by

$$\underline{\epsilon}(n) = \underline{\mathbf{I}}(n) - \hat{\underline{\mathbf{I}}}(n) = \left(\underline{\mathbf{F}}_\Delta - \underline{\mathbf{W}} \underline{\mathcal{H}}_{N_f} \right) \cdot \underline{\mathbf{I}}_{N_b}^\Delta(n) - \underline{\mathbf{W}} \cdot \underline{\mathbf{n}}_{N_b-L}^\Delta(n) \quad (5.90)$$

where the extraction matrix $\underline{\mathbf{F}}_\Delta$ is defined by

$$\underline{\mathbf{F}}_\Delta = \left(\begin{array}{c|c|c} \underline{\mathbf{0}}_{N \times \Delta N} & \underline{\mathbf{E}}_N & \underline{\mathbf{0}}_{N \times N_b N} \end{array} \right). \quad (5.91)$$

The orthogonality principle states that the error variance is minimal when the error is orthogonal to each data sample. Hence the cross-correlation between the error and received vectors is

$$\underline{\underline{R}}_{\underline{\underline{e}}\underline{\underline{r}}} = \underline{\underline{0}}_{N \times N_f M}. \quad (5.92)$$

The expression of the MMSE linear JD becomes

$$\begin{aligned} \underline{\underline{W}} &= \underline{\underline{R}}_{\underline{\underline{I}}\underline{\underline{r}}} \underline{\underline{R}}_{\underline{\underline{r}}\underline{\underline{r}}}^{-1} \\ &= \underline{\underline{F}}_{\Delta} \sigma_I^2 \underline{\underline{H}}_{N_f}^T \left(\sigma_n^2 \underline{\underline{E}}_{N_f M} + \sigma_I^2 \underline{\underline{H}}_{N_f} \underline{\underline{H}}_{N_f}^T \right)^{-1} \end{aligned} \quad (5.93)$$

$$= \underline{\underline{F}}_{\Delta} \left(\sigma_I^{-2} \underline{\underline{E}}_{(N_f+L)N} + \sigma_n^{-2} \underline{\underline{H}}_{N_f}^T \underline{\underline{H}}_{N_f} \right)^{-1} \underline{\underline{H}}_{N_f}^T \sigma_n^{-2} \quad (5.94)$$

where the last equality results from the matrix inversion lemma. The first stage $\underline{\underline{H}}_{N_f}^T \sigma_n^{-2}$ of the FIR MMSE linear JD is equivalent to a bank of N matched filters, each one providing $N_f + L$ outputs corresponding to the $N_f + L$ symbols contributing to the received signal segment. Some of these filters are truncated, due to the limited size of the observation window. The second stage is a square matrix of size $(N_f + L)N$ whose purpose is to minimize the interference. It is given by the inverse of the key matrix

$$\underline{\underline{R}}_{N_f} = \left(\sigma_I^{-2} \underline{\underline{E}}_{(N_f+L)N} + \sigma_n^{-2} \underline{\underline{H}}_{N_f}^T \underline{\underline{H}}_{N_f} \right). \quad (5.95)$$

The last stage $\underline{\underline{F}}_{\Delta}$ selects the appropriate block of N symbols according to the chosen decision delay. A modification of the decision delay Δ for a fixed detector length N_f just requires the modification of the extraction matrix $\underline{\underline{F}}_{\Delta}$ in the JD expression (5.94). The symbol estimates (5.89) are

$$\hat{\underline{\underline{I}}}(n) = \underline{\underline{F}}_{\Delta} \left[\left(\underline{\underline{E}}_{(N_f+L)N} - \underline{\underline{R}}_{N_f}^{-1} \sigma_I^{-2} \right) \underline{\underline{I}}_{N_b}^{\Delta}(n) + \left(\underline{\underline{R}}_{N_f}^{-1} \sigma_n^{-2} \underline{\underline{H}}_{N_f}^T \right) \underline{\underline{u}}_{N_b-L}^{\Delta}(n) \right] \quad (5.96)$$

and the corresponding error covariance matrix becomes

$$\underline{\underline{R}}_{\underline{\underline{e}}\underline{\underline{e}}} = \underline{\underline{F}}_{\Delta} \underline{\underline{R}}_{N_f}^{-1} \underline{\underline{F}}_{\Delta}^T. \quad (5.97)$$

5.5.2 MMSE decision-feedback joint detection

The FIR DF-JD also uses a number of previous decisions to build improved symbol estimates. As in the case of the IIR DF-JD, it is assumed that each

time a new symbol is detected, it is used for the next estimation to be performed. Hence, the estimation is computed as follows :

$$\hat{\underline{\mathbf{I}}}(n) = \underline{\underline{\mathbf{W}}} \cdot \underline{\mathbf{r}}_{N_b-L}^\Delta(n) - \left(\underline{\underline{\mathbf{B}}} - \underline{\underline{\mathbf{F}}}_\Delta \right) \tilde{\underline{\mathbf{I}}}_{N_b}^\Delta(n) \quad (5.98)$$

where the causal feedback matrix $\underline{\underline{\mathbf{B}}}$ has the following structure :

$$\underline{\underline{\mathbf{B}}} = \left(\begin{array}{c|c|c|c|c} \mathbf{0}_{N \times \Delta N} & \underline{\underline{\mathbf{B}}}_0 & \underline{\underline{\mathbf{B}}}_1 & \cdots & \underline{\underline{\mathbf{B}}}_{N_b} \end{array} \right). \quad (5.99)$$

The first block $\underline{\underline{\mathbf{B}}}_0$ is constrained to be a $N \times N$ upper triangular matrix with '1's on the main diagonal. In the next derivations, the decisions are assumed to be correct.

We can use again the orthogonality principle. It comes

$$\begin{aligned} \underline{\underline{\mathbf{W}}} &= \underline{\underline{\mathbf{R}}}_{\mathbf{I}\mathbf{r}} \underline{\underline{\mathbf{R}}}_{\mathbf{r}\mathbf{r}}^{-1} \\ &= \sigma_I^2 \underline{\underline{\mathbf{B}}} \underline{\underline{\mathcal{H}}}_{N_f}^T \left(\sigma_n^2 \underline{\underline{\mathbf{E}}}_{N_f M} + \sigma_I^2 \underline{\underline{\mathcal{H}}}_{N_f} \underline{\underline{\mathcal{H}}}_{N_f}^T \right)^{-1} \end{aligned} \quad (5.100)$$

$$= \underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{R}}}_{N_f}^{-1} \underline{\underline{\mathcal{H}}}_{N_f}^T \sigma_n^{-2}. \quad (5.101)$$

The error covariance matrix becomes

$$\underline{\underline{\mathbf{R}}}_{\epsilon\epsilon} = \underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{R}}}_{N_f}^{-1} \underline{\underline{\mathbf{B}}}^T. \quad (5.102)$$

Let us now derive the optimal feedback matrix $\underline{\underline{\mathbf{B}}}$. First of all, we note that the Cholesky factorization can be applied on the symmetric positive definite matrix $\underline{\underline{\mathbf{R}}}_{N_f}$ defined in (5.95) :

$$\underline{\underline{\mathbf{R}}}_{N_f} = \underline{\underline{\mathbf{L}}}_{N_f} \underline{\underline{\mathbf{D}}}_{N_f} \underline{\underline{\mathbf{L}}}_{N_f}^T \quad (5.103)$$

where $\underline{\underline{\mathbf{L}}}_{N_f}$ is a lower triangular matrix and $\underline{\underline{\mathbf{D}}}_{N_f}$ is diagonal. Using (5.103), the error covariance matrix in (5.102) can be written as follows :

$$\underline{\underline{\mathbf{R}}}_{\epsilon\epsilon} = \left(\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} \right) \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \left(\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} \right)^T. \quad (5.104)$$

The product $\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T}$ has the same structure as the feedback matrix $\underline{\underline{\mathbf{B}}}$ itself, i.e. the structure given by Equation (5.99). As a consequence, the

estimation error variances are lower bounded as follows :

$$\begin{aligned}
 \text{diag}(\underline{\underline{\mathbf{R}}}_{\epsilon\epsilon}) &= \text{diag} \left(\left[\underline{\underline{\mathbf{F}}}_{\Delta} + \left(\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} - \underline{\underline{\mathbf{F}}}_{\Delta} \right) \right] \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \left[\underline{\underline{\mathbf{F}}}_{\Delta} + \left(\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} - \underline{\underline{\mathbf{F}}}_{\Delta} \right) \right]^T \right) \\
 &= \text{diag} \left(\underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T + \left[\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} - \underline{\underline{\mathbf{F}}}_{\Delta} \right] \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \left[\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} - \underline{\underline{\mathbf{F}}}_{\Delta} \right]^T \right) \\
 &\geq \text{diag} \left(\underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T \right)
 \end{aligned} \tag{5.105}$$

with equality if and only if $\underline{\underline{\mathbf{B}}} \underline{\underline{\mathbf{L}}}_{N_f}^{-T} = \underline{\underline{\mathbf{F}}}_{\Delta}^T$. The optimal feedback matrix and the associated error covariance matrix are

$$\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{L}}}_{N_f}^T \quad \underline{\underline{\mathbf{R}}}_{\epsilon\epsilon} = \underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T. \tag{5.106}$$

The estimation error is white, and the error variances are given by a subset of the inverse eigenvalues of the key matrix $\underline{\underline{\mathbf{R}}}_{N_f}$. The subset depends on the selected decision delay Δ . The optimal feedforward matrix (5.101) becomes

$$\underline{\underline{\mathbf{W}}} = \underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{L}}}_{N_f}^{-1} \underline{\underline{\mathcal{H}}}_{N_f}^T \sigma_n^{-2}. \tag{5.107}$$

Finally, it can be demonstrated that, as in the IIR scenario, the product of error variances does not depend on the decision ordering. Let $\underline{\underline{\mathbf{Q}}}$ be the permutation matrix associated with an arbitrary permutation of the decision order. This matrix, obtained by permuting the columns of the size- N unit matrix, is unitary, i.e. $\underline{\underline{\mathbf{Q}}} \underline{\underline{\mathbf{Q}}}^T = \underline{\underline{\mathbf{E}}}_N$. The key matrix $\underline{\underline{\mathbf{R}}}^{(q)}$ associated with the new decision order is given by

$$\underline{\underline{\mathbf{R}}}_{N_f}^{(q)} = \underline{\underline{\mathbf{Q}}} \underline{\underline{\mathbf{R}}}_{N_f} \underline{\underline{\mathbf{Q}}}^T \quad \text{with} \quad \underline{\underline{\mathbf{Q}}} = \begin{pmatrix} \underline{\underline{\mathbf{E}}}_{\Delta N} & & \\ & \underline{\underline{\mathbf{Q}}} & \\ & & \underline{\underline{\mathbf{E}}}_{N_b N} \end{pmatrix}. \tag{5.108}$$

Let $\underline{\underline{\mathbf{D}}}_{N_f}^{(q)}$ be the diagonal eigenvalue matrix associated with $\underline{\underline{\mathbf{R}}}_{N_f}^{(q)}$. The product of estimation error variances obtained with the permuted decision order is given by

$$\prod_{p=1}^N \left(\underline{\underline{\mathbf{R}}}_{\epsilon\epsilon}^{(q)} \right)_{pp} = \det \left(\underline{\underline{\mathbf{F}}}_{\Delta} \left(\underline{\underline{\mathbf{D}}}_{N_f}^{(q)} \right)^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T \right) \tag{5.109}$$

$$= \det \left(\underline{\underline{\mathbf{Q}}} \left[\underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T \right] \underline{\underline{\mathbf{Q}}}^T \right) \tag{5.110}$$

$$= \det \left(\underline{\underline{\mathbf{F}}}_{\Delta} \underline{\underline{\mathbf{D}}}_{N_f}^{-1} \underline{\underline{\mathbf{F}}}_{\Delta}^T \right) = \prod_{p=1}^N \left(\underline{\underline{\mathbf{R}}}_{\epsilon\epsilon} \right)_{pp}. \tag{5.111}$$

The second equality can be demonstrated by using the *nesting property* of the Cholesky factorization. This property states that the lower triangular and diagonal Cholesky factors of the $k \times k$ submatrix of $\underline{\underline{R}}_{N_f}^{(q)}$ formed from its first k rows and columns are equal to the first k rows and columns of $\underline{\underline{L}}_{N_f}^{(q)}$ and $\underline{\underline{D}}_{N_f}^{(q)}$, respectively.

5.5.3 Complexity analysis

In this Section, we analyze in more details the complexity associated with the computation of the filter coefficients.

The FIR linear JD is given by Equations (5.93) and (5.94). Both the direct and indirect expressions can be used to compute the filter coefficients. The best choice depends on the respective size parameters N , M , L , N_f and Δ . As the indirect method (5.94) must be implemented for the derivation of the DF-JD filters, we propose here the analysis of the direct method (5.93) for the computation of the linear JD.

Algorithm 5.1 *Computation of the FIR linear JD coefficients :*

$$\left(\underline{\underline{H}}_{N_f}, \sigma_I^2, \sigma_n^2\right) \rightarrow \underline{\underline{W}}$$

with the direct method. The algorithm proceeds in three steps:

1. *Computation of the channel output correlation matrix:*

$$\underline{\underline{R}}_{rr} = \sigma_n^2 \underline{\underline{E}}_{N_f M} + \sigma_I^2 \underline{\underline{H}}_{N_f} \underline{\underline{H}}_{N_f}^T.$$

The result is a Hermitian block banded matrix with N_f blocks of size $M \times M$ and a bandwidth of L blocks.

2. *Cholesky factorization of the channel output correlation matrix:*

$$\underline{\underline{R}}_{rr} = \underline{\underline{L}}_r \underline{\underline{L}}_r^T.$$

The Cholesky factor $\underline{\underline{L}}_r$ is lower triangular and has the same size, block-size and bandwidth as $\underline{\underline{R}}_{rr}$.

3. *Resolution of the Wiener-Hopf equation:*

$$\left(\underline{\underline{L}}_r \underline{\underline{L}}_r^T\right) \underline{\underline{W}}^T = \underline{\underline{R}}_{Ir}^T.$$

This implies N resolutions of two banded triangular systems.

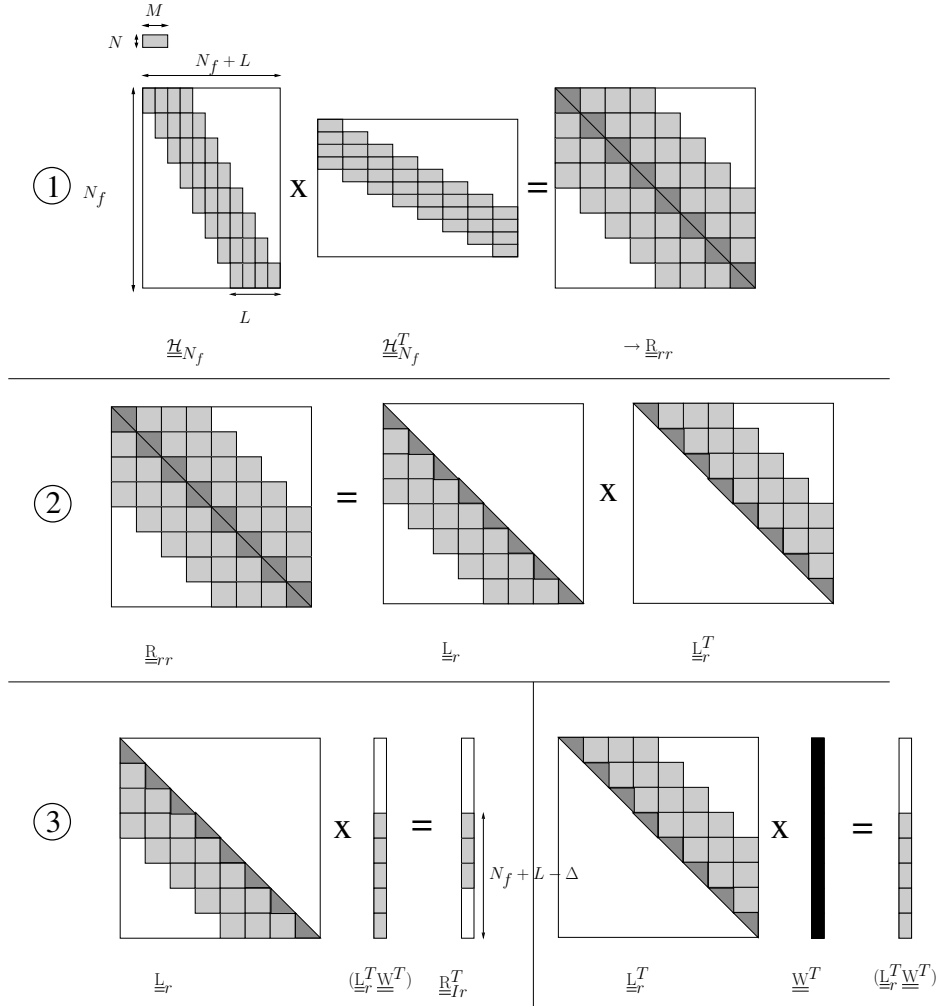


Figure 5.4: Matrix manipulations for the FIR linear JD computation

Figure 5.4 illustrates the matrix manipulations associated with the three steps of Algorithm 5.1.

The FIR DF-JD is given by Equations (5.106) and (5.107). The computation of the feedforward and feedback filters requires the factorization of the key matrix $\underline{\underline{R}}_{N_f}$.

Algorithm 5.2 *Computation of the FIR DF-JD coefficients .*

$$\left(\underline{\underline{H}}_{N_f}, \sigma_I^2, \sigma_n^2\right) \rightarrow (\underline{W}, \underline{B})$$

The algorithm is made up of three steps:

1. *Computation of the MMSE matrix:*

$$\underline{\underline{R}}_{N_f} = \left(\sigma_I^{-2} \underline{\underline{E}}_{(N_f+L)N} + \sigma_n^{-2} \underline{\underline{H}}_{N_f}^T \underline{\underline{H}}_{N_f}\right). \quad (5.112)$$

The result is a Hermitian block banded matrix with $N_f + L$ blocks of size $N \times N$ and a bandwidth of L blocks. The computation may be restricted to the first $N(\Delta + 1)$ columns.

2. *Cholesky factorization of the MMSE matrix:*

$$\underline{\underline{R}}_{N_f} = \underline{\underline{L}}_{N_f} \underline{\underline{L}}_{N_f}^T.$$

The Cholesky factor $\underline{\underline{L}}_{N_f}$ is lower triangular and has the same size, block-size and bandwidth as $\underline{\underline{R}}_{N_f}$. The feedback filters are then immediately obtained with elementary matrix manipulations:

$$\underline{D}_{N_f}^{1/2} = \text{diag}\left(\underline{\underline{L}}_{N_f}\right) \quad (5.113)$$

$$\underline{B} = \underline{F}_{\Delta} \underline{D}_{N_f}^{-1/2} \underline{\underline{L}}_{N_f}. \quad (5.114)$$

3. *Computation of the feedforward filters:*

$$\underline{\underline{L}}_{N_f} \underline{\underline{W}}_{N_f} = \sigma_n^{-2} \underline{\underline{H}}_{N_f}^T. \quad (5.115)$$

Only the first $(\Delta + 1)N$ rows of $\underline{\underline{W}}_{N_f}$ need to be computed. This equation represents N_f times M triangular systems of decreasing size from $(\Delta + 1)N$ down to $(\Delta + 2 - N_f)N$. The feedforward filters are then obtained by:

$$\underline{W} = \underline{F}_{\Delta} \underline{D}_{N_f}^{-1/2} \underline{\underline{W}}_{N_f}. \quad (5.116)$$

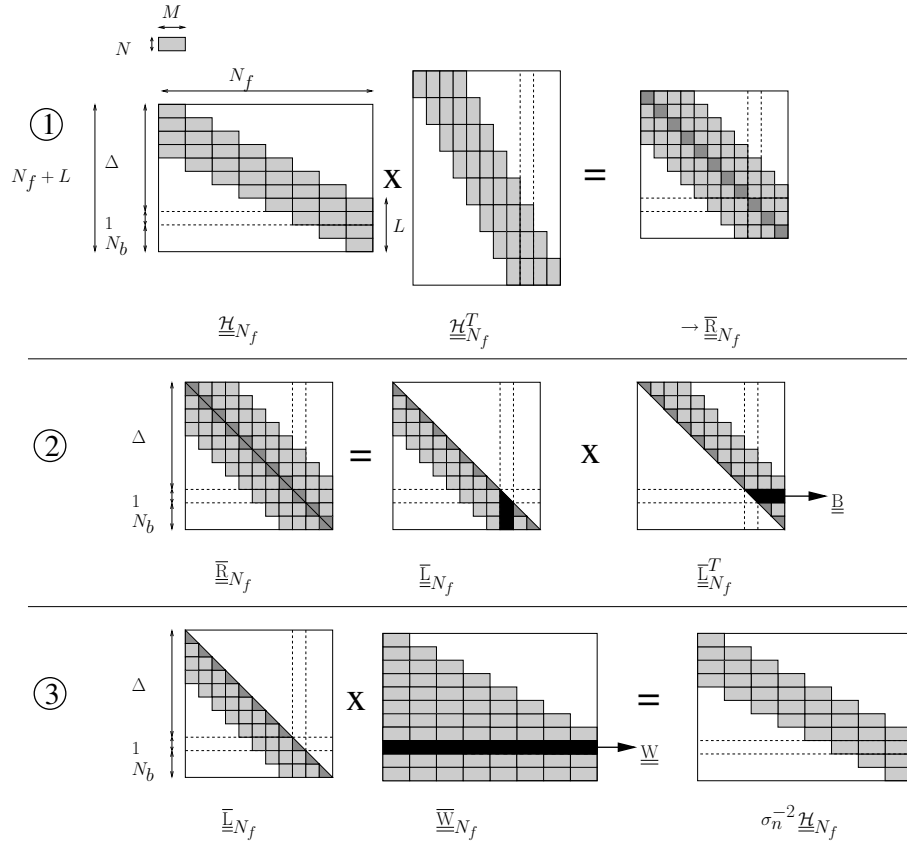


Figure 5.5: Matrix manipulations for the FIR DF-JD computation

Table 5.1: *Approximate complexity (in scalar operations) associated with the full computation of FIR-JD coefficients*

	Full Linear JD	Full DF-JD
1	$NM^2 \quad \frac{1}{2}N_fL^2$	$N^2M \quad \frac{1}{2}N_fL^2$
2	$M^3 \quad (\frac{1}{2}N_fL^2 - \frac{1}{3}L^3)$	$N^3(\frac{1}{2}N_fL^2 - \frac{1}{3}L^3)$
3	$NM^2 \quad 2(N_fL - \frac{1}{2}L^2)$	$N^2M \quad \frac{1}{2}N_f^2L$

Figure 5.5 illustrates the matrix manipulations associated with the three steps of Algorithm 5.2.

Table 5.1 gives a rough order-of-magnitude estimate of the computational complexity associated with each of the three steps of the presented algorithms, assuming $N_f \gg L \gg 1$. The values given here correspond to the approximate number of scalar operations required for each matrix manipulation, taking the banded structure of the matrices into account [60].

Faster algorithms can be investigated when the channel matrix $\underline{\underline{\mathcal{H}}}_{N_f}$ has a *periodic structure* :

- Concerning the FIR linear JD computation with the direct method, the channel output correlation matrix $\underline{\underline{\mathbf{R}}}_{rr}$ is then block-Toeplitz. Its computation (Step 1 in Algorithm 5.1) requires only $\mathcal{O}(NM^2 \frac{1}{2}L^2)$ scalar operations. The resolution of the Wiener-Hopf equation (Steps 2 and 3 in Algorithm 5.1) can be performed by N successive applications of the Levinson algorithm [60], which requires $\mathcal{O}(NM^2 N_f L)$ scalar operations.
- Concerning the FIR DF-JD computation, the key matrix $\overline{\underline{\underline{\mathbf{R}}}}_{N_f}$ is then also block-Toeplitz. The computation of the key Cholesky factor $\underline{\underline{\mathbf{L}}}$ (Steps 1 and 2 in Algorithm 5.2) can be performed very efficiently by using the so-called generalized Schur algorithm. This method is based on the observation by [61] that the block-symmetric matrix

$$\underline{\underline{\mathcal{M}}}_{N_f} \triangleq \begin{bmatrix} \sigma_n^2 \underline{\underline{\mathbf{E}}}_{N_f M} & \underline{\underline{\mathcal{H}}}_{N_f} \\ \underline{\underline{\mathcal{H}}}_{N_f}^T & -\sigma_I^2 \underline{\underline{\mathbf{E}}}_{(N_f+L)N} \end{bmatrix} \quad (5.117)$$

admits a triangular factorization of the following form :

$$\underline{\underline{\mathcal{M}}}_{N_f} = \begin{bmatrix} \underline{\underline{\mathcal{L}}}_1 & \\ \underline{\underline{\mathcal{L}}}_2 & \underline{\underline{\mathcal{L}}}_3 \end{bmatrix} \begin{bmatrix} \underline{\underline{\mathbf{E}}}_{N_f M} & \\ & -\underline{\underline{\mathbf{E}}}_{(N_f+L)N} \end{bmatrix} \begin{bmatrix} \underline{\underline{\mathcal{L}}}_1 & \\ \underline{\underline{\mathcal{L}}}_2 & \underline{\underline{\mathcal{L}}}_3 \end{bmatrix}^T \quad (5.118)$$

where the lower triangular matrix $\underline{\underline{\mathcal{L}}}_3$ satisfies the relation

$$\underline{\underline{\mathbf{R}}}_{N_f} = \underline{\underline{\mathcal{L}}}_3 \underline{\underline{\mathcal{L}}}_3^T, \quad (5.119)$$

so that the desired Cholesky factor $\underline{\underline{\mathbf{L}}}_{N_f}$ is given by

$$\underline{\underline{\mathbf{L}}}_{N_f} = \underline{\underline{\mathcal{L}}}_3. \quad (5.120)$$

The triangular factorization (5.118) can be performed with simple operations (Givens rotations and hyperbolic rotations) on the matrix $\underline{\underline{\mathcal{M}}}_{N_f}$, which requires $\mathcal{O}(N^2(N_f + L)^2)$ scalar operations.

5.6 Joint detection for CAP-CDMA with long codes

The linear and DF IIR MMSE JDs designed for a CAP-CDMA transmission system with long codes are also given by Equations (5.55) and (5.72), respectively, where the channel matrix $\underline{\underline{\mathbf{H}}}(z)$ is now of size $MP \times 2KP$, and corresponds to the observation model proposed in Equation (4.79). Their interpretation deserves some comments. The matched filter receiver $\underline{\underline{\mathbf{H}}}^H(1/z^*)$ is made here of a bank of $2KP$ branches, each one matched to a particular code segment of a particular user. In fact, the $2P$ parallel matched filters associated with a particular user are nothing but a parallel description (with P time invariant branches) of two periodically time varying matched filters. The outputs of these matched filters are downsampled to the symbol rate ($1/PT'$). Then a MIMO equalizer $\underline{\underline{\mathbf{S}}}_c^{-1}(z)$ takes care of both multiple access interference and the possible interference between the different code segments of the different users. In this MIMO filter, $2P$ outputs are associated with each user. Each one of these outputs is obtained by processing the $2KP$ matched filter outputs by means of a specific set of $2KP$ filters. This again appears to be a parallel version of a time varying process. For a given user the P sets of $2KP$ time invariant filters are in fact equivalent to a single set of $2KP$ time varying filters, with period P . Finally, the feedback section $\underline{\underline{\mathbf{B}}}(z) - \underline{\underline{\mathbf{E}}}_{2KP}$ is also a parallel representation of a period P time varying feedback process.

It turns out that, in principle, the detector should account for the possible interference between any pair of segments of any pair of users. This strategy which corresponds to the optimum setup is however very demanding. It is well-known that the complexity of the MMSE JD coefficient computation is roughly proportional to the third power of the spreading factor (see Table 5.1). With the multirate description described above, the spreading factor is virtually multiplied by P , which dramatically increases the computational complexity. The purpose of this section is to investigate FIR solutions of reduced complexity. The idea of reducing the complexity is to explore what happens when a few segments only interfere with each other. In such a situation, it seems logical to use only a limited number of received signal segments rather than all of them.

5.6.1 MMSE linear joint detection

From the IIR linear joint detector described above (see (5.50)), the following FIR version can be obtained :

$$\hat{\mathbf{I}}_0^{P-1}(n) = \begin{bmatrix} \underline{\mathcal{W}}_{-1} & \underline{\mathcal{W}}_0 & \underline{\mathcal{W}}_1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_0^{P-1}(n+1) \\ \mathbf{r}_0^{P-1}(n) \\ \mathbf{r}_0^{P-1}(n-1) \end{bmatrix}. \quad (5.121)$$

Complexity can be further reduced by using only N_f segments of the received signal to build the estimate of the current symbol. Assuming a decision delay of Δ symbol periods, the estimation is now :

$$\hat{\mathbf{I}}_0^{P-1}(n) = \overline{\underline{\mathcal{W}}} \begin{bmatrix} \mathbf{r}_0^{\Delta-1}(n+1) \\ \mathbf{r}_0^{P-1}(n) \\ \mathbf{r}_{P-N_b+L}^{P-1}(n-1) \end{bmatrix} \quad (5.122)$$

where matrix $\overline{\underline{\mathcal{W}}}$ has non-zero entries at the positions given by the shaded area of Figure 5.6.

The number of filter taps associated with this simplified detection scheme is much lower. In fact, the span of the time-varying detector made of modules of size $2K \times M$ may now be chosen with a better granularity: the number of modules does not need to be a multiple of P or, equivalently, the span of the detector does not have to be a multiple of P . A drawback, however, lies in the need for these filters to be computed separately for each of the P segments.

The $2K$ estimates at phase p are now given by:

$$\hat{\mathbf{I}}_p(n) = \underline{\underline{\mathbf{W}}}_p \mathbf{r}_{p-N_b+L}^{p+\Delta}(n) \quad (5.123)$$

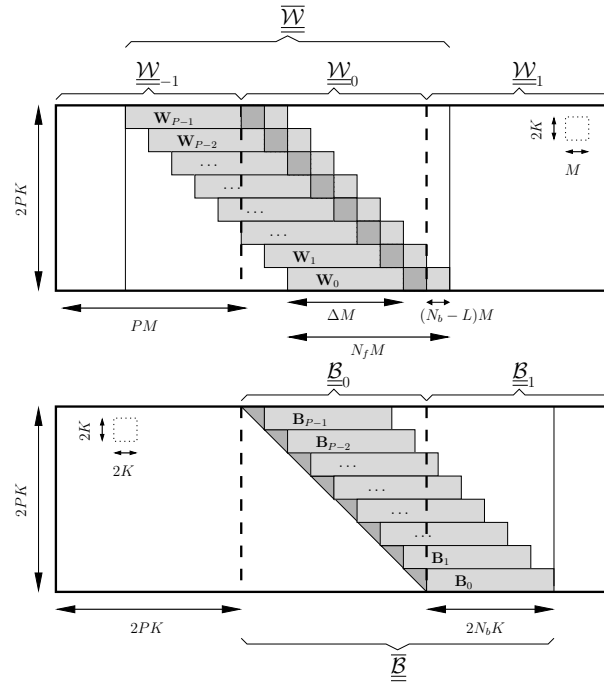


Figure 5.6: Feedforward and feedback time-varying filters

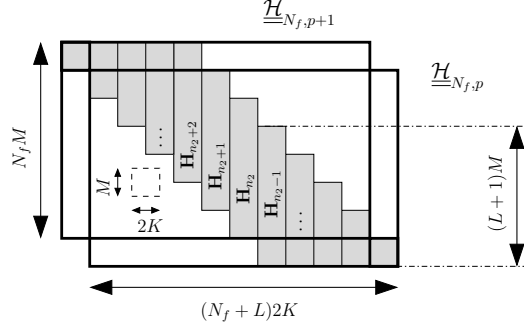


Figure 5.7: Two successive channel matrices

with

$$\underline{\mathbf{r}}_{p-N_b+L}^{p+\Delta}(n) = \underline{\mathcal{H}}_{N_f,p} \underline{\mathbf{I}}_{p-N_b}^{p+\Delta}(n) + \underline{\mathbf{n}}_{p-N_b+L}^{p+\Delta}(n). \quad (5.124)$$

The segment-dependent channel matrix $\underline{\mathcal{H}}_{N_f,p}$ is illustrated in Figure 4.7. The MMSE feedforward estimator for the symbols corresponding to segment p is now (see (5.93) and (5.94)) :

$$\underline{\mathbf{W}}_p = \sigma_I^2 \underline{\mathbf{E}}_{\Delta} \underline{\mathcal{H}}_{N_f,p}^T \left(\sigma_n^2 \underline{\mathbf{E}}_{N_f M} + \sigma_I^2 \underline{\mathcal{H}}_{N_f,p} \underline{\mathcal{H}}_{N_f,p}^T \right)^{-1} \quad (5.125)$$

$$= \underline{\mathbf{E}}_{\Delta} \left(\sigma_I^{-2} \underline{\mathbf{E}}_{(N_f+L)2K} + \sigma_n^{-2} \underline{\mathcal{H}}_{N_f,p}^T \underline{\mathcal{H}}_{N_f,p} \right)^{-1} \underline{\mathcal{H}}_{N_f,p}^T \sigma_n^{-2} \quad (5.126)$$

where the new extraction matrix $\underline{\mathbf{E}}_{\Delta}$ is

$$\underline{\mathbf{E}}_{\Delta} = \left(\underline{\mathbf{0}}_{2K \times \Delta 2K} \mid \underline{\mathbf{E}}_{2K} \mid \underline{\mathbf{0}}_{2K \times N_b 2K} \right). \quad (5.127)$$

In the case of long CIRs and detectors (i.e. $(N_f, L) \gg 1$), it is crucial to design a fast update algorithm for the computation of $\underline{\mathbf{W}}_{p+1}$, on the basis of $\underline{\mathbf{W}}_p$. This is possible thanks to the specific structure of the successive channel matrices $\underline{\mathcal{H}}_{N_f,p}$ (see Figure 5.7) :

$$\begin{aligned} \underline{\mathcal{H}}_{N_f,p} &= \begin{matrix} N_f+L-1 & \xleftrightarrow{1} \\ \xleftrightarrow{N_f-1} & \xleftrightarrow{1} \\ \xleftrightarrow{1} & \end{matrix} \begin{pmatrix} \underline{\mathbf{H}}_0 & \underline{\mathbf{0}} \\ \underline{\mathbf{h}}_1^T & \underline{\mathbf{h}}_0 \end{pmatrix}_p \\ \underline{\mathcal{H}}_{N_f,p+1} &= \begin{matrix} 1 & N_f+L-1 \\ \xleftrightarrow{1} & \xleftrightarrow{N_f-1} \\ \xleftrightarrow{N_f-1} & \end{matrix} \begin{pmatrix} \underline{\mathbf{h}}_0 & \underline{\mathbf{h}}_1^T \\ \underline{\mathbf{0}} & \underline{\mathbf{H}}_0 \end{pmatrix}_{p+1} \end{aligned} \quad (5.128)$$

where $\underline{\underline{H}}_0(p+1) = \underline{\underline{H}}_0(p)$. Using this property, the three steps of Algorithm 5.1 can be replaced by the following updating process.

Algorithm 5.3 *Update of the FIR linear JD coefficients :*

1. *Update the channel output correlation matrix:*

$$\begin{aligned} \underline{\underline{R}}_{rr}(p) &= \begin{matrix} & \begin{matrix} \xleftrightarrow{N_f-1} & \xleftrightarrow{1} \end{matrix} \\ \begin{matrix} N_f-1 \Downarrow \\ 1 \Downarrow \end{matrix} & \left(\begin{array}{cc} \underline{\underline{R}}_0 & \underline{\underline{r}}_1 \\ \underline{\underline{r}}_1^T & \underline{\underline{r}}_0 \end{array} \right)_p \end{matrix} \\ \underline{\underline{R}}_{rr}(p+1) &= \begin{matrix} & \begin{matrix} \xleftrightarrow{1} & \xleftrightarrow{N_f-1} \end{matrix} \\ \begin{matrix} 1 \Downarrow \\ N_f-1 \Downarrow \end{matrix} & \left(\begin{array}{cc} \underline{\underline{r}}_0 & \underline{\underline{r}}_1^T \\ \underline{\underline{r}}_1 & \underline{\underline{R}}_0 \end{array} \right)_{p+1} \end{matrix}. \end{aligned} \quad (5.129)$$

Using (5.128), we get $(\underline{\underline{R}}_0)_{p+1} = (\underline{\underline{R}}_0)_p$. The update reduces to the computation of:

$$\begin{aligned} \left(\begin{array}{c} \underline{\underline{r}}_0 \\ \underline{\underline{r}}_1 \end{array} \right)_{p+1} &= \sigma_n^2 \left(\begin{array}{c} \underline{\underline{E}}_M \\ \underline{\underline{0}} \end{array} \right) \\ &+ \sigma_I^2 \left(\begin{array}{cc} \underline{\underline{h}}_0 & \underline{\underline{h}}_1^T \\ \underline{\underline{0}} & \underline{\underline{H}}_0 \end{array} \right)_{p+1} \left(\begin{array}{c} \underline{\underline{h}}_0^T \\ \underline{\underline{h}}_1 \end{array} \right)_{p+1}. \end{aligned} \quad (5.130)$$

2. *Update the Cholesky factor $\underline{\underline{L}}_{r,p+1}$:*

$$\begin{aligned} \underline{\underline{L}}_{r,p} &= \begin{matrix} & \begin{matrix} \xleftrightarrow{N_f-1} & \xleftrightarrow{1} \end{matrix} \\ \begin{matrix} N_f-1 \Downarrow \\ 1 \Downarrow \end{matrix} & \left(\begin{array}{cc} \underline{\underline{L}}_0 & \underline{\underline{0}} \\ \underline{\underline{L}}_1^T & \underline{\underline{l}}_0 \end{array} \right)_p \end{matrix} \\ \underline{\underline{L}}_{r,p+1} &= \begin{matrix} & \begin{matrix} \xleftrightarrow{1} & \xleftrightarrow{N_f-1} \end{matrix} \\ \begin{matrix} 1 \Downarrow \\ N_f-1 \Downarrow \end{matrix} & \left(\begin{array}{cc} \underline{\underline{l}}_0 & \underline{\underline{0}} \\ \underline{\underline{l}}_1 & \underline{\underline{L}}_0 \end{array} \right)_{p+1} \end{matrix}. \end{aligned} \quad (5.131)$$

Part of the new Cholesky factor has first to be computed from scratch:

$$\left(\begin{array}{c} \underline{\underline{l}}_0 \\ \underline{\underline{l}}_1 \end{array} \right)_{p+1} \left(\begin{array}{c} \underline{\underline{l}}_0^T \\ \underline{\underline{l}}_1 \end{array} \right)_{p+1} = \left(\begin{array}{c} \underline{\underline{r}}_0 \\ \underline{\underline{r}}_1 \end{array} \right)_{p+1}. \quad (5.132)$$

The remaining part, $\underline{\underline{L}}_0$, of the Cholesky factor fulfils:

$$\left(\underline{\underline{L}}_0 \underline{\underline{L}}_0^T + \underline{\underline{l}}_1 \underline{\underline{l}}_1^T \right)_{p+1} = \left(\underline{\underline{L}}_0 \underline{\underline{L}}_0^T \right)_p. \quad (5.133)$$

The updated submatrix may be computed by M successive applications of the Cholesky rank-1 update algorithm [60].²

3. Solve the new Wiener-Hopf equation with the updated $\underline{\underline{R}}_{rr}$ and the new $\underline{\underline{R}}_{Ir}$:

$$\left(\underline{\underline{L}}_{p+1} \underline{\underline{L}}_{p+1}^T \right) \underline{\underline{W}}_{p+1}^T = \underline{\underline{R}}_{Ir}^T(p+1).$$

A detailed complexity analysis will be given and discussed in Section 5.6.3.

5.6.2 MMSE decision-feedback joint detection

The FIR version of the full complexity DF joint detector (see (5.60)) is given by:

$$\begin{aligned} \hat{\underline{\underline{I}}}_0^{P-1}(n) = & \begin{bmatrix} \underline{\underline{W}}_{-1} & \underline{\underline{W}}_0 & \underline{\underline{W}}_1 \end{bmatrix} \begin{bmatrix} \underline{\underline{r}}_0^{P-1}(n+1) \\ \underline{\underline{r}}_0^{P-1}(n) \\ \underline{\underline{r}}_0^{P-1}(n-1) \end{bmatrix} \\ & - \begin{bmatrix} (\underline{\underline{B}}_0 - \underline{\underline{E}}_{2KP}) & \underline{\underline{B}}_1 \end{bmatrix} \begin{bmatrix} \underline{\underline{I}}_0^{P-1}(n) \\ \underline{\underline{I}}_0^{P-1}(n-1) \end{bmatrix}. \end{aligned} \quad (5.134)$$

The reduced complexity decision-feedback estimation becomes:

$$\hat{\underline{\underline{I}}}_0^{P-1}(n) = \underline{\underline{W}} \begin{bmatrix} \underline{\underline{r}}_0^{\Delta-1}(n+1) \\ \underline{\underline{r}}_0^{P-1}(n) \\ \underline{\underline{r}}_{P-N_b+L}^{P-1}(n-1) \end{bmatrix} - (\underline{\underline{B}} - \underline{\underline{F}}) \begin{bmatrix} \underline{\underline{I}}_0^{\Delta-1}(n+1) \\ \underline{\underline{I}}_0^{P-1}(n) \\ \underline{\underline{I}}_{P-N_b}^{P-1}(n-1) \end{bmatrix} \quad (5.135)$$

where $\underline{\underline{W}}$ and $\underline{\underline{B}}$ have non-zero entries at the positions given by the shaded area of Figure 5.6.

²The Cholesky rank-1 update/downdate algorithm is an efficient method of computing a lower triangular matrix $\tilde{\underline{\underline{L}}}_1$ of size N_1 and a rectangular matrix $\tilde{\underline{\underline{L}}}_2$ of size $N_2 \times N_1$ satisfying the relation $(\underline{\underline{L}}_1^T \underline{\underline{L}}_2^T)^T \underline{\underline{L}}_1^T \pm (\underline{\underline{x}}_1^T \underline{\underline{x}}_2^T)^T \underline{\underline{x}}_1^T = (\tilde{\underline{\underline{L}}}_1^T \tilde{\underline{\underline{L}}}_2^T)^T \tilde{\underline{\underline{L}}}_1^T$ where $\underline{\underline{x}}_1$ and $\underline{\underline{x}}_2$ are vector columns. The algorithm applies N_1 appropriate Givens rotations on an augmented version of matrix $(\underline{\underline{L}}_1^T \underline{\underline{L}}_2^T)$. Its complexity is $\mathcal{O}(N_1^2 + 2N_1N_2)$. If $\underline{\underline{x}}_1$ and $\underline{\underline{x}}_2$ are matrices and not vectors, then the result is n consecutive updates using the n columns of $\underline{\underline{x}}_1$ and $\underline{\underline{x}}_2$. $\tilde{\underline{\underline{L}}}_1$ obviously appears as an updated Cholesky factor.

The $2K$ estimates at phase p are now given by:

$$\hat{\mathbf{I}}_p(n) = \underline{\underline{\mathbf{W}}}_p \mathbf{r}_{p-N_b+L}^{p+\Delta}(n) - \left(\underline{\underline{\mathbf{B}}}_p - \underline{\underline{\mathbf{F}}}_\Delta \right) \mathbf{I}_{p-N_b}^{p+\Delta}(n). \quad (5.136)$$

The optimal feedback and feedforward matrices are now (see (5.106) and (5.107)):

$$\underline{\underline{\mathbf{B}}}_p = \underline{\underline{\mathbf{F}}}_\Delta \underline{\underline{\mathbf{L}}}_{N_f,p}^T \quad (5.137)$$

$$\underline{\underline{\mathbf{W}}}_p = \underline{\underline{\mathbf{F}}}_\Delta \underline{\underline{\mathbf{D}}}_{N_f,p}^{-1} \underline{\underline{\mathbf{L}}}_{N_f,p}^{-1} \underline{\underline{\mathcal{H}}}_{N_f,p}^T \sigma_n^{-2} \quad (5.138)$$

with

$$\underline{\underline{\mathbf{R}}}_{N_f,p} = \left(\sigma_I^{-2} \underline{\underline{\mathbf{E}}}_{(N_f+L)2K} + \sigma_n^{-2} \underline{\underline{\mathcal{H}}}_{N_f,p}^T \underline{\underline{\mathcal{H}}}_{N_f,p} \right) = \underline{\underline{\mathbf{L}}}_{N_f,p} \underline{\underline{\mathbf{D}}}_{N_f,p} \underline{\underline{\mathbf{L}}}_{N_f,p}^T. \quad (5.139)$$

The updating algorithm for the computation of $(\underline{\underline{\mathbf{W}}}_{p+1}, \underline{\underline{\mathbf{B}}}_{p+1})$, on the basis of $(\underline{\underline{\mathbf{W}}}_p, \underline{\underline{\mathbf{B}}}_p)$ is based on the following decomposition of the successive channel matrices:

$$\begin{aligned} \underline{\underline{\mathcal{H}}}_{N_f,p} &= \begin{matrix} & \begin{matrix} \xleftrightarrow{\Delta} & \xleftrightarrow{N_b} & \xleftrightarrow{1} \end{matrix} \\ \begin{matrix} N_f-1 \Downarrow \\ 1 \Uparrow \end{matrix} & \left(\begin{array}{ccc} \underline{\underline{\mathbf{H}}}_0 & \underline{\underline{\mathbf{H}}}_1^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{h}}}_1^T & \underline{\underline{\mathbf{h}}}_2^T & \underline{\underline{\mathbf{h}}}_0 \end{array} \right)_p \end{matrix} \\ \underline{\underline{\mathcal{H}}}_{N_f,p+1} &= \begin{matrix} & \begin{matrix} \xleftrightarrow{1} & \xleftrightarrow{\Delta} & \xleftrightarrow{N_b} \end{matrix} \\ \begin{matrix} 1 \Downarrow \\ N_f-1 \Uparrow \end{matrix} & \left(\begin{array}{ccc} \underline{\underline{\mathbf{h}}}_0 & \underline{\underline{\mathbf{h}}}_1^T & \underline{\underline{\mathbf{h}}}_2^T \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{H}}}_0 & \underline{\underline{\mathbf{H}}}_1 \end{array} \right)_{p+1} \end{matrix} \end{aligned} \quad (5.140)$$

where $(\underline{\underline{\mathbf{H}}}_0 \ \underline{\underline{\mathbf{H}}}_1)_{p+1} = (\underline{\underline{\mathbf{H}}}_0 \ \underline{\underline{\mathbf{H}}}_1)_p$. In its updating version, the filter coefficient computation algorithm (Algorithm 5.2) becomes:

Algorithm 5.4 *Update of the FIR DF-JD coefficients :*

1. Compute the first $2K$ columns of the new MMSE matrix $\underline{\underline{\mathbf{R}}}_{N_f,p+1}$:

$$\left(\begin{array}{c} \underline{\underline{r}}_0 \\ \underline{\underline{r}}_1 \\ \underline{\underline{r}}_2 \end{array} \right)_{p+1} = \sigma_I^{-2} \left(\begin{array}{c} \underline{\underline{E}}_{2K} \\ \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} \end{array} \right) + \sigma_n^{-2} \left(\begin{array}{c} \underline{\underline{h}}_0^T \\ \underline{\underline{h}}_1 \\ \underline{\underline{h}}_2 \end{array} \right)_{p+1} (\underline{\underline{\mathbf{h}}}_0)_{p+1} \quad (5.141)$$

with

$$\bar{\underline{\underline{R}}}_{N_f, p+1} = \begin{matrix} & \begin{matrix} \xleftrightarrow{1} & \xleftrightarrow{\Delta} & \xleftrightarrow{N_b} \end{matrix} \\ \begin{matrix} 1 \downarrow \\ \Delta \Downarrow \\ N_b \Downarrow \end{matrix} & \begin{pmatrix} \underline{\underline{r}}_0 & \underline{\underline{r}}_1^T & \underline{\underline{r}}_2^T \\ \underline{\underline{r}}_1 & \underline{\underline{R}}_0 & \underline{\underline{R}}_1^T \\ \underline{\underline{r}}_2 & \underline{\underline{R}}_1 & \underline{\underline{R}}_2 \end{pmatrix}_{p+1} \end{matrix}. \quad (5.142)$$

2. Update the Cholesky factor $\bar{\underline{\underline{L}}}_{N_f, p+1}$:

$$\bar{\underline{\underline{L}}}_{N_f, p} = \begin{matrix} & \begin{matrix} \xleftrightarrow{\Delta} & \xleftrightarrow{N_b} & \xleftrightarrow{1} \end{matrix} \\ \begin{matrix} \Delta \Downarrow \\ N_b \Downarrow \\ 1 \downarrow \end{matrix} & \begin{pmatrix} \underline{\underline{L}}_0 & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{L}}_1 & \underline{\underline{L}}_2 & \underline{\underline{0}} \\ \underline{\underline{L}}_1^T & \underline{\underline{L}}_2^T & \underline{\underline{l}}_0 \end{pmatrix}_p \end{matrix} \quad (5.143)$$

$$\bar{\underline{\underline{L}}}_{N_f, p+1} = \begin{matrix} & \begin{matrix} \xleftrightarrow{1} & \xleftrightarrow{\Delta} & \xleftrightarrow{N_b} \end{matrix} \\ \begin{matrix} 1 \downarrow \\ \Delta \Downarrow \\ N_b \Downarrow \end{matrix} & \begin{pmatrix} \underline{\underline{l}}_0 & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{l}}_1 & \underline{\underline{L}}_0 & \underline{\underline{0}} \\ \underline{\underline{l}}_2 & \underline{\underline{L}}_1 & \underline{\underline{L}}_2 \end{pmatrix}_{p+1} \end{matrix}.$$

The first $2K$ columns have to be computed from scratch:

$$\begin{pmatrix} \underline{\underline{l}}_0 \\ \underline{\underline{l}}_1 \\ \underline{\underline{l}}_2 \end{pmatrix}_{p+1} \begin{pmatrix} \underline{\underline{l}}_0 \end{pmatrix}_{p+1}^T = \begin{pmatrix} \underline{\underline{r}}_0 \\ \underline{\underline{r}}_1 \\ \underline{\underline{r}}_2 \end{pmatrix}_{p+1}. \quad (5.144)$$

The $\Delta 2K$ next columns satisfy the update relation:

$$\begin{pmatrix} \underline{\underline{L}}_0 \\ \underline{\underline{L}}_1 \end{pmatrix}_{p+1} \begin{pmatrix} \underline{\underline{L}}_0 \end{pmatrix}_{p+1}^T = \begin{pmatrix} \underline{\underline{L}}_0 \\ \underline{\underline{L}}_1 \end{pmatrix}_p \begin{pmatrix} \underline{\underline{L}}_0 \end{pmatrix}_p^T - \begin{pmatrix} \underline{\underline{l}}_1 \\ \underline{\underline{l}}_2 \end{pmatrix}_{p+1} \begin{pmatrix} \underline{\underline{l}}_1 \end{pmatrix}_{p+1}^T \\ + \sigma_n^{-2} \begin{pmatrix} \underline{\underline{h}}_1 \\ \underline{\underline{h}}_2 \end{pmatrix}_{p+1} \begin{pmatrix} \underline{\underline{h}}_1 \end{pmatrix}_{p+1}^T - \sigma_n^{-2} \begin{pmatrix} \underline{\underline{h}}_1 \\ \underline{\underline{h}}_2 \end{pmatrix}_p \begin{pmatrix} \underline{\underline{h}}_1 \end{pmatrix}_p^T. \quad (5.145)$$

This problem is solved by $3 \times 2K$ successive applications of the Cholesky rank-1 update/downdate algorithm.

3. Update the feedforward matrix $\overline{\underline{W}}_{N_f, p+1}$:

$$\begin{aligned} \overline{\underline{W}}_{N_f, p} &= \begin{matrix} & \overset{N_f-1}{\overleftrightarrow{\Delta}} & \overset{1}{\overleftrightarrow{\Delta}} \\ \Delta \Downarrow & & \\ 1 \Downarrow & & \\ N_b \Downarrow & & \end{matrix} \left(\begin{array}{cc} \underline{W}_0 & \underline{0} \\ \underline{w}_1 & \underline{w}_0 \\ \underline{W}_1 & \underline{w}_2 \end{array} \right)_p \\ \overline{\underline{W}}_{N_f, p+1} &= \begin{matrix} & \overset{1}{\overleftrightarrow{\Delta}} & \overset{N_f-1}{\overleftrightarrow{\Delta}} \\ 1 \Downarrow & & \\ \Delta \Downarrow & & \\ N_b \Downarrow & & \end{matrix} \left(\begin{array}{cc} \underline{w}_0 & \underline{0} \\ \underline{w}_1 & \underline{W}_0 \\ \underline{w}_2 & \underline{W}_1 \end{array} \right)_{p+1}. \end{aligned} \quad (5.146)$$

There is a triangular system for the first $2K$ columns:

$$\left(\begin{array}{cc} \underline{l}_0 & \underline{0} \\ \underline{l}_1 & \underline{L}_0 \end{array} \right)_{p+1} \left(\begin{array}{c} \underline{w}_0 \\ \underline{w}_1 \end{array} \right)_{p+1} = \sigma_n^{-2} \left(\begin{array}{c} \underline{h}_0 \\ \underline{h}_1 \end{array} \right)_{p+1}. \quad (5.147)$$

The last $(N_f - 1)2K$ columns satisfy the update equation:

$$\left(\underline{W}_0^T \underline{L}_0^T \right)_{p+1} = \left(\underline{W}_0^T \underline{L}_0^T \right)_p \quad (5.148)$$

which can be incorporated into (5.145).

5.6.3 Computational complexity

In this section, we evaluate the number of operations required by the computation of the optimal filter coefficients for each code segment. Table 5.2 gives a rough order-of-magnitude estimate of the computational complexity associated with each of the three steps of the presented algorithms, assuming $N_f \gg L \gg 1$. The values given here correspond to the approximate number of complex scalar operations required for each matrix manipulation, taking the banded structure of the matrices into account [60].

Figures 5.8 and 5.9 give the exact number of real instructions required for the full computation of a new set of JD coefficients, and Figure 5.10 compares the number of instructions required for the full computation to that required for the updating procedure proposed above. To obtain these curves, a specific number of instructions was associated with each type of operation: 2 instructions for a multiply-and-add (which represents the majority of the operations required by the algorithms), and 1 instruction

Table 5.2: Approximate complexity (in scalar operations) associated with the fast update of FIR-JD coefficients

	Update Linear JD	Update DF-JD
1	$(2K)M^2 - \frac{1}{2}L^2$	$(2K)^2M - L$
2	$M^3 - 5(N_fL - \frac{1}{2}L^2)$	$(2K)^3 - 14(N_fL - \frac{1}{2}L^2)$
3	$(2K)M^2 - 2(N_fL - \frac{1}{2}L^2)$	$(2K)^2M - (N_fL - \frac{1}{2}L^2 + 7N_f^2)$

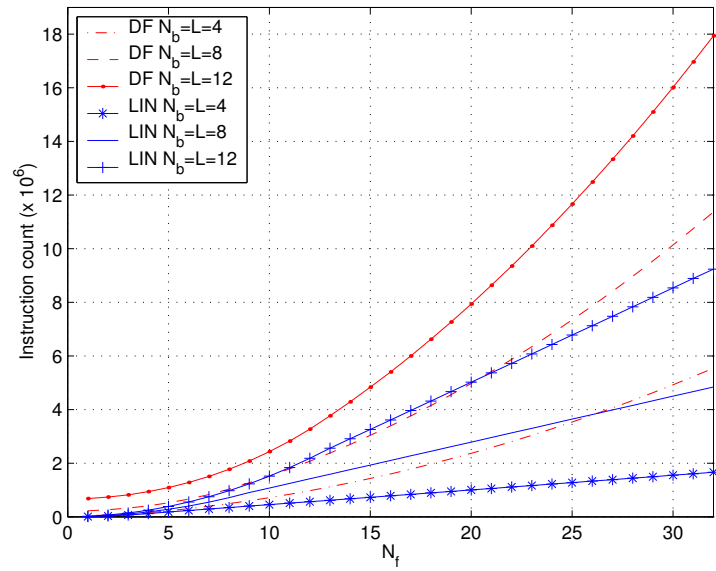


Figure 5.8: Instruction count for computing the JD coefficients vs. N_f and $N_b = L$, with $M = K = 10$.

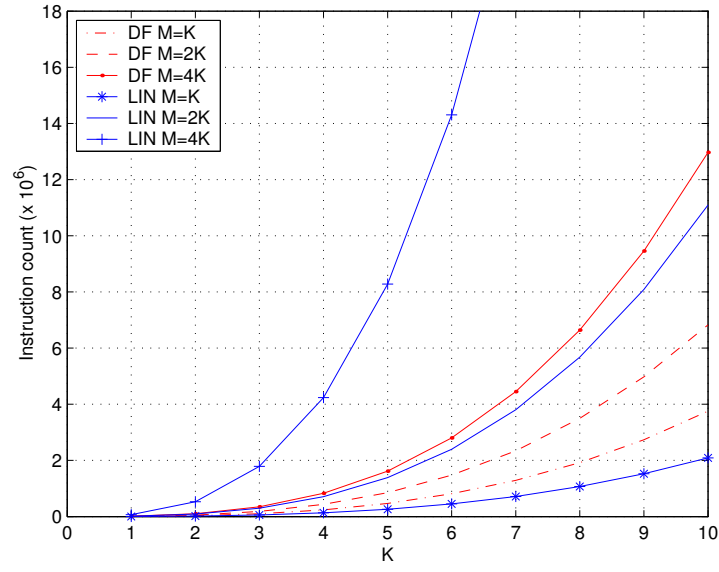


Figure 5.9: Instruction count for computing the JD coefficients vs. K and M , with $L = 8$ and $N_f = 16$.

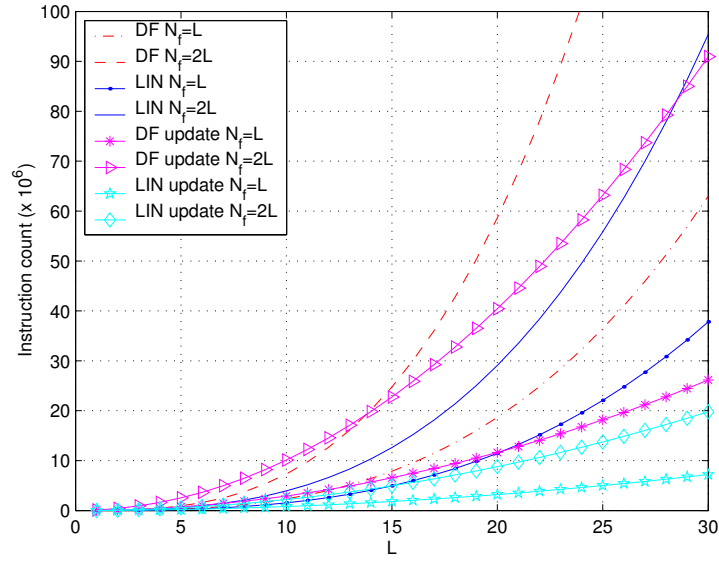


Figure 5.10: Instruction count for computing the JD coefficients vs. $L = N_b$ and N_f , with $K = 10$ and $M = 10$.

for the other operations. The displayed curves thus give approximately 2 times the number of scalar operations.

From these results, we can draw some general conclusions:

- When $M = K$, the computational complexity is higher for the DF-JD (roughly proportional to N_f^2) than for the linear JD (roughly proportional to N_f). This appears in Figure 5.8. The impact of the channel impulse response length L is comparable in the two cases.
- In the case of a long detector with $N_f \gg L$, the complexity associated with the computation of the feedforward section of the DF-JD (proportional to N_f^2) becomes prohibitive (Figure 5.8). The direct Equation (5.100) should be used instead.
- In some scenarios, it may be that $M \gg K$, especially for non-saturated systems ($K < N$) operating at a fractional rate ($N < M$). In that case, the complexity of the linear-JD algorithm is dramatically higher than its DF counterpart. This effect is illustrated in Figure 5.9. Equation (5.126) should be implemented instead of the direct Equation (5.125) to reduce the complexity. It should be noted that the fractional receiver oversampling factor can be chosen such that $M > N(1 + \alpha)$ and hence the classical value of $2N$ is not really necessary. In any case, considering that M is generally proportional to N , the resulting complexity increases with the third power of the spreading factor.
- The proposed update algorithms are effective for long channels only ($L \gg 1$). This is illustrated in Figure 5.10. This is a direct consequence of the Cholesky rank-1 update/downdate algorithm: the number of operations is $\mathcal{O}(n^2)$ instead of $\mathcal{O}(n^3)$ but the number of instructions associated with each of these operations is higher.

5.7 ZF equalization for Cyclic-Prefixed transmission

The cyclic-prefixed block transmission system presented in Section 4.6 is well-suited to low-complexity detection techniques exploiting the circulant channel structure induced by the cyclic prefix extension.

Let us consider a single-user system. The received signal, after removal of the CP, is given by

$$\underline{\mathbf{r}} = \underline{\mathbf{H}}\underline{\mathbf{I}} + \underline{\mathbf{n}} = \underline{\mathbf{C}} [\underline{\mathbf{S}}\underline{\mathbf{I}}] + \underline{\mathbf{n}} \quad (5.149)$$

where $\underline{\underline{S}}$ is the signature matrix and $\underline{\underline{C}}$ is a circulant channel matrix. The channel matrix can be factorized as follows

$$\underline{\underline{C}} = \underline{\underline{W}}_N^T \underline{\underline{\Lambda}} \underline{\underline{W}}_N \quad (5.150)$$

where $\underline{\underline{W}}_N$ is the DFT matrix of size N .

In a zero-forcing context, the effect of the frequency-selective channel can be easily suppressed by equalizing the received signal in the frequency domain :

$$\underline{y} = \underline{\underline{W}}_N \underline{r} = \underline{\underline{\Lambda}} \underline{\underline{W}}_N [\underline{\underline{S}} \underline{I}] + \underline{\underline{W}}_N \underline{n}. \quad (5.151)$$

This equalization only requires the application of a diagonal matrix on the frequency-domain samples $\underline{y}$:

$$\underline{\hat{y}} = \underline{\underline{\Lambda}}^{-1} \underline{y}. \quad (5.152)$$

Returning to the time domain, the equalized signal becomes :

$$\underline{\hat{r}} = \underline{\underline{W}}_N^T \underline{\hat{y}} = \underline{\underline{S}} \underline{I} + \left[\underline{\underline{W}}_N^T \underline{\underline{\Lambda}}^{-1} \underline{\underline{W}}_N \right] \underline{n}. \quad (5.153)$$

The symbols are finally recovered by correlating the equalized signal with the N signature codes $s_p(n)$:

$$\underline{\hat{I}} = \underline{\underline{S}}^T \underline{\hat{r}} = \underline{I} + \underline{\underline{S}}^T \left[\underline{\underline{W}}_N^T \underline{\underline{\Lambda}}^{-1} \underline{\underline{W}}_N \right] \underline{n} = \underline{I} + \underline{e}. \quad (5.154)$$

The error covariance matrix is

$$\underline{\underline{R}}_{ee} = \sigma_n^2 \underline{\underline{S}}^T \left[\underline{\underline{W}}_N^T \underline{\underline{\Lambda}}^{-2} \underline{\underline{W}}_N \right] \underline{\underline{S}}. \quad (5.155)$$

The *arithmetic mean* of the error variances

$$\frac{1}{N} \sum_{p=1}^N \sigma_{e_p}^2 = \frac{1}{N} \text{tr} [\underline{\underline{R}}_{ee}] \quad (5.156)$$

is constant for all orthogonal signature sets $\underline{\underline{S}}$ with $\underline{\underline{S}}^T \underline{\underline{S}} = \underline{\underline{E}}_N$.

What really matters, as far as the bit rate computation is concerned, is the geometrical mean of the error variances. This geometrical mean is actually different for every set of orthogonal signatures. Two extreme cases can be put forward :

- The geometrical mean of the error variances is minimum when the error covariance matrix is diagonal. Indeed, it is known that

$$\prod_{p=1}^N \left(\underline{\underline{R}}_{ee} \right)_{pp} \geq \det \left(\underline{\underline{R}}_{ee} \right) \quad (5.157)$$

with equality if and only if $\underline{\underline{R}}_{ee}$ is diagonal, and the right-hand side of (5.157) does not depend on the selected signature set. A diagonal error covariance matrix is obtained by letting $\underline{\underline{S}} = \underline{\underline{W}}$, i.e. by implementing a DMT system³. Among all CP transmission systems, DMT is then the one that minimizes the geometrical mean of error variances at the output of the linear zero-forcing receiver.

- The sum of error variances being constant, the maximum product of error variances is achieved when all variances are equal. In other words, the geometrical mean of the error variances is maximum when the error covariance matrix has a constant diagonal. This is obtained by letting $\underline{\underline{S}} = \underline{\underline{E}}_N$, i.e. by implementing a single-carrier system. Among all CP transmission systems, single-carrier is then the one that maximizes the geometrical mean of error variances at the output of the linear zero-forcing receiver. It also has the interesting property that all N symbols in the block have an identical SNR at the receiver output.

The other CP-based transmission schemes (like Hadamard transmission for instance) have an intermediate score.

To conclude this section, we propose to show that the fundamental reason for the difference of performance among the various CP transmission schemes is neither related to the CP technique nor to the choice of a zero-forcing equalizer, but is only due to the use of a linear receiver, which is not the optimal structure. Let us consider a zero-forcing decision-feedback (ZF-DF) receiver. In this structure, the feedforward part of the equalizer is no longer made of one tap per subchannel in the frequency domain, but is a full matrix described in the time domain. It is again assumed that the receiver first removes the cyclic prefix, so that it works with the same received vector $\underline{\underline{r}}$. The symbol estimates are now

$$\hat{\underline{\underline{I}}} = \underline{\underline{F}} \underline{\underline{r}} - \left[\underline{\underline{B}} - \underline{\underline{E}}_N \right] \underline{\underline{I}} \quad (5.158)$$

³Actually, baseband DMT can be described by means of *real* signature codes $s_p(n)$. See a discussion about this in Section 4.2.3.

where $\underline{\underline{B}}$ is the causal feedback section, i.e. an upper triangular matrix with ones on the main diagonal. The derivation of the optimal matrices $\underline{\underline{F}}$ and $\underline{\underline{B}}$ under a ZF constraint [62] requires the following Cholesky factorization :

$$\sigma_n^{-2} \underline{\underline{H}}^T \underline{\underline{H}} = \underline{\underline{L}} \underline{\underline{D}} \underline{\underline{L}}^T. \quad (5.159)$$

The optimal matrices are then given by

$$\begin{aligned} \underline{\underline{F}} &= \sigma_n^{-2} \underline{\underline{D}}^{-1} \underline{\underline{L}}^{-1} \underline{\underline{H}}^T \\ \underline{\underline{B}} &= \underline{\underline{L}}^T. \end{aligned} \quad (5.160)$$

From (5.158) and (5.160), the symbol estimates are

$$\hat{\underline{\underline{I}}} = \underline{\underline{I}} + \sigma_n^{-2} \underline{\underline{D}}^{-1} \underline{\underline{L}}^{-1} \underline{\underline{H}}^T \underline{\underline{n}}. \quad (5.161)$$

The error covariance matrix is $\underline{\underline{R}}_{ee} = \underline{\underline{D}}^{-1}$. The geometrical average of the error variances becomes

$$\frac{1}{N} \prod_{p=1}^N (\underline{\underline{R}}_{ee})_{pp} = \frac{1}{N} \det(\underline{\underline{D}})^{-1} \quad (5.162)$$

$$= \frac{1}{N} \det(\underline{\underline{L}} \underline{\underline{D}} \underline{\underline{L}}^T)^{-1} \quad (5.163)$$

$$= \frac{1}{N} \det\left(\sigma_n^{-2} \underline{\underline{S}}^T \underline{\underline{W}}_N^T \underline{\underline{\Lambda}}^2 \underline{\underline{W}}_N \underline{\underline{S}}\right)^{-1} \quad (5.164)$$

$$= \frac{1}{N} \prod_{p=1}^N \left(\frac{\Lambda_p^2}{\sigma_n^2}\right)^{-1}. \quad (5.165)$$

At the output of a DF-ZF receiver, the geometrical mean of the error variances is thus independent of the signature set. Furthermore, it can be easily verified that the DF-ZF receiver is identical to the linear-ZF receiver in the case of DMT transmission. The same level of performance (as measured by the geometrical mean of error variances) is then accessible to all types of CP transmission schemes as soon as ZF decision-feedback equalization is performed in the receiver. This receiver, however, is rather complex, except for the case of DMT transmission, where it is identical to the simple ZF linear receiver.

5.8 Applications to the powerline channel

In this Section, the performance of a complete PLC system is analyzed, combining the multiple access techniques proposed in Chapter 4, the chan-

nel model introduced in Chapter 2, and the MMSE Joint Detector derived in Chapter 5.

The frequency band available for powerline communications is supposed to be in the range of 1 MHz to 10 MHz. Passband transmission is thus required. A maximum power spectral density of $\bar{\gamma}(f) = -60$ dBm/Hz is imposed for the transmitted signals in the whole frequency band.

A CAP-FB transmission system with a central frequency of $f_c = 5.5$ MHz, a chip rate of $1/T'_c = 9$ MHz, and an average transmission power of $P = 9$ mW, is compatible with the proposed transmission constraints. The use of a paraunitary FB with symbols of equal variance on all signatures guarantees a flat power spectral density for the transmitted signal sum. The chip shaping filters are sine- and cosine-modulated half-root Nyquist filters with a roll-off factor of $\alpha = 0.22$. Taking into account the roll-off effect, the effective bandwidth of the transmitted signals is roughly 0 to 11 MHz. The chip duration is $T'_c = 111$ ns.

The powerline network considered in the next computations is given in Figure 2.12. The $K = 5$ users in positions 2, 5, 10, 15 and 20 are simultaneously active, and the differential propagation mode is considered. Unused terminations are assumed to be left open. The computed end-to-end channel responses are illustrated in the time domain in Figure 2.22, and in the frequency domain (0-50 MHz) in the lower part of Figure 2.23. The top of Figure 3.15 also gives the channel frequency responses in the range of 0 to 10 MHz. The minimum delay between two successive paths is approximately 50 ns. Combining the significant channel paths with the cable responses and the modulated shaping filters, the global channel impulse responses have an extension of approximately $9 \mu\text{s}$, i.e. $L_c = 81$ chips.

In the next computations, the AWGN level is chosen as $N_0 = -140$ dBm/Hz, which is a rather optimistic noise scenario.

The receiver operates at a fractional rate M/T'_c with $M = 3$. A finite-length MMSE DF-JD provides noisy estimates $\hat{I}_p(n)$ of the transmitted symbols. The SINR's ρ_p at the N detector outputs are computed assuming perfect past decisions, and the user bit rates are finally obtained according to (4.56). The SNR gap used in these computations is $\Gamma = 1$ (which corresponds to ideal transmission).

5.8.1 CAP-FB systems : OFDMA, CDMA and TDMA

Three types of FBs of length $N = 128$ are firstly compared (see Section 4.2.3) : DFT filters (CAP-OFDMA), Hadamard codes (CAP-CDMA) and single-carrier signatures (CAP-TDMA). For this choice of N , the symbol duration is $T = NT'_c = 14\mu\text{s}$. The global channel impulse responses have thus an extension of less than one symbol ($L = 1$). For this configuration, an observation window of $N_f = 3$ successive symbols, combined with a decision delay of $\Delta = 2$ symbols and a feedback section of length $N_b = 1$, is sufficient to approach the optimal performance.

Tables 5.3, 5.4 and 5.5 give the bit rates (in Mbits/s) obtained with the three CAP-FB systems. The first rows correspond to the single-user configurations, obtained when all of the N signatures are allocated to a single user. As expected, the SU rates are identical for the three filter banks. In a multiuser configuration, a balanced signature allocation should be derived, that would provide each user with a bit rate in proportion with its own potential :

- Using TDM(A), the channel can be dedicated to each user in turn, which corresponds to K successive SU configurations, with time intervals of equal duration for each user. In that scenario, each user gets 20 % of its SU bit rate. The obtained rates are given in the second rows. A better performance can be expected by using a smart signature allocation algorithm.
- In the downlink, the partial bit rates R_{kp} , obtained with the MMSE DF-JD by allocating signature p to user k , can be computed once for all. Algorithm 3.6 introduced in Chapter 3 can then be used to solve the problem of optimal subchannel assignment with balanced rate constraints. The obtained maximum balanced rates are given in the third rows.
- In the uplink, the subchannel assignment problem is much more difficult to solve. Indeed, by using the MMSE DF-JD, the partial bit rates R_{kp} for a given subchannel p actually depend on the allocation of the $N - 1$ other subchannels. The following method was used to obtain a good (if not optimal) signature allocation. Starting from the optimal signature allocation for the downlink (which is easily obtained with Algorithm 3.6), some signatures are successively transferred from one user to another, until the balanced rate constraints are almost satisfied. Five quasi-balanced signature allocations can then be obtained,

Table 5.3: *CAP-OFDMA: maximum balanced rates (in Mbits/s)*

Configuration	User 2	User 5	User 10	User 15	User 20	%
SU-OFDM	218.13	195.90	156.94	123.04	92.98	100.0
TDMA-OFDM	43.63	39.18	31.39	24.61	18.60	20.0
downlink-OFDM	51.42	46.18	37.00	29.00	21.92	23.6
uplink-OFDMA	52.39	47.05	37.69	29.55	22.33	24.0
Allocation 1	52.57	47.08	38.54	26.99	24.05	
Allocation 2	53.12	48.33	36.94	28.90	22.26	
Allocation 3	51.02	46.75	39.99	28.88	22.00	
Allocation 4	51.56	50.02	38.48	27.36	22.01	
Allocation 5	52.75	46.51	36.80	30.61	22.26	

Table 5.4: *CAP-CDMA: maximum balanced rates (in Mbits/s)*

Configuration	User 2	User 5	User 10	User 15	User 20	%
SU-CDMA	218.13	195.90	156.94	123.04	92.98	100.0
TDMA-CDMA	43.63	39.18	31.39	24.61	18.60	20.0
downlink-CDMA	49.92	44.83	35.92	28.16	21.28	22.9
uplink-CDMA	50.81	45.64	36.56	28.66	21.66	23.3
Allocation 1	51.55	44.33	36.00	30.19	21.52	
Allocation 2	51.23	45.47	37.21	28.28	21.58	
Allocation 3	51.27	47.85	35.40	28.11	21.55	
Allocation 4	50.53	44.33	37.07	29.97	20.94	
Allocation 5	50.48	44.49	37.18	27.96	22.65	

Table 5.5: *CAP-TDMA: maximum balanced rates (in Mbits/s)*

Configuration	User 2	User 5	User 10	User 15	User 20	%
SU-SC	218.13	195.90	156.94	123.04	92.98	100.0
TDMA-SC	43.63	39.18	31.39	24.61	18.60	20.0
downlink-TDMA	43.63	39.18	31.39	24.61	18.60	20.0

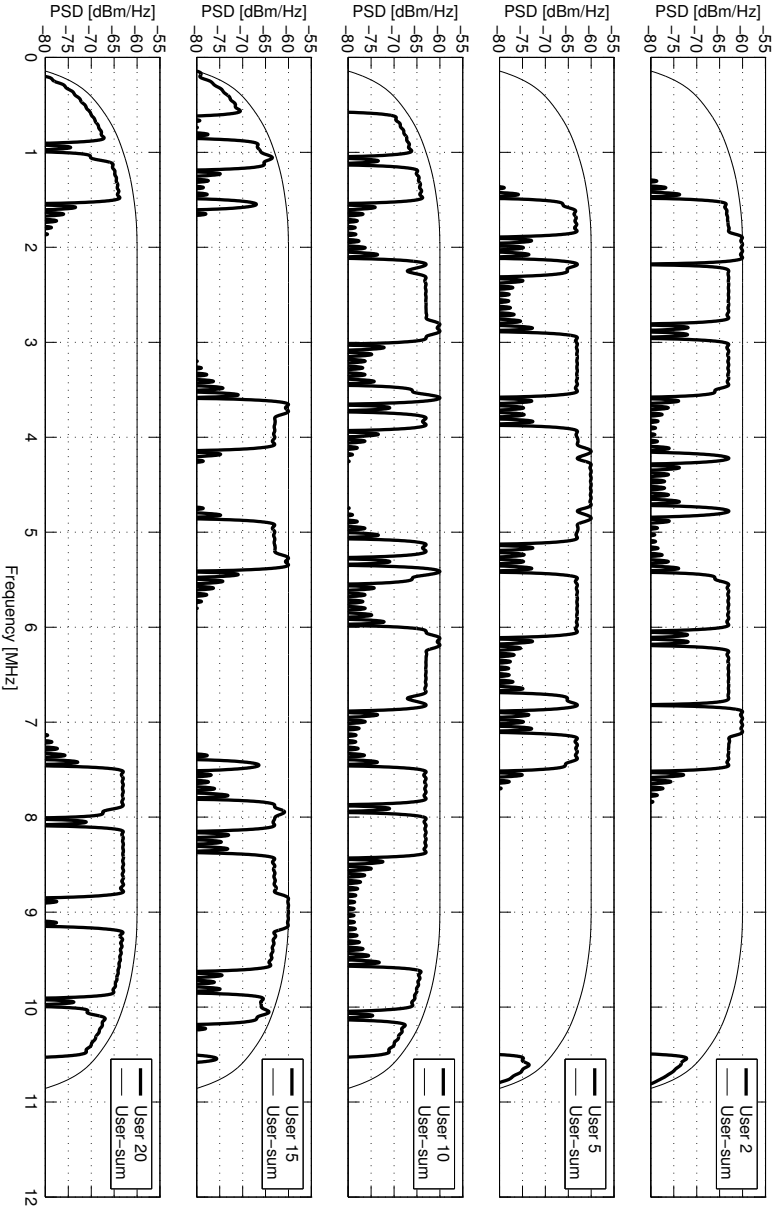


Figure 5.11: *Power spectral density of the transmitted signals, CAP-OFDMA with DFT tones of length $N = 128$.*

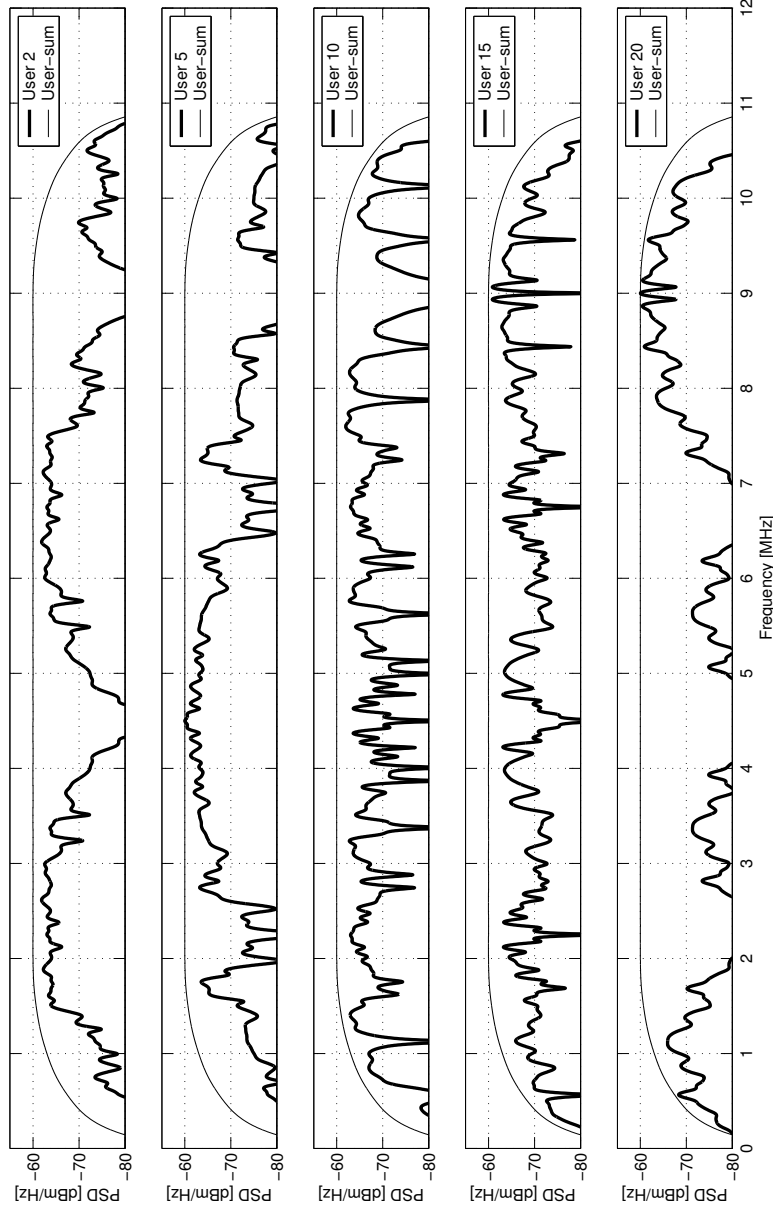


Figure 5.12: Power spectral density of the transmitted signals, CAP-CDMA with Hadamard codes of length $N = 128$.

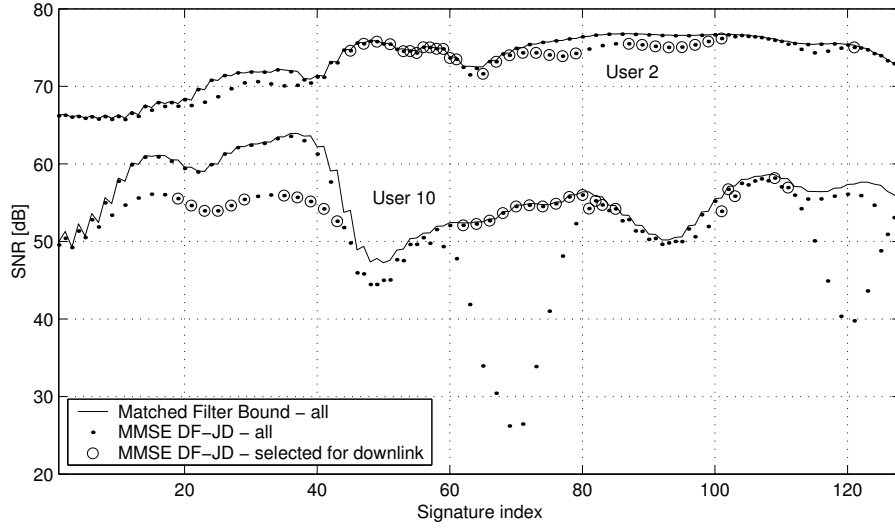


Figure 5.13: *SINR at the output of the MMSE DF-JD vs. tone index, CAP-OFDMA with $N = 128$ tones.*

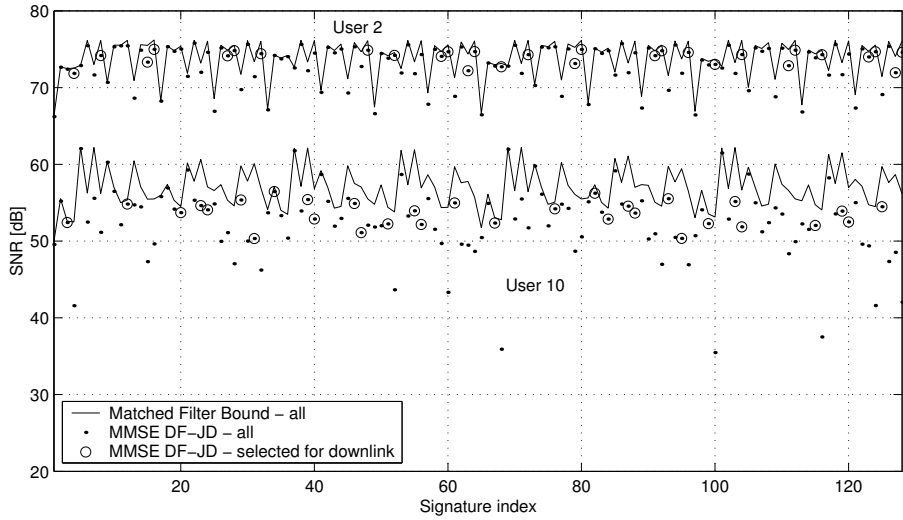


Figure 5.14: *SINR at the output of the MMSE DF-JD vs. code index, CAP-CDMA with $N = 128$ codes.*

Table 5.6: *CAP-CDMA with long codes: maximum balanced rates (in Mbits/s)*

Configuration	User 2	User 5	User 10	User 15	User 20	%
SU-CDMA	218.13	195.90	156.94	123.05	92.99	100
TDMA-CDMA	43.63	39.18	31.39	24.61	18.60	20
uplink-CDMA	44.29	40.10	32.02	25.90	20.85	
	20.31 %	20.47 %	20.40 %	21.05 %	22.43 %	

that differ from each other by only one subchannel. Perfectly balanced rates are finally obtained by time-sharing between the 5 proposed allocations, with appropriate sharing coefficients. The uplink balanced rates obtained with this method are given in the fourth rows, while the last rows give the user rates corresponding to the 5 discrete signature allocations.

In a multiuser configuration, the performance of the MMSE DF-JD now depends on the selected filter bank. No gain can be expected from the CAP-TDMA system (single-carrier signatures), as all signatures have an identical spectrum. A rate gain is achieved with the two other systems as they are able to shape the spectrum of the transmitted signals in accordance with the physical channels.

Figures 5.11 and 5.12 give the power spectral density (in dBm/Hz) of the transmitted signals in uplink, for the selected signature allocations in CAP-OFDMA and CAP-CDMA. These Figures also illustrate how different filter banks are able to shape the spectrum of the transmitted signals with specific degrees of freedom.

Figures 5.13 and 5.14 show the distribution of the SINRs at the output of the MMSE DF-JD vs. the signature index, for both the single-user CAP-OFDMA and CAP-CDMA systems. The continuous lines give the matched filter bound. Circles are used to denote the signatures selected for users 2 and 10 by the allocation algorithm in the downlink.

5.8.2 CAP-CDMA with long codes

For the purpose of comparison, a long-code CAP-CDMA system is considered, with short Hadamard codes of length $N = 20$, and a long-code with a periodicity of $P = 64$ segments obtained by truncating a Gold code of length 1023. For this choice of N , the symbol duration is $T = NT'_c = 2.22\mu\text{s}$. The

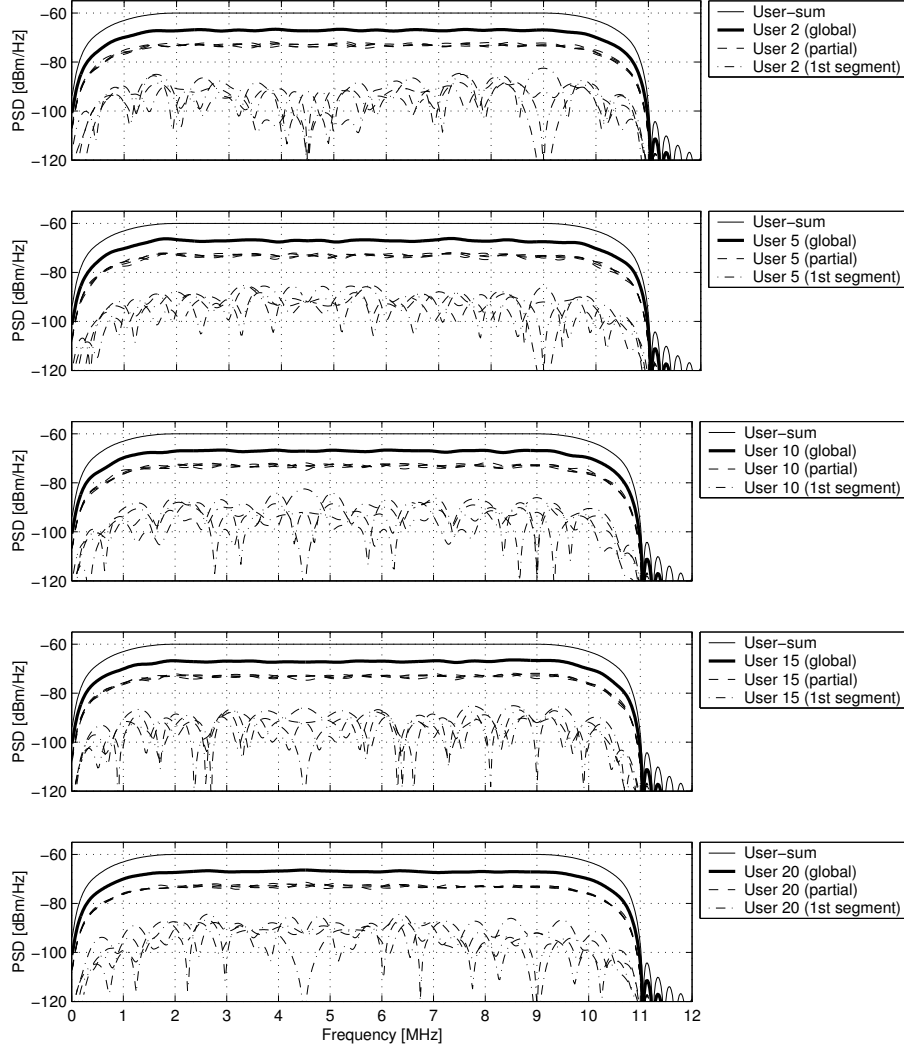


Figure 5.15: Power spectral density of the transmitted signals, CAP-CDMA with long codes (short spreading codes: Hadamard codes of length $N = 20$, long code periodicity: $P = 64$ symbols).

global channel impulse responses have thus an extension of $L = 5$ symbols. For this configuration, an observation window of $N_f = 12$ successive symbols, combined with a decision delay of $\Delta = 11$ symbols and a feedback section of length $N_b = 5$, is sufficient to approach the optimal performance.

Table 5.6 gives the computed user rates (in Mbits/s). The first row gives the single-user rates, obtained by allocating the $N = 20$ spreading codes to the same user. These rates are almost identical to the SU rates obtained with a paraunitary FB (and no long code). The second row gives the balanced rates obtained by time-sharing between single-user configurations. The third row gives the user rates obtained by allocating arbitrarily 4 spreading codes to each of the 5 users. These rates are not exactly balanced : a slight power control could be investigated to obtain perfectly balanced rates. Actually the rate gain with respect to the time-sharing solution is negligible.

Figure 5.15 illustrates the power spectral density of the transmitted signals, which are approximately flat in the whole frequency band. The global PSD of a given user is obtained by combining the PSDs of the 4 subchannels (4 spreading codes). The PSD of a given subchannel is then obtained by combining the PSDs associated with each of the 64 code segments.

Finally, Figure 5.16 gives the distribution of the SINR at the output of the MMSE DF-JD as a function of the code segment, for each of the 4 spreading codes of users 2, 5, 10 and 20. The horizontal dashed lines give the average SINR associated with each spreading code (obtained by considering the 64 code segments). It turns out that the different symbols associated with a given spreading code may have quite different SINRs.

5.9 Conclusion

In this Chapter, fundamental limits have been derived on the performance of the FB-based multiple access scheme presented in Chapter 4. The matched filter bound, on one hand, provides a useful information on the maximum signal to noise ratio that can be expected by allocating a given signature to a given user. This bound can be a starting point for a signature allocation algorithm. This bound, however, is not achievable in a practical system. In particular, it does not account for the mutual interference between signatures.

A more useful bound is the capacity region of the FB-based multiple access system. It differs from the general capacity region defined in Chapter 3

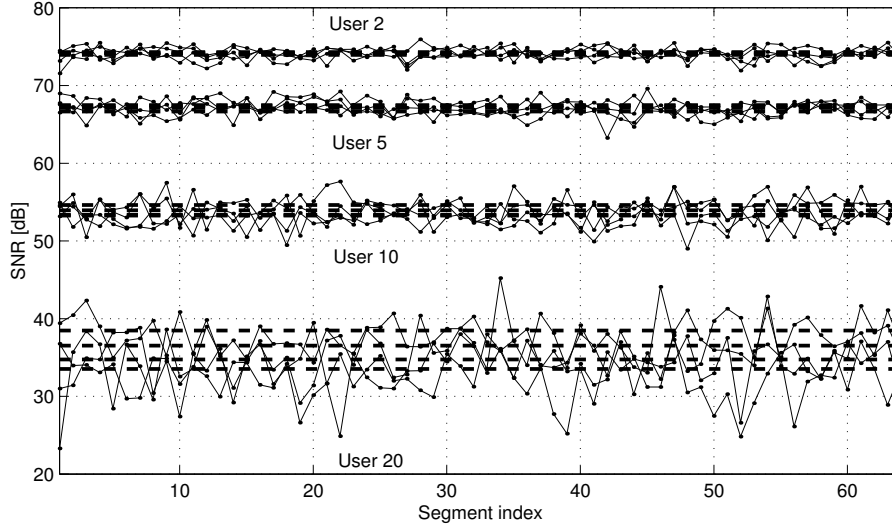


Figure 5.16: *SINR at the output of the MMSE DF-JD vs. code segment, CAP-CDMA with $N = 20$ spreading codes, and a long code periodicity of $P = 64$.*

by the fact that the power spectral density of the transmitted signals is now restricted by the selected FB. In other words, the signature allocation algorithm allows the shaping of the transmitted PSDs, but with a limited flexibility that is closely related to the nature of the FB and the number of available subchannels. For a paraunitary FB, the FB-based single-user capacity has been shown to be independent of the FB, and identical to the true (PSD-constrained) single-user capacity. The boundary of the capacity region, however, differs from one FB to another, as it depends on the flexibility with which a given FB is able to shape the PSD of the individual transmitted signals. For a given FB, the computation of the balanced capacity was shown to be extremely difficult. Lower and upper bounds have been proposed, which are easier to compute.

Linear and decision-feedback MMSE joint detectors suited to the FB-based system have been proposed as a practical and efficient solution. Infinite detectors were firstly introduced. Their performance provide an upper bound on that of the practical finite-length detectors. The arithmetic mean of the estimation error variances (for the linear JD) and the geometrical mean of the estimation error variances (for the DF-JD) were shown to be independent of the FB if the paraunitary property is satisfied. Moreover, it

was shown that the single-user capacities and the sum-capacity of the multiuser channel could be achieved by means of an infinite DF-JD followed by AWGN decoders. This proves that the MMSE DF-JD is an efficient solution to transform the initial FB-based frequency-selective system into a set of parallel and independent subchannels. All points of the capacity region are not achievable with the DF-JD, however, as successive decoding among the users is generally required, which is not feasible with a causal DF-JD.

The practical computation of joint detector coefficients has been analyzed in detail as a function of the system parameters. Complex matrix manipulations are required (e.g. Cholesky factorizations), but the effective complexity of these manipulations is reduced if the banded structure of the matrices is taken into account. Anyway, the resulting complexity is still proportional to the third power of the spreading factor N (which is also the number of signatures), which restricts the feasibility of the proposed system to moderate values of N . The number of subchannels should be sufficient, however, to allow the spectral shaping of the transmitted signals with a good flexibility. The final choice for the number of subchannels is thus a trade-off between performance and complexity issues.

The MMSE JD was shown to be also compatible with the long-code CAP-CDMA system. The price to pay is the need for a systematic update of the detector coefficients for every new symbol, which dramatically increases the computational complexity of the receiver. Fast update algorithms have been proposed to reduce that complexity, but the complexity reduction is rather low. For this reason, long-code CAP-CDMA is not a good candidate, and short-code CAP-CDMA with an efficient code allocation algorithm should be preferred.

Finally, the linear zero-forcing receiver has been shown to be a powerful low-complexity solution suited to the cyclic-prefixed FB-based *downlink* transmission scheme. In each receiver, the signal just needs to be equalized by a single coefficient in the frequency domain. Back to the time domain, the symbols are finally recovered by a simple correlation with the adequate signature code. This solution has a low complexity, even with large values of N . It represents an attractive alternative to the MMSE-JD for the downlink. In order to limit the rate loss induced by the cyclic prefix, however, the number of subchannels N should be very large.

