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Uniform concentration bounds for frequencies of rare events

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Abstract

New Vapnik–Chervonkis type concentration inequalities are derived for the empirical distribution of an independent random sample. Focus is on the maximal deviation over classes of Borel sets within a low probability region. The constants are explicit, enabling numerical comparisons.

Keywords: VC theory, Concentration Inequalities, Rare Events

1. Introduction

Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$. We always let μ_n denote the associated *empirical measure*, i.e., $\mu_n := n^{-1} \sum_{i=1}^n \delta_{X_i}$. We already know, from [Vapnik \(1971, Theorem 2\)](#), the *Vapnik and Chervonenkis (VC) inequality* which states that

$$\mathbb{P}\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t\right) \leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/8}, \quad (1.1)$$

where $\mathbb{S}_{\mathcal{A}}(n) := \max_{x_1, \dots, x_n \in \mathbb{R}^d} |\{x_1, \dots, x_n\} \cap A : A \in \mathcal{A}|$ is the *shattering coefficient* of the class $\mathcal{A}$. For background on VC theory, we refer to [Lugosi \(2002\)](#).

Setting the upper bound equal to $\delta \in (0, 1)$ and solving for t yields that, with probability $1 - \delta$,

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq 2\sqrt{\frac{2}{n} [\log(4/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}. \quad (1.2)$$

In [Appendix A](#), we show an improvement of this inequality that will be used in [Section 2](#).

Our main goal is to develop bounds with *explicit constants* when working with low probability regions ([Section 2](#)), i.e., it will be assumed that there exists some Borel set $\mathbb{A} \subseteq \mathbb{R}^d$ which contains every set $A \in \mathcal{A}$ and has a “small” probability

$$p := \mu(\mathbb{A}). \quad (1.3)$$

For example, consider an empirical risk minimization problem where we use the empirical risk to classify some data. In that situation, p could be 1% if we are looking to understand how the prediction behaves when the input variables take values above the 99th percentile of the distribution of $\|X\|$, where X has the same distribution as the data in the training set (we refer to [Goix et al. \(2015, Remark 5\)](#), [Jalalzai et al. \(2018\)](#) and [Cléménçon et al. \(2021, Section 4.2\)](#) for classification in extreme regions).

The bounds should get smaller since all the encountered events have a small probability; however, the estimation is more difficult since we have less data in those regions. In this extreme events framework, a first improvement of the

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VC inequality (1.2), based on [Anthony and Shawe-Taylor \(1993, Theorem 2.1\)](#) and [Lugosi \(2002, Theorem 1.11\)](#), is given by

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq 2 \sqrt{\frac{2p}{n} [\log(12/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}. \quad (1.4)$$

The details of the statement and the proof are deferred to [Appendix B](#). Even though the factor p in the square root will improve the bound, this result is still not fully satisfactory. The *effective sample size*, i.e., the expected number of data points in the extreme region, is np rather than n . Therefore, the shattering coefficient involved in the bound seems too large.

This work is heavily motivated by [Goix et al. \(2015\)](#) who already introduced a VC-type inequality adapted to extreme events. However, unlike the classical VC bound, the constants appearing in their result are not explicit. A similar remark can be made about [Giné and Koltchinskii \(2006\)](#) where concentration inequalities are also introduced for normalized empirical processes, in a very general setting. In [Section 3](#) and [Section 4](#), we discuss different methods that lead to new, explicit, inequalities. Essentially, there are two possibilities: either we directly apply the standard tools of VC theory on the maximal deviation itself ([Section 3](#)), or we use a variation of the classical *McDiarmid bounded differences inequality* ([McDiarmid, 1998, Theorem 3.8](#)) and then we deal with the expectation $\mathbb{E} [\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|]$ ([Section 4](#)). After deriving the different possible bounds, we compare them in [Section 5](#).

Remark 1 (Pointwise measurability). To ensure measurability of the supremum appearing in (1.1), a common hypothesis is to assume that the class $\mathcal{A}$ is *pointwise measurable*, as suggested by [van der Vaart and Wellner \(1996, Example 2.3.4\)](#), i.e., that there exists a subclass $\mathcal{A}_0 \subseteq \mathcal{A}$, at most countable, such that for every $A \in \mathcal{A}$, there exists a sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{1}_{A_n}(x) = \mathbb{1}_A(x), \quad \text{for every } x \in \mathbb{R}^d.$$

It is easily showed that, under this assumption, $\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| = \sup_{A \in \mathcal{A}_0} |\mu_n(A) - \mu(A)|$ and we may replace $\mathcal{A}$ by $\mathcal{A}_0$ everywhere.

2. Inequalities for extreme events

The main tool to develop adapted bounds is expressed in the following lemma, which already appears in [Goix et al. \(2015, Page 17\)](#) and the idea of which is likely to be found in other places in the literature as well, see, e.g., [Novak \(2011, Equation \(14.6\)\)](#) for a result in the same spirit. The proof is elementary and can be found in [Lhaut \(2021, Chapter 3\)](#).

Lemma 2 (Conditioning trick). *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). Let $Y_1, \dots, Y_n$ be an independent random sample with common distribution*

$$\mu_{\mathbb{A}}(\cdot) = \mu(\cdot | \mathbb{A}) = \frac{\mu(\cdot \cap \mathbb{A})}{\mu(\mathbb{A})} = \frac{\mu(\cdot \cap \mathbb{A})}{p},$$

and independent of $X_1, \dots, X_n$. If, for every $k = 1, \dots, n$ we denote μ_k^Y the empirical measure associated to $Y_1, \dots, Y_k$ and if $K \sim \text{Bin}(n, p)$ denotes the number of data points X_i in $\mathbb{A}$, then the following equality in distribution holds

$$\left[(\mu_n(A))_{A \in \mathcal{A}} \mid K = k \right] \stackrel{d}{=} \left(\frac{k}{n} \mu_k^Y(A) \right)_{A \in \mathcal{A}},$$

in the sense of equality of finite-dimensional distributions: for every $A_1, \dots, A_N \in \mathcal{A}$, the random vectors

$$(\mu_n(A_1), \dots, \mu_n(A_N)) \mid K = k \quad \text{and} \quad \frac{k}{n} (\mu_k^Y(A_1), \dots, \mu_k^Y(A_N))$$

have the same multivariate distribution. In particular, under the pointwise measurability assumption ([Remark 1](#)), this equality still holds when considering the supremum over the class $\mathcal{A}$.

This result provides us with a convenient way to adapt classical VC inequalities to our setting: we start by conditioning μ_n on K , then we apply concentration inequalities on the empirical measure μ_K^Y which takes account of the extreme nature of the encountered events and finally we integrate out on K . In this last step, we will make use of the *Bernstein's inequality* for binomial random variables (Lugosi, 2002, Theorem 1.5) which states that for every $t > 0$,

$$\mathbb{P}(|K - np| \geq t) \leq 2 \exp\left(-\frac{t^2}{2(np(1-p) + t/3)}\right). \quad (2.1)$$

When we will be working with the expectation, we shall also make use of another version of the VC inequality (1.2) for the expected maximal deviation:

$$\mathbb{E} \left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \right] \leq \sqrt{\frac{2 \log(2\mathbb{S}_{\mathcal{A}}(2n))}{n}}, \quad (2.2)$$

the proof of which can be found, for example, in Lugosi (2002, Theorem 1.9).

In the bounds that we propose, we will need to define the shattering coefficient $\mathbb{S}_{\mathcal{A}}(x)$ for a *real* parameter $x > 0$. It is simply understood that $\mathbb{S}_{\mathcal{A}}(x) := \mathbb{S}_{\mathcal{A}}(\lfloor x \rfloor)$, where $\lfloor x \rfloor$ denotes the integer part of x .

3. Working directly with the maximal deviation

Using the improvement of the VC inequality that we show in Appendix A, we derive a first inequality.

Theorem 3. *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). If $np \geq 4 \log(4/\delta)$, where $\delta \in (0, 1)$, then we have, with probability $1 - \delta$,*

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \frac{2}{3n} \log(4/\delta) + \sqrt{\frac{p}{n}} \left(\sqrt{2 \log(4/\delta)} + 2 \sqrt{\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(4np)} + 1 \right).$$

Proof of Theorem 3. Let $t > 0$. If $K = \sum_{i=1}^n \mathbb{1}\{X_i \in \mathbb{A}\} \sim \text{Bin}(n, p)$, then by Lemma 2,

$$\begin{aligned} \mathbb{P} \left(\sup_{A \in \mathcal{A}} \frac{|\mu_n(A) - \mu(A)|}{p} \geq t \right) &= \mathbb{E} \left[\mathbb{P} \left(\sup_{A \in \mathcal{A}} \frac{|\mu_n(A) - \mu(A)|}{p} \geq t \mid K \right) \right] \\ &= \mathbb{E} \left[\mathbb{P} \left(\sup_{A \in \mathcal{A}} \left| \frac{K}{np} \mu_K^Y(A) - \mu_{\mathbb{A}}(A) \right| \geq t \mid K \right) \right] \\ &\leq \mathbb{E} \left[\mathbb{P} \left(\frac{K}{np} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| + \left| \frac{K}{np} - 1 \right| \geq t \mid K \right) \right]. \end{aligned}$$

Let $t = u + s$ for $u, s > 0$. Then, the latter quantity is bounded by

$$\mathbb{P} \left(\left| \frac{K}{np} - 1 \right| \geq s \right) + \mathbb{E} \left[\mathbb{P} \left(\frac{K}{np} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \geq u \mid K \right) \mathbb{1} \left\{ \left| \frac{K}{np} - 1 \right| \leq s \right\} \right]. \quad (3.1)$$

Let $\delta \in (0, 1)$. We will pick $u = u(\delta)$ and $s = s(\delta)$ such that each term in (3.1) is bounded by $\delta/2$.

We deal with the first term of (3.1) directly using the Bernstein inequality (2.1),

$$\mathbb{P} \left(\left| \frac{K}{np} - 1 \right| \geq s \right) \leq \mathbb{P}(|K - np| \geq nps) \leq 2 \exp \left(-\frac{n^2 p^2 s^2}{2(np(1-p) + nps/3)} \right) \leq 2 \exp \left(-\frac{nps^2}{2(1+s/3)} \right).$$

The positive root s_+ associated with the quadratic equation obtained by equaling this last term to $\delta/2$ satisfies

$$s_+ \leq \frac{2}{3np} \log(4/\delta) + \sqrt{\frac{2}{np} \log(4/\delta)} =: s(\delta).$$

To deal with the second term of (3.1), we use the improved VC inequality that we develop in [Appendix A](#) in the form (A.1),

$$\begin{aligned} \mathbb{P}\left(\frac{K}{np} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \geq u \mid K\right) &= \mathbb{P}\left(\sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \geq \frac{np u}{K} \mid K\right) \\ &\leq 4\mathbb{S}_{\mathcal{A}}(2K) \exp\left(-\frac{(np)^2 u^2}{2K} \left(1 - \sqrt{\frac{2K}{(np)^2 u^2}}\right) - \frac{1}{4}\right). \end{aligned}$$

By monotonicity in K on the region of interest ¹, we get

$$\mathbb{E}\left[\mathbb{P}\left(\frac{K}{np} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \geq u \mid K\right) \mathbb{1}\left\{\left|\frac{K}{np} - 1\right| \leq s\right\}\right] \leq 4\mathbb{S}_{\mathcal{A}}(2np(1+s)) \exp\left(-\frac{np u^2}{2(1+s)} \left(1 - \sqrt{\frac{2(1+s)}{np u^2}}\right) - \frac{1}{4}\right).$$

If $np \geq 4 \log(4/\delta)$, we have $s(\delta) \leq \frac{1}{6} + \frac{1}{\sqrt{2}} \leq 1$. Hence, with $s = 1$, the second term is bounded by

$$4\mathbb{S}_{\mathcal{A}}(4np) \exp\left(-\frac{np}{4} u^2 + \frac{\sqrt{np}}{2} u - \frac{1}{4}\right).$$

This latter expression equals $\delta/2$ if

$$u = u(\delta) := \frac{1}{\sqrt{np}} + 2\sqrt{\frac{1}{np} [\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(4np)]}.$$

Regrouping the two terms, we obtain that with probability at least $1 - \delta$,

$$\sup_{A \in \mathcal{A}} \frac{|\mu_n(A) - \mu(A)|}{p} \leq \frac{2}{3np} \log(4/\delta) + \sqrt{\frac{1}{np}} \left(\sqrt{2 \log(4/\delta)} + 2 \sqrt{\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(4np)} + 1 \right).$$

Multiplying both sides by p , we get the result. $\square$

Another possibility consists of symmetrizing the process, based on our improved version of the classical argument (Lemma 10), before using the conditioning trick. It leads to a simpler bound.

Theorem 4. *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). If $np \geq 2 \log(8/\delta)$, where $\delta \in (0, 1)$, then we have, with probability $1 - \delta$,*

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \sqrt{\frac{2p}{n}} \left(2 \sqrt{\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(8np)} + 1 \right).$$

Proof of Theorem 4. Let $a \in (0, 1)$ and $t > 0$. By symmetrization (Lemma 10), if $4(1-a)^2 nt^2 \geq 2$,

$$\mathbb{P}\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t\right) \leq 4 \mathbb{P}\left(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu'_n(A)) \geq at\right),$$

where μ'_n is the empirical measure associated to $X'_1, \dots, X'_n$, an independent ghost sample with common distribution μ and independent of $X_1, \dots, X_n$. We easily verify that, since $\mu(A) \leq p$ for all $A \in \mathcal{A}$, the symmetrization remains true under the weaker hypothesis that

$$(1-a)^2 nt^2 \geq 2p. \quad (3.2)$$

¹The map $x \in (0, +\infty) \mapsto -x(1 - \sqrt{1/x}) = -x + \sqrt{x}$ is decreasing on $(1/4, +\infty)$. Our focus is on that region of the real line since, in our situation, $x = \frac{(np)^2 u^2}{2K}$ with $u = u(\delta) \geq 1/\sqrt{np}$ and $K \leq np(1 + s(\delta)) \leq 2np$.

It suffices to adapt the bound on the variance $\text{Var}[\mathbb{1}\{X_i \in A\}]$ in the proof of Lemma 10.

Let $\tilde{K} = \sum_{i=1}^n \mathbb{1}\{X_i \in \mathbb{A} \text{ or } X'_i \in \mathbb{A}\} = \sum_{i=1}^n \mathbb{1}\{(X_i, X'_i) \in \tilde{\mathbb{A}}\}$, where $\tilde{\mathbb{A}} = (\mathbb{A} \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \mathbb{A})$. Then, if $(Y_i, Y'_i)_{i=1}^n$ is an independent sample, also independent of $(X_i, X'_i)_{i=1}^n$, with common distribution given by the conditional distribution of (X_1, X'_1) when it lies in $\tilde{\mathbb{A}}$, we have by Lemma 2

$$\begin{aligned} & \mathbb{P}\left(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu'_n(A)) \geq at\right) \\ &= \mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{1}{n} \sum_{i=1}^n (\mathbb{1}\{X_i \in A\} - \mathbb{1}\{X'_i \in A\}) \geq at\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{1}{n} \sum_{i=1}^n (\mathbb{1}\{X_i \in A\} - \mathbb{1}\{X'_i \in A\}) \geq at \mid \tilde{K}\right)\right] \\ &= \mathbb{E}\left[\mathbb{P}\left(\frac{\tilde{K}}{n} \sup_{A \in \mathcal{A}} \frac{1}{\tilde{K}} \sum_{i=1}^{\tilde{K}} (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}) \geq at \mid \tilde{K}\right)\right] \\ &= \mathbb{E}\left[\mathbb{P}\left(\sup_{A \in \mathcal{A}} \sum_{i=1}^{\tilde{K}} (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}) \geq nat \mid \tilde{K}\right)\right]. \end{aligned}$$

Randomizing using *Rademacher random variables*, we have

$$\sum_{i=1}^{\tilde{K}} (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}) \stackrel{d}{=} \sum_{i=1}^{\tilde{K}} \sigma_i (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}),$$

where $\tilde{K}$ is fixed and $\sigma_1, \dots, \sigma_{\tilde{K}}$ are i.i.d. sign random variables independent of $(Y_1, Y'_1), \dots, (Y_n, Y'_n)$, i.e., for every $i \in \{1, \dots, \tilde{K}\}$, $\mathbb{P}(\sigma_i = 1) = \mathbb{P}(\sigma_i = -1) = 1/2$. Conditioning on $(Y_i, Y'_i)_{i=1}^n$ and applying Hoeffding's inequality (Lugosi, 2002, Theorem 1.2) to each of the $\mathbb{S}_{\mathcal{A}}(2\tilde{K})$ possible vectors $(\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\})$ that can arise as A ranges over $\mathcal{A}$, we obtain

$$\begin{aligned} & \mathbb{P}\left(\sup_{A \in \mathcal{A}} \sum_{i=1}^{\tilde{K}} (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}) \geq nat \mid \tilde{K}\right) \\ &= \mathbb{E}_Y \left[\mathbb{P}_{\sigma} \left(\sup_{A \in \mathcal{A}} \sum_{i=1}^{\tilde{K}} \sigma_i (\mathbb{1}\{Y_i \in A\} - \mathbb{1}\{Y'_i \in A\}) \geq nat \mid \tilde{K}, Y_1, Y'_1, \dots, Y_n, Y'_n \right) \right] \\ &\leq \min \left\{ 1, \mathbb{S}_{\mathcal{A}}(2\tilde{K}) \exp \left(-\frac{a^2(nt)^2}{2\tilde{K}} \right) \right\}, \end{aligned}$$

where the last inequality follows from the fact that the expectation $\mathbb{E}_Y$ is taken only with respect to the random variables $Y_1, Y'_1, \dots, Y_n, Y'_n$. We deduce

$$\mathbb{P}\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t\right) \leq 4 \mathbb{E} \left[\min \left\{ 1, \mathbb{S}_{\mathcal{A}}(2\tilde{K}) \exp \left(-\frac{a^2(nt)^2}{2\tilde{K}} \right) \right\} \right]. \quad (3.3)$$

To be able to deal with this last quantity, we consider whether $\tilde{K}$ is less than $2np(1+s)$ or not, where $s > 0$ is to be determined. By monotonicity, it gives

$$\mathbb{E} \left[\min \left\{ 1, \mathbb{S}_{\mathcal{A}}(2\tilde{K}) \exp \left(-\frac{a^2(nt)^2}{2\tilde{K}} \right) \right\} \right] \leq \mathbb{P}(\tilde{K} \geq 2np(1+s)) + \mathbb{S}_{\mathcal{A}}(4np(1+s)) \exp \left(-\frac{na^2t^2}{4p(1+s)} \right).$$

Since $a \in (0, 1)$ is not fixed, we may, given n and t , choose the largest $a < 1$ such that the condition (3.2) is respected. It is given by $a = a(n, t) = 1 - \sqrt{2p/(nt^2)}$. For such an a , we have

$$\mathbb{E} \left[\min \left\{ 1, \mathbb{S}_{\mathcal{A}}(2\tilde{K}) \exp \left(-\frac{a^2(nt)^2}{2\tilde{K}} \right) \right\} \right] \leq \mathbb{P}(\tilde{K} \geq 2np(1+s)) + \mathbb{S}_{\mathcal{A}}(4np(1+s)) \exp \left(-\frac{nt^2 - 2\sqrt{2npt} + 2p}{4p(1+s)} \right). \quad (3.4)$$

Let $\delta \in (0, 1)$. We choose $s = s(\delta)$ such that the first term of (3.4) is bounded $\delta/8$. Then, we choose $t = t(\delta)$ such that the second term equals $\delta/8$. As a consequence of (3.3), we will deduce that $P(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t(\delta)) \leq \delta$, or, equivalently, $\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq t(\delta)$ with probability at least $1 - \delta$.

By Bernstein's inequality (2.1), since $\tilde{K} \sim \text{Bin}(n, \tilde{p})$ with $\tilde{p} = p(2 - p)$, we have

$$\begin{aligned} P(\tilde{K} \geq 2np(1 + s)) &= P(\tilde{K} - n\tilde{p} \geq 2np(1 + s) - n\tilde{p}) = P(\tilde{K} - n\tilde{p} \geq np(2s + p)) \\ &\leq \exp\left(-\frac{(np)^2(2s + p)^2}{2(n\tilde{p}(1 - \tilde{p}) + np(2s + p)/3)}\right) \\ &\leq \exp\left(-\frac{np(2s + p)^2}{2((2 - p) + (2s + p)/3)}\right) \leq \exp\left(-\frac{np s^2}{1 + s/3}\right). \end{aligned}$$

Equating this expression to $\delta/8$, we obtain a quadratic equation in s whose positive solution is given by

$$s_+ = s(\delta) := \frac{\log(8/\delta)}{3np} + \sqrt{\frac{\log(8/\delta)}{np}}.$$

Since we assumed $np \geq 2 \log(8/\delta)$, we have $s(\delta) \leq \frac{1}{6} + \frac{1}{\sqrt{2}} \leq 1$. Hence, by taking $s = 1$, the first term of (3.4) is bounded by $\delta/8$.

For such a choice of s , the second term of (3.4) becomes

$$\mathbb{S}_{\mathcal{A}}(8np) \exp\left(-\frac{nt^2 - 2\sqrt{2npt} + 2p}{8p}\right).$$

This last expression equals $\delta/8$ if

$$t = t(\delta) := 2\sqrt{\frac{2p}{n} [\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(8np)]} + \sqrt{\frac{2p}{n}}. \quad \square$$

4. Working with the expected maximal deviation

The variation of the McDiarmid bounded differences inequality that we use is recalled in Goix et al. (2015, Proposition 11). When combined with Goix et al. (2015, Lemma 12), it leads to a convenient Bernstein-type concentration inequality for the maximal deviation that is particularly adapted when working with rare events, i.e., for every $t > 0$,

$$P\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| - E\left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|\right] \geq t\right) \leq \exp\left(-\frac{nt^2}{4p + 2t/3}\right).$$

Setting the upper bound equal to $\delta \in (0, 1)$ and solving for $t > 0$ proves the following result.

Proposition 5 (Concentration of the maximal deviation). *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). Then, for every $\delta \in (0, 1)$, with probability at least $1 - \delta$, we have*

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \frac{2}{3n} \log(1/\delta) + 2\sqrt{\frac{p}{n} \log(1/\delta)} + E\left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|\right].$$

Using this inequality, we may directly work with the expectation $E[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|]$ to obtain a bound with high probability for $\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|$.

We start by applying the conditioning trick (Lemma 2) on the expectation.

Lemma 6. *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). Then,*

$$E\left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|\right] \leq E\left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)|\right] + \sqrt{\frac{p}{n}},$$

where $\mu_{\mathbb{A}}, \mu_K^Y$ and K are as in Lemma 2.

Proof of Lemma 6. By Lemma 2, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \right] &= \mathbb{E} \left[\mathbb{E} \left[\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \mid K \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sup_{A \in \mathcal{A}} \left| \frac{1}{n} \sum_{i=1}^K \mathbb{1}\{Y_i \in A\} - \mu(A) \right| \mid K \right] \right] \\ &= \mathbb{E} \left[\sup_{A \in \mathcal{A}} \left| \frac{1}{n} \sum_{i=1}^K \mathbb{1}\{Y_i \in A\} - \mu(A) \right| \right]. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \mathbb{E} \left[\sup_{A \in \mathcal{A}} \left| \frac{1}{n} \sum_{i=1}^K \mathbb{1}\{Y_i \in A\} - \mu(A) \right| \right] &= \mathbb{E} \left[\sup_{A \in \mathcal{A}} \left| \frac{K}{n} (\mu_K^Y(A) - \mu_{\mathbb{A}}(A)) + \left(\frac{K}{n} - p \right) \mu_{\mathbb{A}}(A) \right| \right] \\ &\leq \mathbb{E} \left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \right] + \mathbb{E} \left[\left| \frac{K}{n} - p \right| \right]. \end{aligned}$$

The second term is easily bounded using Jensen's inequality and the fact that $K \sim \text{Bin}(n, p)$:

$$\left(\mathbb{E} \left[\left| \frac{K}{n} - p \right| \right] \right)^2 \leq \mathbb{E} \left[\left(\frac{K}{n} - p \right)^2 \right] = \text{Var} \left[\frac{K}{n} - p \right] = \frac{\text{Var}[K]}{n^2} = \frac{np(1-p)}{n^2} \leq \frac{p}{n}. \quad \square$$

To deal with the remaining expectation $\mathbb{E} \left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \right]$, we will make use of (2.2). Furthermore, we will also assume that the VC dimension $V_{\mathcal{A}} := \sup\{n \in \mathbb{N} : \mathbb{S}_{\mathcal{A}}(n) = 2^n\}$ of the class $\mathcal{A}$ is finite to be able to apply the famous *Sauer's Lemma* (Bousquet et al., 2004, Lemma 1) which states that for every $n \in \mathbb{N}$,

$$\mathbb{S}_{\mathcal{A}}(n) \leq (n+1)^{V_{\mathcal{A}}}. \quad (4.1)$$

The purpose of this assumption is that, whenever $V_{\mathcal{A}} < +\infty$, we will be able to use Jensen's inequality which is sharper than the monotonicity arguments underlying every proof in Section 3.

Proposition 7. *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). Then, if $V_{\mathcal{A}} < +\infty$,*

$$\mathbb{E} \left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \right] \leq \sqrt{\frac{2p}{n} [\log 2 + V_{\mathcal{A}} \log(2np + 1)]}.$$

Proof of Proposition 7. By the VC inequality for the expectation (2.2) combined with Sauer's Lemma (4.1) and Jensen's inequality, we have

$$\begin{aligned} \mathbb{E} \left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \right] &= \mathbb{E} \left[\mathbb{E} \left[\frac{K}{n} \sup_{A \in \mathcal{A}} |\mu_K^Y(A) - \mu_{\mathbb{A}}(A)| \mid K \right] \right] \\ &\leq \mathbb{E} \left[\frac{K}{n} \sqrt{\frac{2 \log(2\mathbb{S}_{\mathcal{A}}(2K))}{K}} \right] \leq \frac{1}{n} \mathbb{E} \left[\sqrt{2K [\log(2) + V_{\mathcal{A}} \log(2K + 1)]} \right] \\ &\leq \frac{1}{n} \sqrt{2 \mathbb{E}[K] [\log(2) + V_{\mathcal{A}} \log(2 \mathbb{E}[K] + 1)]} = \sqrt{\frac{2p}{n} [\log(2) + V_{\mathcal{A}} \log(2np + 1)]}, \end{aligned}$$

where we made use of the concavity of the map $K \mapsto \sqrt{2K [\log(2) + V_{\mathcal{A}} \log(2K + 1)]}$ (the derivative is clearly decreasing) and the fact that $K \sim \text{Bin}(n, p)$. $\square$

Combining all the ingredients of this section, we finally get a bound on the maximal deviation.

Corollary 8. Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). Then, if $V_{\mathcal{A}} < +\infty$,

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \frac{2}{3n} \log(1/\delta) + \sqrt{\frac{2p}{n}} \left(\sqrt{2 \log(1/\delta)} + \sqrt{\log(2) + V_{\mathcal{A}} \log(2np + 1)} + \frac{\sqrt{2}}{2} \right).$$

5. Comparison of the bounds

We obtained various bounds on the maximal deviation $\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)|$ which hold with probability $1 - \delta \in (0, 1)$. Before comparing them, we summarize the main results of our paper and we recall the bound derived in Theorem 11.

- Theorem 3 (working directly on the maximal deviation and symmetrizing the process after the application of the conditioning trick): whenever $np \geq 4 \log(4/\delta)$,

$$\begin{aligned} \sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \\ \leq \frac{2}{3n} \log(4/\delta) + \sqrt{\frac{2p}{n}} \left(\sqrt{\log(4/\delta)} + \sqrt{2[\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(4np)]} + 1 \right). \end{aligned} \quad (5.1)$$

- Theorem 4 (working directly on the maximal deviation and symmetrizing the process before the application of the conditioning trick): whenever $np \geq 2 \log(8/\delta)$,

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \sqrt{\frac{2p}{n}} \left(2 \sqrt{\log(8/\delta) + \log \mathbb{S}_{\mathcal{A}}(8np)} + 1 \right). \quad (5.2)$$

- Corollary 8 (working on the expected maximal deviation and using Jensen's inequality): whenever $V_{\mathcal{A}} < +\infty$,

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \frac{2}{3n} \log(1/\delta) + \sqrt{\frac{2p}{n}} \left(\sqrt{2 \log(1/\delta)} + \sqrt{\log(2) + V_{\mathcal{A}} \log(2np + 1)} + \frac{\sqrt{2}}{2} \right). \quad (5.3)$$

- Theorem 11 (based on Anthony and Shawe-Taylor (1993, Theorem 2.1)): whenever $np \geq (8/3) \log(3/\delta)$,

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq 2 \sqrt{\frac{2p}{n} [\log(12/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}. \quad (5.4)$$

We observe that the bound (5.3) seems to beat (5.1) in any case. Indeed, both inequalities have the same structure: one term decaying like n^{-1} and one term decreasing like $n^{-1/2}$; however, those terms seem always lower in the concentration bound obtained through the use of McDiarmid's inequality. Even though it could be that the shattering coefficient of $\mathcal{A}$ grows slower than the rate induced by the use of Sauer's Lemma (4.1) in (5.3), the fact that (4.1) enables us to use Jensen's inequality heavily compensates the loss.

The comparison between (5.2) and (5.3) is a bit harder since the structure is not the same anymore. Nevertheless, the coefficient of the leading term in (5.2) seems larger than the one in (5.3). Hence, even though the inequality obtained in Section 4 could be worse for small samples due to its additional term, we think that for practical sample sizes, it remains the best bound.

On Figure 1, we provide a graphical comparison of our results with the more classical VC inequality for relative deviations (5.4) on the simplest, one-dimensional, VC class that we may think off: $\mathcal{A} = \{(-\infty, t] : t \in \mathbb{R}, t \leq Q(10^{-3})\}$, where $Q(p) := \inf\{x \in \mathbb{R} : F(x) \geq p\}$ is the *left-continuous inverse* of $F(x) := \mu((-\infty, x])$, the cumulative distribution function (cdf) associated to μ . In this situation, we easily verify that $\mathbb{S}_{\mathcal{A}}(n) = n + 1$ and that the main assumption of our paper (1.3) is satisfied with $p = 10^{-3}$. Note also that $\mathcal{A}$ is pointwise measurable (Remark 1). We used $\delta = 10^{-2}$.

One clearly observes better performance of our bound (5.3) in this situation, which is valid for every sample size. The bounds coming from Section 3 are closer to the older bound; however, it seems that for samples with a size larger than 10^6 data points, they perform a bit better. More importantly, those bounds take into account the reduced effective sample size np in their measure of size complexity $\mathbb{S}_{\mathcal{A}}$, which is not the case with (5.4).

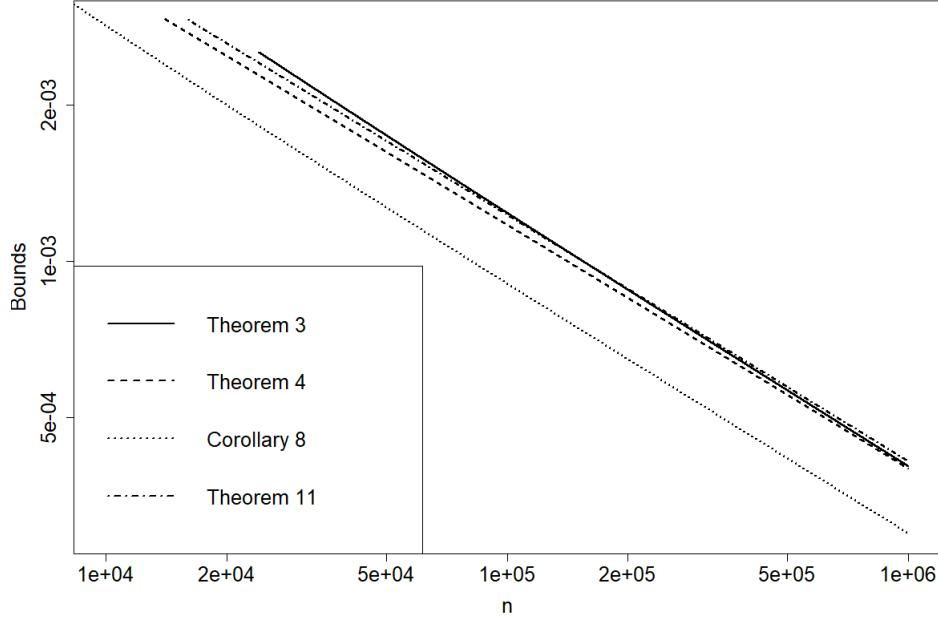


Figure 1: Comparison of the bounds for the empirical cdf on sufficiently large samples (to satisfy the hypothesis of the different theorems). Axes are on logarithmic scale.

Appendix A. An improvement of the Vapnik–Chervonenkis inequality

We show an improvement of the VC inequality (Lhaut, 2021, Theorem 2.17).

Theorem 9. *Let $X_1, \dots, X_n$ be an independent sample with common distribution μ on $\mathbb{R}^d$ and let μ_n be the associated empirical distribution. Consider a non-empty class $\mathcal{A}$ of Borel sets of $\mathbb{R}^d$. Then, for every $\delta \in (0, 1)$, with probability $1 - \delta$, we have*

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq \sqrt{\frac{1}{2n}} + \sqrt{\frac{2}{n} [\log(4/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}.$$

The order of the bound is the same as in the original VC bound (1.2). However, the constant in front of the square root is almost halved. The proof relies on a variation of a classical *symmetrization* argument (Bousquet et al., 2004, Lemma 2 p. 185).

Lemma 10 (Symmetrization). *Let $X_1, \dots, X_n, X'_1, \dots, X'_n$ be an independent sample with common distribution μ on $\mathbb{R}^d$. Let μ_n denote the empirical distribution associated to $X_1, \dots, X_n$ and let μ'_n denote the empirical distribution associated to $X'_1, \dots, X'_n$. Consider a non-empty class $\mathcal{A}$ of Borel sets of $\mathbb{R}^d$. If $t > 0$ and $a \in (0, 1)$ are such that $4(1 - a)^2 nt^2 \geq 2$, then*

$$\left. \begin{aligned} & \mathbb{P} \left(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu(A)) \geq t \right) \\ & \mathbb{P} \left(\sup_{A \in \mathcal{A}} (\mu(A) - \mu_n(A)) \geq t \right) \end{aligned} \right\} \leq 2 \mathbb{P} \left(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu'_n(A)) \geq at \right).$$

The sample $X'_1, \dots, X'_n$ is often referred to as a “ghost sample”. The choice $a = 1/2$ gives back the initial symmetrization. However, as we will see, this more flexible result will give a way to substantially improve the classical bound.

Proof of Lemma 10. Let $A \in \mathcal{A}$ and $t > 0$. We first observe that

$$\mathbb{1}\{\mu_n(A) - \mu(A) \geq t\} \mathbb{1}\{\mu'_n(A) - \mu(A) \leq (1 - a)t\} \leq \mathbb{1}\{\mu_n(A) - \mu'_n(A) \geq at\}.$$

Integrating both sides with respect to the ghost sample, we get

$$\mathbb{1}\{\mu_n(A) - \mu(A) \geq t\} P'(\mu'_n(A) - \mu(A) \leq (1-a)t) \leq P'(\mu_n(A) - \mu'_n(A) \geq at),$$

where we used the notation P' to emphasize that the integration was realized only with respect to $X'_1, \dots, X'_n$. By Bienaymé–Tchebychev inequality and hypothesis, we deduce

$$\begin{aligned} P'(\mu'_n(A) - \mu(A) \leq (1-a)t) &= 1 - P'(\mu'_n(A) - \mu(A) \geq (1-a)t) \\ &\geq 1 - \frac{\text{Var}[\mu'_n(A)]}{(1-a)^2 t^2} = 1 - \frac{\text{Var}[\mathbb{1}\{X'_1 \in A\}]}{n(1-a)^2 t^2} \geq 1 - \frac{1}{4n(1-a)^2 t^2} \geq 1/2, \end{aligned}$$

where we used the fact that the variance of $\mathbb{1}\{X'_1 \in A\}$ is bounded by $1/4$ (since $\max_{p \in [0,1]} \{p(1-p)\} = 1/4$). We obtain

$$\mathbb{1}\{\mu_n(A) - \mu(A) \geq t\} \leq 2 P'(\mu_n(A) - \mu'_n(A) \geq at).$$

Since this relation holds for every $A \in \mathcal{A}$, we finally get that

$$\begin{aligned} \mathbb{1}\{\sup_{A \in \mathcal{A}} \mu_n(A) - \mu(A) \geq t\} &= \sup_{A \in \mathcal{A}} \mathbb{1}\{\mu_n(A) - \mu(A) \geq t\} \leq 2 \sup_{A \in \mathcal{A}} P'(\mu_n(A) - \mu'_n(A) \geq at) \\ &\leq 2 P'(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu'_n(A)) \geq at). \end{aligned}$$

Taking the expectation with respect to the sample $X_1, \dots, X_n$, it follows that

$$P(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu(A)) \geq t) \leq 2 P(\sup_{A \in \mathcal{A}} (\mu_n(A) - \mu'_n(A)) \geq at).$$

The other inequality has a similar proof. $\square$

Proof of Theorem 9. Let $nt^2 \geq 1/2$. Following the same arguments as the ones used to prove the classical VC inequality (Lhaut, 2021, Theorem 2.14) and using our symmetrization, we show that for any $a \in (0, 1)$, whenever $4(1-a)^2 nt^2 \geq 2$, we have

$$P\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t\right) \leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-a^2 nt^2/2}.$$

Choosing the largest $a \in (0, 1)$ such that this relation holds, i.e. $a = a(n, t) = 1 - \sqrt{1/(2nt^2)}$, we get

$$P\left(\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \geq t\right) \leq 4\mathbb{S}_{\mathcal{A}}(2n) \exp\left\{-\frac{nt^2}{2}\left(1 - \sqrt{\frac{2}{nt^2}}\right) - \frac{1}{4}\right\}. \quad (\text{A.1})$$

Setting the upper bound equal to $\delta \in (0, 1)$ and solving for t leads to a quadratic equation for t whose positive solution corresponds to the bound proposed in the theorem.

If $nt^2 < 1/2$, the bound (A.1) is trivial. $\square$

Appendix B. Vapnik–Chervonenkis inequality for relative deviations

Theorem 11 (VC inequality for relative deviations). *Let $X_1, \dots, X_n$ be an independent random sample with common distribution μ on $\mathbb{R}^d$ and μ_n be the associated empirical measure. Let $\mathcal{A}$ be a non-empty class of Borel sets on $\mathbb{R}^d$. Let $\mathbb{A}$ and p be as in (1.3). If $np \geq (8/3) \log(3/\delta)$, where $\delta \in (0, 1)$, then we have, with probability $1 - \delta$,*

$$\sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \leq 2 \sqrt{\frac{2p}{n} [\log(12/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}.$$

Proof. It follows from Lugosi (2002, Theorem 1.11) that

$$\mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{\mu(A) - \mu_n(A)}{\sqrt{\mu(A)}} \geq t\right) \leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/4} \quad (\text{B.1})$$

and

$$\mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{\mu_n(A) - \mu(A)}{\sqrt{\mu_n(A)}} \geq t\right) \leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/4}, \quad (\text{B.2})$$

for every $t > 0$. Therefore, it is convenient to decompose our probability of interest as follows

$$\begin{aligned} \mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{|\mu(A) - \mu_n(A)|}{\sqrt{p}} \geq t\right) &\leq \mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{\mu(A) - \mu_n(A)}{\sqrt{\mu(A)}} \geq t\right) + \mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{\mu_n(A) - \mu(A)}{\sqrt{p}} \geq t\right) \\ &\leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/4} + \mathbb{P}(\mu_n(\mathbb{A}) \geq 2p) + \mathbb{P}\left(\sup_{A \in \mathcal{A}} \frac{\mu_n(A) - \mu(A)}{\sqrt{\mu_n(A)}} \geq t/\sqrt{2}\right) \\ &\leq 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/4} + \mathbb{P}(\mu_n(\mathbb{A}) \geq 2p) + 4\mathbb{S}_{\mathcal{A}}(2n)e^{-nt^2/8}, \end{aligned} \quad (\text{B.3})$$

where we made use of (B.1) and (B.2). By hypothesis, the second term of (B.3) is bounded by $\delta/3$. Indeed, by Bernstein's inequality (2.1),

$$\mathbb{P}(\mu_n(\mathbb{A}) \geq 2p) = \mathbb{P}(K - np \geq np) \leq \exp\left(-\frac{3}{8}np\right),$$

where $K \sim \text{Bin}(n, p)$ as in Lemma 2. Choosing $t = t(\delta) = 2\sqrt{\frac{2}{n}[\log(12/\delta) + \log \mathbb{S}_{\mathcal{A}}(2n)]}$, we get that the first term and last term of (B.3) are also bounded by $\delta/3$. The result follows. $\square$

References

- M. Anthony and J. Shawe-Taylor. A result of Vapnik with applications. *Discrete Applied Mathematics*, 47(3):207–217, 1993.
- O. Bousquet, U. von Luxburg, and G. Rätsch, editors. *Advanced Lectures on Machine Learning: ML Summer Schools 2003, Canberra, Australia, February 2 - 14, 2003, Tübingen, Germany, August 4 - 16, 2003, Revised Lectures*. Springer, Berlin, Heidelberg, 2004.
- S. Cléménçon, H. Jalalzai, A. Sabourin, and J. Segers. Concentration bounds for the empirical angular measure with statistical learning applications, 2021. arXiv:2104.03966.
- E. Giné and Koltchinskii. Concentration inequalities and asymptotic results for ratio type empirical processes. *The Annals of Probability*, 34(3): 1143–1216, 2006.
- N. Goix, A. Sabourin, and S. Cléménçon. Learning the dependence structure of rare events: a non-asymptotic study. *Journal of Machine Learning Research*, 40:1–18, 2015.
- H. Jalalzai, S. Cléménçon, and A. Sabourin. On binary classification in extreme regions. In *Advances in Neural Information Processing Systems*, pages 3092–3100, 2018.
- S. Lhaut. Inégalités de concentration pour événements rares. Master's thesis, Faculté des sciences, Université catholique de Louvain, 2021. Prom. : Segers, Johan.
- G. Lugosi. Pattern classification and learning theory. In L. Györfi, editor, *Principles of nonparametric learning (Udine, 2001)*, volume 434 of *CISM Courses and Lect.*, pages 1–56. Springer, Vienna, 2002.
- C. McDiarmid. Concentration. In M. e. a. Habib, editor, *Probabilistic Methods for Algorithmic Discrete Mathematics*, volume 16 of *Algorithms and Combinatorics*, pages 195–248. Springer Berlin Heidelberg, 1998.
- S. Novak. *Extreme value methods with applications to finance*. CRC Press, 2011.
- A. van der Vaart and J. Wellner. *Weak Convergence and Empirical Processes*. Springer Series in Statistics. Springer-Verlag New York, 1996.
- V. Vapnik. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability and Its Applications*, XVI(2): 264–280, 1971.