MARKOV MOVES, L^2-BURAU MAPS AND LEHMER'S CONSTANTS

FATHI BEN ARIBI

Abstract. We study the effect of Markov moves on L^2-Burau maps of braids, in order to construct link invariants from these maps with a process mirroring the well-known Alexander-Burau formula.

We prove such a Markov invariance for the L^2-Burau maps which descend to the groups of the braid closures or lower, and for which the associated link invariants are twisted L^2-Alexander torsions. When the L^2-Burau map descends to a link group, the corresponding link invariant was known to be the L^2-Alexander torsion of the link by a previous result of A. Conway and the author.

Furthermore, we find two counter-examples to Markov invariance, meaning two families of L^2-Burau maps that cannot yield link invariants with the process described in our paper. The proofs use relations between Fuglede-Kadison determinants, Mahler measures, and random walks on Cayley graphs, as well as works of Boyd, Bartholdi and Dasbach-Lalin.

Along the way, we compute new values for Fuglede-Kadison determinants over non-cyclic free groups. As a consequence, we partially answer a question of Lück, as we provide new upper bounds for Lehmer's constants for all torsionfree groups which have non-cyclic free subgroups.

Our results suggest that twisted L^2-Alexander torsions are the only link invariants we can hope to construct from L^2-Burau maps with the present approach.

1. Introduction

The L^2-Burau maps and reduced L^2-Burau maps of braids were introduced in 2016 in [BAC] by A. Conway and the author. These maps generalise the Burau representation of braid groups and are indexed by a positive real number t and a group epimorphism γ starting at a free group of finite rank. In a sense, all the L^2-Burau maps are contained between the Burau representation and the Artin injection of the braid group B_n in the automorpishm group $\mathrm{Aut}(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$. Moreover, A. Conway and the author proved in [BAC] that for a braid $\beta \in B_n$, its image by the reduced L^2-Burau map $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}$ associated to the epimorphism $\gamma_\beta \colon \mathbb{F}_n \to G_\beta \cong \pi_1\left(S^3 \setminus \hat{\beta}\right)$ yields the L^2*-Alexander torsion* of the braid closure $\hat{\beta}$. The L^2-Alexander torsion is an invariant of links constructed by Li-Zhang and Dubois-Friedl-Lück [LZ, DFL], which detects various topological information. As a consequence of the main result of [BAC], the reduced L^2-Burau map $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}$ thus contains deep topological information about β such as the hyperbolic volume of $\hat{\beta}$ or the genus of $\hat{\beta}$.

It is then natural to wonder whether L^2-Burau maps associated to other epimorphisms can similarly provide link invariants and detect topological information of the braid, and this article provides a partial positive answer to this question.

We first introduce the notion of *Markov-admissibility* of a family $\mathcal{Q}$ of group epimorphisms $Q_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta}$ indexed by braids $\beta \in \sqcup_{n \geqslant 1} B_n$, in Section 4. Roughly speaking, we say that such a family is Markov-admissible when for any two braids α, β that have isotopic closures (and thus are related by Markov moves), the associated epimorphisms Q_α and Q_β descend "to the same depth" and are related by a sequence of commutative diagrams. Markov admissibility appears to be a necessary condition in order to construct Markov functions and link invariants from general families of epimorphisms indexed by braids.

In this paper we focus on a specific candidate for being a Markov function, namely the function

$$F_{\mathcal{Q}} := \left(\begin{array}{ccc} \sqcup_{n\geqslant 1} B_n & \to & \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{>0})/\{t \mapsto t^m, m \in \mathbb{Z}\} \\ \beta & \mapsto & \left[t \mapsto \dfrac{\det^r_{G_{Q_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)} \right)}{\max(1,t)^n} \right] \end{array} \right),$$

where $\mathcal{Q}$ is a Markov-admissible family of epimorphisms and $\det^r_G$ is the *regular Fuglede-Kadison determinant* for the group G, a version of the determinant for infinite-dimensional G-equivariant operators on $\ell^2(G)$ such as the L^2-Burau maps (see Section 2 for a definition).

The first main result of this article is the following theorem (stated here without technical details for readability):

Theorem 1.1 (Theorem 5.1)**.** *Let $\mathcal{Q}$ be a Markov-admissible family of epimorphisms that descends to the groups of the braid closures or deeper. Then $F_{\mathcal{Q}}$ is a Markov function, and thus defines an invariant of links. Moreover, this link invariant is a twisted L^2-Alexander torsion.*

As detailed in Section 5, to prove Theorem 1.1 we study how Markov moves on braids modify reduced L^2-Burau maps and we use properties of the Fuglede-Kadison determinant.

Part of Theorem 1.1 was already proven in [BAC], but without discussing Markov invariance. Indeed, for $\mathcal{Q}$ the family that exactly descends to the groups of the braid closures G_β, [BAC, Theorem 4.9] directly linked $F_{\mathcal{Q}}$ to the L^2-Alexander torsions of the braid closures, as a variant of the well-known Alexander-Burau formula [Bu]; and since the L^2-Alexander torsions were already known link invariants, Markov invariance was thus reciprocally guaranteed, and carefully studying Markov moves was unnecessary.

However, as stated in Theorem 1.1, the methods of the current article provide several new link invariants, which (unsurprisingly) happen to be *twisted* L^2-Alexander torsions of links.

Theorem 1.1 is not an equivalence in the sense that $F_{\mathcal{Q}}$ could theoretically be a Markov function for other families $\mathcal{Q}$, but the specific convenient cancellations that occur in matrix coefficients in the proof of Theorem 1.1 make this seem unlikely. As further evidence, we establish two families $\mathcal{Q}$ such that $F_{\mathcal{Q}}$ is not a Markov function, in the second main result of this article:

Theorem 1.2 (Theorems 6.1 and 6.4)**.** *Let $\mathcal{Q}$ be either the family of identities of the free groups or the family of abelianizations of the free groups. Then $F_{\mathcal{Q}}$ is not a Markov function.*

Our main tools to prove Theorem 1.2 are relations between Fuglede-Kadison determinants (which are technical to define and difficult to compute), Mahler measures of polynomials (notably studied by Boyd [Bo]) and combinatorics on Cayley graphs (developed in [Ba, DL]). Along the way, we compute new values for Fuglede-Kadison determinants of operators over the free groups, which are interesting in their own right :

Theorem 1.3 (Theorem 6.3)**.** *Let $d \geqslant 3$, and let $x_1, \ldots, x_{d-1}$ be $d-1$ generators of the free group $\mathbb{F}_{d-1}$. Then we have:*

$$\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id} + R_{x_1} + \ldots + R_{x_{d-1}}) = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}.$$

Theorem 1.3 has a consequence which is independent of the rest of the themes of this paper, in that it yields new upper bounds for Lehmer's constants $\Lambda_1^w(\mathbb{F}_d)$ of the free groups (see [Lu2] for a survey of Lehmer's constants and Corollary 7.4 for more details). Hence the non-cyclic free groups $\mathbb{F}_d$ (as well as any torsionfree group having non-cyclic free subgroups, such as groups of hyperbolic 3-manifolds) are now the first torsionfree groups G for which Lehmer's constant $\Lambda_1^w(G)$ is known to be lesser than 1.176. The same property was known to be satisfied for the smaller Lehmer's constant $\Lambda^w(G)$ in the case of G being the fundamental group of the Weeks manifold (see Example 7.3).

As the reader will probably agree, the initial question (how to build link invariants from L^2-Burau maps) is still far from answered. We restricted ourselves to studying the most intuitive form of a (potential) Markov function (our function $F_{\mathbb{Q}}$), and we found that twisted L^2-Alexander torsions appeared to be the best link invariants we could obtain with it. This last point is unsurprising considering the form of $F_{\mathbb{Q}}$ and its natural connection with the Alexander polynomial and its variations.

However, there may well be new link invariants to discover via other functions of the L^2-Burau maps, and we hope that our computations of how these maps change under Markov moves can be of use for future research in this vein.

The present article arose as a part of a wider project in collaboration with C. Anghel aiming to construct new knot invariants from L^2-versions of the Burau and Lawrence representations of braid groups. To attain this end, studying the influence of Markov moves on L^2-Burau maps appears to be a natural step.

The article is organised as follows: in Section 2 we recall preliminaries on braid groups and L^2-invariants; in Section 3, we cover some improvements of classical properties of L^2-torsions; in Section 4 we introduce the notion of a Markov-admissible family of group epimorphisms and we study several examples; in Section 5 we state and prove Theorem 5.1 on Markov invariance; in Section 6 we present counter-examples to Markov invariance and new computations of Fuglede-Kadison determinants; finally, in Section 7, we present new upper bounds on Lehmer's constants for a large class of torsionfree groups.

Acknowledgements

The author was supported by the FNRS in his position at UCLouvain. We thank Cristina Anghel-Palmer, Anthony Conway, Jacques Darné and Wolfgang Lück for helpful discussions.

2. Preliminaries

We will mostly follow the conventions of [BAC] and [Lu].

2.1. **Braid groups.** The braid group B_n can be seen as the set of isotopy classes of orientation-preserving homeomorphisms of the punctured disk $D_n := D^2 \setminus \{p_1, ..., p_n\}$ which fix the boundary pointwise. Recall that B_n admits a presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ following the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for each i, and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 2$. Topologically, the generator σ_i is the braid whose i-th component passes over the $(i+1)$-th component.

Fix a base point z of D_n and denote by x_i the simple loop based at z turning once around p_i counterclockwise for $i = 1, 2, \ldots, n$ (see Figure 1). The group $\pi_1(D_n)$ can

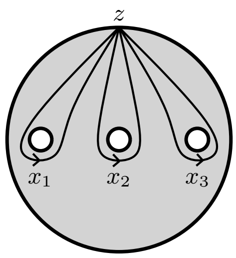

FIGURE 1. The punctured disk D_3.

then be identified with the free group $\mathbb{F}_n$ on the x_i. If H_β is a homeomorphism of D_n representing a braid β, then the induced automorphism h_β of the free group $\mathbb{F}_n$ depends only on β. It follows from the way we compose braids that $h_{\alpha\beta} = h_\beta \circ h_\alpha$, and the resulting *right* action of B_n on $\mathbb{F}_n$ (named the *Artin action*) can be explicitly described by

$$h_{\sigma_i}(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases} \qquad h_{\sigma_i^{-1}}(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

In this paper we will also use a second set of generators of $\mathbb{F}_n$, namely

$$g_1 := x_1, \ g_2 := x_1 x_2, \ \ldots, \ g_n := x_1 \ldots x_n.$$

Looking at Figure 1, g_i represents the class of the loop that circles the first i punctures. On these generators, B_n acts in the following way:

$$h_{\sigma_i}(g_j) = \begin{cases} g_{i+1} g_i^{-1} g_{i-1} & \text{if } j = i, \\ g_j & \text{otherwise,} \end{cases} \qquad h_{\sigma_i^{-1}}(g_j) = \begin{cases} g_{i-1} g_i^{-1} g_{i+1} & \text{if } j = i, \\ g_j & \text{otherwise,} \end{cases}$$

where we use the convention $g_0 := 1$.

2.2. **Fox calculus.** Denoting by $\mathbb{F}_n$ the free group on $x_1, x_2, \ldots, x_n$, and for $i \in \{1, \ldots, n\}$, the i-th *Fox derivative* $\dfrac{\partial}{\partial x_i} : \mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z}[\mathbb{F}_n]$ (first introduced in [Fox]) is the linear extension of the map defined on elements of $\mathbb{F}_n$ by

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \qquad \frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{i,j} x_j^{-1}, \qquad \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}.$$

The following formula is sometimes called the *fundamental formula of Fox calculus*:

Proposition 2.1. *Let $u \in \mathbb{Z}\mathbb{F}_n$ and $\epsilon \colon \mathbb{Z}\mathbb{F}_n \to \mathbb{Z}$ the ring epimorphism defined by $\epsilon \colon x_i \mapsto 1$ for all $i \in \{1, \ldots, n\}$. Then:*

$$u - \epsilon(u) \cdot 1 = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot (x_i - 1).$$

2.3. **Fuglede-Kadison determinant.** In this section we will give short definitions of the von Neumann trace and the Fuglede-Kadison determinant, without going into technical details. More details can be found in [Lu] and [BAC].

Let G be a finitely generated group. The Hilbert space $\ell^2(G)$ is the completion of the group algebra $\mathbb{C}G$, and the space of bounded operators on it is denoted $B(\ell^2(G))$. We will focus on *right-multiplication operators* $R_w \in B(\ell^2(G))$, where $R_.$ denotes the right regular action of G on $\ell^2(G)$ extended to the group ring $\mathbb{C}G$ (and further extended to the rings of matrices $M_{p,q}(\mathbb{C}G)$).

For any element $w = a_0 \cdot 1_G + a_1 g_1 \ldots + a_r g_r \in \mathbb{C}G$, the *von Neumann trace* tr_G of the associated right multiplication operator is defined as

$$\mathrm{tr}_G(R_w) = \mathrm{tr}_G \left(a_0 \mathrm{Id}_{\ell^2(G)} + a_1 R_{g_1} \ldots + a_r R_{g_r} \right) := a_0,$$

and the von Neumann trace for a finite square matrix over $\mathbb{C}G$ is given as the sum of the traces of the diagonal coefficients.

Now the most concise definition of the *Fuglede-Kadison determinant* $\det_G(A)$ of a right-multiplication operator A is probably

$$\det{}_G(A) := \lim_{\varepsilon \to 0^+} \left(\exp \circ \left(\frac{1}{2} \mathrm{tr}_G \right) \circ \ln \right) \left((A_\perp)^* (A_\perp) + \varepsilon \mathrm{Id} \right) \geqslant 0,$$

where $A_\perp$ is the restriction of A to a supplementary of its kernel, $*$ is the adjunction and ln the logarithm of an operator in the sense of the holomorphic functional calculus. Compare with [Lu, Theorem 3.14] and Proposition 2.2 below. We call the operator A *of determinant class* if $\det_G(A) \neq 0$.

The following properties concern the classical Fuglede-Kadison determinant $\det_G$ described in [Lu2, Chapter 3], which is not always multiplicative if one deals with non injective operators. Moreover, this determinant forgets about the influence of the spectral value 0, which surprisingly makes it take the value 1 for the zero operator. More recent articles have used the *regular Fuglede-Kadison determinant* $\det_G^r$ instead, which is defined for square injective operators, is zero for non injective operators, and is always multiplicative. In this paper we will work with both types of determinants, but the reader should be reassured that most of the statements we will make remain unchanged while replacing one determinant with the other (up to assumptions on injectivity usually). Similarly, the statements of the following Proposition 2.2 admit immediate variants with $\det_G^r$.

Proposition 2.2 ([Lu2]). *Let G be a countable discrete group and let*

$$A, B, C, D \in \sqcup_{p,q \in \mathbb{N}} R_{M_{p,q}(\mathbb{C}G)}$$

be general right multiplication operators. The Fuglede-Kadison determinant satisfies the following properties:

(1) *(multiplicativity) If A, B are injective, square and of the same size, then*
$$\det{}_G(A \circ B) = \det{}_G(A) \det{}_G(B).$$
(2) *(block triangular case) If A, B are injective and square, then*
$$\det{}_G \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det{}_G(A) \det{}_G(B),$$
where C has the appropriate dimensions.
(3) *(induction) If $\iota \colon G \hookrightarrow H$ is a group monomorphism, then*
$$\det{}_H(\iota(A)) = \det{}_G(A).$$
(4) *(relation with the von Neumann trace) If A is a positive operator, then*
$$\det{}_G(A) = (\exp \circ \mathrm{tr}_G \circ \ln)(A).$$
(5) *(simple case) If $g \in G$ is of infinite order, then for all $t \in \mathbb{C}$ the operator $\mathrm{Id} - tR_g$ is injective and*
$$\det_G(\mathrm{Id} - tR_g) = \max(1, |t|).$$
(6) *(2×2 trick) For all $A, B, C, D \in N(G)$ such that B is invertible, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is injective if and only if $DB^{-1}A - C$ is injective, and in this case one has:*
$$\det{}_G \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det{}_G(B) \det{}_G(DB^{-1}A - C).$$

(7) *(relation with Mahler measure) Let $G = \mathbb{Z}^d$, and $P \in \mathbb{C}[X_1, \ldots, X_d]$ denote the d-variable polynomial associated to the operator $A \in R_{\mathbb{C}\mathbb{Z}^d}$. Then*

$$\det_{\mathbb{Z}^d}(A) = \mathcal{M}(P) = \exp\left(\frac{1}{(2\pi)^d}\int_0^{2\pi}\ldots\int_0^{2\pi}\ln\left(\left|P(e^{i\theta_1}, \ldots, e^{i\theta_d})\right|\right) d\theta_1 \ldots d\theta_d\right),$$

where $\mathcal{M}$ is the d-variable Mahler measure.

(8) *(limit of positive operators) If A is injective, then*

$$\det{}_G(A) = \lim_{\varepsilon \to 0^+} \sqrt{\det{}_G(A^*A + \varepsilon \mathrm{Id})}.$$

(9) *(dilations) Let $\lambda \in \mathbb{C}^*$. Then:*

$$\det{}_G(\lambda\ \mathrm{Id}^{\oplus n}) = |\lambda|^n.$$

Remark 2.3. If the group G satisfies the *strong Atiyah conjecture* (see [Lu, Chapter 10]), then the right multiplication operator by any non-zero element of $\mathbb{C}G$ is injective, which makes it convenient to apply some parts of Proposition 2.2. Note that free groups and free abelian groups satisfy the strong Atiyah conjecture.

2.4. **L^2-Burau maps on braids.** Let $n \in \mathbb{N}^*$, $t > 0$, let $\Phi_n \colon \mathbb{F}_n \twoheadrightarrow \mathbb{Z}$ denote the projection that sends all free generators to 1, and let $\gamma \colon \mathbb{F}_n \twoheadrightarrow G$ denote an epimorphism such that Φ_n factors through γ. Let $\kappa(t, \Phi_n, \gamma) : \mathbb{Z}\mathbb{F}_n \to \mathbb{R}G$ denote the ring homomorphism that sends $g \in \mathbb{F}_n$ to $t^{\Phi_n(g)}\gamma(g) \in \mathbb{R}G$.

Then, following [BAC], the associated *L^2-Burau map* on B_n is

$$\mathcal{B}^{(2)}_{t,\gamma} \colon B_n \ni \beta \ \mapsto\ R_{\kappa(t,\Phi_n,\gamma)(J)} \in B\left(\ell^2(G)^{\oplus n}\right),$$

where $J = \left(\dfrac{\partial h_\beta(x_j)}{\partial x_i}\right)_{1 \leqslant i,j \leqslant n}$ is the Fox jacobian of h_β for the base of the x_i.

The *reduced L^2-Burau map* on B_n (associated to the same parameters t, γ) is

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma} \colon B_n \ni \beta \ \mapsto\ R_{\kappa(t,\Phi_n,\gamma)(J')} \in B\left(\ell^2(G)^{\oplus(n-1)}\right),$$

where $J' = \left(\dfrac{\partial h_\beta(g_j)}{\partial g_i}\right)_{1 \leqslant i,j \leqslant n-1}$ is the Fox jacobian of h_β for the base of the g_i.

In the remainder of this article we will focus on reduced L^2-Burau maps.

Note that L^2-Burau maps (reduced or not) admit other definitions as maps over a certain homology of a cover of the punctured disk (see [BAC] for details). Although these homological definitions may be more natural and more useful for further generalizations, the remainder of this article will only use the previous definitions via Fox jacobians.

The following proposition is an (anti-) multiplication formula for the reduced L^2-Burau maps.

Proposition 2.4 ([BAC])**.** *For any n, t, γ as above and any two braids $\alpha, \beta \in B_n$, we have:*

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha\beta) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma \circ h_\beta}(\alpha).$$

Note that the unreduced L^2-Burau maps satisfy an identical formula.

It follows from Proposition 2.4 that a reduced L^2-Burau map can be computed for any braid via knowing the values on the generators σ_i of the braid group. For the reader's convenience and since they will be used in the remainder of this article, we now provide the image of the generators σ_i:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_1) = \begin{pmatrix} -tR_{\gamma(g_2 g_1^{-1})} & 0 \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-3)},$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i) = \mathrm{Id}^{\oplus(i-2)} \oplus \begin{pmatrix} \mathrm{Id} & tR_{\gamma(g_{i+1}g_i^{-1})} & 0 \\ 0 & -tR_{\gamma(g_{i+1}g_i^{-1})} & 0 \\ 0 & \mathrm{Id} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-i-2)} \ for\ 1 < i < n-1,$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_{n-1}) = \mathrm{Id}^{\oplus(n-3)} \oplus \begin{pmatrix} \mathrm{Id} & tR_{\gamma(g_n g_{n-1}^{-1})} \\ 0 & -tR_{\gamma(g_n g_{n-1}^{-1})} \end{pmatrix},$$

and the images of the inverses σ_i^{-1} of the generators:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_1^{-1}) = \begin{pmatrix} -\frac{1}{t}R_{\gamma(g_1^{-1})} & 0 \\ \frac{1}{t}R_{\gamma(g_1^{-1})} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-3)},$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i^{-1}) = \mathrm{Id}^{\oplus(i-2)} \oplus \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & -\frac{1}{t}R_{\gamma(g_{i-1}g_i^{-1})} & 0 \\ 0 & \frac{1}{t}R_{\gamma(g_{i-1}g_i^{-1})} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-i-2)} \ for\ 1 < i < n-1,$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_{n-1}^{-1}) = \mathrm{Id}^{\oplus(n-3)} \oplus \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ 0 & -\frac{1}{t}R_{\gamma(g_{n-2}g_{n-1}^{-1})} \end{pmatrix}.$$

Remark 2.5. It follows from what precedes and from Proposition 2.2 that for all $t > 0$, we have $\det_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i^{\pm 1})\right) = t^{\pm 1}$.

2.5. **L^2-torsions.** In this section we follow [Lu] and [BA2].

A *finitely generated Hilbert $\mathcal{N}(G)$-module* is an Hilbert space V on which there is a left G-action by isometries, and such that there exists a positive integer m and an embedding ϕ of V into $\bigoplus_{i=1}^m \ell^2(G)$ (in this paper, such spaces V will always be of the form $\ell^2(G)^{\oplus n}$ for $n \in \mathbb{N}$).

For U and V two finitely generated Hilbert $\mathcal{N}(G)$-modules, we will call $f\colon U \to V$ a *morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules* if f is a linear G-equivariant map, bounded for the respective scalar products of U and V (in this paper, these morphisms will simply be right multiplication operators).

A *finite Hilbert $\mathcal{N}(G)$-chain complex C_** is a sequence of morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$C_* = 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

such that $\partial_p \circ \partial_{p+1} = 0$ for all p (in this paper, n will be at most 3).

The *p-th L^2-homology of C_** $H_p^{(2)}(C_*) := Ker(\partial_p)/\overline{Im(\partial_{p+1})}$ is a finitely generated Hilbert $\mathcal{N}(G)$-module. We say that C_* is *weakly acyclic* if its L^2-homology is trivial. We say that C_* is of *determinant class* if all the operators ∂_p are of determinant class.

Let C_* be a finite Hilbert $\mathcal{N}(G)$-chain complex as above. Its *L^2-torsion* is

$$T^{(2)}(C_*) := \prod_{i=1}^{n} \det{}_{\mathcal{N}(G)}(\partial_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

if C_* is weakly acyclic and of determinant class, and is defined to be $T^{(2)}(C_*) := 0$ otherwise.

Let π be a group and $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ two group homomorphisms. We say that (π, ϕ, γ) *forms an admissible triple* if $\phi\colon \pi \to \mathbb{Z}$ factors through γ. For X a CW-complex, we say that $(X, \phi\colon \pi_1(X) \to \mathbb{Z}, \gamma\colon \pi_1(X) \to G)$ *forms an admissible triple* if $(\pi_1(X), \phi, \gamma)$ forms one. Let (X, ϕ, γ) be such an admissible triple, $\pi = \pi_1(X)$ and $t > 0$. We define a ring homomorphism

$$\kappa(\pi, \phi, \gamma, t) \colon \begin{pmatrix} \mathbb{Z}[\pi] & \longrightarrow & \mathbb{R}[G] \\ \sum_{j=1}^r m_j g_j & \longmapsto & \sum_{j=1}^r m_j t^{\phi(g_j)} \gamma(g_j) \end{pmatrix}$$

and we also denote $\kappa(\pi, \phi, \gamma, t)$ its induction over the $M_{p,q}(\mathbb{Z}[\pi])$.

Assume X is compact. The cellular chain complex of $\widetilde{X}$ denoted $C_*(\widetilde{X}, \mathbb{Z}) = \left(\ldots \to \bigoplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k \to \ldots\right)$ is a chain complex of left $\mathbb{Z}[\pi]$-modules. Here the $\widetilde{e}_i^k$ are lifts of the cells e_i^k of X. The group π acts on the right on $\ell^2(G)$ by $g \mapsto R_{\kappa(\pi,\phi,\gamma,t)(g)}$, an action which induces a structure of right $\mathbb{Z}[\pi]$-module on $\ell^2(G)$. Let

$$C_*^{(2)}(X, \phi, \gamma, t) = \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \mathbb{Z})$$

denote the finite Hilbert $\mathcal{N}(G)$-chain complex obtained by the tensor product associated to these left- and right-actions; we call $C_*^{(2)}(X, \phi, \gamma, t)$ *a* $\mathcal{N}(G)$*-cellular chain complex of* X.

If $C_*^{(2)}(X, \phi, \gamma, t)$ is a $\mathcal{N}(G)$-cellular chain complex of X, then denote

$$T^{(2)}(X, \phi, \gamma)(t) = T^{(2)}\left(C_*^{(2)}(X, \phi, \gamma, t)\right)$$

the *L^2-Alexander torsion of* (X, ϕ, γ) *at* $t > 0$. It is non-zero if and only if $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class.

Let $L = L_1 \cup \ldots \cup L_c$ be a link in S^3, M_L its exterior and $\alpha_L \colon G_L \to \mathbb{Z}^c$ the abelianization of its group. Any homomorphism $\phi \colon G_L \to \mathbb{Z}$ factors through α_L and thus is written $\phi = (n_1, \ldots, n_c) \circ \alpha_L$ where $n_1, \ldots, n_c \in \mathbb{Z}$. Any admissible triple (M_L, ϕ, γ) can thus be written $(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)$, and we will therefore denote

$$T^{(2)}_{L,(n_1,\ldots,n_c)}(\gamma)(t) := T^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t)$$

the *twisted L^2-Alexander torsion* associated to L, the coefficients $(n_1, \ldots, n_c)$, the morphism γ (the *twist*), at the value t. We sometimes omit the *twisted* when $\gamma = \mathrm{id}$.

3. Some useful properties of L^2-torsions

In this section we recall and generalize several natural properties of L^2-torsions, that are used in the proof of Theorem 5.1 (2).

The following Proposition 3.1 is a rephrasing of several results in [Lu], which concern short exact sequences of finite Hilbert $\mathcal{N}(G)$-chain complexes. Recall that $0 \to C_* \overset{\eta_*}{\to} D_* \overset{\rho_*}{\to} E_* \to 0$ is a *short exact sequence of finite Hilbert* $\mathcal{N}(G)$*-chain complexes* if $0 \to C_p \overset{\eta_p}{\to} D_p \overset{\rho_p}{\to} E_p \to 0$ is exact for every p and if η_*, ρ_* commute with the boundary operators of C_*, D_*, E_*.

Proposition 3.1 ([Lu]). *Let $0 \to C_* \overset{\eta_*}{\to} D_* \overset{\rho_*}{\to} E_* \to 0$ be a short exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes, such that for every $p \in \mathbb{Z}$, η_p and ρ_p are of determinant class. Then the following hold:*

1. *If two among C_*, D_*, E_* are weakly acyclic, then the third one is as well.*
2. *If C_*, D_*, E_* are all weakly acyclic, and if two of them are of determinant class, then the third one is as well.*
3. *If either C_*, D_*, E_* are all weakly acyclic and of determinant class, or D_* is not, then the L^2-torsions satisfy*

$$T^{(2)}(D_*) \cdot \left(\prod_{p \in \mathbb{Z}} \left(\frac{\det_G(\rho_p)}{\det_G(\eta_p)} \right)^{(-1)^p} \right) = T^{(2)}(C_*) \cdot T^{(2)}(E_*).$$

Proof. Let us prove (1). If two among C_*, D_*, E_* are weakly acyclic, then the long weakly exact homology sequence of finite Hilbert $\mathcal{N}(G)$-modules

$$LHS_* = \ \ldots \to H^{(2)}_{n+1}(E_*) \to H^{(2)}_n(C_*) \to H^{(2)}_n(D_*) \to H^{(2)}_n(E_*) \to \ldots$$

of [Lu, Theorem 1.21] is trivial, and thus C_*, D_*, E_* are all weakly acyclic.

Now, (2) and (3) follow from [Lu, Theorem 3.35 (1)], the assumptions on ι_*, ρ_*, and the fact that LHS_* is trivial (which implies that LHS_* is of determinant class and of L^2-torsion one). ☐

The following proposition is a slight generalization of [BA, Theorem 2.12], and concerns the invariance under simple homotopy equivalence.

Proposition 3.2. *Let $f : X \to Y$ be a simple homotopy equivalence between two finite CW-complexes inducing the group isomorphism $f_\sharp \colon \pi_1(X) \to \pi_1(Y)$. The triple (Y, ϕ, γ) is an admissible triple if and only if $(X, \phi \circ f_\sharp, \gamma \circ f_\sharp)$ is one. Moreover, for all $t > 0$:*

1. *$C^{(2)}_*(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ is weakly acyclic if and only if $C^{(2)}_*(Y, \phi, \gamma, t)$ is,*
2. *$C^{(2)}_*(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ is weakly acyclic and of determinant class if and only if $C^{(2)}_*(Y, \phi, \gamma, t)$ is,*
3. *$T^{(2)}(X, \phi \circ f_\sharp, \gamma \circ f_\sharp)(t) \ \doteq \ T^{(2)}(Y, \phi, \gamma)(t)$.*

Proof. The proof is almost exactly as the one of [BA, Theorem 2.12]: we study the case where f is an elementary expansion, and we relate $C^{(2)}_*(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ and $C^{(2)}_*(Y, \phi, \gamma, t)$ through an exact sequence. Then we apply Proposition 3.1. ☐

The following Proposition 3.3 is a slight generalization of the gluing formula of [BA, Theorem 3.1] and [BA2, Proposition 3.5] for L^2-Alexander torsions (which only stated that (1) and (2) together imply (3)).

Proposition 3.3. *Let X, A, B, V be finite CW-complexes such that $X = A \cup B$ and $V = A \cap B$. We denote the various inclusions (which are assumed to be cellular) and their inductions on fundamental groups as in the following diagrams:*

$$\begin{tikzcd}
 & A \arrow[dr, "J_A"] & \\
V \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & X \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_A"] & & & \\
\pi_1(V) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(X) \arrow[r, "\gamma"] \arrow[drr, "\phi"'] & G \arrow[d, dotted] \\
 & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}
\end{tikzcd}$$

Let $(\pi_1(X), \phi \colon \pi_1(X) \to \mathbb{Z}, \gamma \colon \pi_1(X) \to G)$ be an admissible triple, and $t > 0$. If any two of the following properties are satisfied,

1. *$C^{(2)}_*(V, \phi \circ i, \gamma \circ i, t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*
2. *$C^{(2)}_*(A, \phi \circ j_A, \gamma \circ j_A, t)$ and $C^{(2)}_*(B, \phi \circ j_B, \gamma \circ j_B, t)$ are weakly acyclic (resp. weakly acyclic and of determinant class),*
3. *$C^{(2)}_*(X, \phi, \gamma, t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*

then the third property is satisfied as well, and we have

$$T^{(2)}(X, \phi, \gamma)(t) \doteq \frac{T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A)(t) \cdot T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B)(t)}{T^{(2)}(V, \phi \circ i, \gamma \circ i)(t)}.$$

Proof. The proof works almost exactly as the one of [BA, Theorem 3.1]. Let us now sketch the modified arguments. Let V_*, X_* denote the finite Hilbert $\mathcal{N}(G)$-chain complexes of properties (1) and (3), and C_* the direct sum of the two finite Hilbert $\mathcal{N}(G)$-chain complexes in property (2). Observe that property (2) can be rephrased as C_* being weakly acyclic (resp. weakly acyclic and of determinant class). As explained in [BA, Theorem 3.1] (via classical arguments), we have an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes $0 \to V_* \to C_* \to X_* \to 0$, where the horizontal operators are of determinant class. The result then follows from Proposition 3.1 and the computation of Fuglede-Kadison determinants of the horizontal operators. ☐

The following Proposition 3.4 is a slight generalization of the L^2-Torres formula of [BA2, Theorem 4.4] (which only stated that (1) implies (2) instead of their equivalence).

Proposition 3.4. *Let $L = L_1 \cup \ldots \cup L_c$ be a c-component link, and $L' = L \cup L_{c+1}$ a $(c+1)$-component link link admitting L as a sublink. Let $M_L, M_{L'}$ denote the exteriors of L and L'. Let $Q\colon \pi_1(M_{L'}) \twoheadrightarrow \pi_1(M_L)$ denote the group epimorphism induced by removing the component L_{c+1}. Let $\lambda \in \pi_1(M_{L'})$ denote the class of a preferred longitude of L_{c+1}.*

Let $\phi : \pi_1(M_L) \to \mathbb{Z}$ and $\gamma : \pi_1(M_L) \to G$ be group homomorphisms such that $(\pi_1(M_L), \phi, \gamma)$ forms an admissible triple. We can write $\phi = (n_1, \ldots, n_c) \circ \alpha_L$ and thus $\phi \circ Q = (n_1, \ldots, n_c, 0) \circ \alpha_{L'}$ for some non zero vector $(n_1, \ldots, n_c) \in \mathbb{Z}^c$.

Assume that $(\gamma \circ Q)(\lambda)$ is of infinite order in G. Then for all $t > 0$, the following are equivalent:

1. *$C_*^{(2)}(M_{L'}, (n_1, \ldots, n_c, 0) \circ \alpha_{L'}, \gamma \circ Q)(t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*
2. *$C_*^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t)$ is weakly acyclic (resp. weakly acyclic and of determinant class).*

Moreover, we have

$$T^{(2)}_{L,(n_1,\ldots,n_c)}(\gamma)(t) \doteq \frac{T^{(2)}_{L',(n_1,\ldots,n_c,0)}(\gamma \circ Q)(t)}{\max(1,t)^{|\mathrm{lk}(L_1,L_{c+1})n_1 + \ldots + \mathrm{lk}(L_c,L_{c+1})n_c|}}.$$

Proof. The proof works almost exactly as in [BA2, Section 4], except that we use the generalized gluing formula of Proposition 3.3 instead of the weaker version of [BA2, Proposition 3.5]. ☐

4. Markov admissibility of a family of epimorphisms

For each $n \geqslant 1$ and each braid $\beta \in B_n$, let us denote

- $n(\beta) := n$ the number of strands,
- h_β the (Artin) group automorphism on $\mathbb{F}_{n(\beta)}$ (note that $\beta \mapsto h_\beta$ is anti-multiplicative),
- $\gamma_\beta\colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_\beta$ the quotient by all relations of the form $\cdot = h_\beta(\cdot)$,
- $\hat\beta$ the closure of β, a link in S^3,
- $G_{\hat\beta} = \pi_1\left(S^3 \setminus \hat\beta\right)$ the group of the link $\hat\beta$,
- $\Phi_n\colon \mathbb{F}_n \twoheadrightarrow \mathbb{Z}$ the epimorphism which sends the n generators to 1,
- $\iota_n\colon \mathbb{F}_n \hookrightarrow \mathbb{F}_{n+1}$ the group inclusion sending the n generators of $\mathbb{F}_n$ to the first n generators of $\mathbb{F}_{n+1}$.

Definition 4.1. A family $\mathcal{Q}$ of group epimorphisms of the form

$$\mathcal{Q} = \left\{Q_\beta\colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta} \mid \beta \in \sqcup_{n\geqslant 1} B_n\right\}$$

will be called *Markov-admissible* if it satisfies the following conditions:

(1) For all $n \geqslant 1$ and all $\alpha, \beta \in B_n$, there exists a group homomorphism $\chi^{\mathcal{Q}}_{\beta,\alpha} \colon G_{Q_\beta} \to G_{Q_{\alpha^{-1}\beta\alpha}}$ such that $Q_{\alpha^{-1}\beta\alpha} \circ h_\alpha = \chi^{\mathcal{Q}}_{\beta,\alpha} \circ Q_\beta$. Observe that this implies that each $\chi^{\mathcal{Q}}_{\beta,\alpha}$ is uniquely defined and an isomorphism.
(2) For all $n \geqslant 1$, $\beta \in B_n$ and $\varepsilon \in \{\pm 1\}$, there exists a group monomorphism $\sigma^{\mathcal{Q}}_{\beta,\varepsilon} \colon G_{Q_\beta} \hookrightarrow G_{Q_{\sigma_n^\varepsilon \beta}}$ such that $Q_{\sigma_n^\varepsilon\beta} \circ \iota_n = \sigma^{\mathcal{Q}}_{\beta,\varepsilon} \circ Q_\beta$. Observe that each $Q_{\sigma_n^\varepsilon\beta}$ is uniquely defined.

These two conditions are illustrated in Figure 2.

$$\begin{tikzcd}
\mathbb{F}_n \arrow[r, "h_\alpha", "\sim"'] \arrow[d, two heads, "Q_\beta"] & \mathbb{F}_n \arrow[d, two heads, "Q_{\alpha^{-1}\beta\alpha}"] \\
G_{Q_\beta} \arrow[r, "\chi^{\mathcal{Q}}_{\beta,\alpha}", "\sim"'] & G_{Q_{\alpha^{-1}\beta\alpha}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}_n \arrow[r, hook, "\iota_n"] \arrow[d, two heads, "Q_\beta"] & \mathbb{F}_{n+1} \arrow[d, two heads, "Q_{\sigma_n^\varepsilon\beta}"] \\
G_{Q_\beta} \arrow[r, hook, "\sigma^{\mathcal{Q}}_{\beta,\varepsilon}"] & G_{Q_{\sigma_n^\varepsilon\beta}}
\end{tikzcd}$$

FIGURE 2. Conditions for $\mathcal{Q}$ to be Markov-admissible

Roughly speaking, the epimorphisms in a Markov-admissible family will be compatible in a way that lets us hope to compute (L^2-)knot invariants by using them. Note that less restrictive definitions may be preferred in the future if we find better ways of computing L^2-objects such as Fuglede-Kadison determinants.

We will now present several examples of Markov-admissible families, which are all displayed in Figure 4 for clarity. Moreover, Figure 4 summarizes the results of Sections 5 and 6 concerning Markov invariance (✓ standing for yes and ✗ for no).

$$\begin{tikzcd}
 & & \mathbb{F}_n\ ✗ \arrow[dll, two heads, "\varphi_n"'] \arrow[ddd, two heads, "\Phi_n"] \arrow[drr, two heads, "\gamma_\beta"] & & & & \\
✗\ \mathbb{Z}^n \arrow[ddrr, dashed, two heads] & & & & G_\beta\ ✓ \arrow[rr, "\Psi_\beta", "\sim"'] \arrow[dr, two heads, "\psi_\beta"'] \arrow[ddll, dashed, two heads] & & G_{\hat\beta}\ ✓ \\
 & & & & & \Gamma_{\psi_\beta}\ ✓ \arrow[dlll, dashed, two heads] & \\
 & & \mathbb{Z}\ ✓ & & & &
\end{tikzcd}$$

FIGURE 3. When does $\overline{\mathcal{B}}^{(2)}_{t,\gamma}$ yield a map on braids which is invariant under Markov moves, for $\gamma \colon \mathbb{F}_n \twoheadrightarrow G$?

Example 4.2. The family of identity morphisms $\mathcal{Q} = \left\{ \mathrm{id}_{\mathbb{F}_{n(\beta)}} \right\}$ is Markov-admissible, with $\chi^{\mathcal{Q}}_{\beta,\alpha} = \mathrm{id}_{\mathbb{F}_{n(\beta)}}$ and $\sigma^{\mathcal{Q}}_{\beta,\varepsilon} = \iota_{n(\beta)}$.

Example 4.3. The family $\mathcal{Q} = \left\{ \Phi_{n(\beta)} \right\}$ is Markov-admissible, with $\chi^{\mathcal{Q}}_{\beta,\alpha} = \sigma^{\mathcal{Q}}_{\beta,\varepsilon} = \mathrm{id}_{\mathbb{Z}}$.

Example 4.4. The family of abelianizations $\mathcal{Q} = \left\{ \varphi_{n(\beta)} \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \mathbb{Z}^{n(\beta)} \right\}$ is Markov-admissible, with $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}$ the inclusion $\mathbb{Z}^{n(\beta)} \hookrightarrow \mathbb{Z}^{n(\beta)+1}$ induced by $\iota_{n(\beta)}$, and $\chi^{\mathcal{Q}}_{\beta,\alpha}$ the permutation on the canonical generators of $\mathbb{Z}^n$ corresponding to the permutation of $n(\alpha)$ strands induced by $\alpha \in B_n$.

The following proposition is an elementary result in group theory, but is stated for the reader's convenience.

Proposition 4.5. *Let $f\colon G \to H$ be a group homomorphism, and N a normal subgroup of G. If f is surjective and $Ker(f) \subset N$ (in particular if f is an isomorphism), then f induces an isomorphism between G/N and $H/f(N)$.*

Let us now consider epimorphisms that descend to the fundamental groups of the braid closure complements.

Proposition 4.6. *The family $\mathcal{Q} = \{\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_\beta\}$ is Markov-admissible. Moreover the monomorphisms $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}$ are isomorphisms.*

Proof. First step: Markov 1:
Take $n \geqslant 1$ and $\alpha, \beta \in B_n$. Note that

$$\begin{aligned} h_\alpha(Ker(\gamma_\beta)) &= h_\alpha\left(\langle\langle h_\beta(x)x^{-1}; x \in \mathbb{F}_n\rangle\rangle \right) = \langle\langle h_{\beta\alpha}(x)h_\alpha(x)^{-1}; x \in \mathbb{F}_n\rangle\rangle \\ &= \langle\langle h_{\alpha^{-1}\beta\alpha}(y)y^{-1}; y \in \mathbb{F}_n\rangle\rangle = Ker(\gamma_{\alpha^{-1}\beta\alpha}), \end{aligned}$$

thus it follows from Proposition 4.5 that the isomorphism $h_\alpha \colon \mathbb{F}_n \xrightarrow{\sim} \mathbb{F}_n$ induces the required isomorphism $\chi^{\mathcal{Q}}_{\beta,\alpha} \colon G_\beta \xrightarrow{\sim} G_{\alpha^{-1}\beta\alpha}$.

Second step: Markov 2:
Take $n \geqslant 1$ and $\beta \in B_n$. Let $\beta_+ = \sigma_n^{-1}\iota(\beta) \in B_{n+1}$. First notice that

$$\begin{aligned} &h_{\beta_+}(x_j)x_j^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_j))x_j^{-1} = h_{\iota(\beta)}(x_j)x_j^{-1} \quad \text{for } 1 \leqslant j \leqslant n-1, \\ &h_{\beta_+}(x_n)x_n^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_n))x_n^{-1} = h_{\iota(\beta)}(x_{n+1})x_n^{-1} = x_{n+1}x_n^{-1}, \\ &h_{\beta_+}(x_{n+1})x_{n+1}^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_{n+1}))x_{n+1}^{-1} = h_{\iota(\beta)}(x_{n+1}^{-1}x_n x_{n+1})x_{n+1}^{-1} = x_{n+1}^{-1}h_{\iota(\beta)}(x_n). \end{aligned}$$

Hence $Ker(\gamma_{\beta_+}) = \langle\langle\ x_{n+1}x_n^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle$.

We can now define $\sigma^{\mathcal{Q}}_{\beta,-1} := \Big(G_\beta \ni [x]_{G_\beta} \mapsto [\iota_n(x)]_{G_{\beta_+}} \in G_{\beta_+}\Big)$, where $x \in \mathbb{F}_n$ and $[\cdot]_G$ is the quotient class in G. Since $\iota_n(Ker(\gamma_\beta)) \subset Ker(\gamma_{\beta_+})$, then $\sigma^{\mathcal{Q}}_{\beta,-1}$ is a well-defined group homomorphism.

Let us prove that $\sigma^{\mathcal{Q}}_{\beta,-1}$ is surjective. Let $[y]_{G_{\beta_+}} \in G_{\beta_+}$, with $y \in \mathbb{F}_{n+1}$. Let $y' \in \iota_n(\mathbb{F}_n)$ be the word constructed from y by replacing all letters x_{n+1} with x_n. Hence $[y]_{G_{\beta_+}} = [y']_{G_{\beta_+}} \in Im(\sigma^{\mathcal{Q}}_{\beta,-1})$ and $\sigma^{\mathcal{Q}}_{\beta,-1}$ is surjective.

Let us prove that $\sigma^{\mathcal{Q}}_{\beta,-1}$ is injective. Let $[\iota_n(x)]_{G_{\beta_+}} \in G_{\beta_+}$ (with $x \in \mathbb{F}_n$) be trivial. Then $\iota_n(x) \in Ker(\gamma_{\beta_+}) = \langle\langle\ x_{n+1}x_n^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle$. Thus $\iota_n(x)$ is a product of conjugates (in $\mathbb{F}_{n+1}$) of terms $(x_{n+1}x_n^{-1})^{\pm 1}$ and/or terms in $\iota_n(Ker(\gamma_\beta))$. But since $\iota_n(x)$ is a free word without the letter x_{n+1}, we conclude that the conjugates of terms $(x_{n+1}x_n^{-1})^{\pm 1}$ in $\iota_n(x)$ cancel each other. Thus $\iota_n(x) \in \iota_n(Ker(\gamma_\beta))$ and $[x]_{G_\beta} = 1$. Hence $\sigma^{\mathcal{Q}}_{\beta,-1}$ is injective.

Third step: Markov 2 again:
Take $n \geqslant 1$, $\beta \in B_n$, and let $\beta_+ = \sigma_n\iota(\beta) \in B_{n+1}$. The proof is similar as in the Second step, with the following differences:

$$\begin{aligned} &h_{\beta_+}(x_n)x_n^{-1} = h_{\iota(\beta)}(x_n)x_{n+1}\left(h_{\iota(\beta)}(x_n)\right)^{-1}x_n^{-1}, \\ &h_{\beta_+}(x_{n+1})x_{n+1}^{-1} = h_{\iota(\beta)}(x_n)x_{n+1}^{-1}, \\ &Ker(\gamma_{\beta_+}) = \langle\langle\ h_{\iota(\beta)}(x_n)x_{n+1}^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle, \end{aligned}$$

and we replace the letters x_{n+1} with $h_{\iota(\beta)}(x_n)$ in the proof of the surjectivity of $\sigma^{\mathcal{Q}}_{\beta,1}$. ☐

Remark 4.7. In the second step of the previous proof, one can alternatively prove that $\sigma^{\mathcal{Q}}_{\beta,-1}$ is an isomorphism by observing that it corresponds to the sequence of Tietze transformations going from the presentation

$$\langle x_1,\dots,x_n | h_\beta(x_1)x_1^{-1},\dots,h_\beta(x_{n-1})x_{n-1}^{-1}, h_\beta(x_n)x_n^{-1}\rangle$$

of G_β to the presentation

$$\langle x_1,\dots,x_n,x_{n+1} | h_{\iota(\beta)}(x_1)x_1^{-1},\dots,h_{\iota(\beta)}(x_{n-1})x_{n-1}^{-1}, x_{n+1}x_n^{-1}, x_{n+1}^{-1}h_{\iota(\beta)}(x_n)\rangle =$$
$$\langle x_1,\dots,x_n,x_{n+1} | h_{\beta_+}(x_1)x_1^{-1},\dots,h_{\beta_+}(x_{n-1})x_{n-1}^{-1}, h_{\beta_+}(x_n)x_n^{-1}, h_{\beta_+}(x_{n+1})x_{n+1}^{-1}\rangle$$

of G_{β_+}.

Finally, let us fix notations for epimorphisms that go lower than the groups of the braid closures.

Example 4.8. Let $\left\{\psi_\beta \colon G_\beta \twoheadrightarrow \Gamma_{\psi_\beta}\right\}_{\beta\in\sqcup_{n\geqslant 1}B_n}$ be a family of epimorphisms such that

- $\Phi_{n(\beta)}$ factors through $\psi_\beta\circ\gamma_\beta$,
- $\chi^{\mathcal{Q}}_{\beta,\alpha}\left(Ker(\psi_\beta)\right) = Ker(\psi_{\alpha^{-1}\beta\alpha})$,
- $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}\left(Ker(\psi_\beta)\right) = Ker(\psi_{\sigma_n^\varepsilon\iota(\beta)})$.

Then it follows from Propositions 4.5 and 4.6 that the family

$$\mathcal{Q} = \left\{\psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\right\}$$

is Markov-admissible.

Note that the first assumption (that $\Phi_{n(\beta)}$ factors through $\psi_\beta\circ\gamma_\beta$) is not necessary for $\mathcal{Q}$ to be Markov-admissible, but is relevant in order to compare the function $F_{\mathcal{Q}}$ of the next section with L^2-Alexander torsions (for which such a factoring property is assumed, see [BAC]).

5. A sufficient condition for Markov invariance

For $\mathcal{Q} = \{Q_\beta\}$ a Markov-admissible family we define the function

$$F_{\mathcal{Q}} := \left(\begin{array}{ccc} \sqcup_{n\geqslant 1}B_n & \to & \mathcal{F}(\mathbb{R}_{>0},\mathbb{R}_{>0})/\{t\mapsto t^m, m\in\mathbb{Z}\} \\ \beta & \mapsto & \left[t\mapsto \dfrac{\det^r_{G_{Q_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^n}\right] \end{array}\right),$$

taking values in equivalence classes $[t\mapsto f(t)]$ of selfmaps of $\mathbb{R}_{>0}$, up to multiplication by monomials with integer exponents.

The following theorem gives a sufficient condition on $\mathcal{Q}$ for $F_{\mathcal{Q}}$ to be a Markov function and thus yield a invariant of links.

Theorem 5.1. *Let $\mathcal{Q} = \{\psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then :*

1. *$F_{\mathcal{Q}}$ is a Markov function, and thus defines an invariant of links, that we will denote $\mathcal{T}_{\mathcal{Q}}$.*
2. *For all $n\geqslant 1$, $t>0$, and $\beta\in B_n$, we have*

$$F_{\mathcal{Q}}(\beta)(t) \doteq T^{(2)}_{\hat\beta,(1,\dots,1)}(\psi_\beta\circ(\Psi_\beta)^{-1})(t).$$

In particular, given a link L, then for any braid β such that $\hat\beta = L$, the invariant $\mathcal{T}_{\mathcal{Q}}(L)$ of L is equal to the equivalence class of selfmaps of $\mathbb{R}_{>0}$

$$\left[t\mapsto T^{(2)}_{L,(1,\dots,1)}(\psi_\beta\circ(\Psi_\beta)^{-1})(t)\right]$$

of twisted L^2-Alexander torsions of the exterior of L, where the twist is the epimorphism $\psi_\beta\circ(\Psi_\beta)^{-1} : G_L \twoheadrightarrow \Gamma_{\psi_\beta}$.

Remark that for $\mathcal{Q} = \{\Psi_\beta \circ \gamma_\beta : \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{\hat{\beta}}\}_\beta$, where Ψ_β denotes an isomorphism between G_β and $G_{\hat{\beta}}$, it follows from [BAC] that the link invariant $\mathcal{T}_\mathcal{Q}$ is the L^2-Alexander torsion. Hence, part (2) of Theorem 5.1 is a generalization of the main result of [BAC].

To prove Theorem 5.1 (1), we will use two lemmas, one for each Markov move.

Lemma 5.2. *Let $\mathcal{Q} = \{\psi_\beta \circ \gamma_\beta : \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then $F_\mathcal{Q}$ is invariant under the first Markov move.*

Proof. Let $\mathcal{Q} = \{Q_\beta := \psi_\beta \circ \gamma_\beta : \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$. Let $n \geqslant 1$ be an integer, $t > 0$, and $\alpha, \beta \in B_n$. We will prove that

$$\det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) = \det{}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right).$$

Remark that, for any epimorphism $\gamma : \mathbb{F}_n \to G$, it follows from Proposition 2.4 that:

$$\mathrm{Id}^{\oplus(n-1)} = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(1) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha\alpha^{-1}) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}}}(\alpha),$$

thus $\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}) = \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}}}(\alpha)\right)^{-1}$.

Consequently, we have for any epimorphism $\gamma : \mathbb{F}_n \to G$:

$$\begin{aligned}\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}\beta\alpha) &= \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\beta\alpha}}(\alpha^{-1}) \\ &= \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1}.\end{aligned}$$

Hence, for any epimorphism $\gamma : \mathbb{F}_n \to G$:

$$\begin{aligned}&\det{}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= \det{}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= \det{}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right),\end{aligned}$$

where the second equality follows from Remark 2.5 and Proposition 2.2 (1).

For $\gamma = Q_{\alpha^{-1}\beta\alpha}$, we thus have:

$$\begin{aligned}&\det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= \det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_\alpha}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= \det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^\mathcal{Q}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right).\end{aligned}$$

Now, since for every braid $\sigma \in B_n$, $\gamma_\sigma \circ h_\sigma = \gamma_\sigma$ and $Q_\sigma = \psi_\sigma \circ \gamma_\sigma$, we conclude that:

$$\begin{aligned}&\det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= \det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^\mathcal{Q}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= \det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^\mathcal{Q}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= \det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^\mathcal{Q}_{\beta,\alpha}\circ Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= \det{}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right),\end{aligned}$$

where the last equality follows from Proposition 2.2 (3). $\square$

Lemma 5.3. *Let $\mathcal{Q} = \{\psi_\beta \circ \gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then $F_\mathcal{Q}$ is invariant under the second Markov move.*

Proof. For any braid β let us denote $Q_\beta := \psi_\beta \circ \gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}$.

First step: negative crossing:

Let $n \geqslant 1$ be an integer, $\beta \in B_n$ and $\beta_- = \sigma_n^{-1}\iota(\beta) \in B_{n+1}$. Let $t > 0$. We will prove that

$$\frac{\det^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right)}{\max(1,t)^{n+1}} = \frac{1}{t} \cdot \frac{\det^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^n}.$$

In order to do this, we will compose $\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}$ with two operators (one named $\mathfrak{G}$ on the left, one named $\mathfrak{D}$ on the right) so that we obtain a block triangular operator with the upper left block "mostly" equal to $\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}$.

First, it follows from Proposition 2.4 that

$$\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) = \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}\left(\sigma_n^{-1}\iota(\beta)\right) = \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\,\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right).$$

Recall that $\overline{\mathcal{B}}^{(2)}_{t,id}\left(\sigma_n^{-1}\right) = \begin{pmatrix} \mathrm{Id} & & & & 0 \\ & \ddots & & & \vdots \\ & & \mathrm{Id} & & 0 \\ & & & \mathrm{Id} & \mathrm{Id} \\ 0 & \dots & 0 & 0 & -\frac{1}{t}R_{g_{n-1}g_n^{-1}} \end{pmatrix}$, thus the operator defined as $\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right)\right)^{-1}$ is equal to:

$$\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right)\right)^{-1} = \begin{pmatrix} \mathrm{Id} & & & & 0 \\ & \ddots & & & \vdots \\ & & \mathrm{Id} & & 0 \\ & & & \mathrm{Id} & tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \\ 0 & \dots & 0 & 0 & -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \end{pmatrix}.$$

We therefore compute:

$$\begin{aligned} \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) \circ \mathfrak{D} &= \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta)) \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right) \circ \mathfrak{D} \\ &= \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta)) \\ &= \begin{pmatrix} & 0 \\ \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) & \vdots \\ & 0 \\ \mathcal{R} & \mathrm{Id} \end{pmatrix}, \end{aligned}$$

where the row $\mathcal{R}$ is the right multiplication operator by the row

$$\left(\kappa(t,\Phi_{n+1},Q_{\beta_-})\left(\frac{\partial h_{\iota(\beta)}(g_j)}{\partial g_n}\right)\right) = \left(\kappa(t,\Phi_{n+1}\circ\iota_{\mathbb{F}_n},Q_{\beta_-}\circ\iota_{\mathbb{F}_n})\left(\frac{\partial h_\beta(g_j)}{\partial g_n}\right)\right)$$

that has $n-1$ coefficients in $\mathbb{R}\Gamma_{\psi_{\beta_-}}$.

Now, the fundamental formula of Fox calculus (Proposition 2.1) implies that:

$$\begin{pmatrix} R_{g_1-1} & \dots & R_{g_n-1} \end{pmatrix} \cdot \begin{pmatrix} R\dfrac{\partial h_{\iota(\beta)}(g_j)}{\partial g_i} \end{pmatrix}_{1\leqslant i,j\leqslant n} = \begin{pmatrix} R_{h_{\iota(\beta)}(g_1)-1} & \dots & R_{h_{\iota(\beta)}(g_n)-1} \end{pmatrix},$$

thus, by applying $\kappa(t,\Phi_{n+1},Q_{\beta_-})$ to the previous equality and by definition of $\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))$, we obtain:

$$\begin{aligned}
&\left(tR_{Q_{\beta_-}(g_1)}-\mathrm{Id} \quad \dots \quad t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}\right)\cdot \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\\
&=\left(tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)}-\mathrm{Id} \qquad \dots \qquad t^nR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n)}-\mathrm{Id}\right)\\
&=\left(tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)}-\mathrm{Id} \quad \dots \quad t^{n-1}R_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_{n-1})}-\mathrm{Id} \quad t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}\right),
\end{aligned}$$

where the second equality follows from the fact that $\beta\in B_n$ leaves g_n unchanged in the Artin action. Let us thus define the following block triangular operator $\mathfrak{G}$:

$$\mathfrak{G}:=\begin{pmatrix}
\mathrm{Id} & & & 0\\
& \ddots & & \vdots\\
& & \mathrm{Id} & 0\\
tR_{Q_{\beta_-}(g_1)}-\mathrm{Id} & \dots & t^{n-1}R_{Q_{\beta_-}(g_{n-1})}-\mathrm{Id} & t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}
\end{pmatrix}.$$

It immediately follows from what precedes that

$$\mathfrak{G}\circ\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-)\right)\circ\mathfrak{D}=\mathfrak{G}\circ\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\right)=$$

$$\begin{pmatrix}
 & & & 0\\
 & \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) & & \vdots\\
 & & & 0\\
tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)}-\mathrm{Id} & \dots & t^{n-1}R_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_{n-1})}-\mathrm{Id} & t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}
\end{pmatrix}.$$

On the other hand, we compute $\mathfrak{G}\circ\left(\mathrm{Id}^{\oplus n}\right)\circ\mathfrak{D}=$

$$\begin{pmatrix}
\mathrm{Id} & & & 0\\
& \ddots & & \vdots\\
& & & 0\\
& & \mathrm{Id} & tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_ng_{n-1}^{-1})}\\
tR_{Q_{\beta_-}(g_1)}-\mathrm{Id} & \dots & t^{n-1}R_{Q_{\beta_-}(g_{n-1})}-\mathrm{Id} & \star
\end{pmatrix},$$

with $\star=t^nR_{Q_{\beta_-}\left(g_nh_{\iota(\beta)}(g_{n-1}^{-1})g_{n-1}\right)}-t^{n+1}R_{Q_{\beta_-}\left(g_nh_{\iota(\beta)}(g_{n-1}^{-1})g_n\right)}$.

Hence

$$\mathfrak{G}\circ\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-)-\mathrm{Id}^{\oplus n}\right)\circ\mathfrak{D}=$$

$$\begin{pmatrix}
 & & & 0\\
 & \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta)-\mathrm{Id}^{\oplus(n-1)} & & \vdots\\
 & & & 0\\
 & & & -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_ng_{n-1}^{-1})}\\
\dots & t^jR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_j)}-t^jR_{Q_{\beta_-}(g_j)} & \dots & \square
\end{pmatrix},$$

where $j\in\{1,\dots,n-1\}$ and

$$\square=t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}-t^nR_{Q_{\beta_-}\left(g_nh_{\iota(\beta)}(g_{n-1}^{-1})g_{n-1}\right)}+t^{n+1}R_{Q_{\beta_-}\left(g_nh_{\iota(\beta)}(g_{n-1}^{-1})g_n\right)}.$$

Now we use the fact that our epimorphism Q_{β_-} descends deeper than the braid closure group: indeed, for every $j\in\{1,\dots,n-1\}$, we have:

$$Q_{\beta_-}(g_j)=\left(Q_{\beta_-}\circ h_{\beta_-}\right)(g_j)=\left(Q_{\beta_-}\circ h_{\iota(\beta)}\circ h_{\sigma_n^{-1}}\right)(g_j)=\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_j).$$

Thus $\mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right) \circ \mathfrak{D}$ is actually upper block triangular and equal to:

$$\begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus(n-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \end{matrix} \\ 0 & -\mathrm{Id} + t^{n+1} R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_n\right)} \end{pmatrix}.$$

By applying $\det^r_{\Gamma_{\psi_{\beta_-}}}$ to the previous equality, it therefore follows from Proposition 2.2 and Remark 2.5 that

$$\begin{aligned} &\max(1,t)^n \cdot \det{}^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right) \cdot t \\ &= \det{}^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right) \cdot \max(1,t)^{n+1}. \end{aligned}$$

We conclude by using the fact that $Q_{\beta_-} \circ \iota_{\mathbb{F}_n} = \sigma^{\mathcal{Q}}_{\beta,-1} \circ Q_\beta$ (by Markov-admissibility of $\mathcal{Q}$) and Proposition 2.2 (3).

Second step: positive crossing:
Let $\beta_+ = \sigma_n \iota(\beta) \in B_{n+1}$. We will proceed almost exactly as in the first step, except for the following differences:

- We now aim to prove that

$$\frac{\det{}^r_{\Gamma_{\psi_{\beta_+}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}}(\beta_+) - \mathrm{Id}^{\oplus n}\right)}{\max(1,t)^{n+1}} = \frac{\det{}^r_{\Gamma_{\psi_{\beta}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta}}(\beta) - \mathrm{Id}^{\oplus (n-1)}\right)}{\max(1,t)^{n}}.$$

- The operator $\mathfrak{D}$ becomes:

$$\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}\circ h_{\iota(\beta)}}(\sigma_n)\right)^{-1} = \begin{pmatrix} \mathrm{Id} & & & & 0 \\ & \ddots & & & \vdots \\ & & \mathrm{Id} & & 0 \\ & & & \mathrm{Id} & \mathrm{Id} \\ 0 & \dots & 0 & 0 & -\frac{1}{t} R_{Q_{\beta_+}(g_n g_{n+1}^{-1})} \end{pmatrix}.$$

- The final lower right coefficient $\star$ of $\mathfrak{G} \circ \mathfrak{D}$ becomes

$$\star = t^{n-1} R_{Q_{\beta_+}(g_{n-1})} - \mathrm{Id} - t^{n-1} R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1} g_n\right)} + \frac{1}{t} R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}.$$

- The final lower right coefficient $\square$ of $\mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}}(\beta_+) - \mathrm{Id}^{\oplus n}\right) \circ \mathfrak{D}$ becomes

$$\square = -t^{n-1} R_{Q_{\beta_+}(g_{n-1})} + t^n R_{Q_{\beta_+}(g_n)} + t^{n-1} R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1} g_n\right)} - \frac{1}{t} R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}.$$

- The simplification

$$\square = t^n R_{Q_{\beta_+}(g_n)} - \frac{1}{t} R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}$$

comes from the following reasoning. Remark that in the ring $\mathbb{Z}\mathbb{F}_{n+1}$ we have the equalities:

$$\begin{aligned} g_{n-1} - g_n g_{n+1}^{-1} g_n &= g_n \left(g_n^{-1} - g_{n+1}^{-1} g_n g_{n-1}^{-1} \right) g_{n-1} \\ &= g_n \left(g_n^{-1} - h_{\sigma_n}^{-1} \left(g_n^{-1} \right) \right) g_{n-1} \\ &= g_n \left(g_n^{-1} - h_{\sigma_n}^{-1} \left(h_{\iota(\beta)}^{-1} (g_n^{-1}) \right) \right) g_{n-1} \\ &= g_n \left(g_n^{-1} - h_{\beta_+}^{-1} \left(g_n^{-1} \right) \right) g_{n-1} \\ &= g_n \left(h_{\beta_+} \left(h_{\beta_+}^{-1} \left(g_n^{-1} \right) \right) - h_{\beta_+}^{-1} \left(g_n^{-1} \right) \right) g_{n-1}. \end{aligned}$$

Hence, by composing with $\gamma_{\beta_+} : \mathbb{Z}[\mathbb{F}^n] \to \mathbb{Z}[G_{\beta_+}]$, the quotient by all relations of the form $h_{\beta_+}(*) = *$, we obtain $\gamma_{\beta_+} \left(g_{n-1} - g_n g_{n+1}^{-1} g_n \right) = 0$. The previous equality to zero remain true when applying any deeper epimorphism $Q_{\beta_+} = \psi_{\beta_+} \circ \gamma_{\beta_+}$ instead.

- Applying $\det^r_{\Gamma_{\psi_{\beta_+}}}$ to the final equality now yields

$$\begin{aligned} &\max(1,t)^n \cdot \det{}^r_{\Gamma_{\psi_{\beta_+}}} \left(\overline{\mathcal{B}}^{(2)}_{t, Q_{\beta_+}} (\beta_+) - \mathrm{Id}^{\oplus n} \right) \cdot \frac{1}{t} \\ &= \det{}^r_{\Gamma_{\psi_{\beta_+}}} \left(\overline{\mathcal{B}}^{(2)}_{t, Q_{\beta_+} \circ \iota_{\mathbb{F}_n}} (\beta) - \mathrm{Id}^{\oplus (n-1)} \right) \cdot \frac{1}{t} \max(1,t)^{n+1}, \end{aligned}$$

as expected.

□

We now have all the tools to prove Theorem 5.1.

Proof of Theorem 5.1. Part (1) follows immediately from Lemmas 5.2 and 5.3. The last part follows immediately from parts (1) and (2). Let us prove part (2).

Let L be a link in S^3, of exterior $M_L = S^3 \setminus \nu L$. Let $n \geqslant 1$ and $\beta \in B_n$ such that $L = \hat{\beta}$. We will mostly follow the way of the proof of [BAC, Theorem 4.9], except for the fact that ψ_β is now a general epimorphism and not the identity. This is why we need the generalized properties of Section 3. We will skip over some details already covered in [BAC].

Recall that

$$P = \langle g_1, \ldots, g_n | r_1 := h_\beta(g_1) g_1^{-1}, \ldots, r_{n-1} := h_\beta(g_{n-1}) g_{n-1}^{-1} \rangle$$

is a presentation of G_β, and also of G_L, through the isomorphism $\Psi_\beta : G_\beta \xrightarrow{\sim} G_L$. Let W_P be the 2-dimensional CW-complex constructed from P. Recall that W_P has a single 0-cell, one 1-cell for each generator of P, and one 2-cell for each relator of P, each 2-cell being glued on the wedge of circles that is the 1-skeleton following the word in the generators formed by the relator in question.

Now denote $L' = L \cup C_\beta$, where C_β is the boundary circle of D_n not coming from one of the punctures, when drawing L as the closure of β. Then

$$P' = \langle g_1, \ldots, g_n, y | r_1' := h_\beta(g_1) y g_1^{-1} y^{-1}, \ldots, r_n' := h_\beta(g_n) y g_n^{-1} y^{-1} \rangle$$

is a presentation of $G_{L'}$, with y a meridian of C_β. Let $W_{P'}$ be the 2-dimensional CW-complex constructed from P'. Since L' is not split, $W_{P'}$ and $M_{L'}$ are therefore $K(G_{L'}, 1)$ spaces, and since the Whitehead group of $G_{L'}$ is trivial, we have that $W_{P'}$ is simple homotopy equivalent to $M_{L'}$.

To simplify notations, let us denote $G := \Gamma_{\psi_\beta}$, $\psi := \psi_\beta \circ (\Psi_\beta)^{-1} : G_L \twoheadrightarrow G$ and $\phi_L := (1, \ldots, 1) \circ \alpha_L \colon G_L \twoheadrightarrow \mathbb{Z}$. Let $t > 0$. Let $Q \colon \pi_1(M_{L'}) \twoheadrightarrow \pi_1(M_L)$ denote the group epimorphism induced by removing the component C_β. Let us also denote as follows the five finite Hilbert $\mathcal{N}(G)$-chain complexes we will use in the proof:

- $E_* := C_*^{(2)}(M_L, \phi_L, \psi, t)$,

- $E'_* := C_*^{(2)}(M_{L'}, \phi_L \circ Q, \psi \circ Q, t)$,
- $W_* := C_*^{(2)}(W_P, \phi_L, \psi, t) =$
$$\bigoplus_{j=1}^{n-1} \ell^2(G)\widetilde{r_j} \xrightarrow[\begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,\psi}(\beta) - \mathrm{Id}^{n-1} \\ * \end{pmatrix}]{\partial_2} \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{g_i} \xrightarrow[R_{(t\psi(g_1)-1,\ldots,t^n\psi(g_n)-1)}]{\partial_1} \ell^2(G),$$
- $W'_* := C_*^{(2)}(W_{P'}, \phi_L \circ Q, \psi \circ Q, t) =$
$$\bigoplus_{j=1}^{n} \ell^2(G)\widetilde{r'_j} \xrightarrow[\begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,\psi}(\beta) - \mathrm{Id}^{n-1} & & 0 \\ * & & 0 \\ tR_{\psi(h_\beta(g_1))} - \mathrm{Id} & \ldots & t^n R_{\psi(g_n)} - \mathrm{Id} \end{pmatrix}]{\partial_2} \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{g_i} \oplus \ell^2(G)\widetilde{y}$$
$$\xrightarrow[R_{(t\psi(g_1)-1,\ldots,t^n\psi(g_n)-1,0)}]{\partial_1} \ell^2(G),$$
- $D_* := 0 \to \ell^2(G)\widetilde{r'_n} \xrightarrow[\mathrm{Id}-t^n R_{\psi(g_n)}]{\partial_2} \ell^2(G)\widetilde{y} \xrightarrow[0]{\partial_1} 0 \to 0.$

The forms of W_* and W'_* follow from the fact that we can use Fox calculus to describe the boundary operators in celullar chain complexes associated to the universal covers $\widetilde{W_P}$ and $\widetilde{W'_P}$.

Observe that we have $T^{(2)}(W_*) = F_{\mathcal{Q}}(\beta)(t)$. Moreover, we have $T^{(2)}(E_*) = T^{(2)}_{L,(1,\ldots,1)}(\psi)(t)$ by definition. Thus we only need to prove that $T^{(2)}(W_*) = T^{(2)}(E_*)$. Let us state the two following facts.

<u>Fact 1:</u> For λ denoting the class of a preferred longitude of C_β in $G_{L'}$, $\psi(Q(\lambda))$ has infinite order in G. This Fact is a consequence of ϕ_L factoring through ψ and $\phi(Q(\lambda)) = n \neq 0$.

<u>Fact 2:</u> There exists
$$0 \to W_* \xrightarrow{\eta_*} W'_* \xrightarrow{\rho_*} D_* \to 0,$$
a short exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes, with η_2, η_1 the obvious inclusions, $\eta_0 = 0$, ρ_2, ρ_1 the obvious projections, and $\rho_0 = \mathrm{Id}$. This Fact follows from comparing the cells of W_P and $W_{P'}$, and from the definitions of L^2-Burau maps with Fox calculus.

We can now establish the following equalities of L^2-torsions:
$$T^{(2)}(E_*) \doteq \frac{T^{(2)}(E'_*)}{\max(1,t)^n} \doteq \frac{T^{(2)}(W'_*)}{\max(1,t)^n} \doteq T^{(2)}(W_*),$$
where the first equality follows from Proposition 3.4 and Fact 1, the second one follows from Proposition 3.2, and the third one follows from Proposition 3.1, Fact 2 and Proposition 2.2 (5) and (8).

Note that if any one of the four E_*, E'_*, W'_*, W_* is not weakly acyclic (resp. weakly acyclic but not determinant class), then they all are, and in this case all four previous L^2-torsions are zero by convention (thus the previous equalities still stand). □

Remark 5.4. In the previous proof, the exact sequence in Fact 2 is reversed from the one in the proof of [BAC, Theorem 4.9], and the latter is incorrect. Our proof of Theorem 5.1 (2), which generalizes the one of [BAC, Theorem 4.9], can thus be considered as an erratum of this mistake.

Proof. The proof of Theorem 5.1 (2) can be shortened if L is non split, directly by using the simple homotopy equivalence between M_L and W_P, like in the proof of [BAC, Theorem 4.9]. □

6. Counter-examples to Markov invariance

Theorem 5.1 established that $F_{\mathcal{Q}}$ is a Markov function when the Markov-admissible family of epimorphisms $\mathcal{Q}$ descends to the link groups or lower. It is now natural to ask if those families $\mathcal{Q}$ are the only ones for which $F_{\mathcal{Q}}$ is a Markov function. Unfortunately, we do not have a clear answer to this question at the time of writing.

However, by looking at the details of the proof of Theorem 5.1 (more precisely those of Lemmas 5.2 and 5.3), it appears that applying the epimorphism γ_β (or a strictly deeper epimorphism) is necessary in order to obtain the cancellations in the matrices that yield Markov invariance for $F_{\mathcal{Q}}$. As additional evidence for this hypothesis, we discovered two families $\mathcal{Q}$ such that $F_{\mathcal{Q}}$ is not a Markov function: one family (the identities of the free group) lives strictly higher than the γ_β, and the other one (the abelianizations of the free groups) is not comparable to the γ_β (Figure 4 can help visualizing this). See the following Theorems 6.1 and 6.4.

These two counter-examples illustrate some of the difficulties in computing general Fuglede-Kadison determinants, and some transversal techniques one might need to use in order to do so (techniques such as computation of Mahler measures of polynomials, or combinatorics of closed paths on Cayley graphs).

Of course, appeareances might still deceive, and it could happen that unexpected identities of Fuglede-Kadison determinants occur even with families of epimorphisms not encompassed by Theorem 5.1, especially since computing such determinants remains a daunting task today.

This section is split into two parts, which cover Theorems 6.1 and 6.4 respectively.

6.1. Descending to the free abelian group.

In the following theorem, we prove that when the family of epimorphisms $\mathcal{Q}$ descends to the free abelian groups, $F_{\mathcal{Q}}$ is not a Markov function and thus cannot yield link invariants. The proof uses properties of the Fuglede-Kadison determinant and a value of the Mahler measure due to Boyd [Bo].

Theorem 6.1. *For the family of abelianizations $\mathcal{Q} = \left\{\varphi_{n(\beta)} \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \mathbb{Z}^{n(\beta)}\right\}$, the value at $t=1$ of the function $F_{\mathcal{Q}}$ is not invariant under Markov moves. In particular $F_{\mathcal{Q}}$ is not a Markov function.*

Proof. Let $t>0,\ n=2,\ \beta = \sigma_1^{-1} \in B_2$, and $\beta_+ = \sigma_1^{-1}\sigma_2 \in B_3$. Following Section 2.1, we compute that β_+ acts on $\mathbb{F}_3$ by

$$\begin{cases} g_1 \\ g_2 \end{cases} \overset{h_{\beta_+}}{\mapsto} \begin{cases} g_1^{-1}g_2 \\ g_3g_2^{-1}g_1^{-1}g_2 \end{cases},$$

where g_1, g_2, g_3 denote the generators as in Section 2.1. Thus Fox calculus gives:

$$\overline{\mathcal{B}}^{(2)}_{t,id}(\beta_+) - \mathrm{Id}^{\oplus 2} = \begin{pmatrix} -\frac{1}{t}R_{g_1^{-1}} - \mathrm{Id} & -R_{g_3g_2^{-1}g_1^{-1}} \\ \frac{1}{t}R_{g_1^{-1}} & -\mathrm{Id} - tR_{g_3g_2^{-1}} + R_{g_3g_2^{-1}g_1^{-1}} \end{pmatrix}.$$

Hence, from Remark 2.3 and Proposition 2.2 (6), we have:

$$\begin{aligned}
&\det{}_{\mathbb{F}_3}\left(\overline{\mathcal{B}}^{(2)}_{t,id}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) \\
&= \det{}_{\mathbb{F}_3}\left(\left(-\mathrm{Id} - tR_{g_3g_2^{-1}} + R_{g_3g_2^{-1}g_1^{-1}}\right)\left(-R_{g_1g_2g_3^{-1}}\right)\left(-\frac{1}{t}R_{g_1^{-1}} - \mathrm{Id}\right) - \frac{1}{t}R_{g_1^{-1}}\right) \\
&= \det{}_{\mathbb{F}_3}\left(-\left(tR_{g_1} + R_{g_1g_2g_3^{-1}} + \frac{1}{t}R_{g_2g_3^{-1}}\right)\right) \\
&= \frac{1}{t}\det{}_{\mathbb{F}_3}\left(\mathrm{Id} + tR_{g_1} + t^2 R_{g_1g_3g_2^{-1}}\right).
\end{aligned}$$

Let z_1, z_2, z_3 denote the canonical generators of $\mathbb{Z}^3$. Then it follows from the same arguments as in the previous computation that:

$$\det{}_{\mathbb{F}_3}\left(\overline{\mathcal{B}}^{(2)}_{t,\varphi_3}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) = \frac{1}{t}\det{}_{\mathbb{Z}^3}\left(\mathrm{Id} + tR_{z_1} + t^2 R_{z_1 z_3}\right).$$

Let H be the subgroup of $\mathbb{Z}^3$ isomorphic to $\mathbb{Z}^2$ and generated by $\lambda = z_1$ and $\mu = z_1 z_3$. Recall that $F_{\mathcal{Q}}(\beta_+)$ is an equivalence class of functions of $t > 0$ up to multiplication by a monomial, thus the value at $t = 1$ is always the same regardless of the representant of the equivalence class. Let us denote this value $F_{\mathcal{Q}}(\beta_+)(1)$. We then have:

$$\begin{aligned} F_{\mathcal{Q}}(\beta_+)(1) &= \det{}_{\mathbb{Z}^3}\left(\mathrm{Id} + R_{z_1} + R_{z_1 z_3}\right) \\ &= \det{}_H\left(\mathrm{Id} + R_\lambda + R_\mu\right) \\ &= \mathcal{M}(1 + X + Y) = 1.38135... \neq 1. \end{aligned}$$

where $\mathcal{M}$ is the Mahler measure of a polynomial, the second equality follows from Proposition 2.2 (3), the third one from Proposition 2.2 (7) and the fourth one from [Bo, Section 4].

Now, since by Proposition 2.2 (5) we have:

$$F_{\mathcal{Q}}(\beta)(1) = \det{}_{\mathbb{Z}}\left(-R_{\varphi_2(g_1^{-1} g_2)} - \mathrm{Id}\right) = 1,$$

we conclude that $\beta \mapsto F_{\mathcal{Q}}(\beta)(1)$ is not invariant under Markov moves. □

6.2. **Remaining at the free group.** The following lemma gives a method to compute Fuglede-Kadison determinants using the generating series associated to the number of closed paths on a regular graph (see [Ba, DL] and [Lu, Section 3.7]).

Lemma 6.2. *Let G be a finitely presented group and let $A \in R_{\mathbb{C}G}$ denote an injective right multiplication operator on $\ell^2(G)$ by a non-zero element of the group algebra. Then for any $\lambda \in \left(0, \|A\|^{-2}\right)$, we have:*

$$\det{}_G(A) = \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{\lambda}} \exp\left(-\frac{1}{2}\int_0^1 \frac{w_{\lambda,\varepsilon}(t) - 1}{t} dt\right),$$

where

$$w_{\lambda,\varepsilon}(t) = \sum_{n=0}^{\infty} \mathrm{tr}_G\left(\left((1 - \lambda\varepsilon)\mathrm{Id} - \lambda A^* A\right)^n\right) t^n$$

is a well-defined power series for ε small enough and $t < 1$.

Proof. Let $\lambda \in \left(0, \|A\|^{-2}\right)$. Since A is injective, it follows from Proposition 2.2 (8) that

$$\det{}_G(A) = \lim_{\varepsilon \to 0^+} \sqrt{\det{}_G(A^* A + \varepsilon \mathrm{Id})}.$$

Let $\varepsilon > 0$ such that $\lambda < \dfrac{1}{\|A\|^2+\varepsilon} < \dfrac{1}{\|A\|^2}$. Since $A^*A+\varepsilon\mathrm{Id}$ is a positive operator, we have:

$$\begin{aligned}\det{}_G(A^*A+\varepsilon\mathrm{Id}) &= (\exp\circ\mathrm{tr}_G\circ\ln)\left(A^*A+\varepsilon\mathrm{Id}\right)\\ &= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\circ\ln\right)\left(\lambda A^*A+\lambda\varepsilon\mathrm{Id}\right)\\ &= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\circ\ln\right)\left(\mathrm{Id}-\left(\mathrm{Id}-(\lambda A^*A+\lambda\varepsilon\mathrm{Id})\right)\right)\\ &= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\right)\left(-\sum_{n=1}^{\infty}\frac{1}{n}(\mathrm{Id}-(\lambda A^*A+\lambda\varepsilon\mathrm{Id}))^n\right)\\ &= \frac{1}{\lambda}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)\right),\end{aligned}$$

where the first equality follows from Proposition 2.2 (4), the fourth one from holomorphic functional calculus and the fact that the spectrum of the positive operator $\lambda A^*A+\lambda\varepsilon\mathrm{Id}$ is inside $(0,1)$ (since $\lambda < \dfrac{1}{\|A\|^2+\varepsilon}$).

Now, since the series $\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)$ converges, we therefore have that

$$w_{\lambda,\varepsilon}(t) := \sum_{n=0}^{\infty}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)t^n$$

is a well-defined power series for $t<1$, and moreover that for all $T<1$,

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)T^n = \int_0^T\frac{w_{\lambda,\varepsilon}(t)-1}{t}dt.$$

Finally, once again since $\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)$ converges, we can apply Abel's theorem and make $T\to 1^-$ in the previous equality. Hence:

$$\begin{aligned}\det{}_G(A) &= \lim_{\varepsilon\to 0^+}\sqrt{\det{}_G(A^*A+\varepsilon\mathrm{Id})}\\ &= \lim_{\varepsilon\to 0^+}\sqrt{\frac{1}{\lambda}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id}-\lambda A^*A)^n\right)\right)}\\ &= \lim_{\varepsilon\to 0^+}\sqrt{\frac{1}{\lambda}\exp\left(-\int_0^1\frac{w_{\lambda,\varepsilon}(t)-1}{t}dt\right)},\end{aligned}$$

and the lemma follows. □

In the following theorem, we compute Fuglede-Kadison determinants of basic operators on the free groups, using the explicit value (given in [Ba, DL]) of the generating series counting paths on a regular infinite tree.

Theorem 6.3. *Let $d \geqslant 3$, and let $x_1,\ldots,x_{d-1}$ be $d-1$ generators of the free group $\mathbb{F}_{d-1}$. Then we have:*

$$\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id}+R_{x_1}+\ldots+R_{x_{d-1}}) = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}.$$

In particular, for any two generators x,y of the free group $\mathbb{F}_2$, we have:

$$\det{}_{\mathbb{F}_2}(\mathrm{Id}+R_x+R_y) = \frac{2}{\sqrt{3}} = 1.15...$$

The following proof is somewhat technical but only uses classical calculus arguments.

Proof. Let $d \geqslant 3$. First we use Lemma 6.2, by taking $G = \mathbb{F}_{d-1}$, $A = \mathrm{Id} + R_{x_1} + \ldots + R_{x_{d-1}}$, $\lambda \in (0, \frac{1}{d^2})$, and we obtain:

$$\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id}+R_{x_1}+\ldots+R_{x_{d-1}}) = \det{}_G(A) = \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\lambda}} \exp\left(-\frac{1}{2}\int_0^1 \frac{w_{\lambda,\varepsilon}(t)-1}{t}dt\right),$$

where

$$w_{\lambda,\varepsilon}(t) = \sum_{n=0}^{\infty} \mathrm{tr}_G \left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) t^n$$

for $\varepsilon > 0$ small enough and $t \in [0,1)$.

Now we compute, for $\varepsilon > 0$ small enough and $t \in [0,1)$:

$$\begin{aligned}
w_{\lambda,\varepsilon}(t) &= \sum_{n=0}^{\infty} \mathrm{tr}_G \left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) t^n \\
&= \sum_{n=0}^{\infty} \mathrm{tr}_G \left(\sum_{k=0}^{n} \binom{n}{k} (1-\lambda\varepsilon)^{n-k}(-\lambda)^k (A^*A)^k\right) t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} (1-\lambda\varepsilon)^{n-k}(-\lambda)^k \mathrm{tr}_G\left((A^*A)^k\right) t^n \\
&= \sum_{k=0}^{\infty} \left(\frac{-\lambda}{1-\lambda\varepsilon}\right)^k \mathrm{tr}_G\left((A^*A)^k\right) \sum_{n=k}^{\infty} \binom{n}{k} (1-\lambda\varepsilon)^n t^n \\
&= \sum_{k=0}^{\infty} \left(\frac{-\lambda}{1-\lambda\varepsilon}\right)^k \mathrm{tr}_G\left((A^*A)^k\right) \frac{((1-\lambda\varepsilon)t)^k}{(1-(1-\lambda\varepsilon)t)^{k+1}} \\
&= \frac{1}{1-(1-\lambda\varepsilon)t} \sum_{k=0}^{\infty} \left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right)^k \mathrm{tr}_G\left((A^*A)^k\right),
\end{aligned}$$

where the fifth equality follows from the binomial formula. Thus, by denoting $u(t) := \sum_{k=0}^{\infty} \mathrm{tr}_G\left((A^*A)^k\right) t^k$, we have

$$w_{\lambda,\varepsilon}(t) = \frac{1}{1-(1-\lambda\varepsilon)t}\; u\left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right).$$

Since it suffices to prove that

$$\lim_{\varepsilon\to 0^+} \int_0^1 \frac{w_{\lambda,\varepsilon}(t)-1}{t}dt = \ln\left(\frac{d^{d-2}}{(d-1)^{d-1}\lambda}\right),$$

we will study the integral $I_{\lambda,\varepsilon} := \int_0^1 \frac{w_{\lambda,\varepsilon}(t)-1}{t}dt$.

It follows from [DL] and [Ba] that

$$u(t) = \frac{2d-2}{d-2+d\sqrt{1-4(d-1)t}},$$

thus

$$I_{\lambda,\varepsilon} = \int_0^1 \frac{\frac{1}{1-(1-\lambda\varepsilon)t}\; \frac{2d-2}{d-2+d\sqrt{1+\frac{4(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}}} - 1}{t}dt.$$

An antiderivative of the integrand is

$$F(t) := (d-2)\ln\left(1-\left(1-\frac{d^2\lambda}{1-\lambda\varepsilon}\right)(1-\lambda\varepsilon)t\right)+$$
$$\frac{d}{2}\ln\left(\frac{1-(1-(1-\lambda\varepsilon)t)\sqrt{1+\frac{4(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}}+\left(\frac{2(d-1)\lambda}{1-\lambda\varepsilon}-1\right)(1-\lambda\varepsilon)t}{(1-\lambda\varepsilon)^2t^2}\right)+$$
$$\left(1-\frac{d}{2}\right)\ln\Bigg(-d(d-2)(1-(1-\lambda\varepsilon)t)\left(\sqrt{1+\frac{4(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}}-1\right)$$
$$2d^2(d-1)\lambda t+2(1-(1-\lambda\varepsilon)t)\Bigg).$$

It remains to compute $I_{\lambda,\varepsilon}=F(1)-F(0)$. At $t=0$, we first remark that:

$$\sqrt{1+\frac{4(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}}\underset{t\to 0^+}{=}1+\frac{2(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}-\frac{2(d-1)^2\lambda^2t^2}{(1-(1-\lambda\varepsilon)t)^2}+o_{t\to 0^+}(t^2),$$

thus, from an immediate calculation, the fraction inside the ln in the second term of $F(t)$ above behaves like

$$\frac{2\lambda^2(d-1)^2}{1-(1-\lambda\varepsilon)t}+o_{t\to 0^+}(1).$$

Hence

$$F(0)=(d-2)\ln(1)+\frac{d}{2}\ln\left(2\lambda^2(d-1)^2\right)+\left(1-\frac{d}{2}\right)\ln(2)=\ln\left(2\lambda^d(d-1)^d\right).$$

Furthermore, at $t=1$ we compute:

$$F(1)=(d-2)\ln(d^2\lambda+\lambda\varepsilon)+$$
$$\frac{d}{2}\ln\left(\frac{\lambda}{(1-\lambda\varepsilon)^2}\left(2(d-1)+\varepsilon-\sqrt{\varepsilon}\sqrt{\varepsilon+4(d-1)}\right)\right)+$$
$$\left(1-\frac{d}{2}\right)\ln\left(2d^2(d-1)\lambda+2\lambda\varepsilon-d(d-2)\lambda\sqrt{\varepsilon}\left(\sqrt{\varepsilon+4(d-1)}-\sqrt{\varepsilon}\right)\right).$$

Taking $\varepsilon\to 0^+$, we find:

$$\begin{aligned}F(1)\underset{\varepsilon\to 0^+}{\longrightarrow}&(d-2)\ln(d^2\lambda)+\frac{d}{2}\ln\left(2(d-1)\lambda\right)+\left(1-\frac{d}{2}\right)\ln\left(2d^2(d-1)\lambda\right)\\&=\ln\left(2\lambda^{d-1}d^{d-2}(d-1)\right).\end{aligned}$$

Hence we obtain the limit

$$I_{\lambda,\varepsilon}=F(1)-F(0)\underset{\varepsilon\to 0^+}{\longrightarrow}\ln\left(2\lambda^{-1}d^{d-2}(d-1)^{1-d}\right),$$

and the result follows. □

We can now prove that we cannot expect Markov invariance by remaining at the level of the free group, as the following theorem shows:

Theorem 6.4. *For the family of identities $\mathcal{Q}=\{\mathrm{id}_{\mathbb{F}_{n(\beta)}}\}$, the function $F_{\mathcal{Q}}$ is not invariant under Markov moves.*

Proof. Let us take $t = 1$, $n = 2$, $\beta = \sigma_1^{-1} \in B_2$, $\beta_+ = \sigma_1^{-1}\sigma_2 \in B_3$. Then, as in the proof of Theorem 6.1, we obtain

$$\begin{aligned} F_{\mathbb{Q}}(\beta_+)(1) &= \det{}_{\mathbb{F}_3}\left(\mathcal{B}^{(2)}_{1,id}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) \\ &= \det{}_{\mathbb{F}_3}\left(\mathrm{Id} + R_{g_1} + R_{g_1 g_3 g_2^{-1}}\right) \\ &= \det{}_F\left(\mathrm{Id} + R_x + R_y\right), \end{aligned}$$

where F is the free group on two generators x, y that embeds in $\mathbb{F}_3$ via $x \mapsto g_1, y \mapsto g_1 g_3 g_2^{-1}$, and the last equality follows from Proposition 2.2 (3). Now, since

$$F_{\mathbb{Q}}(\beta)(1) = \det{}_{\mathbb{F}_2}\left(-R_{g_1^{-1}g_2} - \mathrm{Id}\right) = 1,$$

it then suffices to prove that $\det{}_F\left(\mathrm{Id} + R_x + R_y\right) \neq 1$. This is the case thanks to Theorem 6.3. ☐

7. New upper bounds for Lehmer's constants

Given a group G, its *Lehmer's constants*, as defined by Lück in [Lu2], are:

- $\Lambda(G) := \inf\left\{\det_G(A) \mid A \in \sqcup_{p,q\in\mathbb{N}} R_{M_{p,q}(\mathbb{Z}G)},\ \det_G(A) > 1\right\}$,
- $\Lambda^w(G) := \inf\left\{\det_G(A) \mid A \in \sqcup_{n\in\mathbb{N}} R_{M_n(\mathbb{Z}G)},\ A \text{ injective},\ \det_G(A) > 1\right\}$,
- $\Lambda_1(G) := \inf\left\{\det_G(A) \mid A \in R_{\mathbb{Z}G},\ \det_G(A) > 1\right\}$,
- $\Lambda_1^w(G) := \inf\left\{\det_G(A) \mid A \in R_{\mathbb{Z}G},\ A \text{ injective},\ \det_G(A) > 1\right\}$.

Observe that $1 \leqslant \Lambda(G) \leqslant \Lambda^w(G) \leqslant \Lambda_1^w(G)$ and $1 \leqslant \Lambda(G) \leqslant \Lambda_1(G) \leqslant \Lambda_1^w(G)$.

Lehmer's constants are inspired by the well-known Lehmer problem, that we re-state as a conjecture as follows:

Conjecture 7.1 (Lehmer's problem, [Lu2] Problem 1.3)**.** *$\Lambda_1^w(\mathbb{Z})$ is equal to the Mahler measure of Lehmer's polynomial $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$:*

$$\Lambda_1^w(\mathbb{Z}) = \mathcal{M}(L) = \mathcal{M}(z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1) = 1.176...$$

As explained in [Lu2], Lehmer's constants are especially interesting to study for torsionfree groups. For this class of groups, Lück proposed the following question:

Question 7.2 ([Lu2], Introduction). For which torsionfree groups G do we have

$$\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = \mathcal{M}(L) = 1.176...\ ?$$

Example 7.3 ([Lu2], Example 13.2). Lück provided a partial negative answer to Question 7.2 by proving that $\Lambda(G) = \Lambda^w(G) \leqslant 1.06 < \mathcal{M}(L)$ for G the fundamental group of the hyperbolic Weeks manifold. It follows from Proposition 2.2 (3) that the same is true for any group G' containing G as a subgroup.

Now, as a consequence of Theorem 6.3, we obtain new upper bounds on Lehmer's constants and a negative answer to Question 7.2 for a large class of torsionfree groups:

Corollary 7.4. *For every $d \geqslant 2$, Lehmer's constants $\Lambda(\mathbb{F}_d), \Lambda_1(\mathbb{F}_d), \Lambda^w(\mathbb{F}_d)$ and $\Lambda_1^w(\mathbb{F}_d)$ do not depend on d.*

Moreover we have

$$\Lambda(\mathbb{F}_d) \leqslant \Lambda^w(\mathbb{F}_d) \leqslant \Lambda_1^w(\mathbb{F}_d) = \Lambda_1(\mathbb{F}_d) \leqslant \frac{2}{\sqrt{3}} = 1.15... < \mathcal{M}(L) = 1.176...$$

for every $d \geqslant 2$.

In particular, any torsionfree group G containing a free subgroup $\mathbb{F}_d$ for some $d \geqslant 2$ (such as the fundamental group of a hyperbolic 3-manifold, see [AFW, C.3, C.26]*) also satisfies*

$$\Lambda(G), \Lambda_1(G), \Lambda^w(G), \Lambda_1^w(G) \in \left[1, \frac{2}{\sqrt{3}}\right].$$

Proof. The first statement follows from Proposition 2.2 (3) and the fact every free group $\mathbb{F}_d$ (for $d \geqslant 2$) injects in $\mathbb{F}_2$ and vice-versa.

In the second statement, the first two inequalities follow immediately from the definitions, and the first equality from the fact that free groups satisfy the Strong Atiyah Conjecture (see Remark 2.3 and [Lu, Theorem 10.19]). The third inequality follows from Theorem 6.3 and the first statement (observe that $d \mapsto \dfrac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}$ is increasing for $d \geqslant 3$, thus $\dfrac{2}{\sqrt{3}}$ is the best upper bound available).

Finally, the third statement follows from the second statement and Proposition 2.2 (3): for any torsionfree group G containing a non-cyclic free subgroup $\mathbb{F}_d$, we have $\lambda(G) \leqslant \lambda(\mathbb{F}_d)$ for any $\lambda \in \{\Lambda, \Lambda_1, \Lambda^w, \Lambda_1^w\}$. □

Remark 7.5. The fundamental group G of the Weeks manifold contains non-cyclic free subgroups (see [AFW, C.3, C.26]), and thus is covered by the third statement of Corollary 7.4. In this sense, Corollary 7.4 can be seen as a generalization of the class of counterexamples to Question 7.2 mentioned in Example 7.3. However, for this same class, the upper bound 1.06... in Example 7.3 is better than the bound $\dfrac{2}{\sqrt{3}}$ of Corollary 7.4, for the two constants $\Lambda(G), \Lambda^w(G)$.

From what precedes, it seems reasonable to expect that the combinatorial and analytical techniques used to prove Theorem 6.3 will help find new advancements on problems such as Question 7.2.

References

[AFW] M. Aschenbrenner, S. Friedl and H. Wilton, 3-*manifold groups*, EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015. xiv+215 pp.

[Ba] L. Bartholdi, *Counting paths in graphs*, Enseign. Math. (2) 45 (1999), no. 1-2, 83–131.

[BA] F. Ben Aribi, *A study of properties and computation techniques of the L^2-Alexander invariant in knot theory*, PhD thesis, Université Paris Diderot (2015).

[BA2] F. Ben Aribi, *Gluing formulas for the L^2-Alexander torsions* Commun. Contemp. Math. 21 (2019), no. 3, 1850013, 31 pp.

[BAC] F. Ben Aribi and A. Conway, *L^2-Burau maps and L^2-Alexander torsions* (2018), Osaka J.Math. 55, 529-545.

[Bo] D. Boyd, *Speculations concerning the range of Mahler's measure*, Canad. Math. Bull. 24 (1981), no. 4, 453—469.

[Bu] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen*, Abh. Math. Sem. Univ. Hamburg 11 (1935), 179–186.

[DL] O. Dasbach and M. Lalin, *Mahler measure under variations of the base group*, Forum Math. 21 (2009), no. 4, 621–637.

[DFL] J. Dubois, S. Friedl and W. Lück, *The L^2-Alexander torsion of 3-manifolds*, Journal of Topology 9, No. 3 (2016), 889–926.

[Fox] R.H. Fox, *Free differential calculus. II. The isomorphism problem of groups*, Ann. of Math. (2), 59 (1954), 196–210.

[LZ] W. Li and W. Zhang, *An L^2-Alexander invariant for knots*, Commun. Contemp. Math. **8** (2006), no. 2, 167–187.

[Lu] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 44. Springer-Verlag, Berlin, 2002.

[Lu2] W. Lück, *Lehmer's Problem for arbitrary groups*, to appear in Journal of Topology and Analysis, arXiv:1901.00827.

UCLouvain, IRMP, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
Email address: fathi.benaribi@uclouvain.be