SNAIL LEMMA IN A POINTED REGULAR CATEGORY

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Abstract. In this paper we explore the snail lemma in a pointed regular category. In particular, we show that under the presence of cokernels of kernels, the validity of the snail lemma is equivalent to subtractivity of the category. As a corollary, this gives that in the more restrictive context of a normal category the validity of the snail lemma is equivalent to the validity of the snake lemma.

1. Introduction

In studying the Hochschild-Serre exact sequence associated with a surjective homomorphism $p\colon E \to G$ of groups, and a G-module M, the second author noticed that such a sequence can be obtained by embedding p and M in the 2-category of categorical groups and then using a general form of the Brown exact sequence associated with a fibration of pointed groupoids (see [8, 9, 10]). The advantage of this unusual technique is that it becomes apparent that the Hochschild-Serre exact sequence exists even if p is not surjective, and that the surjectivity of p is only needed to identify the middle point of the sequence with the group of equivariant homomorphisms from the kernel of p to M.

This suggests to look for the *snail lemma*, a new diagram lemma similar to the classical snake lemma and having as starting point any commutative square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

where α and β satisfy suitable conditions but where, in contrast with the snake lemma, we do not require that f is surjective (recall that the fact that f is surjective enters in the snake lemma in order to construct the homomorphism connecting the "kernel part" and the "cokernel part" of the sequence). In [19] the second author proved that the snail lemma holds true in any pointed regular category [4] which is at the same time a protomodular category [6], and in this context naturally subsumes the snake lemma (the snake lemma was established in this context by D. Bourn in [7]). In this paper we show that

- already in a normal category [13], i.e. in a pointed regular category where every regular epimorphism is a normal epimorphism (that is, a cokernel), which is less

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than being protomodular, the validity of the snake lemma can be deduced from that of the snail lemma,

- and moreover, in a normal category the snail lemma is equivalent to the snake lemma, as well as to the 3×3 lemma;

- we deduce these two facts from a more general technical result that in a pointed regular category having cokernels of kernels, the snail lemma is equivalent to subtractivity of the category in the sense of [12].

Note that subtractivity is a much weaker property than protomodularity. It arises naturally in the theory of ideals and related topics in universal algebra — see [11, 17, 1, 2, 3, 18]. At the same time, subtractivity can be seen as an implicit ingredient for diagram chasing arguments, appearing already in [16, 5] (see [14]).

2. Preliminaries

In this paper we will use the technique of "chasing subobjects", which goes back to Mac Lane [15, 16], and which, in the context of a pointed regular category was made explicit in [13]. Let $\mathbb{C}$ be such a category. Then there is a *pointed subobject functor* $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ from $\mathbb{C}$ to the category $\mathbf{Set}_*$ of pointed sets. It sends an object X of $\mathbb{C}$ to the pointed set of subobjects of X, with the base point given by the "null subobject", i.e. the subobject given by the null morphism $0 \to X$ from the null object 0 to X. This functor preserves and reflects null objects, null morphisms, and regular epimorphisms (which in $\mathbf{Set}_*$ are the same as surjective morphisms between pointed sets), and preserves monomorphisms and weak pullbacks. Moreover, the pointed subobject functor also preserves and reflects exactness of a sequence, when exactness is defined in the following "non-symmetric" way: call a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of morphisms in a pointed regular category *exact* at Y when, in any (and hence every) decomposition $f = m \cdot e$ of f into a regular epimorphism e followed by a monomorphism m, the monomorphism m is a kernel of g. The category $\mathbf{Set}_*$ is itself a pointed regular category, and in this category the above condition simply states that an element of Y is mapped by g to the base point of Z, if and only if some element from X is mapped to it by f. These properties of the pointed subobject functor allow to reduce proving diagram lemmas in an abstract pointed regular category to proving them in $\mathbf{Set}_*$ (see [13, 14]). In fact, even the additional conditions on $\mathbb{C}$ which are sufficient (and which, in many cases also turn out to be necessary) to prove some of the standard diagram lemmas can also be conveniently captured by the pointed subobject functor. For example, the condition of subtractivity [12] of $\mathbb{C}$, which (as shown in [13]) is necessary and sufficient for proving the "lower 3×3 lemma", is equivalent to the following property of the pointed subobject functor: for any span

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X & & Y \end{tikzcd} \tag{1}$$

in $\mathbb{C}$, the corresponding span

$$\begin{tikzcd} & \mathcal{S}(S) \arrow[dl, "\mathcal{S}(s_1)"'] \arrow[dr, "\mathcal{S}(s_2)"] & \\ \mathcal{S}(X) & & \mathcal{S}(Y) \end{tikzcd}$$

in $\mathbf{Set}_*$ is such that for all $a, b \in \mathcal{S}(S)$, if $\mathcal{S}(s_1)(a) = \mathcal{S}(s_1)(b)$ and $\mathcal{S}(s_2)(b) = 0$, then there exists $c \in \mathcal{S}(S)$ such that $\mathcal{S}(s_1)(c) = 0$ and $\mathcal{S}(s_2)(c) = \mathcal{S}(s_2)(a)$. We could think of c as the result of subtraction $c = a - b$. Note however that we are not assuming either that this subtraction is uniquely determined or that it is canonically defined in any way. Such spans in $\mathbf{Set}_*$ are called *subtractive spans* of pointed sets, and a span in $\mathbb{C}$ whose image under the pointed subobject functor is a subtractive span is called a *subtractive span* in $\mathbb{C}$. This notion also has an equivalent internal definition in $\mathbb{C}$ (see [13]), which clarifies that a subtractive span in $\mathbb{C} = \mathbf{Set}_*$ is the same as a subtractive span of pointed sets. Repeating what we said above in this terminology, we get the following characterisation of subtractivity of a pointed regular category, which we adopt in this paper as its definition:

2.1. Theorem. [13] *A pointed regular category is subtractive if and only if any span in it is subtractive.*

As an illustration of the property of subtractivity of a span, let us show that the pointed subobject functor $\mathcal{S} : \mathbf{Grp} \to \mathbf{Set}_*$ carries every span (1) of groups to a subtractive span of pointed sets. We will use additive notation for group operations. Let A and B be subgroups of S such that $s_1(A) = s_1(B)$ and $s_2(B) = 0$. Then take

$$C = \mathsf{Ker}(s_1) \cap s_2^{-1}(s_2(A)).$$

Clearly $s_1(C) = 0$ and $s_2(C) \subseteq s_2(s_2^{-1}(s_2(A))) = s_2(A)$. To show that $s_2(A) \subseteq s_2(C)$ pick any $s_2(a) \in s_2(A)$ where $a \in A$. Then $s_1(a) = s_1(b)$ for some $b \in B$. We have $s_1(a - b) = s_1(a) - s_1(b) = 0$, and so $a - b \in \mathsf{Ker}(s_1)$. At the same time, $s_2(a) = s_2(a) - 0 = s_2(a) - s_2(b) = s_2(a - b)$, which shows that $a - b \in s_2^{-1}(s_2(A))$. Thus, $a - b \in C$, and $s_2(a) = s_2(a - b)$ gives $s_2(a) \in s_2(C)$.

Notice that in the argument above we only used the identities $x - x = 0$ and $x - 0 = x$ in a group. This is not surprising, since a pointed variety of universal algebras is a subtractive category if and only if its algebraic theory contains a binary operation satisfying these identities (see [12]). Such varieties (also in the non-pointed case) were first considered in [11], and later in [17] they were called *subtractive varieties*. The notion of a subtractive category introduced in [12] was obtained by extending the notion of a subtractive variety to categories. Subtractivity also has roots in categorical algebra, as explained in [14] (see the same paper for further examples of subtractive categories). Subtractivity was rediscovered by the second author with Lemma 4.1 in [19], whose formulation is very similar to Theorem 2.2 below.

In this paper we will also rely on the following characterisation of subtractivity:

2.2. THEOREM. *Let $\mathbb{C}$ be a pointed finitely complete category in which any kernel has a cokernel. Then $\mathbb{C}$ is a subtractive category if and only if for any internal relation*

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}$$

in it, if $r_1 \cdot \mathsf{ker} r_2$ is an isomorphism and r_2 is a normal epimorphism, then $r_2 \cdot \mathsf{ker} r_1$ is also an isomorphism.

Here it is necessary to make the following remarks:

- The same theorem with "any morphism has a cokernel" in the place of "any kernel has a cokernel" is proved in [12]. However, the proof given there only relies on the existence of cokernels of kernels, and so the same proof would be valid in this case. Hence we omit the proof in this paper.

- In this theorem, as usual in categorical algebra, by an "internal relation" is meant a span where the two arrows of the span are jointly monomorphic. This in particular implies that the composites $r_1 \cdot \mathsf{ker} r_2$ and $r_2 \cdot \mathsf{ker} r_1$ are monomorphisms. So, we could replace both occurrences of "isomorphism" in the theorem with "regular epimorphism".

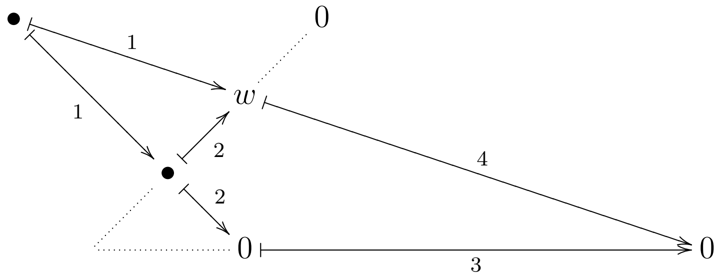
3. The incomplete snail lemma

Let us now illustrate the technique of chasing subobjects on the following diagram lemma, which we will call the "incomplete snail lemma", and which is a predecessor to the "complete snail lemma" (or simply, "snail lemma") — the main subject of this paper dealt in the next section.

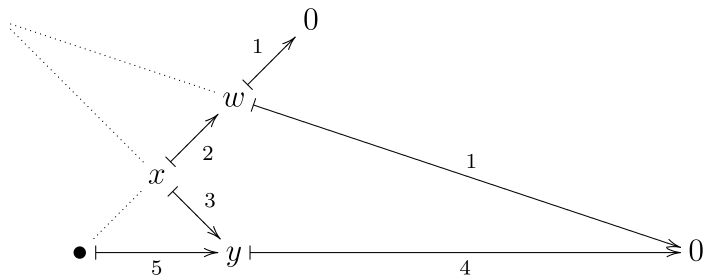
In a pointed regular category $\mathbb{C}$, the *incomplete snail lemma* states that given any commutative diagram

$$\begin{tikzcd}[column sep=small] W_1 \arrow[rr] \arrow[dr] & & W_2 \arrow[r] \arrow[ddrrrr, two heads] & 0 & & & \\ & X \arrow[ur] \arrow[dr] & & & & & \\ Y_1 \arrow[ur] \arrow[rr] & & Y_2 \arrow[rrrr, two heads] & & & & Z \end{tikzcd}$$

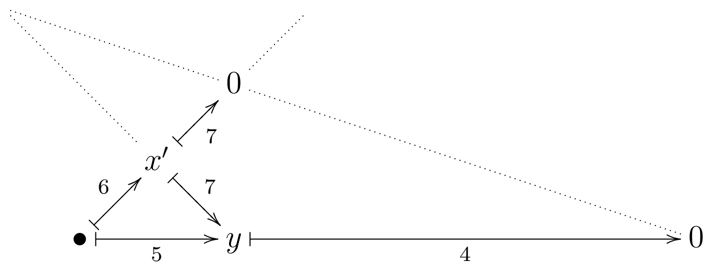
if all sequences (of arrows lying on a straight line) apart from the sequence $W_1 \to W_2 \to Z$ are exact, then this sequence is also exact. Here, as usual, 0 stands for a null object, and exactness of a sequence means exactness at each object inside the sequence. We will now show that the incomplete snail lemma holds as soon as $\mathbb{C}$ is a subtractive category. By the virtue of the preservation and reflection properties of the pointed subobject functor, it is sufficient to show that the incomplete snail lemma holds for pointed sets, in those cases when all spans in the incomplete snail diagram are subtractive. In fact, we will only use subtractivity of the span $Y_2 \leftarrow X \to W_2$. First, we show that in the sequence $W_1 \to W_2 \to Z$, if an element of W_2 comes from an element of W_1, then it is mapped to the 0 of Z (i.e. the base point of Z). This can be obtained by chasing elements from W_1 to Z via Y_2, as summarised in the following display, where numbers indicate steps in the chase:



It remains to show that if an element of W_2 is mapped to 0 in Z, then it comes from an element of W_1. For this, we first perform the following diagram chase:



Then, removing the elements x of X and w of W_2 from the display, but keeping the others, we can continue with the diagram chase as follows:



Next, we use subtractivity of the span $Y_2 \leftarrow X \rightarrow W_2$; subtraction $x - x'$ yields:

$$\begin{tikzcd}
\bullet \arrow[rr, mapsto, "10"] \arrow[dr, mapsto, "9"'] & & w \\
 & x - x' \arrow[ur, mapsto, "8"'] \arrow[dr, mapsto, "8"] & \\
 & & 0
\end{tikzcd}$$

This completes the proof.

We have proved that in a pointed regular category subtractivity implies validity of the incomplete snail lemma. We now show that the converse of this is also true: subtractivity can be deduced from the snail lemma. To show this, we will rely on a result obtained in [14] that a pointed regular category $\mathbb{C}$ is subtractive if and only if the following diagram lemma holds in it, which is called the "cross lemma" in [14], but we will refer to it as the *spider lemma* in this paper: if all sequences in

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[dr] & & W_2 \arrow[r] \arrow[ur] & 0 \\
 & & X \arrow[ur] \arrow[dr] & & \\
0 \arrow[r] & Y_1 \arrow[r] \arrow[ur] & & Y_2 \arrow[r] \arrow[dr] & 0 \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

except the top horizontal sequence are exact, then the top horizontal sequence is also exact. Thus, it is sufficient to deduce the spider from the snail. In the spider, exactness at W_1 can be established in any pointed regular category, and exactness at W_2 can be deduced from the snail diagram

$$\begin{tikzcd}
W_1 \arrow[drrr] \arrow[ddrr] & & & & 0 & \\
 & & & W_2 \arrow[ur] \arrow[ddrr] & & \\
 & & X \arrow[ur] \arrow[dr] & & & \\
 & Y_1 \arrow[ur] \arrow[rr] & & Y_2 \arrow[rr] & & 0
\end{tikzcd}$$

obtained from the spider diagram.

This brings us to the first result of this paper:

3.1. Theorem. *A pointed regular category is subtractive if and only if the incomplete snail lemma holds in it.*

4. The complete snail lemma

In a pointed regular category $\mathbb{C}$, the *complete snail lemma* states that given a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\gamma=\langle\alpha,f\rangle"] \arrow[dd, "\alpha"'] & & B \arrow[dd, "\beta"] \\
& A_0 \times_{f_0,\beta} B \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}$$

where the regular images of α, β, and γ are kernels, the morphisms α, β, and γ have cokernels, and moreover, in the following extended commutative diagram, determined by the one above,

$$\begin{tikzcd}
\mathsf{Ker}\gamma \arrow[r, "i"] \arrow[dr, "\mathsf{ker}\gamma"'] & \mathsf{Ker}\alpha \arrow[rrr, "f'"] \arrow[d, "\mathsf{ker}\alpha"] & & & \mathsf{Ker}\pi_1 \approx \mathsf{Ker}\beta \arrow[d, "\mathsf{ker}\beta"] \arrow[ddll, "\mathsf{ker}\pi_1"'] \arrow[dddl, "\mathsf{coker}\gamma\cdot\mathsf{ker}\pi_1"] \\
& A \arrow[rrr, "f"] \arrow[dr, "\gamma"] \arrow[ddd, "\alpha"'] & & & B \arrow[ddd, "\beta"] \\
& & A_0 \times_{f_0,\beta} B \arrow[urr, "\pi_2"] \arrow[dr, "\mathsf{coker}\gamma"] \arrow[ddl, "\pi_1"'] & & \\
& & & \mathsf{Coker}\gamma \arrow[ddll, "\pi_1'"] & \\
& A_0 \arrow[rrr, "f_0"'] \arrow[d, "\mathsf{coker}\alpha"'] & & & B_0 \arrow[d, "\mathsf{coker}\beta"] \\
& \mathsf{Coker}\alpha \arrow[rrr, "f_0'"'] & & & \mathsf{Coker}\beta
\end{tikzcd}$$

the sequence

$$\mathsf{Ker}\gamma \to \mathsf{Ker}\alpha \to \mathsf{Ker}\beta \to \mathsf{Coker}\gamma \to \mathsf{Coker}\alpha \to \mathsf{Coker}\beta$$

is exact.

4.1. Theorem. *Let $\mathbb{C}$ be a pointed regular category. The complete snail lemma holds in $\mathbb{C}$ if and only if $\mathbb{C}$ is a subtractive category in which any kernel has a cokernel.*

Proof. Suppose the complete snail lemma holds in $\mathbb{C}$. Then, every kernel in $\mathbb{C}$ has a cokernel. Indeed, given a kernel $k : K \to X$, apply the complete snail lemma to the diagram

$$\begin{tikzcd}
K \arrow[rr, "1_K"] \arrow[dr, "1_K"] \arrow[dd, "k"'] & & K \arrow[dd, "k"] \\
& K \arrow[ur, "1_K"'] \arrow[dl, "k"] & \\
X \arrow[rr, "1_X"'] & & X
\end{tikzcd}$$

to conclude that k has a cokernel. After this, to show that $\mathbb{C}$ is subtractive we use Theorem 2.2 and apply the complete snail lemma to the following diagram:

$$\begin{tikzcd}
\mathsf{Ker}r_2 \arrow[rr, "\mathsf{ker}r_2"] \arrow[dr, "\mathsf{ker}r_2"] \arrow[dd, "r_1\cdot\mathsf{ker}r_2"'] & & R \arrow[dd, "r_1"] \\
& R \arrow[ur, "1_R"'] \arrow[dl, "r_1"] & \\
X \arrow[rr, "1_X"'] & & X
\end{tikzcd}$$

Since as given in Theorem 2.2, $r_2 : R \to Y$ is a normal epimorphism, it is the cokernel of its kernel, and so the resulting exact sequence is

$$0 \longrightarrow 0 \longrightarrow \mathsf{Ker}r_1 \xrightarrow{r_2\cdot\mathsf{ker}r_1} Y \longrightarrow 0 \longrightarrow 0$$

which shows that $r_2 \cdot \mathsf{ker}r_1$ is a regular epimorphism and hence, being a monomorphism, it is an isomorphism.

To show the converse, first we note that the existence of cokernels of kernels implies that all desired cokernels in the formulation of the snail lemma will exist. Next, we can use the technique of chasing subobjects to show that we have exactness everywhere except $\mathsf{Coker}\gamma$. For this, subtractivity is not needed. We do not give explicit chasing arguments since they can be trivially recovered, especially after the following hints:

- To show exactness at $\mathsf{Ker}\alpha$, we will need to use the fact that since the morphisms $\mathsf{ker}\alpha$ and $\mathsf{ker}\beta$ are monomorphisms, their images under the pointed subobject functor will be injective.

- Exactness at $\mathsf{Ker}\beta$ uses again the fact that $\mathsf{ker}\beta$ is a monomorphism.

- Exactness at $\mathsf{Coker}\alpha$ uses the fact that the cokernels $\mathsf{coker}\alpha$ and $\mathsf{coker}\gamma$ are regular epimorphisms, and hence by the pointed subobject functor they are mapped to surjections. It also relies on the fact that the pullback in the diagram will be mapped by the pointed subobject functor $\mathcal{S}$ to a weak pullback, since every pullback is a weak pullback and the pointed subobject functor preserves weak pullbacks. The effect of this in diagram chasing is that given elements $a \in \mathcal{S}(A_0)$ and $b \in \mathcal{S}(B)$ which are mapped to the same element in $\mathcal{S}(B_0)$ (by $\mathcal{S}(f_0)$ and $\mathcal{S}(\beta)$, respectively), there will exist an element $c \in \mathcal{S}(A_0 \times_{f_0,\beta} B)$ (although not unique) which will map to a and b (by $\mathcal{S}(\pi_1)$ and $\mathcal{S}(\pi_2)$, respectively).

We only need subtractivity to show exactness at $\mathsf{Coker}\gamma$, which can be also deduced from the incomplete snail lemma applied to the following diagram:

$$\begin{tikzcd}
\mathsf{Ker}\pi_1 \approx \mathsf{Ker}\beta \arrow[ddr, "\mathsf{ker}\pi_1"'] \arrow[drr, "\mathsf{coker}\gamma\cdot\mathsf{ker}\pi_1"] & & & 0 & & \\
& & \mathsf{Coker}\gamma \arrow[ur] \arrow[ddrrr, "\pi_1'"] & & & \\
& A_0 \times_{f_0,\beta} B \arrow[ur, "\mathsf{coker}\gamma"'] \arrow[dr, "\pi_1"] & & & & \\
A \arrow[ur, "\gamma"] \arrow[rr, "\alpha"'] & & A_0 \arrow[rrr, "\mathsf{coker}\alpha"'] & & & \mathsf{Coker}\alpha
\end{tikzcd}$$

■

Together with the equivalence of the incomplete snail and subtractivity established in the previous section, we can conclude:

4.2. Corollary. *Let $\mathbb{C}$ be a pointed regular category. The complete snail lemma holds in $\mathbb{C}$ if and only if the incomplete snail lemma holds in $\mathbb{C}$, and any kernel in $\mathbb{C}$ has a cokernel.*

Further interesting corollaries can be stated as immediate consequences of the theorem above and the results obtained in [13, 14]. In particular, we point out the following, which shows that many of the diagram lemmas are equivalent to each other in any "normal category" [13]. Recall that a *normal category* is a pointed regular category in which any regular epimorphism is a normal epimorphism.

4.3. Corollary. *For a normal category $\mathbb{C}$, the following conditions are equivalent to $\mathbb{C}$ being a subtractive category:*

(a) The snake lemma holds.

(b) The complete snail lemma holds.

(c) The incomplete snail lemma holds.

(d) *The* 3×3 *lemma holds.*

(e) *The spider lemma holds.*

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