On actions and split extensions in varieties of hoops

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New Trends in Many-Valued Logic and Categorical Algebra
Palermo, 18/03/2026

Joint work with Giuseppe Metere, Federica Piazza and Marco Elio Tabacchi

Actions / Split extensions of groups

- Let B and X be groups. An *external action* of B on X is a map

$$\xi\colon B \times X \to X\colon (b, x) \mapsto b * x$$

such that

$$(bb') * x = b * (b' * x), \quad 1_B * x = x, \quad b * (xx') = (b * x)(b * x').$$

- The *semi-direct product* $X \rtimes_\xi B$ is $X \times B$ with multiplication

$$(x, b) \cdot (x', b') = (x(b * x'), bb'),$$

where $1_{X \rtimes B} = (1_X, 1_B)$ and $(x, b)^{-1} = (b^{-1} \cdot x^{-1}, b^{-1})$.

- We can associate with ξ the split extension

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & X \rtimes_\xi B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

- Conversely, given

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B,$$

  we can define the action

$$B \times X \to X : (b, x) \mapsto s(b)k(x)s(b)^{-1}.$$

- There is a bijection $\mathrm{EAct}(B, X) \cong \mathrm{SplExt}(B, X)$.
- This bijection extends to a natural isomorphism of functors

$$\mathrm{EAct}(-, X) \cong \mathrm{SplExt}(-, X).$$

- This construction can be generalised to any *category of groups with operations*, such as rings, associative algebras, Lie algebras, Leibniz algebras, Jordan algebras, ...
- Let B and X be rings. An *external action* of B on X consists of a pair of maps

$$f\colon B\times X\to X\colon (b,x)\mapsto b*x,$$
$$g\colon X\times B\to X\colon (x,b)\mapsto x*b$$

such that

$$b*(xx')=(b*x)x',\quad (x*b)x'=x(b*x'),\quad (xx')*b=x(x'*b),$$

$$(bb')*x=b*(b'*x),\quad (b*x)*b'=b*(x*b'),\quad x*(bb')=(x*b)*b'.$$

- **Can we do the same for other algebraic structures?**

Hoops

Definition

A *hoop* is an algebra $H = (H, \cdot, \to, 1)$ of type $(2, 2, 0)$ such that

H1. $(H, \cdot, 1)$ is a commutative monoid;

H2. $x \to x = 1$;

H3. $x \cdot (x \to y) = y \cdot (y \to x)$;

H4. $(x \cdot y) \to z = x \to (y \to z)$,

for any $x, y, z \in H$.

A hoop H is *bounded* if there is a constant $0 \in H$ such that

H5. $0 \to x = 1$,

for any $x \in H$.

- $1 \to x = x$ and $x \to 1 = 1$.

Definition

Let H be a hoop. H is a

- *Basic hoop* if it satisfies the identity

$$((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1,$$

  for any $x, y, z \in H$.

- *Wajsberg hoop* if it satisfies the identity

$$(x \to y) \to y = (y \to x) \to x,$$

  for any $x, y \in H$.

- *Gödel hoop* if it is a basic hoop satisfying $x \cdot x = x$, for any $x \in X$.
- *Product hoop* if it is a basic hoop satisfying

$$(y \to z) \vee ((y \to (x \cdot y)) \to x) = 1,$$

  for any $x, y, z \in H$.

BCIRLs

Definition

A *commutative integral residuated lattice* (or CIRL) is an algebra $A = (A, \vee, \wedge, \cdot, \to, 1)$ of type $(2, 2, 2, 2, 0)$ such that

(1) $(A, \cdot, 1)$ is a commutative monoid.

(2) $(A, \vee, \wedge, 1)$ is a lattice with top element 1.

(3) The following *residuation property* holds in A:

$$x \cdot y \leq z \quad \text{if and only if} \quad x \leq y \to z,$$

(4) A *bounded* CIRL (or BCIRL) is a CIRL with an extra constant 0 that is the bottom element in the lattice order.

BL-algebras

Definition

A *BL-algebra* is an algebra $A = (A, \vee, \wedge, \cdot, \to, 0, 1)$ such that

(1) $(A, \vee, \wedge, \cdot, \to, 0, 1)$ is a BCIRL.

(2) $x \wedge y = x \cdot (x \to y)$, for any $x, y \in A$. (*divisibility condition*)

(3) $(x \to y) \vee (y \to x) = 1$, for any $x, y \in A$. (*prelinearity condition*)

Definition

Let A be a BL-algebra. A is a

- *MV-algebra* if it satisfies *involutivity*: $\neg\neg x = x$, where $\neg x = x \to 0$.
- *Gödel algebra* if it satisfies *idempotency*: $x \cdot x = x$.
- *Product algebra* if it satisfies the identity

$$\neg x \vee ((x \to x \cdot y) \to y) = 1.$$

Examples

1. The algebra

$$[0,1]_{\mathrm{MV}} = ([0,1], \cdot_L, \to_L, \max, \min, 0, 1),$$

   where $x \cdot_L y = \max\{x+y-1, 0\}$ and $x \to_L y = \min\{1-x+y, 1\}$, is called the *standard MV-algebra*.

2. The algebra

$$[0,1]_G = ([0,1], \cdot_G, \to_G, \max, \min, 0, 1),$$

   where $x \cdot_G y = \min\{x, y\}$ and

$$x \to_G y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise} \end{cases}$$

   is called the *standard Gödel algebra*.

Examples

3. The algebra

$$[0,1]_P = ([0,1], \cdot_P, \to_P, \max, \min, 0, 1),$$

where $x \cdot_P y = xy$ and

$$x \to_P y = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{otherwise} \end{cases}$$

is called the *standard product algebra.*

Some results

- Bounded basic hoops are term-equivalent to BL-algebras.
- Bounded Wajsberg hoops, bounded Gödel hoops, and bounded product hoops are term-equivalent, respectively, to MV-algebras, Gödel algebras, and product algebras.
- The varieties $\mathsf{BHoops}, \mathsf{WHoops}, \mathsf{GHoops}$ and PHoops are *semi-abelian* categories, as is the category of rings.
- The varieties $\mathsf{BLAlg}, \mathsf{MVAlg}, \mathsf{GAlg}$ and PAlg are *ideally exact* categories, and their initial object is the two-element Boolean algebra $\{0, 1\}$. The same holds for the category of unital rings, where the inital object is the ring $\mathbb{Z}$.

Split extensions with strong section

Definition (W. Rump, 2009)

A split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

in the variety Hoops has *strong section* if $a \to s(b) = sp(a) \to s(b)$.

Example (Double negation)

Let A be a BL-algebra. The split extension

$$\mathrm{D}(A) \xrightarrow{i} A \underset{j}{\overset{p}{\rightleftarrows}} \mathrm{MV}(A)$$

where $\mathrm{MV}(A) = \{x \in A \mid \neg\neg x = x\}$, $\mathrm{D}(A) = \{x \in A \mid \neg\neg x = 1\}$, $p(a) = \neg\neg a$, and i and j are canonical inclusions, has strong section.

Strong external actions

Definition (M. M., G. Metere, F. Piazza, M. E. Tabacchi, 2025)

Let B, X be hoops. A *strong external action* of B on X consists of a pair of maps

$$f\colon B \times X \to X \colon (b, x) \mapsto f_b(x),$$
$$g\colon B \times X \to X \colon (b, x) \mapsto g_b(x)$$

such that

E1. $f_b(1) = g_b(1) = 1$;

E2. $f_1 = g_1 = \mathrm{id}_X$;

E3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;

E4. $g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to f_{b_3}(z))$
$= g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to f_{b_3}(z)),$

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

Proposition (M. M., G. Metere, F. Piazza, M. E. Tabacchi, 2025)

Let B, X be hoops and let $f, g\colon B \times X \to X$ be a strong external action. Then the set

$$Y' = \{(x, b) \in X \times B \mid f_b(x) = x\}$$

endowed with the operations

$$(x, b) \to (y, b') = (g_{b' \to b}(x \to y), b \to b'),$$
$$(x, b) \cdot (y, b') = (f_{b \cdot b'}(x \cdot y), b \cdot b')$$

and unit $1_{Y'} = (1, 1)$ is a hoop. Furthermore, the split extension

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & Y' \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

where $\iota_1(x) = (x, 1)$, $\pi_2(x, b) = b$ and $\iota_2(b) = (1, b)$, has strong section.

Proposition (M. M., G. Metere, F. Piazza, M. E. Tabacchi, 2025)

Let

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B,$$

be a split extension with strong section in Hoops. *Then the pair of maps*

$$f, g \colon B \times X \to X$$

defined by

$$f_b(x) = s(b) \to (s(b) \cdot x) \quad \text{and} \quad g_b(x) = s(b) \to x$$

give rise to a strong external action of B on X.

Proposition (M. M., G. Metere, F. Piazza, M. E. Tabacchi, 2025)

Let B, X be hoops. There is a bijection τ_B between the sets $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ *and* $\mathrm{EAct}_{\mathrm{ss}}(B, X)$.

Theorem (M. M., G. Metere, F. Piazza, M. E. Tabacchi, 2025)

There is a natural isomorphism

$$\tau \colon \mathrm{SplExt}_{\mathrm{ss}}(-, X) \cong \mathrm{EAct}_{\mathrm{ss}}(-, X).$$

Example (Double negation)

Let A be a BL-algebra. The strong external action associated with the *double negation* split extension is

$$f, g \colon \mathrm{MV}(A) \times \mathrm{D}(A) \to \mathrm{D}(A)$$

defined by

$$f_b(x) = b \to (b \cdot x) \quad \text{and} \quad g_b(x) = b \to x.$$

What about subvarieties?

- The result extends to the subvarieties of basic, Wajsberg, Gödel and product hoops.
- We have to require more axioms: for instance, a strong external action (f, g) in WHoops must satisfy

$$g_{b_2 \to b_1}(x \to y) \to y = g_{b_1 \to b_2}(y \to x) \to x.$$

Grazie per l'attenzione!

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