Can we use CART[1] to evaluate the age of energy-containing eddies in a turbulent flow?

Eric Deleersnijder, November 18, 2008

Evaluating timescales (age and residence time) related to constituents of seawater helps understand the results of numerical simulations of complex marine flows. There are several tools for doing this. The Constituent-oriented Age and Residence time Theory (CART) is one of them. So far CART has been applied essentially to marine problems (see Appendix), but its equations are sufficiently general for them to be applicable to any flow model resting on the Boussinesq approximation[2].

The age of a particle of a constituent of seawater is the time that has elapsed since this particle left the region in which the age is prescribed to be zero. Thus, as in everyday life, the age of a constituent is the measure of an elapsed time. Consider a constituent whose concentration $C(t,\mathbf{x})$ obeys the following partial differential equation

$$\frac{\partial C}{\partial t} + \nabla \bullet (\mathbf{u}C) = P - D + \nabla \bullet (\mathbf{K} \bullet \nabla C) , \qquad (1)$$

where $\mathbf{u}$ and $\mathbf{K}$ denote the velocity vector, which is divergence-free[3], and the diffusivity tensor, which is symmetric and positive-definite (e.g. Deleersnijder et al. 2001); the production and destruction rates of the constituent under study are the positive functions P and D, respectively. According to CART, the mean age $a(t,\mathbf{x})$ of the constituent particles that are at time t in an arbitrarily-small water volume located at $\mathbf{x}$ is

$$a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} , \qquad (2)$$

where $\alpha(t,\mathbf{x})$ is the age concentration. Assuming that the age of newly-produced constituent particles is zero, the age concentration obeys the equation (Delhez et al. 1999, Deleersnijder et al. 2001)

$$\frac{\partial \alpha}{\partial t} + \nabla \bullet (\mathbf{u}\alpha) = C - Da + \nabla \bullet (\mathbf{K} \bullet \nabla \alpha) . \qquad (3)$$

Herein an attempt is made to apply a similar approach in order to estimate the age of energy-containing eddies in a turbulent flow, hoping that this timescale will lead to useful diagnoses of the dynamics of turbulent fluctuations. Accordingly, the age of the turbulence kinetic energy $k(t,\mathbf{x})$ will be estimated by means of a CART-like method, assuming that this timescale is approximately equivalent to the age of the energy-containing eddies. To do so, a turbulence closure scheme is needed. For the present study the classical k-ε closure model is believed to be sufficient.

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[1]: CART: Constituent-oriented Age and Residence time Theory (www.climate.be/CART)
[2]: Jongen T., 2004, Extension of the age-of-fluid method to unsteady and closed-flow sytems, *American Institute of Chemical Engineers Institute Journal*, 50, 2020-2037
[3]: The Boussinesq approximation is in use.

The classical k-ε closure model

In the closure scheme selected, the eddy viscosity is expressed as follows:

$$\nu_t \;=\; c_\mu \frac{k^2}{\varepsilon} \,, \tag{4}$$

where $c_\mu \approx 0.09$ is a dimensionless constant, while $\varepsilon(t,\mathbf{x})$ denotes the rate of dissipation of the turbulence kinetic energy. The turbulence kinetic energy and its dissipation rate obey the equations

$$\frac{\partial k}{\partial t} \;+\; \nabla \bullet (\mathbf{u}k) \;=\; P_k \;-\; \varepsilon \;+\; \nabla \bullet \left(\frac{\nu_t}{\sigma_k} \nabla k \right) , \tag{5}$$

$$\frac{\partial \varepsilon}{\partial t} \;+\; \nabla \bullet (\mathbf{u}\varepsilon) \;=\; c_{1\varepsilon} \frac{\varepsilon}{k} P_k \;-\; c_{2\varepsilon} \frac{\varepsilon}{k} \varepsilon \;+\; \nabla \bullet \left(\frac{\nu_t}{\sigma \varepsilon} \nabla \varepsilon \right) , \tag{6}$$

where $\sigma_k \approx 1$, $c_{1\varepsilon} \approx 1.44$, $c_{2\varepsilon} \approx 1.92$ and $\sigma_\varepsilon \approx 1.3$ are dimensionless constants; the production rate of turbulence kinetic energy is

$$P_k \;=\; \nu_t (\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{t}}) : \nabla \mathbf{u} \;. \tag{7}$$

For this closure model to be able to represent a logarithmic layer, it is necessary that four of the abovementioned dimensionless constants satisfy the following relation

$$c_{1\varepsilon} \;=\; c_{2\varepsilon} \;-\; \frac{\kappa^2}{\sigma_\varepsilon \sqrt{c_\mu}} \,, \tag{8}$$

where $\kappa \approx 0.4$ is the von Karman constant.

Age of the energy-containing eddies

Resorting to an approach similar to CART's, the age of the energy containing eddies $a_k(t,\mathbf{x})$ may be as estimated as the ratio of the age concentration of the eddies $\alpha_k(t,\mathbf{x})$ to the turbulence kinetic energy, i.e.

$$a_k(t,\mathbf{x}) \;=\; \frac{\alpha_k(t,\mathbf{x})}{k(t,\mathbf{x})} \,. \tag{9}$$

Assuming that the age of the newly-generated eddies is zero, the equation governing the age concentration of the eddies reads

$$\frac{\partial \alpha_k}{\partial t} \;+\; \nabla \bullet (\mathbf{u}\alpha_k) \;=\; k \;-\; \varepsilon a_k \;+\; \nabla \bullet \left(\frac{\nu_t}{\sigma_k} \nabla \alpha_k \right) . \tag{10}$$

Below relations (9)-(10) are applied to three simple flows (grid turbulence, shear layer, logarithmic layer) in order to see if the age derived in this manner seems to be acceptable, which would help gain confidence in the present developments.

Grid-generated turbulence

Let x denote the distance to the grid. Turbulent diffusion processes are usually neglected, so that, at a steady-state, the turbulence kinetic energy equation reduces to

$$U\frac{dk}{dx} = -\varepsilon \ . \tag{11}$$

Then, the equation for the age concentration simplifies to

$$U\frac{d\alpha_k}{dx} = k - \varepsilon a_k \ . \tag{12}$$

Combining (9), (11) and (12) yields the equation obeyed by the age:

$$U\frac{da_k}{dx} = 1 \ . \tag{13}$$

It seems natural to prescribe that the age of the eddies are zero at the grid, i.e. $a_k(0) = 0$. Then, the age of the eddies downstream of the grid simply is

$$\boxed{a_k(x) = \frac{x}{U}} \tag{14}$$

The age of the eddies at a distance x to the grid is the time needed to travel this distance at speed U. As turbulent diffusion is assumed to be negligible, this result is in accordance with the most elementary intuition.

Local-equilibrium shear layer

In a local-equilibrium shear layer, the turbulent energy production by shear is balanced by the viscous dissipation, i.e. $P_k = \varepsilon$. Accordingly, the equation (10) for the age concentration simplifies to

$$k - \varepsilon a_k = 0 \ , \tag{15}$$

immediately yielding the mean age of the energy-containing eddies

$$\boxed{a_k = \frac{k}{\varepsilon}} \tag{16}$$

Logarithmic layer

In a logarithmic layer, we have

$$k = \frac{u_*^2}{\sqrt{c_\mu}} \ , \quad \varepsilon = \frac{u_*^3}{\kappa z} \ , \quad \nu_t = \kappa u_* z \ , \tag{17}$$

where z is the distance to the wall and u_* is the relevant friction velocity. The equation for the age concentration (10) becomes

$$\frac{d}{dz}\left(\frac{\nu_t k}{\sigma_k}\frac{da_k}{dz}\right) + k - \varepsilon a_k = 0 \ . \tag{18}$$

Combining (17) an d(18) yields

$$\boxed{a_k(z) \;=\; \frac{\sigma_k\,\kappa}{1-\kappa^2}\,\frac{z}{u_*}} \tag{19}$$

i.e. $a_k(z) \approx 0.48\, z/u_*$. Not surprisingly, the larger the eddies, the larger their mean age.

Perspectives

I believe the above results are acceptable or, at least, not counterintuitive. However, additional tests are needed. In this respect, I am willing to consider any relevant suggestion.

Appendix: CART-related publications

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