

Length of astronomical seasons, total and average insolation over seasons

André Berger^{*}, Qiuzhen Yin^{*}, Zhipeng Wu

Earth and Climate Research Center, Earth and Life Institute, Université catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

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ABSTRACT

Daily insolation at a given latitude is often used in the explanation of the long-term variations of paleoclimate records. However, the length of astronomical seasons, which has been paid less attention, could also be important. This paper provides an original calculation of the length of these seasons using the eccentric anomaly. The past and future long-term variations as well as the spectral characteristics of the length of astronomical seasons, of the total and average insolation over the astronomical seasons and of the caloric insolation are shown and compared. The length of astronomical seasons is only a function of climatic precession. The total irradiation for a given latitude and an astronomical season is mainly a function of obliquity except that it is mainly a function of eccentricity in a latitudinal band that is season dependent and from where the phase between obliquity and total irradiation reverses (e.g. $\sim 11.5^\circ\text{N}$ for the astronomical northern hemisphere summer half-year). The spectral characteristic of the mean insolation over the astronomical summer half-year varies in latitude and time. It contains stronger obliquity signal in high latitudes than in low latitudes, but the obliquity signal already gets very weak at 45°N/S leaving the dominance of climatic precession between 45°N and 45°S . The variation of the mean summer half-year insolation over the last 1 Ma is characterized by a Mid-Brunhes transition at ~ 450 ka BP, with a dominant role of climatic precession at all latitudes before ~ 450 ka BP but significantly enhanced role of obliquity in high latitudes after this time. The simultaneous weakening of the variations of climatic precession and obliquity around 800 ka ago, a time corresponding to the Mid-Pleistocene Transition, leads to weak variations in both total and mean seasonal insolation. The insolation of a specific case, the Holocene, is finally discussed. Depending on the sensitivity to different types of insolation, different climate variables and regions could show different evolution and trend during the Holocene.

1. Introduction

Paleoclimate reconstructions based on various geological records such as ocean and lake sediments, ice cores, loess, speleothems help us elucidate the natural variability of the climate system over different time scales. Spectral analysis of paleoclimate records shows significant periods of ~ 100 ka, 41 ka, 23 ka and 19 ka over the last one million years (Hays et al., 1976). These periods correspond to the major periodicities of the long-term variations of the three astronomical parameters, eccentricity, obliquity and climatic precession (Milankovitch, 1941; Berger, 1978a), a strong evidence that shows the fundamental role of the astronomical forcing on the climate system (Hays et al., 1976; Berger, 1977). Astronomical forcing has been used not only to explain the climate variations on the timescale of tens and hundreds of thousands of years, but also to explain some centennial and millennial scale events, such as the 8.2 ka BP event (Renssen et al., 2007), the Heinrich Events (Crucifix and Berger, 2002; Jiang et al., 2023), the Dansgaard-Oeschger

events (Claussen et al., 2003; Sanchez Goñi et al., 2008; Kuniyoshi et al., 2022) and the ice age terminations (Yuan et al., 2004; Cheng et al., 2009, 2016). The slow-varying astronomical forcing has also been found able to trigger abrupt cooling at the end of interglacials (Yin et al., 2021).

Astronomical forcing drives the Earth's climate mainly through its control on the latitudinal and seasonal distribution of insolation (incoming solar radiation) which variations are driven by the long-term (Berger, 1978a) or short term (Bretagnon, 1982) variations of eccentricity, obliquity and climatic precession (which is defined by $e \sin \tilde{\omega}$; Berger and Loutre, 1991). Different types of insolation have been calculated and they have different relationship with climatic precession, obliquity and eccentricity and therefore show different spectral characteristics (Berger et al., 1993). The instantaneous insolation or the daily insolation for some astronomical positions of the Earth on its orbit, like the perihelion, the aphelion, the solstices and the equinoxes is probably the easiest to calculate (Berger, 1978a; Berger et al., 1993). The daily

^{*} Corresponding authors.

E-mail addresses: andre.berger@uclouvain.be (A. Berger), qiuzhen.yin@uclouvain.be (Q. Yin).

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insolation for a given latitude and a given day, for example the summer solstice insolation at 65°N (as in Milankovitch, 1941), is mostly used in paleoclimate study to explain the variations of proxy records. The variations of the daily insolation are dominated by climatic precession (except close to the polar night), with obliquity playing a secondary role but more important in high than in low latitudes (Berger et al., 1993). Besides the classical daily insolation, another particular interesting calculation, although more complicated, concerns the total irradiation received at a given latitude over any time interval of the year such as the astronomical seasons (Berger et al., 2010). Different from the daily insolation, the long-term variations of the total irradiation depend almost exclusively upon obliquity with a very small contribution of eccentricity but with no contribution of climatic precession (Berger et al., 2010). To attempt to solve the difficulty of having to deal with both the total irradiation received during a given season and the length of the season, which both vary in time, the caloric insolation (i.e., the total solar energy received during a half-year caloric season) was introduced by Milankovitch (1941), recalculated by Vernekar (1972) and by Berger (1978b) using a more accurate astronomical solution. The caloric seasons divide the year into two equally long seasons, with the caloric summer comprising all the days during which the irradiation at a given latitude is stronger than on any day of the other half-year – the caloric winter. However, the caloric insolation does not solve the problem completely because the beginning and end of the caloric seasons change with time and also because it is impossible to define them in the equatorial regions where the seasonal march of insolation has two minima and two maxima. In addition to daily insolation, total irradiation and caloric insolation at a given latitude, the latitudinal insolation gradient (e.g. Yin, 2013) and the integrated daily insolation above a threshold (e.g. Huybers, 2006; Raynaud et al., 2007) have also been used to explain past climate changes.

The different types of insolation mentioned above are not the only important characteristics that may have a control on climate. For example, the total irradiation depends on both the intensity of insolation received per unit of time and also the length of the interval during which it is received. Some climate components may be more controlled by the total irradiation, and some more by the mean insolation received during a season.

In this paper we will discuss the length of any time interval of the year using the eccentric anomaly to obtain a more accurate value than when using the true anomaly as in Milankovitch (1941). Section 2 provides the calculation of the length of the quarterly and half-year astronomical seasons. Section 3 presents the long-term variations and the spectral characteristics over the past 1 Ma and future 100 ka of the length of the astronomical seasons, of the total and mean insolation over the astronomical seasons for a given latitude as well as of the caloric summer half-year insolation. In Section 4, an example case is given, and we present the evolution of different types of insolation over the Holocene especially in relation to the Holocene climate trend. A summary is given in Section 5.

2. Length of the astronomical seasons

2.1. Definition

There are different kinds of definitions of seasons and years. The most familiar are the calendar year and seasons. The common calendar year has 365 days going from January 1 to December 31. It is an approximation of the number of days of the Earth's orbital period which is slightly less than 365.25 days. The calendar seasons are also called meteorological seasons and are for the Northern Hemisphere: spring (March, April, May), summer (June, July, August), autumn (fall) (September, October, November) and winter (December, January, February). They contain respectively 92, 92, 91, and 90 days (or 91 during the leap year). The calendar summer half-year gathers spring and summer with 184 days, while the calendar winter half-year is 181 days

long (or 182 during the leap year). The meteorological (calendar) seasons have the advantage of a constant length, but because of the changing length of the astronomical seasons, they do not always correspond to the same astronomical configuration.

The astronomical year is the time for the Earth to revolve around the Sun completing 360°. The (quarterly) astronomical seasons are the periods during which the Sun transverses the quadrants of the ecliptic counted from the spring equinox or vernal point in a geocentric system of coordinates (Fig. 1). They are defined in terms of the true longitude of the Sun, λ , with $\lambda = 0^\circ$ at the spring (vernal or March) equinox, $\lambda = 90^\circ$ at the summer (June) solstice, $\lambda = 180^\circ$ at the fall (Autumn or September) equinox and $\lambda = 270^\circ$ at the winter (December) solstice. Consequently, the four astronomical seasons (with a lower-case letter) can be defined by:

NH spring (*spring*) and SH fall (*fall**): $0 \leq \lambda < 90^\circ$

NH summer (*summer*) and SH winter (*winter**): $90 \leq \lambda < 180^\circ$

NH fall (*fall*) and SH spring (*spring**): $180 \leq \lambda < 270^\circ$

NH winter (*winter*) and SH summer (*summer**): $270 \leq \lambda < 360^\circ$

NH and SH stand respectively for Northern Hemisphere and Southern Hemisphere. By convention in this paper, the seasons will refer to the NH, although occasionally the season with an asterisk will refer to the local season in the SH.

Since the areas of the seasonal sectors of the ecliptic are not equal, and since the radius vector of the Sun sweeps out equal areas during equal time intervals (second law of Kepler), the four astronomical seasons have different durations. The astronomical northern hemisphere summer half-year is defined for λ going from 0° to 180° and

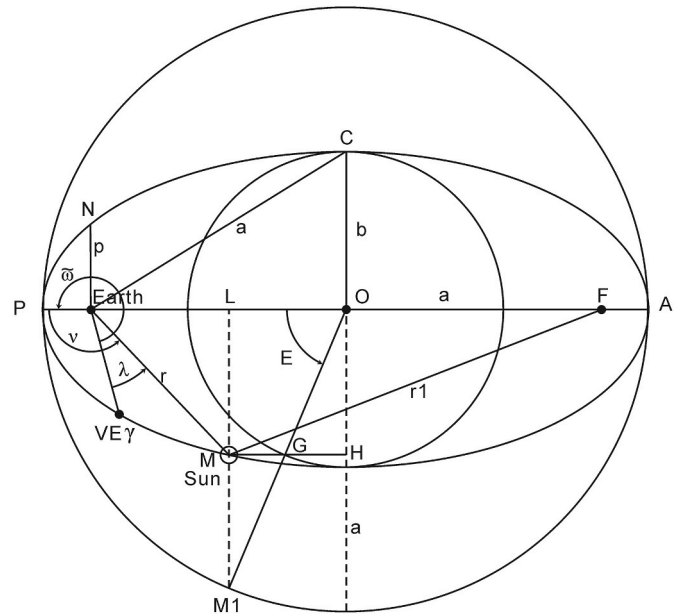


Fig. 1. Relative motion of the Sun around the Earth on the ecliptic (the elliptical orbit). P is the perigee, A the apogee, VE the vernal (spring or March) equinox (in this geocentric system, it is also the γ point, the direction of the Sun seen from the Earth at spring equinox and the point from which the longitudes are measured). λ : the longitude of the Sun; ω : the longitude of the perigee; ν : the true anomaly of the Sun; E: the eccentric anomaly of the Sun; the distance O-A is a , the semi-major axis of the ecliptic; and the distance Earth-P is equal to $a(1-e)$, e being the eccentricity of the elliptical orbit equal to $\frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$ with the distance O-Earth = OF = ae being the focus distance c or linear eccentricity. p is the semi-latus rectum equal to $\frac{b^2}{a} = a(1 - e^2)$. This figure is not at scale, and the eccentricity has been exaggerated for a clear view.

astronomical northern hemisphere winter half-year is defined for λ going from 180° to 360° . Therefore, it is clear that the astronomical summer and winter half-years do not divide the calendar year into two intervals of the same length measured in calendar days. The advantage of the astronomical seasons is that they always correspond to the same astronomical configuration. The astronomical summer and winter half-years are exactly half-years in astronomical units: each are 180° long.

As the time during which the Sun leaves the vernal point and returns to it, is called the tropical year and since the seasons are reckoned from the vernal point, the astronomical seasons are fractions of the tropical year. The tropical year will therefore be used in the calculation of the seasons. However, the length of the tropical year changes with time because of the motion of the vernal point on the ecliptic (the so-called precession of the equinoxes) and for very accurate calculations, the difference between the tropical and the sidereal years must be considered (the sidereal year corresponds to the revolution of the Earth around the Sun with respect to the fixed stars; it is used in the second law of Kepler and is presently about 20 minutes longer than the tropical year).

2.2. Calculation of the length of the seasons from the true anomaly ν (or the true longitude λ)

If T_1 and T_2 are the time elapsed while the radius vector is sweeping over the area A_1 and A_2 respectively, we have from the second law of Kepler

$$T_1 = \frac{A_1}{A_2} T_2 \quad (1)$$

For any interval (ν_1, ν_2) given in terms of the true anomaly ν , the area $A_{(\nu_1, \nu_2)}$ is given by

$$A_{(\nu_1, \nu_2)} = \frac{1}{2} \int_{\nu_1}^{\nu_2} r^2 d\nu \quad (2)$$

where $r = \frac{a(1-e^2)}{1+e \cos \nu}$, $\nu = \lambda - \tilde{\omega}$, with λ (the true longitude) and $\tilde{\omega}$ (the longitude of the perigee). These longitudes are measured from the vernal equinox (which is also the vernal point γ in a geocentric system of coordinates), whereas the true anomaly is measured from the perigee (Fig. 1). According to the definition of ν , the true anomaly of the vernal equinox for which $\lambda = 0$, is $\nu_0 = -\tilde{\omega}$.

This formulation is classical, but it must be stressed that in this paper $\tilde{\omega}$ is the longitude of the perigee which is equal to the longitude of the perihelion, denoted Π_γ in Milankovitch (1941) and calculated in Berger (1978a), plus 180° . The reason why the longitude of the perigee is used (which means that the geocentric system is used) is related to the origin of the seasons. This origin in paleoclimate research is set at the vernal (spring or March) equinox for which $\lambda = 0$. The γ point from which the longitudes are measured in celestial mechanics and which is defined by the direction in which the Sun is seen from the Earth at the spring equinox coincides indeed with the spring equinox in a geocentric system (Fig. 1).

Let us now consider the particular case of the quarterly astronomical seasons (defined as in section 2.1):

$$A_{\text{year}} = \frac{1}{2} \int_{\nu=-\tilde{\omega}}^{\nu=2\pi-\tilde{\omega}} r^2 d\nu$$

$$A_{\text{spring}} = \frac{1}{2} \int_{\nu=-\tilde{\omega}}^{\nu=\pi/2-\tilde{\omega}} r^2 d\nu$$

$$A_{\text{summer}} = \frac{1}{2} \int_{\nu=\pi/2-\tilde{\omega}}^{\nu=\pi-\tilde{\omega}} r^2 d\nu$$

$$A_{\text{fall}} = \frac{1}{2} \int_{\nu=\pi-\tilde{\omega}}^{\nu=3\pi/2-\tilde{\omega}} r^2 d\nu$$

$$A_{\text{winter}} = \frac{1}{2} \int_{\nu=3\pi/2-\tilde{\omega}}^{\nu=2\pi-\tilde{\omega}} r^2 d\nu$$

It will be assumed that variations of the eccentricity and of the longitude of the perigee over one year are sufficiently small to be ignored and the terms in e^2 and higher in r^2 will be neglected as in Milankovitch (1941). Therefore with $r^2 \simeq a^2(1 - 2e \cos \nu)$, the area swept over one year is given by:

$$A_{\text{year}} = \frac{1}{2} \int_{\nu_0}^{\nu_0+2\pi} r^2 d\nu \simeq \frac{a^2}{2} [1 - 2e \sin \nu]_{\nu_0}^{\nu_0+2\pi} \simeq \pi a^2 \quad (3)$$

with an accuracy to the first degree of eccentricity (the correct area of an ellipse being $ab = a^2 \sqrt{1-e^2}$).

For any interval of time (ν_1, ν_2) or (λ_1, λ_2) expressed in λ , we have from (1), (2), and (3):

$$\begin{aligned} A_{(\nu_1, \nu_2)} &\simeq \frac{a^2}{2} [1 - 2e \sin \nu]_{\nu_1}^{\nu_2} = \frac{a^2}{2} [\lambda_2 - \lambda_1 - 2e(\sin(\lambda_2 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}))] \\ &= \frac{a^2}{2} [\lambda_2 - \lambda_1 - 2e(\cos \tilde{\omega} (\sin \lambda_2 - \sin \lambda_1) - \sin \tilde{\omega} (\cos \lambda_2 - \cos \lambda_1))] \end{aligned} \quad (4)$$

$$T_{(\lambda_1, \lambda_2)} = \frac{T}{2\pi} [\lambda_2 - \lambda_1 - 2e(\cos \tilde{\omega} (\sin \lambda_2 - \sin \lambda_1) - \sin \tilde{\omega} (\cos \lambda_2 - \cos \lambda_1))] \quad (5)$$

and the lengths of the quarterly astronomical seasons are finally given by:

$$\begin{aligned} T_{\text{spring}} &\simeq \frac{T}{4} \left(1 - \frac{4e}{\pi} (\sin \tilde{\omega} + \cos \tilde{\omega}) \right) = T_{\text{fall}}^* \\ T_{\text{summer}} &\simeq \frac{T}{4} \left(1 - \frac{4e}{\pi} (\sin \tilde{\omega} - \cos \tilde{\omega}) \right) = T_{\text{winter}}^* \\ T_{\text{fall}} &\simeq \frac{T}{4} \left(1 + \frac{4e}{\pi} (\sin \tilde{\omega} + \cos \tilde{\omega}) \right) = T_{\text{spring}}^* \\ T_{\text{winter}} &\simeq \frac{T}{4} \left(1 + \frac{4e}{\pi} (\sin \tilde{\omega} - \cos \tilde{\omega}) \right) = T_{\text{summer}}^* \end{aligned} \quad (6a)$$

where T is the length of the tropical year and the $*$ refers to the season of the SH. This clearly shows that the length of the astronomical seasons is only a function of the climatic precession. It follows also that for any value of $\tilde{\omega}$:

$$T_{\text{spring}} + T_{\text{fall}} = T_{\text{summer}} + T_{\text{winter}} = \frac{T}{2}$$

which holds also for the local seasons of the SH.

At the present-day with $e = 0.0167$, $\tilde{\omega} = 282^\circ$ and $T = 365.2422$ days, spring (92.8 days) and summer (93.6 days) have about the same length and are longer than fall (89.8 days) and winter (89 days) which are also about equally long. These values can also be obtained from the formulas providing the mean longitudes in terms of the true longitudes (the mean longitude is the longitude of a “mean” Sun revolving around the Earth with a constant speed equal to $\frac{2\pi}{T}$): the mean longitude at the

spring equinox is -1.9° , at the summer solstice it is 89.6° , at the fall equinox 181.9° and at the winter solstice 270.4° (Berger et al., 2010). The length of the astronomical spring calculated from these mean longitudes is therefore $(89.6 + 1.9 = 91.5) \times \frac{365.2422}{360} = 92.8$ days, as just obtained above. Similarly, the length of the astronomical summer is 93.6 days, of the astronomical fall is 89.8 days and of the astronomical winter 89 days, as expected.

For the half-year astronomical seasons of the NH (Summer and Winter with an uppercase letter), we can write:

$$T_{Summer} = T_{spring} + T_{summer}$$

$$T_{Winter} = T_{fall} + T_{winter}$$

And therefore,

$$\begin{aligned} T_{Summer} &= \frac{T}{2} \left(1 - \frac{4e \sin \tilde{\omega}}{\pi} \right) = T_{Winter}^* \\ T_{Winter} &= \frac{T}{2} \left(1 + \frac{4e \sin \tilde{\omega}}{\pi} \right) = T_{Summer}^* \end{aligned} \quad (6b)$$

$Winter^*$ and $Summer^*$ (with an uppercase letter and an *) refer to the half-year astronomical seasons of the SH.

For the present, we obtain

$$T_{Summer} = 186 \text{ days } 10 \text{ hours}$$

$$T_{Winter} = 178 \text{ days } 20 \text{ hours}$$

The summer half-year of the NH is consequently 7 days 14 hours longer than the winter half-year; in the SH, the opposite is true. Because $\tilde{\omega} = 282^\circ$, this is about the maximum value of the difference between the lengths of the summer and winter half-year astronomical seasons. The maximum actually amounts 7 days 18 hours and is reached for $\tilde{\omega} = 270^\circ$ (winter solstice at perigee implying that the winter season is shorter) and eccentricity kept at its present-day value. Winter half-year astronomical season is longer than summer for the summer solstice at perigee/winter solstice at apogee.

The date of the summer solstice, fall equinox, winter solstice and perigee/perihelion can easily be computed from the number of days, NT , elapsed since the vernal equinox taken as being the origin of the true longitude:

$$NT \text{ at summer solstice} = T_{spring} = \frac{T}{2\pi} \left(\frac{\pi}{2} - 2e(\sin \tilde{\omega} + \cos \tilde{\omega}) \right)$$

$$NT \text{ at fall equinox} = T_{Summer} = \frac{T}{2\pi} (\pi - 4e \sin \tilde{\omega})$$

$$NT \text{ at winter solstice} = T_{Summer} + T_{fall} = \frac{T}{2\pi} \left(\frac{3\pi}{2} + 2e(\cos \tilde{\omega} - \sin \tilde{\omega}) \right)$$

$$NT \text{ at perihelion} = \frac{T}{2\pi} (\tilde{\omega} - 2e \sin \tilde{\omega})$$

NT is therefore a function of e and $\tilde{\omega}$, which means climatic precession. As e and $\tilde{\omega}$ vary in time, the calendar dates of the summer solstice, fall equinox and winter solstice and the length of seasons vary in time and depend on climatic precession.

At the present-day ($e = 0.016724$, $\tilde{\omega} = 282^\circ$), the winter solstice occurs at about perigee and the calendar dates of the summer solstice, fall equinox, winter solstice are: 21.8 June, 23.4 September and 22.2 December. More examples are given in Table 1 for the time slices when the summer solstice occurs at about perigee/perihelion. At 12 ka BP, the calendar dates of the fall equinox and the solstices are 15.1 September, 17.7 June and 18 December, respectively 5, 8 and 4 days earlier than the present-day.

Fig. 2 shows that during the last one million years, the calendar dates varied (for a year of 365.25 days) between 11.6 and 28.9 June for the summer solstice, between 7.5 September and 1.9 October for the fall equinox, and between 11.3 and 28.7 December for the winter solstice. It is clear that the variation of the dates of the fall equinox and of the solstices depends on climatic precession. In the next 100 ka, the variation of climatic precession is relatively small due to a small eccentricity, so the variations of the dates of the solstices and fall equinox are also relatively small, varying between 17.9 and 22.7 June, 16.3 and 23.4 September, and between 17.6 and 22.5 December.

2.3. Calculation from the eccentric anomaly

Another way to compute the length of the seasons or of any time interval during the year is to start from the mean anomaly: $M = n(t - \tau)$ where n is the mean motion, τ the passage at perigee and t time elapsed since the passage at perigee. This is a more accurate procedure than using the true anomaly.

The mean anomaly is related to the eccentric anomaly, E (Fig. 1), by the Kepler equation (see Annex 1):

$$M = n(t - \tau) = E - e \sin E \quad (7)$$

$$\text{Therefore: } \frac{dM}{dt} = n \text{ and } \frac{dM}{dE} = 1 - e \cos E \text{ or } \frac{dE}{dt} = \frac{n}{1 - e \cos E}$$

The length of any interval of time during the year defined by the eccentric anomaly is therefore given by:

$$T_{1,2} = \int_{t_1}^{t_2} dt = \int_{E_1}^{E_2} \frac{dt}{dE} dE = \int_{E_1}^{E_2} \frac{1 - e \cos E}{n} dE = \frac{1}{n} [E - e \sin E]_{E_1}^{E_2} \quad (8)$$

and $E_2 - E_1$ and $\sin E_2 - \sin E_1$ must be calculated as a function of ν or λ .

Table 1

Eccentricity, longitude of perihelion/perigee (Berger, 1978a) and calendar dates for the fall equinox and the solstices for the present-day and selected times in the past assuming the vernal equinox is on 21.0 March. 0 ka is for the present-day and represents the year 1950. The time slices chosen as examples were those used by Yin and Berger (2012) to simulate the climate of the past interglacials.

time (ka BP)	eccentricity	longitude of perihelion (°)	longitude of perigee (°)	calendar dates			
				vernal equinox March	summer solstice June	fall equinox September	winter solstice December
0	0.016724	102.04	282.04	21	21.8	23.4	22.2
12	0.019608	262.02	82.02	21	17.7	15.1	18
127	0.039378	275.41	95.41	21	16.2	10.5	14.9
242	0.034033	274.78	94.78	21	16.7	11.7	15.7
334	0.031539	281.08	101.08	21	17.4	12.4	15.6
409	0.019322	276.67	96.67	21	18.3	15.2	17.4
506	0.034046	274.05	94.05	21	16.6	11.7	15.7
579	0.047152	263.91	83.91	21	14.3	8.7	15.1
693	0.045154	274.96	94.96	21	15.5	9.2	14.3
788	0.026196	272.86	92.86	21	17.4	13.5	16.7

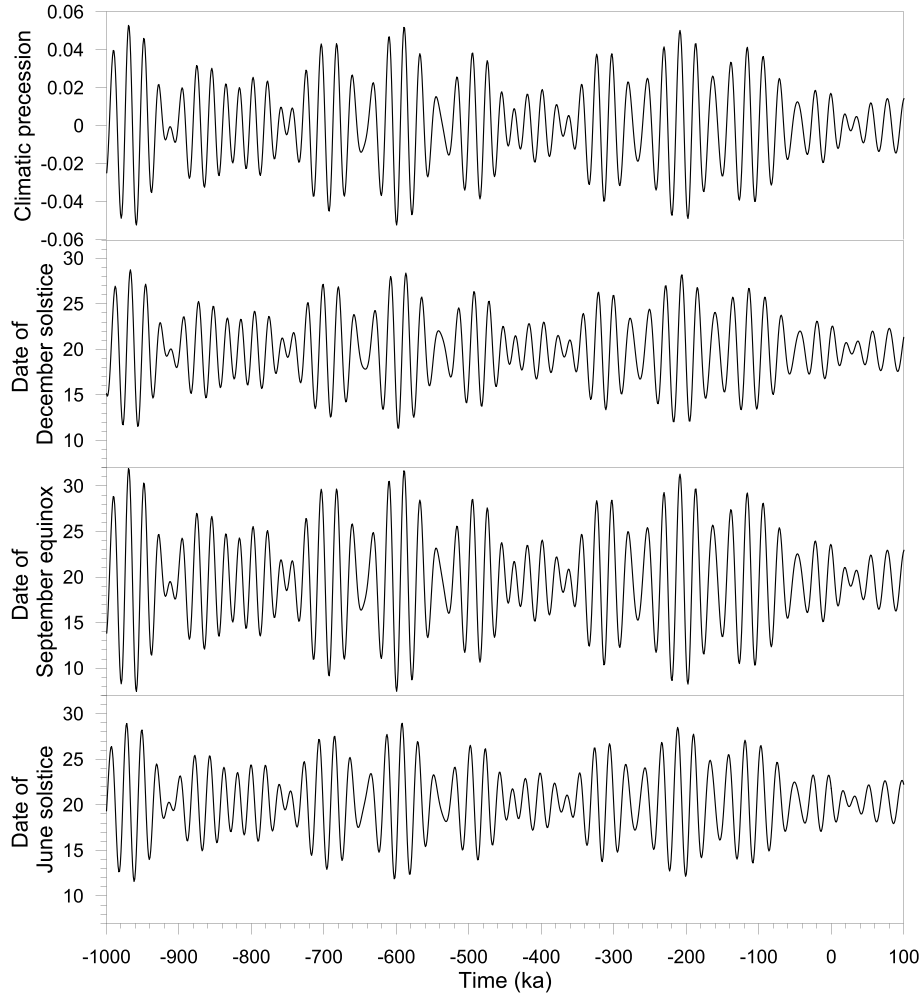


Fig. 2. Climatic precession and the calendar dates of the winter solstice, fall equinox and summer solstice during the last one million years (negative time) and the future 100 ka (positive time), calculated based on the solution of Berger (1978a). The length of the year is 365.25 days and the vernal equinox is on 21 March.

CALCULATION of $E_2 - E_1$:

From Fig. 1, it can be shown that $\overline{OG} = b$ and $\overline{LM} = \overline{OH} = b \sin E$. As in addition:

$$\overline{EarthL} = ae - a \cos E \quad (9)$$

the Pythagorean theorem applied to the triangle $EarthLM$ leads to

$$r = a(1 - e \cos E) \quad (10)$$

As we have also in this triangle

$$\overline{EarthL} = r \cos(\pi - \nu) = -r \cos \nu \quad (11)$$

from (9) equal to (11), and r given by (10) we obtain

$$\cos \nu = \frac{a(\cos E - e)}{r} = \frac{\cos E - e}{1 - e \cos E} \quad (12)$$

With $\tan \frac{\nu}{2} = \sqrt{\frac{1 - \cos \nu}{1 + \cos \nu}}$ and (12), we have

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \text{ and } \tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2}.$$

This is allowing to calculate $E_2 - E_1$ considering

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$\text{and } \tan a - \tan b = \frac{\sin(a - b)}{\cos a \cos b}$$

leading to

$$\tan\left(\frac{E_2 - E_1}{2}\right) = \sqrt{\frac{1-e}{1+e}} \frac{\tan \frac{\nu_2}{2} - \tan \frac{\nu_1}{2}}{1 + \frac{1-e}{1+e} \tan \frac{\nu_2}{2} \tan \frac{\nu_1}{2}} = \sqrt{1-e^2} \frac{\sin \frac{\lambda_2 - \lambda_1}{2}}{\cos \frac{\lambda_2 - \lambda_1}{2} + e \cos \frac{\lambda_2 + \lambda_1 - 2\omega}{2}} \quad (13)$$

CALCULATION of $\sin E_2 - \sin E_1$:

From $\cos \nu = 1 - 2 \sin^2 \frac{\nu}{2} = 2 \cos^2 \frac{\nu}{2} - 1$ and (10) and (12):

$$2r \sin^2 \frac{\nu}{2} = r(1 - \cos \nu) = a(1+e)(1 - \cos E) \quad (14a)$$

$$2r \cos^2 \frac{\nu}{2} = r(1 + \cos \nu) = a(1-e)(1 + \cos E) \quad (14b)$$

From (14b-14a), with r given by (10):

$$\cos E = \frac{e + \cos \nu}{1 + e \cos \nu}$$

Then $\sin^2 E = 1 - \cos^2 E$ leads to

$$\sin E = \frac{\sqrt{1-e^2} \sin \nu}{1 + e \cos \nu}$$

and

$$\sin E_2 - \sin E_1 = \sqrt{1-e^2} \frac{\sin \nu_2 - \sin \nu_1 + e \sin(\nu_2 - \nu_1)}{(1 + e \cos \nu_1)(1 + e \cos \nu_2)} \quad (15a)$$

From $\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$ we have:

$$\sin \nu_2 - \sin \nu_1 = 2 \cos \frac{\lambda_1 + \lambda_2 - 2\tilde{\omega}}{2} \sin \frac{\lambda_2 - \lambda_1}{2}.$$

And (15a) becomes:

$$\sin E_2 - \sin E_1 = \sqrt{1 - e^2} \frac{2 \cos \frac{\lambda_1 + \lambda_2 - 2\tilde{\omega}}{2} \sin \frac{\lambda_2 - \lambda_1}{2} + e \sin(\lambda_2 - \lambda_1)}{(1 + e \cos(\lambda_1 - \tilde{\omega}))(1 + e \cos(\lambda_2 - \tilde{\omega}))} \quad (15b)$$

Finally from (8), (13) and (15b), the length of any time interval (λ_1 , λ_2) can be written:

$$T_{1,2} = \frac{T}{2\pi} \left[2 \operatorname{arccot} \frac{\cos \frac{\lambda_2 - \lambda_1}{2} + e \cos \frac{\lambda_2 + \lambda_1 - 2\tilde{\omega}}{2}}{\sqrt{1 - e^2} \sin \frac{\lambda_2 - \lambda_1}{2}} + e \sqrt{1 - e^2} \frac{2 \cos \frac{\lambda_1 + \lambda_2 - 2\tilde{\omega}}{2} \sin \frac{\lambda_1 - \lambda_2}{2} + e \sin(\lambda_2 - \lambda_1)}{(1 + e \cos(\lambda_1 - \tilde{\omega}))(1 + e \cos(\lambda_2 - \tilde{\omega}))} \right] \quad (16)$$

$E_2 - E_1$ is unequivocally given by $\sin E_2 - \sin E_1$ and $\tan(\frac{E_2 - E_1}{2})$.

As $0 \leq E_2 - E_1 \leq 360^\circ$, $0 \leq \frac{E_2 - E_1}{2} \leq 180^\circ$ and if $\tan(\frac{E_2 - E_1}{2}) < 0$ and the calculated arctan is such that $-\frac{\pi}{2} \leq \arctan(\frac{E_2 - E_1}{2}) \leq \frac{\pi}{2}$, π must be added to $\arctan(\frac{E_2 - E_1}{2})$.

For any season with an interval of π : $\lambda_2 - \lambda_1 = \pi$, we have from (15b) and (13):

$$\sin E_2 - \sin E_1 = -2\sqrt{1 - e^2} \frac{\sin(\lambda_1 - \tilde{\omega})}{1 - e^2 \cos^2(\lambda_1 - \tilde{\omega})} \quad (17a)$$

$$\tan\left(\frac{E_2 - E_1}{2}\right) = \frac{-\sqrt{1 - e^2}}{e \sin(\lambda_1 - \tilde{\omega})} \text{ and } E_2 - E_1 = 2 \operatorname{arccot} \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}} \quad (17b)$$

From (16) and $T_{1,2} \equiv T_{\lambda_1, \lambda_1 + \pi}$, we have

$$T_{\lambda_1, \lambda_1 + \pi} = \frac{T}{\pi} \left[\operatorname{arccot} \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}} + \sqrt{1 - e^2} \frac{e \sin(\lambda_1 - \tilde{\omega})}{1 - e^2 \cos^2(\lambda_1 - \tilde{\omega})} \right] \quad (18)$$

In Annex 2, this equation is written in a simpler form introducing a variable $E_2 - E_1 = 2\psi$.

Or $E_2 - E_1 = \pi - 2\psi$.

For any season distant by $\frac{\pi}{2}$: $\lambda_2 - \lambda_1 = \frac{\pi}{2}$.

In (15b) with $\sin \frac{\lambda_2 - \lambda_1}{2} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$,

$$\begin{aligned} 2 \cos \frac{\lambda_1 + \lambda_2 - 2\tilde{\omega}}{2} \sin \frac{\lambda_2 - \lambda_1}{2} &= 2 \cos\left(\lambda_1 - \tilde{\omega} + \frac{\pi}{4}\right) \frac{\sqrt{2}}{2} \\ &= 2 \left(\cos \frac{\pi}{4} \cos(\lambda_1 - \tilde{\omega}) - \sin \frac{\pi}{4} \sin(\lambda_1 - \tilde{\omega}) \right) \frac{\sqrt{2}}{2} \\ &= 2 \frac{\sqrt{2}}{2} (\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega})) \frac{\sqrt{2}}{2} = \cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}) \end{aligned}$$

And (15b) leads to

$$\sin E_2 - \sin E_1 = \sqrt{1 - e^2} \frac{\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}) + e}{(1 + e \cos(\lambda_1 - \tilde{\omega}))(1 - e \sin(\lambda_1 - \tilde{\omega}))}$$

From (13) we have

$$\tan \frac{E_2 - E_1}{2} = \sqrt{1 - e^2} \frac{\sqrt{2}/2}{\sqrt{2}/2 + e\sqrt{2}/2(\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}))}$$

$$\cot\left(\frac{E_2 - E_1}{2}\right) = \frac{1 + e(\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}))}{\sqrt{1 - e^2}}$$

and therefore with $T_{1,2} \equiv T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$:

$$T_{\lambda_1, \lambda_1 + \frac{\pi}{2}} = \frac{T}{2\pi} \left[2 \operatorname{arccot} \frac{1 + e(\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}))}{\sqrt{1 - e^2}} - e\sqrt{1 - e^2} \frac{\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}) + e}{(1 + e \cos(\lambda_1 - \tilde{\omega}))(1 - e \sin(\lambda_1 - \tilde{\omega}))} \right] \quad (19)$$

In general we can add the components of $T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$ and of $T_{\lambda_1 + \frac{\pi}{2}, \lambda_1 + \pi}$ to obtain $T_{\lambda_1, \lambda_1 + \pi}$ (see Annex 3).

Half-year astronomical seasons

From these formulas, the length of the astronomical summer and winter half-years can easily be calculated with $\lambda_1 = 0$ and $\lambda_2 = 180^\circ$ respectively:

$$\begin{aligned} T_{\text{Summer}} &= \frac{T}{\pi} \left[\operatorname{arccot} \frac{e \sin \tilde{\omega}}{\sqrt{1 - e^2}} - \frac{e\sqrt{1 - e^2} \sin \tilde{\omega}}{1 - e^2 \cos^2 \tilde{\omega}} \right] \\ T_{\text{Winter}} &= \frac{T}{\pi} \left[\operatorname{arccot} \frac{-e \sin \tilde{\omega}}{\sqrt{1 - e^2}} + \frac{e\sqrt{1 - e^2} \sin \tilde{\omega}}{1 - e^2 \cos^2 \tilde{\omega}} \right] \end{aligned} \quad (20)$$

For the present-day, these formula give $T_{\text{Summer}} = 186.4$ days and $T_{\text{Winter}} = 178.8$ days, numerical values in very good agreement with the values found above from the true anomaly. This agreement can also be demonstrated analytically as here below:

Being given that: $\operatorname{arccot} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \dots$, these lengths are, with an approximation to the second degree of eccentricity, indeed given by:

$$T_{\text{Summer}} = \frac{T}{\pi} \left(\frac{\pi}{2} - e \sin \tilde{\omega} - e \sin \tilde{\omega} \right) = \frac{T}{2} \left(1 - \frac{4e \sin \tilde{\omega}}{\pi} \right)$$

$$T_{\text{Winter}} = \frac{T}{\pi} \left(\frac{\pi}{2} + e \sin \tilde{\omega} + e \sin \tilde{\omega} \right) = \frac{T}{2} \left(1 + \frac{4e \sin \tilde{\omega}}{\pi} \right)$$

exactly what was found with the approximate development using the true anomaly and therefore demonstrating that the approximate formula is acceptable. We draw attention that in Milankovitch (1941) it is the longitude of the perihelion which is used (Milankovitch, 1941 English edition p240 and p246), with $\nu_0 = 180^\circ - \Pi$, leading to (Milankovitch, 1941 English edition p273) $T_s = \frac{T}{2} \left(1 + \frac{4e \sin \Pi_p}{\pi} \right)$.

Quarterly astronomical seasons

From the general formula for $T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$ (above), the lengths of the 4 quarterly astronomical seasons are given by:

$$T_k = \frac{T}{2\pi} \left[2 \operatorname{arccot} \frac{1 + e(i \cos \tilde{\omega} + j \sin \tilde{\omega})}{\sqrt{1 - e^2}} - e\sqrt{1 - e^2} \frac{i \cos \tilde{\omega} + j \sin \tilde{\omega} + e}{(1 + ei \cos \tilde{\omega})(1 + ej \sin \tilde{\omega})} \right] \quad (21)$$

$k = 1$	spring	$i = 1$	$j = 1$
$k = 2$	summer	$i = -1$	$j = 1$
$k = 3$	fall	$i = -1$	$j = -1$
$k = 4$	winter	$i = 1$	$j = -1$

These formulas give values in good agreement with those found above with, for example, for present-day: $T_{\text{spring}} = 92.8$ days and $T_{\text{winter}} = 89$ days.

From quarterly to half-year astronomical seasons

As done in Annex 3 for the general case, it can also be demonstrated that $T_{\text{Summer}} = T_{\text{spring}} + T_{\text{summer}}$:

$$\text{If } \cot x = \frac{1 + e(\cos \tilde{\omega} + \sin \tilde{\omega})}{\sqrt{1 - e^2}} \text{ and } \cot y = \frac{1 + e(-\cos \tilde{\omega} + \sin \tilde{\omega})}{\sqrt{1 - e^2}},$$

$$\cot(x + y) = \frac{e \sin \tilde{\omega}}{\sqrt{1 - e^2}}$$

And as:

$$-e\sqrt{1-e^2} \left[\frac{\cos\tilde{\omega} + \sin\tilde{\omega} + e}{(1+e\cos\tilde{\omega})(1+e\sin\tilde{\omega})} + \frac{-\cos\tilde{\omega} + \sin\tilde{\omega} + e}{(1-e\cos\tilde{\omega})(1+e\sin\tilde{\omega})} \right] =$$

$$-2e\sqrt{1-e^2} \frac{e\sin\tilde{\omega}}{1-e^2\cos^2\tilde{\omega}}$$

we obtain indeed: $T_{Summer} = \frac{T}{\pi} \left[\arccot \frac{e\sin\tilde{\omega}}{\sqrt{1-e^2}} - \frac{e\sqrt{1-e^2}\sin\tilde{\omega}}{1-e^2\cos^2\tilde{\omega}} \right]$.

3. Long-term variations of the length of the astronomical seasons and insolation

3.1. Long-term variations of the length of the quarterly astronomical seasons

As shown in (20) and (21), the length of the astronomical seasons is a function of climatic precession which varies in time, so the length of the seasons also varies in time. Over the last one million years, the length of astronomical summer varied between 82.7 and 100 days with a mean of 91.3 days (Fig. 3). This range of variations, about 20 days, is quite large, an extreme length being 10% different from the average.

When the major axis of the Earth orbit is perpendicular to the equinoctial line ($\tilde{\omega} = 90^\circ$ or 270°) (Fig. 4), the duration of the spring is equal to the duration of the summer, and the duration of the fall to that of the winter, such as the situation in 1250 CE when the winter solstice was occurring simultaneously with perigee. If these two lines coincide ($\tilde{\omega} = 0^\circ$ or 180°) (Fig. 4), the duration of the astronomical spring is equal to that of the winter and the duration of the summer is equal to that of the fall, such as the situation about 4500 years into the future when the NH spring and winter will have the same short length and consequently summer and fall will be equally long. These can also be observed in Fig. 3.

Fig. 4 and Table 2 summarize the behavior of the length of the quarterly astronomical seasons during a $\tilde{\omega}$ cycle assuming e is constant. They show:

For $\tilde{\omega} = 0^\circ$ $T_{winter} = T_{spring} < T_{summer} = T_{fall}$.
 For $\tilde{\omega} = 45^\circ$ $T_{spring} < T_{summer} = T_{winter} < T_{fall}$.
 For $\tilde{\omega} = 90^\circ$ $T_{spring} = T_{summer} < T_{fall} = T_{winter}$.
 For $\tilde{\omega} = 135^\circ$ $T_{summer} < T_{spring} = T_{fall} < T_{winter}$.

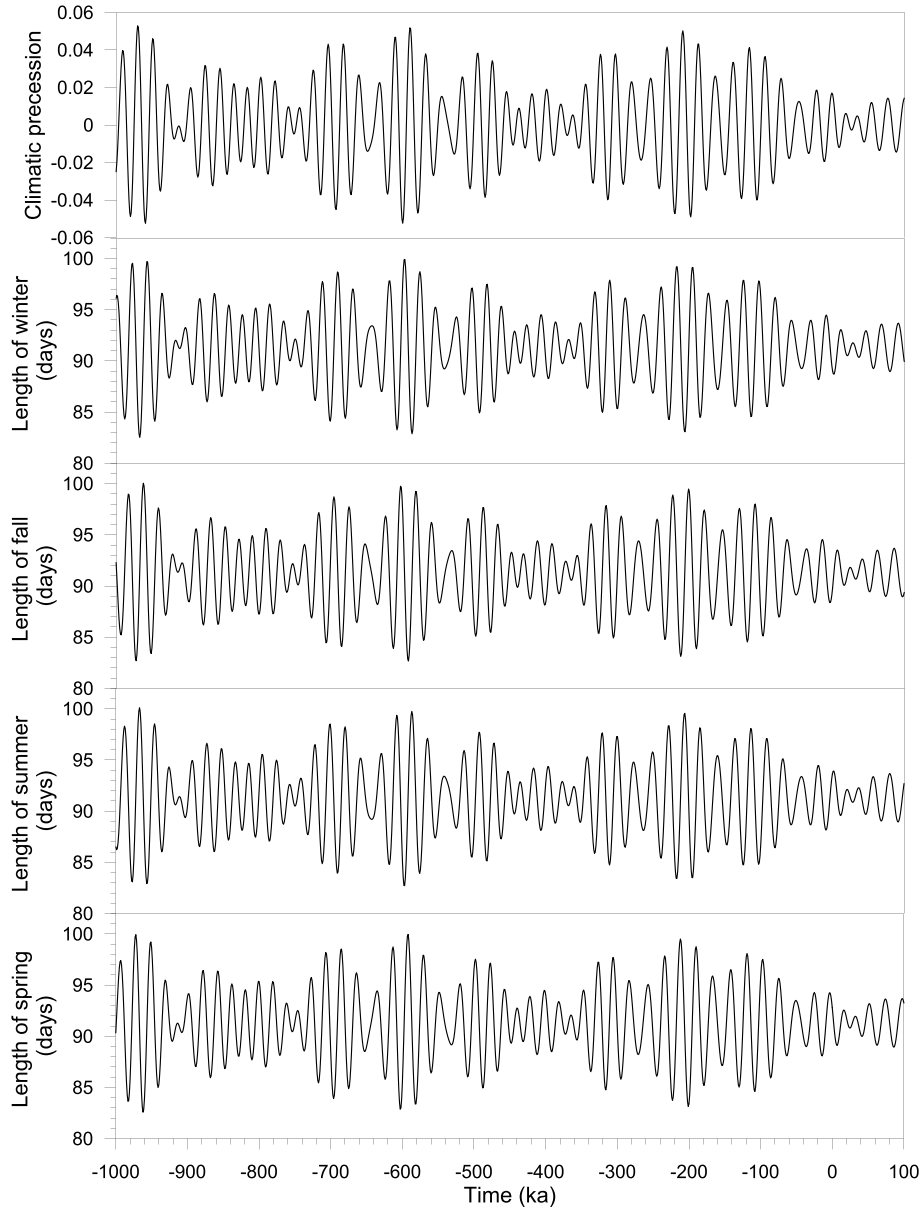


Fig. 3. Climatic precession and the length of astronomical winter, fall, summer and spring calculated based on the solution of Berger (1978a).

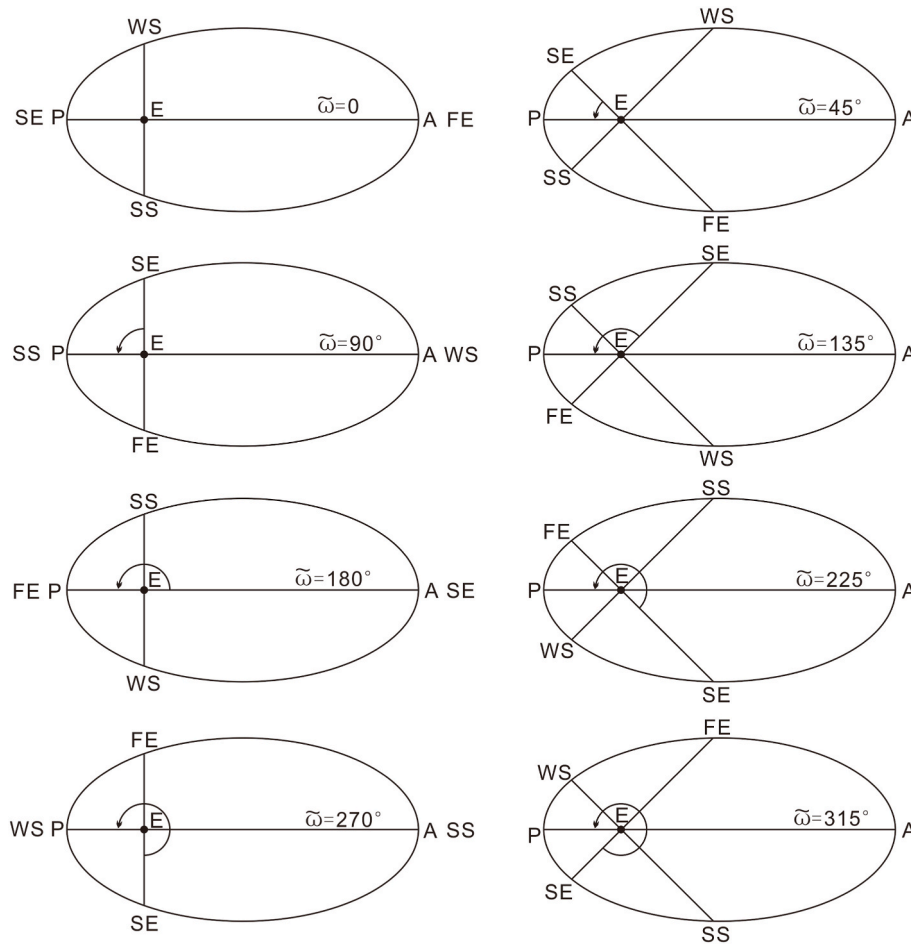


Fig. 4. Localization of the seasons for different values of the longitude of the perigee $\tilde{\omega}$. For $\tilde{\omega}$ equal to 0° , 90° , 180° , 270° , the respective following seasons occur at perigee: spring equinox, summer solstice, fall equinox and winter solstice.

Table 2

Long-term variations of the length of the quarterly astronomical seasons, T_i , during a cycle of the longitude of perigee, $\tilde{\omega}$, assuming a constant eccentricity, e . $i=1, 2, 3, 4$ denotes spring, summer, fall and winter, respectively. The values $+1$, -1 , $\sqrt{2}$ and $-\sqrt{2}$ are those of Δ defined by: $T_i = \frac{T}{4} \left(1 + \frac{4e}{\pi} \Delta \right)$ where T is the length of the tropical year. $\searrow$ and $\nearrow$ mean that the length of seasons decreases and increases between two $\tilde{\omega}$ values, respectively.

$T_i \setminus \tilde{\omega}$	0	45	90	135	180	225	270	315	360
1	-1	$\searrow$	$-\sqrt{2}$	$\nearrow$	-1	$\nearrow$	0	$\searrow$	-1
2	+1	$\searrow$	0	$\searrow$	-1	$\nearrow$	$-\sqrt{2}$	$\nearrow$	-1
3	+1	$\nearrow$	$\sqrt{2}$	$\searrow$	+1	$\searrow$	0	$\searrow$	+1
4	-1	$\nearrow$	0	$\nearrow$	+1	$\searrow$	$\sqrt{2}$	$\searrow$	+1

For $\tilde{\omega} = 180^\circ$ $T_{summer} = T_{fall} < T_{winter} = T_{spring}$.

For $\tilde{\omega} = 225^\circ$ $T_{fall} < T_{summer} = T_{winter} < T_{spring}$.

For $\tilde{\omega} = 270^\circ$ $T_{fall} = T_{winter} < T_{spring} = T_{summer}$.

For $\tilde{\omega} = 315^\circ$ $T_{winter} < T_{spring} = T_{fall} < T_{summer}$.

Two contiguous seasons can therefore have the same length only for $\tilde{\omega} = k\frac{\pi}{2}$ ($k = 0, 1, 2, 3$). $k = 0$ and 2 correspond to equinoxes at perigee (respectively spring and fall), with $T_{spring} = T_{winter}$ and $T_{summer} = T_{fall}$. $k = 1$ and 3 correspond to solstices at perigee (respectively summer and winter), with $T_{spring} = T_{summer}$ and $T_{fall} = T_{winter}$. Among these 4 particular values of $\tilde{\omega}$, the shorter length is $\frac{T}{4} \left(1 - \frac{4e}{\pi} \right)$ and the longest length is $\frac{T}{4} \left(1 + \frac{4e}{\pi} \right)$ (these are not the extreme values as shown below).

The values of $\tilde{\omega}$ for which spring and fall have the same length ($\sin \tilde{\omega} = -\cos \tilde{\omega}$), or summer and winter have the same length ($\sin \tilde{\omega} = \cos \tilde{\omega}$), are $\tilde{\omega} = (2k + 1)\frac{\pi}{4}$ ($k = 0, 1, 2, 3$); for $k = 0$ or 2, $T_{summer} = T_{winter}$

and for $k = 1$ or 3, $T_{spring} = T_{fall}$. In these four cases, the length of the quarterly astronomical seasons is exactly equal to one fourth of the tropical year.

As shown in Table 3, the minimum and maximum lengths of the four successive astronomical seasons are also reached for these values of $\tilde{\omega}$ ($\tilde{\omega} = (2k + 1)\frac{\pi}{4}$ ($k = 0, 1, 2, 3$)):

$T_{min} = \frac{T}{4} \left(1 - \frac{4\sqrt{2}}{\pi} e \right)$ for $\tilde{\omega}$ equal 45, 135, 225, 315° occurring respectively at spring ($\tilde{\omega} = 45^\circ$), summer ($\tilde{\omega} = 135^\circ$), fall ($\tilde{\omega} = 225^\circ$) and winter ($\tilde{\omega} = 315^\circ$).

$T_{Max} = \frac{T}{4} \left(1 + \frac{4\sqrt{2}}{\pi} e \right)$ for $\tilde{\omega}$ equal to 225, 315, 45, 135° occurring respectively at spring ($\tilde{\omega} = 225^\circ$), summer ($\tilde{\omega} = 315^\circ$), fall ($\tilde{\omega} = 45^\circ$) and winter ($\tilde{\omega} = 135^\circ$).

Table 3

Variations of the length of the astronomical summer and winter half-years as a function of the longitude of the perigee, $\tilde{\omega}$, assuming eccentricity and obliquity are kept constant.

Season	$\tilde{\omega}$	0	90	180	270	360
$T_s = T_{s^*}$		$T/2$	min	$T/2$	MAX	$T/2$
$T_w = T_{w^*}$		$T_s = T_w$	MAX	$T_s = T_w$	min	$T_s = T_w$
NH season contrast		$T/2$		$T/2$		$T/2$
$w_s; w_{s^*}$			MAX		min	
		$w_s = w_{s^*}$		$w_s = w_{s^*}$		$w_s = w_{s^*}$
		$w_w = w_{w^*}$		$w_w = w_{w^*}$		$w_w = w_{w^*}$
$w_w; w_{w^*}$			min		MAX	
SH season contrast						

Where

$$w_s = W_s/T_s, w_{s^*} = W_{s^*}/T_{s^*} = W_s/T_w$$

$$w_w = W_w/T_w, w_{w^*} = W_{w^*}/T_{w^*} = W_w/T_s$$

T_s = length of northern hemisphere summer half-year.

T_w = length of northern hemisphere winter half-year.

T_{s^*} = length of southern hemisphere summer half-year.

T_{w^*} = length of southern hemisphere winter half-year.

W total irradiation during a given season.

w mean irradiance during a given season.

3.2. The long-term variations of the length of the half-year astronomical seasons

About the half-year astronomical seasons, if $0 \leq \tilde{\omega} \leq 180^\circ$ (perigee moving from spring equinox to summer solstice and fall equinox) (Fig. 4), the NH summer half-year is shorter than winter; it is longer for $180 < \tilde{\omega} < 360^\circ$ (perigee moving from fall equinox to winter solstice and spring equinox).

The seasonal contrast between the length of the astronomical summer half-year and winter half-year $T_{\text{Summer}} - T_{\text{Winter}}$ is given by:

$$T_{\text{Summer}} - T_{\text{Winter}} = -T \frac{4e \sin \tilde{\omega}}{\pi}$$

For $e = 0$, this difference vanishes ($T_{\text{Summer}} = T_{\text{Winter}}$) and the contribution of climatic precession to the length of the seasons disappears.

For $\tilde{\omega} = 0$ or 180° (equinoxes at perigee), the astronomical summer half-year and winter half-year are exactly half-year long:

$$T_{\text{Summer}} = T_{\text{Winter}} = \frac{T}{2}$$

The extreme values of T_{Summer} and T_{Winter} are reached for the extreme values of $\sin \tilde{\omega}$. Leaving aside the variations of eccentricity, the minimum value for these half-year astronomical seasons is equal to:

$$T_{\text{Summer}}(\tilde{\omega} = 90^\circ) = T_{\text{Winter}}(\tilde{\omega} = 270^\circ) = \frac{T}{2} \left(1 - \frac{4e}{\pi} \right)$$

and their maximum to:

$$T_{\text{Summer}}(\tilde{\omega} = 270^\circ) = T_{\text{Winter}}(\tilde{\omega} = 90^\circ) = \frac{T}{2} \left(1 + \frac{4e}{\pi} \right)$$

The seasonal contrast in the length of the NH half-year seasons ($T_{\text{Summer}} - T_{\text{Winter}}$) is maximum for $\tilde{\omega} = 270^\circ$ (winter solstice at perigee) and equal to $\frac{4e}{\pi} T$. This seasonal contrast is minimum for $\tilde{\omega} = 90^\circ$ (summer solstice at perigee) and equal to $-\frac{4e}{\pi} T$ (summer is shorter than winter).

For large values of the eccentricity (e.g. 0.05) the departure from a half-year ($T/2$) reaches about 11 days (the difference between the half-year astronomical seasons is then more than 22 days).

Over a climatic precession cycle, we observe that:

- When $\tilde{\omega}$ increases from 0° to 90° , the summer half-year of the NH becomes shorter and shorter at the expense of the winter half-year (Fig. 4 and Table 3). Thus if we set aside the variations in the

obliquity, the total summer half-year irradiation (which is only a function of obliquity, Berger et al., 2010) would be unchanged, but distributed over shorter and shorter intervals; therefore the mean summer radiation per unit time would increase from year to year. The opposite applies for the winter; its length becomes longer and its daily radiation less intense.

- When $\tilde{\omega}$ reaches the value of 90° , the length of the NH summer half-year is very near to its minimum (Fig. 4, Table 3) (again, the minimum does not coincide strictly with $\tilde{\omega} = 90^\circ$ because of the variability of e). The mean summer radiation rises therefore to its maximum (again assuming obliquity is kept constant) and the mean winter radiation drops to its minimum. The seasonal contrast is at its maximum. This however applies only for the NH. In the SH, the local summer half-year (which coincides with the winter half-year of the NH) has reached its maximum; the quantities of total solar radiation W_{s^*} which is received during this long SH summer by the southern latitudes are equal to those which the northern latitudes receive during their short northern summer. In the SH consequently, the summer will be long and cool, the winter short and warm. At the period when the seasonal contrasts in the NH achieve their greatest intensity, they undergo their largest reduction in the southern one (Table 3).

- When $\tilde{\omega}$ reaches the value of 180° , T_s and T_w (T_s for NH summer half-year and T_w for NH winter half-year) are equal and the mean irradiances are equal in both hemispheres during their local season: $W_s/T_s = W_{s^*}/T_{s^*}$.

- At $\tilde{\omega} = 270^\circ$ the length of the NH summer half-year has reached its maximum, that of the winter half-year has dropped to its minimum. The seasonal contrasts are tempered by a long and cool summer and a short and mild winter. In the SH, the contrasts experience their greatest intensification.

- At $\tilde{\omega} = 360^\circ$ the lengths of the seasons are equal and, as for $\tilde{\omega} = 180^\circ$ the mean solar irradiances received during the local seasons are the same in both hemispheres. Everything then starts again as above, with the difference that, because of the variability of e , the seasonal contrasts in the following cycle during which $\tilde{\omega}$ again grows from 0 to 360° are not equal to those of the preceding period. In the interval of increasing e , the contrasts will be either more intense or more reduced; and when e decreases, the maximum and minimum will be less pronounced.

The variations of the irradiation of the seasons due to obliquity will

however overlap those due to e and $\tilde{\omega}$ related to their length (Berger et al., 2010). As the variations of ε have a period different from those of e and $\tilde{\omega}$ and have the same impact in both hemispheres during the same local season (but with a different degree according to the latitudes), the variations of the seasonal irradiation become more complicated to document. Milankovitch (1941) tried to overcome this difficulty by introducing the caloric seasons.

3.3. Long-term variations of the insolation during astronomical and caloric seasons

Fig. 5 shows the long-term variations of the total irradiation over the astronomical summer half-year ($0^\circ \leq \lambda < 180^\circ$) for different latitudes

between the North and South Poles. As predicted by the calculation of Berger et al. (2010), they are dominated by obliquity at all latitudes except at 11.5°N where the obliquity signal disappears and only an eccentricity signal is left although with a weak amplitude of variations. North of 11.5°N , these long-term variations are positively correlated with obliquity, while south of 11.5°N , they are negatively correlated with obliquity. As far as the astronomical half-year winter ($180^\circ \leq \lambda < 360^\circ$) (Fig. 5), which is the half-year summer of the SH, is concerned, the variations of the total irradiation over the local summers and winters of the two hemispheres separated by 11.5°N and 11.5°S respectively are symmetric. Note that this special latitude where the obliquity signal disappears and where the phase between total irradiation and obliquity changes is a function of the interval over which the

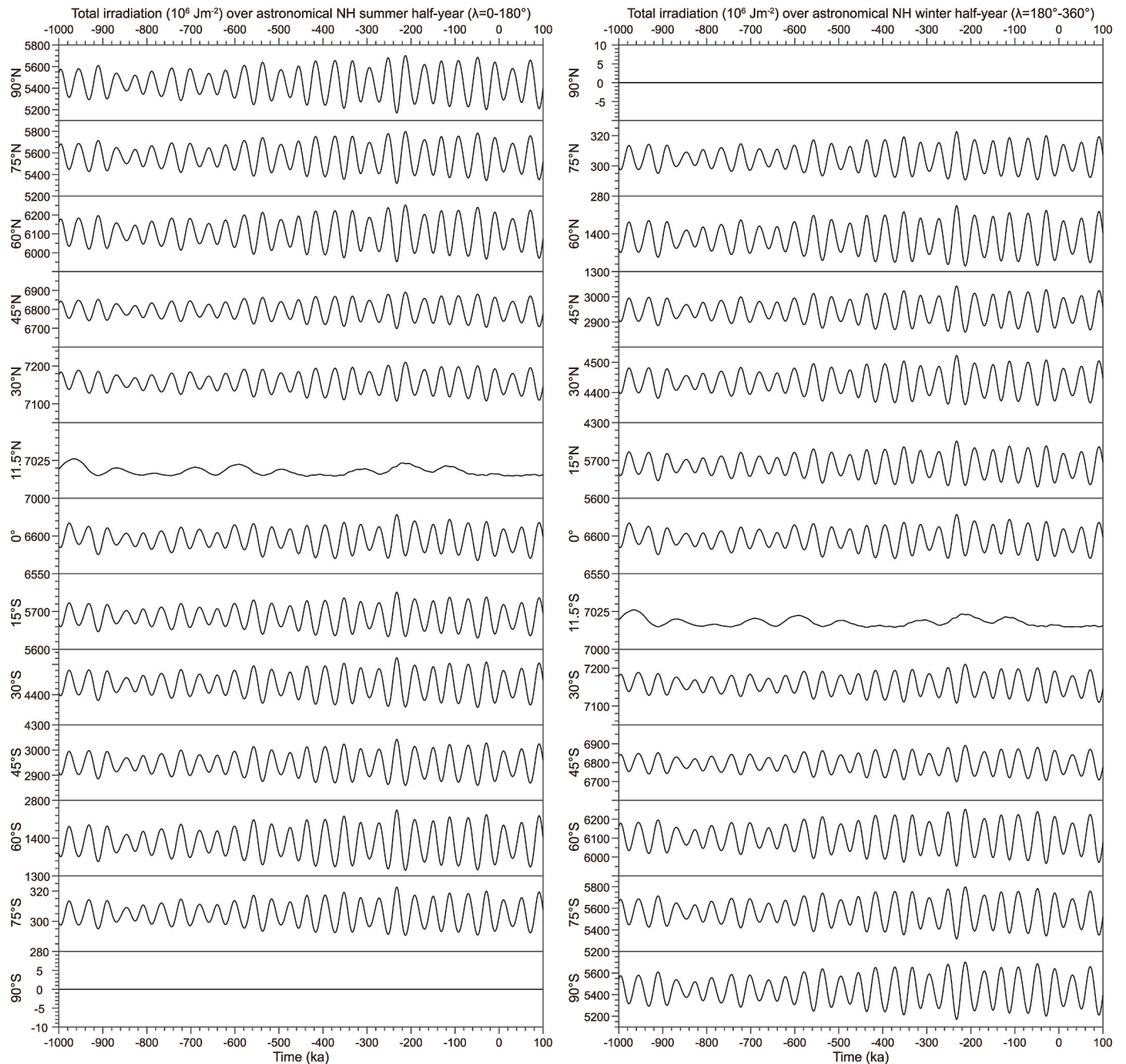


Fig. 5. Long-term variations of the total irradiation (10^6 J/m^2) over the astronomical NH summer half-year (SH winter half-year) ($\lambda = 0-180^\circ$) (left) and NH winter half-year (SH summer half-year) ($\lambda = 180-360^\circ$) (right) for latitudes between the North and South Poles. The total irradiation is calculated using the elliptical integrals of Berger et al. (2010), and the astronomical solution of Berger and Loutre (1991) is used.

total irradiation is calculated. For example, it is 13.5°N for a shorter summer defined by $60^\circ \leq \lambda < 120^\circ$ and it is 44°N/S for total annual insolation (Berger et al., 2010).

If the quarterly astronomical seasons are considered, the properties of the elliptical integrals used in Berger et al. (2010) to compute the total irradiation W at a latitude φ over an interval going from the true longitude λ' to λ'' must be taken into account and lead to:

$$W(\varphi, \lambda', \lambda'') = W(-\varphi, \pi + \lambda', \pi + \lambda'')$$

Therefore, for latitudes and seasons of the NH, we obtain:

$$W_{spring} = W_{summer} = \frac{W_{Summer}}{2} \quad \text{and} \quad W_{fall} = W_{winter} = \frac{W_{Winter}}{2}$$

with similar relationships for the SH. As a consequence, the pattern of the variations of the quarterly seasons is the same as for the half-year seasons.

Given the distinctly different variations of the total irradiation of the half-year seasons north and south of 11.5°N(S), we select 65°N, 11.5°N and 75°S to illustrate the variations of the mean astronomical summer half-year insolation and of the caloric summer insolation (Figs. 6–8). 65°N is the classical latitude used by Milankovitch in the understanding of the climate response to insolation forcing, and 75°S is the latitude of

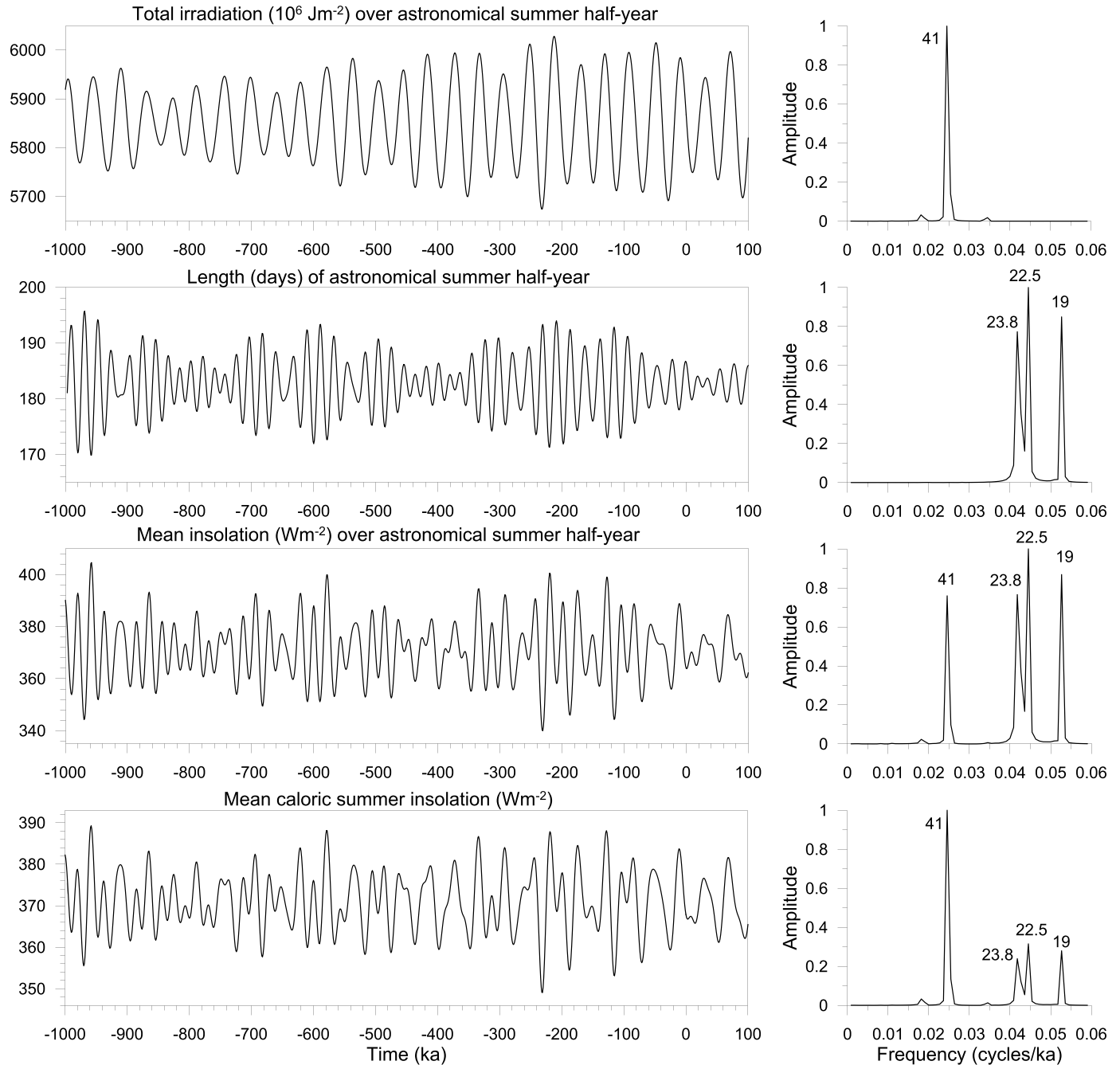


Fig. 6. From top to bottom: Total irradiation, length and mean insolation of the astronomical summer half-year, the mean caloric summer insolation, and their spectra, all for 65°N. The total irradiation is calculated using the elliptical integral of Berger et al. (2010); the length of the astronomical summer half-year is calculated using Equation (20); the caloric summer insolation is calculated based on Berger (1978b); the astronomical solution of Berger and Loutre (1991) is used. The major periodicities are indicated in thousands of years.

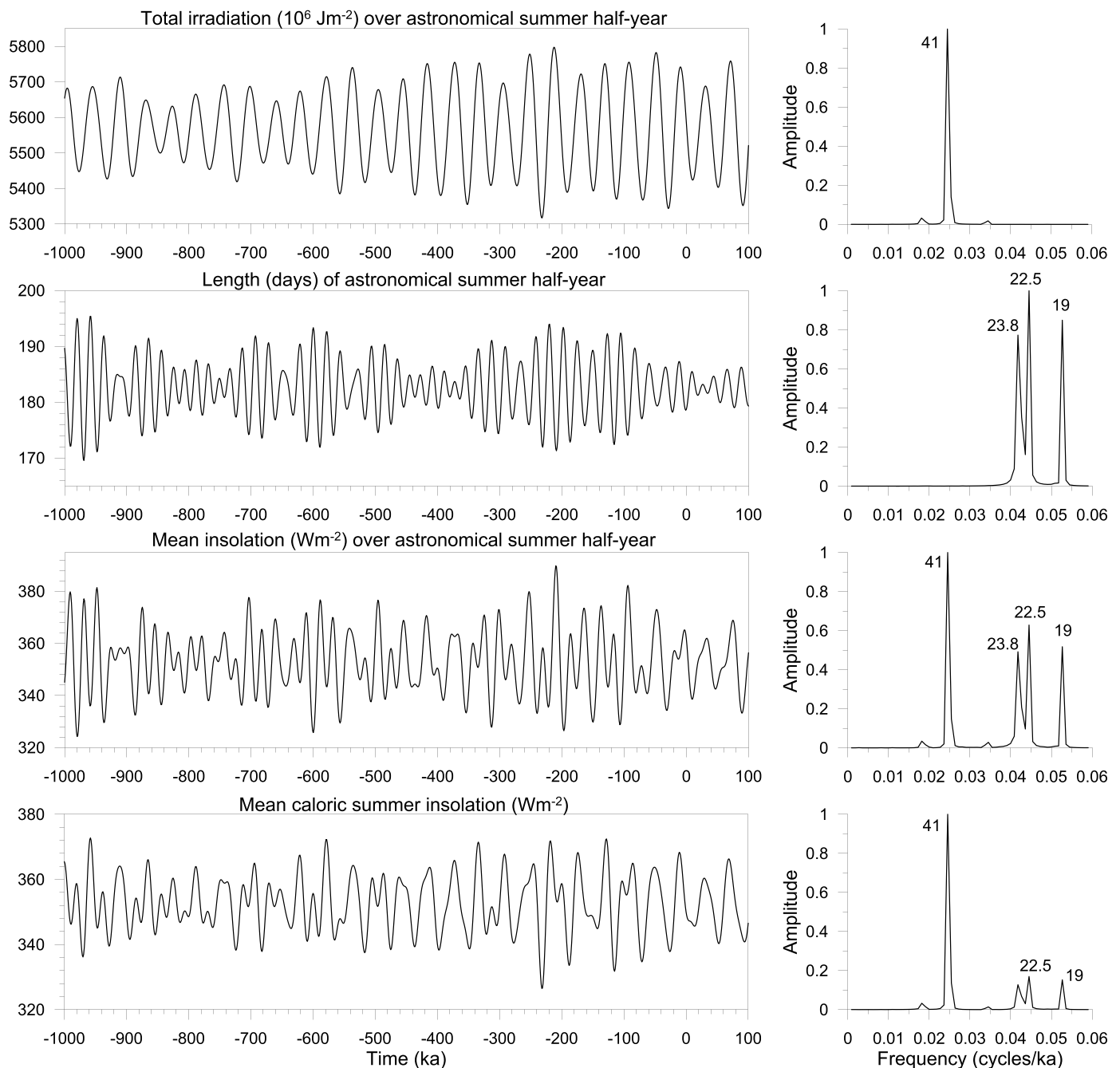


Fig. 7. Same as Fig. 6 but for 75°S, and local season is referred.

the EPICA Dome C ice core. The mean astronomical summer half-year insolation is obtained by dividing the total astronomical summer half-year irradiation by the length of the astronomical summer half-year. As both the total irradiation and the length of astronomical seasons vary in time, to accommodate this duality, Milankovitch introduced the caloric seasons which length do not vary in time. The calculation of the caloric summer and winter insolation as well as their long-term variations at all latitudes in both hemispheres shows that the caloric insolation is mainly a function of climatic precession in low latitudes and of obliquity in high latitudes (Berger, 1978b). Here the caloric summer insolation is compared with the mean insolation of the astronomical summer half-year for the three selected latitudes.

As the total summer irradiation at 65°N is dominated by obliquity and the length of astronomical summer half-year is only a function of climatic precession, the mean insolation of the astronomical summer

half-year is expectedly co-controlled by obliquity and climatic precession. This mean summer insolation shows an equal power in the 41-ka and ~21-ka periodicities if the variations over the past 1 Ma and the future 100 ka are considered (Fig. 6). In comparison, the caloric summer insolation for the same time interval contains stronger obliquity and weaker climatic precession signals (Fig. 6). If local summer is considered, the total and mean insolation at 75°S show similar variations as at 65°N, except that the mean summer insolation and the caloric summer insolation contain stronger obliquity signal due to a higher latitude (Fig. 7). Completely different from 65°N and 75°S, at 11.5°N, as the obliquity signal disappears in the total summer irradiation where only the eccentricity signal is left, the variations of the mean summer insolation show only a climatic precession cycle (Fig. 8), which is the same for the caloric summer insolation as already shown in Berger (1978b).

In Figs. 6–8, the spectra of the variations between 1 Ma BP and 100

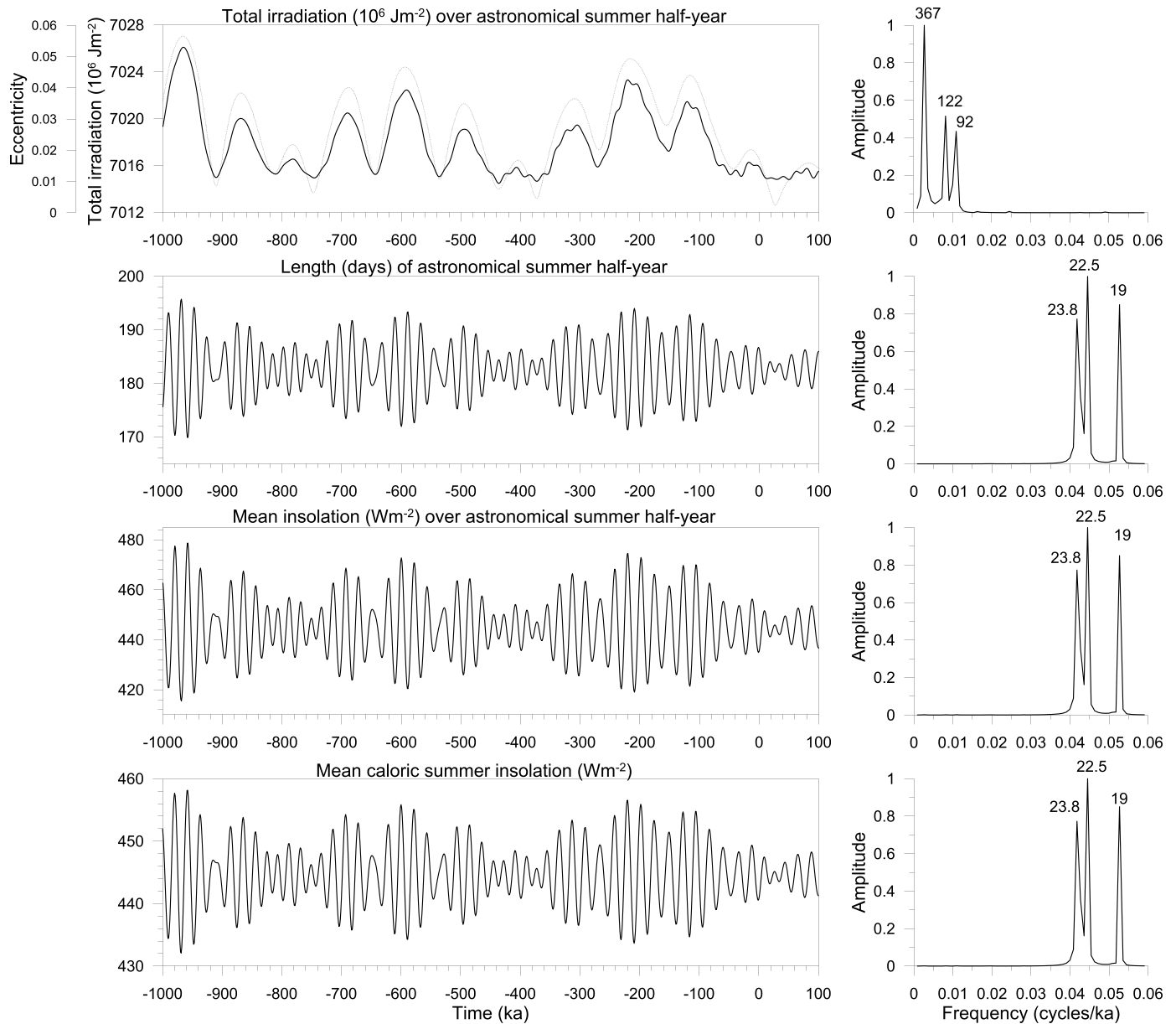


Fig. 8. Same as Fig. 6, but for 11.5°N, and eccentricity (dash) is additionally added in top panel.

ka are shown. However, it is worth noting that the characteristics of the spectra and the relative weight of climatic precession and obliquity vary not only in latitude but also in time. The variations of the mean insolation at the corresponding latitudes in the two hemispheres during their local summer or winter show similar spectral characteristics (Fig. 9). Obliquity has more weight in high latitudes than in low latitudes in both summer and winter. In the summer hemisphere, its signal already gets very weak at 45°N/S and almost disappears between 30°N and 30°S with a complete disappearance at 11.5°N/S (Fig. 9). In the winter hemisphere, the obliquity signal is stronger than at the corresponding latitudes of the summer hemisphere, but the energy received is much lower (Fig. 5).

In time, the relative weight of climatic precession and obliquity in the mean seasonal insolation strongly depends on their amplitude of variations. Due to the ~1.3-Ma cycle of the amplitude modulation of obliquity (Berger et al., 1998), the amplitude of obliquity variations is smaller before ~450 ka BP than after, as far as the time interval between 1 Ma BP and 100 ka AP is concerned. This leads generally to a more important contribution of climatic precession than of obliquity in the

mean seasonal insolation variations before ~450 ka BP at all latitudes and significantly enhanced contribution of obliquity after ~450 ka BP in high latitudes (Fig. 9). As this time corresponds to the timing of the so-called Mid-Brunhes Event (MBE) or Mid-Brunhes Transition (MBT, Yin, 2013), the distinct change of the relative contributions of climatic precession and obliquity in the mean seasonal insolation variations at ~450 ka BP could have contributed to the occurrence of this climate transition. The contribution of climatic precession is also more important than obliquity in the mean summer insolation after ~450 ka BP except at the very high latitudes such as 75°N/S and the Poles and also except during ~350–450 ka BP and 0–100 ka AP, two periods which are characterized by small variations of climatic precession due to small eccentricity but large variations of obliquity (Fig. 9). During these two periods, obliquity dominates the variations of the mean summer insolation between 60°N and 90°N and between 60°S and 90°S.

The interval around 800 ka BP is also characterized by small variation of climatic precession like at around 400 ka BP and the future 100 ka due to the ~400-ka cycle of eccentricity, but differently, it is also characterized by small variation of obliquity. This explains the almost

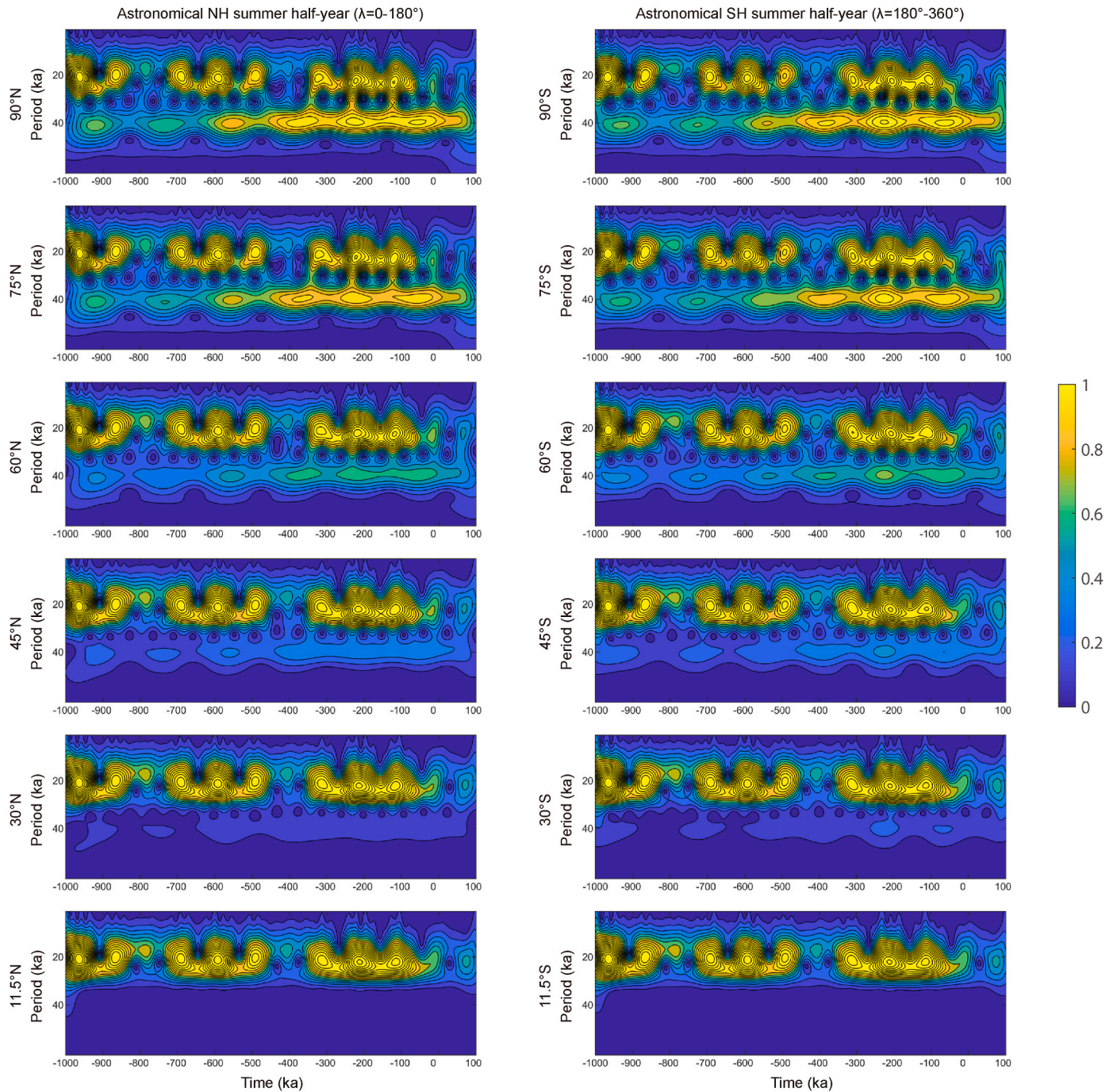


Fig. 9. Wavelet analysis of the mean insolation over the astronomical summer half-year of the NH (left) (which is winter for the SH latitudes) and the summer half-year of the SH (which is winter for the NH latitudes).

equal contributions of climatic precession and obliquity at 75°N/S and 90°N/S around 800 ka BP but the more important contribution of climatic precession at 60°N/S. The difference in the relative contributions of climatic precession and obliquity between ~800 ka BP and ~400 ka BP has been used to explain the different sub-precessional climate and vegetation variations between the MIS-19 and MIS-11 interglacials (Su et al., 2023). Both MIS-11 and MIS-19 have been suggested to be good analogues of the present interglacial MIS-1 and its future due to small value of eccentricity which leads to small variations of climatic precession (e.g. Berger and Loutre, 2002; Loutre and Berger, 2003; Tzedakis et al., 2009; Yin and Berger, 2015). However, if the relative contribution of climatic precession and obliquity on the mean summer insolation is considered and if the total irradiation over a season is considered,

MIS-11 remains a decent analogue of the present interglacial and the next 100 ka, but MIS-19 is not as good due to its smaller variation of obliquity. The simultaneous weakening of the climatic precession and obliquity variations around 800 ka BP leads to weak variations in the total and mean insolation over seasons. In the meantime, abnormality in the glacial conditions between 900 and 700 ka BP has been suggested by proxy records from Asia (Guo et al., 1998) and Europe (Sanchez Goñi et al., 2023). Studies based on marine sediments also suggest that the time interval at ~900 ka BP was marked by an abrupt increase in glacial ice volume and/or a reorganization of deep-ocean circulation, which could be accompanied by a perturbation in global carbon cycle and major ice sheet changes in Antarctica or Eurasia (Elderfield et al., 2012; Hodell and Channell, 2016; Ford and Raymo, 2020). This period

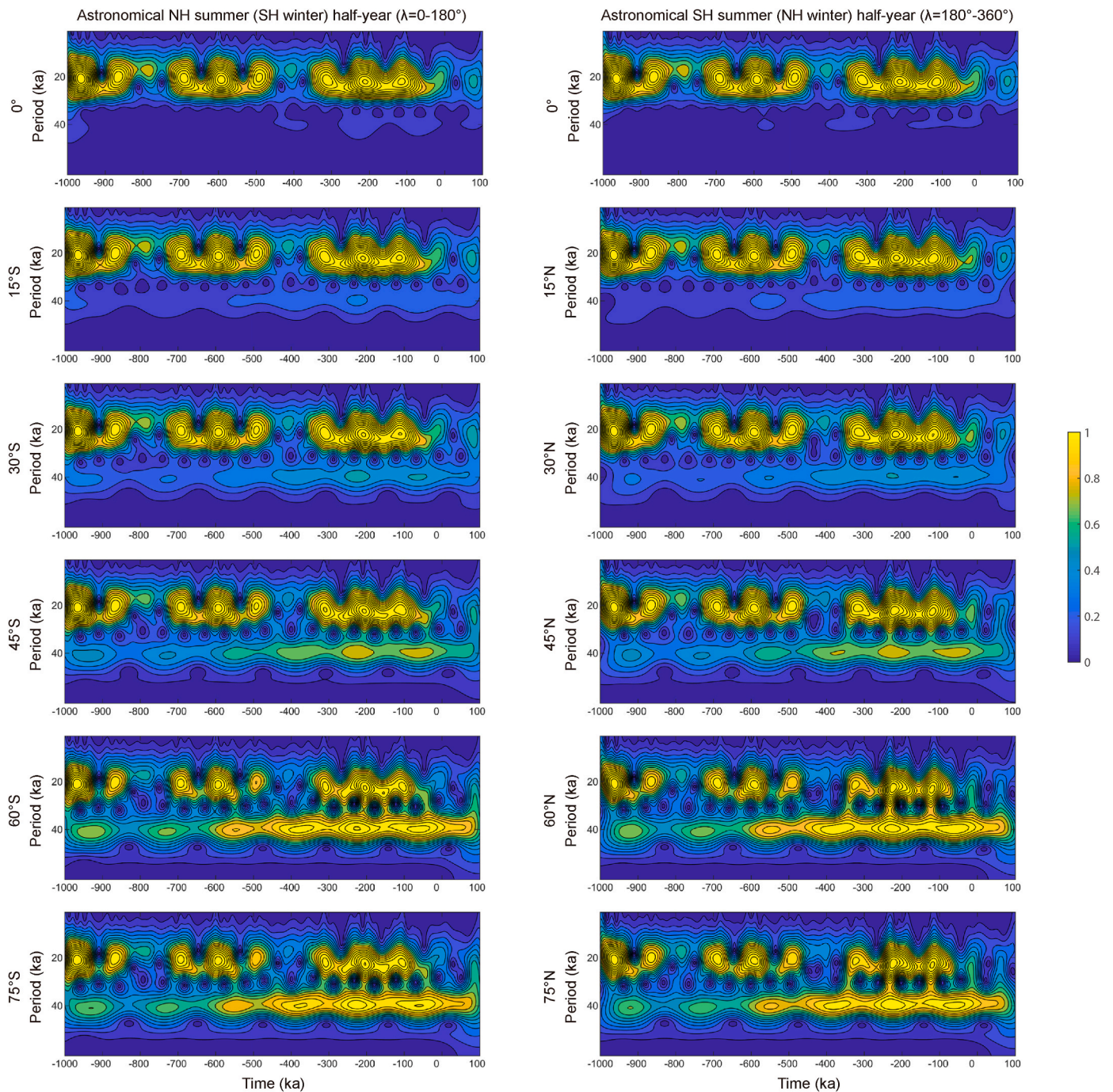


Fig. 9. (continued).

corresponds to the Mid-Pleistocene Transition (MPT), so to which extent this transition could be related to the simultaneous weakening of the climatic precession and obliquity variations deserves further investigation.

4. Variations of different insolation metrics over the Holocene

Roughly twelve thousand years ago, just after the last glacial period, the climate entered the current warm period named the Holocene interglacial (MIS-1). Proxy-based reconstructions generally show a maximum warming in early-mid Holocene, the so called Holocene Thermal Maximum or Holocene Climatic Optimum, which is followed by a cooling trend during the late Holocene until the Industrial

Revolution and the present-day global warming. Although the late-Holocene cooling trend was confirmed by simulations with some Earth system models of intermediate complexity (EMICs) such as MoBiDiC (Crucifix et al., 2002), Ecilt-Clio-Vecode (Renssen et al., 2006) and LOVECLIM1.3 (Askjær et al., 2022) which have a relatively low climate sensitivity to CO_2 , it has been questioned by simulations with many general circulation models (GCMs) (e.g. Liu et al., 2014; Askjær et al., 2022; Wanner et al., 2008; Wanner, 2021 and more references in there). The cooling trend shown by proxy records and the abovementioned EMICs is consistent with the decreasing summer insolation in the NH, suggesting the major role of orbital forcing relative to CO_2 which slowly increases by ~ 20 ppmv from the mid-Holocene to pre-industrial time, whereas the warming trend simulated by many GCMs seems to suggest

the major role of the slow increase of CO_2 . This is also indicated by another LOVECLIM simulation where the model did simulate a Holocene warming trend after the climate sensitivity to CO_2 was artificially increased (Liu et al., 2014). Therefore, a critical factor that determines the Holocene climate trend would be the relative climate sensitivity to orbital forcing and CO_2 . In literature, the insolation that is mostly used to compare with the reconstructed or simulated Holocene climate is the daily insolation at June or December solstice, or annual mean insolation, with other insolation metrics seldomly being paid attention. As other insolation metrics might also be important for some components of the climate system, in this section we give an analysis of the evolution of several insolation metrics during the Holocene in hoping to provide a broader view on insolation change during this period.

Fig. 10 shows the evolutions of several insolation metrics as a function of latitudes and time. At summer (winter) solstice, the NH (SH) extra-tropical latitudes receive the maximum daily insolation during a year. From the early to late Holocene, the daily insolation at summer solstice continuously decreases at all latitudes especially in the NH (Fig. 10a). This has often been used to explain the late-Holocene cooling trend. On the contrary, the daily insolation at winter solstice increases from the early to late Holocene at all latitudes (Fig. 10b). However, the variation pattern is totally different if the total irradiation over the astronomical local summer half-year is considered for both hemispheres (Fig. 10c). As demonstrated in Berger et al. (2010), this total irradiation is only a function of obliquity and its variation is the same in the NH and SH extra-tropical latitudes. Due to the decreasing obliquity from the early to late Holocene, the total summer irradiation also decreases in the extra-tropical latitudes of both hemispheres with the fastest decrease occurring in high latitudes but very small change in the tropics. Differently, the total irradiation over the astronomical winter half-year increases from the early to late Holocene in both hemispheres with the largest change occurring in the mid latitudes (Fig. 10d). If the length of the season is also considered, the mean summer half-year insolation in the NH latitudes decreases from the early to late Holocene, but it increases in the SH (Fig. 10e), and the opposite happens in the mean winter half-year insolation (Fig. 10f). Being a function of only obliquity and due to the decreasing obliquity, the total annual irradiation

decreases from the early to late Holocene in mid and high latitudes of both hemispheres with the largest change in polar regions, but a slow and small increase in the low latitudes (Fig. 10g).

Therefore, in terms of the trend from the early to late Holocene, different insolation metrics behave differently. Although the Milankovitch theory proposes that the Quaternary glacial cycles were primarily driven by the summer insolation at northern high latitudes (Milankovitch, 1941), how different components of the climate system respond to changes of insolation is still far from being well understood. For example, speleothem records indicate that the low-latitude hydroclimate is more sensitive to low-latitude insolation and climatic precession (e.g. Cheng et al., 2022). Some studies suggest that the Antarctica climate is controlled by NH summer insolation (e.g. Kawamura et al., 2007) whereas some suggest that it is more controlled by local summer insolation (e.g. Raymo et al., 2006; Raynaud et al., 2007) or local summer duration (Huybers and Denton, 2008). A recent study combining ice core records and model simulations indicates that the Antarctica summer temperature is highly correlated with the local mean insolation over the astronomical austral summer half-year (Raynaud et al., 2023). Model results show that the NH high-latitude climate is more sensitive to climatic precession whereas the SH climate is more sensitive to obliquity (Yin and Berger, 2012, 2015; Wu et al., 2020).

If different climate variables and regions have different sensitivity to different insolation, they may exhibit different trend during the Holocene in response to orbital forcing. For example, those which are more sensitive to the daily insolation at summer solstice, the total summer insolation or total annual insolation in polar regions, would show a cooling trend; those which are more sensitive to total winter insolation in the mid-latitudes or to total annual insolation in the tropics, would show a warming trend. These could be further investigated with climate models.

5. Summary and conclusion

The calculation and long-term variations of the daily insolation which is used to force climate models and is often used in the explanation of paleoclimate records are well known (Berger, 1978a; Berger and Loutre, 1991; Laskar et al., 2004). The total irradiation received during

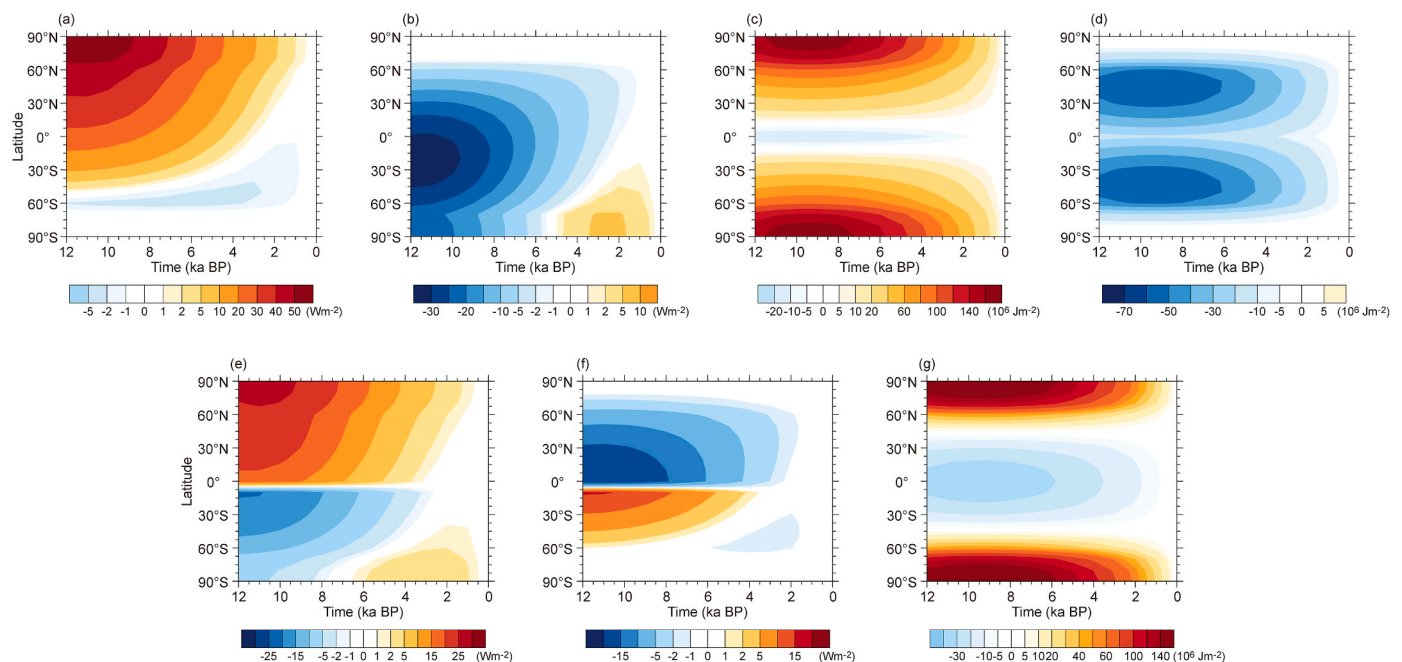


Fig. 10. Evolution of different types of insolation during the Holocene. (a) Daily insolation at summer solstice (Wm^{-2}), (b) daily insolation at winter solstice (Wm^{-2}), (c) total insolation over the astronomical local summer half-year (10^6 Jm^{-2}), (d) total insolation over the astronomical local winter half-year (10^6 Jm^{-2}), (e) mean insolation over the astronomical local summer half-year (Wm^{-2}), (f) mean insolation over the astronomical local winter half-year (Wm^{-2}), and (g) total annual insolation (10^6 Jm^{-2}). All are deviations from their present-day values, and are calculated by using the astronomical solution of Berger (1978a).

any time interval of the year has also been calculated by using elliptic integrals and its long-term variations are demonstrated being mainly a function of obliquity (Berger et al., 2010). In addition to the total energy, the length of the interval during which the energy is received could also be important for the climate system. In this paper, an accurate calculation of the length of any time interval of the year is given by using the eccentric anomaly. Specifically, the calculation of the length of the quarterly and half-year astronomical seasons is provided for any given values of eccentricity, obliquity and climatic precession. It shows that the length of the astronomical seasons is only a function of climatic precession. Over the last one million years, the length of astronomical summer varied between 82.7 and 100 days, with the range of variations being quite significant. The difference between the maximum and minimum length of the quarterly seasons amounts only about 5 days over the next 100,000 years due to the small value of eccentricity and therefore small variations of climatic precession over that period. The mean insolation over a season being calculated by dividing the total insolation received over the season by the length of the season, is thus a function of both obliquity and climatic precession.

The daily insolation, total insolation and mean insolation have different relationship with climatic precession, obliquity and eccentricity and therefore show different spectral characteristics. The variations of the daily insolation are dominated by climatic precession (except close to the polar night), with obliquity playing a secondary role but more important in high than in low latitudes (Berger et al., 1993). In contrast, the total insolation over the NH astronomical summer half-year is dominated by obliquity at all latitudes except at 11.5°N where it contains only an eccentricity signal. It is positively (negatively) correlated with obliquity north (south) of 11.5°N.

As far as the mean insolation of the astronomical summer half-year is concerned, its spectral characteristics vary in both latitude and time (Fig. 9). Obliquity has more weight in high latitudes than in low latitudes, but its signal already gets very weak at 45°N/S leaving the absolute dominance of climatic precession between these two latitudes, and almost disappears between 30°N and 30°S with a complete disappearance at 11.5°N/S. In time, the relative weight of climatic precession and obliquity in the mean summer insolation strongly depends on their amplitude of variations. Before ~450 ka BP, the variation of obliquity is weak due to the ~1.3 Ma cycle of its amplitude modulation, climatic precession is therefore more important in the variations of the mean summer insolation at all latitudes. During the time periods of small eccentricity, which leads to small variation of climatic precession, and large variation of obliquity, such as during ~350–450 ka BP and 0–100 ka AP, obliquity dominates the variations of the mean summer insolation between 60°N and 90°N and between 60°S and 90°S. The variations of climatic precession and obliquity simultaneously get weakened around 800 ka ago, leading to weak variations in both total and mean season insolation. Whether this has an effect on Mid-Pleistocene Transition could be further investigated by climate models.

Different metrics related to insolation and astronomical forcing exist, such as the caloric summer half-year and summer solstice insolation (Milankovitch, 1941), daily insolation (Berger et al., 1993), total insolation over any time interval of the year (Berger et al., 2010), integrated

daily insolation above a threshold (Huybers, 2006), latitudinal insolation gradient (Yin, 2013), and the length of astronomical seasons and mean summer insolation presented in this study. They have different contributions from climatic precession and obliquity and therefore exhibit distinct long-term variations and spectral characteristics. In the analysis and interpretation of paleoclimate records, the choice of the astronomical metrics often implies a hypothesis on the relationship between the records and insolation. For instance, if summer solstice insolation is chosen, the records are assumed to be sensitive to maximum summer insolation; if mean summer insolation is chosen, the record is assumed to be sensitive to both the total insolation received during the astronomical summer and the length of summer.

In terms of the Holocene that is taken here as an example, different insolation metrics have different trend over the Holocene. Therefore, depending on the sensitivity to different insolation metrics, different components in the climate system and/or different regions could show different trend in response to astronomical forcing. In general, slow components such as deep ocean, ice sheets and soil development would be more affected by cumulated energy and thus total or integrated insolation, while fast components such as Arctic sea ice would be more affected by maximum summer insolation, and vegetation would be affected by the energy received during the growing season and the length of growing season. However, which component of the climate system is more sensitive to which astronomical metric is not that evident. This is not only because of the highly variable nature of insolation in seasons and in latitudes, but also due to the complex interactions within the climate system and the possible nonlinear response of a climate component/variable to insolation. Combining modelling and proxy studies would contribute to a better understanding of this complex topic.

Author contribution

A. Berger made the mathematical formulations and Fortran programs to calculate the length of astronomical seasons and insolation; A. Berger and Q.Z. Yin designed the study, analyzed the data and drafted the manuscript with the contribution of Z.P. Wu.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data submitted as supplements.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at <https://doi.org/10.1016/j.quascirev.2024.108620>.

Annex 1: Kepler Equation

The mean anomaly is related to the eccentric anomaly, E , by the Kepler equation (Fig. 1) as explained here.

In Fig. 1, the abscissa of M is $(\overline{OL})$: $x = a \cos E$ and the ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ leads to $y = \overline{OH} = b \sin E$. As $\overline{OH} = \overline{LM} = b \sin E$ and $\overline{LM} = \overline{OG} \sin E$, $\overline{OG} = b$ and G belongs to the circle of radius b .

As we also have $\frac{\overline{LM}}{\overline{LM}_1} = \frac{\overline{OG}}{\overline{OM}_1} = \frac{b}{a}$, the area of the sectors $PEarthM$ of the ellipse and $PEarthM_1$ of the circle of radius a are in the same proportion: $\text{area } PEarthM = \frac{b}{a} \text{ area } PEarthM_1$.

Moreover

$\text{Area } PEarthM_1 = \text{area } POM_1 (\pi a^2 \frac{E}{2\pi}) - \text{area } EarthOM_1$ (triangle with base ae and height $\overline{LM}_1$) $= \frac{1}{2}a^2E - \frac{1}{2}ae a \sin E$.

Therefore, $\text{area } PEarthM = \frac{1}{2}ab (E - e \sin E)$ and $\frac{\text{area } PEarthM}{\text{area ellipse}} = \frac{E - e \sin E}{2\pi}$.

The areal velocity being $\frac{\pi ab}{T}$, if t is the instant when the planet passes through M and τ the time of its perigee passage, the Second Law of Kepler leads to $\text{area } PEarthM = \frac{\pi ab}{T} (t - \tau)$, $\frac{\text{area } PEarthM}{\text{area ellipse}} = \frac{t - \tau}{T}$.

Defining the mean anomaly M by $\frac{M}{2\pi} = \frac{\text{area } PEarthM}{\text{area ellipse}}$, we have $M = n(t - \tau)$.

This is the angle described by the radius vector going from M to P in an interval $(t - \tau)$ with a constant angular velocity $n = \frac{2\pi}{T}$, the mean angular velocity of the planet on its orbit.

Finally

$$\frac{2\pi}{T} (t - \tau) = E - e \sin E, n(t - \tau) = E - e \sin E, M = n(t - \tau) = E - e \sin E \quad (7)$$

which is called the Kepler's equation.

Annex 2: $T_{\lambda_1, \lambda_1 + \pi}$ as a function of ψ

If ψ is defined by $2\psi = E_2 - E_1$ as in Berger (1973) (doctoral dissertation) and in Berger (1975) (Annales Soc Sc Brux), we obtain a simpler expression. Indeed, from (17b), we have

$$\tan \psi = \tan \left(\frac{E_2 - E_1}{2} \right) = \frac{-\sqrt{1 - e^2}}{e \sin(\lambda_1 - \tilde{\omega})}$$

From $\sin \psi = \frac{\tan \psi}{\sqrt{1 + \tan^2 \psi}}$ (times the sign of $\tan \psi$), we have

$$\sin \psi = \frac{\frac{-\sqrt{1 - e^2}}{e \sin(\lambda_1 - \tilde{\omega})}}{\sqrt{1 + \frac{1 - e^2}{e^2 \sin^2(\lambda_1 - \tilde{\omega})}}} \text{ (times the sign of } \tan \psi \text{ or of } (-\sin \nu_1) \text{)}$$

and $\sin \psi = \frac{\sqrt{1 - e^2}}{\sqrt{1 - e^2 \cos^2 \nu_1}}$, as $e \sin \nu_1$ simplifies from numerator and denominator with its sign.

From $\cos \psi = \frac{\cot \psi}{\sqrt{1 + \cot^2 \psi}}$ which for $0 \leq \psi \leq \pi$ gives $\sin \psi$ always positive, we have

$$\cos \psi = \frac{-e \sin \nu_1}{\sqrt{1 - e^2 \cos^2 \nu_1}}$$

Therefore with $\sin 2\psi = 2 \sin \psi \cos \psi$, $\sin 2\psi = \sin(E_2 - E_1) = \frac{-2e\sqrt{1 - e^2} \sin \nu_1}{1 - e^2 \cos^2 \nu_1}$.

From (17a), we obtain $\sin(E_2 - E_1) = e(\sin E_2 - \sin E_1)$ and finally

$$T_{\lambda_1, \lambda_1 + \pi} = \frac{T}{2\pi} (\pi - 2\psi - \sin 2\psi)$$

where ψ is given by $\tan \psi = \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}}$.

Notice that sign of $\tan \psi$ is the sign of $-\sin \nu_1$:

If $0 \leq \lambda_1 - \tilde{\omega} \leq \pi$ the sign is $-$

If $\pi \leq \lambda_1 - \tilde{\omega} \leq 2\pi$ the sign is $+$

If $\lambda_1 - \tilde{\omega} \leq 0, 0 \leq 2\pi + \lambda_1 - \tilde{\omega} \leq \pi$ the sign is $-$

If $\lambda_1 - \tilde{\omega} \leq 0, \pi \leq 2\pi + \lambda_1 - \tilde{\omega} \leq 2\pi$ the sign is $+$

Later ψ was also defined by: $E_2 - E_1 = \pi - 2\psi$ which is equivalent to $\tan \psi = \cot \frac{E_2 - E_1}{2}$ and because $0 \leq E_2 - E_1 \leq 360^\circ$, we have $-\pi \leq 2\psi \leq \pi$, $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$ and $\cos \psi$ is always positive. In this case:

From (17b): $\tan \psi = \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}}$.

From $\sin \psi = \frac{\tan \psi}{\sqrt{1 + \tan^2 \psi}}$ we have: $\sin \psi = \frac{-e \sin \nu_1}{\sqrt{1 - e^2 \cos^2 \nu_1}}$.

From $\cos \psi = \frac{1}{\sqrt{1 + \tan^2 \psi}}$ we have: $\cos \psi = \frac{\sqrt{1 - e^2}}{\sqrt{1 - e^2 \cos^2 \nu_1}}$.

From $\sin 2\psi = 2 \sin \psi \cos \psi$ we have: $\sin 2\psi = \sin(E_2 - E_1) = \frac{-2e\sqrt{1 - e^2} \sin \nu_1}{1 - e^2 \cos^2 \nu_1}$.

And from (17a): $\sin(E_2 - E_1) = e(\sin E_2 - \sin E_1)$, and finally:

$$T_{\lambda_1, \lambda_1 + \pi} = \frac{T}{2\pi} (\pi - 2\psi - \sin 2\psi)$$

where ψ is given by $\tan \psi = \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}}$ and $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$.

Annex 3: From $T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$ to $T_{\lambda_1, \lambda_1 + \pi}$

In general we can add the components of $T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$ and of $T_{\lambda_1 + \frac{\pi}{2}, \lambda_1 + \pi}$ to obtain $T_{\lambda_1, \lambda_1 + \pi}$:

For $T_{\lambda_1, \lambda_1 + \frac{\pi}{2}}$:

$$\sin E_2 - \sin E_1 = \sqrt{1 - e^2} \frac{\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}) + e}{(1 + e \cos(\lambda_1 - \tilde{\omega}))(1 - e \sin(\lambda_1 - \tilde{\omega}))} = \sqrt{1 - e^2} \frac{C - S + e}{(1 + eC)(1 - eS)}$$

where $S = \sin(\lambda_1 - \tilde{\omega})$ and $C = \cos(\lambda_1 - \tilde{\omega})$.

$$\text{For } T_{\lambda_1 + \frac{\pi}{2}, \lambda_1 + \pi} : \sin E_2 - \sin E_1 = \sqrt{1 - e^2} \frac{\cos(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}) - \sin(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}) + e}{(1 + e \cos(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}))(1 - e \sin(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}))} = \sqrt{1 - e^2} \frac{-S - C + e}{(1 - eS)(1 - eC)}$$

This gives, when added, the value of $\sin E_2 - \sin E_1$ for the interval $(\lambda_1, \lambda_1 + \pi)$:

$$\sin E_2 - \sin E_1 = \sqrt{1 - e^2} 2 \frac{-S(1 - eS)}{(1 - eS)(1 - e^2 C^2)} = -2\sqrt{1 - e^2} \frac{\sin(\lambda_1 - \tilde{\omega})}{1 - e^2 \cos^2(\lambda_1 - \tilde{\omega})}$$

As we have also

$$\arccot \frac{1 + e(\cos(\lambda_1 - \tilde{\omega}) - \sin(\lambda_1 - \tilde{\omega}))}{\sqrt{1 - e^2}} + \arccot \frac{1 + e(\cos(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}) - \sin(\lambda_1 + \frac{\pi}{2} - \tilde{\omega}))}{\sqrt{1 - e^2}}$$

which leads for the interval $(\lambda_1, \lambda_1 + \pi)$ to:

$$\arccot \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}}$$

We obtain finally: $T_{\lambda_1, \lambda_1 + \pi} = \frac{T}{\pi} \left[\arccot \frac{-e \sin(\lambda_1 - \tilde{\omega})}{\sqrt{1 - e^2}} + \sqrt{1 - e^2} \frac{e \sin(\lambda_1 - \tilde{\omega})}{1 - e^2 \cos^2(\lambda_1 - \tilde{\omega})} \right]$ which is as (18).

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