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The unreasonable effectiveness of dimension reduction in complex geophysical flow modelling

How to reduce the number of degrees of freedom of a complex model by a factor of 10^3 or larger

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(with many thanks to Chris Mooers, Anne Mouchet, François Primeau and Jia Wang)

The article of Eugene P. Wigner

- In 1960 Princeton University physicist Eugene P. Wigner authored an article that did not go unnoticed:

The unreasonable effectiveness of mathematics
in the natural sciences

(Communications on Pure and Applied Mathematics, XIII, 1-14)

- Wigner argued that the astonishing efficiency of the mathematical modelling of natural phenomena must reflect some deeper, albeit somewhat elusive, truth about mathematics and physics.

- Though the ambition of the present study is much more modest than Wigner's, a similar statement holds true here too: dimension reduction works well, but the reason thereof has yet to be found.

Dimension reduction

- Consider a system described by N ordinary differential equations:

$$\frac{d}{dt}\psi_n(t) = f_n(t,\vec{\psi}) \ , \quad n = 1,2...N, \quad \text{with} \ \ \vec{\psi} = (\psi_1,\psi_2...\psi_N)$$

Let $g_i(t,\vec{\psi})$, $i = 1,2,...I$, with $I << N$, be a series of indicators of the state of the system.

- Consider another model of the same system with less equations:

$$\frac{d}{dt}\tilde{\psi}_m(t) = \tilde{f}_m(t,\vec{\tilde{\psi}}) \ , \quad m = 1,2...M, \quad \text{with} \ \ \vec{\tilde{\psi}} = (\tilde{\psi}_1,\tilde{\psi}_2...\tilde{\psi}_M)$$

with $M < N$. Then, estimate the same indicators $\tilde{g}_i(t,\vec{\tilde{\psi}})$, $i = 1,2,...I$.

Dimension reduction is efficient if
$M << N$ **and** $\tilde{g}_i(t,\vec{\tilde{\psi}}) \approx g_i(t,\vec{\psi})$

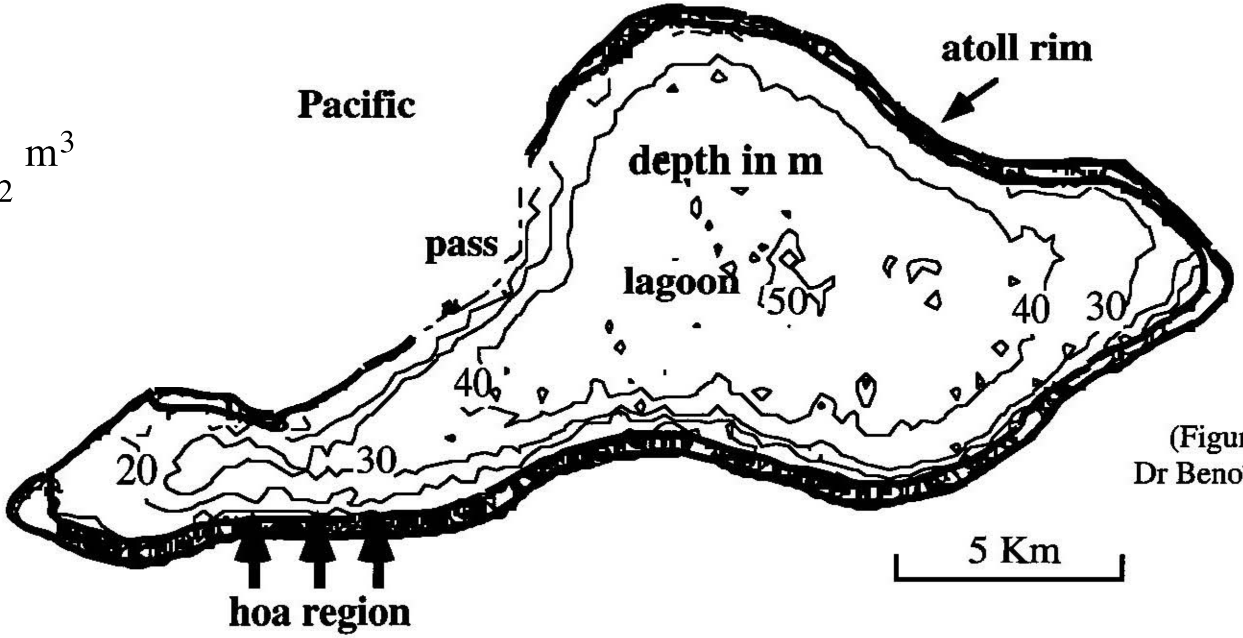
Applications

- Herein, no theoretical developments on dimension reduction; only applications.

- Problems dealt with:
  - Tracer transport from the Mururoa Atoll lagoon to the Pacific
  - Tracer transport in the Prince William Sound, Alaska
  - Ventilation of the World Ocean
  - Timescale to tracer equilibrium in the World Ocean

Mururoa Atoll lagoon (I)

Mouth/Pass:
length: ≈ 5 km
depth: ≈ 8 m
Lagoon:
volume: $4.7 \times 10^9 \ \mathrm{m}^3$
area: $1.4 \times 10^8 \ \mathrm{m}^2$
mean depth: 33 m



(Figure courtesy Dr Benoît Tartinville)

Mururoa Atoll is part of the Tuamotu Archipelago (French Polynesia) and is located in the Pacific at (longitude, latitude) = (138°55' E, 21°50° S).

- How to model the flux of (radioactive) tracers to the Pacific through the atoll mouth?

Mururoa Atoll lagoon (II)

- B. Tartinville developed a complex 3D model for the lagoon hydrodynamics and tracer transport, with N=33,510 wet grid points.

Residence time*:

blue particle: 13 days

yellow particle: 38 days

red particle: > 50 days

green particle: > 50 days

*: time needed to leave
the lagoon
by crossing the pass

50-day numerical simulation

(Figure courtesy
Dr Benoît Tartinville)

4 particles released
at the same time

- This suggests that an intense mixing is taking place in the lagoon.

Mururoa Atoll lagoon (III)

- The tracer budget of the lagoon can be described as follows:

$$\frac{d}{dt}m(t) = \underbrace{\phi_{in}(t) - \phi_{out}(t)}_{\text{transport}} \underbrace{- \gamma m}_{\substack{\text{radioactive}\\ \text{decay}}} , \quad m(t) = \text{tracer mass}$$

with $\phi_{in(t)}$: incoming flux (through lagoon bottom)

$\phi_{out(t)}$: outgoing flux (through the atoll mouth)

$\log 2/\gamma$: half-life of the tracer

- Key hypothesis: mixing throughout the lagoon is infinitely efficient.

$$\boxed{\Rightarrow \quad \phi_{out} = \frac{m}{\theta}}$$

where θ is the mean residence time, a timescale that is to be estimated by means of the 3D model.

Mururoa Atoll lagoon (IV)

- Then, the evolution of the tracer mass may be modelled by a “box model” consisting of a single ordinary differential equation (M=1):

$$\frac{dm}{dt} + \frac{m}{\theta^*} = \phi_{in} \quad , \quad \text{with} \;\; \theta^* = \frac{\theta}{1+\gamma\theta}$$

General solution: $m(t) = m(0)\, e^{-t/\theta^*} + \int_0^t \phi_{in}(t')\, e^{-(t-t')/\theta^*} \, dt'$.

- How well does this mass estimate compare with that computed by means of the 3D model $m^{3D}(t)$?

- Instantaneous, pointwise release of a unit mass of a passive tracer:

$$m(t) = e^{-t/\theta} \quad \text{and} \quad m^{3D}(t) = \int_\Omega C(t,\mathbf{x}) \, d\mathbf{x} \quad \text{with } C(0,\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_0)$$

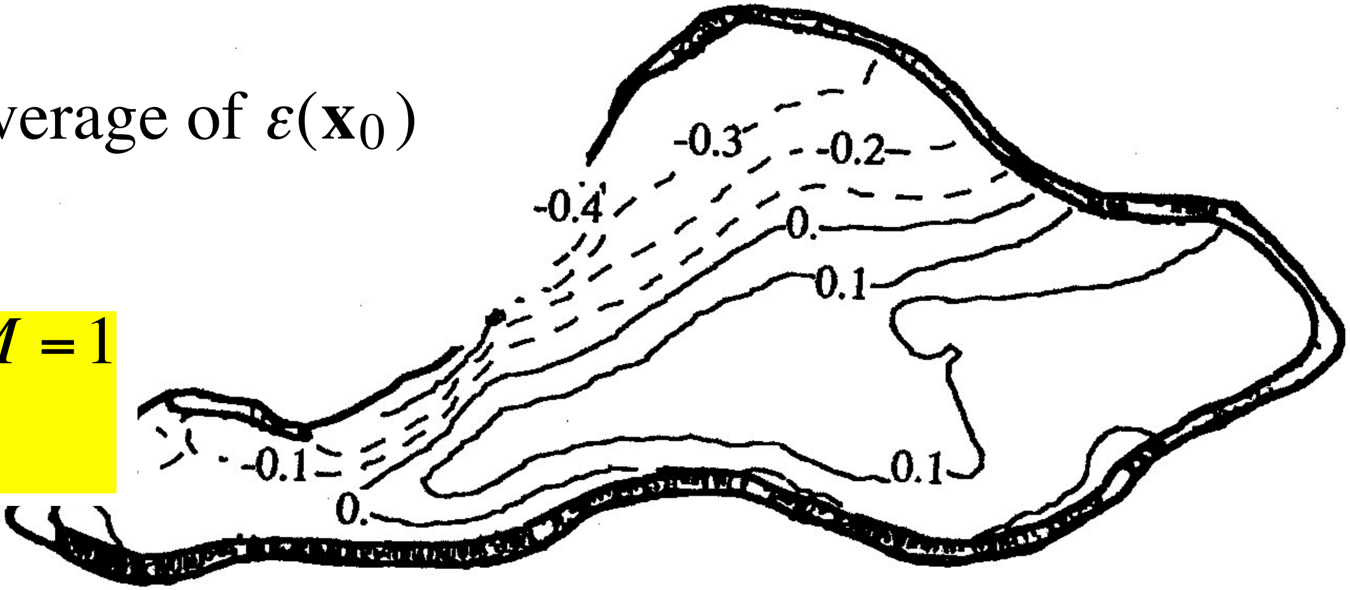
Mururoa Atoll lagoon (V)

- Error measure: $\varepsilon(\mathbf{x}_0) = \frac{1}{\theta}\int_0^{\infty}[m^{3D}(t) - m(t)]\, e^{-t/\theta}\, dt$

depth-average of $\varepsilon(\mathbf{x}_0)$

$N = 33{,}510$ and $M = 1$

$|\varepsilon| \approx 12\%$

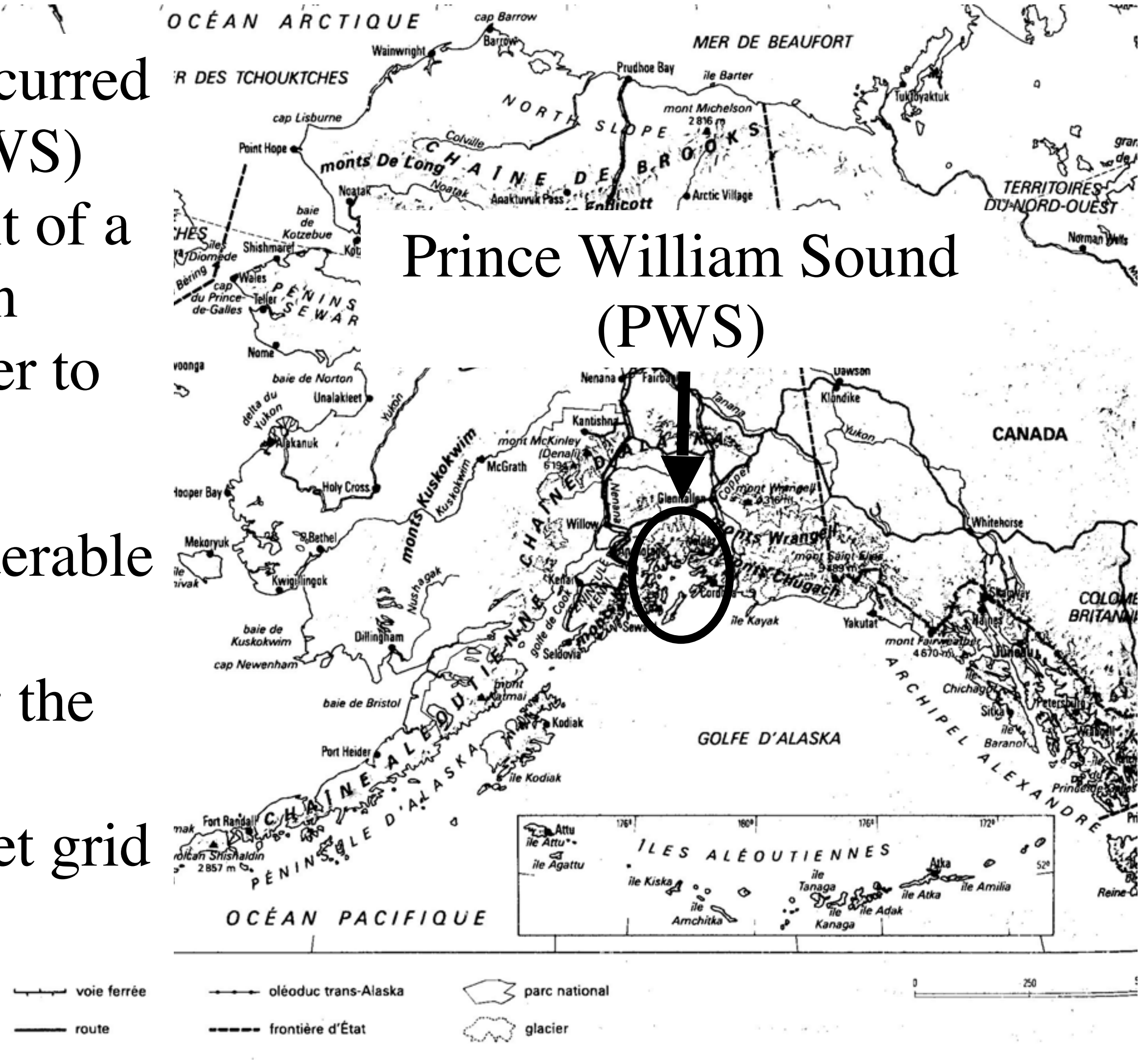


As $|\varepsilon(\mathbf{x}_0)| \approx 0.12$, the one-equation box model (M=1) provides a good estimate of the tracer mass — the indicator of the state of the system — simulated by means of a complex 3D model (N=33,510).

Prince William Sound, Alsaka (I)

America's largest oil spill occurred in Prince William Sound (PWS) on 24 March 1989, as a result of a navigation error of the Exxon Valdez, causing this oil tanker to run onto Bligh Reef.

Then, during the 90's, considerable efforts were devoted to the modelling of the PWS, using the 3D Princeton Ocean Model (POM), with $N = 83,762$ wet grid points.

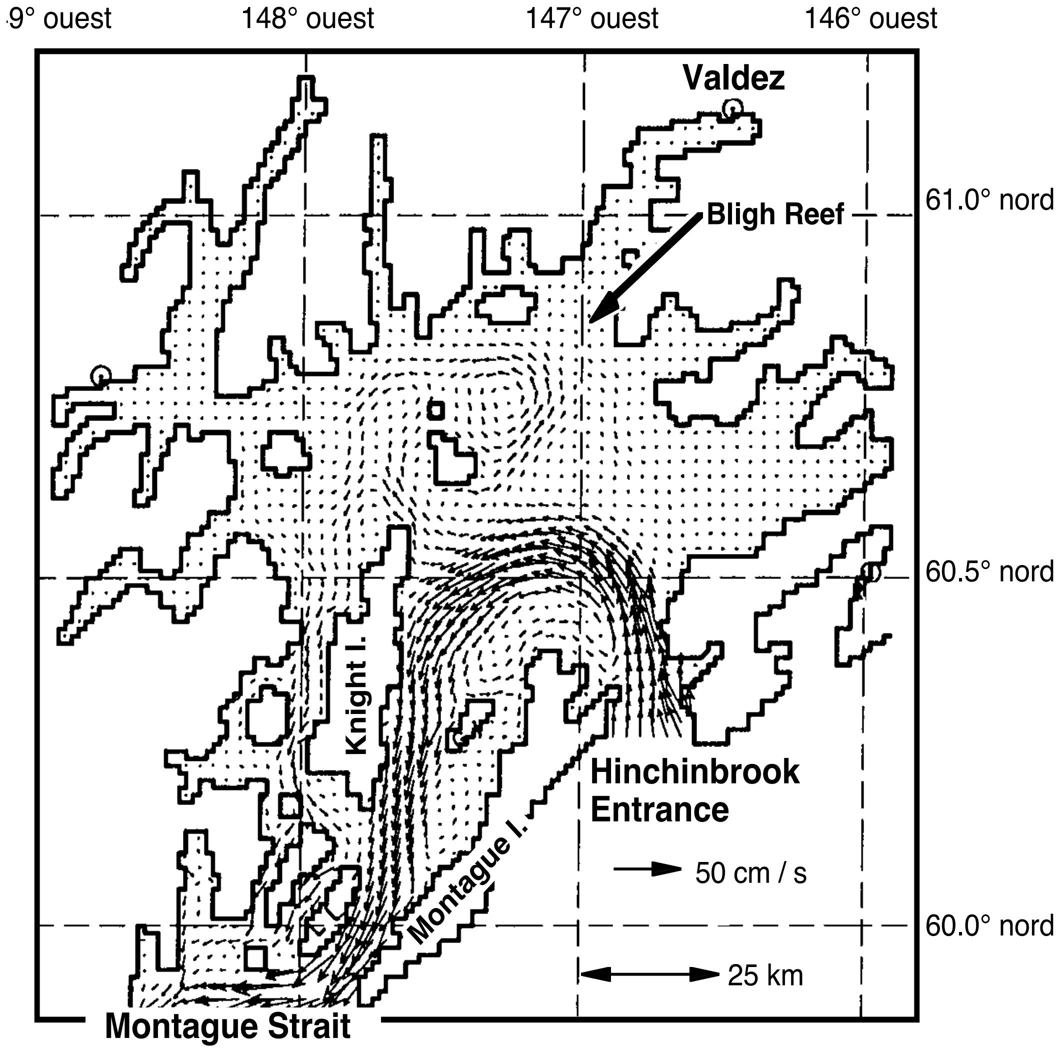


Prince William Sound, Alsaka (II)

Surface current in the PWS as simulated by POM in the zero-wind case (Mooers and Wang, 1998)

The hydrodynamics of the southern part of the domain is clearly different from that prevailing in the northern part.

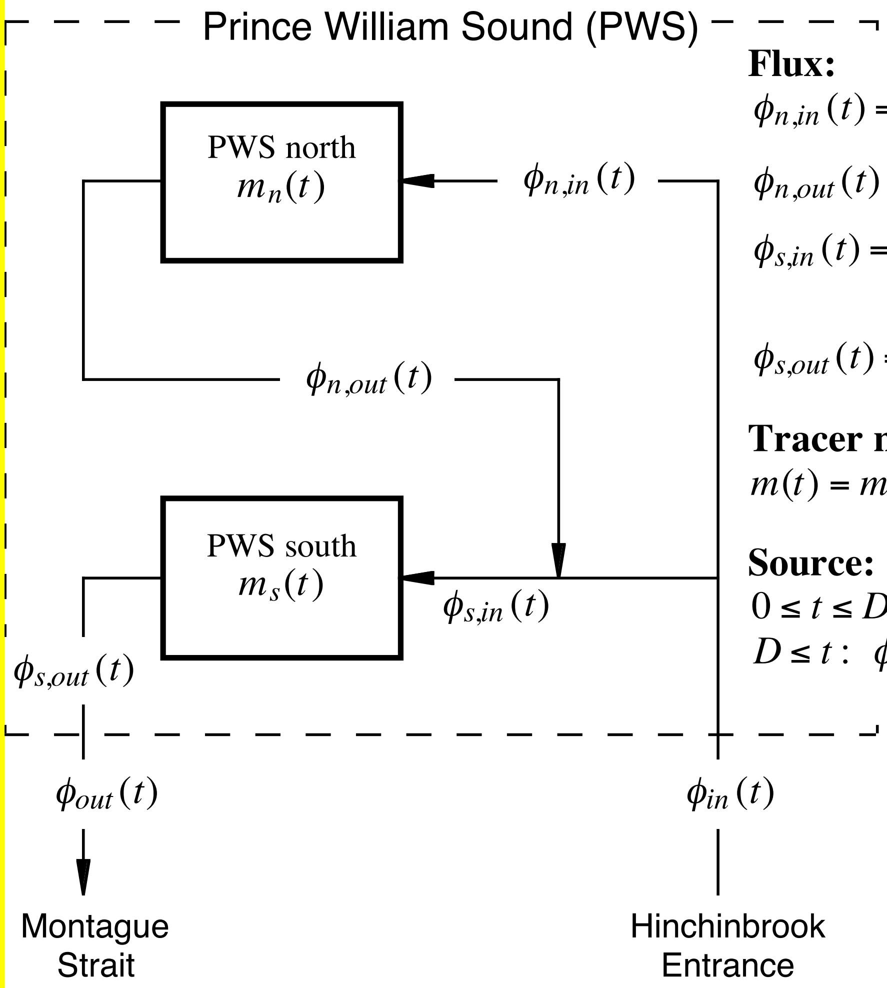
This suggest a way for achieving dimension reduction: dividing PWS into 2 boxes, i.e. northern PWS and southern PWS.



Prince William Sound, Alsaka (III)

Key hypotheses:

- Northern box is well-mixed, i.e. similar to Mururoa lagoon, with residence time θ.
- Southern box behaves as a pipe: $\phi_{s,out}(t) = \phi_{s,in}(t-\tau)$, with $\tau =$ transit time.
- $\alpha =$ fraction of the incoming flux at Hinchinbrook Entrance that enters directly the northern box.



Flux:

$$\phi_{n,in}(t) = \alpha\phi_{in}(t)$$

$$\phi_{n,out}(t) = \frac{m_n(t)}{\theta}$$

$$\phi_{s,in}(t) = (1-\alpha)\phi_{in}(t) + \phi_{n,out}(t)$$

$$\phi_{s,out}(t) = \phi_{s,in}(t-\tau)$$

Tracer mass:

$$m(t) = m_n(t) + m_s(t)$$

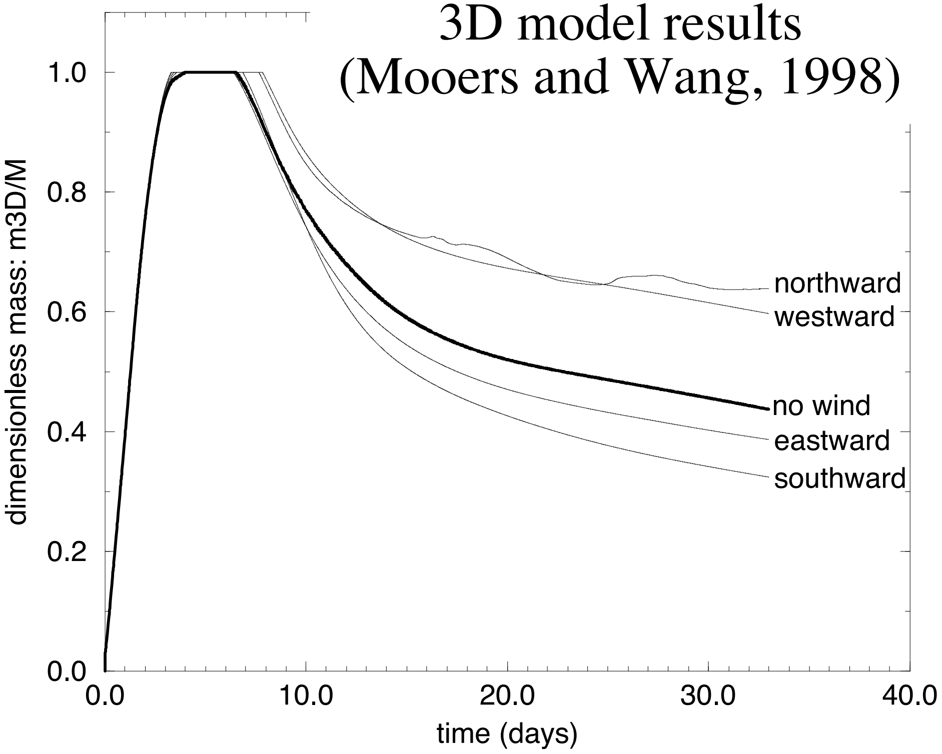
Source:

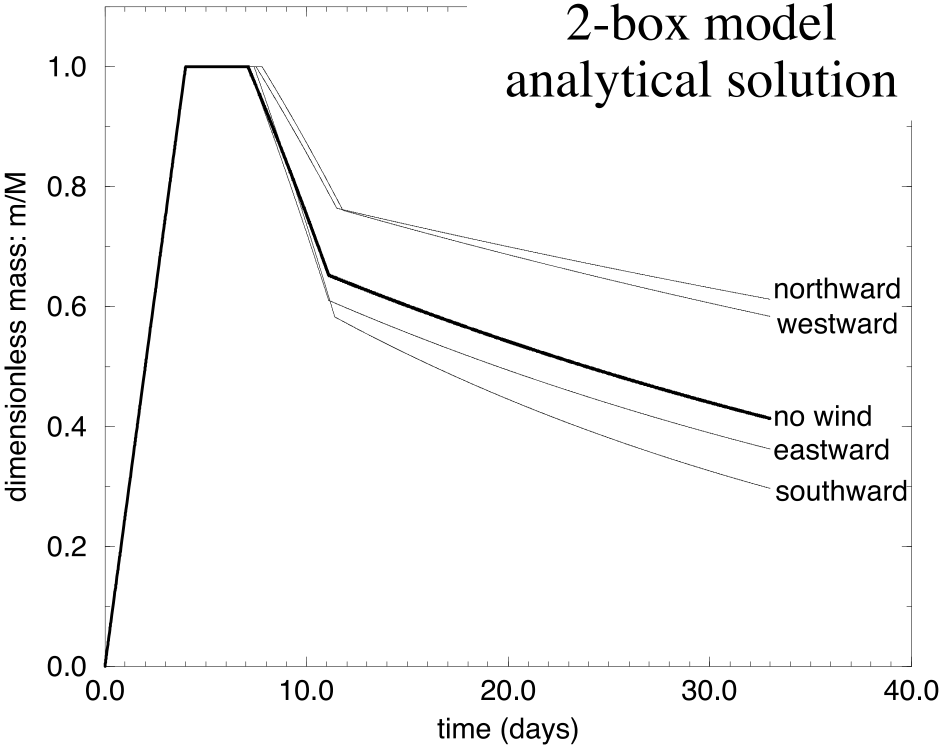
$0 \le t \le D: \ \phi_{in}(t) = \Phi$

$D \le t: \ \phi_{in}(t) = 0$

Passive tracer source in Hinchinbrook Entrance active for $0 \le t \le D = 4$ days.

Prince William Sound, Alsaka (IV)


3D model results
(Mooers and Wang, 1998)


2-box model
analytical solution

Parameters α, θ and τ have a clear physical meaning

$$0 \le t \le D: \quad m(t) = \Phi t$$

$$D \le t \le \tau: \quad m(t) = \Phi D = M$$

$$\tau \le t \le D + \tau: \quad m(t) = \Phi(D + \tau - t) + \alpha \Phi \theta [1 - e^{-(t-\tau)/\theta}]$$

$$D + \tau \le t: \quad m(t) = \alpha \Phi \theta (e^{D/\theta} - 1) e^{-(t-\tau)/\theta}$$

Prince William Sound, Alsaka (V)

Parameters α, θ and τ are determined so as to minimize the error measure:

$$\varepsilon = \Phi D(t_{\max})^{-1/2} \left[\int_0^{t_{\max}} [m^{3D}(t) - m(t)]^2 \, dt \right]^{1/2}$$

$N = 83{,}762$ and $M = 2$

$\varepsilon \approx 7\%$

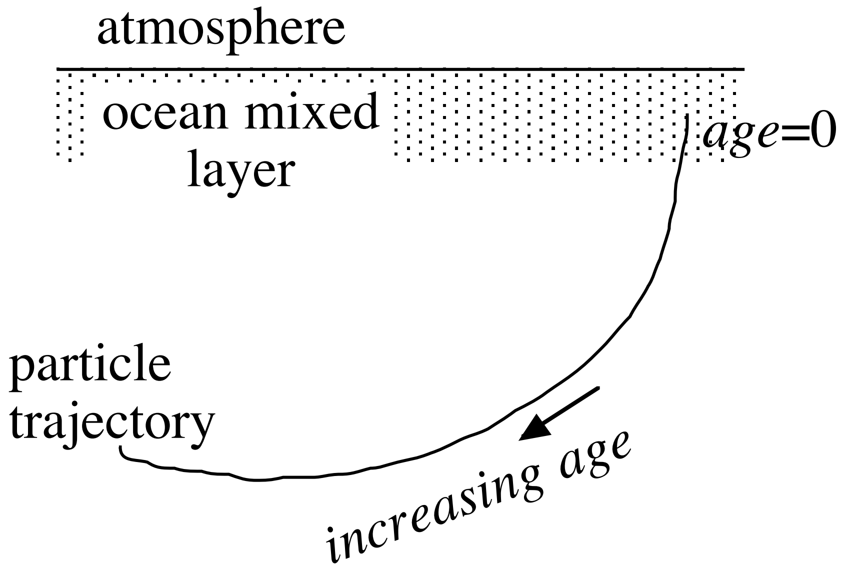
	no wind	westward	southward	eastward	northward
τ (days)	7.1	7.8	7.4	7.1	7.5
θ (days)	48	80	32	42	97
α	0.68	0.78	0.62	0.62	0.78
ε	0.065	0.066	0.067	0.066	0.066

Ventilation of the World Ocean (I)

According to England (JPO, 1995), the “World Ocean circulation at its largest scale can be thought of as a gradual renewal or ventilation of the deep ocean by water that was once at the sea surface.”

Therefore, the age, a measure of the time since leaving the ocean upper mixed layer, is a popular diagnostic tool in the World Ocean.

estimating ocean ventilation rate



age = time elapsed since leaving surface mixed layer

Ventilation of the World Ocean (II)

- At a steady state, the *water age distribution* $c(\mathbf{x},\boldsymbol{\tau})$ is satisfies

$$\frac{\partial c}{\partial \boldsymbol{\tau}} = -\nabla \bullet (\mathbf{u}c - \mathbf{K} \bullet \nabla c) \ , \ \left[c(\mathbf{x},\tau)\right]_{\Gamma} = \delta(\tau - 0) \ , \ \left[c(\mathbf{x},0)\right]_{\Omega} = 0$$

with $\boldsymbol{\tau}$ = the age, Γ = the ocean surface and Ω = the ocean interior.

- *Global water age distribution* $\mu(\tau)$: the volume of the water whose age lies in the interval $[\tau, \tau + \Delta\tau]$ $(\Delta\tau \to 0)$ is $\Omega\mu(\tau)\Delta\tau$, with

$$\mu(\tau) = \frac{1}{\Omega}\int_{\Omega} c(\mathbf{x},\tau)\, d\mathbf{x}$$

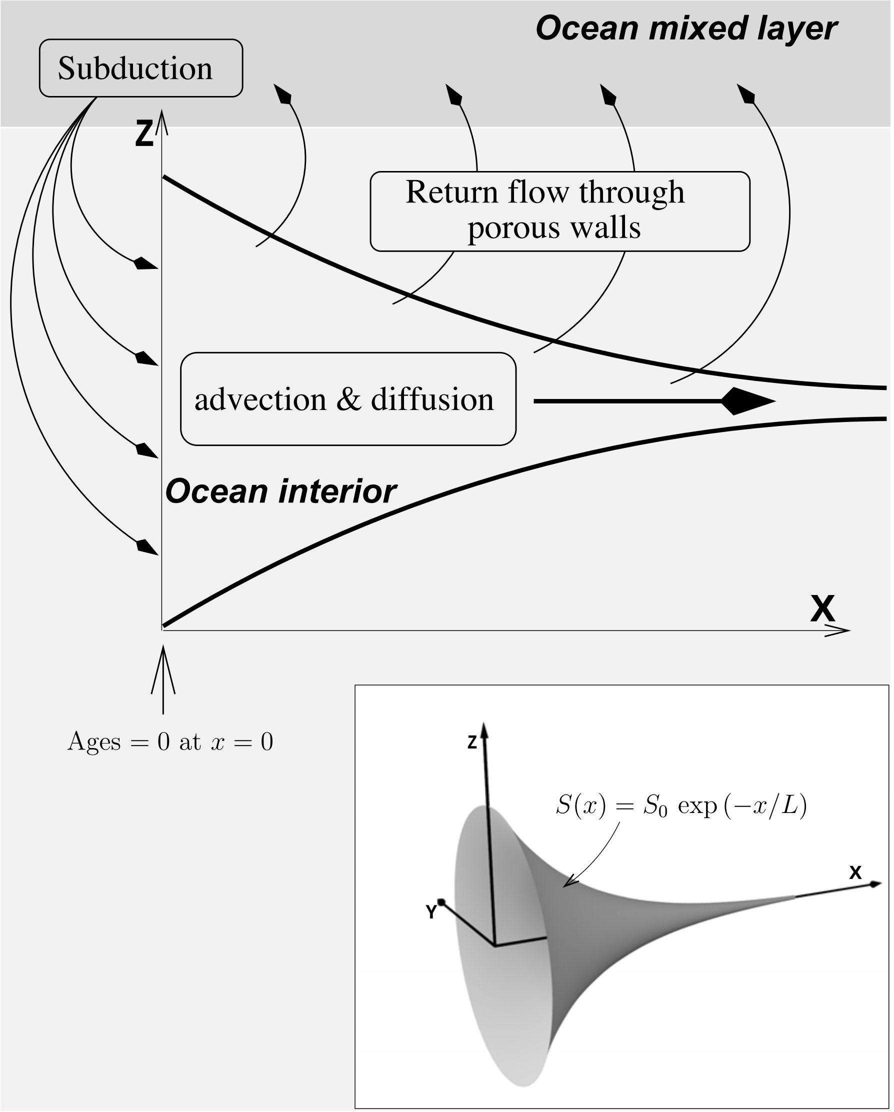
- *Global mean water age*: $\bar{a} = \int_{0}^{\infty} \tau\, \mu(\tau)\, d\tau = \frac{1}{\Omega}\int_{0}^{\infty}\int_{\Omega} \tau\, c(\mathbf{x},\tau)\, d\mathbf{x}\, d\tau$.

- The global water age distribution was derived from the results of A. Mouchet's off-line version of the LLN 3D ocean model with *N*=50,863 wet grid points.

Ventilation of the World Ocean (III)

The leaky-funnel model,
a World Ocean idealization,
is based on the following
key assumption:

The horizontal circulation in the
actual ocean may be thought to
be a consequence of
localized sinking
and
generalized upwelling.
(Warren, 1981)


Ocean mixed layer

Subduction

Return flow through porous walls

advection & diffusion

Ocean interior

Ages $= 0$ at $x = 0$

$S(x) = S_0 \exp(-x/L)$

Ventilation of the World Ocean (IV)

- Parameters of the leaky funnel model:

U = water velocity, K = diffusivity

L = e-folding length scale for the section: $S(x) = S_0\, e^{-x/L}$

L is also the mean length of the water parcel trajectories in the funnel

- The leaky funnel water age distribution is

$$\mu(\tau) \;=\; \sqrt{\frac{K}{\pi L^2 \tau}}\exp\left(-\frac{U'^2\tau}{4K}\right) \;+\; \frac{1}{\theta}\left[1+\operatorname{erf}\left(\frac{1}{\theta}\sqrt{\frac{L^2\tau}{K}}\right)\right]\exp\left(-\frac{U\tau}{L}\right)$$

with $Pe = UL/K$, $\dfrac{1}{\theta} = \dfrac{U'}{2L}\left(1-\dfrac{2}{Pe'}\right)$, $Pe' = U'L/K$ and $U' = U + K/L$.

- The leaky funnel mean age is $\bar{a} = \dfrac{L}{U + K/L}$.

Ventilation of the World Ocean (V)

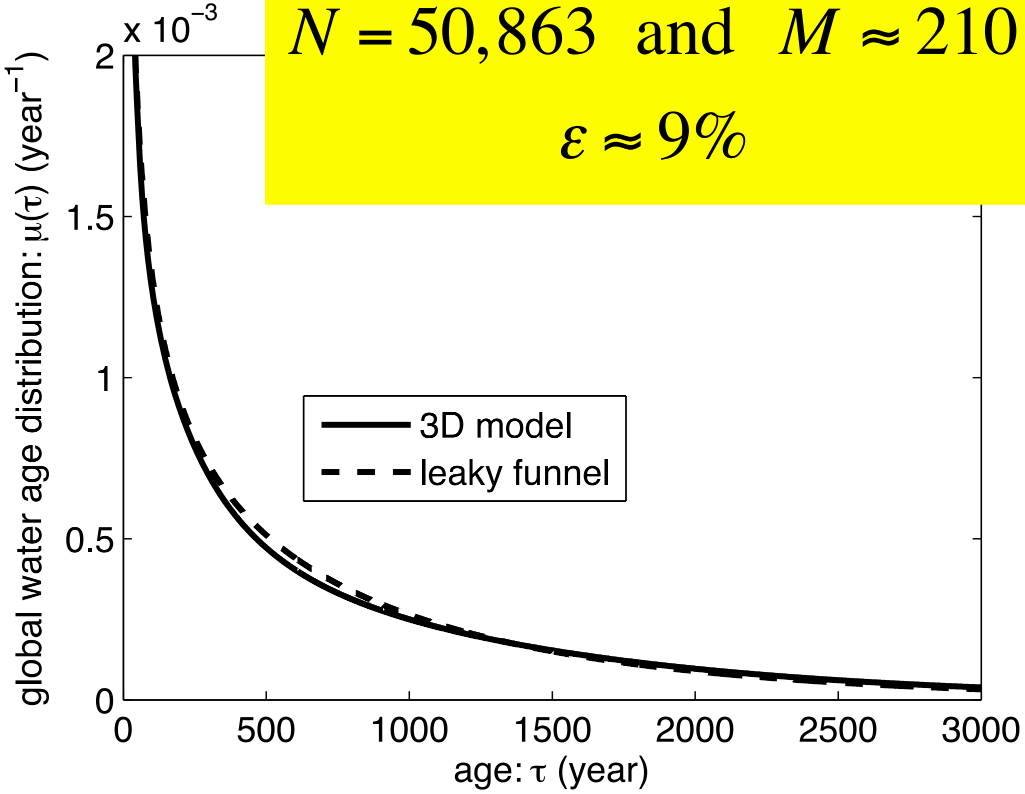
The parameters of the leaky funnel are optimised so as to minimise the following error measure:

$$\varepsilon = \frac{\int_0^\infty \left|\mu^{3D}(\tau) - \mu(\tau)\right| e^{-U\tau/L}\, d\tau}{\int_0^\infty \mu^{3D}(\tau)\, e^{-U\tau/L}\, d\tau}$$

$$U \approx 3\times 10^{-3}\,\mathrm{ms^{-1}} \;,\quad L \approx 10^{8}\,\mathrm{m}$$

$$K \approx 1.4\times 10^{5}\,\mathrm{m^2\,s^{-1}}$$

$$\bar{a}^{3D} = 764\,\mathrm{y} \;,\quad \bar{a} = 721\,\mathrm{y}$$



$$N = 50{,}863 \;\text{ and }\; M \approx 210$$

$$\varepsilon \approx 9\%$$

$$M \approx \frac{10\,L}{\Delta x} \;,\quad \text{with}\;\; \Delta x \approx \frac{\min(L, K/U)}{10}$$

Timescale to tracer equilibrium in the World Ocean (I)

- In paleoclimate studies, one has to consider tracers entering the World Ocean through a fraction α $(0 < \alpha \leq 1)$ of the ocean-atmosphere interface Γ_α, the rest of the ocean surface being assumed to be impermeable.

- Exact expression of the relevant surface boundary condition (SBC) is little known, and is often controversial...

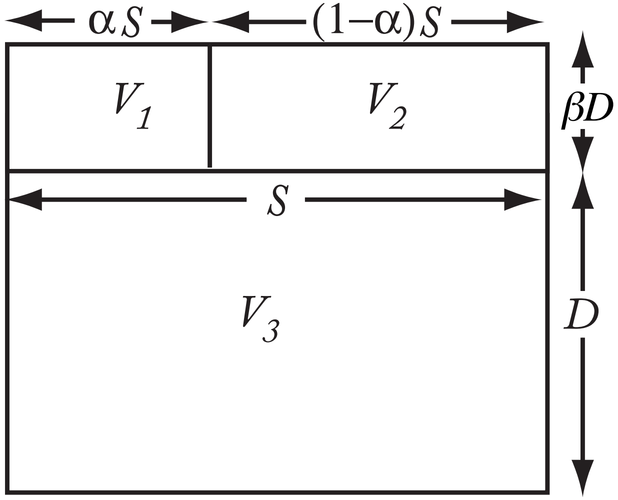
- Primeau and Delersnijder (2008) examined the two ends of the spectrum of the possible SBCs on Γ_α:
  - Dirichlet condition: tracer concentration imposed on Γ_α;
  - Neumann condition: flux imposed as Dirac impulse.

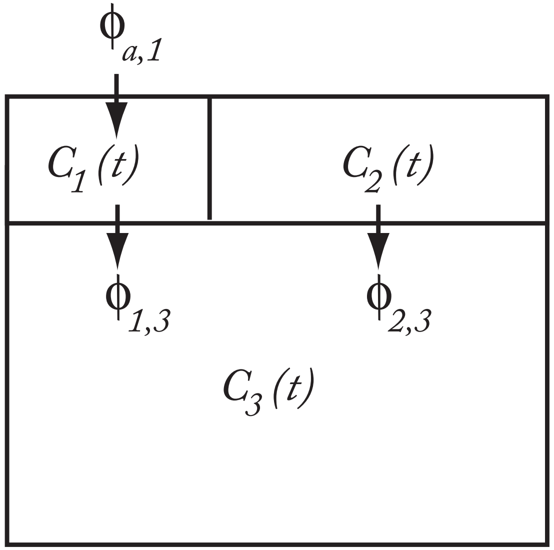
- In both cases, the SBCs are such that $\lim\limits_{t \to \infty} C(t, \mathbf{x}) = 1$.

Timescale to tracer equilibrium in the World Ocean (II)

- F. Primeau computed the eigenvalues of the discrete advection-diffusion operator of his off-line tracer model ($N = 63{,}090$) under Dirichlet and Neumann SBCs, yielding the timescale of decay of the most slowing decaying mode, τ_D^{3D} and τ_N^{3D}, as a function of α.

- A 3-box idealization of this problem:

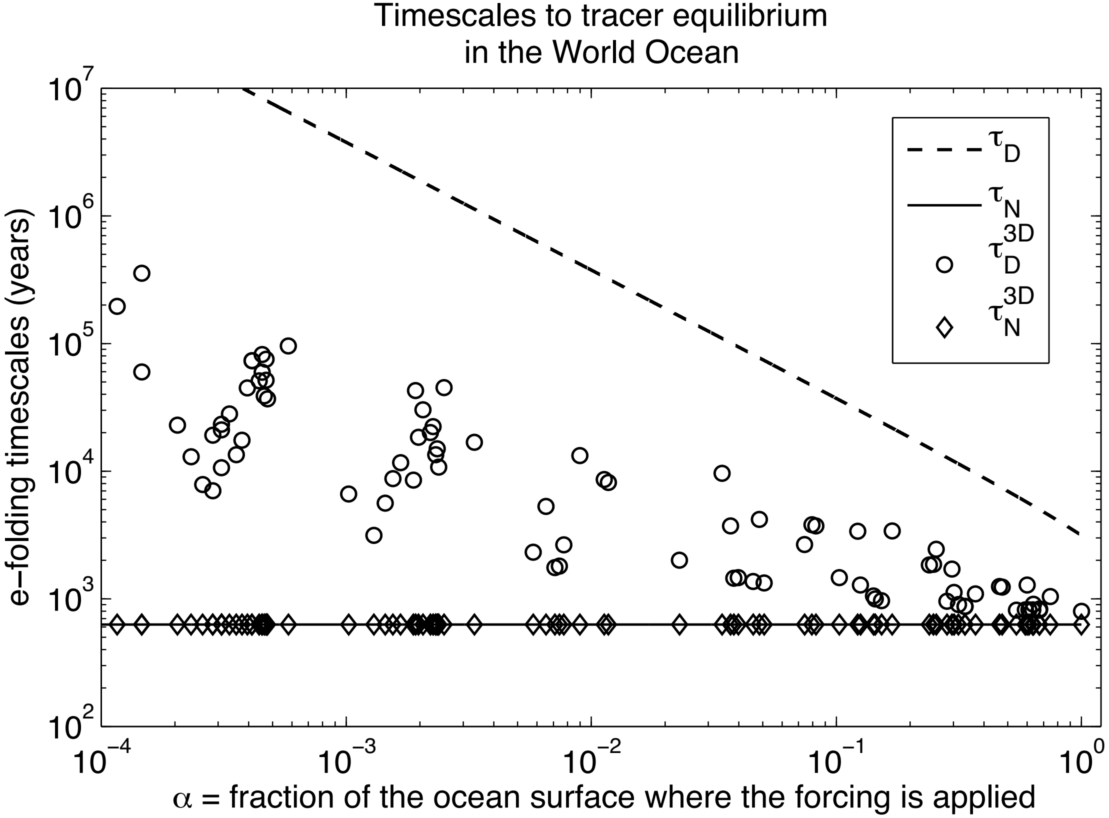

Geometry


Concentrations / Fluxes

Timescale to tracer equilibrium in the World Ocean (III)

- For the 3-box model: $\tau_N = \gamma^{-1}, \quad \tau_D = \dfrac{2\gamma^{-1}}{1+\beta-\sqrt{(1+\beta)^2-4\alpha\beta}}$,

where γ^{-1} is a timescale associated with the fluxes between boxes.



- τ_N and τ_N^{3D} are strictly independent of α!
- τ_D and τ_D^{3D} decrease as α increases, but exhibit largely different values.
- 3-box model is quantitatively unsatisfactory.
- Can we improve it?

Conclusion

- Dimension reduction efficiency:

	N	M	N/M	ε
Mururoa	33,510	1	33,510	12%
PWS	83,762	2	41,881	7%
Ventilation	50,863	210	242	9%
Tracer equil.	63,090	2 / 3	31,545 / 21,030	big / small

- Dimension reduction is of use for the interpretation of the results of complex models, mainly because the parameters of the simple model have a clear physical meaning (residence time, transit time, etc).

- Simple models (box, pipe, etc) presumably work well because they are meant to capture the largest space- and time-scales, the smaller scales contributing essentially to mixing/dissipation.

References

DELEERSNIJDER E., A. MOUCHET and F. PRIMEAU, The global water age distribution of the World Ocean as simulated by an OGCM and idealised by the leaky funnel model (in preparation)

DELEERSNIJDER E., B. TARTINVILLE and J. RANCHER, 1997, A simple model of the tracer flux from the Mururoa lagoon to the Pacific, *Applied Mathematics Letters*, 10 (5), 13-17

DELEERSNIJDER E., J. WANG and C.N.K. MOOERS, 1998, A two-compartment model for understanding the simulated three-dimensional circulation in Prince William Sound, Alaska, *Continental Shelf Research*, 18, 279-287

MOOERS C.N.K. and J. WANG, 1998, On the implementation of a three-dimensional model for Prince William Sound, Alaska, *Continental Shelf Research*, 18, 253-277

MOUCHET A. and E. DELEERSNIJDER, 2008, The leaky funnel model, a metaphor of the ventilation of the World Ocean as simulated in an OGCM, *Tellus*, 60A, 761-774

PRIMEAU F. and E. DELEERSNIJDER, 2008, On the time to tracer equilibrium in the global ocean, *Ocean Science* (under revision, http://www.ocean-sci-discuss.net /5/471/2008/)

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