Impact of volatility clustering on equity indexed annuities

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Abstract

This study analyses the impact of volatility clustering in stock markets on the evaluation and risk management of equity indexed annuities (EIA). To introduce clustering in equity returns, the reference index is modelled by a diffusion combined with a bivariate self-excited jump process. We infer a semi-closed form or parametric expression of the moment generating functions in this framework for the equity return and the intensities of jumps. An econometric procedure is proposed to fit the model to a time series. Next, we develop a method, based on a normal inverse Gaussian approximation of the index return, to evaluate options embedded in simple variable annuities. To conclude, we compare prices, one-year value at risks, and tail value at risks of simple EIAs, computed with different models.

KEYWORDS: Variable annuities, Hawkes process, self-exciting process.

1 Introduction

Significant volumes of equity indexed annuities (EIA) are sold in the US insurance market. They provide a guaranteed annual return combined with some participation in equity market appreciation. Participation in an equity index is offered by guaranteeing a specified proportion (the participation rate) of the return on an index, with a floor and a cap. The valuation of options embedded in EIAs and the wide variety of EIAs is at the origin of abundant literature. Bacinello (2003) and Bauer et al. (2008) proposed a unified framework to evaluate variable annuities. Milevsky and Salisbury (2006), and Dai et al. (2008), studied the valuation of guaranteed minimum withdrawal benefits. Hardy (2003) presented an overview of various investment guarantees. Much attention has been given to the EIAs valuation under the Black–Scholes framework, including studies by Tiong (2000), Lee (2003), and Lin and Tan (2003). Recently, researchers investigated the influence of alternative dynamics on EIA prices. For example, Gerber et al. (2013) appraised equity-linked death benefits in a jump diffusion setting. Siu et al. (2014) and Fan et al. (2015) evaluated EIA when the reference index was a switching Brownian motion. Kelani and Quittard Pinon (2014), and Ballotta (2009) studied ratchet options when the underlying asset is led by Lévy processes.

Jumps or Lévy processes capture stylized features of stock market dynamics like negative skewness and an excess of kurtosis. However, they have stationary increments and thus do not exhibit any clustering of large jumps. On the other hand, empirical analysis conducted in Aït-Sahalia et al. (2015) and Embrechts et al. (2011) emphasize the importance of this effect in financial markets.

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Clustering is not characterized by a single jump but by the amplification of movement that takes place subsequently over several days. However, to the best of our knowledge, most of the models commonly used for the pricing of EIAs do not integrate this characteristic. This observation motivates the developments proposed in this work.

We model the EIA index by a jump diffusion combined with a mechanism of mutual excitation between positive and negative shocks to introduce clustering of volatility in equity returns. Such an approach was introduced in the econometric literature by Aït-Sahalia et al. (2015), Giot (2005), Chavez-Demoulin et al. (2005), and Chavez-Demoulin and McGill (2012), and is based on Hawkes processes (see Hawkes (1971a), (1971b), or Hawkes and Oakes (1974)). These are parsimonious self-exciting point processes for which the intensity jumps in response to a shock and reverts to a target level in the absence of an event. Hawkes processes are increasingly integrated in high frequency finance. Examples include modelling the duration between trades (Bauwens and Hautsch, 2009), and the arrival process of buy and sell orders, as in Bacry et al. (2013).

This study contributes to the literature on variable annuities in several ways. Firstly, the proposed model allows for volatility clustering, a feature that is absent from most of the previous studies on EIAs. Secondly, we find semi-closed form expressions for the moment generating function (MGF) of the index, and of the intensities of jump processes. When the positive and negative jumps are self-excited but mutually independent, we show that these MGFs admit parametric closed form expressions. Thirdly, an econometric method to fit the model to the time series is developed. This study also introduces a method to evaluate options embedded in simple variable annuities based on a normal inverse Gaussian approximation of the index return. Finally, we conduct a complete numerical illustration in which prices and risk of EIA linked to the S&P 500 are assessed. The results are then compared to those obtained with pure jump diffusion and lognormal models.

The rest of the paper is organized as follows. The second section presents the dynamics of the reference index and its characteristics. The third section details the econometric procedure to estimate parameters from a time series. Section 4 focuses on the pricing of EIAs. The last section is a numerical application in which we quantify the impact of volatility clustering on prices and risk of EIAs.

2 The mutually excited jumps diffusion model (MEJD)

EIAs are saving instruments that provide some participation in equity market appreciation. This section describes the dynamics of the stock index that serves as a reference for the determination of the EIA participation rate. The mechanism of the EIA is detailed later in section 4. We consider a complete probability space $(\Omega, \mathcal{F}, P)$ that is equipped with the filtration $\mathcal{F}_t$, generated by an equity index, S_t . The dynamics of S_t is defined by the following stochastic differential equation SDE:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dW_t + (e^{J_1} - 1)dN_t^1 + (e^{J_2} - 1)dN_t^2 \tag{1}$$

where W_t is a Brownian motion and σ is the related constant volatility. N_t^1 and N_t^2 are two independent point processes with intensities denoted by λ_t^1 and λ_t^2. J^1 and J^2 are positive and negative exponential Jumps, respectively. The drift of S_t is such that the expected instantaneous return is constant and equal to μ:

$$\mu_t = \mu - \lambda_t^1 \mathbb{E}(e^{J_1} - 1) - \lambda_t^2 \mathbb{E}(e^{J_2} - 1)\,. \tag{2}$$

The densities of exponential jumps, denoted by $(\nu_i(z))_{i=1,2}$, are defined by two parameters $\rho_1,\ \rho_2 \in \mathbb{R}^+$ as follows:

$$\begin{aligned} \nu_1(z) &= \rho_1 e^{-\rho_1 z} 1_{\{z\geq 0\}} \,, \\ \nu_2(z) &= \rho_2 e^{+\rho_2 z} 1_{\{z<0\}} \,. \end{aligned} \tag{3}$$

The average sizes of jumps $(J_i)_{i=1,2}$ are equal to $\mathbb{E}(J_1) = \frac{1}{\rho_1}$ and $\mathbb{E}(J_2) = -\frac{1}{\rho_2}$. In the following developments, the moment generating function of these jumps are noted:

$$\begin{aligned} \psi^1(z) &:= \mathbb{E}\left(e^{zJ_1}\right) = \frac{\rho_1}{\rho_1 - z} \quad z < \rho_1 \\ \psi^2(z) &:= \mathbb{E}\left(e^{zJ_2}\right) = \frac{\rho_2}{\rho_2 + z} \quad -\rho_2 < z \,. \end{aligned}$$

The clustering of shocks is modelled by the assumption that the frequencies depend on the jumps themselves:

$$\begin{aligned} d\lambda_t^1 &= \kappa_1(c_1 - \lambda_t^1)dt + \delta_{1,1} J_1 dN_t^1 + \delta_{1,2} J_2 dN_t^2 \\ d\lambda_t^2 &= \kappa_2(c_2 - \lambda_t^2)dt + \delta_{2,1} J_1 dN_t^1 + \delta_{2,2} J_2 dN_t^2 \end{aligned} \tag{4}$$

and revert to a constant level $c_{1,2}$ at a speed $k_{1,2}$. The constraints, $\delta_{1,1}$, $\delta_{2,1} > 0$ and $\delta_{2,2}$, $\delta_{1,2} < 0$, are added to ensure the positivity of λ_t^1 and λ_t^2. We call this model the mutually excited jumps diffusion model (MEJD) in the remainder of the article. Equations (4) ensure the presence of contagion between positive and negative jumps.

The intensities are assumed to be observable. This assumption is not penalizing because jumps are detectable by a "peak over threshold" method and the paths of λ_t^1 and λ_t^2 can be retrieved by a log likelihood maximization. These points are detailed and illustrated in section 3. In the next developments, the infinitesimal generator of S_t of a real function $f(t, S_t, \lambda_t^1, \lambda_t^2)$ is defined by the following expression:

$$\begin{aligned} \mathcal{A}f(t, S_t, \lambda_t) = f_t + \mu\, S_t\, f_S + \frac{1}{2}\sigma^2 S_t^2\, f_{SS} + \sum_{i=1,2} \kappa_i (c_i - \lambda_t^i)\, f_{\lambda^i} \\ +\lambda_t^1 \int_{-\infty}^{-\infty} f\left(t,\ S_t + S_t\,(e^z - 1)\,,\ \lambda_t^1 + \delta_{1,1} J_1,\ \lambda_t^2 + \delta_{2,1} J_1\right) - f - (e^z - 1) S_t\, f_S\, \nu_1(dz) \\ +\lambda_t^2 \int_{-\infty}^{-\infty} f\left(t,\ S_t + S_t\,(e^z - 1)\,,\ \lambda_t^1 + \delta_{1,2} J_2,\ \lambda_t^2 + \delta_{2,2} J_2\right) - f - (e^z - 1) S_t\, f_S\, \nu_2(dz) \end{aligned} \tag{5}$$

where f_t , f_S , f_{SS}, and f_{λ_i} are partial derivatives with respect to time, to S_t, and to $\lambda_t^{1,2}$, respectively. On the other hand, the infinitesimal variation of f is ruled by a SDE:

$$\begin{aligned} df &= \left(f_t + \mu\, S_t\, f_S - \sum_{i=1,2} \lambda_t^i \mathbb{E}(e^{J_i} - 1) S_t\, f_S + \frac{1}{2}\sigma^2 S_t^2\, f_{SS} + \sum_{i=1,2} \kappa_i (c_i - \lambda_t^i)\, f_{\lambda_i} \right) dt + \\ & \quad f_S \sigma\, dW_t + \left[f\left(t,\ S_t + S_t\left(e^{J_i} - 1\right),\ \lambda_t^1 + \delta_{1,1} J_1,\ \lambda_t^2 + \delta_{2,1} J_1\right) - f \right] dN_t^1 \\ & \quad + \left[f\left(t,\ S_t + S_t\left(e^{J_2} - 1\right),\ \lambda_t^1 + \delta_{1,2} J_2,\ \lambda_t^2 + \delta_{2,2} J_2\right) - f \right] dN_t^2 \,. \end{aligned} \tag{6}$$

It is then easy to show that if $f = \ln S_t$, the dynamics of the log stock index is provided by:

$$d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t + \sum_{i=1,2} \left(J_i dN_t^i - \mathbb{E}(e^{J_i} - 1) \lambda_t^i dt \right), \tag{7}$$

and that S_t is the product of two exponentials, one related to the Brownian motion and one to the jump processes of the log return:

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu - \frac{1}{2}\sigma^2 ds + \int_0^t \sigma\, dW_s\right) && (8) \\ &\times \exp\left(\sum_{i=1,2}\int_0^t J_i dN_s^i - \sum_{i=1,2}\int_0^t \mathbb{E}(e^{J_i}-1)\lambda_s^i ds\right). \end{aligned}$$

On the other hand, we can show by direct integration of equations (4) that λ_t^1 and λ_t^2 are given by:

$$\begin{aligned} \lambda_t^1 &= c_1 + e^{-\kappa_1 t}\left(\lambda_0^1 - c_1\right) + \int_0^t e^{-\kappa_1(t-s)}\delta_{1,1} J_1 dN_s^1 + \int_0^t e^{-\kappa_1(t-s)}\delta_{1,2} J_2 dN_s^2\,, && (9) \\ \lambda_t^2 &= c_2 + e^{-\kappa_2 t}\left(\lambda_0^2 - c_2\right) + \int_0^t e^{-\kappa_2(t-s)}\delta_{2,1} J_1 dN_s^1 + \int_0^t e^{-\kappa_2(t-s)}\delta_{2,2} J_2 dN_s^2\,. \end{aligned}$$

These expressions allow us to calculate their expectation, detailed in the next proposition:

Proposition 2.1. *The expected values of λ_t^1 and λ_t^2 are given by*

$$\begin{aligned} \begin{pmatrix} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right) \end{pmatrix} &= V \begin{pmatrix} \frac{1}{\gamma_1}\left(e^{\gamma_1 t}-1\right) & 0 \\ 0 & \frac{1}{\gamma_2}\left(e^{\gamma_2 t}-1\right) \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} && (10) \\ &+ V \begin{pmatrix} e^{\gamma_1 t} & 0 \\ 0 & e^{\gamma_2 t} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_0^1 \\ \lambda_0^2 \end{pmatrix} \end{aligned}$$

where V is a matrix

$$V = \begin{pmatrix} \frac{\delta_{1,2}}{\rho_2} & \frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{1,1}}{\rho_1} - \kappa_1 - \gamma_1 & \frac{\delta_{1,1}}{\rho_1} - \kappa_1 - \gamma_2 \end{pmatrix}. \tag{11}$$

If we denote

$$\Delta = \left(\left(\frac{\delta_{2,2}}{\rho_2} + \kappa_2\right) + \left(\frac{\delta_{1,1}}{\rho_1} - \kappa_1\right)\right)^2 - 4\frac{\delta_{2,1}}{\rho_1}\frac{\delta_{1,2}}{\rho_2}$$

then γ_1, γ_2 are constant and defined by

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}\left(\frac{\delta_{2,2}}{\rho_2} - \frac{\delta_{1,1}}{\rho_1} + (\kappa_1+\kappa_2)\right) + \frac{1}{2}\sqrt{\Delta}\,, && (12) \\ \gamma_2 &= -\frac{1}{2}\left(\frac{\delta_{2,2}}{\rho_2} - \frac{\delta_{1,1}}{\rho_1} + (\kappa_1+\kappa_2)\right) - \frac{1}{2}\sqrt{\Delta}\,. \end{aligned}$$

Proof. We first calculate the expectation of equations (9):

$$\begin{aligned} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right) &= c_1 + e^{-\kappa_1 t}\left(\lambda_0^1 - c_1\right) + \frac{\delta_{1,1}}{\rho_1}\int_0^t e^{-\kappa_1(t-s)}\mathbb{E}\left(\lambda_s^1|\mathcal{F}_0\right) ds \\ &\quad - \frac{\delta_{1,2}}{\rho_2}\int_0^t e^{-\kappa_1(t-s)}\mathbb{E}\left(\lambda_s^2|\mathcal{F}_0\right) ds\,, \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right) &= c_2 + e^{-\kappa_2 t}\left(\lambda_0^2 - c_2\right) + \frac{\delta_{2,1}}{\rho_1}\int_0^t e^{-\kappa_2(t-s)}\mathbb{E}\left(\lambda_s^1|\mathcal{F}_0\right) ds \\ &\quad - \frac{\delta_{2,2}}{\rho_2}\int_0^t e^{-\kappa_2(t-s)}\mathbb{E}\left(\lambda_s^2|\mathcal{F}_0\right) ds\,. \end{aligned}$$

Next, we derive these expressions with respect to time and find that $\mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right)$ and $\mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right)$ are solutions of a system of ordinary differential equations (ODE's):

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right) \end{pmatrix} = \underbrace{\begin{pmatrix} -\kappa_1+\frac{\delta_{1,1}}{\rho_1} & -\frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{2,1}}{\rho_1} & -\kappa_2-\frac{\delta_{2,2}}{\rho_2} \end{pmatrix}}_{M}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right) \end{pmatrix}+\begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} \tag{13}$$

Solving this system requires the determination of the eigenvalues γ and eigenvectors (v_1, v_2) of the matrix M. We know that these eigenvalues cancel the following determinant

$$det\begin{pmatrix} -\kappa_1+\frac{\delta_{1,1}}{\rho_1}-\gamma & -\frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{2,1}}{\rho_1} & -\kappa_2-\frac{\delta_{2,2}}{\rho_2}-\gamma \end{pmatrix} = 0$$

and are a solution of a second order polynomial

$$0 = \left(\left(\frac{\delta_{1,1}}{\rho_1}-\kappa_1\right)-\gamma\right)\left(\left(-\kappa_2-\frac{\delta_{2,2}}{\rho_2}\right)-\gamma\right)+\frac{\delta_{2,1}}{\rho_1}\frac{\delta_{1,2}}{\rho_2}$$

if the discriminant Δ is positive

$$\Delta=\left(\frac{\delta_{2,2}}{\rho_2}-\frac{\delta_{1,1}}{\rho_1}+(\kappa_1+\kappa_2)\right)^2-4\left(\frac{\delta_{2,1}}{\rho_1}\frac{\delta_{1,2}}{\rho_2}-\left(\frac{\delta_{1,1}}{\rho_1}-\kappa_1\right)\left(\kappa_2+\frac{\delta_{2,2}}{\rho_2}\right)\right)>0$$

then γ_1 and γ_2 are given by equations (12). Eigenvectors of the matrix M are orthogonal and such that

$$\begin{pmatrix} -\kappa_1+\frac{\delta_{1,1}}{\rho_1}-\gamma & -\frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{2,1}}{\rho_1} & -\kappa_2-\frac{\delta_{2,2}}{\rho_2}-\gamma \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Then we infer that

$$\begin{pmatrix} v_1^i \\ v_2^i \end{pmatrix}=\begin{pmatrix} \frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{1,1}}{\rho_1}-\kappa_1-\gamma_i \end{pmatrix} \qquad i=1,2.$$

Finally, if $D = diag(\gamma_1, \gamma_2)$ and V is the matrix of eigenvectors, the matrix M in equation (10) admits the following decomposition

$$\begin{pmatrix} -\kappa_1+\frac{\delta_{1,1}}{\rho_1} & -\frac{\delta_{1,2}}{\rho_2} \\ \frac{\delta_{2,1}}{\rho_1} & -\kappa_2-\frac{\delta_{2,2}}{\rho_2} \end{pmatrix} = VDV^{-1}.$$

If we define

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V^{-1}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right) \end{pmatrix},$$

the system (10) can be rewritten as two independent ODEs

$$\begin{pmatrix} \frac{\partial}{\partial t}u_1 \\ \frac{\partial}{\partial t}u_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}+V^{-1}\begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix}. \tag{14}$$

If we introduce the following notations,

$$V^{-1}\begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

the solutions of ODEs (14) are given by

$$\begin{aligned} u_1(t) &= \frac{\epsilon_1}{\gamma_1}\left(e^{\gamma_1 t}-1\right)+d_1 e^{\gamma_1 t} \\ u_2(t) &= \frac{\epsilon_2}{\gamma_2}\left(e^{\gamma_2 t}-1\right)+d_2 e^{\gamma_2 t} \end{aligned}$$

where $d=(d_1,d_2)'$ is such that $d=V^{-1}\begin{pmatrix}\lambda_0^1\\ \lambda_0^2\end{pmatrix}$. Then we can conclude the solutions of ODE's (13) are given by (10).

□

According to this result, the model is stable, in the sense that the limits of λ_t^1 and λ_t^2 exist when $t\to+\infty$, iff γ_1 and γ_2 are negative. In this case, the asymptotic expectations of intensities are equal to

$$\lim_{t\to\infty}\begin{pmatrix}\mathbb{E}\left(\lambda_t^1|\mathcal{F}_0\right)\\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_0\right)\end{pmatrix} = V\begin{pmatrix}-\frac{1}{\gamma_1} & 0\\ 0 & -\frac{1}{\gamma_2}\end{pmatrix}V^{-1}\begin{pmatrix}\kappa_1 c_1\\ \kappa_2 c_2\end{pmatrix}. \tag{15}$$

The probability density function of the MEJD has no closed form expression. However, the moments of S_t are computable by the next proposition.

Proposition 2.2. *The β moment of S_T for $T>t$, is an affine function of intensities $\left(\lambda_t^i\right)_{i=1,2}$:*

$$\mathbb{E}\left(\left(S_T\right)^\beta|\mathcal{F}_t\right) = S_t^\beta exp\left(A(t,T,\beta)+\sum_{i=1,2}B_i(t,T,\beta)\lambda_t^i\right) \tag{16}$$

where $A(t,T,\beta)$, $B_1(t,T,\beta)$ and $B_2(t,T,\beta)$ are solutions of a system of ODEs:

$$\frac{\partial}{\partial t}A = -\mu\beta-\frac{1}{2}\beta(\beta-1)\sigma^2-\sum_{i=1,2}\kappa_i c_i\, B_i \tag{17}$$

$$\frac{\partial}{\partial t}B_1 = \kappa_1\, B_1-\left(\frac{\rho_1}{\rho_1-(\beta+B_1\,\delta_{1,1}+B_2\,\delta_{2,1})}-1-\beta\left(\psi_1(1)-1\right)\right) \tag{18}$$

$$\frac{\partial}{\partial t}B_2 = \kappa_2\, B_2-\left(\frac{\rho_2}{\rho_2+(\beta+B_1\,\delta_{1,2}+B_2\,\delta_{2,2})}-1-\beta\left(\psi_2(1)-1\right)\right)$$

with the terminal conditions:

$$A(T,T,\beta)=B_1(T,T,\beta)=B_2(T,T,\beta)=0\,.$$

Proof. Let us define $Y_t:=\mathbb{E}\left(\left(S_T\right)^\beta|\mathcal{F}_t\right)$. As $\mathcal{F}_t\subset\mathcal{F}_u$ for any $u\geq t$, applying the rule of conditional expectations leads to:

$$\begin{aligned} Y_t &= \mathbb{E}\left(\mathbb{E}\left(\left(S_T\right)^\beta\,|\,\mathcal{F}_u\right)\,|\,\mathcal{F}_t\right) \\ &= \mathbb{E}\left(Y_u\,|\,\mathcal{F}_t\right). \end{aligned}$$

Then, by assuming enough regularity to allow one to take the limit within the expectation, the following limit converges to zero:

$$\lim_{u\to t}\frac{\mathbb{E}\left(Y_u\,|\,\mathcal{F}_t\right)-Y_t}{u-t} = 0, \tag{19}$$

in which the left term is the infinitesimal generator of the MGF. Let us denote $f(t, S_{t,}, \lambda_t^1, \lambda_t^2) := Y_t$ and f_t ,f_{λ^i} ,f_S ,f_{SS} the partial derivatives of f with respect to time, intensities and the risk asset. The equation (19) is equal to $\mathcal{A}f = 0$ or after developments to:

$$
\begin{aligned}
0 = {} & f_t + \mu\, S_t\, f_S + \frac{1}{2}\sigma^2 S_t^2\, f_{SS} + \sum_{i=1,2} \kappa_i (c_i - \lambda_t^i)\, f_{\lambda^i} \\
& + \lambda_t^1 \int_{-\infty}^{-\infty} f\left(t,\, S_t e^z,\, \lambda_t^1 + \delta_{1,1} z,\, \lambda_t^2 + \delta_{2,1} z\right) - f - (e^z - 1) S_t\, f_S\, \nu_1(dz) \\
& + \lambda_t^2 \int_{-\infty}^{-\infty} f\left(t,\, S_t e^z\, \lambda_t^1 + \delta_{1,2} z,\, \lambda_t^2 + \delta_{2,2} z\right) - f - (e^z - 1) S_t\, f_S\, \nu_2(dz)
\end{aligned}
\tag{20}
$$

and f satisfies the terminal condition:

$$
f(T, S_T, \lambda_T) = S_T^{\beta}. \tag{21}
$$

In the remainder of this proof, f is assumed to be an exponential affine function of λ_t^i:

$$
f \;=\; S_t^{\beta} exp \left(A(t,T,\beta) + \sum_{i=1,2} B_i(t,T,\beta) \lambda_t^i \right).
$$

where $A(t,T,\beta)$ and $B_{i=1,2}(t,T,\beta)$ are functions of time and of β. Under this assumption, the partial derivatives of f are given by:

$$
f_t \;=\; \left(\frac{\partial}{\partial t} A + \sum_{i=1,2} \frac{\partial}{\partial t} B_i\, \lambda_t^i \right) g,
$$

$$
f_S = \frac{\beta}{S_t} f \qquad f_{SS} = \frac{\beta(\beta-1)}{S_t^2} f \qquad\qquad f_{\lambda^i} = B_i\, f,
$$

and the integrals in the equation (20) are rewritten as:

$$
\begin{aligned}
& \int_{-\infty}^{-\infty} f\left(t,\, S_t e^z,\, \lambda_t^1 + \delta_{1,1} z,\, \lambda_t^2 + \delta_{2,1} z\right) - f - (e^z - 1) S_t\, f_S\, \nu_1(dz) \\
& \qquad = f\left(\psi_1(\beta + B_1\, \delta_{1,1} + B_2\, \delta_{2,1}) - 1 - \beta\left(\psi_1(1) - 1\right)\right) \\
& \int_{-\infty}^{-\infty} f\left(t,\, S_t e^z\, \lambda_t^1 + \delta_{1,2} z,\, \lambda_t^2 + \delta_{2,2} z\right) - f - (e^z - 1) S_t\, f_S\, \nu_2(dz) \\
& \qquad = f\left(\psi_2(\beta + B_1\, \delta_{1,2} + B_2\, \delta_{2,2}) - 1 - \beta\left(\psi_2(1) - 1\right)\right)
\end{aligned}
$$

Inserting these expressions in the equation (20) leads to the following SDE that is satisfied only if the system of equations (17) and (18) holds:

$$
\begin{aligned}
0 = {} & \frac{\partial}{\partial t} A + \mu\beta + \frac{1}{2}\beta(\beta-1)\sigma^2 + \sum_{i=1,2} \kappa_i c_i\, B_i \\
& + \lambda_t^1 \left[\frac{\partial}{\partial t} B_1 - \kappa_1\, B_1 + \left(\frac{\rho_1}{\rho_1 - (\beta + B_1\, \delta_{1,1} + B_2\, \delta_{2,1})} - 1 - \beta\left(\psi_1(1) - 1\right) \right) \right] \\
& + \lambda_t^2 \left[\frac{\partial}{\partial t} B_2 - \kappa_2\, B_2 + \left(\frac{\rho_2}{\rho_2 + (\beta + B_1\, \delta_{1,2} + B_2\, \delta_{2,2})} - 1 - \beta\left(\psi_2(1) - 1\right) \right) \right]
\end{aligned}
\tag{22}
$$

□

Note that when $\beta = 1$, the equations (17) and (18) simplify as follows:

$$\frac{\partial}{\partial t}A = -\mu - \sum_{i=1,2} \kappa_i c_i \, B_i \tag{23}$$
$$\frac{\partial}{\partial t}B_1 = \kappa_1 \, B_1 - \left(\frac{\rho_1}{\rho_1 - (1 + B_1\,\delta_{1,1} + B_2\,\delta_{2,1})} - 1 - (\psi_1(1) - 1) \right)$$
$$\frac{\partial}{\partial t}B_2 = \kappa_2 \, B_2 - \left(\frac{\rho_2}{\rho_2 + (1 + B_1\,\delta_{1,2} + B_2\,\delta_{2,2})} - 1 - (\psi_2(1) - 1) \right)$$

and it is easy to check that $B = 0$ is a natural solution. In this case, as we could expect, $A(t,T) = \mu(T-t)$ and $\mathbb{E}(S_T) = e^{\mu(T-t)}$. For any other value of β, equations (17) and (18) can be solved numerically by an Euler discretization method. If there is no cross dependence between positive and negative jumps ($\delta_{1,2} = \delta_{2,1} = 0$), there exist parametric closed form expressions for A , B_1 and B_2. They are given by the next proposition:

Proposition 2.3. *Let us assume that $\delta_{1,2} = \delta_{2,1} = 0$ and define the following constants:*

$$\alpha_{1,i} \;=\; \left(\kappa_i\rho_i + \kappa_i(-1)^i\beta - (-1)^i\delta_{i,i}\left(1 + \beta\left(\psi_i(1) - 1\right)\right)\right) \qquad i = 1,2$$

, $\alpha_{2,i} = \rho_i(-1)^i\delta_{i,i}$, $\gamma_{1,i} = \frac{\alpha_{2,i}\kappa_i}{\alpha_{1,i}^2}$ and $\gamma_{2,i} = \frac{\alpha_{2,i}}{\alpha_{1,i}}$ for $i = 1,2$. Then the functions A and $B_{i=1,2}$ admit a parametric form:

$$B_i(t_i(u),T,\beta) = \frac{1}{(-1)^i\delta_{i,i}}\left[\frac{\alpha_{2,i}}{\alpha_{1,i}(u-1)} - \left(\rho_i + (-1)^i\beta\right)\right], \tag{24}$$

$$A(t,T,\beta) \;=\; \left(\mu\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right)(T-t) + \sum_{i=1,2}\int_{u_{t,i}}^{u_{0,i}} \kappa_i c_i B_i(t_i(u),T,\beta)dt_i(u) \tag{25}$$

where $t_i(u)$ is a function linking the time t to a parameter u as follows:

$$\begin{aligned} t_i(u) \;=\; & T - \frac{1}{\kappa_i}\ln\left(\frac{\alpha_{1,i}}{\alpha_{2,i}}(u-1)\left(\rho_i + (-1)^i\beta\right)\left(\frac{u_{0,i}^2 - u_{0,i} - \gamma_{1,i}}{u^2 - u - \gamma_{1,i}}\right)^{\frac{1}{2}}\right), \\ & -\frac{1}{\kappa_i}\left(\frac{\tan^{-1}\left(\frac{2u_{0,i}-1}{\sqrt{-4\gamma_{1,i}-1}}\right) - \tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\sqrt{-4\gamma_{1,i}-1}}\right). \end{aligned} \tag{26}$$

The constants $u_{0,i} = 1 + \frac{\gamma_{2,i}}{(\rho_i + (-1)^i\beta)}$ and $u_{t,i}$ are such that $t_i(u_{0,i}) = T$ and $t_i(u_{t,i}) = t$. The integrals in the definition (25) of the function $A(t,T,\beta)$ have the following closed form expressions:

$$\begin{aligned} & \int_{u_{t,i}}^{u_{0,i}} \kappa_i c_i B_i(t_i(u),T,\beta)dt(u) = \\ & -\frac{1}{\kappa_i\frac{\alpha_{1,i}}{\alpha_{2,i}}\left(\rho_i + (-1)^i\beta\right)}\left(\frac{\kappa_i c_i\alpha_{2,i}}{(-1)^i\delta_{i,i}\alpha_{1,i}}\left(\frac{1}{1-u_{0,i}} - \frac{1}{1-u_{t,i}}\right) + \frac{\kappa_i c_i\left(\rho_i + (-1)^i\beta\right)}{(-1)^i\delta_{i,i}}\ln\frac{1-u_{t,i}}{1-u_{0,i}}\right) \\ & -\frac{\kappa_i c_i\alpha_{2,i}}{(-1)^i\delta_{i,i}\alpha_{1,i}}\frac{1}{\kappa_i}\left[\frac{1}{2\gamma_{1,i}}\ln\left(\frac{(u-1)^2}{(u-1)u - \gamma_{1,i}}\right) - \frac{(2\gamma_{1,i}+1)\tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\gamma_{1,i}\sqrt{-4\gamma_{1,i}-1}}\right]_{u=u_{t,i}}^{u=u_{0,i}} \\ & -\frac{\kappa_i c_i\left(\rho_i + (-1)^i\beta\right)}{(-1)^i\delta_{i,i}}\frac{1}{\kappa_i}\left[\frac{1}{2}\ln\left(u^2 - u - \gamma_{1,i}\right) + \frac{1}{\sqrt{-4\gamma_{1,i}-1}}\tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)\right]_{u=u_{t,i}}^{u=u_{0,i}} \end{aligned} \tag{27}$$

Proof. From the previous proposition, we know that B_i's are solutions of ODEs:

$$\frac{\partial}{\partial t} B_i = \kappa_i\, B_i - \left(\frac{\rho_i}{\rho_i + (-1)^i\,(\beta + B_i\,\delta_{i,i})} - 1 - \beta\,(\psi_i(1) - 1)\right)\; i = 1, 2\,. \tag{28}$$

Let us define $D_i(t) := \left(\rho_{i,i} + (-1)^i\beta\right) + (-1)^i\delta_{i,i}B_i(t, T, \beta)$. Then $\frac{\partial}{\partial t}B_i = \frac{1}{(-1)^i\delta_{i,i}}\frac{\partial}{\partial t}D_i$ and the terminal condition on B_i becomes $D_i(T) = \rho_i + (-1)^i\beta$. The equations (28) are then rewritten as follows:

$$\begin{aligned}
\frac{\partial}{\partial t} D_i &= \kappa_i(-1)^i\delta_{i,i}\, B_i - \left(\frac{\rho_i(-1)^i\delta_{i,i}}{D_i} - (-1)^i\delta_{i,i}\,(1 + \beta\,(\psi_i(1) - 1))\right)\; i = 1, 2\,. \\
&= \kappa_i\,\left(D_i - \left(\rho_i + (-1)^i\beta\right)\right) - \left(\frac{\rho_i(-1)^i\delta_{i,i}}{D_i} - (-1)^i\delta_{i,i}\,(1 + \beta\,(\psi_i(1) - 1))\right)\; i = 1, 2\,.
\end{aligned}$$

if we define $\alpha_{1,i} := \left(\kappa_i\rho_i + \kappa_i(-1)^i\beta - (-1)^i\delta_{i,i}\,(1 + \beta\,(\psi_i(1) - 1))\right)$ and $\alpha_{2,i} := \rho_i(-1)^i\delta_{i,i}$, then the dynamics of D_t is rewritten as:

$$D_i\frac{\partial}{\partial t}D_i = \kappa_i\, D_i^2 - \,\alpha_{1,i}\, D_i - \alpha_{2,i}\; i = 1, 2 \tag{29}$$

We operate another change of variable: $D_i(t) = e^{-\kappa_i(T-t)}y_i(t)$. Then $\frac{\partial}{\partial t}D_i = \kappa_i D_i + e^{-\kappa_i(T-t)}\frac{\partial}{\partial t}y_i(t)$ and the terminal condition becomes $y_i(T) = \rho_i + (-1)^i\beta$. The equation (29) is then equivalent to

$$y_i\frac{\partial}{\partial t}y_i(t) = -\,\alpha_{1,i}e^{+\kappa_i(T-t)}\, y_i - \alpha_{2,i}e^{+2\kappa_i(T-t)}\; i = 1, 2 \tag{30}$$

On the other hand, let us define $x_i(t) = \alpha_{1,i}\int_t^T e^{\kappa_i(T-u)}du = \frac{\alpha_{1,i}}{\kappa_i}\left(e^{\kappa_i(T-t)} - 1\right)$. When $t = T$, $x_i(T) = 0$, and $y_i(T) = \rho_i + (-1)^i\beta$. As $\frac{dx_i}{dt} = -\alpha_{1,i}e^{\kappa_i(T-t)}$, then

$$\frac{\partial y_i}{\partial t} = \frac{\partial y_i}{\partial x_i}\frac{\partial x_i}{\partial t} = -\alpha_{1,i}e^{\kappa_i(T-t)}\frac{\partial y_i}{\partial x_i}\,. \tag{31}$$

Combining equations (30) and (31) leads to

$$y_i\frac{\partial y_i}{\partial x_i} = y_i + \frac{\alpha_{2,i}}{\alpha_{1,i}}e^{+\kappa_i(T-t)}\; i = 1, 2 \tag{32}$$

and the left hand term can be reformulated as a function of $x(t)$:

$$y_i\frac{\partial y_i}{\partial x_i} - y_i = \frac{\alpha_{2,i}\kappa_i}{\alpha_{1,i}^2}x_i + \frac{\alpha_{2,i}}{\alpha_{1,i}}\; i = 1, 2 \tag{33}$$

or

$$y_i\frac{\partial y_i}{\partial x_i} - y_i = \gamma_{1,i}x_i + \gamma_{2,i}\; i = 1, 2 \tag{34}$$

where $\gamma_{1,i} := \frac{\alpha_{2,i}\kappa_i}{\alpha_{1,i}^2}$ and $\gamma_{2,i} := \frac{\alpha_{2,i}}{\alpha_{1,i}}$. This last ODE is an Abel's equation of the second kind. It is possible to find a parametric closed form solution using a method detailed in Zaitsev and Polyanin (2003). The equation (34) is indeed equivalent to

$$F(x_i, y_i, u) = y_i u - y_i - \gamma_{1,i}x_i - \gamma_{2,i} = 0 \qquad u = \frac{\partial y_i}{\partial x_i}\,. \tag{35}$$

If we look for a solution of the form $x_i = x_i(u)$ and $y_i = y_i(u)$, in accordance with the first relation, we infer that:

$$\frac{\partial x_i}{\partial u} = -\frac{F_u}{F_{x_i} + uF_{y_i}} \qquad \frac{\partial y_i}{\partial u} = -\frac{uF_u}{F_{x_i} + uF_{y_i}},$$

where $F_{x_i} = \frac{\partial F}{\partial x_i} = -\gamma_{1,i}$, $F_{y_i} = \frac{\partial F}{\partial y_i} = u - 1$ and $F_u = \frac{\partial F}{\partial u} = y_i$. The solution of this system of ODE is

$$y_i(u) \;=\; \gamma_{3,i} \exp\left(-\int_{-\infty}^{u} \frac{z}{z^2 - z - \gamma_{1,i}} dz\right). \tag{36}$$

where $\gamma_{3,i}$ is a constant retrieved later from the terminal condition. On the other hand, from the equation (35) $(u-1)\,y_i - \gamma_{2,i} = \gamma_{1,i} x_i$, we can infer that

$$x_i(u) \;=\; \frac{(u-1)}{\gamma_{1,i}} \gamma_{3,i} \exp\left(-\int_{-\infty}^{u} \frac{z}{z^2 - z - \gamma_{1,i}} dz\right) - \frac{\gamma_{2,i}}{\gamma_{1,i}}. \tag{37}$$

At time T, by construction, we have that

$$\left(\rho_i + (-1)^i \beta\right) u_{0,i} - \left(\rho_i + (-1)^i \beta\right) - \gamma_{2,i} = 0$$

Let us assume $u_{0,i} = 1 + \frac{\gamma_{2,i}}{(\rho_i + (-1)^i \beta)}$, such that $x_i(u_{0,i}) = 0$ (this value of x corresponds to time T), then $y_i(u_{0,i}) = \rho_i + (-1)^i \beta$ and $\gamma_{3,i}$ must be equal to:

$$\gamma_{3,i} = \left(\rho_i + (-1)^i \beta\right) \exp\left(\int_{-\infty}^{u_{0,i}} \frac{z}{z^2 - z - \gamma_{1,i}} dz\right). \tag{38}$$

And if we invert the order of integration (we will see later that if $u \to 1$, $t(u) \to +\infty$ and then, $u < u_{0,i} = 1 + \frac{\gamma_{2,i}}{(\rho_i + (-1)^i \beta)}$), $x_i(t)$ and $y_i(t)$ become:

$$\begin{aligned}
y_i(u) &= \left(\rho_i + (-1)^i \beta\right) \exp\left(\int_{u}^{u_{0,i}} \frac{z}{z^2 - z - \gamma_{1,i}} dz\right), \\
x_i(u) &= \frac{(u-1)}{\gamma_{1,i}} \left(\rho_i + (-1)^i \beta\right) \exp\left(\int_{u}^{u_{0,i}} \frac{z}{z^2 - z - \gamma_{1,i}} dz\right) - \frac{\gamma_{2,i}}{\gamma_{1,i}}.
\end{aligned}$$

where (this result can be checked by direct differentiation):

$$\begin{aligned}
\int_{u}^{u_{0,i}} \frac{z}{z^2 - z - \gamma_{1,i}} dz &= \left[\frac{1}{2} \ln\left(z^2 - z - \gamma_{1,i}\right) + \frac{\tan^{-1}\left(\frac{2z-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\sqrt{-4\gamma_{1,i} - 1}}\right]_{z=u}^{z=u_{0,i}} \\
&= \ln\left(\frac{u_{0,i}^2 - u_{0,i} - \gamma_{1,i}}{u^2 - u - \gamma_{1,i}}\right)^{\frac{1}{2}} + \frac{\tan^{-1}\left(\frac{2u_{0,i}-1}{\sqrt{-4\gamma_{1,i}-1}}\right) - \tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\sqrt{-4\gamma_{1,i} - 1}}
\end{aligned}$$

As by construction $x_i(u) = \frac{\alpha_{1,i}}{\kappa_i}\left(e^{\kappa_i (T - t(u))} - 1\right)$ the value of time that corresponds to a value of u is given by:

$$t_i(u) \;=\; T - \frac{1}{\kappa_i} \ln\left(1 + \frac{\kappa_i}{\alpha_{1,i}} x_i(u)\right), \tag{39}$$

which corresponds to the equation (26). The expression (24) is derived from the definition of $y_i(u)$:

$$B_i(t_i(u),T,\beta)=\frac{1}{(-1)^i\delta_{i,i}}\left[e^{-\kappa_i(T-t)}y_i(t_i(u))-\left(\rho_i+(-1)^i\beta\right)\right]$$

and from $x_i(t)=\frac{\alpha_{1,i}}{\kappa_i}\left(e^{\kappa_i(T-t)}-1\right)$. Thus, it remains to calculate the function $A(t,T,\beta)$ that is the solution of equation (17):

$$A(t,T,\beta) = \left(\mu\beta+\frac{1}{2}\beta(\beta-1)\sigma^2\right)(T-t)+\sum_{i=1,2}\int_{u_{t,i}}^{u_{0,i}}\kappa_i c_i B_i(t_i(u),T,\beta)dt(u)\,.$$

The integral present in this last expression is developed as follows:

$$\begin{aligned}
&\int_{u_{t,i}}^{u_{0,i}}\kappa_i c_i B_i(t_i(u),T,\beta)dt(u)\\
&=\int_{u_{t,i}}^{u_{0,i}}\kappa_i c_i\frac{1}{(-1)^i\delta_{i,i}}\left[\frac{\alpha_{2,i}}{\alpha_{1,i}(u-1)}-\left(\rho_i+(-1)^i\beta\right)\right]\frac{dt_i(u)}{du}du\\
&=-\frac{\kappa_i c_i\alpha_{2,i}}{(-1)^i\delta_{i,i}\alpha_{1,i}}\int_{u_{t,i}}^{u_{0,i}}\frac{1}{(1-u)}\frac{dt_i(u)}{du}du-\frac{\kappa_i c_i\left(\rho_i+(-1)^i\beta\right)}{(-1)^i\delta_{i,i}}\int_{u_{t,i}}^{u_{0,i}}\frac{dt_i(u)}{du}du\,,
\end{aligned}$$

and from the equation (39), we infer that

$$\begin{aligned}
\frac{d}{du}t_i(u) &= -\frac{d}{du}\left(\frac{1}{\kappa_i}\ln\left(\frac{\alpha_{1,i}}{\alpha_{2,i}}(u-1)\left(\rho_i+(-1)^i\beta\right)\right)+\frac{1}{\kappa_i}\int_u^{u_{0,i}}\frac{z}{z^2-z-\gamma_{1,i}}dz\right)\\
&= \frac{1}{\kappa_i\frac{\alpha_{1,i}}{\alpha_{2,i}}\left(\rho_i+(-1)^i\beta\right)}\frac{1}{(1-u)}+\frac{1}{\kappa_i}\frac{u}{u^2-u-\gamma_{1,i}}
\end{aligned}$$

as we have that

$$\int_{u_{t,i}}^{u_{0,i}}\frac{u}{(1-u)\left(u^2-u-\gamma_{1,i}\right)}=\left[\frac{1}{2\gamma_{1,i}}\ln\left(\frac{(u-1)^2}{(u-1)u-\gamma_{1,i}}\right)-\frac{(2\gamma_{1,i}+1)\tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\gamma_{1,i}\sqrt{-4\gamma_{1,i}-1}}\right]_{u=u_{t,i}}^{u=u_{0,i}}$$

$$\int_{u_{t,i}}^{u_{0,i}}\frac{u}{\left(u^2-u-\gamma_{1,i}\right)}du = \left[\frac{1}{2}\ln\left(u^2-u-\gamma_{1,i}\right)+\frac{1}{\sqrt{-4\gamma_{1,i}-1}}\tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)\right]_{u=u_{t,i}}^{u=u_{0,i}}$$

□

In later developments, the MGF of intensities is needed for the pricing of variable annuities.

Proposition 2.4. *For any real vector $\omega=(\omega_1,\omega_2)$, the moment generating of $\omega_1\lambda_T^1+\omega_2\lambda_T^2$ is given by the following expression*

$$\mathbb{E}\left(\exp\left(\sum_{i=1,2}\omega_i\lambda_T^i\right)|\mathcal{F}_t\right)=\exp\left(C(t,T,\omega)+\sum_{i=1,2}D_i(t,T,\omega)\lambda_T^i\right)$$

where the functions $C(t,T,\omega)$ and $D_{i=1,2}(t,T,\omega)$ are solutions of a system of ODEs:

$$\begin{aligned}
\frac{\partial C}{\partial t} &= -\sum_{i=1,2}\kappa_i c_i\, D_i\,, & (40)\\
\frac{\partial D_1}{\partial t} &= \kappa_1 D_1 - \left(\frac{\rho_1}{\rho_1-(D_1\delta_{1,1}+D_2\delta_{2,1})}-1\right) & (41)\\
\frac{\partial D_2}{\partial t} &= \kappa_2 D_2 - \left(\frac{\rho_2}{\rho_2+D_1\delta_{1,2}+D_2\delta_{2,2}}-1\right)
\end{aligned}$$

with the terminal conditions

$$C(T,T,\omega) \;=\; 0\ ,\ D_i(T,T,\omega)=\omega_i\ i=1,2\,.$$

Proof. Let us denote $f(t,\lambda_t^1,\lambda_t^2)=\mathbb{E}\left(\exp\left(\sum_{i=1,2}\omega_i\lambda_T^i\right)|\mathcal{F}_t\right)$, then f satisfies the next ODE,

$$\begin{aligned}
0 = f_t + \sum_{i=1,2}\kappa_i(c_i-\lambda_t^i)\,f_{\lambda^i} &+ \int \lambda_t^1\left(f\left(t,\ \lambda_t^1+\delta_{1,1}z,\lambda_t^2+\delta_{2,1}z\right)-f\left(t,\lambda_t^1\right)\right)\nu_1(z)\\
&+\int \lambda_t^2\left(f\left(t,\ \lambda_t^1+\delta_{1,2}z,\lambda_t^2+\delta_{2,2}z\right)-f\left(t,\lambda_t^2\right)\right)\nu_2(z)
\end{aligned}$$

If we assume that f is the exponential of an affine function of intensities,

$$f(t,\lambda_t^1,\lambda_t^2)=\exp\left(C(t,T,\omega)+\sum_{i=1,2}D_i(t,T,\omega)\lambda_t^i\right)$$

then we can conclude in a similar way to the proof of proposition 2.2.

□

When there is no cross excitation between positive and negative jumps ($\delta_{1,2}=\delta_{2,1}=0$), the functions C and $D_{i=1,2}$ also admit a parametric form, which is described in the next proposition.

Corollary 2.5. *Let us assume that $\delta_{1,2}=\delta_{2,1}=0$ and define the following constants: $\alpha_{1,i}=\left(\kappa_i\rho_i-(-1)^i\delta_{i,i}\right)$, $\alpha_{2,i}=\rho_i(-1)^i\delta_{i,i}$, $\gamma_{1,i}=\frac{\alpha_{2,i}\kappa_i}{\alpha_{1,i}^2}$ and $\gamma_{2,i}=\frac{\alpha_{2,i}}{\alpha_{1,i}}$. Then the functions C and $D_{i=1,2}$ are equal to:*

$$D_i(t_i(u),T,\omega)=\frac{1}{(-1)^i\delta_{i,i}}\left(\frac{\alpha_{2,i}}{\alpha_{1,i}(u-1)}-\rho_i\right) \tag{42}$$

$$C(t,T,\omega) \;=\; =\sum_{i=1,2}\int_{u_{t,i}}^{u_{0,i}}\kappa_i c_i D_i(t_i(u),T,\omega)dt(u)\,. \tag{43}$$

where $t_i(u)$ is a function linking the time t to a parameter u as follows:

$$\begin{aligned}
t_i(u) \;&=\; T-\frac{1}{\kappa_i}\ln\left(\frac{\alpha_{1,i}}{\alpha_{2,i}}(u-1)\left(\rho_i+(-1)^i\delta_{i,i}\omega_i\right)\left(\frac{u_{0,i}^2-u_{0,i}-\gamma_{1,i}}{u^2-u-\gamma_{1,i}}\right)^{\frac{1}{2}}\right)\\
&-\frac{1}{\kappa_i}\frac{\tan^{-1}\left(\frac{2u_{0,i}-1}{\sqrt{-4\gamma_{1,i}-1}}\right)-\tan^{-1}\left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}}\right)}{\sqrt{-4\gamma_{1,i}-1}} \qquad (44)
\end{aligned}$$

The constants $u_{0,i} = 1 + \frac{\gamma_{2,i}}{(\rho_i + (-1)^i \delta_i \omega_i)}$ *and* $u_{t,i}$ *satisfy the relation* $t_i(u_{0,i}) = T$ *and* $t_i(u_{t,i}) = t$. *The integrals in the definition (43) of the function* $C(t, T, \omega)$ *have the following closed form expressions:*

$$\int_{u_{t,i}}^{u_0} \kappa_i c_i D_i(t_i(u), T, \omega) dt(u) =$$

$$-\frac{1}{\kappa_i \frac{\alpha_{1,i}}{\alpha_{2,i}} \left(\rho_i + (-1)^i \delta_{i,i} \omega_i\right)} \left(\frac{\kappa_i c_i \alpha_{2,i}}{(-1)^i \delta_{i,i} \alpha_{1,i}} \left(\frac{1}{1-u_{0,i}} - \frac{1}{1-u_{t,i}} \right) - \frac{\kappa_i c_i \rho_i}{(-1)^i \delta_{i,i}} \ln \frac{1-u_{t,i}}{1-u_{0,i}} \right)$$

$$-\frac{\kappa_i c_i \alpha_{2,i}}{(-1)^i \delta_{i,i} \alpha_{1,i}} \frac{1}{\kappa_i} \left[\frac{1}{2\gamma_{1,i}} \ln \left(\frac{(u-1)^2}{(u-1)u - \gamma_{1,i}} \right) - \frac{(2\gamma_{1,i}+1) \tan^{-1} \left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}} \right)}{\gamma_{1,i} \sqrt{-4\gamma_{1,i}-1}} \right]_{u=u_{t,i}}^{u=u_{0,i}}$$

$$-\frac{\kappa_i c_i \rho_i}{(-1)^i \delta_{i,i}} \frac{1}{\kappa_i} \left[\frac{1}{2} \ln \left(u^2 - u - \gamma_{1,i}\right) + \frac{1}{\sqrt{-4\gamma_{1,i}-1}} \tan^{-1} \left(\frac{2u-1}{\sqrt{-4\gamma_{1,i}-1}} \right) \right]_{u=u_{t,i}}^{u=u_0}$$

The proof is similar to the one of proposition 2.3.

3 Econometric calibration of the MEJD

In the MJED model, jumps, and their intensities are not directly observable. On the other hand, the density function of returns does not admit any closed form expression. Thus, the parameters cannot be estimated by maximization of the log-likelihood. However, several alternatives exist with different levels of accuracy. One approach to fit the model consists of employing GMM methods, as done by Aït-Sahalia et al. (2015), to measure the contagion in stock markets. When high frequency data are available, Mancini (2009) uses asymptotic properties of the Brownian motion to filter jumps. In addition, with high frequency data, Barndorff-Nielsen & Shephard (2004, 2006) define and use the bipower and the multipower variation processes to detect jumps.

As we work with low frequency data, we instead opt for an asymmetric "peaks over threshold" (POT) procedure. This empirical approach, which is inspired from the methodology used in extreme value theory, was successfully applied by Embrechts et al. (2011) and Hainaut (2016 a, 2016 b) to fit Hawkes processes to stock indices and to interest rates.

The dataset consists of a discrete record $\{S_{t_0}, S_{t_1}, ..., S_{t_n}\}$ of $n+1$ observations of the reference stock index, equally spaced with $t_j = jh$ for a given lag h. We assume that when the return of this index, denoted by $\Delta s_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}$, exceeds some predetermined thresholds, it is likely that a jump occurred. These thresholds, denoted by $b(\alpha_1)$ and $b(\alpha_2)$, are the percentiles of a normal distribution, with confidence levels α_1, α_2. The first step to choose the thresholds consists of fitting, by log-likelihood maximization, a Gaussian process to the time series data. The drift and volatility of this process are respectively denoted by μ and σ_{th}, such that $\Delta s_i \sim N\left(\mu h \,,\, \sigma_{th}^2 h\right)$. If $\Phi(.)$ denotes the PDF of a standard normal, $b(\alpha_1)$ and $b(\alpha_2)$ are assumed equal to the α_1 and α_2 percentiles of the Brownian term: $b(\alpha) = \mu h + \sigma_{th} \sqrt{h} \Phi^{-1}(\alpha)$. The confidence levels play, of course, a crucial role and particular attention must be paid to their estimation. For any pair of values (α_1, α_2), potential jumps can be filtered and removed from the sample of observations. If (α_1, α_2) are adequate, the log-returns in this reduced sample are then exclusively ruled by a Brownian motion. In this case, the skewness and kurtosis of the sample from which the jumps are removed, should then be close to 0 and 3 (skewness and kurtosis of a normal random variable). In addition, the Jarque-Bera test applied to this sample should not reject the assumption of normality. Based on this observation, the best levels of confidence should minimize the following function:

$$\alpha_1, \alpha_2 = \arg \min_{Sample\, without\, jumps} \left((Skewness)^2 + (Kurtosis - 3)^2 \right) \tag{45}$$

To assess the robustness of this choice, the procedure is applied to the daily observations of the S&P, from the 16th of May 2006 to the 11th of June 2016 (2515 daily observations). For this dataset, $\mu = 4.73\%$ and $\sigma_{th} = 21.06\%$. The skewness, kurtosis, and p-value of the Jarque Bera test for different couples of confidence levels are reported in tables 1, 2, and 3, respectively. In these tables, the closest skewness to zero is obtained with $\alpha_1 = 94\%$ and $\alpha_2 = 10\%$. Whereas the closest kurtosis to zero is computed with $\alpha_1 = 94\%$ and $\alpha_2 = 6\%$. The Jarque-Bera test does not reject the hypothesis of normality for these two couples of confidence levels. The confidence levels that minimize the objective function (45) are $\alpha_1 = 94.93\%$ and $\alpha_2 = 7.86\%$. With these confidence levels, 235 jumps are detected: 86 are positive and 149 are negative. The skewness and kurtosis of the sample of log-returns from which jumps are removed, are respectively equal to -0.0016 and 2.99. The first graph of figure 1 presents a QQ plot of log-returns versus a normal distribution, when jumps are removed from the dataset. Thus, the normality of the sample is clearly confirmed.

		α_1				
	Skewness	98%	96%	94%	92%	90%
α_2	2%	-0.19	-0.33	-0.44	-0.52	-0.57
	4%	0.00	-0.14	-0.26	-0.33	-0.39
	6%	0.11	-0.03	-0.14	-0.22	-0.28
	8%	0.21	0.07	-0.05	-0.13	-0.19
	10%	0.29	0.15	0.03	-0.05	-0.11

Table 1: This table reports the skewness of samples of daily S&P log-returns from which jumps are removed, for different levels of confidence.

		α_1				
	Kurtosis	98%	96%	94%	92%	90%
α_2	2%	3.53	3.44	3.41	3.41	3.42
	4%	3.35	3.22	3.13	3.10	3.10
	6%	3.29	3.12	3.01	2.96	2.94
	8%	3.26	3.07	2.93	2.87	2.84
	10%	3.27	3.04	2.89	2.81	2.78

Table 2: This table reports the kurtosis of samples of daily S&P log-returns from which jumps are removed, for different levels of confidence.

			α_1				
		p-value	98%	96%	94%	92%	90%
	α_2	2%	0.00	0.00	0.00	0.00	0.00
		4%	0.00	0.00	0.00	0.00	0.00
		6%	0.00	0.45	0.02	0.00	0.00
		8%	0.00	0.32	0.50	0.02	0.00
		10%	0.00	0.01	0.49	0.12	0.01

Table 3: This table reports the P-values of the Jarque Bera statistics computed with samples of daily S&P log-returns, from which jumps are removed, for different levels of confidence.

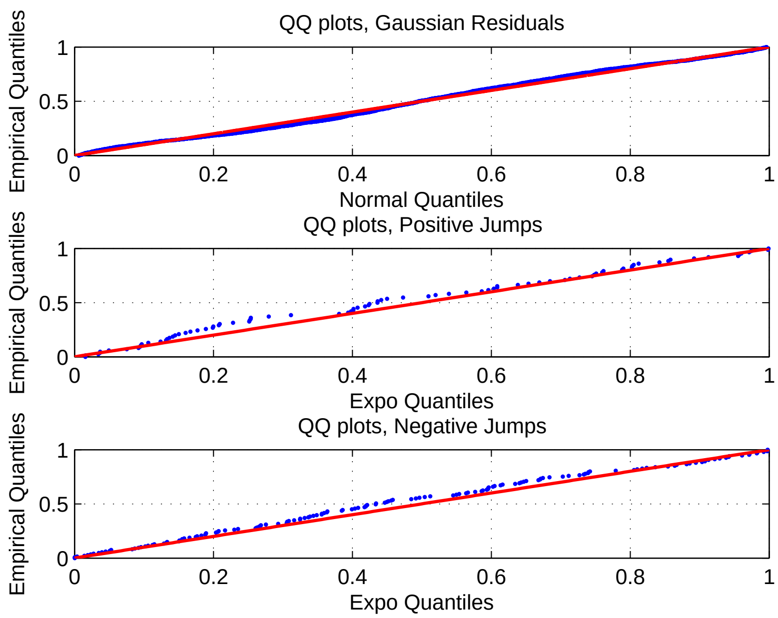

Figure 1: The first graph is a QQ plot of log-returns versus a normal distribution, for the sample of daily S&P log-returns, from which jumps are removed. The last two graphs present QQ plots of filtered jumps versus censored exponential distributions.

Once the confidence levels are determined, we can identify log-returns that include a jump. Under the assumption that Brownian increments are not significant with respect to jumps, the dynamics of the index log return can be described as:

$$\begin{cases} \Delta s_i \approx \mu h + (e^{J_1} - 1) & \Delta s_i > b(\alpha_1)\,, \\ \Delta s_i \approx \mu h + (e^{J_2} - 1) & \Delta s_i < b(\alpha_2)\,, \\ \Delta s_i \approx \mu h + \sigma W_h & b(\alpha_2) \leq \Delta s_i \leq b(\alpha_1)\,, \end{cases}$$

for $i = 0$ to n. Then, σ is assessed by the standard deviation of the sample of log-returns from which jumps are excluded. Whereas the parameters defining the distributions of J_1 and J_2 are obtained by maximizing the following two log-likelihoods of filtered jumps:

$$\begin{cases} \rho_1 & = \arg\max \sum_{i=1}^{n} \log \nu_1 \left(\ln\left(\Delta s_i - \mu h\right) + 1\, ,\, \rho_1\right) I_{\Delta s_i > b(\alpha_1)} \\ \rho_2 & = \arg\max \sum_{i=1}^{n} \log \nu_2 \left(\ln\left(\Delta s_i - \mu h\right) + 1\, ,\, \rho_2\right) I_{\Delta s_i < b(\alpha_2)} \end{cases}$$

Parameters ρ_1 and ρ_2 estimated using this method from the S&P 500 time series data, are presented in table 4. The average sizes of positive and negative jumps are equal to 3.43% and -3.12%, respectively. As jumps smaller (in absolute value) than thresholds $b(\alpha_1)$ or $b(\alpha_2)$ are censored, the two last graphs in figure 1 report the QQ plots of filtered jumps with respect to censored exponential distributions ($\nu_1\left(\Delta s_i\, ,\, \rho_1 - b(\alpha_1)\right)$ and $\nu_2\left(\Delta s_i\, ,\, \rho_2 + b(\alpha_2)\right)$). These graphs clearly confirm the quality of the fit.

At this stage, it remains to estimate the parameters defining the intensities of jumps. For this purpose, we first construct the discretized sample paths of λ_t^1 and λ_t^2. Let us denote the variations of these intensities over the time interval $[t_{i-1}, t_i]$ by $\Delta\lambda_i^k = \lambda_{t_i}^k - \lambda_{t_{i-1}}^k$, where $k = 1, 2$. For a given set of parameters $(\kappa_1, \kappa_2, c_1, c_{2,}, \delta_{1,1}, \delta_{1,2}, \delta_{2,1}, \delta_{2,2})$, these variations are approached by the following rules:

$$\begin{cases} \Delta\lambda_i^1 \approx \kappa_1(c_1 - \lambda_{i-1}^1)h + \delta_{1,1}\left(\ln\left(\Delta s_i - \mu h\right) + 1\right) & \Delta s_i > b(\alpha_1)\,, \\ \Delta\lambda_i^2 \approx \kappa_2(c_2 - \lambda_{i-1}^2)h + \delta_{2,1}\left(\ln\left(\Delta s_i - \mu h\right) + 1\right) & \Delta s_i > b(\alpha_1)\,, \\ \Delta\lambda_i^1 \approx \kappa_1(c_1 - \lambda_{i-1}^1)h + \delta_{2,1}\left(\ln\left(\Delta s_i - \mu h\right) + 1\right) & \Delta s_i < b(\alpha_2)\,, \\ \Delta\lambda_i^2 \approx \kappa_2(c_2 - \lambda_{i-1}^2)h + \delta_{2,2}\left(\ln\left(\Delta s_i - \mu h\right) + 1\right) & \Delta s_i < b(\alpha_2)\,, \\ \Delta\lambda_i^k \approx \kappa_k(c_k - \lambda_{i-1}^k)h\; k = 1, 2 & b(\alpha_2) \leq \Delta s_i \leq b(\alpha_1)\,. \end{cases}$$

If we set λ_0^1 and λ_0^2 to their mean reversion level c_1 and c_2, respectively, the values of intensities at times t_i are obtained by summing up the variations:

$$\begin{aligned} \lambda_i^1 &= \lambda_i^1 + \sum_{j=1}^{i} \Delta\lambda_j^1 \quad i = 0, ..., n \\ \lambda_i^2 &= \lambda_0^2 + \sum_{j=1}^{i} \Delta\lambda_j^2 \quad i = 0, ..., n. \end{aligned}$$

When h is small, the probability of observing a positive (resp. negative) jumps in the i^{th} interval of time is equal to $\lambda_i^1 h$ (resp. $\lambda_i^2 h$). The set of parameters defining the dynamics of intensities is finally obtained by maximization of two independent log-likelihoods:

$$\begin{cases} (\kappa_1, c_1, \delta_{1,1}, \delta_{1,2}) & = \arg\max \sum_{i=1}^{n} \log\left(\left(\lambda_i^1 h\right) I_{\Delta s_i > b(\alpha_1)} + \left(1 - \lambda_i^1 h\right) I_{\Delta s_i \leq b(\alpha_1)}\right) \\ (\kappa_2, c_2, \delta_{2,1}, \delta_{2,2}) & = \arg\max \sum_{i=1}^{n} \log\left(\left(\lambda_i^2 h\right) I_{\Delta s_i < b(\alpha_2)} + \left(1 - \lambda_i^2 h\right) I_{\Delta s_i \geq b(\alpha_2)}\right)\,, \end{cases} \tag{46}$$

under the constraints: $\delta_{1,1} \geq 0$, $\delta_{1,2} \leq 0$, $\delta_{2,1} \geq 0$ and $\delta_{2,2} \leq 0$. These constraints are required to ensure the positivity of intensities. The parameters of the MEJD fitted by this method are reported in table 4. The speeds of mean reversion κ_1 and κ_2 are similar and close to 20. For the intensity of positive jump process, the level of reversion is nearly null. Whereas this level is equal to 3.76 for the intensity of negative jumps. The asymptotic expectations of intensities are, however, much higher: 8.64 and 14.95 for positive and negative jumps, respectively. The difference between these asymptotic expectations and mean reversion levels is explained by the fact that jumps in the dynamics of intensities are exclusively positive. This also explains why, whatever the maturity considered, $\mathbb{E}\left(\lambda_t^1\right)$ and $\mathbb{E}\left(\lambda_t^2\right)$ are always greater than c_1 and c_2, which are the lowest values that λ_t^1 and λ_t^2 can take.

Because $\delta_{1,1}$ is null, there is no self-excitation between positive jumps. On the contrary, the contagion between negative and positive shocks measured by $\delta_{1,2}$ is significantly high. This confirms a fact that is well known amongst traders: after a large fall, the market tends to bounce back, even if only temporarily. On the other hand, $\delta_{2,1}$ is null. This indicates that the occurrence of a positive jump does not raise the probability of a negative shock. However, the level of self-excitation of negative jumps is significant. Notice that if we remove the constraints $\delta_{1,1} \geq 0$ and $\delta_{2,1} \geq 0$, the log-likelihoods (46) for λ_t^1 and λ_t^2 improve by $+0.05\%$ and $+0.85\%$, respectively, which is not a significant difference. In this case, $\delta_{1,1}$ is slightly negative ($\delta_{1,1} = -9.16$) but still very close to zero compared to $\delta_{1,2}$ ($\delta_{1,1} = -371.60$). This confirms the low level of self-excitation between positive jumps. On the contrary, in absence of the constraint $\delta_{2,1} \geq 0$, the estimate of $\delta_{2,1}$ ($\delta_{1,1} = -60.44$) is not negligible with respect to $\delta_{2,2}$ ($\delta_{2,2} = -249.04$). This suggests that the occurrence of a positive jump seems to reduce the instantaneous probability of observing a negative shock. However, this set of parameters does not ensure the positivity of intensities and as such, is not adapted for simulations and pricing purposes.

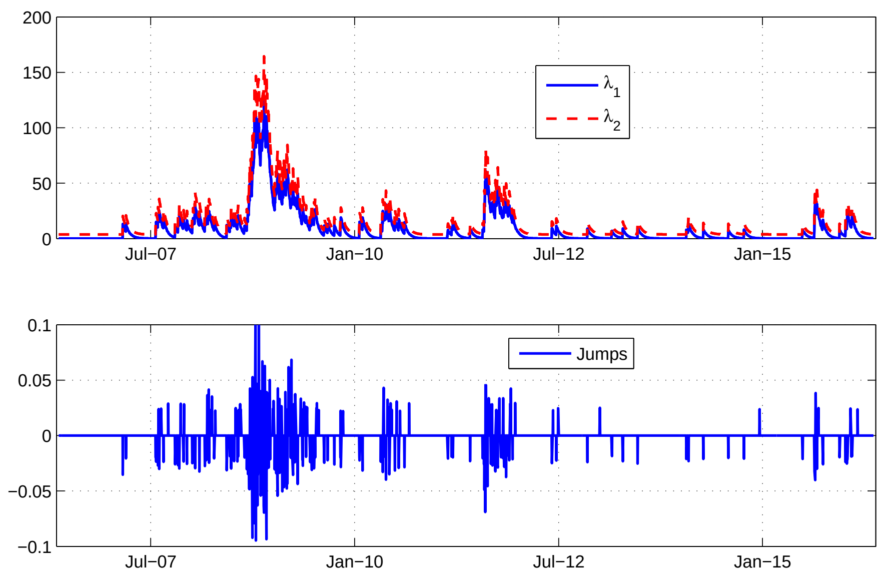

Figure 2: These graphs show filtered intensities and detected jumps for the S&P 500 from May 2006 to June 2016.

The standard deviation of the Brownian motion is equal to 12.49%. A comparison with the volatility of log-return ($\sigma_{th} = 21.06\%$) allows us to infer that jump processes explain 8.57% of the overall standard deviation. Parameters from table 4 are used in the next paragraph to evaluate variable annuities. The sample paths of λ_t^1 , λ_t^2, and jumps, over the period 2006-2016, are shown in figure 2. We clearly observed grouped jumps during the credit crunch crisis of 2008, the second dip of the double-dip recession in 2012, and the European debt crisis of 2015. This confirms the analysis of Ait Sahalia et al. (2015) who suggested that intensities are a good indicator of market stress.

In section 5, to evaluate the impact of the mutual and self-excitation on option prices, a pure jump diffusion model (JD) is adjusted to the time series data of S&P 500 log-returns. The considered process has a dynamic that is defined by the equation (1).But intensities of N_t^1 and N_t^2 are constant and denoted by λ^1 and λ^2. Details are provided in appendix B. This process does not admit any closed form expression for its PDF. The calibration is hence done by the POT method developed in this section. The parameter estimates of the JD model are provided in table 5. Notice that λ^1 and λ^2 are comparable to the asymptotic expectation of intensities of the model with self-excitation. We will come back to the importance of this observation in section 5.

	Value	Std Error		Value	Std Error
μ	4.73%	8.70e-4	σ	12.42%	3.83e-5
ρ_1	29.14	3.38e-1	ρ_2	32.07	2.15e-1
κ_1	20.01	4.67e-4	κ_2	19.48	2.98e-4
c_1	0.09	6.23e-6	c_2	3.76	5.88e-6
$\delta_{1,1}$	0.00	9.5e-5	$\delta_{1,2}$	-367.05	5.42e-4
$\delta_{2,1}$	0.00	1.81e-6	$\delta_{2,2}$	-467.56	7.82e-4
λ_0^1	$= c_1$		λ_0^1	$= c_2$	
$\lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$	8.64		$\lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$	14.95	

Table 4: This table presents the MEJD parameters fitting the S&P 500 log-returns, over the period 2006-2016.

	Value	Std Error		Value	Std Error
μ	4.73%	8.70e-4	σ	12.42%	3.83e-5
ρ_1	29.14	3.38e-1	ρ_2	32.07	2.15e-1
λ^1	8.72		λ^2	15.11	

Table 5: This table reports the parameters of the jump diffusion process without self-excitation (JD model), fitting the S&P 500 log-returns over the period 2006-2016.

4 Risk neutral measure and simple equity indexed annuities

The absence of arbitrage implies the existence of an equivalent risk neutral measure Q, under which the discounted value of all assets is a martingale. By construction, the MEJD model is incomplete because there are no traded securities linked to intensities λ_t^1 and λ_t^2. The uniqueness of Q is hence not warranted. In this case, there exist several approaches to determine a suitable risk neutral measure, like the Esscher or the minimum entropy measures. However, we opt here for an alternative method. As options embedded in variable annuities have long term expiry dates compared to those effectively traded, there is no market from which we can guess the parameters defining S_t over the long run. It is then reasonable to assume that the dynamics of jump processes are identical under the real and the risk neutral measures. In practice, this assumption is commonly made by actuaries to evaluate long term commitments. On the other hand, S_t is according to equation (8), the product of a Brownian exponential and an independent martingale related to jump processes. The drift of S_t can then be adjusted to the risk free rate by considering only a change of measure for the Brownian motion. If r denotes the constant risk free rate and $\theta := \frac{\mu-r}{\sigma}$, the following Radon Nykodym derivative

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s \, dW_s\right)$$

does not modify the dynamics of the jump components under Q. Furthermore, as

$$\begin{aligned}\left(\frac{dQ}{dP}\right)_t e^{-\int_0^t r ds} S_t &= \exp\left(\int_0^t \mu - r - \frac{1}{2}\left(\sigma^2+\theta^2\right) ds + \int_0^t \sigma + \theta \, dW_s\right) \\ &\quad \times \exp\left(\int_0^t J dN_s - \int_0^t \mathbb{E}(e^J-1)\lambda_s ds\right)\end{aligned}$$

and the two factors in this product are independent, we can check that the expected discounted value of S_t under Q is well equal to its current value:

$$\mathbb{E}\left(\left(\frac{dQ}{dP}\right)_t e^{-\int_0^t r(s)ds} S_t | \mathcal{F}_0\right) = S_0 \, .$$

In the remainder of this section, we introduce the specifications of EIAs and a pricing method. Two types of EIAs exist: simple and compound. This work focuses on the first category, mainly because there is no other alternative than Monte Carlo simulations for pricing compound EIAs. We consider a simple annuity, purchased by an individual of age x, with a participation rate indexed on S_t, at the end of each period of length Δ. Times at which the annuity is indexed are denoted by t_k for $k=1$ to n and t_n. If the individual passes away between t_{k-1} and t_k or survives until the annuity expires, the following cash-flow (per unit of invested capital) is settled:

$$G(t_k) = 1 + \sum_{j=1}^{k}\left(\max\left\{\min\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}}, e^{\gamma\Delta}\right), e^{g\Delta}\right\} - 1\right) \tag{47}$$

where η is the participation rate, g is a minimum guarantee, and γ is a cap. The time of exit (death or expiry) is noted τ. ${}_{t_k}p_x$ and q_{x+t_k} denote the probability of survival until time t_k, and the probability of death between times, t_{k-1} and t_k. Under the assumption that the time of payment is independent from the filtration of the assets (as it is when the exit is caused by the death of the insured), the market value of the simple indexed annuity is equal to

$$P_0 = C\sum_{k=1}^{n}\mathbb{E}^Q\left(e^{-\int_0^{t_k} r\,ds} G(t_k)\,|\,\mathcal{F}_0\right) {}_{t_k-\Delta}p_x \, q_{x+t_k} + C \, {}_{t_n}p_x \mathbb{E}^Q\left(e^{-\int_0^{t_n} r\,ds} G(t_n)\,|\,\mathcal{F}_0\right) \tag{48}$$

where C is the capital that is indexed on the asset S_t. The discounted expected value of $G(t_k)$ under the risk neutral measure can be developed as follows:

$$\begin{aligned}&\mathbb{E}^Q\left(e^{-\int_0^{t_k} rds} G(t_k)\,|\,\mathcal{F}_0\right) = \\ &\quad e^{-\int_0^{t_k} rds} + \mathbb{E}^Q\left(\sum_{j=1}^{k} e^{-\int_0^{t_k} rds}\left(\max\left\{\min\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}}, e^{\gamma\Delta}\right), e^{g\Delta}\right\} - 1\right) \,|\, \mathcal{F}_0\right) .\end{aligned} \tag{49}$$

The term in this last expectation is the sum of two payoffs of European options:

$$\begin{aligned}&\max\left\{\min\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}}, e^{\gamma\Delta}\right), e^{g\Delta}\right\} = \\ &e^{g\Delta} + \max\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}} - e^{g\Delta}, 0\right) - \max\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}} - e^{\gamma\Delta}, 0\right)\end{aligned}$$

and the equation 49 becomes:

$$\begin{aligned}\mathbb{E}^Q\left(e^{-\int_0^{t_k} rds}G(t_k)\,|\,\mathcal{F}_0\right) &= e^{-\int_0^{t_k} rds}+\sum_{j=1}^{k}e^{-\int_0^{t_k} r\,ds}\left(e^{g\Delta}-1\right) \\ &+\sum_{j=1}^{k}e^{-\int_0^{t_k} rds}\mathbb{E}^Q\left(\max\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}}-e^{g\Delta},0\right)\,|\,\mathcal{F}_0\right) \\ &-\sum_{j=1}^{k}e^{-\int_0^{t_k} rds}\mathbb{E}^Q\left(\max\left(\eta\frac{S_{t_j}}{S_{t_{j-1}}}-e^{\gamma\Delta},0\right)\,|\,\mathcal{F}_0\right).\end{aligned} \tag{50}$$

The conditional distribution of forward returns, $\frac{S_{t_j}}{S_{t_{j-1}}}\,|\,\mathcal{F}_0$ in the MEJD model, is unknown. In theory, it would be possible to infer its probability density function (PDF) by a Fourier transform of its MGF, with three dimensions. However, in practice, this solution is computationally intensive and unstable, as the MGF is evaluated numerically. In many other circumstances, the statistical distribution of variables of interest is unknown. As underlined in Eriksson et al. (2004), there have been historically three different ways of approximating a density. The first one uses the Pearson (1895) family of frequency curves to approximate a distribution via moment matching. The most useful Pearson functions represent densities only with two parameters and match then only two moments.

The second method to approach a statistical distribution is the Gram-Charlier expansion (Charlier 1905) or the Edgeworth expansion (1907). These methods are based on the expansion of the Gaussian density in terms of Hermite polynomials. The main drawback of such expansions is that they do not always provide positive definite density. This issue is discussed by Draper and Tierny (1972), and more recently by Jondeau and Rockinger (2001).

The last method of approximation consists of working with a monotonic transform of a known distribution, matching moments. However, Johnson (1949) shows that the moment structure is often too complicated to make moment matching possible.

Eriksson et al. (2004) propose an alternative to use the family of normal inverse Gaussian (henceforth NIG). As shown in Barndorff-Nielsen (1997), NIG densities have a simple analytical structure of cumulants, which are particularly useful for the purpose of moment matching. Furthermore, Eriksson et al. (2004) show that NIG approximations perform better than Gram-Charlier and Edgeworth expansions for a wide variety of skewed and fat-tailed distributions.

On the other hand, several research papers, as in Jönsson et al. (2010), demonstrated the usefulness of NIG distributions in financial applications. Barndorff-Nielsen (1995) and Raible (2000) show that replacing the Brownian motion present in diffusion models by a Lévy process of the type normal inverse Gaussian provides a better fit of stock or bond log-returns. Carr and Madan (1998) proposed a Fast Fourier method to price options when the stock return is an NIG process. Eriksson et al. (2009) use a NIG distribution so as to bootstrap the risk neutral density of stock returns from option prices. Lai and Laurier (2007) showed that NIG-based models outperformed the Black and Scholes model for options pricing. Hainaut and Macgilchrist (2010) used a NIG mean reverting process to model the term structure of interest rates.

Based on this literature review, the forward return Y_{t_j} for $j = 1...n$ are approached by a set of

NIG random variables, fitted by moment matching. As NIG distributions are defined by four parameters, only the first four moments of Y_{t_j} are required to fit them. Those moments of $\frac{S_{t_j}}{S_{t_{j-1}}} \mid \mathcal{F}_0$ may be inferred by combining propositions 2.2 and 2.3, as stated in the next corollary:

Corollary 4.1. *If we denote by ω_B the vector of functions $(B_1(t_{j-1}, t_j, \beta), B_2(t_{j-1}, t_j, \beta))$, as defined in the proposition 2.2 and calculated using μ replaced by the risk free rate r, the β non-centred moment of the forward return under Q, is given by*

$$\begin{aligned}\mathbb{E}^Q\left(\left(\frac{S_{t_j}}{S_{t_{j-1}}}\right)^{\beta} |\mathcal{F}_0\right) &= \mathbb{E}^Q\left(\frac{1}{S^{\beta}_{t_{j-1}}}\mathbb{E}^Q\left(S^{\beta}_{t_j}|\mathcal{F}_{t_{j-1}}\right)|\mathcal{F}_0\right) \\ &= \exp\left(A(t_{j-1}, t_j, \beta) + C(0, t_{j-1}, \omega_B) + \begin{pmatrix} D_1(0, t_{j-1}, \omega_B) \\ D_2(0, t_{j-1}, \omega_B)\end{pmatrix}' \begin{pmatrix}\lambda_0^1 \\ \lambda_0^2\end{pmatrix}\right)\end{aligned} \tag{51}$$

where $C(0, t_{j-1}, \omega_B)$, $D_1(0, t_{j-1}, \omega_B)$ and $D_2(0, t_{j-1}, \omega_B)$ are functions defined in the proposition 2.3.

If the conditional forward returns are denoted by $Y_{t_j} = \frac{S_{t_j}}{S_{t_{j-1}}}|\mathcal{F}_0$ for $j = 1...n$, it is well known that their variances, skewness and kurtosis are related to moments by the following relations:

$$\mathbb{V}\left(Y_{t_j}\right) = \mathbb{E}\left(Y^2_{t_j}\right) - \mathbb{E}\left(Y_{t_j}\right)^2,$$

$$\mathbb{S}\left(Y_{t_j}\right) = \frac{\mathbb{E}\left(Y^3_{t_j}\right) - 3\mathbb{E}\left(Y_{t_j}\right)\mathbb{V}\left(Y_{t_j}\right) - \mathbb{E}\left(Y_{t_j}\right)^3}{\mathbb{V}(Y_{t_j})^{\frac{3}{2}}},$$

$$\mathbb{K}(Y_{t_j}) = \frac{1}{\left(\mathbb{V}(Y_{t_j})\right)^2}\left(\mathbb{E}\left(Y^4_{t_j}\right) - 4\mathbb{E}\left(Y_{t_j}\right)\mathbb{E}\left(Y^3_{t_j}\right) + 6\mathbb{E}\left(Y_{t_j}\right)^2\mathbb{E}\left(Y^2_{t_j}\right) - 3\mathbb{E}\left(Y_{t_j}\right)^4\right) - 3.$$

The set of NIG random variables and their parameters are denoted by $\tilde{Y}_{t_j}$ and μ_j^{nig},α_j^{nig}, β_j^{nig}, δ_j^{nig}, for $j = 1...n$. The parameters α_j^{nig} and β_j^{nig} must satisfy the constraint, $\alpha_j^{nig\,2} - \beta_j^{nig\,2} \geq 0$. If $\gamma_j^{nig} := \sqrt{\alpha_j^{nig\,2} - \beta_j^{nig\,2}}$, the variance, skewness and excess of kurtosis of $\tilde{Y}_{t_j}$ are equal to:

$$\mathbb{E}\left(\tilde{Y}_{t_j}\right) = \mu_j^{nig} + \frac{\delta_j^{nig}\beta_j^{nig}}{\sqrt{\alpha_j^{nig\,2} - \beta_j^{nig\,2}}}, \tag{52}$$

$$\mathbb{V}\left(\tilde{Y}_{t_j}\right) = \frac{\delta_j^{nig}(\beta_j^{nig\,2} + \gamma_j^{nig\,2})}{\gamma_j^{nig\,3}}, \tag{53}$$

$$\mathbb{S}\left(\tilde{Y}_{t_j}\right) = 3\frac{\beta_j^{nig}}{\alpha_j^{nig}\sqrt{\delta_j^{nig}\gamma_j^{nig}}}, \tag{54}$$

$$\mathbb{K}(\tilde{Y}_{t_j}) = 3\frac{\alpha_j^{nig\,2} + 4\beta_j^{nig\,2}}{\delta_j^{nig}\alpha_j^{nig\,2}\gamma_j^{nig}} - 3. \tag{55}$$

The density function of $\tilde{Y}_{t_j}$, that is denoted by $m_j(y,\, \mu_j^{nig}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig})$, has a closed form expression:

$$
\begin{aligned}
m_j(.) \;&=\; a(\alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig}) q \left(\frac{y - \mu_j^{nig}}{\delta_j^{nig}} \right)^{-1} \\
&\quad \times K_1 \left(\delta_j^{nig} \alpha_j^{nig} q \left(\frac{y - \mu_j^{nig}}{\delta_j^{nig}} \right) \right) e^{\beta_j^{nig}(y - \mu_j^{nig})}
\end{aligned} \tag{56}
$$

where $q(x) = \sqrt{1 + x^2}$, $K_1(x)$ is the third order Bessel function and

$$
a(\alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig}) \;=\; \pi^{-1} \alpha_j^{nig} e^{\delta_j^{nig} \sqrt{(\alpha_j^{nig\,2} - \beta_j^{nig\,2})}}.
$$

If the n sets of parameters $(\mu_j^{nig}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig})$ are found by the moments matching procedure, European options in the equation (50) are evaluated numerically by discretizing the following integrals:

$$
\begin{aligned}
\mathbb{E}^Q \left(\max \left(\eta \frac{S_{t_j}}{S_{t_{j-1}}} - e^{g\Delta}, 0 \right) | \mathcal{F}_0 \right) \;&=\; \int_0^\infty \max \left(\eta\, y - e^{g\Delta}, 0 \right) m_j(y,\, \mu_j^{nig}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig})\, dy \\
\mathbb{E}^Q \left(\max \left(\eta \frac{S_{t_j}}{S_{t_{j-1}}} - e^{\gamma\Delta}, 0 \right) | \mathcal{F}_0 \right) \;&=\; \int_0^\infty \max \left(\eta\, y - e^{\gamma\Delta}, 0 \right) m_j(y,\, \mu_j^{nig}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig})\, dy
\end{aligned}
$$

Prices obtained by this approach are compared in the next section to those computed using a Monte Carlo simulation, and with JD and lognormal models. Annuities in the JD model are priced with the NIG approximation fitted by moment matching as for the MEJD. The moments of forward return in the JD model have a closed form expression that is provided in appendix B.

5 Numerical application

To check the capacity of the NIG PDF to approximate the MEJD distribution, we simulate 10 000 sample paths of S_T with parameters of table 4. Next, we estimate the NIG parameters matching the first four moments of $\frac{S_T}{S_0}$, computed with the corollary 4.1. The time horizon is set to $T = 2$ years. The left plot of figure 3 shows these PDFs and reveals that the two distributions (curves "MC PDF" and "NIG PDF") are similar. A comparison with a lognormal random variable fitted to the same dataset reveals that both NIG and Monte Carlo distributions have fatter negative tails and a negative skewness.

In the second plot, we compare the PDFs of S_T (built with the NIG approximation), with (MEJD) and without a self-excitation mechanism (JD). The maturity T is still 2 years. The parameters employed to build the PDF of the JD and of the MEJD are reported in tables 5 and 4, respectively. Two cases are considered for the MEJD. First, initial intensities are set to their asymptotic expectation: $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$. Such values are observed during periods of moderate to low activity of jumps. Second, intensities are equal to five times their asymptotic expectation. Such intensities are observed in periods of severe recession.

The JD distribution is very close to the PDF of the MEJD, when $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$. This similarity is explained by the proximity of λ^1 and λ^2 to $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$, as emphasized in section 3. However, the skewness of the JD is slightly more pronounced than that of the MEJD. Post-analysis, it was found that the reason behind this observation is that in the MEJD model, λ_t^1 and λ_t^2 revert to levels c_1 and c_2, respectively, that are significantly lower than the asymptotic expected intensities. In this sense, the JD model is slightly more conservative than the MEJD, as it does not capture the fact that intensities converge to lower levels between two successive periods of intense jump activities. When $\lambda_0^1 = 5 \times \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = 5 \times \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$, the MEJD density exhibits a fatter right tail than that of the JD.

To understand the impact of the mutual excitation on option prices, we evaluate European calls and puts with the MEJD, JD, and lognormal (Black&Scholes) models. For the MEJD process, initial intensities are set to their asymptotic expectation. The options expire in 2 years and strikes, $K = e^{gT}$,, range from $g = -10\%$ to $g = 10\%$. Call prices obtained with the JD and MEJD models are lower than those computed using a lognormal model. The reason is that JD and MEJD PDFs both have fatter right tails than that of the lognormal PDF. Therefore, the probability to exercise the call is lower. The spread between prices yielded by MEJD and JD models comes from the difference of skewness, as explained in the previous paragraph. For the same reason, puts are clearly more expensive in the MEJD or JD models than in the Black & Scholes model.

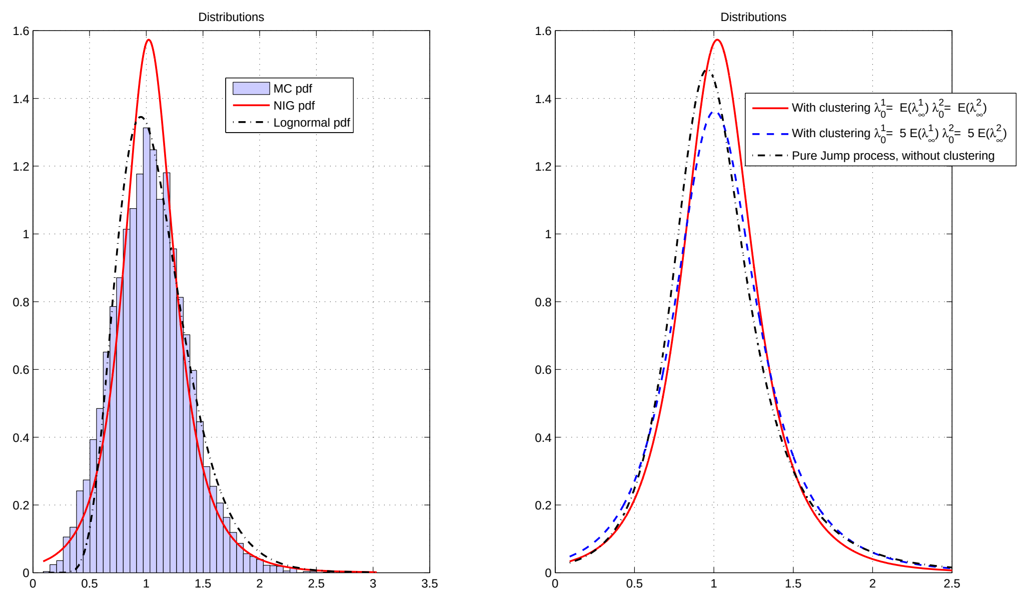

Figure 3: Left plot: comparison of the NIG PDF, lognormal PDF and the PDF computed by Monte Carlo simulations for S_T. This PDF is obtained with 10 000 runs and a time step of 0.005. The time horizon T is 2 years. The intensities of the model with clustering of volatility are set to their asymptotic expectations: $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$. Other parameters used in this simulation are listed in table 4. Right plot: comparison of S_T PDFs , using the NIG approximation, with and without clustering effects (JD and MEJD models). The maturity is $T = 2$ years and the parameters are listed in tables 4 and 5.

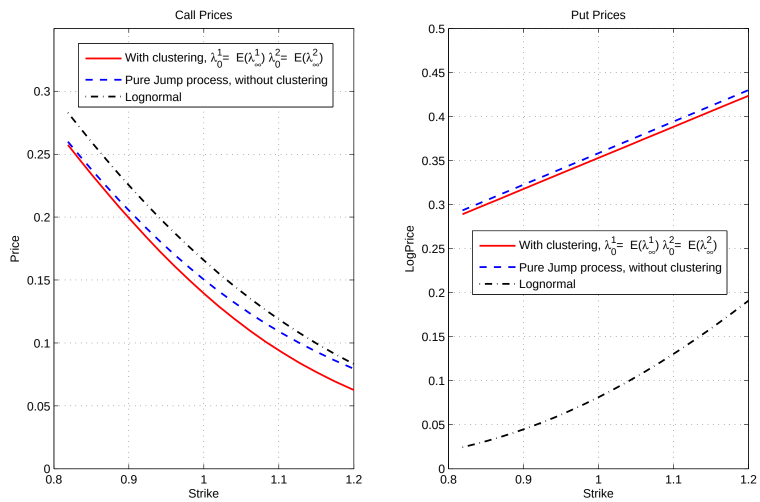

Figure 4: Left plot: comparison of European call option prices computed with MEJD, JD, and lognormal models. The maturity T is 2 years and the strike rate, g, varies from -10% to 10%. The risk free rate is set to 2%. Right plot: prices of European put options, with the same features as call options. For the model with clustering of volatility, The intensities of jump processes are set to their asymptotic expectations: $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$. Other parameters are reported in tables 4 and 5.

The table 6 reports the prices of several variable annuities, computed with the MEJD, JD, and lognormal (B&S) models. The considered maturities are 5, 10, and 15 years. The purchaser of the annuity is a 60 year old man and mortality rates are those yielded by a Gompertz-Makeham adjustment, as described in appendix A.

We observe that whatever the specifications of the EIA, prices obtained with the MEJD and JD models are close to those obtained within the B&S framework. The maximum spread between MEJD and B&S prices (expressed in % of the B&S price) is -2.07%. Whereas the maximum spread between JD and B&S prices is -2.78%. However, for most of EIAs, the spread is between -1% and 0%. To explain this counter-intuitive observation, notice that MEJD and JD models provide a better fit of tails than the B&S model. However, this effect is not reflected by EIA prices because, according to equation (50), these prices depend upon the difference between call options, of same maturity, with different strikes. The impact of extreme shocks on EIA values is then offset by this difference. What is more surprising is that MEJD or JD models always yield slightly lower prices than those obtained with the B&S model. However, parameters used to evaluate EIAs are calibrated using the “peak over threshold” procedure and options are priced numerically. We cannot then guarantee that this trend is not related to small numerical inaccuracies of these methods.

	MEJD	JD	B&S	Spread (%) MEJD-B&S	Spread (%) JD-B&S
T	$g = 0\%$				
5	1.0301	1.0181	1.0302	-0.0112	-1.1713
10	1.0588	1.0371	1.0592	-0.0353	-2.0869
15	1.0767	1.0473	1.0773	-0.0536	-2.7836
T	$g = 2\%$				
5	1.0735	1.0661	1.0745	-0.0978	-0.7803
10	1.1383	1.1250	1.1403	-0.1798	-1.3468
15	1.1846	1.1666	1.1874	-0.2388	-1.7547
	$\gamma = 4\%$				
5	1.0320	1.0280	1.0320	-0.0001	-0.3918
10	1.0625	1.0551	1.0625	-0.0054	-0.6971
15	1.0817	1.0718	1.0818	-0.0094	-0.9289
	$\gamma = 8\%$				
5	1.1110	1.1013	1.1142	-0.2850	-1.1602
10	1.2070	1.1894	1.2130	-0.4965	-1.9521
15	1.2778	1.2540	1.2861	-0.6422	-2.4980
	$\eta = 80\%$				
5	1.0036	1.0083	1.0104	-0.6804	-0.2121
10	1.0104	1.0191	1.0230	-1.2336	-0.3837
15	1.0111	1.0229	1.0282	-1.6661	-0.5179
	$\eta = 90\%$				
5	1.0301	1.0316	1.0392	-0.8734	-0.7298
10	1.0589	1.0618	1.0757	-1.5588	-1.2916
15	1.0769	1.0808	1.0997	-2.0742	-1.7141

Table 6: Prices of variable annuities for a 60 year old man and maturities up to $T = 15$ years. The specifications of annuities are the following: $\Delta = 1$, $\eta = 1$, $g = 2\%$, $\gamma = 6\%$ and $r = 2\%$. The intensities of jumps processes are set to their asymptotic expectation: $\lambda_0^1 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^1\right)$ and $\lambda_0^2 = \lim_{t\to\infty} \mathbb{E}\left(\lambda_t^2\right)$. Other parameters are reported in tables 4 and 5.

Adding a feedback mechanism in the dynamics of the S&P 500 has a limited impact on the pricing of EIA. However, we will see that it has heavy consequences on the risk management policy of the company. To emphasize this point, we calculate the value at risk (VaR) and the tail value at risk (TVaR) of the net asset value (NAV). According to Solvency II, the NAV is the difference between the market values of assets and liabilities. The solvency capital is the difference between the expected NAV and its 99.5% percentile, at the end of a one-year time horizon. In our tests, we consider only one annuity, bought by a 60 year old man and expiring in 10 years. The other features of the EIA are: $C = 1$, $\Delta = 1$, η=1, $g = 2\%$, $\gamma = 6\%$. The NAV at time $t = 0$ is null and we assume that the premium paid to the insurance company is totally invested in the reference index, without any hedge. Three dynamics are considered for this index: MEJD, JD and B&S models.

The VaR and TVaR of the NAV are computed with 1000 Monte Carlo simulations, and for confidence levels of 1% 5% and 10%. The results are reported in table 7 in % of the annuity premium and

$g = 0\%$		VaR$_\alpha$ (%)			TVaR$_\alpha$ (%)	
α	MEJD	B&S	JD	MEJD	B&S	JD
10%	-23.44	-19.11	-25.94	-36.81	-26.47	-33.22
5%	-32.50	-25.37	-31.05	-45.73	-31.40	-38.20
1%	-52.57	-36.14	-44.07	-62.98	-39.12	-48.58
$g = 2\%$		VaR$_\alpha$ (%)			TVaR$_\alpha$ (%)	
α	MEJD	B&S	JD	MEJD	B&S	JD
10%	-24.64	-20.30	-27.08	-38.03	-27.66	-34.36
5%	-33.69	-26.56	-32.19	-46.96	-32.59	-39.34
1%	-53.80	-37.32	-45.21	-64.22	-40.31	-49.72
$\gamma = 8\%$		VaR$_\alpha$ (%)			TVaR$_\alpha$ (%)	
α	MEJD	B&S	JD	MEJD	B&S	JD
10%	-23.84	-19.69	-26.33	-36.77	-27.07	-35.83
5%	-34.34	-25.85	-34.67	-44.82	-31.58	-41.46
1%	-51.12	-34.82	-46.30	-60.12	-37.79	-50.48
$\eta = 80\%$		VaR$_\alpha$ (%)			TVaR$_\alpha$ (%)	
α	MEJD	B&S	JD	MEJD	B&S	JD
10%	-25.53	-21.33	-27.84	-38.56	-28.71	-37.33
5%	-36.09	-27.49	-36.17	-46.63	-33.22	-42.96
1%	-53.07	-36.46	-47.81	-61.99	-39.43	-51.98

Table 7: Age=60 years, initial maturity 10 years, del=1; eta=1; γ=6%; r=2%. This table presents the VaR and TVaR of the NAV of one annuity (as a percentage of the premium), in one year. The contract is purchased by a 60 year old man with the initial features: $\Delta = 1$, $\eta = 1$, $g = 2\%$, $\gamma = 6\%$ and $r = 2\%$.

compared with these obtained with a B&S model. We observe that whatever the set of parameters, the 1% VaR and TVaR computed with the MEJD are significantly higher than those calculated using the JD model or a lognormal distribution. This observation is the direct consequence of the presence of a contagion mechanism in the MEJD model.

6 Conclusions

This study is among the first to evaluate the impact of volatility clustering on pricing and risk management of variable annuities. The novelty of our approach consists of modelling the stock index and determining the participation rate of the annuity by a mutually excited jumps diffusion (MEJD). In this framework, the intensities of positive and negative shocks are amplified proportionally to recent historical jumps. Moreover, they revert exponentially to a target level in the absence of an event. The first part focuses on the theoretical properties of the MEJD. In particular, we find analytical and semi-analytical expressions for the MGFs of intensities and stock returns. In absence of mutual excitation between positive and negative jumps, new parametric closed form expressions for these MGFs are found.

The second part of the paper introduces an econometric procedure to fit the MEJD process to time series data. This approach, inspired from the peaks over threshold method used in extreme value theory, is applied to measure the level of self and mutual excitations between positive and negative jumps in the S&P 500 daily returns. The model fitted to this sample of observations captures stylized features of stock markets. First, after a large fall, the market tends to bounce back, even if only temporarily. Second, the level of self-excitation of negative jumps is significantly important.

The third part of this work presents the specifications of equity indexed annuities and disentangles their value in a sum of call options. The evaluation of these instruments requires the PDF of forward stock returns. As an exact calculation is not possible, the forward returns are approached by a set of NIG random variables, fitted by moment matching. The numerical application reveals that annuity prices computed with MEJD, JD, and lognormal models do not differ by more than one or two percent. In fact, MEJD and JD models exhibit better fit tails of distribution than the B&S model. But this effect is not reflected by EIA prices, mainly because these prices depend upon the difference between call options, of the same maturity, with different strikes. The impact of extreme shocks on EIA values is then offset by this difference. However, this takes into account the fact that the feedback mechanism between jumps in the dynamics of reference index has a huge impact on risk management. The risk that grouped negative jumps hit the reference index return over a short period of time, increasing the VaR and TVaR of the NAV by at least 5% , compared to the yields by a lognormal model.

Appendix A, mortality assumptions

In the examples presented in this paper, the real mortality rates $\mu(x+t)$ are assumed to follow a Gompertz Makeham distribution. The chosen parameters are those defined by the Belgian regulator ("Arrêté Vie 2003") for the pricing of life annuities purchased by males. For an individual of age x, the mortality rate is given by:

$$\mu(x) = a_\mu + b_\mu c_\mu^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = -\ln(g_\mu)\ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values given in Table 8. As an example, Table 9 presents the progression of mortality rates with age for the male individual. The survival probability is computed as follows: ${}_tp_x = 1 - \exp\left(-\int_0^t \mu(x+s)ds\right)$.

Table 8: Belgian legal parameters for modelling mortality rates, for life insurance products, targeting a male population.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.101077536030

Table 9: Mortality rates, predicted by the Gompertz Makeham model based on parameters of table 8.

Age x	$\mu(x)$
30	0.10%
40	0.18%
50	0.37%
60	0.88%
70	2.23%
80	5.74%

Appendix B, moment generating function of a jump diffusion process

In numerical applications, EIA price yields by the MEJD are compared with these computed with a jump diffusion (JD) model without mutual and self-excitation. The dynamics of this JD model is similar to the one of the MEJD:

$$\frac{dS_t}{S_t} = \mu_J dt + \sigma dW_t + (e^{J_1} - 1)dN_t^1 + (e^{J_2} - 1)dN_t^2 \tag{57}$$

excepted that N_t^1 and N_t^2 are here Poisson processes with constant intensities λ^1 and λ^2. J_1 and J_2 are still exponential jumps. In this case, the drift is not random and equal to

$$\mu_J = \mu - \lambda^1 \mathbb{E}\left(e^{J_1} - 1\right) - \lambda^2 \mathbb{E}\left(e^{J_2} - 1\right) .$$

In this framework, stock prices admit the following representation under the real measure,

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2 - \sum_{i=1,2} \mathbb{E}(e^{J_i} - 1)\lambda^i\right) t + \int_0^t \sigma\, dW_s + \sum_{i=1,2} \int_0^t J_i dN_s^i\right) \tag{58}$$

whereas under the risk neutral measure, the drift μ is replaced by the constant risk free rate. To build the NIG approximation of the JD process, by moment matching, the next corollary is used:

Corollary 6.1. *In the JD model, the moments of forward return under Q admit a closed form expression:*

$$\begin{aligned} \mathbb{E}\left(\left(\frac{S_{t_j}}{S_{t_{j-1}}}\right)^\beta |\mathcal{F}_0\right) &= \exp\left(\left(\beta\left(r - \frac{1}{2}\sigma^2\right) + \frac{\beta^2\sigma^2}{2}\right)\Delta\right. \\ &\quad \left. - \left(\beta \sum_{i=1,2} \lambda^i \left(\psi^i(1) - 1\right) - \sum_{i=1,2} \lambda^i \left(\psi^i(\beta) - 1\right)\right)\Delta\right) \end{aligned}$$

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