

Chapter 5

Precursors of decimal fractions processing

*Abstract*¹

Decimal fractions (for instance 0.5) are rational numbers ($5/10$) represented using the decimal place-value system. Regarding epistemology and history, a plausible developmental prediction would be that the understanding of rational numbers represented by common fractions precedes the understanding of decimal fractions represented using the place-value system. This study investigated the precursors of decimal fraction performance through a longitudinal study. Performance in rational numbers and place-value system understanding on natural numbers were measured before the teaching of decimal fractions in Grade 4. The decimal fraction performance was measured before, during and after the teaching of decimal fractions. Results show that the decimal system understanding was the strongest predictor of further performance on decimal fractions.

1. This research has been conducted in collaboration with Julie Fanuel, Jacques Grégoire and Christophe Mussolin. It will be published as "Predicting the acquisition of decimal fractions: when the understanding of the place-value system counts more than the understanding of fractions".

5.1 Introduction

It is widely assumed that mathematical knowledge is hierarchical and that the learning of basic concepts provides the foundation for mastering more complex mathematics (e.g., Aunola, Leskinen, Lerkkanen, & Nurmi, 2004 ; Hecht & Vagi, 2010 ; Jordan, Kaplan, Ramineni, & Locuniak, 2009 ; Moeller, Pixner, Zuber, Kaufmann, & Nuerk, 2011). For instance, basic numerical knowledge such as number magnitude understanding predicts, or is related to, the acquisition of arithmetic skills (Butterworth, 2005 ; De Smedt, Verschaffel, & Ghesquière, 2009 ; Holloway & Ansari, 2009). Whereas many authors have examined the numerical precursor competencies of arithmetic skills for whole numbers, the investigation of the predictors of more complex numbers such as fractions has only begun (Hecht, Close, & Santisi, 2003 ; Hecht & Vagi, 2010).

From a mathematical point of view, decimal fractions are the subset of rational numbers that can be written as common fractions with a power of ten as denominator (e.g., $2/10$) or using the place-value system (e.g., 0.2). Historically, some rational numbers were first represented by common fraction. This kind of representation was used from the second millennium BC onwards (O'Connor & Robertson, 2000a). A second kind of representation, using the place-value system, has been suggested hundreds of years later, from the tenth century AC onwards (O'Connor & Robertson, 1999a ; Salah Ould Moulaye, 2004). This last notation is more convenient for calculations and is thus often used now in daily life. However, the use of common fractions historically precedes the use of decimal fractions represented with the place-value system. From a developmental point of view, one could ask how children acquire knowledge about decimal fractions and whether this development is modeled on epistemology or history. Undoubtedly, the learning of decimal fractions written using the place-value system (e.g., 0.21) involves both the understanding of natural numbers and place-value system (e.g., 21) and the knowledge of common fractions (e.g., $2/10$ and $1/100$). In the following sections, we will discuss successively the learning of the decimal number system, rational numbers, and finally decimal fractions.

5.1.1 The learning of the decimal place-value number system

In the written decimal system, numbers are represented by ten Arabic numerals (referred to as digits hereafter) that can be assembled by the so called place-value principle. According to this principle, the position of a digit in the sequence determines its value. Starting from the right, the value begins by the units and increases by a power of 10 at each step to the left. The mastery of the single-digits and of the place-value principle is essential to perform number comparison and multi-digit arithmetic tasks. The understanding of number requires knowing, for instance, that the 5 in 152 represents five sets of ten.

Nevertheless, children could experience difficulties in the learning of the place-value principle. These difficulties could be reflected by errors in transcoding between verbal number words and Arabic numbers for three-digit or more digit numbers (e.g., Camos, 2008 ; Power & Dal Martello, 1990 ; Zuber, Pixner, Moeller, & Nuerk, 2009), in number comparison (Mann, Moeller, Pixner, Kaufmann, & Nuerk, 2011 ; Pixner et al., 2009) or in multi-digit arithmetic problems (Deschuyteneer, De Rammelaere, & Fias, 2005 ; Fuson et al., 1997). The lack of conceptual understanding of the place-value principle could also be illustrated by children's difficulty to identify the tens, hundreds, and so forth in a multi-digit number, or to explain 10-for-1 trading with concrete representations (Fuson et al., 1997). Furthermore, about 80% of first- and second-grade children showing errors related to place-value were also impaired in other domains like counting and arithmetic (Gervasoni & Sullivan, 2007). Moreover, this representation requires in fact a complete understanding of the role of the zero, to know that 0.5 is equal to 0.50 for instance. A strong understanding of the place-value system could thus be important for future competencies, especially for the representation of decimal fractions.

5.1.2 The learning of rational numbers

Since Kieren's mathematical analysis of rational numbers (1976, 1980), the concept of rational number has been considered not as a single concept but as several interrelated subconstructs, namely part-whole, ratio, operator, quotient and measure subconstruct. Later, a theoretical model linking these subconstructs has been proposed (Behr, Harel, Post, & Lesh, 1992). The part-whole subconstruct seems to have a particular status. Connected to the ability to partition either a continuous quantity or a set of discrete objects into equal-sized subparts or sets, it was considered as permeating the four other subconstructs (Kieren, 1976, 1980) or as being fundamental to the learning of the four other subconstructs of rational number (Behr, Lesh, Post, & Silver, 1983). More recently, an attempt to provide empirical validity to the theoretical model showed that the tasks measuring the part-whole subconstruct were the most succeeded by fifth- and sixth-grade children, supporting the assumption that the part-whole subconstruct constitutes a foundation to the understanding of the four remaining subconstructs (Charalambous & Pitta-Pantazi, 2007).

The learning of rational numbers is one of the most challenging for students (Carpenter, Fennema, & Romberg, 1993 ; Mack, 1995 ; Mazzocco & Devlin, 2008 ; Ni, 2001). One factor leading to difficulties lies in the fact that rational numbers were multifaced as explained above. Another factor results from the conceptual change needed in the acquisition of rational numbers due to the interference of whole number knowledge (Desmet et al., 2010 ; Merenluoto & Lehtinen, 2004 ; Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004). Regarding the precursors of fraction skills, it appeared that proficiency in simple arithmetic correlates consistently and uniquely with accuracy on fraction computation when measured at one time point (Hecht et al., 2003). However, results

from a longitudinal study with fourth- and fifth-graders showed that arithmetic fluency did not mediate group differences in emerging fraction skills (Hecht & Vagi, 2010). The conceptual knowledge about fractions (part-whole conceptual knowledge and adding rational numbers by means of pictures) was the stronger predictor of subsequent fraction computation, fraction estimation and word problem solving with fraction quantities. The part-whole subconstruct seems thus to be a reliable precursor of subsequent fraction competencies and could also be a precursor of decimal fraction competencies.

5.1.3 The learning of decimal fractions

The learning of decimal fractions as well as the learning of rational numbers is notoriously difficult because it requires conceptual change (Desmet et al., 2010 ; Lehtinen et al., 1997 ; Merenluoto & Lehtinen, 2002 ; Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004). Various misconceptions about decimal fractions have been observed (Desmet et al., 2010 ; Nesher, 1986 ; Resnick et al., 1989 ; Sackur-Grisvard & Leonard, 1985). Some of these misconceptions are based on natural number knowledge, like 1.14 considered as larger than 1.7 because 14 is larger than 7, and others are based on fraction knowledge, like the decimal fraction with the shorter decimal part considered as the bigger (for instance, $1.2 > 1.35$) because “a tenth is always bigger than a hundredth” (Resnick et al., 1989). Natural number knowledge and fraction knowledge could thus lead to misconceptions for decimal fractions, but they are also required to understand these new numbers.

5.1.4 The present study

The present study investigated three competing hypotheses about the learning of decimal fractions. First, the competency in decimal fractions would be equally predicted by competency in natural numbers (and place-value system) and by competency in fractions (Hypothesis 1). Alternatively, competency in natural numbers and place-value system (Hypothesis 2) or competency in fractions (Hypothesis 3) would be the best predictor of knowledge about decimal fractions. To test these hypotheses, we conducted a longitudinal study in children from the beginning of Grade 4. Traditionally, in the French-speaking community of Belgium, some simple fractions like $\frac{1}{2}$ or $\frac{1}{4}$ are taught from Grade 2 and decimal fractions are taught in Grade 4. Children were assessed on natural numbers, place-value system, fraction knowledge and prior conception of decimal fractions before the teaching of decimal fractions (Time 1), and on decimal fractions during (Time 2) and after (Time 3) the teaching of decimal fractions.

5.2 Method

5.2.1 Participants

Fifty-five children of Grade 4 (29 boys and 26 girls; mean age 111.58 months, $SD = 3.73$) participated to the present study. Children came from three voluntary classes of the French-speaking community of Belgium. Exclusionary criteria concern children who repeated Grade 4 and children absent at one of the assessment sessions. Some additional children were excluded because they repeated Grade 4 or were absent at one of the sessions.

5.2.2 Measures

Fraction knowledge

Two tasks assessed the part-whole subconstruct of fraction knowledge. In a first task (10 items), participants were presented a figure with a part shaded (e.g., 1:2) and indicated the fraction that was shaded. In a second task (12 items), participants were presented with a fraction symbol and had to shade the part of the figure corresponding to the fraction (e.g., 1:2) beside it. Another task assessed the operator subconstruct of fraction knowledge (10 items) by asking children to compute a fraction of an Arabic number (e.g., 1:4 of 8 = ...).

Decimal place-value system understanding

Three tasks were used to assess the understanding of the decimal place-value system with natural numbers. The first one (20 items) was a transcoding task of natural numbers from written number words to Arabic numbers, and vice-versa (e.g., 1002 $\leftrightarrow$ one thousand and two). In the second task (10 items), participants answered questions reflecting the place-value understanding, for instance, “which number comes after 99999?”. The third task (20 items) involved placement of natural numbers on number lines or identification of natural numbers on number lines.

Arithmetic knowledge

Arithmetic knowledge was assessed by a task requiring children to solve simple (e.g., $54 + 32$) and complex (e.g., $37 + 19$) additions on one-, two- and three-digit natural numbers. A timed version (4 minutes) has been used for the kinds of addition.

Decimal fraction knowledge

Three tasks assessed knowledge about decimal fractions. The first one was a comparison task on pairs of decimal fractions (70 items). Children were presented with two decimal fractions (e.g., 0.2 vs. 0.06) and were asked to “circle the larger number or to

place an equals sign between them if the numbers are the same in value". The second task concerned the order and betweenness of decimal fractions (32 items). Children were asked to write a number included between two decimal fractions (e.g., $0.2 < \dots < 0.3$). The third task implied transcoding of decimal fractions from written number words to Arabic numbers and inversely (e.g., $0.01 \leftrightarrow$ one hundredth) (20 items).

Decimal fraction computation

Two tasks assessed the decimal fraction computation. In these tasks, children were required to solve simple (e.g., $0.2 + 0.5$) and complex (e.g., $0.5 + 0.7$ or $0.3 + 0.04$) additions on decimal fractions in both timed and untimed versions.

5.2.3 Procedure

The tasks were presented at three time points throughout a longitudinal study. They were administrated by the experimenter collectively in the classroom. The first time point (T_1) was at the autumn term (October) of Grade 4, before any formal teaching of decimal fractions. At this time point, all tasks were administrated at three different days. The second time point (T_2) was at the winter term (February) of Grade 4, during the first teaching of decimal fractions. The third time (T_3) was at the summer term (June) of the same year, after the teaching of decimal fractions. At the second and third time points, the tasks including decimal fractions were presented, but not the tasks including natural numbers or fractions.

5.3 Results

5.3.1 Factorial analyses

To investigate whether our tasks measured the fraction knowledge, decimal system understanding, arithmetic knowledge, decimal fraction knowledge and decimal fraction computation factors, we conducted factor analysis using principal components extraction on the mean accuracies².

Three components emerged with eigenvalues larger than one (4.08, 1.43 and 1.13 respectively) and accounted for 60.28% (37%, 13% and 10% respectively) of the common variability. Loadings on each component after varimax rotation were presented for each task in Table 1. The component loading criterion of .5 or higher was used to observe task that loaded strongly and singularly on a latent component.

The component matrix after rotation indicated that all tasks implying natural numbers showed strong saturations on the first component. The task involving fraction to

2. These analyses were conducted on a larger sample including the participants of the present study and 50 additional children. In all, 105 children of Grade 4 (54 boys and 51 girls; mean age 111.76 months, $SD = 3.68$) participated.

operate on natural numbers also showed higher saturation on the first component, while the two other tasks involving fractions showed higher saturation on the third component. All the tasks on decimal fractions showed higher saturations on the second component, but the betweenness task also showed a high saturation on component 1.

The first component seemed to reflect the natural numbers knowledge. A composite variable, labelled Decimal System Understanding, was created with three tasks: transcoding, place-value problems and number lines. Although the addition task showed high saturation on the same component than these three tasks, we choose to consider it as a different factor, labelled Arithmetic Knowledge, because it could be interesting to distinguish arithmetic knowledge (for instance, $5 + 9 = 14$) from place-value understanding in the case of decimal fractions addition (for instance, $0.05 + 0.09 = 0.14$ and not 0.014).

The second component seemed to be more related to decimal fraction competencies. Although the addition task showed saturation on the same component than the others tasks implying decimal fraction, we consider it a different factor, labelled Decimal Fraction Computation, because it implies more procedural knowledge than the other tasks. A composite variable, labelled Decimal Fraction Knowledge, was created with the three other tasks.

Finally, the third component appeared to be associated with part-whole fraction knowledge. A composite variable, labelled Fraction Knowledge, was created with the two tasks assessing part-whole knowledge. The task involving the operator sub-construct of fraction was excluded from further analysis, because it involved natural numbers and thus showed higher saturation on the first component than on the third component.

Tabl. 1
Component loadings of each task.

Task	Component 1	Component 2	Component 3
Fraction			
Part-whole (figure shaded)	.265	.192	.758
Part-whole (figure to shade)	.154	.042	.835
Operator on numbers	.607	.251	.372
Natural numbers			
Transcoding	.691	-.008	.187
Place-value problems	.812	.029	.192
Number lines	.726	.124	.220
Addition	.604	.376	-.213
Decimal Fractions			
Addition	.103	.747	.231
Transcoding	-.074	.743	.096
Betweenness	.448	.509	.346
Comparison	.265	.679	-.063

5.3.2 Achievement at the first measurement point

The means and standard deviations for each measure at T_1 were reported in Table 2. The main effect of Task was significant, $F(2.50, 135.18) = 78.60$, $p < .001$, partial $\eta^2 = .58$. The correct response rates were higher for Decimal System Understanding and Fraction Knowledge tasks than for Arithmetic Knowledge and Decimal Fraction Knowledge tasks. The Decimal Fraction Computation task yielded the lowest correct response rates ($p < .001$ for all comparisons). Table 2 also provides an overview of the bivariate correlations among these measures. The correlations between Decimal Fraction Computation and Arithmetic Knowledge or between Decimal Fraction Computation and Decimal Fraction Knowledge were positive and significant, while the correlations between Decimal Fraction Knowledge and other factors were not significant. All the correlations between Decimal System Understanding, Fraction Knowledge and Arithmetic Knowledge were positive and significant.

Tabl. 2

Means and standard deviation for each measure, and bivariate correlations among measures.

	Correlations									
Measures	M	SD	1.	2.	3.	4.	5.	6.	7.	8.
Before the teaching (T_1)										
1. Fraction knowledge	68.5	20.2	—							
2. Decimal system understanding	71.9	16.6	.503**	—						
3. Arithmetic knowledge	43.6	20.1	.422**	.643***	—					
4. Decimal fraction knowledge	35.8	12.3	.214	.250	.200	—				
5. Decimal fraction computation	26.6	25.7	.215	.153	.265 ^{-.05}	.513***	—			
During the teaching (T_2)										
6. Decimal fraction knowledge	58.9	13.4	.394**	.476***	.377**	.271 ^a	.210	—		
7. Decimal fraction computation	66.2	26.5	.284*	.402**	.462***	.343*	.383**	.514***	—	
After the teaching (T_3)										
8. Decimal fraction knowledge	83.3	15.7	.441**	.566***	.398**	.339**	.203*	.834***	.596***	—
9. Decimal fraction computation	77.1	27.4	.357**	.455**	.443***	.285	.331*	.540***	.874***	.656***

Note. *** = $p < .001$, ** = $p < .01$, * = $p < .05$, ^a = $p = .05$.

5.3.3 Longitudinal analyses

Changes in decimal fraction performance

We first investigated the improvement of performance for decimal fractions (knowledge and computation measures) across time points (see Table 2). Two-way repeated measures ANOVA were conducted on the mean accuracy with the Measure (1-2) and the Time (1-3) as the within-subject factors. The main effect of Measure was not significant, $F(1, 54) = 3.12$. The main effect of Time was significant, $F(1.28, 69.07) = 208.33$, $p < .001$, partial $\eta^2 = .79$, showing an increase in accuracy from T_1 to T_3 . Post hoc t-tests, adjusted with the Bonferroni correction at alpha level $0.05/3 = 0.017$, showed that all time measures differed significantly from each other (in all cases, $ps < .001$). The interaction between Measure and Time, $F(1.46, 78.94) = 19.93$, $p < .001$, partial $\eta^2 = .27$, indicated that the improvement of performance from T_1 to T_2 was higher for Decimal Fraction Computation ($p < .01$) compared to Decimal Fraction

Knowledge ($p < .05$) and that the reverse was true from T_2 to T_3 ($p < .001$ for the two improvements).

Prediction of decimal fraction performance

The main question of the present study was whether Fraction Knowledge, Decimal System Understanding and Simple Arithmetic Knowledge equally predicted Decimal Fraction Knowledge and Computation or whether one factor was a better predictor than the others. Simple regressions were conducted for each predictor and for prior ability on decimal fractions. The proportion of variance accounted by each measure is reported in Table 3. The results showed that each predictor significantly contributed to the improvement in decimal fraction performance, except Decimal Fraction Computation for Decimal Fraction Knowledge. However, the proportion of variance explained by each measure differed. The Decimal System Understanding explained a greater part of variance than Fraction Knowledge and Arithmetic Knowledge for Decimal Fraction Knowledge, while Decimal System Understanding and Arithmetic explained a similar part of variance for Decimal Fraction Computation.

Tabl. 3

Proportion of variance in Decimal Fraction Knowledge and Decimal Fraction Computation explained by each predictor at Time 2 and Time 3 ($N = 105$) using simple regression analyses

Decimal Fraction	Time 2				Time 3			
	Knowledge		Computation		Knowledge		Computation	
	R^2	p	R^2	p	R^2	p	R^2	p
Predictors:								
Fraction knowledge	.15	< .01	.08	< .05	.19	= .001	.13	< .01
Decimal system understanding	.23	< .001	.16	< .01	.32	< .001	.21	< .001
Arithmetic knowledge	.14	< .01	.21	< .001	.16	< .01	.20	= .001
Prior ability								
Decimal fraction knowledge	.07	< .05	.12	= .01	.12	= .01	.08	< .05
Decimal fraction computation	.04	ns.	.15	< .01	.04	ns.	.11	< .05

To investigate further the influence of Decimal System Understanding and Fraction Knowledge as well as prior abilities in decimal fractions on further decimal fractions competencies in T_2 and T_3 , stepwise regression analyses were conducted. For Decimal Fraction Knowledge, the final model for T_2 ($R = .476$, adjusted $R^2 = .227$, $F(1, 53) = 15.54$, $p < .001$) and for T_3 ($R = .566$, adjusted $R^2 = .307$, $F(1, 53) = 24.95$, $p < .001$) included three predictors (see Table 4). The variance inflation factors (VIF's) were always less than 1.8^3 . For T_2 as well as for T_3 the models showed that the Decimal System Understanding was the strongest predictor of Decimal Fraction Knowledge. As indicated

3. Most authors assumed that a VIF value of less than 10 indicates an inconsequential collinearity (Hair, Anderson, Tatham, Black, 1995; Menard, 1995).

by the positive beta weights, a higher success rate in Decimal System Understanding in T_1 was associated with a better performance in decimal fractions in T_2 and T_3 .

The final model for Decimal Fraction Computation included two factors for T_2 ($R = .535$, adjusted $R^2 = .259$, $F(1, 52) = 10.44$, $p < .001$) and for T_3 ($R = .527$, adjusted $R^2 = .277$, $F(1, 52) = 9.97$, $p < .001$) (see Table 4). At T_2 , the Arithmetic Knowledge explained the greater part of variance, and adding the prior ability in the Decimal Fraction Computation increased the explained variance. As showing by the beta weights, children who performed better in Arithmetic Knowledge and Decimal Fraction Computation at T_1 exhibited a higher success rate for Decimal Fraction Computation at T_2 . At T_3 , the best predictor was the Decimal System Understanding and adding the prior ability in the Decimal Fraction Computation increased the explained variance. Children who performed better in Decimal System Understanding and who had a higher prior ability in Decimal Fraction Computation at T_1 showed higher accuracy in Decimal Fraction Computation at T_3 .

Tabl. 4

Stepwise regression analysis predicting Decimal Fraction Computation and Decimal Fraction Knowledge at T_2 and T_3

	Time 2					Time 3			
	Measure	β	R^2	ΔR^2		Measure	β	R^2	ΔR^2
Decimal Fraction Knowledge									
Step 1. Decimal System		.476	.212	.227***	Decimal System	.566	.307	.320***	
Decimal Fraction Computation									
Step 1. Arithmetic		.388	.199	.214**	Decimal System	.414	.192	.207***	
Step 2. Prior ability		.280	.259	.073*	Prior ability	.268	.277	.070*	

Note. Decimal System = Decimal System Understanding, Arithmetic = Arithmetic Knowledge, Prior ability = Decimal Fraction Computation at T_1 .

5.4 Discussion

The present study investigated precursors of performance in manipulating decimal fractions in children of Grade 4. At the second time point, children have received some formal teaching about decimal fraction. The results showed that the understanding of decimal fractions improved, and that the Decimal System Understanding based on natural numbers was the strongest predictor of this first progression (Hypothesis 2) for Decimal Fraction Knowledge, but not for Decimal Fraction Computation. For Decimal Fraction Computation, Arithmetic Knowledge was the strongest predictor and prior ability in Decimal Fraction Computation explained a small but significant part of the variance. At the third time point, children still progressed in processing decimal fractions. Again Decimal System Understanding was the stronger predictor of the improvement in

Decimal Fraction Knowledge and in Decimal Fraction Computation (Hypothesis 2). The prior knowledge still had an impact for Decimal Fraction Computation. These results did not fit with the view that the competency in decimal fractions was equally predicted by Decimal System Understanding and Fraction Knowledge (Hypothesis 1), or strongly by Fraction Knowledge (Hypothesis 3).

Mathematically, decimal fractions are a subset of rational numbers : they are rational numbers that can be written by a fraction where the denominator is a power of ten (e.g., $5/100$). Historically, decimal fractions represented using the place value system appeared hundreds of years after the representation using common fractions. A plausible developmental prediction would thus be that the understanding of rational numbers represented by fractions, for instance $1/2$ or $1/4$, precedes the understanding of decimal fractions represented using the decimal system, for instance 0.15. Accordingly, performance in rational numbers represented by fractions would be a strong predictor of further performance in decimal fractions (Hypothesis 3). The results of the current study go against this view. Regression analysis showed that fraction knowledge was a significant predictor of further performance in decimal fraction knowledge and computation, but explained a smaller part of variance when the decimal system understanding was also taken into account. The accuracy on decimal fractions was better predicted by the mastery of the decimal system understanding. Accuracy in Decimal System Understanding i.e., the understanding of the representation system appeared to be more important for Decimal Fraction Knowledge and Computation than accuracy in Fraction Knowledge. This stresses the importance of the representation format and fits with the fact that for adults and even more for Grade 6 children, the fraction notation led to more errors than the decimal notation in a number line task (Iuculano & Butterworth, 2011), and that the fraction notation seemed also to be mastered later than the decimal notation in a comparison task performed by children from Grade 3 to 6 (see chapter 4).

Arithmetic Knowledge was also a predictor of Decimal Fraction Computation. The computation of decimal fraction indeed implies knowledge about simple arithmetic. For instance, the problem $5 + 3$ is useful to compute $0.5 + 0.3$. But, at T_3 , performance in decimal fraction addition was better explained by the place-value understanding than by simple arithmetic. To compute $0.03 + 0.2$, arithmetic knowledge is not enough, place-value knowledge is more important. This is certainly due to the fact that our addition task included complex additions that implied the understanding of the place-value system.

Our findings are thus in accordance with the hierarchical view of mathematical knowledge and learning (e.g. Aunola et al., 2004 ; De Smedt et al., 2009 ; Hecht et al., 2003 ; Hecht & Vagi, 2010 ; Jordan et al., 2009 ; Moeller et al., 2011). However, all plausible precursors from a mathematical point of view had not the same weight. The present results indicated that competencies in whole numbers and decimal system understanding were stronger precursors for the learning of decimal fractions than fraction knowledge. This stress the importance to confront mathematical predictions with developmental

empirical data. It also indicated that, even if the learning of decimal fractions requires conceptual change (Desmet et al., 2010 ; Lehtinen et al., 1997 ; Merenluoto & Lehtinen, 2002 ; Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004) all prior conceptions does not impeded the further learning, some of them are essential for the further learning and could be considered as foundations.

Regarding practical implications, decimal system understanding seems to be essential to the basic understanding of decimal fractions and to compute with decimal fractions. We can so suppose that the teaching of decimal fractions would gain to be based on the understanding of the place-value system rather than on rational numbers represented by common fractions, even if future work requiring didactical intervention is needed to test this prediction. Furthermore, the teaching of decimal fractions, would include a cognitive conflict Posner et al. (1982), but would also help the students to differentiate relevant prior conceptions from prior conceptions impeding the understanding of decimal fractions (diSessa, 1988 ; Clement et al., 1989).

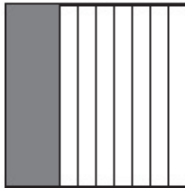
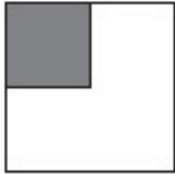
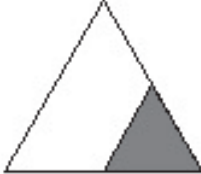
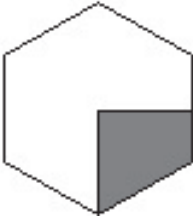
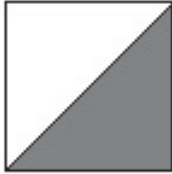
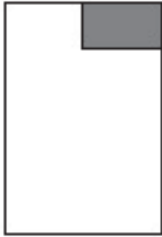
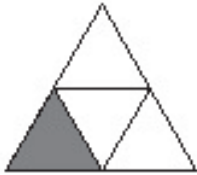
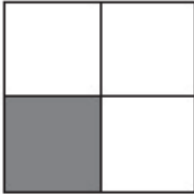
5.5 Appendix

The tasks used to measure fraction knowledge, decimal place-value system understanding, arithmetic knowledge, decimal fraction knowledge and decimal fraction computation were presented hereafter.

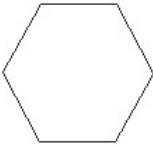
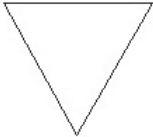
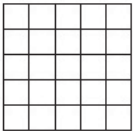
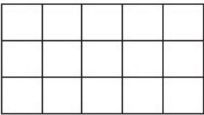
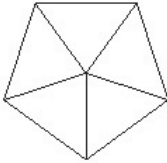
5.5.1 Fraction knowledge

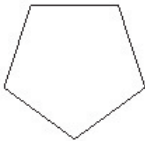
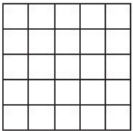
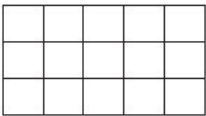
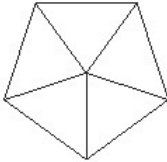
Part-whole subconstruct

Quelle fraction de chaque forme est coloriée [Which fraction is shaded for each figure]?
(10 items)

Colorie la fraction demandée dans chaque série de 6 formes [Colour the asked fraction for each set of 6 figures]. (12 items)

$\frac{1}{2}$			
			

$\frac{1}{10}$			
			

Operator subconstruct

Résous [Solve]. (10 items)

- | | |
|--------------------------|---------------------------|
| $\frac{1}{2}$ de 6=... | $\frac{1}{10}$ de 100=... |
| $\frac{1}{2}$ de 24=... | $\frac{1}{100}$ de 80=... |
| $\frac{1}{2}$ de 13=... | $\frac{1}{4}$ de 10=... |
| $\frac{1}{10}$ de 80=... | $\frac{1}{10}$ de 15=... |
| $\frac{1}{4}$ de 8=... | $\frac{1}{4}$ de 18=... |

5.5.2 Decimal place-value understanding

Transcoding

Écriture de nombres arabes sous dictée [Transcoding verbal numbers into arabic numbers]. (20 items)

Treize
Quatre-vingt trois
Cent neuf
Trois cent cinq
Mille deux
Trois mille seize
Seize cent cinquante quatre
Dix huit cent trente
Trente mille
Vingt-six mille sept cent quarante cinq

Écris ces nombres à l'aide de lettres [Transcoding arabic numbers into verbal numbers].
(10 items)

18
32
56
115
105
300
420
670
567
281

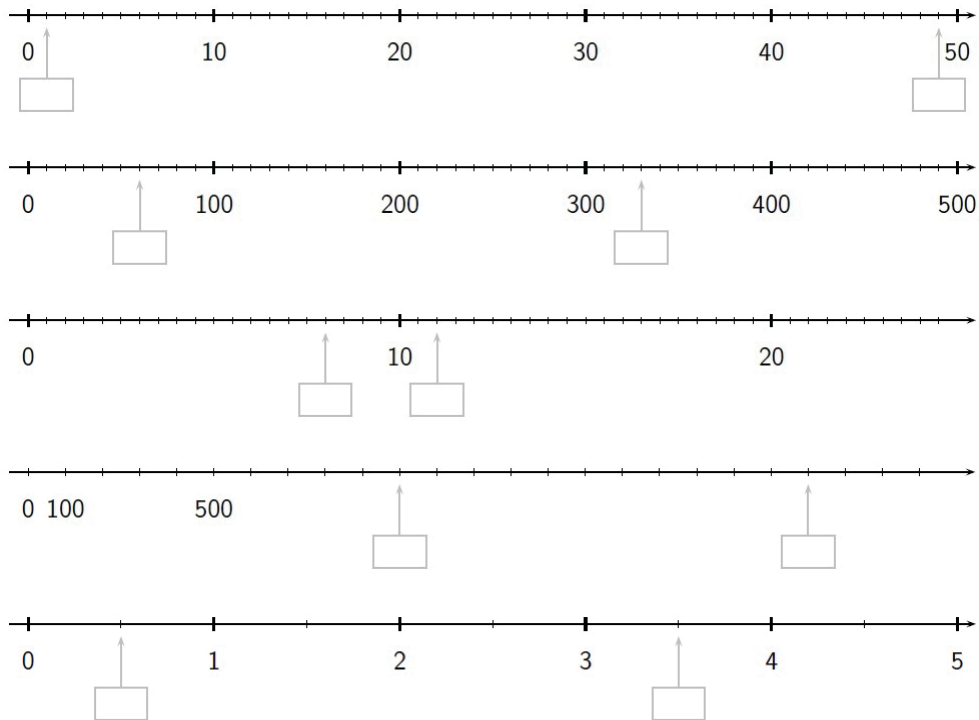
Place-value

Complète [Complete]. (10 items)

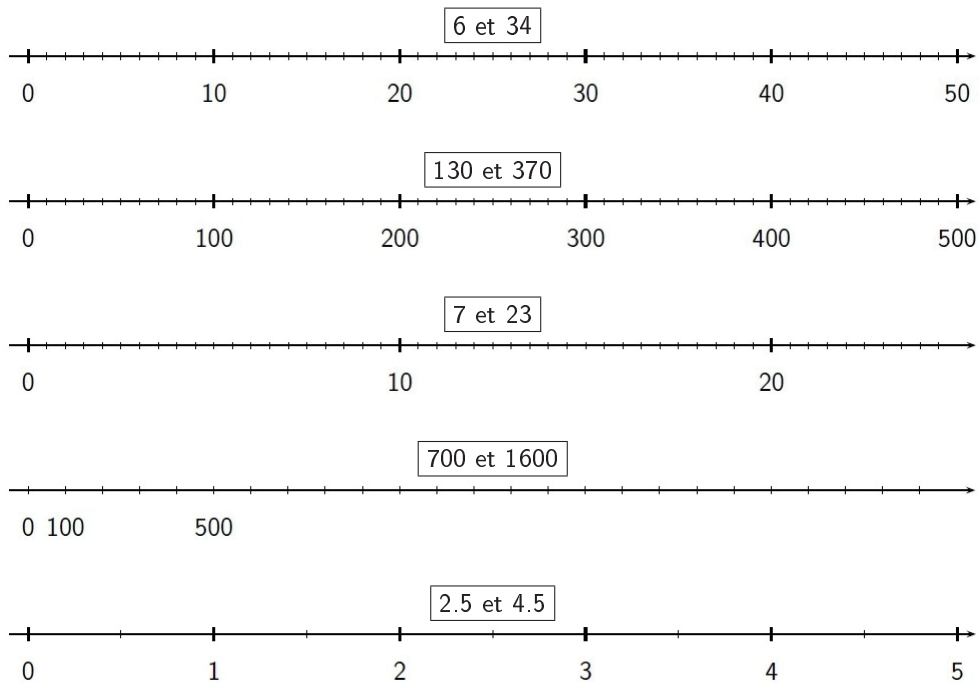
Dix en moins que mille, cela fait ...
Le nombre qui suit 99899 est ...
Une unité en moins que 2 centaines, cela fait ...
Le nombre qui contient 2 dizaines en plus que 145 est ...
Le plus grand nombre de trois chiffres différents c'est ...
Dix en moins que cinq centaines, cela fait ...
Deux dizaines en plus d'une centaine, cela fait ...
Le nombre qui est juste avant 10000 est ...
Le plus petit nombre de deux chiffres c'est ...
Le nombre qui contient deux centaines en plus de 809 est ...

Number lines

Complète les cases [Complete]. (10 items)



Place les nombres suivants [Place these numbers]. (10 items)



5.5.3 Arithmetic knowledge

Simple additions

$67 + 12 = \dots$	$382 + 15 = \dots$	$72 + 6 = \dots$
$527 + 31 = \dots$	$237 + 41 = \dots$	$912 + 73 = \dots$
$91 + 6 = \dots$	$35 + 42 = \dots$	$673 + 25 = \dots$
$25 + 41 = \dots$	$21 + 5 = \dots$	$734 + 51 = \dots$
$52 + 6 = \dots$	$82 + 5 = \dots$	$26 + 3 = \dots$
$38 + 41 = \dots$	$371 + 425 = \dots$	$653 + 241 = \dots$
$67 + 21 = \dots$	$521 + 354 = \dots$	$72 + 6 = \dots$
$713 + 252 = \dots$	$427 + 31 = \dots$	$823 + 41 = \dots$
$835 + 21 = \dots$	$472 + 16 = \dots$	$623 + 142 = \dots$
$642 + 53 = \dots$	$531 + 64 = \dots$	$54 + 32 = \dots$
$632 + 147 = \dots$	$72 + 4 = \dots$	$562 + 314 = \dots$
$326 + 142 = \dots$	$671 + 316 = \dots$	$16 + 23 = \dots$
$152 + 346 = \dots$	$46 + 3 = \dots$	$87 + 21 = \dots$
$72 + 14 = \dots$	$28 + 41 = \dots$	$65 + 31 = \dots$
$71 + 6 = \dots$	$92 + 5 = \dots$	$54 + 3 = \dots$
$63 + 24 = \dots$	$64 + 5 = \dots$	$78 + 21 = \dots$
$12 + 34 = \dots$	$482 + 15 = \dots$	$471 + 523 = \dots$
$341 + 126 = \dots$	$534 + 62 = \dots$	$631 + 42 = \dots$
$42 + 5 = \dots$	$521 + 68 = \dots$	$713 + 254 = \dots$
$148 + 532 = \dots$	$42 + 51 = \dots$	$56 + 3 = \dots$
$87 + 12 = \dots$	$627 + 351 = \dots$	$45 + 2 = \dots$
$56 + 3 = \dots$	$83 + 4 = \dots$	
$562 + 34 = \dots$	$63 + 25 = \dots$	
$621 + 45 = \dots$	$43 + 5 = \dots$	
$83 + 14 = \dots$	$731 + 142 = \dots$	
$563 + 215 = \dots$	$813 + 245 = \dots$	
$451 + 317 = \dots$	$612 + 57 = \dots$	
$72 + 24 = \dots$	$762 + 17 = \dots$	
$462 + 317 = \dots$	$47 + 51 = \dots$	

Complex additions

$37 + 28 = \dots$	$36 + 5 = \dots$	$261 + 357 = \dots$
$326 + 457 = \dots$	$538 + 27 = \dots$	$725 + 167 = \dots$
$529 + 34 = \dots$	$37 + 19 = \dots$	$381 + 56 = \dots$
$42 + 9 = \dots$	$257 + 416 = \dots$	$194 + 53 = \dots$
$64 + 18 = \dots$	$36 + 18 = \dots$	$467 + 219 = \dots$
$382 + 57 = \dots$	$46 + 15 = \dots$	$784 + 52 = \dots$
$348 + 271 = \dots$	$53 + 28 = \dots$	$765 + 82 = \dots$
$673 + 29 = \dots$	$83 + 9 = \dots$	$583 + 174 = \dots$
$781 + 164 = \dots$	$871 + 42 = \dots$	$235 + 47 = \dots$
$346 + 281 = \dots$	$23 + 38 = \dots$	$56 + 27 = \dots$
$14 + 9 = \dots$	$346 + 28 = \dots$	$319 + 45 = \dots$
$63 + 29 = \dots$	$45 + 17 = \dots$	$236 + 149 = \dots$
$532 + 94 = \dots$	$826 + 49 = \dots$	$629 + 146 = \dots$
$68 + 5 = \dots$	$654 + 293 = \dots$	$365 + 54 = \dots$
$463 + 291 = \dots$	$374 + 218 = \dots$	$673 + 54 = \dots$
$782 + 63 = \dots$	$47 + 5 = \dots$	$46 + 9 = \dots$
$289 + 315 = \dots$	$867 + 15 = \dots$	$628 + 157 = \dots$
$65 + 7 = \dots$	$562 + 174 = \dots$	$73 + 18 = \dots$
$281 + 63 = \dots$	$134 + 248 = \dots$	$319 + 47 = \dots$
$765 + 128 = \dots$	$134 + 49 = \dots$	$273 + 81 = \dots$
$462 + 275 = \dots$	$79 + 14 = \dots$	$29 + 4 = \dots$
$35 + 8 = \dots$	$736 + 58 = \dots$	$573 + 61 = \dots$
$491 + 256 = \dots$	$173 + 285 = \dots$	
$419 + 237 = \dots$	$45 + 9 = \dots$	
$572 + 163 = \dots$	$57 + 6 = \dots$	

5.5.4 Decimal fraction knowledge

Comparison

Pour chaque paire, entoure le nombre le plus grand, ou place un signe = si les nombres sont égaux [Circle the larger number or place an = sign between them if the numbers are the same in value]. (70 items)

0,02	0,1
------	-----

0,10	0,2
------	-----

0,1	0,2
-----	-----

0,01	0,2
------	-----

0,1	0,01
-----	------

0,20	0,1
------	-----

0,3	0,30
-----	------

0,4	0,3
-----	-----

0,3	0,04
-----	------

0,4	0,03
-----	------

0,02	0,2
------	-----

0,40	0,4
------	-----

0,4	0,30
-----	------

0,3	0,40
-----	------

0,4	0,05
-----	------

0,5	0,04
-----	------

0,5	0,50
-----	------

0,50	0,4
------	-----

0,5	0,40
-----	------

0,5	0,4
-----	-----

0,03	0,3
------	-----

0,60	0,5
------	-----

0,5	0,6
-----	-----

0,50	0,6
------	-----

0,4	0,04
-----	------

0,06	0,5
------	-----

0,05	0,6
------	-----

0,60	0,6
------	-----

0,6	0,7
-----	-----

0,07	0,6
------	-----

0,70	0,7
------	-----

0,06	0,7
------	-----

0,5	0,05
-----	------

0,7	0,60
-----	------

0,6	0,70
-----	------

0,04	0,1
------	-----

0,40	0,1
------	-----

0,1	0,4
-----	-----

0,10	0,4
------	-----

0,06	0,6
------	-----

0,01	0,4
------	-----

0,80	0,8
------	-----

0,5	0,2
-----	-----

0,20	0,5
------	-----

0,05	0,2
------	-----

0,9	0,90
-----	------

0,02	0,5
------	-----

0,7	0,07
-----	------

0,2	0,50	0,8	0,80
0,6	0,03	0,7	0,04
0,6	0,30	0,4	0,07
0,3	0,06	0,70	0,4
0,3	0,30	0,5	0,08
0,3	0,60	0,60	0,9
0,6	0,3	0,50	0,5
0,08	0,8	0,3	0,03
0,4	0,7	0,8	0,5
0,09	0,9	0,8	0,05
0,7	0,40	0,5	0,80

Order and betweenness

Écris un nombre plus grand que celui de gauche et plus petit que celui de droite [Write a number larger than the number on the left and smaller than the number on the right].
(32 items)

0,07	<		<	0,7	0,2	<		<	0,30
0,5	<		<	0,6	0,3	<		<	0,40
0,5	<		<	0,7	2	<		<	4
0,03	<		<	0,3	2,7	<		<	4,7
6,5	<		<	7,5	0,01	<		<	0,1
0,2	<		<	0,4	0,7	<		<	0,9
4,3	<		<	5,3	5,1	<		<	7,1
0,3	<		<	0,4	1,2	<		<	3,2
0,7	<		<	0,8	0,05	<		<	0,5
0,6	<		<	0,70	6	<		<	7
0,1	<		<	0,3	4	<		<	5
6	<		<	8	8	<		<	9
0,7	<		<	0,80	3,6	<		<	5,6
5	<		<	7	2	<		<	3
5,8	<		<	6,8	2	<		<	3
0,2	<		<	0,3	1	<		<	3

Transcoding

Écris ces nombres à l'aide de lettres [Transcoding arabic numbers into verbal numbers].
(10 items)

0,1
0,5
0,25
0,32
0,07
0,04
0,002
0,006
2,5
3,4

Écriture de nombres arabes sous dictée [Transcoding verbal numbers into arabic numbers]. (10 items)

Deux dixièmes
Quatre dixièmes
Vingt sept centièmes
Quarante trois centièmes
Cinq centièmes
Huit centièmes
Trois millièmes
Sept millièmes
Une unité quatre dixièmes
Cinq unités trois dixièmes

5.5.5 Decimal fraction computation

Simple additions

5 + 0,4 =
0,4 + 0,1 =
0,05 + 0,03 =
0,4 + 0,25 =
0,31 + 0,47 =
0,4 + 0,02 =
1 + 0,8 =
0,5 + 0,04 =
0,45 + 0,32 =
0,3 + 0,45 =
0,06 + 0,01 =
0,6 + 0,1 =
0,3 + 0,4 =
4 + 0,2 =
0,04 + 0,03 =
0,3 + 0,01 =
0,7 + 0,12 =
0,21 + 0,53 =
0,04 + 0,01 =
0,56 + 0,12 =
0,2 + 0,5 =
0,5 + 0,32 =
0,5 + 0,02 =
7 + 0,1 =
0,14 + 0,51 =
0,2 + 0,06 =
0,02 + 0,05 =
0,2 + 0,56 =
3 + 0,6 =
0,5 + 0,3 =

Complex additions

0,53 + 0,29 =
0,9 + 0,5 =
0,05 + 0,07 =
0,9 + 0,4 =
0,06 + 0,07 =
0,36 + 0,57 =
0,7 + 0,5 =
0,08 + 0,05 =
0,43 + 0,38 =
0,25 + 0,37 =
0,4 + 0,8 =
0,06 + 0,08 =
0,03 + 0,09 =
0,5 + 0,8 =
0,65 + 0,17 =
0,04 + 0,07 =
0,4 + 0,9 =
0,54 + 0,27 =
0,43 + 0,29 =
0,3 + 0,8 =
0,09 + 0,04 =
0,08 + 0,09 =
0,13 + 0,38 =
0,6 + 0,7 =
0,6 + 0,5 =
0,06 + 0,09 =
0,45 + 0,19 =
0,7 + 0,7 =
0,07 + 0,08 =
0,27 + 0,19 =

Chapter 6

The mixed impact of the euro symbol

*Abstract*¹

One way of making decimal fractions meaningful to children would be to include everyday mathematics, for instance knowledge of money, in the classroom. This study investigated the impact of the euro symbol on the processing of decimal fractions. Fourth- to seventh-grade children performed tasks involving comparison and betweenness on different types of decimal fractions including or not including the euro symbol. The results show that euro had an effect on the accuracy modulated by the types of decimal fractions and by the grade or score levels. The euro symbol stressed the processing of the fractional part of the decimal as a natural number. It provided some help to fourth graders and children with poor overall scores, but reduced the correct response rates for older children and children with high overall scores. The inclusion of everyday mathematics related to money in the classroom has therefore to be carefully considered, because it could hinder the development of an accurate understanding of decimal fractions.

1. This research has been conducted in collaboration with Jacques Grégoire and has been submitted as "The mixed impact of the euro symbol on the processing of decimal fractions".

6.1 Introduction

One of the main challenges of mathematics teaching is to make the topic meaningful to children. It is widely assumed that the inclusion of mathematical tasks related to children's everyday lives in the classroom is helpful for this. Various labels have been used to discuss the real-world and circumstantial knowledge which enable children to solve mathematical problems outside of school context (Baranes, Perry, & Stigler, 1989 ; D. W. Carraher & Schliemann, 2002 ; T. N. Carraher, Carraher, & Schliemann, 1987 ; Guberman, 1996 ; Leinhardt, 1988 ; Mack, 1990, 1995 ; Nunes, Schliemann, & Carraher, 1993 ; Saxe, 1988). Throughout this paper this type of knowledge will be referred as to *everyday mathematics*.

In a seminal study, T. N. Carraher, Carraher, et Schliemann (1985) showed that Brazilian street vendors aged from 9 to 15 performed significantly better on arithmetical problems in the context of their work than on the same problems presented symbolically in a school-like context. These results revealed that children learn some mathematics through their everyday experiences, but that success in everyday mathematics does not necessarily lead to success in school-like activities. In another study, T. N. Carraher et al. (1987) showed that the situation in which the problems were presented also had a strong impact on the technique used to solve them. Computation exercises seemed to elicit written algorithmic solutions (symbol manipulation) while children retained the meaning of a simulated store and verbal problems and engaged in the manipulation of quantities with a greater probability of success (see also Nunes et al., 1993). T. N. Carraher et al. (1987) argued that the children do better on verbal and simulated store problems because they call to mind everyday mathematics. This suggests that mathematics teachers can take advantage of everyday mathematics for formal learning. Furthermore, there is some evidence of similarities between everyday mathematics and formal mathematics. Some everyday mathematics includes arithmetical properties, such as additive composition and the commutative and associative laws of addition (D. W. Carraher & Schliemann, 2002). For instance, when subtracting 135 from 200, a street vendor might take away 100, then 30, then 5, reflecting an implicit knowledge of the associative property of addition (i.e., the sum $100 + 30 + 5$).

Nevertheless, these findings need to be interpreted with caution. Some of the children in these studies (such as the street vendors) had limited amounts of schooling, but frequently needed everyday mathematics. Moreover, the effect of context on either the success rate or the strategy used was not replicated in a study of U.S. third graders (Baranes et al., 1989). Second, the activation of everyday mathematics through word problems seems to be more complex than expected. A lot of studies have shown that students around the world are bad at activating their everyday mathematics when solving word problems (e.g. Reusser & Stebler, 1997 ; Verschaffel, De Corte, & Lasure, 1994 ; Verschaffel, Greer, & De Corte, 2000). In the context of school, arithmetical word problems were perceived as different from those in the real world.

Furthermore, everyday mathematics can be quite different from the formal mathematics presented in the classroom (T. N. Carraher et al., 1987 ; Masingila, Davidenko, & Prus-Wisniowska, 1996) and may even constrain the formal learning of mathematics (e.g. Brenner, 1998). For instance, while a real-life object (such as a pie) can be divided into approximately equal parts, in formal mathematics, a fraction of a magnitude (a disk) or a number gives exactly equal parts. The change from everyday mathematics to formal mathematics involves going beyond an intuitive understanding and expressing understanding in new ways (D. W. Carraher & Schliemann, 2002 ; Schliemann & Carraher, 2002) and this process is gradual (Gravemeijer, 1999). It is difficult to give a definitive answer to questions about the use of everyday mathematics in the classroom: each mathematical topic has its specificity and the impact of everyday mathematics on its learning has to be carefully investigated.

The present study focused on one of the most important everyday mathematics topics: money. Little is known about what exactly children understand about money. A longitudinal study showed that young Hawaiian children have some knowledge of money from the beginning of the kindergarten (Brenner, 1998). Preschool children (3;6 to 4 years old) were able to identify dollars and quarters. At this age, money was essentially unquantified, but children already thought that silver coins were less valuable than dollars, while pennies were seen as useless and separated from *real* money². Kindergarten children were more aware than preschoolers that nickels and dimes could be useful, and had more knowledge of the relative values of the various components of the money system. However, none of them were able to say how many cents were equal to one dollar, although they did know that a dollar was worth more than any of the coins and that various combinations of coins could add up to a dollar. In fact, at this age, many children considered the words *cent* and *coin* as synonymous, which lead to confusion when they were asked how many cents there were in a dollar. By the end of Grade 2, nearly all children knew the names and values of all the coins, but most of them were still unable to give the number of cents in a dollar (Brenner, 1998). The majority of children (80%) knew that four quarters equaled a dollar. Interestingly, they interpreted the conventional notation (\$ - . - -) as the conjunction of two types of currency (dollars and cents) rather than as a number of units (dollars) and a fractional part (cents). A more recent study (Guberman, 2004) also showed that most children in Grades 1, 2 and 3 could correctly name the components of the money (one-dollar bills, pennies, nickels, dimes, and quarters), providing either their names (e.g., dime) or values (e.g., ten cents).

Do these abilities with money reflect a deep understanding of the numerical relations between the fractional parts of the currency unit? When a child knows that two nickels are equivalent to one dime, does he or she necessarily understand that this is based on the

2. The U.S. coins in general circulation making up the dollar are minted in denominations of 1 cent (penny, copper colored), 5 cents (nickel, silvered), 10 cents (dime, silvered), 25 cents (quarter, silvered), 50 cents (half, silvered). One dollar is made up of 100 cents.

fact that five hundredths multiplied by two is equivalent to one tenth? The identification of the numerical value of coins as well as their manipulation in solving some basic problems could in fact be made on the basis of their different figurative characteristics (e.g., colors, sizes) rather than on the understanding of the decimal relations between money components. Chandler et Kamii (2009) showed that American children from Grades K to 4 thought, for instance, that the dime (10 cents) was different from 10 pennies, even though they knew that the value of a dime is 10 cents.

Nevertheless, it is widely believed that the use of money improves the learning of decimal fractions because some coins represent fractional values of the euro (or the dollar) and some prices are indicated by a number with a decimal point. The use of money in the classroom is widespread and some studies support this view. Play-money systems have been developed to teach children about the decimal system (e.g. T. Carraher & Schliemann, 1988), although it has been shown that the use of realistic (perceptually rich) bills and coins can detract student's attention from the mathematical ideas being represented (McNeil, Uttal, Jarvin, & Sternberg, 2008). Irwin (2001) argued that the everyday mathematics of decimal fractions³, including monetary exchange between countries, enhances the learning of decimal fractions by sixth- and seven-graders. However, in Irwin's study it is not possible to disentangle the effect of money from that of other variables (for instance the type of number notation). In an exploratory teaching experiment, the use of baking powder label and supermarket receipts using decimal numbers for the weight of some items seemed to improve the learning of the multiplication of decimal fractions by fourth-graders (Bonotto, 2005). However, the performance of the experimental group was not compared to that of a control group in this study. Overall, the conclusion of this review must be that there is a lack of empirical evidence on the impact of familiarity with money on the learning and processing of decimal fractions.

The acquisition of rational numbers, including decimal fractions, is particularly difficult for children. It requires a conceptual change (Desmet et al., 2010 ; Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004), i.e. a reorganization of children's prior knowledge based on natural numbers, which can constrain the learning of decimal fractions and lead to misconceptions. Some of these misconceptions, at an intermediate level of understanding, are synthetic models (Vosniadou, 1994), which result from the learner's attempts to reconcile the new concepts about decimal fractions with their pre-existing knowledge of natural numbers. An important question is how to help children through the conceptual change process. Everyday mathematical knowledge, such as knowledge of money, could be used to give more meaning to decimal fractions and to support the conceptual change process.

3. In the rest of this article, the label decimal fractions will be used to refer to positive rational numbers in the place-value system, e.g., 0.25.

6.1.1 The present study

The present study focused on the euro, the currency of most of the European Union. To the best of our knowledge, European children's competencies with the euro in the context of everyday activities have not previously been investigated. Like the dollar, the euro money system uses coins to represent fractional parts of the unit. One euro is made up of 100 cents. Euro coins are minted in denominations of 1, 2, and 5 cents (bronze colored), 10, 20 and 50 cents (brass colored), and 1 and 2 euro (bicolored).

The aim of the present study was to investigate the impact of children's knowledge of the euro money system on their accuracy in tasks including decimal fractions. In order to control for possible confounding variables, euro knowledge was only activated by the euro symbol (€), and not by word problems or concrete objects. Tasks requiring comparison and betweenness of decimal fractions were performed by children from Grades 4 to 7. In each task, a control condition in which decimal fractions were presented without any reference to money (for instance the comparison 2.12 vs. 2.121) was compared to an experimental condition in which the euro was mentioned (for instance the comparison 2.12 € vs. 2.121 €)⁴.

The knowledge of the euro is important for everyday mathematics, and have so to be included in the mathematical curriculum. However, the use of the euro as a foundation for the teaching of decimal fractions have to be investigated. The impact of some features of the euro, like the lack of thousandths, on the decimal fractions conceptions have to be carefully analyzed.

Hypotheses

It was first hypothesized that the non-existence of thousandths and the lack of coins for tenths in the euro system would encourage children to treat the fractional part as a number of hundredths (a number of cents), i.e. as a natural number (Hypothesis 1). The impact of the euro would thus be modulated by the type of decimal fraction, providing help when the processing of the decimal part as a natural number could lead to the correct answer, but leading to more difficulties in other cases. For instance, knowledge of the euro would not help children to write a number bigger than 1.35 but smaller than 1.36. An alternative hypothesis was that the lack of thousandths for the euro would encourage children to truncate the thousandths, or to round the number to the nearest hundredth (Hypothesis 2).

We also hypothesized that the impact of the euro would be modulated by grade (Hypothesis 3). The presence of the euro symbol was expected to help fourth grade children with a little knowledge of decimal fractions, because the existence of cents in the money system would help them to realize that others numbers than natural numbers exist. For instance, young children might be able to conceptualize a number bigger than

4. The conventional notation for the euro (- . - €) differs from the conventional notation for the dollar (\$ -.-).

2 but smaller than 3 more easily when the euro symbol was present than when it was absent. On the other hand, older children (Grades 5, 6 and 7) who still had intermediate levels of understanding about decimal fractions, might be muddled by the presence of the euro symbol.

Finally, we hypothesized that the impact of the euro symbol could be modulated by the child's knowledge of decimal fractions, independently of their grade. Children with a limited knowledge of, or difficulties in learning about, decimal fractions, might find some support from their knowledge of the euro, while other children would not (Hypothesis 4).

6.2 Method

6.2.1 Participants

The participants in this study were 151 children in Grade 4 (78 girls and 73 boys; mean age 115.17 months, $SD = 6.35$), 148 in Grade 5 (62 girls and 85 boys; mean age 128.80 months, $SD = 7.26$), 160 in Grade 6 (77 girls and 83 boys; mean age 141.52 months, $SD = 7.35$) and 183 in Grade 7 (88 girls and 95 boys; mean age 153.38 months, $SD = 6.79$). The children came from different schools of the French-speaking community of Belgium. These schools represented various socio-economic groups, but the sample is only representative and voluntary, not randomly selected. Exclusionary criteria concern children who repeated any Grade and children with learning disabilities.

6.2.2 Task and stimuli

Comparison of decimal fractions

A paper-and-pencil task was constructed which involved 8 pairs of decimal fractions in each of five different categories (40 pairs in all). In each pair, the two numbers had the same digit in the unit position, but differed in the fractional part. To avoid a ceiling effect, in each pair, the two numbers had a different number of digits in the fractional part (see Desmet et al., 2010).

In the category labeled Round Smaller, the smaller number of the pair had three digits in the fractional part while the larger number had only two digits. The smaller number was therefore the longer number. Both numbers had the same digit in the tenths position. The two digits in the hundredth position differed by one. The smaller number had 7, 8 or 9 as the digit in the thousandth position for four pairs (e.g., 2.13 vs. 2.129), and 1, 2 or 3 for the other four pairs (e.g., 2.13 vs. 2.121). When the smaller number ended in 1, 2 or 3, the correct decision was not influenced by a rounding procedure, but the two numbers could erroneously be considered as equal if the smaller number ending in 7, 8 or 9 was rounded.

In the category Round Bigger, the larger number of the pair had three digits in the fractional part while the smaller number had two digits. The two numbers involved the same digits in the tenth and the hundredth positions. In the thousandth position the larger number had 1, 2 or 3 as digit for four items (e.g., 1.45 vs. 1.452) and 7, 8 or 9 for the other four items (e.g., 1.45 vs. 1.458). So, the correct decision will not be influenced by a rounding procedure when the longer number ended in 7, 8 or 9, but a rounding procedure when the longer number ended in 1, 2 or 3 would lead to a faulty equality.

In the category Equal, the two numbers were equal and had the same digit in the tenths position, but varied in length because one of them ended with a zero (e.g., 1.3 vs. 1.30). In the category Ending in a Zero, the number with two digits in the fractional part was the smaller, and had a zero in the hundredths position and a digit in the tenths position which was smaller than the digit in the tenths position of the shorter number (e.g., 0.2 vs. 0.10). Finally in the category labeled Incongruent Digit, the larger number had only a single digit in the fractional part. The longer number was also the smaller and was constructed with a digit bigger than the digit in the tenth position of the larger number in the hundredths position, preceded by a zero (e.g., 0.5 vs. 0.06). The processing of the digits independently of their place-values would not lead to the correct decision.

Between decimal fractions

A paper-and-pencil test of children's understanding of the betweenness property of decimal fractions was constructed involving thirty-two pairs. Four categories of pairs were distinguished, with 8 pairs of decimal fractions in each category. In the category labeled Successive Naturals, the two numbers were two successive natural numbers (e.g., 1 and 2). In the category labeled Non-Successive Tenths, the two numbers had the same digit in the unit position and two digits separated by a distance of two in the tenths position (e.g., 1.2 and 1.4). In the category labeled Successive Tenths, the two numbers had the same digit in the unit position and two successive digits separated by a distance of one in the tenths position (e.g., 1.2 and 1.3). In the category labeled Successive Hundredths, the two numbers had the same digit in the unit position, the same digit in the tenths position, and two digits separated by a distance of one in the hundredths position (e.g., 1.35 and 1.36).

6.2.3 Procedure

The task was administrated collectively to the children in their classrooms during the spring term. Each child worked individually, in silence but without any time constraint. In order to prevent an influence of the euro knowledge on the control condition, all the participants performed the control condition before the experimental condition. In the first block, the decimal fractions were presented without any reference to the euro. After

a short break, decimal fractions were presented in a second block, with the symbol € added to activate the children's euro knowledge. All the children spent less than twenty minutes on the two blocks together.

It should be noted that this design did not lead to an effect of task order or a fatigue effect. This is confirmed by the lack of decrease in accuracy in children of 10 years across the two blocks⁵, as well as by the relative stability in attention and executive functions at this age (Anderson, Anderson, Northam, Jacobs, & Catroppa, 2001 ; Klenberg, Korkman, & Lahti-Nuuttila, 2001 ; Korkman, Kemp, & Kirk, 2001 ; Rueda et al., 2004).

To avoid the children being able to copy from their neighbors, the random order assigned to children seated on the left of a double school desk differed from that assigned to children seated on the right. The pairs of decimal fractions were presented in the same order to all the children seated on the same side.

Comparison of decimal fractions

For each pair to compare, the children were asked to "circle the larger number or to place an equals sign between them if the numbers have the same value". The position of the correct response was counterbalanced across the test.

Between decimal fractions

The procedure was the same as for the comparison of decimal fractions, except that the child was asked to "write a number which is bigger than the number on the left and smaller than the number on the right".

6.2.4 Results

Comparison of decimal fractions

The mean rate of correct responses in each category of decimal fraction pairs was calculated for each participant, for the condition without and the condition with the € symbol. Gender did not significantly influence performance, $F(1, 639) < 1$, $p > .1$, and was therefore not retained for further analyses. Each category had very good reliability, as shown by the values of Cronbach's alpha given in Table 1.

5. We conducted an additional study on a new sample of 78 children in Grade 5 (38 girls and 40 boys; mean age 124.22 months, $SD = 4.09$) who performed comparison of decimal fractions and between decimal fractions during the fall term. To test for an effect of fatigue, the order of the two blocks was counterbalanced across the children. Some children started with the block without the symbol € and the others started with the block with the symbol €. Results indicated that the order of the two blocks had no impact on children's accuracy neither when the euro symbol was present (comparison task: $F(1, 77) = .20$, $p = .65$, partial $\eta^2 < .01$; between task: $F(1, 77) = .15$, $p = .70$, partial $\eta^2 < .01$), nor when it was absent (comparison task: $F(1, 77) = .61$, $p = .44$, partial $\eta^2 = .01$; between task: $F(1, 77) = .11$, $p = .74$, partial $\eta^2 < .01$).

Tabl. 1

Scale reliabilities (Cronbach's α) and mean rates of correct responses (CR) (in percent) by grade for each category of decimal fractions being compared, without and with the euro symbol.

			Rates of CR by grade					
	Category	Example of pair	α	4	5	6	7	Mean
Without €	Round Smaller	2.13 vs. 2.128	.98	38.0	84.8	96.6	95.7	79.7
	Round Bigger	2.45 vs. 2.452	.96	88.2	89.5	98.4	95.0	92.9
	Equal	1.3 vs. 1.30	.98	58.7	93.5	99.2	98.4	87.4
	Ending in Zero	1.2 vs. 1.10	.98	52.5	93.0	98.0	97.4	85.2
	Incongruent Digit	1.5 vs. 1.06	.99	52.4	90.1	98.5	97.7	84.7
			Mean	59.4	89.3	98.0	96.4	
With €	Round Smaller	2.13 € vs. 2.128 €	.98	36.4	77.0	92.6	89.8	74.9
	Round Bigger	2.45 € vs. 2.452 €	.96	93.7	92.4	96.9	93.8	94.2
	Equal	1.3 € vs. 1.30 €	.98	58.8	87.0	96.6	92.9	83.8
	Ending in Zero	1.2 € vs. 1.10 €	.98	51.6	85.6	95.6	92.9	81.4
	Incongruent Digit	1.5 € vs. 1.06 €	.99	53.7	89.7	97.7	95.5	84.2
			Mean	60.6	85.9	95.6	92.6	

To test Hypotheses 1 and 3, a three-way repeated-measures analysis of variance (ANOVA) was conducted on the correct response rates with Category (1 to 5) and Euro (symbol present or absent) as within-subject factors and Grade (4 to 7) as a between-subjects factor. As the goal of this study was to investigate the impact of Euro, we focused on the main effect of Euro and on its interactions. Other main effects and interactions are reported in Table 2.

Tabl. 2

Comparison task: Statistics for main effects and interactions not including Euro.

Effect/interaction	<i>df</i>	<i>F</i>	<i>p</i>	η^2	Post hoc t-tests
Grade	3, 638	145.69	< .001	.41	Grade 4 < Grade 5 < Grade 6 = Grade 7
Category	2.71, 1731.62	66.77	< .001	.10	Round Smaller < Equal = Ending in Zero = Incongruent Digit < Round Bigger
Grade $\times$ Category	8.14, 1731.62	38.87	< .001	.16	The effect of Category was significant in Grade 4 ($F(2.56, 383.40) = 68.96$, $p < .001$, partial $\eta^2 = .32$), Grade 5 ($F(2.67, 392.29) = 5.89$, $p = .001$, partial $\eta^2 = .04$) and Grade 6 ($F(2.89, 460.00) = 4.73$, $p < .01$, partial $\eta^2 = .03$), but decreased with grade, and was not significant in Grade 7.

The main effect of Euro was significant, $F(1, 640) = 13.32$, $p < .001$, partial $\eta^2 = .02$, in that the correct response rates were generally lower when the euro symbol was present. This effect, was however modulated by Grade, $F(3, 640) = 3.23$, $p < .05$, partial $\eta^2 = .02$, and by Category, $F(2.92, 1866.45) = 9.48$, $p < .001$, partial $\eta^2 = .02$.

Separate analyses by Grade revealed that there was no main effect of Euro in Grade 4, $F(1, 150) < 1$, $p > .05$, partial $\eta^2 < .01$, but that this effect was significant in Grade 5, $F(1, 147) = 6.99$, $p < .01$, partial $\eta^2 = .05$, Grade 6, $F(1, 159) = 7.99$, $p < .01$, partial $\eta^2 = .05$, and Grade 7, $F(1, 182) = 12.75$, $p < .001$, partial $\eta^2 = .07$. From Grades 5 to 7, the correct response rates were increasingly affected by the presence of the euro. To decompose the Category $\times$ Euro interaction, analyses were conducted separately for each category (see Figure 1). The presence of the euro symbol led to lower correct response rates for the categories Round Smaller, $F(1, 638) = 21.20$, $p < .001$, partial $\eta^2 = .03$, Equal, $F(1, 638) = 10.94$, $p = .001$, partial $\eta^2 = .02$, and Ending in Zero, $F(1, 638) = 10.75$, $p = .001$, partial $\eta^2 = .02$, but had no impact for the categories Round Bigger and Incongruent Digit. The triple interaction Category $\times$ Grade $\times$ Euro was not significant, $F(8.75, 1866.45) > 1$, $p > .1$, partial $\eta^2 < .01$.

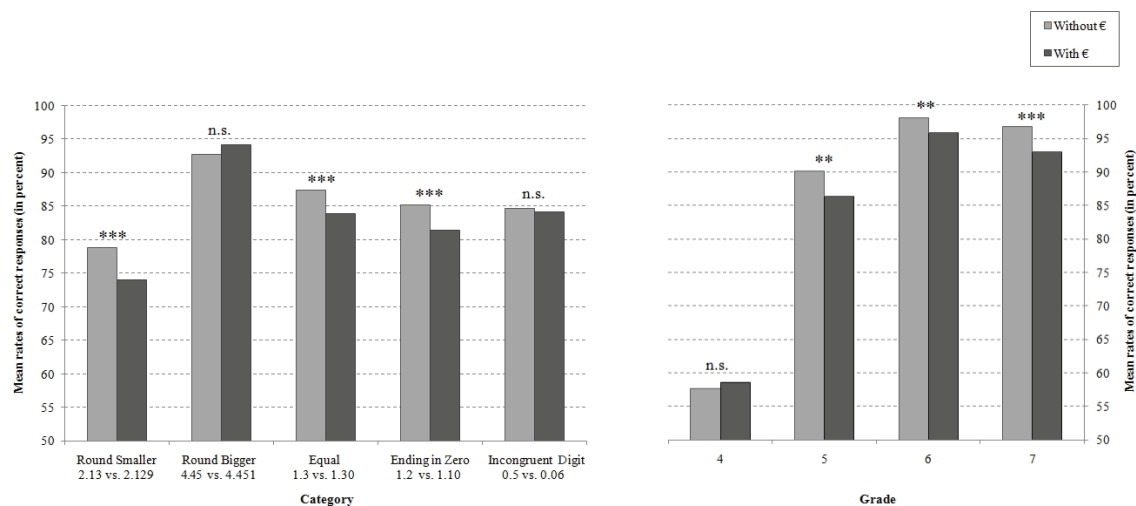


Fig. 1. The Euro $\times$ Category (left) and Euro $\times$ Grade (right) interactions for the comparison task.

To test Hypothesis 2, a four-way repeated ANOVA was conducted on the correct response rates for the categories Round Smaller and Round Bigger with Category (1 and 2), Euro (symbol present or not) and the appropriateness of Rounding (leading to the correct or incorrect answer) as the within subject factors, and the Grade (4 to 7) as the between subjects factor. The main effect of Rounding was not significant, and neither were its interactions with Category, Euro or Grade.

A two-step cluster analysis was conducted on the correct response rates for each Category, with and without Euro, to determine which groups of children were affected by the presence of the euro sign, and to test Hypothesis 4. A three-cluster solution emerged. The cluster analysis was repeated using random start points and yielded exactly the same solution, supporting the three-cluster solution. The mean rates of correct responses for each category with and without euro as a function of cluster membership are reported in Table 3.

Tabl. 3

Mean correct response rates (in percent) for each category of decimal fractions being compared, without and with the euro symbol, as a function of cluster membership.

	Category											
	Round Smaller		Round Bigger		Equal		Ending in Zero		Incong. Digit		Means	
	2.13 vs. 2.129		4.45 vs. 4.451		1.3 vs. 1.30		1.2 vs. 1.10		0.5 vs. 0.06			
	-	€	-	€	-	€	-	€	-	€	-	€
Cluster 1	97.7	97.3	99.1	99.3	99.7	99.4	99.3	98.9	99.5	99.4	99.1	98.9
Cluster 2	73.3	46.5	65.6	72.2	90.7	64.3	91.6	62.5	88.5	80.4	81.9	65.2
Cluster 3	2.4	3.9	97.7	97.7	28.3	37.4	14.0	25.7	14.6	21.2	31.4	37.2

Cluster 1 corresponded to children who were not influenced by the euro symbol and had high correct response rates for all categories of decimal fractions, whether they were presented with or without the €. By contrast, the correct response rates of children in Cluster 2 were modulated by categories. For them, the presence of the euro symbol led to noticeably lower correct response rates, $F(1, 114) = 43.36$, $p < .001$, partial $\eta^2 = .28$, and Euro did interact with Category, $F(2.60, 296.63) = 16.59$, $p < .001$, partial $\eta^2 = .13$. Euro had no impact for the category Round Bigger, but led to lower correct response rates for all other categories ($p = .01$ for the category Incongruent Digit, $p < 0.001$ for others). The children in Cluster 3 had very low levels of correct responses, except in the category Round Bigger. The presence of the euro symbol led to higher correct response rates in this group ($F(1, 90) = 7.368$, $p < .01$, partial $\eta^2 = .08$, although the correct response rates were still below 40%, and did interact with Category, $F(2.61, 234.95) = 3.13$, $p < .05$, partial $\eta^2 = .03$. Euro had no impact for the categories Round Bigger and Round Smaller, but led to higher correct response rates for the other categories ($p = .05$ for category Equal, $p < .05$ for category Incongruent Digit, $p < .01$ for category Ending in Zero). The cluster membership distribution by grade differed significantly from the distribution expected by chance, $\chi^2(6, N = 642) = 268.92$, $p < .001$, with the older children over-represented in Cluster 1, and the younger ones in Cluster 3 (see Table 4).

Tabl. 4

Cluster repartition by grade.

	N	Grade			
		4	5	6	7
Cluster 1	436	40	99	144	153
Cluster 2	115	33	39	15	28
Cluster 3	91	78	10	1	2

These results have shown that the impact of the euro symbol on the comparison of decimal fractions depends on the type of decimal fraction, as predicted by Hypothesis 1. The presence of the euro symbol had no impact when the processing of the fractional part as a natural number led to the correct response (as in the category Round Bigger: e.g., 4.45 vs. 4.451). By contrast, children performed less well when the euro symbol was present when the processing of the fractional part as a natural number led to errors (as in the categories Round Smaller (e.g., 2.13 vs. 2.129), Equal (e.g., 1.3 vs. 1.30) and Ending in Zero (e.g., 1.2 vs. 1.10)). However the euro symbol had no significant impact for the category Incongruent Digit (e.g., 0.5 vs. 0.06). Contrary to the prediction in Hypothesis 2, the presence of euro symbol did not produce any specific behavior for the thousandths.

In any grade did the presence of the euro symbol improve the mean rate of correct responses for the comparison task. Nevertheless, this effect was modulated by grade, as predicted by Hypothesis 3. Surprisingly, the youngest children (Grade 4) did not show any improvement due to the presence of the euro symbol, but the symbol led to lower correct response rates for older children (Grades 5, 6 and 7) who already have some knowledge of decimal fractions. Regarding Hypothesis 4, the cluster analysis showed that children with a high level of correct responses (Cluster 1) were not influenced by the euro symbol, but that children with an intermediate level of understanding (Cluster 2) were affected by its presence (which led to lower correct response rates), while children with the lowest correct response rates seemed to benefit from the presence of the euro symbol. However, their correct response rates were still very low.

To summarize, the presence of the euro symbol did not significantly improve children's performance in the comparison of decimal fractions. Indeed, it led to more errors for some categories of comparison, for older children, and for children with an intermediate level of understanding of decimal fractions. Based on the results for this task, it seems that the presence of the euro symbol encourages children to treat the fractional part of decimal fractions like natural numbers. However, this could be true only for the comparison of decimal fractions and not for other tasks involving them. To test this, the same children were asked to perform a task focusing on the 'betweenness' of decimal fractions.

Between decimal fraction

The mean rate of correct responses in each category of number pairs was calculated for each participant, with and without the euro symbol. The difference between boys and girls in performance was not significant, $F(1, 639) = 2.36$, $p > .1$, and so this variable was excluded from subsequent analyses. The mean percentage of correct responses in each Category and each Grade are shown in Table 5. Table 5 also shows that the categories have good reliability, as measured by Cronbach's alpha. The low alpha in the Non-Successive Tenths with the euro category can be explained by the high level of

correct responses and the low variance.

Tabl. 5

Scale reliabilities (Cronbach's α) and mean rates of correct responses (in percent) by grade for each category of betweenness, without and with the euro symbol

	Category	Example of pair	α	Grades				Mean
				4	5	6	7	
Without €	Successive Naturals	1 ... 2	.99	38.0	81.5	90.0	96.1	76.4
	Non-Successive Tenths	1.2 ... 1.4	.91	90.4	96.7	98.7	98.9	96.2
	Successive Tenths	1.2 ... 1.3	.97	21.1	73.2	88.8	93.2	69.1
	Successive Thousandths	1.35 ... 1.36	.97	16.9	63.9	82.5	86.3	62.4
		Mean		41.6	78.8	90.0	93.6	
With €	Successive Naturals	1 € ... 2 €	.99	47.2	83.2	92.2	96.2	79.7
	Non-Successive Tenths	1.2 € ... 1.4 €	.79	92.8	97.5	98.7	98.3	96.9
	Successive Tenths	1.2 € ... 1.3 €	.99	23.8	73.0	88.2	91.8	69.2
	Successive Thousandths	1.35 € ... 1.36 €	.99	19.8	62.3	71.8	79.9	58.5
		Mean		45.9	79.0	87.7	91.5	

A three-way repeated-measures ANOVA was conducted on the correct response rates with the Category (1 to 4) and the Euro (symbol present or absent) as within-subject factors and the Grade (4 to 7) as a between-subjects factor. As for the comparison task, we will focus in this paper on the main effect of Euro and on interactions involving this variable. Other main effects and interactions are reported in Table 6.

Tabl. 6

Betweenness task: Statistics for main effects and interactions not including Euro.

Effect/interaction	df	F	p	η^2	Post hoc t-tests
Grade	3, 636	149.99	< .001	.41	Grade 4 < Grade 5 < Grade 6 = Grade 7
Category	2.39, 1521.76	332.65	< .001	.34	Successive Thousandths < Successive Naturals < Successive Tenths < Non Successive Tenths
Grade $\times$ Category	7.18, 1521.76	46.54	< .001	.22	The main effect of Category was significant in Grade 4 ($F(2.16, 322.32) = 248.29, p < .001, \text{partial } \eta^2 = .63$), Grade 5 ($F(2.27, 330.94) = 53.42, p < .001, \text{partial } \eta^2 = .27$), Grade 6 ($F(2.41, 380.70) = 33.48, p < .001, \text{partial } \eta^2 = .18$), and Grade 7 ($F(2.07, 378.97) = 35.57, p < .001, \text{partial } \eta^2 = .16$), but decreased with Grade until Grade 6.

The main effect of Euro was not statistically significant, $F(1, 635) < 1, p > .9$, partial $\eta^2 < .01$, but the triple interaction Euro $\times$ Grade $\times$ Category was significant, $F(7.40, 1565.64) = 2.42, p < .05$, partial $\eta^2 = .01$, and so were the Euro $\times$ Category, $F(2.47, 1565.64) = 14.32, p < .001$, partial $\eta^2 = .02$, and the Euro $\times$ Grade, $F(3, 635) = 7.44, p < .001$, partial $\eta^2 = .03$ interactions.

To explore the interactions further, we analyzed the results separately for each grade. In Grade 4, the presence of the euro symbol significantly improved the correct response rates, $F(1, 149) = 17.90$, $p < .001$, partial $\eta^2 = .11$, and did interact with the Category, $F(2.31, 344.22) = 4.25$, $p = .01$, partial $\eta^2 = .03$; it significantly improved the correct response rates for Successive Naturals ($p < .001$), Non-Successive Tenths and Successive Hundredths (both $p < .05$), but had no impact on Successive Tenths. The presence of the euro symbol had no overall impact in Grade 5, $F(1, 146) < 1$, $p > .8$, partial $\eta^2 < .01$, and did not interact with Category, $F(2.67, 389.45) < 1$, $p > .4$, partial $\eta^2 < .01$. In Grade 6, the presence of the euro symbol reduced the correct response rate, $F(1, 158) = 4.67$, $p < .05$, partial $\eta^2 = .03$, and did interact with Category, $F(1.64, 258.71) = 12.79$, $p < .001$, partial $\eta^2 = .08$, decreasing performance on Successive Hundredths ($p < .001$), although it had no impact on Non-Successive Tenths or Successive Tenths and improved performance on Successive Naturals ($p < .05$). In Grade 7, the presence of the euro symbol decreased performance overall, $F(1, 182) = 3.89$, $p = .05$, partial $\eta^2 = .02$, and did interact with Category, $F(2.06, 374.58) = 3.61$, $p < .05$, partial $\eta^2 = .02$, indicating a significant decrease in the correct response rate for Successive Hundredths only ($p < .05$).

To test Hypothesis 4, a two-step cluster analysis was conducted on the correct response rates for each category with and without the euro symbol to locate groups of children who were affected by the symbol in the same way. A two-cluster solution emerged; this was supported by repeated cluster analyses using random start points. The mean rates of correct response for each category with and without the euro symbol, as a function of cluster membership are reported in Table 7.

Tabl. 7

Mean correct response rates (in percent) for each category of betweenness, without and with the euro symbol, as a function of cluster membership and cluster repartition by grade.

	Category									
	Successive Naturals		Non-Successive Tenths		Successive Tenths		Successive Hundredths		Means	
	1 ... 2		1.2 ... 1.4		1.2 ... 1.3		1.35 ... 1.36			
	- €	-	€	-	€	-	€	-	€	
Cluster 1	97.4	97.7	99.3	98.9	97.3	96.2	90.0	82.8	96.0	93.9
Cluster 2	30.5	40.3	89.3	92.2	7.0	9.7	1.7	4.9	32.1	36.8

The children in Cluster 1 ($n = 450$) had high correct response rates for all categories. The presence of the euro symbol led to lower correct response rates, $F(1, 449) = 12.17$, $p = .001$, partial $\eta^2 = .03$. Euro did interact significantly with Category, $F(1.74, 780.96) = 14.07$, $p < .001$, partial $\eta^2 = .03$, and was in fact only significant for the Successive Hundredths category. Cluster 2 ($n = 189$) corresponded to children who had low correct response rates. For them, the presence of the euro symbol led to higher correct response

rates, $F(1, 188) = 15.70$, $p < .001$, partial $\eta^2 = .08$). In this cluster too, Euro did interact significantly with Category, $F(2.33, 437.57) = 5.41$, $p < .01$, partial $\eta^2 = .03$). The presence of the euro symbol had no impact on success in the Successive Tenths category, but a positive impact in the three other categories. The distribution of the cluster membership differed significantly by grade, $\chi^2(3, N = 639) = 255.35$, $p < .001$, with the older children disproportionately represented in Cluster 1 and the younger children in Cluster 2 (see Table 8).

Tabl. 8
Cluster repartition by grade.

		Grade			
	<i>N</i>	4	5	6	7
Cluster 1	450	30	108	142	170
Cluster 2	189	120	39	17	13

The impact of the euro symbol on the rate of correct responses to problems involving betweenness in decimal fractions was modulated by the grade. As predicted by Hypothesis 3, the presence of the euro symbol led to higher correct response rates for young children (in Grade 4), but was a disadvantage in Grades 6 and 7. Moreover, as predicted by Hypothesis 1, the euro symbol did not influence all categories in the same way. It improved the correct response rates when a number between two successive naturals had to be found, particularly for fourth-graders, but had no effect, except for fourth-graders, when a number between two tenths, successive or not, had to be located. When children, especially sixth- and seventh-graders, had to find a number between two successive hundredths, the presence of the euro symbol actually induced more errors. This result supports Hypothesis 2, that the lack of thousandths in the euro system has an impact on the processing of decimal fractions presented as euro problems.

The cluster analysis showed that for children with high levels of correct responses without the euro symbol (Cluster 1), the presence of the symbol led to lower correct response rates. The reverse was true for children with lower correct response rates without the euro symbol (Cluster 2), although their levels of correct responses were still very low. These results support Hypothesis 4.

To summarize, the presence of the euro symbol enhanced performance on betweenness problems for fourth-graders and for children with very low levels of understanding, but this advantage quickly disappeared as performance improved with age. The euro became a constraint from Grade 6 onwards, and for children with high levels of correct responses. However, betweenness is a fairly artificial concept. Some of the most frequent daily-life activities with money involved operations on numbers expressed as euros, for instance adding them, and Experiment 3 was designed to explore how well children performed this task on decimal fractions with and without the euro symbol.

6.3 Discussion

This study has investigated the impact of the euro on the processing of decimal fractions. Children from Grades 4 to 7 had to perform tasks of comparison and betweenness with decimal fractions in two conditions, one without any reference to the euro and the other including the euro symbol (€). The present finding showed that the presence of the euro symbol stressed the processing of the fractional part as a natural number (Hypothesis 1). Contrary to our predictions, the presence of the euro symbol did not lead to a specific processing (such as a rounding procedure) or to a truncation of the thousandths during the comparison task, but was a constraint for the betweenness task, when it is necessary to go beyond the hundredths (Hypothesis 2). Our results also indicated that the impact of the euro was modulated by Grade, providing some assistance to fourth-graders, but reducing the correct response rates for children from Grade 5 onwards (Hypothesis 3). Moreover, the presence of the euro symbol had no effect on children with high scores overall when they were comparing decimal fractions, but impaired their performance on the betweenness task (Hypothesis 4). For children with an intermediate level of understanding (i.e., those who succeeded with some but not all categories of decimal fractions) the presence of the euro symbol acted as a constraint. Finally, children with low overall scores showed a slight improvement in their correct response rates when the euro symbol was present, although their performance remained poor.

The results are discussed below with respect to our hypotheses. Hypotheses 1 and 2 suggested that the presence of the euro symbol would encourage children to treat the fractional part of a decimal fraction as a natural number, and to round or truncate the number to the nearest hundredth. Previous studies have suggested that money could be usefully used to teach children about decimal fractions (e.g., Bonotto, 2005; T. Carraher Schliemann, 1988; Irwin, 2001), but the current findings indicate that this could be problematic. What children have to understand when they learn decimal fractions is, first, that the decimal number system may be extended to values lower than the unit, and, second, that rational numbers are characterized by a density property (namely that there is always at least one rational number between any given two rational numbers). However, we showed throughout the comparison and the betweenness tasks that using the euro symbol encourages children to treat the fractional part of a decimal as a natural number. This is not in line with the two goals described above. In the comparison task, when the processing of the fractional part as a natural number led to the correct response, including the euro symbol had no impact on children's performance. When the processing of the fractional part as a natural number was inconsistent with the correct response, the presence of the euro symbol either led to lower correct response rates or had no impact. For the betweenness task, which can be considered as a first step towards understanding the density property, the results were mixed. The euro symbol helped children appreciate that there are numbers between two natural numbers. It had no impact on their ability

to think of numbers between tenths, but impeded their thinking about numbers between hundredths (possibly due to the lack of thousandths in the euro system).

Taken together, these results indicate that the decimal fractions presented with a euro symbol were considered by the children as combinations of a number of euros and a separate number of cents. Brenner (1998) reached a similar conclusion for the dollar. Prices in euro also reinforce this way of thinking as they often show the number of cents by two digits, even when it is not necessary (e.g., 2.00 € or 1.50 €). So, prices in euro are invariably presented as numbers with two digits in the fractional part and children are encouraged to process this fractional part as a natural number. Thinking of decimal fractions as money therefore cannot help children when treating the fractional part as a natural number leads to the incorrect response; nor can it help them to understand the density property of rational numbers. Although the presence of the euro symbol helps young children (fourth-graders) and children with very weak performance in the betweenness task, their scores remained very low.

Our findings also provided evidence that, as predicted by Hypotheses 3 and 4, presence of the euro symbol reduced the correct response rates for children in Grades 5, 6 and 7 and children who already had a good understanding of decimal fractions. It was also predicted that the euro would help the young children in Grade 4 and children with poor performance overall, but this effect was restricted to the betweenness and comparison tasks and was smaller than expected. The symbol could thus be used as a temporary, motivating instrument when decimal fractions are first introduced, particularly to form the comprehension that there are numbers between two naturals. But it is unlikely that a full understanding of decimal fractions can be built on knowledge of the euro system. If we consider decimal fractions as a set of numbers, numbers expressed as euros are necessarily a subset of the numbers in the set of decimal fractions. So, it is inherently unlikely that children would understand the complexity of decimal fractions from the set of numbers in the euro system. With the euro, children will never meet a number like 1.5 and understand that it is bigger than 1.43, or discover that 1.06 is smaller than 1.6.

The acquisition of decimal fractions requires a conceptual change, namely the reorganization of children's prior knowledge based on natural numbers (Desmet et al., 2010 ; Stafylidou & Vosniadou, 2004). For instance, children have a presupposition of discreteness based on their knowledge of natural numbers, which constrains their understanding of the density property of rational numbers (Vamvakoussi & Vosniadou, 2004). As a first step in this process of conceptual change, children should be presented with a learning situation whose resolution requires them to go beyond their prior knowledge (even if it is not a sufficient condition) (Mason et al., 2008 ; Merenluoto & Lehtinen, 2004). Such a learning situation cannot be built on the euro, particularly when children are trying to understand the density property.

This study on the impact of the euro symbol on the processing of decimal fractions has indicated that the inclusion of everyday mathematics in the classroom has to be considered with caution. Everyday situations can motivate children and help them make mathematics meaningful. However, each formal mathematical topic has its own specificity and complexity, which may be quite different from the related everyday mathematics (T. N. Carraher et al., 1987 ; Masingila et al., 1996). The inclusion of everyday mathematics may even restrict such mathematical learning. For instance, the euro system includes operations with decimal fractions, but cannot be used to illustrate the complexity of formal decimal fractions. The development of everyday mathematics into formal mathematics (D. W. Carraher & Schliemann, 2002 ; Gravemeijer, 1999 ; Schliemann & Carraher, 2002) has to be explicitly tackled with children, and has to include more examples than can be derived from the initial conceptions based on everyday mathematics.

6.4 Appendix

The pairs of decimal fractions for the comparison task.

Category

Round Smaller	2,13	2,128
	1,42	1,419
	4,739	4,74
	6,577	6,58
	3,41	3,402
	5,24	5,231
	7,561	7,57
	8,143	8,15
Round Bigger	2,45	2,452
	1,67	1,671
	4,351	4,35
	6,523	6,52
	3,45	3,458
	5,82	5,829
	7,429	7,42
	8,347	8,34
Equal	1,3	1,30
	2,4	2,40
	3,1	3,10
	4,2	4,20
	5,70	5,7
	6,80	6,8
	7,50	7,5
	8,60	8,6
Ending in Zero	1,4	1,30
	2,5	2,40
	3,2	3,10
	4,3	4,20
	5,70	5,8
	6,80	6,9
	7,50	7,6
	8,60	8,7
Incongruent Digit	1,5	1,06
	2,4	2,05
	3,1	3,02
	4,2	4,03
	5,08	5,7
	6,09	6,8
	7,06	7,5
	8,07	8,6

The pairs of decimal fractions for the betweenness task.

Category			
Successive naturals	1	...	2
	2	...	3
	3	...	4
	4	...	5
	5	...	6
	6	...	7
	7	...	8
	8	...	9
Non-Successive Tenths	1,2	...	1,4
	2,3	...	2,5
	3,4	...	3,6
	4,5	...	4,7
	5,6	...	5,8
	6,7	...	6,9
	7,1	...	7,3
	8,2	...	8,4
Successive Tenths	1,2	...	1,3
	2,6	...	2,7
	3,5	...	3,6
	4,7	...	4,8
	5,3	...	5,4
	6,7	...	6,8
	7,4	...	7,5
	8,3	...	8,4
Successive Thousandths	1,35	...	1,36
	2,54	...	2,55
	3,64	...	3,65
	4,12	...	4,13
	5,23	...	5,24
	6,81	...	6,82
	7,42	...	7,43
	8,24	...	8,25

Chapter 7

A didactical intervention

*Abstract*¹

This study examined the effects of an intervention designed to enhance the decimal fractions understanding of fourth graders ($N = 111$). The intervention mainly conveys conceptual knowledge and was based on measures activities without conventional units and on the approximation of real numbers using the decimal system. Effects of the intervention were investigated in pretest-posttest design with control group, and with follow-up. Results indicated that the intervention led to improvement in both conceptual and procedural knowledge which is still observable at 4-month follow-up assessment. Compared to the control group, the experimental group also showed a better procedural flexibility and a better understanding of the density property of decimal fractions. The results highlight the potential of the approximation of real numbers in the decimal system in the learning of decimal fractions.

1. This research has been conducted in collaboration with Philippe Skilbecq, Julie Fanuel, Cécile Ouvrier-Bufferet and Jacques Grégoire, and will be submitted for publication as “From real numbers to decimal fractions: a didactical intervention”.

7.1 Introduction

Some rational numbers, labeled decimal fractions, could be represented by common fractions with a power of ten as denominator (e.g., $5/10$), but also using the decimal system (e.g., 0.5). This last notation using the decimal system is convenient for calculation and often used in daily life. However, the learning of decimal fractions is laborious for children.

7.1.1 Misconceptions about decimal fractions

A lot of misconceptions about decimal fractions have been described (Desmet et al., 2010 ; Nesher & Peled, 1986 ; Resnick et al., 1989 ; Sackur-Grisvard & Leonard, 1985). Some of these misconceptions are based on common fraction knowledge, like the decimal fraction with the shorter decimal part considered as the bigger (e.g., $1.2 > 1.35$) because “a tenth is always bigger than a hundredth”. Other more frequent misconceptions are based on the knowledge about natural numbers, like 1.14 considered as larger than 1.7 because 14 is larger than 7 (Resnick et al., 1989). Knowledge of the decimal system based on natural numbers and knowledge of rational numbers seem so to be both implied in the learning of decimal fractions. This is in line with the hierarchical view of mathematical knowledge (Aunola et al., 2004 ; Hecht & Vagi, 2010 ; Moeller et al., 2011) considering that basic concepts provide the foundation for further learning of more complex concepts. But research on precursors of the learning of common fractions (Hecht et al., 2003 ; Hecht & Vagi, 2010) and decimal fractions has only begun. The understanding of the part-whole relation might be a precursor of subsequent common fractions competencies (Hecht & Vagi, 2010), while the understanding of the decimal system could be essential for subsequent decimal fractions (see chapter 5).

However, prior conceptions can also impede further learning when the new concepts are incompatible with the existing knowledge. It is well known that the learning of decimal fractions represented using the decimal system (Desmet et al., 2010 ; Vamvakoussi & Vosniadou, 2004) as well as the learning of rational numbers represented by common fractions require a conceptual change (Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004), which results in an interference of natural numbers on the new numbers. The numerical knowledge has to be reorganized and generalized. Consequently, these learning are laborious and time demanding for children. For instance, only 57.5% of Grade 9 students succeeded in ordering the numbers set $5/6$, 1, $1/7$, $4/3$ from the smallest one to the biggest one. Children considered the fraction notation as a juxtaposition of two natural numbers rather than a ratio (Stafylidou & Vosniadou, 2004).

7.1.2 Conceptual and procedural knowledge

In the field of mathematical learning, the distinction between conceptual and procedural knowledge is widely assumed (Rittle-Johnson & Siegler, 1998 ; Schneider & Stern, 2010). Conceptual knowledge refers to knowledge of the concepts that govern a domain, such as place-value for multi-digit number, whereas procedural knowledge refers to step-by-step action sequences to solve problems, such as procedures for adding and subtracting (e.a., Rittle-Johnson & Koedinger, 2009). Concepts and procedures are in fact strongly associated (Hallett, Nunes, & Bryant, 2010 ; Hecht, 1998), it is hard to measure them validly and independently of each other. There is no consensus on the way to measure each type of knowledge independently of the other one, but it is not impossible (Schneider & Stern, 2010). Different theoretical models on the causal interrelations between conceptual and procedural knowledge have therefore been proposed (for a discussion, see Rittle-Johnson & Siegler, 1998 ; Schneider, Rittle-Johnson, & Star, 2011). An approach considered that procedural and conceptual knowledge develop together, in an iterative process (Canobi & Bethune, 2008 ; Rittle-Johnson & Koedinger, 2009 ; Rittle-Johnson, Siegler, & Alihali, 2001 ; Schneider et al., 2011) with bidirectional and even symmetrical (Schneider et al., 2011) causal relations. Increase in conceptual knowledge leads to subsequent increase in procedural knowledge and vice-versa. Explicit teaching on one type of knowledge led to improvements in the other type of knowledge (Rittle-Johnson & Koedinger, 2009). The acquisition of conceptual and procedural knowledge also appeared to be positively influenced by a teaching contrasting incorrect examples with correct examples (Durkin & Rittle-Johnson, 2012). Finally, both conceptual and procedural knowledge contribute to procedural flexibility, that is the learner's knowledge of multiples procedure and ability to choice them adaptively to solve a given problem (Schneider et al., 2011). To sum up, both conceptual and procedural knowledge need to be represented in instructional practices.

Children's knowledge about decimal fractions has been classified as procedural rather than conceptual knowledge (Hiebert & Wearne, 1985) and this could be due to the teaching methods. Several explanations have been proposed (Moss & Case, 1999) : too much time devoted to teaching procedures and too little time to teaching the conceptual meaning, encouraging children to adopt an approach based on the application of rules, use of representations in which rational and whole numbers are easily confused, ignoring the problems with the notation in its own right. It has been shown that a teaching with a greater emphasis on the meaning of rational numbers rather than on procedures to handle them, gives rise to a deeper understanding for the experimental group than for the control group, as well as an improvement on procedures (Moss & Case, 1999).

7.1.3 Teaching of decimal fractions

The teaching of decimal fractions traditionally appears in Grade 4. Various methods have been used to teach decimal fractions. Some of them mainly included the conceptual knowledge about rational numbers, working on the understanding of part-whole relation, ratio and percent (Lachance & Confrey, 2001 ; Moss & Case, 1999) and on equivalence to decimal fractions. Others based their intervention on concrete referents, like Dienes base ten blocks (Hiebert & Wearne, 1988 ; Hiebert et al., 1991), mathematical everyday knowledge, like the knowledge of money (Bonotto, 2005) or metric measurement (Irwin, 2001). Teachers also used the same contexts to teach decimal fractions. On twenty-eight third-grade and thirty fourth-grade U.S. teachers, nearly 90% of the teachers used the context of money (Glasgow et al., 2000), while others used pictorial, metric system, equivalence to fractions, or base 10 blocks contexts. But these suggestions for the teaching of decimal fraction can be debated in various ways. First, a single concrete referent or an everyday knowledge, like the money, may not represent all features of decimal fractions. The use of one referent improved the performance of children on tasks involving the same features than the referent, but not on tasks involving other features (Hiebert et al., 1991). Regarding the use of the metric system, for instance the meter and its subunits to measure length, it could reinforce a conception of decimal fraction as being a juxtaposition of two natural numbers separate by a dot (Brousseau, 1980), like 1 m and 25 cm instead of 1.25 m. Second, the learning of conceptual knowledge about rational numbers represented by common fractions is notoriously laborious for children (Vamvakoussi & Vosniadou, 2004). Furthermore, these interventions only introduced decimal fractions with two (Hiebert & Wearne, 1988 ; Irwin, 2001) or three (Bonotto, 2005) digits in the decimal part, and often in a sequential way, like two-digits decimal fractions first and three- and one-digit next (Moss & Case, 1999). Children have difficulties to understand the density property of rational numbers (Vamvakoussi & Vosniadou, 2010), and limiting the number of digits in the decimal part cannot help children to understand this property.

7.1.4 The present study

In the present study, we investigate the impact of an problem-centered intervention based on the approximation of real numbers in the decimal system. This intervention shares several features with the interventions suggested by other investigators, like (1) a greater emphasis on the conceptual knowledge of decimal fractions than on procedural knowledge, and (2) a greater emphasis on children's spontaneous conceptions and misconceptions. However, our intervention also has important distinctive features, like (1) the use of real numbers as an introduction to the properties of decimal fractions, (2) an emphasis on the principles underlying the decimal system principles rather than on the understanding of rational numbers represented by common fractions, and (3) the use of length measure without the meter as conventional unit, to avoid a conception of

decimal fraction as being the juxtaposition of two natural numbers separate by a dot. Furthermore, our intervention is based on problem solving (e.a., Hiebert et al., 1996 ; Gijbels, Dochy, Bossche, & Segers, 2005), on a problematic situation requiring decimal numbers to be resolved.

We postulate that our intervention would have a positive impact on conceptual knowledge about decimal fractions and in turn on procedural knowledge (Hypothesis 1). We also predict that our intervention would lead to a greater understanding of the density property of decimal fractions (Hypothesis 2). The impact of the intervention will be investigated during and after the intervention in Grade 4, but also in a follow-up assessment 4 months later in Grade 5.

7.2 Didactical intervention in Grade 4

In the present experiment, we aim at investigating the impact of our didactical approach on fourth-grade children. An experimental group was compared to a control group in a pretest – intervention – posttest design.

7.2.1 Method

Participants and design

A total of 111 children of Grade 4 (51 boys, mean age = 9 years 4 months; $SD = 3.70$ months) participated in the study. They came from three voluntary classes of the French-speaking community of Belgium. Exclusionary criteria included children with learning disabilities, children who repeated Grade 4 and children absent at one of the assessment sessions. Forty eight children were assigned to the experimental group (21 boys, mean age = 9 years 4 months, $SD = 3.67$ months) and 63 children were assigned to the control group (30 boys, mean age = 9 years 4 months, $SD = 3.74$ months). Group assignment was not done randomly (each child remains in its classroom), but no statistical differences were found between the two groups for accuracy at baseline (T_1) (see Table 1). Children of the experimental group received the experimental didactical intervention as teaching of decimal fractions. Children of control group received no particular intervention but followed typical teaching of decimal fractions (conceived by their teachers). Children participated in a pretest - intervention - posttest design. The pretest (T_1) was administrated at the beginning of the school year (October) before the intervention started and before any formal teaching of decimal fractions. The posttest (T_3) was administrated at the end of the school year (June) after the end of the intervention. An intermediate test (T_2) was administrated during the intervention (February).

Didactical intervention

The intervention was designed to convey mainly conceptual knowledge such as place-value understanding, before starting the procedural knowledge teaching. The conceptual knowledge intervention started with a problematic situation of measure without conventional units and involved approximation of real numbers using the decimal system, as previously proposed by (Douady, 1980). The intervention was done as a whole class activity.

The preliminary intervention introduced the concept of area equivalence by working on geometric shapes to dissect (cut) them into pieces and rearrange the pieces by translating, rotating, or reflecting, to form different geometric shapes of the same area. This preliminary intervention also involved the measurement of an area with a non-standard area unit. In the two first lessons of the intervention, children were asked to construct squares using a unit square. The constructed squares were then ordered from the smallest to the greatest (see Figure 1). The teacher emphasized that if the length of the sides of the small square and the area of this square are taken as non-standard units, the sequence of sides length is 1, 2, 3, 4, ..., the perimeters sequence is 4, 8, 12, 16, ... and the areas sequence is 1, 4, 9, 16, ... (corresponding to the square number sequence).

Children had to decide whether a square with an area of 8 non-standard units exists or not. The subsequent lesson involved the construction of this 8-square using eight small squares, cutting them and rearranging them, like in the preliminary intervention (see Figure 2). Children must then acknowledge that the 8-square exists since they constructed it.

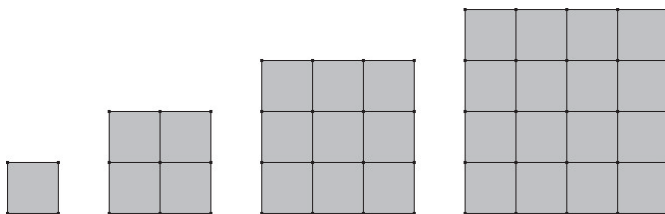


Fig. 1. Squares sequence.

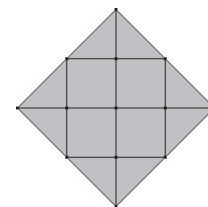


Fig. 2. One 8-square.

The following lessons are based on problem solving. The teacher asked the measure of the side length for the 8-square, which is the real number $2\sqrt{2}$. Children did not have to learn real numbers, but to realize that natural numbers were not sufficient to resolve the problem and to discover that other numbers exist. To resolve the problem, children have previously observed that the side length number is multiplied by itself to obtain the area number (for instance, 2 times 2 equals 4). They used this information to work on the approximation of the real number $2\sqrt{2}$ using the decimal system and rational numbers represented in this system (decimal fractions).

The approximation of the real number $2\sqrt{2}$ was in fact a work on the decimal system principles. First, children discover that the number have to be included between the two natural numbers, 2 and 3. They quickly think to the 2 and a half and that it have to be written 2.5 with the decimal system. The multiplication is then used to find, using a calculator, the number which equals 8 when multiplied by himself. For instance, children of one of the experimental classes suggested the numbers presented in the Table 1 and spontaneously, one child also try to find, at home, the number which equals 10 when multiplied by himself ($\sqrt{10}$) (see Fig.3). During the search, the teacher emphasized the successive decimal subdivisions of decimal fractions in the decimal system and the role of zero. The multiplication allows to observe that 2.10 times 2.10 gives the same result (4.41) than 2.1 times 2.1, and that this result is a smaller number than the result of 2.9 times 2.9 (8.41). The teacher emphasizes that the zero in 2.10 is a place-holder and does not modify the value of the number. Finally, children find the number 2.828... which approximates 8 when multiplied by himself.

Tabl. 1
Some attempts to approximate the
real number $2\sqrt{2}$.

2.5×2.5	$=$	6.25
2.7×2.7	$=$	7.29
2.8×2.8	$=$	7.84
2.80×2.80	$=$	7.84
2.9×2.9	$=$	8.41
2.900×2.900	$=$	8.41
2.81×2.81	$=$	7.8961
2.820×2.820	$=$	7.9524
2.83×2.83	$=$	8.0089
2.830×2.830	$=$	8.0089

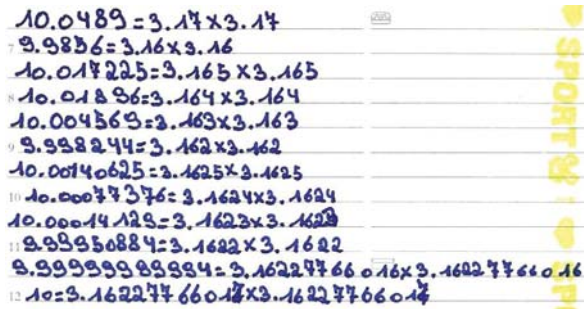


Fig. 3. Attempts to approximate $\sqrt{10}$.

Children were then asked to draw the square with the 2.828 sides length on a (1 line per centimeter) grid sheet representing the superposition of the small square (the non-standard unit) and the squares of 4 units and 9 units areas on a graduated number line (see Figure 5). The interval size of the graduate number line is 20 cm (see Fig. 4).

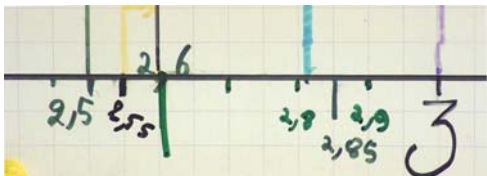


Fig. 4. A section of the number line.

The two lessons allowed to emphasize again the decimal system principles, mainly the place-value principle and the role of zero, and to highlight some misconceptions. For instance, the placement of the number 2.82 on a 20 cm interval between 2 and 3 was

problematic for some children. They started with the number 2, placed the 2.1, 2.2, 2.3, ... sequence with one number per line, and thought that 2.10 is the number following 2.9. Consequently, the interval between 2 and 3 was not long enough and 2.20 was superposed to 3 and children realized that the sequence 2.8, 2.9, 2.10, ... is incorrect. The intermediate posttest was administrated at this moment of the intervention.

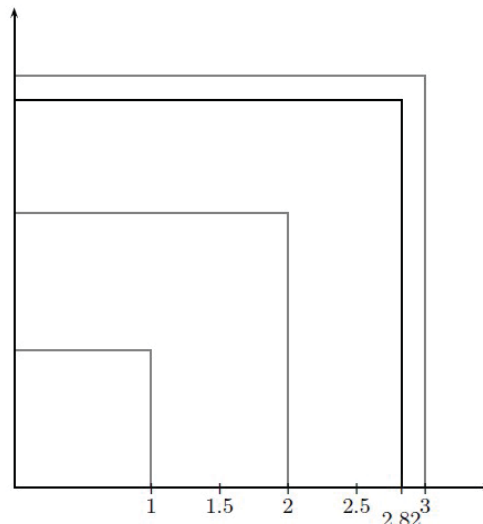


Fig. 5. Superposition of the squares on a graduated number line.

The preceding lessons were mainly based on problematic situations requiring to discover decimal fractions and emphasizing the principles of the decimal system. In the following lessons, the learning of decimal fractions was reinforced with comparison exercises. New situations involving more procedural knowledge such as the addition of decimal fractions were introduced. The procedural knowledge was then explicitly connected to conceptual knowledge about decimal fractions and decimal system. The whole intervention took about 10 lessons.

Control teaching

The teachers of the control group devoted less time to conceptual knowledge of decimal fractions. Decimal fractions were classically taught using metric measurement, number lines and knowledge of money. Tenths were introduced first, and the hundredths next. Only decimal fractions with 1 or 2 digits were taught. Decimal fractions were placed on number lines and then compared. Operations with decimal fractions were introduced next. The procedures for addition and subtraction were explicitly taught.

Dependent measures

Conceptual and procedural knowledge are strongly associated (Hallett et al., 2010 ; Hecht, 1998). It is hard to measure them independently of each other (Schneider & Siegler, 2010). In the present study, we create some tasks assessing mainly conceptual

knowledge and as little procedural knowledge as possible, and other tasks assessing mainly procedural knowledge and as little conceptual knowledge as possible.

Three tasks assessed mainly conceptual knowledge about decimal fractions : comparison, order (and betweenness), and transcoding. In the comparison task (70 items), children were presented with pairs of decimal fractions (e.g., 0.2 vs. 0.06) and were asked to “circle the larger number or to place an equals sign between them if the numbers are the same in value”. Both timed (two minutes) and untimed versions were presented in order to take both processing accuracy and speed into account. In the order and betweenness task (28 items), children were asked to write a number included between two decimal fractions (e.g., $0.2 < \dots < 0.3$). The transcoding task (20 items) implied translating decimal fractions from number words to Arabic numbers and from Arabic numbers to number words (e.g., $0.01 \leftrightarrow$ one hundredth).

An addition task assessed mainly the procedural knowledge about decimal fractions. In this task, children were required to solve simple (e.g., $0.2 + 0.5$) or complex additions, including decimal fractions of different length or trading of hundreds for tenths, or trading of tenths for units (e.g., $0.3 + 0.04$ or $0.5 + 0.7$), in both timed (two minutes) and untimed versions (45 items).

Procedure

The tasks to assess decimal fractions competencies were presented at three time points for a longitudinal study. They were administrated in a group setting by the experimenter. The first time point (T_1) was at the autumn term (October) of Grade 4, before any formal teaching of decimal fractions. The second time point (T_2) was at the winter term (February) of Grade 4, during the first teaching of decimal fractions. The third time (T_3) was at the summer term (June) of the same year, after the teaching of decimal fractions. The tasks were presented in the same order to control an experimental groups. For each time point, children took less than forty minutes for all tasks together.

7.2.2 Results

Reliability and factor analysis

To investigate whether our tasks with decimal fractions measured various types of knowledge, we conducted factor analysis using principal components extraction with varimax rotation on the mean accuracies. Two components emerged with eigenvalues larger than one (3.52 and 1.06) and accounted for 76.75% (59% and 14%) of the common variability. The two components accounted for 65.75% of the variability and best described the data. However, the third component with eigenvalue of .78 accounted for 11% of the variance and was also considered to explain the data. Loadings on each component after rotation were presented for each task in Table 2. The component loading criterion of .6 or higher was used to observe task that loaded strongly and

singularly on a latent component.

The component matrix after rotation indicated that comparison and betweenness tasks showed more saturations on the first component, while all addition tasks showed higher saturation on the second component, and the transcoding task more saturation on the third component. The first component appeared to capture children's conceptual knowledge. The second component seemed to be more related to children's procedural knowledge implied in addition. Finally, the third component appeared to be associated with number words knowledge and transcoding.

Tabl. 2
Component loadings of each task.

Task	Component 1	Component 2	Component 3
Comparison	.844	.121	.136
Comparison (TC)	.690	.389	-.044
Betweenness	.754	.185	.204
Addition	.468	.657	.237
Addition S (TC)	.426	.806	.039
Addition C (TC)	.054	.920	.018
Transcoding	.156	.071	.967

Impact of the intervention

For each participant, the accuracy for each task was computed as the mean rate of correct responses. For tasks with time constraint, the accuracy was computed as the ratio between the score and the maximum score for the task observed at T_3 . The mean accuracy for each task for both groups is given in Table 3.

Three-way repeated measures ANOVA was conducted on the mean accuracy with the Task (1-7) and the Time (1-3) as the within-subject factor, and the Group (1-2) as the between-subjects factor. The Greenhouse-Geisser corrections were applied when violations of sphericity were observed. The main effect of Group was not significant, $F(1, 109) < 1$. The main effect of Task was significant ($F(4.22, 460.11) = 269.36$, $p < .001$, $\eta^2 = .71$) as reported in Table 2. The main effect of Time was also significant ($F(1.76, 191.58) = 700.35$, $p < .001$, $\eta^2 = .87$), indicating that the accuracy increased across time points (33.5 %, 61.2 % and 75.2 % for T_1 , T_2 and T_3 respectively) ($p < .001$ for all comparisons). The interaction between Time, Task, and Group was significant ($F(8.65, 942.50) = 4.32$, $p < .001$, $\eta^2 = .04$), including all double interactions (Time $\times$ Group: $F(1.76, 191.58) = 8.98$, $p < .001$, $\eta^2 = .08$, Time $\times$ Task: $F(8.65, 942.50) = 21.91$, $p < .001$, $\eta^2 = .17$, and Task $\times$ Group: $F(4.22, 460.11) = 4.62$, $p = .001$, $\eta^2 = .04$). The interaction between Time and Task is described in Table 3.

Here after, we focused on significant interactions including Group. To understand these interactions, two-way repeated measures ANOVAs, with Time as the within subject factor and Group as the between subjects factor were conducted for each task.

Tabl. 3
Mean accuracy in each task by group for each time point.

	Control group			Experimental group			Both groups	
	Time 1		Time 2		Time 3		Time 3	
	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)
Conceptual knowledge								
Comparison	51.2 ₊ (18)	77.2 (24)	87.0 (21)	71.8 _d (17)	48.6 ₊ (16)	87.5 (16)	97.5 (11)	74.4 _a (14)
Comparison (TC)	25.7 ₊ (8)	41.8 (18)	55.9 (21)	41.2 _b (13)	25.0 ₊ (8)	47.6 (17)	67.2 (20)	43.5 _b (13)
Betweenness	46.5 ₊ (20)	68.3 (25)	75.0 (22)	63.3 _c (19)	50.8 ₊ (21)	70.8 (26)	91.9 (17)	71.2 _c (18)
Procedural knowledge								
Addition	33.7 ₊ (27)	75.0 (20)	83.9 (21)	64.2 _c (19)	34.2 ₊ (26)	66.7 (21)	86.3 (14)	63.4 _c (17)
Addition S (TC)	19.1 ₊ (15)	51.0 (22)	68.4 (24)	46.2 _b (17)	18.2 ₊ (18)	44.3 (23)	74.8 (22)	46.1 _b (18)
Addition C (TC)	15.0 ₊ (14)	35.4 (18)	38.4 (21)	29.6 _a (15)	15.0 ₊ (14)	30.7 (15)	39.5 (18)	29.1 _d (14)
Number words								
Transcoding	41.7 ₊ (25)	87.6 (17)	93.1 (15)	74.1 _d (13)	44.6 ₊ (27)	73.2 (28)	94.5 (15)	72.8 _{a,c} (17)
Mean (SD)	33.3 (12)	62.3 (15)	71.7 (15)		33.8 (13)	60.1 (16)	78.8 (11)	

Note. TC = with Time Constraint, S = Simple, C = Complex. The means in the same column that do not share subscript letters differ at $p < .05$ regarding the post hoc t-tests. The means in the same row for T_1 that do not share subscript symbol differ at $p < .05$ in the MANOVA analysis.

Tabl. 4
Statistics and description for the Time x Task interaction.

	Mean			Statistics for the main effect of Time			
	Time 1	Time 2	Time 3	F	p	partial η^2	Post hoc t-tests
	Mean (SD)	Mean (SD)	Mean (SD)	df			
Comparison	50.1 _e (18)	81.6 _d (22)	91.6 _e (18)	1.81, 199.50	210.60	< .001	.66 in all cases ***
Comparison (TC)	25.4 _c (8)	44.3 _b (17)	60.8 _b (21)	1.85, 203.70	216.33	< .001	.66 in all cases ***
Betweenness	48.4 _e (20)	69.4 _c (26)	82.3 _d (22)	1.96, 215.66	135.36	< .001	.55 in all cases ***
Addition	33.9 _{c,d} (27)	71.4 _c (21)	84.9 _d (18)	1.56, 171.54	277.71	< .001	.72 in all cases ***
Addition S (TC)	18.7 _b (16)	48.1 _b (23)	71.2 _c (23)	2, 220	397.93	< .001	.78 in all cases ***
Addition C (TC)	15.0 _a (14)	33.3 _a (17)	38.9 _a (17)	1.83, 200.72	125.86	< .001	.53 in all cases ***
Transcoding	42.9 _{d,e} (26)	81.4 _d (23)	93.7 _e (15)	1.87, 205.15	208.58	< .001	.66 in all cases ***

Note. *** = $p < .001$. The means in the same column that do not share subscripts differ at $p < .05$ regarding the post hoc t-tests.

For the addition tasks with time constraint, the Time $\times$ Group interaction was significant for simple additions ($F(1.85, 202.03) = 6.36, p < .01, \eta^2 = .05$), but not for complex additions ($F(1.80, 196.27) = 1.94, ns.$). For simple additions, both groups did not differ significantly at time 1, 2 or 3, but the slope was stronger for control group than for experimental group between T_1 and T_2 , while it was stronger for experimental group than for control group between T_2 and T_3 (see Fig. 6). For the addition task without time constraint, the interaction between Time and Group approached significance threshold ($F(1.51, 164.40) = 3.24, p = .06, \eta^2 = .03$).

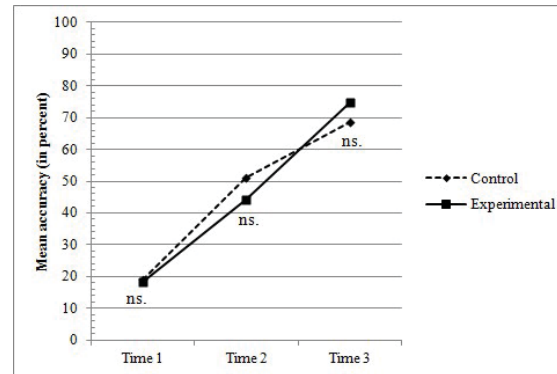


Fig. 6. Mean accuracy for simple additions with time constraint.

Note. ns. = not significant, * = $p < .05$, ** = $p < .01$, *** = $p < .001$.

For the comparison task with time constraint, the interaction of Time with Group was significant ($F(1.89, 205.86) = 6.41, p < .01, \eta^2 = .06$). Both groups did not differ at T_1 and T_2 , but the accuracy of the experimental group was significantly higher than the accuracy of the control group at T_3 ($p < .01$) (see Fig.5). For the comparison task without time constraint, the effect of Time was interact with Group ($F(1.85, 201.53) = 6.66, p < .05, \eta^2 = .06$). For both groups, the accuracy increased with time (in all cases $p < .001$). The accuracy for the experimental group did not differ from the accuracy for the control group at T_1 and T_2 , but was significantly higher than for the control group at T_3 .

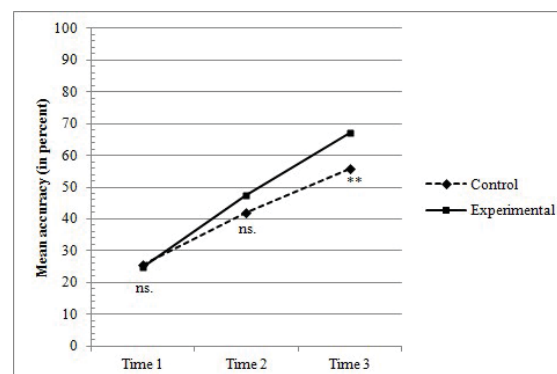


Fig. 7. Mean accuracy for the comparison task with time constraint by group for each time point.

Note. ns. = not significant, * = $p < .05$, ** = $p < .01$, *** = $p < .001$.

For the transcoding task, the Time $\times$ Group interaction was significant ($F(1.85, 201.99) = 7.07, p = .001, \eta^2 = .06$). The accuracy increased with time (all comparisons at $p < .001$), but groups did not progress in the same way. The groups did not differ at T_1 and T_3 , but the accuracy of the control group was higher than the accuracy of the experimental group at T_2 ($p = .001$). Finally, for the task concerning the order and betweenness of decimal fractions, the effect of Time interacted with Group ($F(1.92, 209.53) = 7.34, p = .001, \eta^2 = .06$). Both groups did not differ at T_1 and T_2 , but the experimental group showed an higher accuracy than the control group at T_3 ($p < .001$).

In brief, both groups showed improvement across T_1 , T_2 , and T_3 in tasks involving conceptual knowledge like comparison and betweenness and in tasks involving mainly procedural knowledge. Regarding addition tasks and procedural knowledge, both groups progressed similarly for all tasks except for the task of simple addition with time constraint, in which the experimental group compared to the control group showed a lower progression between T_1 and T_2 , and a greater progression between T_2 and T_3 . Regarding conceptual knowledge, the improvement of the experimental group, mainly between T_2 and T_3 , was greater than the improvement of the control group in comparison and betweenness tasks. However, both groups showed the same improvement for the transcoding task.

7.3 Follow-up assessment in Grade 5

In this experiment, we again examined the effect of our didactical intervention. A follow-up assessment was conducted 4 months after the end of the intervention (and after a 2 months vacation), at the beginning of Grade 5. We also investigated the impact of the intervention with novel tasks not included in the intervention.

7.3.1 Method

Participants

Ninety-two participants in Experiment 1 remained and took part in Experiment 2 (43 boys, mean age = 10 years 3 months; $SD = 4.05$ months). Forty-six children of Grade 5 were respectively assigned to the experimental group (21 boys, mean age = 10 years 3 months, $SD = 3.83$ months) and 46 to the control group (22 boys, mean age = 10 years 3 months, $SD = 4.30$ months) in Grade 4.

Measures

Five tasks without any time constraint assessed the decimal fraction competencies. Tasks involving comparison (20 items), order (and betweenness) (20 items) and addition (20 items) of decimal fractions were similar to those used in Experiment 1. Two other tasks assessed subtraction (20 items) (e.g., $0.7 - 0.04$) and multiplication (20 items)

(e.g., 0.5×0.7) of decimal fractions and implied more procedural knowledge than conceptual knowledge.

Procedure

The tasks were presented by the experimenter in a group setting at the autumn term (October) of Grade 5, before any decimal fraction revision. The tasks were presented in the same order for both control and experimental groups. All testing lasted about thirty minutes.

7.3.2 Results

For each participant, the accuracy was computed as the mean rate of correct responses for each task. The mean accuracies for each group are given in Table 5.

Two-way repeated measures ANOVA was conducted on the mean accuracy with the Task (1-5) as the within-subject factor and the Group (1-2) as the between-subjects factor. The Greenhouse-Geisser corrections were applied when violations of sphericity were observed. The main effect of Group was significant ($F(1, 90) = 4.50, p < .05, \eta^2 = .05$), indicating that the experimental group had higher correct response rates than the control group. The main effect of Task ($F(2.90, 260.94) = 215.41, p < .001, \eta^2 = .71$) was also significant, as reported in Table 4.

Tabl. 5
Mean accuracy in each task by group.

	Control group	Experimental group	Both groups
	Mean (<i>SD</i>)	Mean (<i>SD</i>)	Mean (<i>SD</i>)
Comparison	90.8 (17)	96.7 (10)	93.8 _a (14)
Addition	85.7 (17)	83.3 (16)	84.5 _b (16)
Subtraction	70.3 (28)	74.3 (28)	72.3 _c (28)
Multiplication	30.4 (13)	36.2 (14)	33.3 _d (14)
Betweenness	80.7 (22)	93.9 (11)	87.3 _b (18.5)
Mean (<i>SD</i>)	71.6 (14)	76.9 (10)	

Note. The means in the same column that do not share subscripts differ at $p < 0.001$ regarding the post hoc *t*-tests.

The interaction between Task and Group was also significant ($F(2.90, 260.94) = 2.88, p < .05, \eta^2 = .03$). To understand this interaction, analysis of variance for multiple dependent variables by one factor (MANOVA) was conducted with Group as the independent factor and Tasks as dependent factor. As showed in Fig. 6, the effect of Group was significant for the comparison ($F(1, 90) = 4.311, p < .041, \eta^2 = .05$), multiplication ($F(1, 90) = 3.99, p < .05, \eta^2 = .04$) and betweenness ($F(1, 90) = 13.444, p < .001, \eta^2 = .13$) tasks, but not for the addition and subtraction tasks ($F_s < 1$). For the comparison, multiplication and betweenness tasks, the accuracy was higher for the experimental group than for the control group.

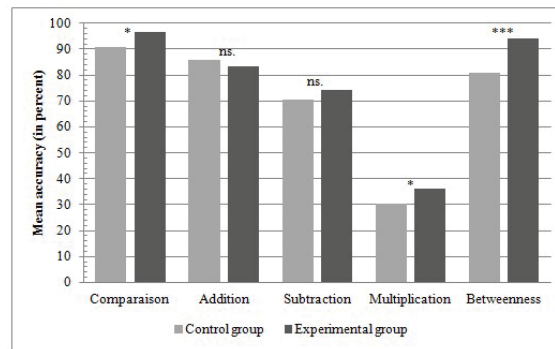


Fig. 8. Mean accuracy for each task by group.

Note. ns. = not significant, * = $p < .05$, ** = $p < .01$, *** = $p < .001$.

To sum up, the experimental group was still better than the control group in the two tasks assessing conceptual knowledge (comparison and betweenness). Better performance for the experimental group was also found for one of the three tasks assessing procedural knowledge (multiplication), but not for the two others (addition or subtraction).

7.4 Discussion

The learning of decimal fractions expressed in the place-value system is laborious. Children have many misconceptions about these numbers. The present study investigated the impact of an experimental intervention involving (1) the approximation of real numbers by decimal fractions in the place-value system; (2) emphasizing conceptual knowledge about the place-value system and decimal fractions; and (3) taking into account children's spontaneous conceptions and misconceptions. The results indicated that the intervention led to immediate as well as long-range improvement on both conceptual and procedural knowledge (Hypothesis 1). Compared to control group, children in the experimental group reached higher accuracy in comparison and betweenness tasks (Hypothesis 2), but not in addition and transcoding tasks. The follow-up data showed that the differences remained 4 months later and that the experimental group were still better even in a multiplication task that was not taught.

Comparison and betweenness tasks are supposed to depend on the same latent component, as demonstrated by our factor analysis. This component seemed to include mainly the place-value concept understanding and so the conceptual knowledge. Another latent component implied all addition tasks, which certainly reflects the procedural knowledge needed in the addition of decimal fractions. Finally, the transcoding task seemed to depend on neither the first nor the second component, but rather on a third component relying on number words knowledge.

After the intervention including mainly conceptual knowledge, the experimental group showed higher conceptual knowledge and the same level of procedural knowledge than the control group. The intervention improved the conceptual knowledge and subsequently (between T_2 and T_3 and in the follow-up assessment) procedural knowledge, like in

an iterative process previously suggested (Canobi & Bethune, 2008 ; Rittle-Johnson & Koedinger, 2009 ; Rittle-Johnson et al., 2001 ; Schneider et al., 2011). However, the causal relation between conceptual and procedural knowledge did not appear to be symmetrical as suggested before (Schneider et al., 2011). Our results showed that the control group receiving a teaching involving mainly procedural knowledge did not improve as strongly as the experimental group in conceptual knowledge. The causal relation seemed so to be stronger from conceptual knowledge to procedural than from procedural knowledge to conceptual knowledge. Moreover, the experimental group showed higher accuracy than the control group in a novel task, not included in the intervention. This suggests that a deeper conceptual knowledge allows resolving new problems, as previously observed (Moss & Case, 1999), while procedural knowledge is strongly related to the problems included in the teaching sequence. Children receiving a teaching including mainly conceptual knowledge, seemed to show a better procedural flexibility, but further investigation is needed to explore this assumption.

The intervention used here was also characterized by the absence of rational numbers expressed in common fractions. The intervention included decimal subdivision of a length in equal-size subparts, and thus the rational number concept, but the representation of rational numbers by common fractions was not used. The representation of decimal fractions using the place-value system were introduced before their representation in common fractions. The results indicated a deep conceptual understanding and a good procedural knowledge about decimal fractions when they were introduced mainly by the extension of the place-value system. To our opinion, it is easier for Grade 4 children to discover decimal fractions through the place-value system than by common fraction. Understanding the fractional notation is specially difficult for children. Introducing decimal fractions in the place-value system by common fractions could thus be laborious.

Furthermore, the understanding of density is laborious for children (Vamvakoussi, Konstantinos, Mertens, & Van Dooren, 2011 ; Vamvakoussi & Vosniadou, 2004, 2010), but the experimental group showed a stronger improvement in the betweenness task than the control group. The intervention emphasizing the place-value system thus seemed to be a good introduction to the density property of rational numbers. Indeed, it could be easier to find a number comprised between 0.3 and 0.4 than a number between $\frac{1}{4}$ and $\frac{1}{3}$. Children have of course to understand both representations of rational numbers (decimal fractions in the place-value system and common fraction) and the conversion between the two representations. But the extension of the place-value system to rational numbers seemed to be a promising starting point.

To conclude, the present study indicated that an introduction to decimal fractions by the approximation of real numbers in the place-value system emphasized conceptual knowledge about the place-value system and conceptual knowledge about decimal fractions. This conceptual knowledge leads to good procedural knowledge and constitutes a first approach of the density property of rational numbers.

Discussion générale

Ci-après, nous reprendrons une synthèse des résultats obtenus dans les différentes études présentées dans cette thèse pour les discuter selon quatre thèmes : la place de l'histoire, les erreurs et leurs origines, le changement conceptuel, et enfin les perspectives didactiques.

Entre histoire, apprentissage et enseignement

En analysant l'histoire des nombres décimaux, nous avons pu constater qu'il a fallu des milliers d'années à l'être humain pour élaborer les principes du système décimal de position, en particulier le principe de position et le zéro. Il a également fallu une très longue période avant que les nombres rationnels apparaissent et, ensuite, avant que certains d'entre eux soient représentés dans le système décimal de position. Suivant l'histoire des nombres décimaux, nous avons émis deux suppositions. La première était que l'apprentissage des nombres décimaux pourrait être modelé par l'histoire des nombres décimaux et que la compréhension des nombres rationnels exprimés en fractions précéderait alors la compréhension des nombres décimaux. La seconde hypothèse, orientée vers la didactique, stipulait que l'enseignement des nombres décimaux pourrait s'ancrer sur la compréhension des nombres rationnels exprimés en fractions, celles-ci ayant historiquement précédé les nombres décimaux.

La compréhension des nombres rationnels représentés par des fractions est-elle indispensable à la compréhension des nombres décimaux ? Les résultats de l'étude transversale, présentée au chapitre 4, investiguant le développement de la compréhension des trois formats de présentation des nombres rationnels (fraction, fraction décimale et nombre décimal), indiquent qu'en 3^e année primaire le traitement des fractions et encore plus celui des nombres décimaux sont sujets à des taux d'erreurs très importants. En 4^e année primaire, même si les taux de réponses correctes augmentent, le profil de performance reste identique (voir la figure p.89). Ensuite, une fois l'enseignement des nombres décimaux initié, en fin de 4^e année primaire, les nombres décimaux deviennent plus faciles à traiter que les fractions, surtout lorsqu'il s'agit de nombres égaux (ou de fractions équivalentes). Il est intéressant aussi de constater qu'après l'enseignement des nombres décimaux, les fractions décimales deviennent elles aussi plus

faciles à traiter que les fractions.

Même si les taux de réponses correctes en 3^e année primaire sont extrêmement faibles, la compréhension des fractions, enseignées dès la 2^e année primaire, débiterait donc quelque peu avant la compréhension des nombres décimaux. Ces résultats sont néanmoins indissociables de l'enseignement reçu. En Communauté française, l'enseignement des fractions les plus courantes telles que le demi ou le quart débute en effet en 2^e année primaire, bien avant celui des nombres décimaux. Ces fractions les plus courantes étaient fréquemment présentées dans notre tâche de comparaison de fractions, ce qui lie encore plus les résultats obtenus à l'enseignement reçu. Le profil développemental observé est, de plus, loin d'une situation où les taux de réponses correctes dans le traitement des fractions dépassent le hasard avant que les taux de réponses correctes dans le traitement des nombres décimaux n'augmentent eux aussi. En réalité, les différences dans les taux de réponses correctes entre chaque format restent relativement faibles, quelle que soit l'année scolaire. Enfin, les nombres décimaux sont plus proches des fractions décimales que des autres fractions et les résultats montrent que les fractions décimales ne deviennent plus simples à traiter que les autres fractions qu'après l'enseignement des nombres décimaux. Il se pourrait donc que ce soient les nombres décimaux qui facilitent la compréhension des fractions décimales, par un processus de transcodage, et non l'inverse. Cependant, il se pourrait aussi qu'à partir de la 5^e année primaire les fractions décimales soient plus présentes dans le curriculum.

Sur base des résultats de cette étude, nous pouvons donc constater, qu'après leur enseignement, les nombres décimaux deviennent plus faciles à traiter que les fractions, mais la question concernant la compréhension des fractions qui serait nécessaire ou non à la compréhension des nombres décimaux reste ouverte. Une étude longitudinale réalisée auprès de participants sur le point de recevoir le premier enseignement des nombres décimaux serait plus à même de donner des éléments de réponses quant à d'éventuels liens de causalité entre la compréhension des fractions et celle des nombres décimaux.

Une telle étude longitudinale a été présentée au chapitre 5. Elle poursuivait deux objectifs. Un premier objectif était d'observer si la compréhension des fractions prédisait l'apprentissage des nombres décimaux. Un second objectif était de savoir si la compréhension du système décimal de position, initialement destiné exclusivement à la représentation des nombres naturels, prédisait elle aussi la compréhension des nombres décimaux. Ensuite, si ces deux facteurs étaient des prédicteurs de la compréhension des nombres décimaux, nous souhaitions savoir s'ils expliquaient autant l'un que l'autre les futures performances sur les nombres décimaux. Les résultats indiquent qu'en début de 4^e année primaire, avant l'apprentissage des nombres décimaux, les tâches portant sur les fractions et sur le système décimal de position ont des taux de réussite semblables et sont relativement bien réussies. Ces deux variables sont corrélées et prédisent la compréhension des nombres décimaux. Il est difficile de totalement exclure la possibilité que ces résultats proviennent d'une variable latente intrinsèque aux mathématiques, telle *un niveau global de réussite en mathématiques* ou d'un facteur extrinsèque, tel que

l'intelligence. Les résultats montrent cependant qu'à un même niveau de difficulté, la compréhension des fractions est un moins bon prédicteur de l'apprentissage des nombres décimaux que la compréhension du système décimal. Cela semble difficilement explicable par une variable latente.

Les résultats de ces deux études tendent donc à montrer que même si la compréhension des fractions précède légèrement la compréhension des nombres décimaux, celle-ci n'est pas un prédicteur aussi puissant que la compréhension du système décimal pour l'apprentissage des nombres décimaux. Néanmoins, pour comprendre les nombres décimaux, le système décimal ne suffit pas. Il faut également accepter qu'une unité puisse être fractionnée.

Perspectives... D'autres études pourraient être menées pour approfondir les liens entre la compréhension des fractions et celle des nombres décimaux. D'une part, les tâches que nous avons utilisées pour mesurer la compréhension des fractions dans notre étude longitudinale nécessitaient le traitement d'une représentation symbolique des fractions. Or, de nombreuses études ont montré que le traitement non-symbolique de rapports était possible dès le plus jeune âge (Singer-Freeman & Goswami, 2001 ; Sophian et al., 1997 ; Sophian, 2000), mais que ces capacités contrastaient avec les performances lorsque les fractions sont représentées sous la forme d'un rapport entre deux nombres entiers (e.a., Stafylidou & Vosniadou, 2004 ; Vamvakoussi & Vosniadou, 2004). Il serait donc intéressant de voir si une compréhension de la relation partie/tout, indépendante de tout symbole, serait un bon prédicteur de l'apprentissage des nombres décimaux.

D'autre part, lorsque nous avons comparé les formats de présentation des nombres rationnels, nous nous sommes limitée à une tâche de comparaison. Les résultats pourraient être différents dans d'autres tâches. Par exemple, dans une tâche mettant en œuvre la propriété de densité, la représentation sous forme de fraction pourrait être plus simple, comme cela a été suggéré par Vamvakoussi et Vosniadou (2004). Néanmoins cette supposition ne se base que sur un entretien et il semble plus simple a priori, lorsque l'on a compris le système décimal, de trouver un nombre entre 0,3 et 0,4 que de trouver un nombre entre $\frac{1}{3}$ et $\frac{1}{4}$. En ce qui concerne les opérations, la représentation sous forme de nombres décimaux devrait également être plus facile. Par contre, pour opérer sur une grandeur, par exemple une aire, il serait sans doute plus aisé d'obtenir $\frac{1}{4}$ d'un disque que d'en obtenir 0,25. Vraisemblablement, chaque format de présentation de nombres rationnels a ses forces et ses faiblesses.

Pour conclure à propos de ce que l'histoire peut nous apporter concernant l'apprentissage, nous pourrions dire que la réalité de l'apprentissage est bien plus complexe qu'un *simple mimétisme* de l'histoire. La compréhension des fractions ne semble pas être première à la compréhension des nombres décimaux. Les fractions seraient au contraire encore plus difficiles à traiter que les nombres décimaux, ces derniers étant également nettement plus fréquents dans l'environnement, pour la manipulation de la

monnaie notamment. Néanmoins, les difficultés apparues au cours de l'histoire, telles que la mise au point du principe de position et du rôle si particulier du zéro, se retrouvent dans les difficultés d'apprentissage des nombres décimaux. L'histoire apporte donc un éclairage sur certaines difficultés d'apprentissage, mais ne trace pas nécessairement les différentes étapes de l'apprentissage pour les élèves d'aujourd'hui.

Traitement des nombres décimaux et origines des erreurs

Un autre objectif de cette thèse était d'en savoir un peu plus sur le traitement des nombres décimaux, de mieux comprendre les erreurs qui sont commises lors de ce traitement, d'en observer l'évolution et de tenter d'en comprendre les origines.

Les nombres rationnels représentés sous forme de fractions semblent pouvoir être représentés et traités de manière holistique (Meert et al., 2009, 2010a) et ce dès 10 ans (Meert et al., 2010b). Vraisemblablement, il devrait pouvoir en être de même pour les nombres décimaux. Nous ne nous sommes pas donné les moyens de répondre à cette question dans cette thèse. Nous avons néanmoins tenu compte, lorsque c'était possible, d'un facteur tel que la distance entre les nombres des paires de nombres décimaux que nous avons utilisées dans nos études. Il semble que la distance entre les nombres à comparer n'explique pas certaines différences observées. Par exemple, dans l'étude dont les résultats apparaissent à la page 50, nous avons utilisé la paire 1,39 vs. 1,7. La distance globale entre les deux nombres est de 0,31, ce qui est exactement la même distance globale qu'entre les nombres de la paire 0,4 vs. 0,71. Pourtant, pour les élèves de la 3^e à la 6^e année primaire, le taux moyen de réussite pour la paire 0,4 vs. 0,71 est de 99% alors qu'il n'est que de 46% pour la paire 1,39 vs. 1,7. Des facteurs autres que la distance globale entre les nombres semblent donc intervenir. Comme pour les naturels (Nuerk et al., 2001) ou les fractions (Meert et al., 2009, 2010b), un effet de congruence entre les composants du nombre et sa magnitude globale pourrait intervenir.

Quels sont les facteurs influençant le traitement des nombres décimaux ? De précédentes études (Resnick et al., 1989 ; Sackur-Grisvard & Leonard, 1985 ; Nesher & Peled, 1986 ; Irwin, 2001) ont montré que les erreurs observées lors de la comparaison des nombres décimaux pourraient s'expliquer par différentes conceptions. Selon une première conception, issue des connaissances liées aux nombres naturels, *le nombre le plus grand serait celui qui est le plus long, ou qui contient le plus grand naturel dans sa partie décimale*, par exemple $1,25 > 1,3$. Selon, une deuxième conception, issue des connaissances liées aux fractions, et ne tenant compte que du dénominateur de la fraction décimale, *le nombre le plus grand serait celui qui est le plus court*, par exemple, $1,3 > 1,45$. Ces deux conceptions conduisent tantôt à des réponses correctes, tantôt à des erreurs. Enfin, selon une troisième conception, conduisant elle systématiquement à une réponse correcte, *le*

nombre le plus grand serait celui dont la partie décimale ne commence pas par un zéro, par exemple, $2,5 > 2,06$.

Néanmoins ces différentes études sont critiquables d'un point de vue méthodologique. Nous avons détaillé ces biais précédemment (voir page 31). Rappelons ici que l'impact de la longueur de la partie décimale n'a pas été différencié de celui de la valeur des chiffres composant la partie décimale, que l'impact du zéro dans la partie décimale n'a pas été spécifiquement analysé et que les consignes utilisées ne permettent pas aux participants de répondre que les nombres sont égaux. De plus, les conceptions initiales avant tout enseignement des nombres décimaux n'ont pas été investiguées. Les conceptions décrites ne peuvent donc pas être indépendantes de l'enseignement reçu.

Nous avons alors formulé différentes hypothèses quant aux facteurs influençant le traitement des nombres décimaux. Premièrement, la longueur de la partie décimale et la valeur des chiffres pourraient avoir un impact distinct, l'impact de la valeur des chiffres étant plus important. Deuxièmement, un zéro présent dans la partie décimale, de par son statut différent de celui des autres chiffres, devrait être acquis plus tardivement. Dans une étude, présentée au chapitre 3, nous avons donc analysé l'impact de trois facteurs : la longueur du nombre, la valeur des chiffres et la présence d'un zéro dans la partie décimale, en évitant les biais méthodologiques des études précédentes et en incluant parmi les participants, des élèves de 3^e primaire n'ayant pas encore reçu d'enseignement des nombres décimaux.

Les résultats de cette étude montrent qu'avant l'enseignement des nombres décimaux, en 3^e et au début de la 4^e année primaire, les élèves sont fortement affectés par la valeur des chiffres constituant la partie décimale. Lorsque ceux-ci sont incongruents avec la magnitude globale du nombre, les taux de réponse correcte chutent radicalement. Dans une moindre mesure, les élèves sont également affectés par le nombre de chiffres apparaissant dans la partie décimale. Il est difficile pour eux de considérer que le nombre le plus court puisse être le plus grand. Enfin, la présence d'un zéro dans la partie décimale les perturbe, surtout lorsque celui-ci termine la partie décimale sans modifier la valeur globale du nombre.

Après l'enseignement des nombres décimaux, en 5^e année primaire, la valeur des chiffres a toujours un impact semblable à celui observé chez les plus jeunes. La longueur a, elle aussi, encore un impact, mais le pattern est inversé par rapport aux élèves plus jeunes. À cet âge, le nombre le plus court a tendance à être choisi comme étant le plus grand. De plus, le rôle du zéro est toujours en cours d'acquisition. Les élèves comprennent à présent que le zéro indique un rang inoccupé lorsqu'il débute la partie décimale (par exemple, 0,02), mais plus difficilement lorsqu'il la termine (par exemple, 0,20). Enfin, en 6^e année primaire, l'ensemble de ces difficultés semble être surmonté.

Ces résultats nous indiquent que les trois facteurs que sont la valeur des chiffres, la longueur et le zéro ont un impact sur le traitement des nombres décimaux.

Commençons par discuter l'impact de la **valeur des chiffres**. Celle-ci crée un effet de congruité majeur sur les taux de réponses correctes. Par exemple, une paire telle que 0,06

vs. 0,7 est comparée beaucoup plus facilement qu'une paire telle que 0,06 vs. 0,4, surtout au début de l'apprentissage. Cela n'a rien de surprenant lorsque l'on sait que, dès 6 ou 7 ans, la représentation mentale est activée automatiquement à partir des chiffres arabes (Zhou et al., 2007 ; Girelli et al., 2000). Lorsque les élèves de 8 à 11 ans comparent les nombres de la paire 0,06 vs. 0,5, ils accèdent certainement aux représentations mentales de 5 et 6. Ils devraient ensuite inhiber le fait que 6 est plus grand que 5 et interpréter les chiffres selon leur valeur de position afin de répondre que 0,5 est plus grand que 0,06. Des effets de congruence entre les composants d'une fraction et sa magnitude globale ont également été observés chez l'adulte (Meert et al., 2009) et chez l'enfant de 10 - 12 ans (Meert et al., 2010b). Une paire telle que $\frac{3}{5}$ vs. $\frac{3}{7}$ est plus facilement comparée qu'une paire de type $\frac{10}{11}$ vs. $\frac{10}{13}$. Nous ne l'avons pas expérimenté, mais étant donné que les adultes sont sensibles à la congruité entre les dizaines et les unités pour les nombres naturels (Nuerk et al., 2001) et à la congruité entre les composants d'une fraction et sa valeur, vraisemblablement un tel effet de congruité pourrait aussi être observé dans le traitement des nombres décimaux chez l'adulte.

Concernant l'impact de la **longueur**, les résultats sont particulièrement interpellants. Si les élèves les plus jeunes ont tendance à choisir le nombre le plus long comme étant le plus grand, en 5^e année primaire ils ont tendance à choisir le nombre le plus court. Pourquoi cette différence ? La théorie de Brousseau (1998) peut ici nous apporter un éclairage intéressant. Selon lui, il convient en effet de distinguer trois types d'obstacles : obstacle ontogénétique lié au stade de développement cognitif de l'élève, obstacle épistémologique lié aux caractéristiques du savoir à acquérir, et obstacle didactique lié à l'enseignement reçu. Selon nous, l'impact de la longueur observé en 5^e année primaire est un obstacle didactique, c'est-à-dire un obstacle issu de l'enseignement reçu. Différents éléments appuient cette hypothèse. Premièrement, cet effet n'apparaît qu'après l'enseignement et est inversé par rapport à l'effet observé chez les élèves n'ayant pas encore reçu cet enseignement. Deuxièmement, cet effet varie selon le curriculum et selon le pays (Nesher & Peled, 1986 ; Sackur-Grisvard & Leonard, 1985 ; Resnick et al., 1989). Dans les pays où le curriculum fait apparaître les fractions avant les nombres décimaux et base l'enseignement de ces derniers sur les fractions, cet effet est plus fréquent. Des entretiens individuels avec les élèves révèlent aussi que cet effet provient d'une implication des connaissances liées aux fractions dans le traitement des nombres décimaux (Resnick et al., 1989). Par exemple, considérer que 0,2 est plus grand que 0,24 semble provenir du fait que $\frac{2}{10}$ est considéré comme étant plus grand que $\frac{24}{100}$ parce que $\frac{1}{10}$ est plus grand que $\frac{1}{100}$. Le raisonnement serait donc basé exclusivement sur le dénominateur de la fraction et la partie décimale serait traitée comme une fraction. Nous pouvons en déduire d'une part que les choix didactiques ont un impact sur les conceptions des élèves, et d'autre part, que dans ce cas précis, les conceptions des fractions mobilisées lors du traitement des nombres décimaux sont immatures. Cela rejoint partiellement les résultats des études concernant l'impact du format de présentation des nombres rationnels et dont nous avons déjà discuté. Au moment d'apprendre les nombres décimaux, les

fractions ne sont en effet pas encore maîtrisées.

Enfin, l'impact du **zéro**, ainsi que l'impact de la longueur chez les plus jeunes, semblent liés à la compréhension du système décimal initialement utilisé pour représenter les nombres naturels. Au début de l'apprentissage, les nombres décimaux sont traités comme s'ils étaient des nombres naturels, ou selon les principes du système décimal de position tels qu'ils ont été compris. Pour comprendre les erreurs lors du traitement des nombres décimaux, il semble en effet important de distinguer les principes du système décimal de la façon dont ils ont été acquis. Par exemple, il est un fait que 10 est dix fois plus grand que 1. Pour autant, on ne peut pas en déduire que le zéro placé à droite « rend dix fois plus grand ». Il indique en réalité qu'il n'y a pas d'unité et que le chiffre 1 est relatif aux dizaines. Un même raisonnement peut s'appliquer en ce qui concerne l'addition. Lorsque l'on opère sur des nombres naturels, par exemple lorsque l'on additionne 14 et 5, additionner 4 et 5 et ajouter ensuite une dizaine est une stratégie efficace. Il n'y a qu'un pas pour en déduire qu'il faut « commencer l'opération avec les chiffres qui sont à droite » et ce que semblent indiquer les résultats de l'étude pilote sur l'addition présentée au chapitre 2. Qu'il s'agisse de connaissances conceptuelles ou procédurales, ces « déductions » semblent faire partie des connaissances des élèves. Sont-elles incontournables ? Sont-elles propres aux connaissances numériques qui immanquablement évoluent des nombres naturels vers les nombres rationnels ? On retrouve ici, au sens de Bachelard (1938) et Brousseau (1998), la notion d'obstacle épistémologique. Cet obstacle est certainement inévitable lorsque l'on progresse des nombres naturels aux nombres décimaux. Il faudrait cependant ne pas l'accentuer par des obstacles didactiques. Nous en discuterons dans la section dédiée à l'enseignement.

Perspectives... Comme pour les fractions, nous pourrions tenter de savoir si les nombres décimaux peuvent être représentés et traités de façon holistique. Au début de l'apprentissage, le traitement pourrait être décomposé et basé essentiellement sur l'intégration des chiffres selon leur valeur de position, mais au fil du temps ces étapes pourraient être de plus en plus automatisées et un accès à une représentation holistique pourrait devenir possible. Si c'est le cas, le traitement de paires telles que 0,08 vs. 0,9 (distance globale de 0,82) devrait être plus rapide et moins sujet aux erreurs que celui de paires telles que 0,01 vs. 0,2 (distance globale de 0,19), tout autre facteur influençant le traitement des nombres étant fixé. Comme pour les nombres naturels et les fractions, il faudrait veiller à utiliser une tâche qui nécessite le recours à une représentation holistique, c'est-à-dire un stockage en mémoire de travail (Zhang & Wang, 2005 ; Zhou et al., 2008). Les nombres utilisés pourraient également avoir un impact. Il se pourrait en effet que nous ayons une représentation approximative des nombres décimaux les plus courants (par exemple, 0,50 ; 0,25 ; 0,10), ou de ceux qui ne dépassent pas le millième, alors que les nombres décimaux de plus de deux ou trois chiffres après la virgule, par exemple 0,5789512, nécessiteraient un recours à un traitement décomposé et séquentiel.

Nous pourrions aussi, comme cela a été fait concernant les nombres naturels, observer

un éventuel **effet de compatibilité entre dixièmes et centièmes**. Pour les nombres naturels, l'effet de compatibilité entre les dizaines et les unités semble indiquer que le traitement des composants (dizaines et unités) se réalise au départ séquentiellement et devient ensuite parallèle (Nuerk et al., 2004 ; Pixner et al., 2009). Les chiffres sont intégrés à leur valeur de position de plus en plus automatiquement. Un design semblable pourrait être utilisé pour savoir si un effet de compatibilité entre dixièmes et centièmes apparaît et évolue de façon semblable pour les nombres décimaux. Les paires telles que 0,7 vs. 0,06 pourraient être traitées plus rapidement et avec moins d'erreurs que les paires telles que 0,6 vs. 0,07. Les pourcentages de réponses correctes observés dans l'étude présentée au chapitre 3 vont dans ce sens. Cependant, la présence d'un zéro dans les paires utilisées pourrait avoir un impact. On pourrait dès lors observer si la comparaison de paires telles que 0,37 vs. 0,52 nécessite des temps de réaction plus élevés que celle de paires telles que 0,32 vs. 0,47, reflétant également un effet de compatibilité entre les dixièmes et les centièmes. Par contre, il serait très difficile de différencier un traitement des nombres décimaux dans leur globalité (0,37 vs. 0,52) d'un traitement de la partie décimale comme si elle était un nombre naturel (37 vs. 52). Pour ce faire, nous pourrions éventuellement utiliser des paires telles que 0,4 vs. 0,23 et 0,9 vs. 0,73. Ces paires sont identiques en ce qui concerne l'incongruence de la longueur et de la valeur des chiffres, la distance globale et la distance entre les dixièmes. Avec de tels nombres décimaux, il serait impossible de créer des paires présentant des centièmes incongruents avec les dixièmes tout en respectant les autres variables. La distance entre les dixièmes et les centièmes pourrait cependant être petite ou grande et cela pourrait se répercuter sur les temps de réaction. La neuroimagerie pourrait elle aussi être utilisée. L'activation du sillon intrapariétal pourrait être modulée par la distance globale entre deux nombres décimaux, comme c'est le cas pour les fractions (Ischebeck et al., 2009).

D'autre part, il a été montré récemment chez l'adulte que, pour les nombres naturels, le traitement de la valeur de position d'un chiffre se faisait automatiquement (Kallai & Tzelgov, 2012). Une paire telle que 003 vs. 050 pour laquelle la position des chiffres est congruente avec la magnitude des chiffres qui doivent être jugés conduit à des taux de réussite supérieurs et des temps de réaction moins élevés qu'une paire incongruente telle que 005 vs. 030. Dans cette étude, tous les nombres utilisés étaient de trois chiffres et donc de même longueur. Cet effet pourrait aussi être investigué avec les nombres décimaux. Les magnitudes de deux chiffres pourraient être comparées avec plus d'aisance dans une paire congruente telle que 0,003 vs. 0,050 que dans une paire incongruente telle que 0,005 vs. 0,030. Si c'était le cas, cela pourrait signifier qu'un traitement automatique des valeurs de position est aussi appliqué à la partie décimale des nombres. Il resterait alors à se demander si les valeurs de position se définissent en prenant la droite du nombre comme point de départ. Le fait que la comparaison de paires de nombres décimaux de longueurs différentes, telles que 0,1 vs. 0,10 ou 0,2 vs. 0,10, soit si délicate pourrait alors s'expliquer par un traitement automatique des valeurs de position qui se définirait erronément en partant de la droite du nombre.

Enfin, étant donné le nombre de facteurs qui peuvent créer une interférence lors du traitement des nombres décimaux, les liens entre les capacités d'inhibition et l'apprentissage des nombres décimaux pourraient être étudiés.

Les questions concernant la représentation et le traitement des nombres décimaux sont donc encore nombreuses.

À propos du développement de la compréhension des nombres décimaux...

Concernant le développement de la compréhension des nombres naturels de plusieurs chiffres, nous savons que différentes conceptions peuvent déterminer les relations entre les mots nombres, les nombres écrits et les quantités (Fuson et al., 1997 ; Fuson, 1998) et que, dans un premier temps, les enfants ont une conception unitaire des nombres à deux chiffres (Fuson et al., 1997 ; Fuson, 1998 ; Collet, 2004). Ils conçoivent, par exemple, le nombre *trente deux* comme une quantité correspondant à un ensemble de trente deux éléments non structuré, et l'écriture en code arabe de cette quantité, 32, comme un tout et non comme deux chiffres exprimant des puissances de dix différentes. Ensuite, grâce notamment à la structure décimale de la chaîne numérique verbale, les dizaines sont distinguées des unités et l'enfant comprend progressivement qu'une dizaine et dix unités sont interchangeables. Il pourrait en être de même pour la compréhension des nombres décimaux. Elle pourrait elle aussi débiter par une conception unitaire de la partie décimale, sans connaissance des différentes puissances négatives de dix et des valeurs de position. Cette hypothèse est compatible avec certains résultats de l'étude présentée au chapitre 3 indiquant que les participants les plus jeunes, de 3^e et 4^e années primaires, pensent majoritairement que 0,1 est égal à 0,01 ou que 0,10 est plus grand que 0,1. Cette conception pourrait par ailleurs être renforcée par une lecture des nombres décimaux qui transformerait 0,15 en *zéro virgule quinze* plutôt qu'en *quinze centièmes* ou par l'utilisation de la monnaie, accentuant systématiquement un nombre de centièmes, de cents, après la virgule et faisant l'impasse sur les dixièmes (voir le chapitre 6). Plus tard dans le développement, les différentes puissances de dix seraient progressivement différenciées et les valeurs de position seraient prises en compte, comme cela a été proposé dans le modèle de Fuson (1997 ; 1998) pour les nombres naturels. Dans le cas des nombres décimaux, la chaîne numérique verbale ne serait cependant plus un support (Vosniadou et al., 2008 ; Resnick et al., 1989). À la lumière du modèle de Fuson, les aspects développementaux concernant les nombres naturels devraient néanmoins être approfondis avant d'aborder les nombres décimaux.

Pour conclure à propos du traitement des nombres décimaux, nous pouvons considérer que, chez l'enfant, deux difficultés se combinent. La première, et probablement la plus robuste, serait l'accès automatique à la magnitude à partir des symboles numériques, créant une interférence en cas d'incongruence avec la valeur globale du nombre. La seconde difficulté serait quant à elle liée à la compréhension du système décimal initialement destiné exclusivement à la représentation des nombres naturels. La

partie décimale pourrait être considérée de façon unitaire, comme c'est le cas pour les nombres naturels de deux chiffres. Cette difficulté s'estompe néanmoins relativement vite. En 6^e année primaire, elle semble surmontée, alors que l'interférence des chiffres a toujours un impact.

Changement conceptuel

Le fait que, dans l'apprentissage des mathématiques, des conceptions initiales puissent interférer avec l'acquisition de nouvelles connaissances est connu depuis des années (e.a., Fischbein, 1971). Plus récemment, la théorie du changement conceptuel, qui s'inscrit dans une approche constructiviste tout en étant spécifique à l'apprentissage des sciences, a été appliquée aux apprentissages mathématiques. Selon cette théorie : (a) les apprenants peuvent avoir des conceptions initiales qui contraignent l'apprentissage, (b) le processus de changement est lent et graduel, et (c) des niveaux intermédiaires de compréhension, c'est-à-dire des modèles synthétiques, peuvent être observés (Vosniadou, 1994, 2002). L'apprentissage des nombres décimaux semble effectivement nécessiter un changement conceptuel. Les connaissances numériques construites à partir de l'expérience quotidienne et des nombres naturels doivent être réorganisées et généralisées. Des modèles synthétiques ont pu être mis en évidence dans une de nos études, présentée au chapitre 3.

Qu'apportent nos résultats à propos des théories du changement conceptuel ? La théorie proposée par Vosniadou (1994, 2002) semble résister à nos résultats. L'apprentissage des nombres décimaux paraît en effet correspondre à un cas de changement conceptuel tel que ces auteurs le définissent. Les apprenants ont effectivement certaines conceptions initiales, basées sur les nombres naturels, qui contraignent l'apprentissage des nombres décimaux. Cet apprentissage est lent et graduel, et des stades intermédiaires de compréhension sont effectivement observés. Comme le soulignaient Greer et Verschaffel (2007), la théorie du changement est un outil facilement exploitable pour analyser certains apprentissages mathématiques.

Toutefois, plusieurs précisions doivent être apportées. D'une part, selon nous, toutes les difficultés liées à l'apprentissage des nombres décimaux ne sont pas conceptuelles. Reprenant Keil et Newman (2008), il se pourrait que des comportements observables, qui pendant une longue période semblent ne pas se modifier, peuvent en réalité masquer des changements conceptuels majeurs. Inversement, des changements radicaux dans les comportements observables, qui peuvent faire penser qu'un changement conceptuel a eu lieu, peuvent en réalité être trompeurs. Thagard (1992) parle également d'un spectre dans le degré de changement allant d'un détail de surface au cœur du concept. Concernant l'apprentissage des nombres décimaux, Vosniadou et al. (2008) (voir le tableau présenté à la page 42) proposent plusieurs aspects différenciant le concept de nombre naturel de celui

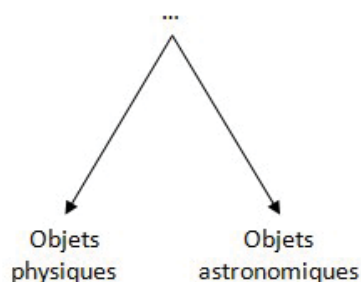
de nombre rationnel. La propriété de densité notamment marque une différence conceptuelle essentielle entre les nombres naturels et les nombres rationnels. Par contre, selon nous, d'autres aspects ne sont pas réellement liés au concept de nombre. C'est le cas de la compréhension du système décimal de position qui est, avant tout, un système de représentation. Les aspects conceptuels pourraient même tout à fait se différencier de ceux liés à la compréhension du système de représentation, ceux-ci pouvant davantage être procéduraux. Il est par exemple tout à fait plausible de rencontrer un élève capable, grâce à sa compréhension du système décimal, d'intercaler un nombre entre 1,34 et 1,35, mais ne comprenant pas réellement la propriété de densité des nombres décimaux. Au cours de l'apprentissage des nombres décimaux, les connaissances procédurales et les connaissances conceptuelles pourraient se différencier, même si leurs développements seraient bien entendu étroitement liés (Rittle-Johnson et al., 2001 ; Rittle-Johnson & Koedinger, 2009 ; Schneider et al., 2011). Cette distinction entre connaissances conceptuelles, connaissances procédurales et les connaissances liées au système symbolique devrait être prise en compte dans l'analyse des changements conceptuels.

D'autre part, concernant les théories du changement conceptuel, un débat existe toujours entre certains auteurs, influencés par la théorie piagétienne de l'apprentissage, et supposant que les connaissances initiales sont organisées de façon structurée, sous la forme de *théories* (Carey, 2009 ; Chi, 2008 ; Vosniadou et al., 2008) et d'autres auteurs défendant l'idée selon laquelle les connaissances initiales se composent d'éléments quasiment indépendants (diSessa, 2008 ; Clark, 2006 ; Harrison et al., 1999). Les connaissances initiales, organisées sous forme de *théories*, auraient une ontologie et une causalité permettant de formuler des explications et des prédictions face à un problème non familier. Les connaissances sous formes d'éléments relativement indépendants seraient, quant à elles, appliquées de façon consistante, mais hautement sensibles au contexte. Selon ce point de vue, des idées contradictoires pourraient aussi coexister à un moment donné du développement. Nos données ne permettent pas de trancher définitivement entre ces deux points de vue, mais pourraient alimenter quelque peu le débat.

Dans les diverses tâches que nous avons proposées aux élèves et dans les entretiens réalisés auprès de ceux-ci, nous avons pu constater une certaine cohérence dans les connaissances des élèves. Prenons un entretien en exemple. Un élève de 3^e année primaire, qui n'a pas encore reçu d'enseignement des nombres décimaux, choisit erronément 0,60 comme étant plus grand que 0,7. Il précise ensuite qu'il choisit 0,60 parce que « *dans 0,60, il y a un chiffre en plus* ». Lorsque qu'il rencontre plus tard les nombres 0,50 vs. 0,2, il choisit, à raison, 0,50 comme étant plus grand en précisant que « *dans 0,50, il y a plus de chiffres* ». Il semble jusqu'ici assez cohérent. Il choisit alors 0,05 comme étant plus grand que 0,2, parce que « *cinq est plus grand que deux, cinq vient après deux* » et 0,8 comme étant égal à 0,008, parce que ce sont « *deux chiffres les mêmes, sauf qu'il y a un peu plus de 0 au deuxième nombre* ». En surface, ses justifications peuvent paraître incohérentes, puisqu'il fait tantôt appel au nombre de chiffres, tantôt à la valeur des chiffres. Néanmoins, ces choix peuvent tous s'expliquer par le fait que cet

élève accorde de l'importance à un zéro placé à droite d'un chiffre, mais pas à gauche. Les choix de cet élève paraissent alors consistants et liés à des conceptions cohérentes, même si ses capacités méta-cognitives ne lui permettent pas de relater explicitement ses conceptions. Nous pouvons aussi considérer que ses connaissances initiales lui permettent de faire des prédictions face à une nouvelle situation non familière et de formuler des explications. Les nombres décimaux sont ici assimilés aux nombres naturels, et l'ensemble des connaissances liées aux nombres naturels leur sont appliquées.

Toujours selon ce point de vue considérant les connaissances initiales organisées sous la forme de *théories*, certains considèrent le processus du changement conceptuel comme un changement dans la catégorisation *ontologique* des connaissances (Chi, 1992, 2008). Dans ce cadre, nous pouvons noter que les apprentissages mathématiques se différencient certainement des apprentissages scientifiques.



Dans l'apprentissage des sciences, on imagine aisément des catégories ontologiques latérales distinctes et mutuellement exclusives, par exemple celles des objets physiques et des objets astronomiques (voir ci-contre). Au cours de l'apprentissage, le concept de Terre est attribué au départ erronément aux objets physiques au lieu des objets astronomiques (Vosniadou et al., 2008), dans ce cas, le changement conceptuel correspond à un *saut de branche* (Thagard, 1988), un déplacement latéral (Chi, 2008).

Dans l'apprentissage des mathématiques, et dans le domaine des nombres plus particulièrement, on imagine difficilement des catégories latérales mutuellement exclusives, puisque les nombres naturels, décimaux, rationnels et réels, ne sont non pas mutuellement exclusifs, mais bien de généralité croissante (Rouche, 2010), les nombres naturels étant, par exemple, des cas particuliers de nombres décimaux. L'arbre ontologique serait alors vertical (voir ci-contre) et le changement conceptuel consisterait non pas à un saut de branche, mais à la création d'une catégorie ontologique plus générale et d'un ordre supérieur.



Les données que nous venons de mettre en évidence semblent donc indiquer que l'on peut considérer les connaissances initiales dans le domaine des nombres comme un ensemble relativement cohérent de connaissances permettant de faire des prédictions consistantes, ce qui serait en accord avec la vision des connaissances initiales sous la forme de *théories* (Carey, 2009 ; Chi, 2008 ; Vosniadou et al., 2008). Une autre partie de nos données indique pourtant une sensibilité au contexte, ce qui pourrait être en accord avec la vision opposée, considérant les connaissances initiales comme un ensemble d'élé-

ments relativement indépendants (diSessa, 2008 ; Clark, 2006 ; Harrison et al., 1999). Les nombres décimaux sont en effet considérés différemment par les élèves lorsqu'ils sont placés dans le contexte de la monnaie, c'est-à-dire de l'euro (voir l'étude présentée au chapitre 6). L'euro, par sa sous-unité particulièrement saillante, le cent, accentue l'idée que la virgule sépare deux nombres naturels, ici un nombre d'euro et un nombre de cent. Lors d'une activité en classe, des élèves de 4^e année primaire indiquent que 7,80 est égal à 7,8 mais que 7,80€ n'est pas égal à 7,8€. Ils précisent même que « *dans les virgules sans les euro, c'est la même chose, mais dans les virgules avec les euro, ce n'est pas pareil* ». Ce qui est compatible avec l'hypothèse selon laquelle, pour les nombres d'euro, ce qui se trouve dans la partie décimale est systématiquement un nombre (entier) de cent. On pourrait donc penser que leurs connaissances sont fortement influencées par le contexte. C'est certainement le cas. Néanmoins, dans le cas des nombres décimaux, ce constat pourrait ne pas avoir d'impact concernant la structure des connaissances initiales. Nous pourrions toujours considérer les connaissances initiales comme étant un ensemble cohérent, construit à partir des nombres naturels. La différence se marquerait en fonction du contexte d'apprentissage, ou du contexte d'application du nouveau savoir, qui rendrait cet ensemble de connaissances initiales plus ou moins obsolète. Comme nous le mentionnions plus haut, le débat entre des connaissances initiales sous forme de *théories* ou sous forme d'éléments relativement indépendants n'est pas totalement clos, mais les données concernant les nombres décimaux tendent à appuyer l'hypothèse d'un ensemble de connaissances initiales organisé sous forme d'une *théorie* construite à partir des nombres naturels plutôt que l'hypothèse d'un ensemble de connaissances indépendantes.

Des connaissances hiérarchiques ou nécessitant un changement conceptuel ?

Initialement le changement conceptuel tel que présenté par Kuhn (1962) supposait une incommensurabilité entre l'ancienne théorie et la nouvelle théorie la remplaçant. On retrouve partiellement cette idée dans les déplacements latéraux entre les branches d'arbres ontologiques représentant des catégories mutuellement exclusives (Thagard, 1988 ; Chi, 2008) (voir la première figure page 162). En mathématiques, et lors de l'apprentissage des nombres en particulier, les choses sont différentes. D'une part, les différents types de nombres ne sont pas des concepts mutuellement exclusifs, mais plutôt un même concept devenant de plus en plus général (voir la seconde figure page 162). On peut dès lors considérer que les connaissances sont hiérarchiques, comme c'est souvent le cas en mathématiques (Aunola et al., 2004 ; Hecht & Vagi, 2010 ; Jordan et al., 2009 ; Moeller et al., 2011). Les connaissances initiales sont avant tout une base pour les apprentissages ultérieurs. Il semble en effet impossible de comprendre les nombres décimaux sans avoir compris les nombres naturels. Les résultats de l'étude longitudinale, présentée au chapitre 5, confirment en effet qu'il est indispensable de maîtriser le système décimal de position avec les nombres naturels pour acquérir les nombres décimaux. Toutes les connaissances initiales construites à partir des nombres naturels ne sont donc pas en contradiction avec les connaissances à acquérir concernant les nombres décimaux. Le

changement conceptuel à réaliser n'est alors que partiel. Nous pourrions même dire que si les principes du système décimal de position sont compris en profondeur et ne sont pas limités à des connaissances *de surface*, si l'on est conscient par exemple que le zéro marque un rang inoccupé et qu'on ne se limite pas à l'idée que *le zéro à droite rend dix fois plus grand*, le seul changement conceptuel réel lors du passage des nombres naturels aux nombres décimaux concernerait la propriété de densité.

Perspectives... Dans de futures études concernant le changement conceptuel, et en particulier son application à l'apprentissage des nombres décimaux, les liens entre la compréhension du système de représentation qu'est le système décimal d'une part, et d'autre part les connaissances procédurales et conceptuelles relatives aux nombres décimaux pourraient être étudiés. L'étude longitudinale présentée au chapitre 5 nous a montré que les connaissances du système décimal de position pour les nombres naturels avaient un plus grand pouvoir de prédiction sur l'apprentissage des nombres décimaux que les connaissances liées aux nombres rationnels exprimés sous forme de fractions. Ces premiers résultats, comme nous l'avons déjà suggéré plus haut, pourraient être approfondis. Dans une nouvelle étude, une mesure des connaissances conceptuelles à propos des nombres rationnels (telles que le fractionnement en parts égales, la densité) la plus indépendante possible de la maîtrise des symboles pourrait être contrastée avec la compréhension des systèmes symboliques permettant de représenter les nombres rationnels (les fractions et le système décimal) et avec les connaissances procédurales. Les liens développementaux entre ces différentes variables pourraient également être étudiés.

Pour conclure à propos du changement conceptuel, nous pouvons retenir que l'apprentissage des nombres décimaux est à la fois un cas de changement conceptuel et un apprentissage de type hiérarchique. La compréhension du système symbolique qu'est le système décimal de position joue un rôle majeur tant dans la compréhension des nombres décimaux que dans les profils d'erreurs qui peuvent être observés lors du traitement de ces nombres. Il serait donc intéressant d'approfondir la distinction entre les réels aspects conceptuels qui évoluent entre les nombres naturels et les nombres décimaux, des aspects procéduraux et davantage liés à la maîtrise des symboles.

Un unique cadre théorique permettant de rendre compte des difficultés d'apprentissage des nombres décimaux ? Si la théorie du changement conceptuel permet d'appréhender l'acquisition des nombres décimaux plutôt comme un processus développemental long et graduel, ayant les connaissances des nombres naturels comme point de départ, et pourrait même laisser penser qu'une fois les connaissances expertes acquises, les connaissances initiales ne sont plus une source de difficultés, d'autres cadres théoriques sont plus à même de rendre compte des processus cognitifs en jeu à chaque moment du développement et également lorsque les connaissances expertes semblent acquises.

Dans la partie de discussion consacrée au traitement cognitif des nombres décimaux (voir page 154) nous avons mentionné que même chez l'adulte, lorsque les nombres décimaux sont maîtrisés, certains facteurs peuvent créer une interférence dans le traitement des nombres décimaux. Des capacités d'inhibition peuvent en effet être nécessaires pour indiquer, par exemple, que 0,1 est plus grand que 0,02 alors que les magnitudes de 1 et 2 sont automatiquement activées et incongruentes avec la magnitude globale des deux nombres. Dans un même ordre d'idées, le traitement et l'apprentissage des nombres décimaux pourraient être envisagés selon la théorie du *double traitement* [dual process theory] (Vamvakoussi, Wim, & Verschaffel, 2012 ; Gillard, Van Dooren, Schaeken, & Verschaffel, 2009a, 2009b). Selon cette théorie, nous disposons d'un système de traitement intuitif/heuristique et d'un système de traitement analytique. Le système intuitif est rapide, automatique, associatif et peu coûteux en capacité de mémoire de travail, alors que le système analytique est lent, contrôlé, conscient et coûteux. Le système intuitif peut conduire à des réponses correctes, mais parfois ce n'est pas le cas et le système analytique intervient pour outrepasser la réponse initiale fournie par défaut par le système intuitif. Empiriquement, les temps de réaction sont plus longs quand le système analytique intervient, et les capacités de mémoire de travail sont davantage mobilisées. La gestion du conflit entre la réponse par défaut fournie par le système intuitif serait essentiellement assurée par une inhibition de cette réponse (Neys & Glumicic, 2008).

Dans le cas des nombres décimaux, on peut supposer que le système intuitif met en œuvre les connaissances liées aux nombres naturels, ce qui parfois conduit à des réponses correctes, mais d'autres fois le système analytique doit intervenir pour dépasser les réponses fournies par le système intuitif. Nous avons largement décrit des résultats mettant en évidence des différences de taux de réponses correctes entre des conditions congruentes et incongruentes. La théorie du *double traitement* pourrait éclairer ces résultats de différentes façons. D'une part, des mesures chronométriques pourraient être utilisées pour distinguer les réponses issues d'un système intuitif de celles issues d'un système analytique, plus coûteux en temps de réaction et en mémoire de travail. Le développement de ces deux systèmes et de leur interaction pourrait également être observé. Il est probable que les élèves en tout début d'apprentissage, c'est-à-dire en 3^e ou début de 4^e année primaire, ne disposent pas encore des connaissances permettant de mobiliser le système analytique. Il est probable également que les élèves plus âgés, de 5^e et 6^e années primaires, utilisent un système analytique pour obtenir des réponses correctes dans les conditions incongruentes, même lorsque celles-ci ne se différencient plus des conditions congruentes en termes de taux de réponses correctes. Un autre élément à l'appui de l'hypothèse selon laquelle les élèves plus âgés obtiennent des réponses correctes dans les conditions incongruentes grâce au système analytique, est que lorsqu'il faut comparer cinq nombres au lieu de trois, ce qui augmente la charge en mémoire de travail, les réponses intuitives liées aux nombres naturels réapparaissent chez des élèves disposant pourtant des connaissances expertes (Sackur-Grisvard & Leonard, 1985).

D'autre part, les effets de congruence observés à partir des taux de réponses cor-

rectes pourraient être analysés à partir de la théorie du *double traitement*. Nous pensons notamment à l'effet de longueur qui s'inverse entre la 4^e et la 5^e année primaire. Les taux de réponses correctes indiquent qu'en 5^e année primaire, il devient plus facile de choisir le nombre le plus court comme étant le plus grand alors que l'inverse était observé auparavant. Une tâche imposant une contrainte de temps, ou un design en double tâche mobilisant les ressources de la mémoire de travail, permettraient de savoir si le choix du nombre le plus court observé chez les élèves plus âgés est lié ou non au système analytique qui se baserait sur des connaissances des fractions et outrepasserait les connaissances intuitives basées sur les nombres naturels. Le fait que ce type de réponse n'apparaît pas chez les élèves plus jeunes semble appuyer cette hypothèse.

L'utilisation de la théorie du *double traitement* permettrait donc un regard plus précis d'un point de vue cognitif sur le développement du traitement des nombres décimaux que la théorie du changement conceptuel. Ces deux champs théoriques ne nous semblent cependant pas incompatibles.

Perspectives didactiques

S'intéresser à la didactique signifie s'intéresser à une interface efficace qu'un enseignant pourrait mettre en place entre un savoir à acquérir et un apprenant, de façon à ce que ce dernier apprenne au mieux. Dans le cas qui nous préoccupe, l'enseignement des nombres décimaux, cela nécessite donc une maîtrise du savoir mathématique propre aux nombres décimaux, une connaissance de l'apprentissage de ces nombres par les élèves, le tout étant orchestré par l'art d'enseigner. Dans plusieurs de nos études, nous avons souhaité être au plus proche des besoins des enseignants.

Par exemple, parmi les enseignants que nous avons eu l'occasion de rencontrer, nombreux étaient ceux qui étaient convaincus que l'apparition de l'euro dans notre vie quotidienne aide les élèves à apprendre les nombres décimaux. L'étude au sujet de l'impact de l'euro, présentée au chapitre 6, avait comme objectif de pouvoir de mettre cette intuition à l'épreuve des faits. Les résultats ont montré que l'impact de l'euro sur l'apprentissage était nuancé. Si l'euro aide par exemple les élèves à concevoir qu'il peut y avoir des nombres entre les naturels, comme c'est le cas entre 1€ et 2€, il ne les aide pas à imaginer qu'il existe une infinité de nombres entre 1,44 et 1,45 puisqu'il n'y a *rien* entre 1,44€ et 1,45€, ou en tout cas, rien dont nous tenons compte dans la vie quotidienne. Ce type d'étude, somme toute assez simple, s'avère particulièrement utile et devrait, selon nous, fournir une base tant pour les recherches en didactique des mathématiques, que pour les choix didactiques des enseignants. Trop souvent en didactique, des intuitions sont prises pour des vérités.

Une autre intuition est celle qui concerne la compréhension des fractions. Si nous n'excluons pas que certains aspects conceptuels propres aux nombres rationnels, tels que la relation partie/tout ou le fractionnement en parts égales, devraient être abordés avant

l'apprentissage des nombres décimaux, nous sommes nettement plus mitigée quant à l'ancrage de l'enseignement des nombres décimaux sur la compréhension des fractions. Nos résultats montrent en effet que, d'une part, la représentation des nombres rationnels sous forme de fractions semble encore plus difficile à maîtriser que celle sous forme de nombres décimaux et que, d'autre part, la compréhension des fractions n'est pas le meilleur prédicteur de l'apprentissage des nombres décimaux. Selon nous, le fait de choisir, parmi deux nombres décimaux, le nombre le plus court comme étant le plus grand n'indique pas uniquement que les connaissances des fractions sont impliquées dans l'apprentissage des nombres décimaux (Resnick et al., 1989), mais aussi qu'une compréhension incomplète des fractions peut donner lieu à des erreurs dans le traitement des nombres décimaux. La compréhension des fractions est tellement délicate pour les élèves, qu'il semble difficile, en 4^e année primaire, de construire un nouvel apprentissage uniquement à partir de celle-ci.

Il ne suffit cependant pas de mettre le doigt sur les écueils didactiques, il est aussi important, si pas plus important, de proposer des pistes didactiques pertinentes.

Quelle serait une didactique pertinente pour l'enseignement des nombres décimaux ? Certaines de nos études avancent des éléments de réponses. Les résultats de l'étude longitudinale présentée au chapitre 5 montrent à quel point la compréhension du système décimal de position est essentielle à la compréhension des nombres décimaux. Il serait donc logique d'inclure un approfondissement du système décimal dans l'enseignement des nombres décimaux. Considérer la compréhension du système décimal comme étant acquise grâce aux nombres naturels est délicat étant donné que certains *traits de surface* diffèrent lorsque le système décimal est utilisé pour représenter les nombres décimaux. Or, les élèves se basent fréquemment sur des traits de surface. Pour effectuer une addition par exemple, il est fréquent qu'ils expliquent qu'il faut additionner les chiffres à partir de la droite sans comprendre qu'il s'agit d'additionner des puissances de dix identiques. Il serait donc pertinent de les guider vers une compréhension plus en profondeur du système décimal et de travailler explicitement les différences entre la représentation des nombres naturels et celle des nombres décimaux.

Nous avons expérimenté une telle séquence didactique dans une étude présentée au chapitre 7. Dans cette séquence, outre une attention toute particulière aux spécificités de l'apprentissage des nombres décimaux, à savoir l'extension du système décimal, nous avons aussi été attentive aux caractéristiques mathématiques des nombres décimaux, ou plus exactement à leur place parmi les autres nombres. Les nombres décimaux ont donc été introduits pour représenter, dans le système décimal, un nombre réel de façon approchée. Ce choix a aussi la particularité de ne pas limiter le nombre de chiffres après la virgule. Les résultats ont montré que les élèves ayant suivi cette séquence didactique ont de meilleures connaissances conceptuelles des nombres décimaux, notamment de la densité de ces nombres.

Perspectives... Comme le souligne Greer (2004), tenir compte des changements conceptuels que devront accomplir les élèves implique l'utilisation de mécanismes spécifiques pour favoriser le changement conceptuel, mais aussi un curriculum réfléchi dans la continuité. Ce principe pourrait être appliqué à l'enseignement des nombres décimaux. Un des concepts majeurs lié aux nombres rationnels, et donc aux nombres décimaux, est la propriété de densité. Il nous semble difficilement concevable que les élèves puissent comprendre la densité lorsque les nombres décimaux qui leur sont proposés ne contiennent qu'un petit nombre de chiffres après la virgule. Par exemple, travailler avec des nombres qui contiennent uniquement un ou deux chiffres après la virgule, ce qui est le cas en 4^e et 5^e années primaires, renforce l'idée que les nombres décimaux sont discrets et que, par exemple, il n'y a aucun nombre entre 1,26 et 1,27. De plus, avec de tels nombres les élèves peuvent développer des connaissances procédurales inappropriées. Il n'est pas rare, notamment en 5^e année primaire, que des additions telles que $0,2 + 0,15$ soient proposées et qu'il soit suggéré aux élèves d'ajouter un zéro à 0,2 de sorte à opérer sur des nombres de longueur identique. Nous avons eu l'occasion de rencontrer des élèves qui pensent alors que l'on peut ajouter un zéro aux nombres décimaux, mais pas plus d'un, ce qui les rend démunis face à une addition telle que $0,2 + 0,153$. Dans la même idée, puisque nous savons que tant la valeur des chiffres que la longueur des nombres peuvent avoir un impact sur le traitement des nombres décimaux, il faudrait veiller à ce que des nombres faisant apparaître tantôt la congruence, la neutralité et l'incongruence de ces facteurs soient utilisés. Les occasions de créer des obstacles didactiques (Brousseau, 1998) étant nombreuses, il faudrait, tant que faire se peut, les éviter. Cela implique évidemment un énorme savoir tant mathématique que didactique dans le chef des enseignants et cela n'est pas toujours chose aisée ...

Pour conclure à propos de l'enseignement des nombres décimaux, il apparaît qu'il faudrait tenir compte à la fois des caractéristiques que devrait avoir un enseignement favorisant le changement conceptuel, d'une analyse mathématique de ce que sont les nombres décimaux et des particularités cognitives de cet apprentissage pour les élèves. Un vaste programme ! D'autant qu'il faudrait également éviter les obstacles didactiques. Une piste prometteuse semble être un enseignement essentiellement basé sur une compréhension approfondie du système décimal et qui accompagne les élèves des concepts qu'ils maîtrisent vers les nuances apportées par les nombres décimaux ...

Conclusions

Mieux comprendre l'apprentissage des nombres décimaux et ses difficultés, et en tirer certaines conclusions pour l'enseignement de ces nombres étaient les objectifs principaux de cette thèse. Nous avons choisi de nous situer à l'interface des sciences cognitives, des théories du changement conceptuel et de la didactique des mathématiques. De ce travail, nous retenons tout particulièrement l'intérêt d'un tel angle pluridisciplinaire. D'une part,

il apparaît que les choix en didactique des mathématiques ne peuvent être que le résultat d'un travail pluridisciplinaire. S'il appartient au mathématicien de savoir quels sont les concepts essentiels à enseigner et quels sont les éventuels obstacles épistémologiques dans la construction de ces concepts, il revient au psychologue d'analyser la façon dont les élèves abordent la construction de ces concepts, quels sont les obstacles qu'ils rencontrent effectivement et quels sont les processus cognitifs. Ensuite, différents choix didactiques devraient être mis à l'épreuve des faits pour tenter de trouver celui qui répond le mieux aux exigences tant des mathématiques que de la psychologie de l'apprentissage.

Nous pourrions cependant noter un travers propre à ce type d'approche. À vouloir combiner les disciplines, il devient en réalité difficile d'approfondir chacune d'entre elles. Chacune des questions que nous avons abordée dans cette thèse pourrait donc être encore approfondie. Nous restons néanmoins convaincue que pour fournir aux enseignants des pistes didactiques qui tiennent compte de leur réalité de terrain, cette articulation entre sciences cognitives, théorie de l'apprentissage et didactique des mathématiques est une voie prometteuse ...

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