

A New Basis for QED Bound State Computations

J. Pestieau and C. Smith

*Institut de Physique Théorique, Université catholique de Louvain,
Chemin du Cyclotron, 2, B-1348, Louvain-la-Neuve*

November 28, 2001

Abstract

A simple method to compute QED bound state properties is presented, in which binding energy effects are treated non-perturbatively. It is shown that to take the effects of all ladder Coulomb photon exchanges into account, one can simply perform the derivative of standard QED amplitudes with respect to the external momentum. For example, the derivative of the light-by-light scattering amplitude gives an amplitude for orthopositronium decay to three photons where any number of Coulomb photon exchanges between the e^+e^- is included.

Various applications are presented. From them, it is shown that binding energy must be treated non-perturbatively in order to preserve the analyticity of positronium decay amplitudes.

Interesting perspectives for quarkonium physics are briefly sketched.

1 Introduction

The properties of positronium, the bound state made of an electron and a positron, provide some of the most precise tests of QED [1]. Both the experimental [2] and theoretical considerations have reach a very high level of precision, requiring for the latter the computations of many-loop diagrams ([3]-[6], and references cited there).

In the present note, we will address one particular aspect of the current QED bound state models, namely the factorization between the bound state dynamics and the decay processes [7]. As explained in [8], the present study is motivated by the recurrent contradiction between standard positronium models and Low's theorem [9]. Our goal is to show that binding energy effects cannot be treated perturbatively if sensible theoretical results are sought at the present level of precision. The central result of the present note is a simple method that allows such an exact non-perturbative treatment of binding energy effects. The lowest order results we shall obtain sum infinite class of diagrams, usually expanded perturbatively. Also, our method is formally relativistic and has a correct analytical behavior.

In section 2, our simple method is introduced by a computation of the rate for parapositronium to two photons, and then demonstrated in general. In the next sections, we apply our result to various decay modes chosen to illustrate some specific aspects.

First, to show explicitly the contradiction between Low's theorem and binding energy perturbative expansions, we consider the paradimuonium ($p\text{-}Dm$, a bound $\mu^+\mu^-$ [10]) decay to γe^+e^- . Then, interesting connections between bound state decay amplitudes and the photon vacuum polarization are exemplified by the orthodimuonium decay $o\text{-}Dm \rightarrow e^+e^-$.

The fourth section is devoted to the interesting decay mode orthopositronium to three photons. We obtain the differential spectrum and decay rate for any value of the binding energy, from the amplitude for light-by-light scattering. We discover that both the spectrum and rate are highly sensitive to the binding energy when it is approaching zero. The implication of this result is the well-known slow convergence of the orthopositronium radiative correction perturbative series.

In the fifth section, we present two extensions of our method. The application to orthopositronium n th radial excitations $o\text{-}Ps(nS)$ is introduced by recomputing the decay rate $\pi^0 \rightarrow \gamma o\text{-}Ps(nS)$ [11]. Finally, it is shown how the positronium hyperfine splitting, i.e. the mass difference of the ortho- and parapositronium, can be obtained from vacuum polarization graphs through finite mass renormalizations.

2 Lowest Order Decay Amplitudes

Positronium amplitudes are built as loop amplitudes: the positronium couples to a virtual e^+e^- loop, to which a given number of photons are attached. The coupling of the positronium to its constituents is essentially determined by the positronium wavefunction. In the following, we will take a form factor inspired from the Schrödinger wavefunction, i.e. a form factor containing the effects of the exchange of Coulomb photons among the constituents (in the ladder approximation) [12].

A very simple method can be used to compute such loop amplitudes: *any lowest order loop amplitude with a Coulomb form factor is the derivative with respect to the positronium mass of the corresponding point-like amplitude*. By point-like amplitude is meant the loop amplitude obtained by replacing the complicated Coulomb form factor for the bound state by a constant from factor. Symbolically, for parapositronium (orthopositronium) decay to an even (odd) number of real or virtual photons, the amplitude is

$$\begin{array}{c} \text{p-Ps} \\ \text{(o-Ps)} \end{array} \begin{array}{c} \text{e}^- \\ \text{e}^+ \end{array} \begin{array}{c} \text{---} \phi \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \propto \frac{\partial}{\partial P^2} \left[\begin{array}{c} \text{p-Ps} \\ \text{(o-Ps)} \end{array} \begin{array}{c} \text{e}^- \\ \text{e}^+ \end{array} \begin{array}{c} \text{---} \gamma^5 (\gamma^\mu) \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \right]$$

where ϕ represents the Schrödinger wavefunction, i.e.

$$\begin{array}{c} \text{p-Ps} \\ \text{(o-Ps)} \end{array} \begin{array}{c} \text{e}^- \\ \text{e}^+ \end{array} \begin{array}{c} \text{---} \phi \text{---} \end{array} \begin{array}{c} \text{---} \end{array} = \begin{array}{c} \text{p-Ps} \\ \text{(o-Ps)} \end{array} \begin{array}{c} \text{e}^- \\ \text{e}^+ \end{array} \begin{array}{c} \text{---} \gamma^5 (\gamma^\mu) \text{---} \end{array} \begin{array}{c} \text{---} \end{array}$$

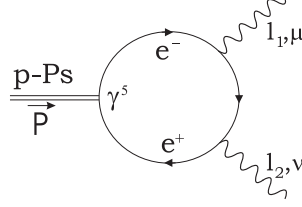
and dashed photon lines stand for Coulomb photons.

In the next subsection, we give an example to illustrate explicitly how the method works, and in the following one, we present the general demonstration.

2.1 An Example: Parapositronium to two photons

The point-like amplitude for the decay $p\text{-Ps} \rightarrow \gamma\gamma$ is obtained by replacing the Coulomb form factor by an elementary, point particle, with a pseu-

doscalar γ_5 coupling to the electron current (with unit coupling constant)



plus the crossed process. The point bound state amplitude is fully relativistic, and standard loop integration techniques can be used. Explicitly, the point-like amplitude is (the subscript γ_5 is a reminder for point-like):

$$\begin{aligned}\mathcal{M}_{\gamma_5}(p\text{-}Ps \rightarrow \gamma\gamma) &= ie^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr} \{ \Gamma^{\mu\nu} \} \varepsilon_\mu(l_1) \varepsilon_\nu(l_2) \\ \Gamma^{\mu\nu} &= \gamma_5 \frac{1}{\not{q} + \not{l}_1 - m} \gamma_\mu \frac{1}{\not{q} - m} \gamma_\nu \frac{1}{\not{q} - \not{l}_2 - m} \\ &\quad + \gamma_5 \frac{1}{\not{q} + \not{l}_2 - m} \gamma_\nu \frac{1}{\not{q} - m} \gamma_\mu \frac{1}{\not{q} - \not{l}_1 - m}\end{aligned}$$

Carrying the trace, we readily obtain

$$\mathcal{M}_{\gamma_5}(p\text{-}Ps \rightarrow \gamma\gamma) = -8me^2 \varepsilon^{\mu\nu\rho\sigma} l_{1,\rho} l_{2,\sigma} \varepsilon_\mu(l_1) \varepsilon_\nu(l_2) \frac{\mathcal{I}_{\gamma_5}(M^2)}{M^2}$$

where the **dimensionless** loop integral form factor is

$$\mathcal{I}_{\gamma_5}(P^2) = \frac{-i}{(4\pi)^2} F\left[\frac{4m^2}{P^2}\right] \quad \text{with } F[a] = 2 \arctan^2(a-1)^{-1/2} \quad (1)$$

From the amplitude, the decay width is

$$\Gamma_{\gamma_5}(p\text{-}Ps \rightarrow \gamma\gamma) = 16\pi\alpha^2 \frac{m^2}{M} |\mathcal{I}_{\gamma_5}(M^2)|^2 = \frac{\pi M\alpha^2}{256} \left(\frac{4\gamma^2}{M^2} + 1 \right) \left(\frac{4}{\pi^2} \arctan^2 \frac{M}{2\gamma} \right)^2 \quad (2)$$

where $\gamma^2 = m^2 - M^2/4$ is related to the binding energy $E_B = M - 2m$.

To get the physical positronium decay amplitude and rate, simply replace in (2) the loop form factor $\mathcal{I}_{\gamma_5}(M^2)$ by its derivative

$$\boxed{\mathcal{I}_{Coul}(M^2) = (32\pi C\phi_o\gamma) \frac{\partial}{\partial M^2} \mathcal{I}_{\gamma_5}(M^2)} \quad (3)$$

where ϕ_o is the S -wave fundamental state Schrödinger wavefunction at zero separation, and $C = \sqrt{M}/m$ is obtained by matching the static limit ($\gamma \rightarrow 0$) with the well-known lowest order result $\Gamma_{p-Ps} = m\alpha^5/2$ [7]. This gives

$$\mathcal{I}_{Coul}(M^2) = -i \frac{C\phi_o}{M} \left[\frac{2}{\pi} \arctan \frac{M}{2\gamma} \right]$$

The factor in square brackets is equal to 1 in the limit $\gamma \rightarrow 0$. Using $|\phi_o|^2 = \alpha^3 m^3 / 8\pi$, the decay rate into two-photon is

$$\Gamma(p-Ps \rightarrow \gamma\gamma) = \frac{\alpha^5 m}{2} \left(\frac{4m^2}{M^2} \right) \left| \frac{2}{\pi} \arctan \frac{M}{2\gamma} \right|^2 \quad (4)$$

The result (4) contains the effects of Coulomb interactions among the constituents, at all orders in α . Indeed, using $\gamma^2 = m^2 - M^2/4 \approx m^2 \alpha^2/4$, the form factor can be expanded as (see the discussion in [7])

$$\Gamma(p-Ps \rightarrow \gamma\gamma) = \frac{\alpha^5 m}{2} \left(1 - 2\frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) \right) \approx \frac{\alpha^5 m}{2} (1 - 0.637\alpha + \mathcal{O}(\alpha^2))$$

In other words, the binding energy effects included in our lowest order computation already account for a great deal of the relativistic and radiative corrections as presented in the literature [3]:

$$\Gamma_{p-Ps} = \frac{\alpha^5 m}{2} (1 - \delta\Gamma_{p-Ps}) \quad (5)$$

where

$$\delta\Gamma_{p-Ps} = A_p \frac{\alpha}{\pi} + 2\alpha^2 \ln \frac{1}{\alpha} + B_p \frac{\alpha^2}{\pi^2} - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + C_p \frac{\alpha^3}{\pi} \ln \frac{1}{\alpha} + \delta_{4\gamma} \frac{\alpha^2}{\pi^2} \quad (6)$$

with $\begin{array}{ll} A_p = 5 - \pi^2/4 \approx 2.5326 & C_p = -7.919(1) \\ B_p = 5.14(30) & \delta_{4\gamma} = 0.274(1) \end{array}$

Numerically, we can write (5) as

$$\Gamma_{p-Ps} \approx \frac{\alpha^5 m}{2} \frac{4m^2}{M^2} \left(\frac{2}{\pi} \arctan \frac{M}{2\gamma} \right)^2 (1 - \delta\Gamma'_{p-Ps})$$

with the same series 6, but with the reduced coefficients

$$\begin{aligned} A'_p &\approx 0.5326 \\ B'_p &\approx 0.607 \\ C'_p &\approx -3.919 \end{aligned}$$

This last form is very interesting because binding energy corrections (i.e. γ -dependent) are singled out, while the remaining radiative corrections are much reduced. This means that one could, at least in principle, express the decay rate as non-perturbative binding energy corrections times a rapidly converging perturbation series of radiative corrections. This is exactly what we have achieved to order α .

2.2 Generalization

The proof of (3) is straightforward using the language of dispersion relations. Indeed, the point-like loop amplitude can be computed from its imaginary part

$$\text{Im } \mathcal{T}_{\gamma_5}(P^2) \equiv \text{Im } \mathcal{M}_{\gamma_5}(p-Ps(P^2) \rightarrow \gamma\gamma)$$

using an unsubtracted dispersion relation with $s = P^2$:

$$\mathcal{T}_{\gamma_5}(M^2) = \text{Re } \mathcal{T}_{\gamma_5}(M^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - M^2} \text{Im } \mathcal{T}_{\gamma_5}(s) \quad (7)$$

($\mathcal{T}(M^2) = \text{Re } \mathcal{T}(M^2)$ because $M < 2m$). The Schrödinger form factor accounting for the non-trivial coupling of the bound state to its constituent is of the form

$$F_B(0, \mathbf{q}) = C\phi_o \mathcal{F}(\mathbf{q}^2) (\mathbf{q}^2 + \gamma^2) \quad \text{with} \quad \mathcal{F}(\mathbf{q}^2) = \frac{8\pi\gamma}{(\mathbf{q}^2 + \gamma^2)^2}$$

where one can recognize $\phi(\mathbf{q}^2) \equiv \phi_o \mathcal{F}(\mathbf{q}^2)$ as the fundamental ($n=1$) S -wave Schrödinger momentum space wavefunction. When expressed in terms of the dispersion relation variable, this form factor is only a function of s , the initial energy:

$$F_B(s) = C\phi_o \frac{32\pi\gamma}{s - M^2}$$

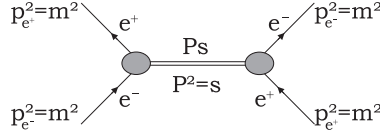
The core of the derivative approach emerges from the observation that inserting F_B in (7) is equivalent to taking the derivative with respect to M^2

$$\begin{aligned} \mathcal{T}_{Coul}(M^2) &= \frac{1}{\pi} \int_{4m^2}^{+\infty} \frac{ds}{s - M^2} F_B(s) \text{Im } \mathcal{T}_{\gamma_5} \\ &= (32\pi C\phi_o\gamma) \frac{1}{\pi} \int_{4m^2}^{+\infty} \frac{ds}{(s - M^2)^2} \text{Im } \mathcal{T}_{\gamma_5} \\ &= (32\pi C\phi_o\gamma) \frac{\partial}{\partial M^2} \mathcal{T}_{\gamma_5}(M^2) \end{aligned} \quad (8)$$

which is the desired result. The case of other parapositronium decay channels, or orthopositronium decay modes is similarly treated (simply replace γ^5 by $\not{e}$ with e^μ the orthopositronium polarization vector).

As a first remark, we repeat here the conclusion of [7], which is that standard positronium result are recovered from the above dispersion integral (7) provided only the vertical cuts are taken into account in the imaginary part. This proves that some contributions are missed in those computations. For instance, in the case of $p\text{-}Ps \rightarrow \gamma\gamma$, there is only the vertical cut, but this is not the case in general: if one of the photon is virtual, or if there is three or more photons in the final state, oblique cuts contribute (see next sections).

The second remark concerns the insertion of the form factor, not in Feynman amplitudes, but directly into the dispersion integral. These two approaches are equivalent only in the punctual case (consider the momentum flow through the form factor in each case). Working at the level of dispersion relations is much more in the spirit of the Bethe-Salpeter equation. Indeed, the wavefunction is extracted from the four-point Green's function, i.e. in configurations with an off-shell bound state (above threshold), and on-shell constituents:



The picture shows that both Bethe-Salpeter and the dispersion integral (7) make use of the form factor F_B with the same kinematical configuration (diagrams with bremsstrahlung radiation off the electron lines are treated similarly).

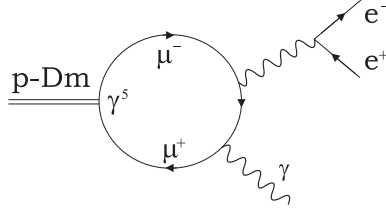
3 Application to decay rate computations

We will now review some applications of the result (8). We leave the interesting case of $o\text{-}Ps \rightarrow \gamma\gamma\gamma$ to the next section.

3.1 Paradimuonium and Low's Theorem

The paradimuonium decay $p\text{-}Dm \rightarrow \gamma e^+ e^-$ is the simplest QED bound state decay process where Low's theorem implications can be illustrated [7], [8], [9]. Following the same steps as for $p\text{-}Ps \rightarrow \gamma\gamma$, we first consider the

point-like amplitude



(plus the crossed diagram) with the result

$$\Gamma_{\gamma^5} (p\text{-}Dm \rightarrow e^+ e^- \gamma) = \frac{16\alpha^3}{3} \frac{m^2}{M} \int_0^{1-a_e} dx_\gamma |\mathcal{I}_{\gamma^5} (M^2, x_\gamma)|^2 \rho(x_\gamma, a_e) \quad (9)$$

$$\text{with } \rho(x_\gamma, a_e) = \sqrt{1 - \frac{a_e}{1-x_\gamma}} [2(1-x_\gamma) + a_e] \frac{x_\gamma^3}{(1-x_\gamma)^2}$$

$$\mathcal{I}_{\gamma^5} [P^2, x_\gamma] = \frac{-i}{(4\pi)^2} \frac{1}{x_\gamma} \left(F \left[\frac{4m^2}{M^2} \right] - F \left[\frac{4m^2}{M^2} \frac{1}{1-x_\gamma} \right] \right)$$

where m is the muon mass, M the dimuonium mass, x_γ the reduced photon energy $2E_\gamma/M$, $a_e = 4m_e^2/M^2$, m_e the electron mass, and with the function F defined in (1). In the limit $x_\gamma \rightarrow 0$, the spectrum $d\Gamma_{\gamma^5}/dx_\gamma$ goes to zero as x_γ^3 as predicted by Low's theorem (the amplitude behaves as x_γ , and an additional factor x_γ comes from phase-space).

Taking the derivative of $\mathcal{I}_{\gamma^5}$ to get the corresponding Coulomb form factor, we find

$$\begin{aligned} \mathcal{I}_{Coul} (M^2, x_\gamma) &= (32\pi C\phi_o\gamma) \frac{\partial}{\partial M^2} \mathcal{I}_{\gamma^5} (M^2, x_\gamma) \\ &= -i \frac{C\phi_o}{M} \frac{1}{x_\gamma} \left(\frac{2}{\pi} \arctan \frac{M}{2\gamma} - \frac{4\gamma y_\gamma}{\pi M} \arctan y_\gamma \right) \\ \text{where } y_\gamma &\equiv \left(\frac{4m^2}{M^2(1-x_\gamma)} - 1 \right)^{-1/2} \end{aligned}$$

The decay rate is then obtained by replacing $\mathcal{I}_{\gamma^5}$ by $\mathcal{I}_{Coul}$ in (9)

$$\Gamma (p\text{-}Dm \rightarrow e^+ e^- \gamma) = \frac{\alpha^6 m}{6\pi} \frac{4m^2}{M^2} \int_0^{1-a_e} dx_\gamma \left| \frac{2}{\pi} \arctan \frac{M}{2\gamma} - \frac{4\gamma y_\gamma}{\pi M} \arctan y_\gamma \right|^2 \frac{\rho(x_\gamma, a_e)}{x_\gamma^2}$$

It is very instructive to analyze in some details this result. Consider the limit $\gamma \rightarrow 0$ for the form factor:

$$\lim_{\gamma \rightarrow 0} \mathcal{I}_{Coul}(M^2, x_\gamma) = -i \frac{C\phi_o}{Mx_\gamma} \quad (10)$$

In that limit, the standard result for the decay rate is recovered

$$\Gamma(p-Dm \rightarrow e^+e^-\gamma) \stackrel{\gamma \rightarrow 0}{=} \frac{\alpha^6 m}{6\pi} \int_0^{1-a_e} dx_\gamma \frac{\rho(x_\gamma, a_e)}{x_\gamma^2}$$

However, the differential rate $d\Gamma/dx_\gamma$ has a wrong behavior when $x_\gamma \rightarrow 0$. The spectrum is linear (in x_γ) in the limit $\gamma \rightarrow 0$, in contradiction with Low's theorem. Therefore, it appears that, contrary to the two real photon case, the limit $\gamma \rightarrow 0$ is far from smooth. It is inconsistent to consider both the soft-photon limit and the on-shell limit simultaneously. Explicitly, the incompatibility of the two limits is obvious if $x_\gamma \rightarrow 0$ is taken first

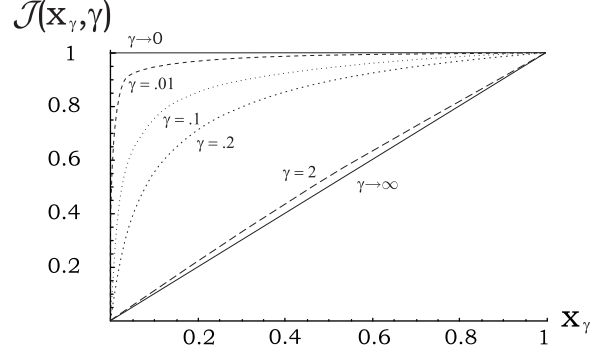
$$\begin{aligned} \lim_{x_\gamma \rightarrow 0} \mathcal{I}_{Coul}(M^2, x_\gamma) &= -i \frac{C\phi_o}{M\pi} \left(\frac{M}{2\gamma} + \left(1 + \frac{M^2}{4\gamma^2} \right) \arctan \frac{M}{2\gamma} \right) \\ &= -i \frac{C\phi_o}{M} \left(\frac{M^2}{8\gamma^2} + \frac{1}{2} - \frac{4\gamma}{3M\pi} + \dots \right) \end{aligned} \quad (11)$$

Mathematically, what these considerations show is that the limit $\gamma \rightarrow 0$ does not exist at $x_\gamma = 0$. Consequences of this result were discussed in [8], but it should be clear that one of the main virtue of our approach is its non-perturbative treatment of γ , leading to correct photon spectra.

To close this section, we just mention that the second term of $\mathcal{I}_{Coul}(M^2, x_\gamma)$ can be traced to the presence of oblique cuts in the imaginary part (i.e. processes like $p-Dm \rightarrow \mu^+\mu^-\gamma$ times $\mu^+\mu^-(\gamma) \rightarrow e^+e^-(\gamma)$, see [7], [8]). Those are the processes neglected in standard approaches, which is an inconsistent approximation since it is the sum of the two terms of $\mathcal{I}_{Coul}(M^2, x_\gamma)$ that enforce Low's theorem. This can be seen by plotting

$$\mathcal{J}(x_\gamma, \gamma) = 1 - \frac{\frac{2\gamma}{M} y_\gamma \arctan y_\gamma}{\arctan \frac{M}{2\gamma}}$$

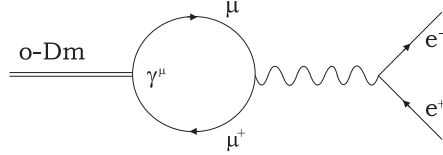
for $M = 1$ and various values of γ



As the picture shows, even if $\gamma/M \ll 1$ for QED bound states, the limit $\gamma \rightarrow 0$ is not to be taken because it is singular.

3.2 Orthodimuonium and Photon Vacuum Polarization

We now consider the decay $o\text{-}Dm \rightarrow \gamma^* \rightarrow e^+ e^-$



The point-like amplitude is easily obtained in terms of the (divergent) photon vacuum polarization function

$$\mathcal{M}_{\gamma^\mu} (o\text{-}Dm \rightarrow e^+ e^-) = \varepsilon_\mu (P) (P^2 g^{\mu\nu} - P^\mu P^\nu) \Pi_{\gamma^\mu} (P^2) \frac{1}{P^2} \{ \bar{u}(p) \gamma_\nu v(p') \}$$

where

$$\Pi_{\gamma^\mu} (P^2) = -\frac{e}{4\pi^2} \left[\frac{D}{3} + \frac{5}{9} + \frac{4}{3\zeta} + \frac{2}{3} \left(1 - \frac{4}{\zeta} \right) \left(1 + \frac{2}{\zeta} \right) \frac{\arctan \frac{1}{\sqrt{4/\zeta - 1}}}{\sqrt{4/\zeta - 1}} \right] \quad (12)$$

with $\zeta = P^2/m^2$, m the muon mass and $D = 2/\varepsilon - \gamma_{Euler} + \log 4\pi\mu^2/m^2$ in dimensional regularization.

The Coulomb form factor will be obtained from the derivative of the vacuum polarization, with respect to $P^2 = M^2$:

$$\begin{aligned}\Pi_{Coul}(M^2) &= (32\pi C\phi_o\gamma) \frac{\partial}{\partial M^2} \Pi_{\gamma^\mu}(M^2) \\ &= \frac{eC\phi_o}{M} \left[\frac{8}{\pi} \left(\gamma \frac{6m^2 + M^2}{3M^3} - \frac{4m^4}{M^4} \arctan \frac{M}{2\gamma} \right) \right] \\ &= \frac{eC\phi_o}{M} \left[1 - \frac{32}{3\pi} \frac{\gamma}{M} + 8 \frac{\gamma^2}{M^2} - \frac{128}{3\pi} \frac{\gamma^3}{M^3} + \dots \right]\end{aligned}\quad (13)$$

This has the effect of removing the divergence, as it should. The decay rate is

$$\begin{aligned}\Gamma(o-Dm \rightarrow e^+e^-) &= \frac{\alpha}{3} \frac{M^2 + 2m_e^2}{M} \sqrt{1 - a_e} |\Pi_{Coul}(M^2)|^2 \\ &= \frac{\alpha^5 m}{6} \left(1 + \frac{a_e}{2} \right) \sqrt{1 - a_e} \left| 1 - \frac{32}{3\pi} \frac{\gamma}{M} + 8 \frac{\gamma^2}{M^2} - \dots \right|^2\end{aligned}$$

To leading order, we recover the standard result $\Gamma(o-Dm \rightarrow e^+e^-) \approx \alpha^5 m/6$ when $a_e \ll 1$. For electromagnetic bound states, the binding energy is related to the fine structure constant, hence the corrections can be cast into

$$\begin{aligned}\Gamma(o-Dm \rightarrow e^+e^-) &= \frac{\alpha^5 m}{6} \left(1 + \frac{a_e}{2} \right) \sqrt{1 - a_e} \left(1 - \frac{16}{3} \frac{\alpha}{\pi} + \left(\pi^2 + \frac{64}{9} \right) \frac{\alpha^2}{\pi^2} + \dots \right) \\ &\approx \frac{\alpha^5 m}{6} \left(1 + \frac{a_e}{2} \right) \sqrt{1 - a_e} (1 - 1.70\alpha + 1.72\alpha^2 + \dots)\end{aligned}$$

Compared to the binding energy corrections to the parapositronium two-photon decay rate, the present corrections are much bigger. Note also that the binding energy correction obtained here, at order α , again account for a great deal of the correction obtained using the standard approach (we consider only part of the total $\mathcal{O}(\alpha)$ correction, see Eq. 47 in [10]), which are

$$\Gamma^{LO+NLO}(o-Dm \rightarrow e^+e^-) = \frac{\alpha^5 m}{6} \left(1 + \frac{a_e}{2} \right) \sqrt{1 - a_e} \left(1 - 4 \frac{\alpha}{\pi} + \dots \right)$$

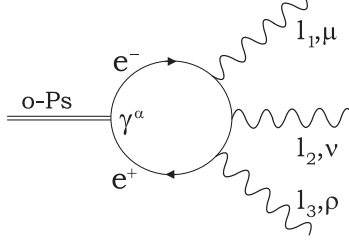
4 Orthopositronium Decay

We now apply our method to the orthopositronium decay to three photons. This is a very interesting decay process. As discussed in [8], the lowest order basis chosen in standard computations, namely the Ore-Powell amplitude

[13], is in contradiction with Low's theorem. What we will now show is that taking binding energy into account, i.e. integrating the effects of all the Coulomb photon exchanges at lowest order through the appropriate form factor, is necessary in order to get a correct spectrum. Also, we will again find that the bulk of the radiative corrections is accounted for already at our lowest order.

4.1 Point-like Amplitude

The point-like amplitude is the standard light-by-light box diagram



plus five other ordering of the photon insertions. The amplitude can be found in many places [6], [14]. The tensor $G_{\alpha}^{\lambda_1 \lambda_2 \lambda_3}$ describing the transition from an off-shell photon to three on-shell photons of helicity states $\lambda_1 \lambda_2 \lambda_3$ is such that

$$\sum_{\lambda_1 \lambda_2 \lambda_3} \left(G_{\alpha}^{\lambda_1 \lambda_2 \lambda_3} G^{* \lambda_1 \lambda_2 \lambda_3, \alpha} \right) = \frac{2^4 \alpha^3}{\pi} [R(123) + R(213) + R(312)] \quad (14)$$

where $R(123) \equiv R(x_1, x_2, x_3, a)$ (x_i is the reduced energy of the photon i and $a = 4m^2/M^2$). The $R(ijk)$ are given in terms of individual dimensionless helicity amplitudes as

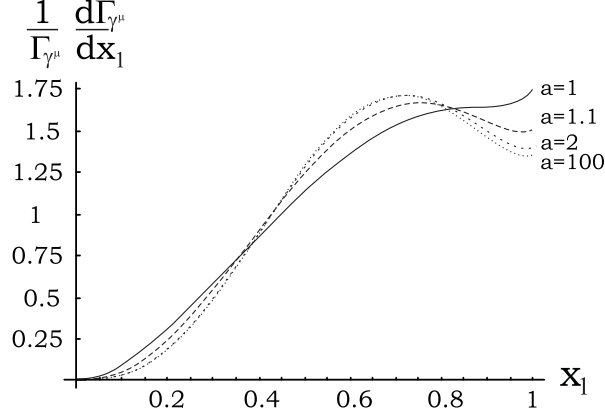
$$\begin{aligned} R(123) = & \frac{1}{3} \left| E_{-++}^{(2)}(123) \right|^2 + \left| E_{+++}^{(2)}(123) \right|^2 \\ & + \frac{x_1}{x_2 x_3 (1-x_1)} \left| E_{-++}^{(1)}(213) \right|^2 + \frac{1}{x_1^2} \left| E_{+++}^{(1)}(123) + E_{+++}^{(1)}(132) \right|^2 \\ & + \frac{(1-x_2)(1-x_3)}{x_1^2 (1-x_1)} \left| \frac{1}{1-x_2} E_{+++}^{(1)}(123) - \frac{1}{1-x_3} E_{+++}^{(1)}(132) \right|^2 \end{aligned} \quad (15)$$

The helicity amplitudes $E_{\pm++}^{(n)}$ are complicated functions of x_i and a , and we do not reproduce them here ([6], [14]). The decay rate of a point-like vector positronium to three photons is then

$$\Gamma_{\gamma\mu} (o-Ps \rightarrow \gamma\gamma\gamma) = \frac{1}{9} \frac{\alpha^3}{2^5 \pi^4} M \int dx_1 dx_2 [R(123) + R(213) + R(312)]$$

with the three-body phase space written in terms of reduced photon energies as $\int d\Phi_3 = M^2 \int dx_1 dx_2 / 2^7 \pi^3$ [6].

Of special interest is the behavior of the low-energy end of the differential rate. For various values of the ratio $a = 4m^2/M^2$ (i.e. of the binding energy $E_B = M - 2m$), the photon spectrum, normalized to the total rate is



Sufficiently close to zero, the behavior is always like x_1^3 , as required by Low's theorem [8]. As a increases, the x^3 behavior is getting more and more pronounced. In the limit $a \rightarrow \infty$, the normalized spectrum is

$$\frac{1}{\Gamma_{\gamma^\mu}} \frac{d\Gamma_{\gamma^\mu}}{dx_1} \xrightarrow{a \rightarrow \infty} \frac{5}{17} x_1^3 \left(\frac{343}{3} - 207x_1 + \frac{973}{10} x_1^2 \right)$$

which is, as expected, the spectrum obtained from the Euler-Heisenberg effective theory [15]

$$\mathcal{L}_{E-H} = \frac{\alpha^2}{90m^4} \left[(F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]$$

4.2 Coulomb Form factor and Ore-Powell spectrum

To insert the Schrödinger wavefunction form factor, we define modified helicity amplitudes according to (8)

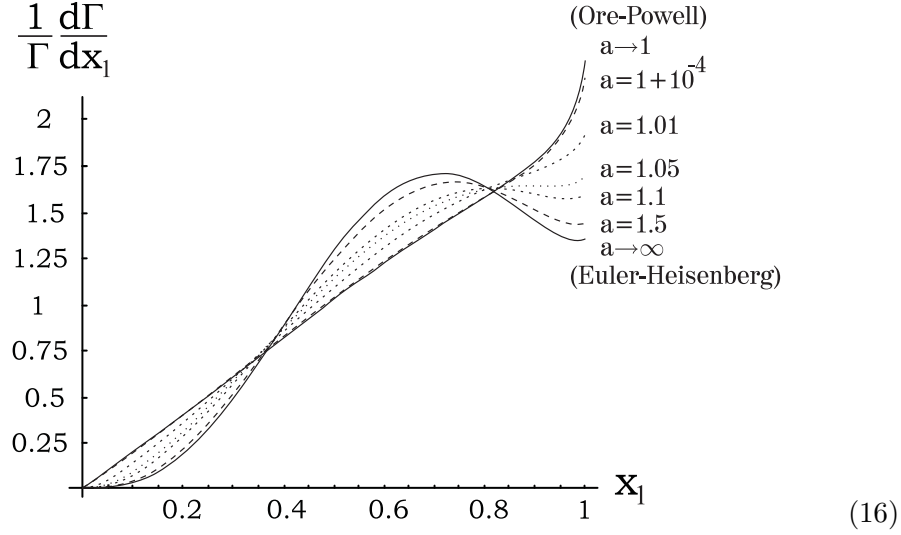
$$E_{\pm\pm\pm, Coul}^{(n)}(ijk, M^2) = (32\pi C \phi_o \gamma) \frac{\partial}{\partial M^2} E_{\pm\pm\pm}^{(n)}(ijk, M^2)$$

The resulting decay amplitude, and decay rate are constructed as in the punctual case, and we reach

$$\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma) = \alpha^6 m \frac{2}{9\pi} \left(\frac{4m^2}{M^2} \right)^2 \left(\frac{\mathcal{R}}{2\pi^2} \right)$$

with $\mathcal{R} \equiv \int dx_1 dx_2 (R_{Coul}(123) + R_{Coul}(213) + R_{Coul}(312))$

R_{Coul} is given by (15) with $E_{\pm\pm\pm, Coul}^{(i)}$ in place of $E_{\pm\pm\pm}^{(i)}$. Integrating over x_2 , we get the photon spectrum for various values of a



As long as $a \neq 1$, the spectrum behavior is in x_1^3 close to zero, i.e. roughly in the range $x_1 \in [0, a - 1]$. The properties of this spectrum as a varies are completely similar to that of $p\text{-}Dm \rightarrow \gamma e^+ e^-$, and one can show that the limits $\gamma \rightarrow 0$ and $x_1 \rightarrow 0$ are again incompatible.

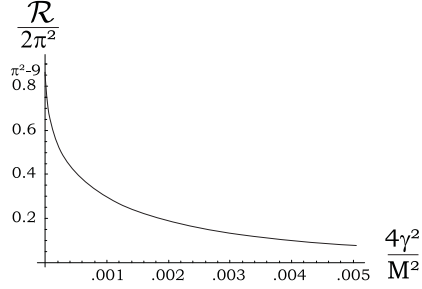
In the context of dispersion relations, the contributions originating in the oblique cuts (processes like $o\text{-}Ps \rightarrow e^+ e^- \gamma$ times $e^+ e^- (\gamma) \rightarrow \gamma \gamma (\gamma)$ in the imaginary part) are essential to maintain a physical spectrum (i.e. in agreement with Low's theorem). Again, those oblique cuts are neglected in standard approaches. Here, they are automatically accounted for since they are included in the point-like result.

Note also that the spectrum for a point-like bound state at threshold corresponds to the spectrum for $a \sim 1.05$ in the Coulomb form factor case. The bound state decay spectra for $a < 1.05$ are unattainable in the point bound state case.

Concerning the total integrated rate, we recover the Ore-Powell result at threshold ($a = 1$)

$$\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma) = \alpha^6 m \frac{2}{9\pi} \left(\frac{17.16}{2\pi^2} \right) = 4.75 \times 10^{-15} \text{ MeV} = \alpha^6 m \frac{2}{9\pi} (\pi^2 - 9)$$

As a increases, the total rate quickly decreases (remember $a = 4\gamma^2/M^2 + 1$)



By a numerical analysis, we found

$$\frac{\mathcal{R}}{2\pi^2} = (\pi^2 - 9) \left(1 - 15.412 \frac{\gamma}{M} + 122 \frac{\gamma^2}{M^2} - 889 \frac{\gamma^3}{M^3} + 1.92 \cdot 10^4 \frac{\gamma^4}{M^4} - \dots \right)$$

For orthopositronium, we can express γ in terms of α , and we find the binding energy corrections to the total rate

$$\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma) = \alpha^6 m \frac{2(\pi^2 - 9)}{9\pi} \left(1 - 12.1 \frac{\alpha}{\pi} + 80.2 \frac{\alpha^2}{\pi^2} - 502 \frac{\alpha^3}{\pi^3} + \dots \right) \quad (17)$$

This series is to be compared to the one presented in the literature [3], which is

$$\Gamma_{o\text{-}Ps} = \alpha^6 m \frac{2(\pi^2 - 9)}{9\pi} \left(1 - A_o \frac{\alpha}{\pi} - \frac{\alpha^2}{3} \ln \frac{1}{\alpha} + B_o \frac{\alpha^2}{\pi^2} - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + C_o \frac{\alpha^3}{\pi} \ln \frac{1}{\alpha} + \delta_{5\gamma} \frac{\alpha^2}{\pi^2} \right)$$

with coefficients

$$\begin{aligned} A_o &= 10.286606(10) & C_o &= 5.517(1) \\ B_o &= 44.52(26) & \delta_{5\gamma} &= 0.19(1) \end{aligned}$$

Again, the exchanges of Coulomb photon appear responsible for the bulk of the radiative corrections at order α . Note that the origin of the slowness

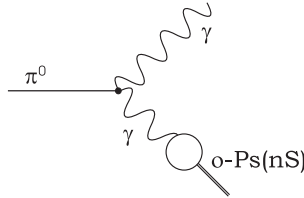
in the convergence of the corrections to Γ_{o-Ps} is clearly identified as coming from the perturbative expansion of binding energy effects (see (17)). In other words, the four-point fermionic loop with a Coulomb form factor is not well-behaved for $\gamma = 0$, and that limit appears as an inappropriate basis for perturbation theory.

5 Application to other processes

In this final section, we present two extensions of the method. First, we accommodate the derivative formula (8) for radial excitations, by considering the pion decay $\pi^0 \rightarrow \gamma o-Ps$. For the second, we compute the hyperfine splitting using the language of mass renormalization.

5.1 Pion Decay to Orthopositronium and Radial Excitations

The technique of taking the derivative of point-like amplitude can be extended to other spherically symmetric wavefunctions. The decay $\pi^0 \rightarrow \gamma o-Ps$ is a good example [11]



To get a sensible theoretical prediction, one must sum the decay rates over the infinite tower of radial excitations $o-Ps (nS)$ in the final state ($o-Ps (nS)$ means n th radial excitation of the S -wave $J = 1$ positronium state). From the standard Schrödinger wavefunction for hydrogen states, we can write a general expression for the form factor for nS radial excitations as

$$\mathcal{I}_{Coul,n}(M_n^2) = (32\pi C\phi_{noo}) \left[{}_1F_1 \left(1-n, 2, 16\gamma_n^2 \frac{\partial}{\partial M_n^2} \right) \right] \left[\gamma_n(M_n^2) \frac{\partial}{\partial M_n^2} \mathcal{I}_{\gamma^\mu}(M_n^2) \right] \quad (19)$$

with

$$|\phi_{noo}| = \sqrt{\frac{m^3 \alpha^3}{8\pi n^3}} \quad \gamma_n = \sqrt{m^2 - M_n^2/4}$$

Where we have denoted M_n the mass of the $o\text{-}Ps(nS)$ state. The hypergeometric functions are essentially the well-known Laguerre polynomials ((19) is equally valid for parapositronium).

For the case at hand, $\mathcal{I}_{\gamma^\mu}$ is the photon vacuum polarization (12), where m refers now to the electron mass. The pion decay rate into orthopositronium states can be written as

$$R_{oPs(nS)} = \frac{\Gamma(\pi^0 \rightarrow \gamma o\text{-}Ps(nS))}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 2e^2 |\Pi_{Coul,n}(M_n^2)|^2 \left(1 - \frac{M_n^2}{m_\pi^2}\right)^3 \left[1 + \mathcal{O}\left(\frac{M_n^2}{m_{\rho,\omega}^2}\right)\right]$$

The $M_n^2/m_{\rho,\omega}^2 \approx 10^{-6}$ corrections arise from the form factor for pion to two photons. The mass ratio $M_n^2/m_\pi^2 \approx 10^{-4}$ is also negligible compared to binding energy corrections, to which we now turn. Up to corrections of order γ_n^2 , we can write using $\gamma_n \approx m\alpha/2n$

$$R_{oPs(nS)} = \frac{\alpha^4}{2} \frac{1}{n^3} \left(1 - 2A_n \frac{\gamma_n}{M_n} + \dots\right) = \frac{\alpha^4}{2} \frac{1}{n^3} \left(1 - \frac{1}{2n} A_n \alpha + \dots\right)$$

(the A_n are the numerical coefficients found by expanding $\Pi_{Coul,n}(M_n^2)$). Summing over n , we find

$$\sum_n R_{oPs(nS)}^{LO} = \frac{\alpha^4}{2} \zeta(3) (1 - 1.66\alpha) \approx 1.684 \times 10^{-9}$$

where $\alpha^4/2 \approx (1.418 \times 10^{-9})$ is obtained from the contribution of the $o\text{-}Ps(1S)$ only. For comparison, [11] found the radiative correction to be $(1 - 0.92\alpha)$. The experimentally quoted branching fraction is [16]

$$\left. \frac{\Gamma(\pi \rightarrow \gamma Ps)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} \right|_{\text{exp}} = (1.9 \pm 0.3) \times 10^{-9}$$

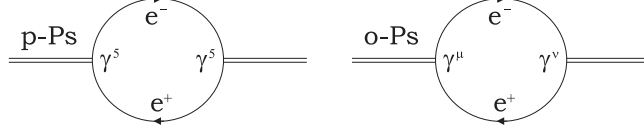
so the agreement is good.

What this little exercise shows is the power of our method as a mean to partially compute higher order corrections. More importantly, it is by now apparent that other, more complicated wavefunctions can easily be accommodated for. Any wavefunction that can be expressed, even approximately, from derivatives of the Coulomb one can fit in our scheme. This may open the way to many applications in QCD, as we will comment in the conclusion.

5.2 Hyperfine Splitting and Mass Renormalization

We implement here the renormalization of the positronium state. The idea is to consider the bare positronium mass M as equal to twice the electron

mass m , and then to carry a (finite) mass renormalization. The diagrams contributing, at lowest order, to this mass shift will be obtained as the second derivatives of the vacuum polarization loops:



Let us first obtain the mass renormalization equations, and then discuss the results numerically.

5.2.1 Mass Renormalization

For a pseudoscalar parapositronium state, the resummation of the Dyson series is trivial. The parapositronium propagator is then

$$G(q^2) = \frac{i}{q^2 - M_o^2 - \Pi_{para}(q^2)}$$

Of interest to us is the mass shift, defined from the pole of $G(q^2)$

$$M_{R,para}^2 - M_o^2 = \Pi_{para}(M_R^2) \rightarrow M_{R,para} - M_o = \frac{\Pi_{para}(M_{R,para}^2)}{2M_o} \quad (20)$$

For the orthopositronium, the transverse part of the bare propagator is

$$G_0^{\mu\nu}(q^2) = \frac{-i \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)}{q^2 - M_0^2 + i\varepsilon}$$

The self-energy of the vector positronium will be of the form

$$\Pi^{\mu\nu}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi_{ortho}(q^2)$$

Proceeding with the resummation of the Dyson series, we end up with

$$G^{\mu\nu}(q^2) = \frac{-i \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)}{q^2 (1 - \Pi_{ortho}(q^2)) - M_o^2}$$

Again, the propagator pole is at $q^2 = M_R^2$, hence

$$M_{R,ortho}^2 - M_o^2 = M_R^2 \Pi_{ortho}(M_R^2) \rightarrow M_{R,ortho} - M_o = \frac{M_{R,ortho}}{2} \Pi_{ortho}(M_{R,ortho}^2) \quad (21)$$

The two equations (20) and (21) are self-consistent equations. By identifying $M_o = 2m$ and $M_{R,para(ortho)} = M_{para(ortho)}$, they are of the form

$$\begin{cases} E_B^{para} \equiv M_{para} - 2m = \frac{\Pi_{para}(M_{para}^2)}{4m} \equiv f^{para}(E_B^{para}) \\ E_B^{ortho} \equiv M_{ortho} - 2m = \frac{M_{ortho}}{2} \Pi_{ortho}(M_{ortho}^2) \equiv f^{ortho}(E_B^{ortho}) \end{cases}$$

Our approach will be to generate corrections to $E_B^{para(ortho)}$ by plugging the non-relativistic result $E_B = -m\alpha^2/4$ into $f^{para(ortho)}$.

5.2.2 Second Derivatives and Hyperfine Splitting

The double derivatives of the vacuum polarization loops are

$$\Pi_{para(ortho)}(M^2) = \frac{1}{2} (32\pi C \phi_o \gamma)^2 \frac{1}{2} \left(\frac{\partial}{\partial M^2} \right)^2 \Pi_{\gamma^5(\gamma^\mu)}(M^2)$$

where a factor of $1/2$ corrects for the factor 2 appearing in the derivative of $1/(s - M^2)^2$, while the other accounts for the double counting of Coulomb photon exchanges.

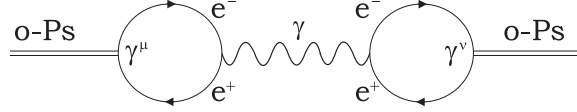
The computations are straightforward. For parapositronium, the point-like quadratic divergence disappears and we find

$$\begin{aligned} E_B^{para} &\equiv M_{para} - 2m = \frac{\Pi_{para}(M_{para}^2)}{4m} \\ &= -\frac{4|\phi_o|^2}{Mm} \left[\left(\frac{4\gamma^2}{M^2} + 1 \right) \frac{\arctan \frac{M}{2\gamma}}{2\gamma/M} - \frac{\frac{4\gamma^2}{M^2} - 1}{\frac{4\gamma^2}{M^2} + 1} \right]_{M=M_{para}} \\ &= -\frac{m\alpha^2}{4} - \frac{9m\alpha^4}{128} + \frac{m\alpha^5}{6\pi} - \frac{115m\alpha^6}{8192} + \dots \end{aligned}$$

Where we have used $|\phi_o|^2 = m^3\alpha^3/8\pi, \gamma^2 = m^2 - M_{para}^2/4$ and $M_{para} = 2m - m\alpha^2/4$ in the right hand side. Proceeding similarly with the orthopositronium renormalization, we consider the second derivative of the photon vacuum polarization function:

$$\begin{aligned} E_{B(1)}^{ortho} &\equiv M_{ortho} - 2m = \frac{M_{ortho}}{2} \Pi_{ortho}(M_{ortho}^2) \\ &= \frac{4|\phi_o|^2}{M^2} \left[\left(\frac{4\gamma^2}{M^2} + 1 \right) \left(\frac{20\gamma^2}{M^2} - 1 \right) \frac{\arctan \frac{M}{2\gamma}}{\gamma/M} - \frac{2 + \frac{176}{3} \frac{\gamma^2}{M^2} + 160 \frac{\gamma^4}{M^4}}{\frac{4\gamma^2}{M^2} + 1} \right]_{M=M_{ortho}} \\ &= -\frac{m\alpha^2}{4} + \frac{27m\alpha^4}{128} - \frac{2m\alpha^5}{3\pi} + \frac{1301m\alpha^6}{8192} + \dots \end{aligned} \tag{22}$$

At the order α^4 , there is also the annihilation diagram



whose contribution is $\Pi_{ortho}^{Ann,\mu\nu}(M^2) = -i(P^2 g^{\mu\nu} - P^\mu P^\nu) (\Pi_{Coul}(P^2))^2$. Using $\Pi_{Coul}(P^2)$ as given by (13), the dominant contribution in the limit $\gamma \rightarrow 0$ is

$$\Pi_{ortho}^{Ann}(P^2) = \frac{\alpha^4}{4} + \dots \rightarrow E_{B(2)}^{ortho,Ann} = \frac{M_{ortho}}{2} \Pi_{ortho}^{Ann}(M_{ortho}^2) = \frac{m\alpha^4}{4} + \dots$$

Hence the lowest loop contributions to the hyperfine splitting are, to order α^4 :

$$\Delta E_{hf} = (E_{B(1)}^{ortho} + E_{B(2)}^{ortho,Ann}) - E_B^{para} = m\alpha^4 \left[\frac{17}{32} + \dots \right] = m\alpha^4 [0.5313 + \dots]$$

This is to be compared to the result (see [1], [17] and references cited there)

$$\begin{aligned} \Delta E_{hf} = m\alpha^4 & \left\{ \frac{7}{12} - \frac{\alpha}{\pi} \left(\frac{8}{9} + \frac{\ln 2}{2} \right) \right. \\ & + \frac{\alpha^2}{\pi^2} \left(-\frac{5}{24}\pi^2 \ln \alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2} \right) \ln 2 - \frac{53}{32}\zeta(3) \right) \\ & + \frac{\alpha^3}{\pi^3} \left(-\frac{7}{8}\pi^2 \ln^2 \alpha + \left(\frac{17}{3} \ln 2 - \frac{217}{90} \right) \pi^3 \ln \alpha \right) + \mathcal{O}(\alpha^3) \Big\} \\ & \approx m\alpha^4 [0.5833 - 0.3933\alpha - 0.2083\alpha^2 \ln \alpha - 0.3928\alpha^2 - \dots] \end{aligned}$$

where the correction of order $\mathcal{O}(\alpha^7 \ln \alpha)$ has been obtained very recently [17].

It is not surprising that our method does not reproduce exactly the above result, because we have neglected many diagrams (like electron self-energy insertions for example), and because we used the lowest order non-relativistic binding energy $E_B = -m\alpha^2/4$ and wavefunction $|\phi_o|^2 = m^3\alpha^3/8\pi$ as a basis. Taken individually, the corrections of $\mathcal{O}(\alpha^4)$ to the para- and orthopositronium masses are off by more than 50%. On the other hand, the difference between both corrections, giving the hyperfine splitting, is surprisingly good. Again, it seems that all the effects contained in the lowest order loop (i.e. all the ladder Coulomb photon exchanges) suffice to account for most of the radiative corrections. In conclusion, further studies of the application of our method to hyperfine splitting appear as necessary.

6 Conclusion

We can characterize our method as a change of basis for perturbation theory: some binding effects are treated at all orders in γ , at each order in α . With such non-perturbative results, the behavior of the usual expansion in the binding energy can be analyzed. *We find that such expansions are problematic when soft-photons are present* (i.e. when a photon's energy is of the order of γ). In other words, expansions in the binding energy are incompatible with analyticity. In all cases, unphysical contributions violating analyticity are subleading, being of the order of $2m - M \approx m\alpha^2/4$. However small, such effects become important in view of the precision achieved in both experimental and theoretical descriptions. In addition, standard positronium models neglect some contributions (corresponding, in the context of dispersion relations, to oblique cuts contributions to the imaginary part of the loop diagrams). Therefore, they cannot be correct at order α^2 for decay processes [7].

In our view, obtaining QED bound state amplitudes from relativistic point-like amplitudes has many advantages. This may lead to simplifications in the computation of higher order corrections, especially concerning issues of gauge invariance and infrared divergence. For instance, two-loop graphs will introduce some $O(\alpha^2)$ and $O(\alpha^2 \ln \alpha)$ corrections (see for example [18]). We should point-out, however, that the technique of taking the derivative will not work as it stands for higher order diagrams, because of binding graphs, and some refinements will be necessary.

As a final comment, the present method can lead to interesting advances in quarkonium physics. Of course, some knowledge of the quarkonium wavefunctions will be necessary. No matter the form of these wavefunctions, it should be possible to expand them in the basis made of the Coulomb wavefunction and its derivatives, to which the present method apply. This is left to further studies. For now, as a first result, it appear quite obvious that the Ore-Powell spectrum used as a lowest order approximation for the photon spectrum in the quarkonium inclusive decays into hadrons + photon is incorrect. If quarkonia were Coulomb bound state, the spectrum would be the curve of (16) with $a \approx 1.2 \rightarrow 1.5$ (since $4m_B^2/M_{\Upsilon(1S)}^2 \approx 1.25$ and $4m_D^2/M_{J/\psi}^2 \approx 1.46$), i.e. a spectrum which is suppressed at high energy compared to the Ore-Powell one. QCD effects should not change much our conclusion. We think that the suppression at high energy observed by CLEO [19] is, at least in part, a manifestation of binding energy effects, giving us confidence in our results.

Acknowledgments: We are very pleased to acknowledge useful discussions with Gabriel Lopez Castro and Stéphanie Trine. C. S. acknowledges financial support from IISN (Belgium).

References

- [1] T. Kinoshita (*Editor*), "Quantum Electrodynamics", World Scientific, 1990.
- [2] C. Westbrook, D. Gidley, R. Conti, A. Rich, Phys. Rev. **A40**, 5489 (1989); J. Nico, D. Gidley, A. Rich, P. Zitzewitz, Phys. Rev. Lett. **65**, 1344 (1990); A. Al-Ramadhan, D. Gidley, Phys. Rev. Lett. **72**, 1632 (1994); S. Asai, S. Orito, N. Shinohara, Phys. Lett. **B357**, 475 (1995); O. Jinnouchi, S. Asai, T. Kobayashi, *hep-ex/0011011*.
- [3] W. E. Caswell, G. P. Lepage, J. Sapirstein Phys. Rev. Lett. **38**, 488 (1977); W. E. Caswell, G. P. Lepage, Phys. Rev. **A20**, 36 (1979); G. Adkins, Ann. Phys. **146**, 78 (1983); A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys. Rev. Lett. **83**, 1135 (1999); G. Adkins, R. Fell, J. Sapirstein, Phys. Rev. Lett. **84**, 5086 (2000); B. Kniehl, A. Penin, Phys. Rev. Lett. **85**, 1210 (2000); K. Melnikov, A. Yelkhovsky, Phys. Rev. **D62**, 116003 (2000); R. Hill, G.P. Lepage, Phys. Rev. **D62**, 111301 (2000).
- [4] W.E. Caswell and G.P. Lepage, Phys. Lett. **B167**, 437 (1986); P. Labelle, MRST Meeting 1992 (CLNS-92-1161).
- [5] I.B. Khriplovich, A.S. Yelkhovsky, Phys. Lett. **B246**, 520 (1990); I.B. Khriplovich, A.I. Milstein, J. Exp. Theor. Phys. **79**, 379 (1994); R.N. Faustov, A.P. Martynenko, V.A. Saleev, Phys. Rev. **A51**, 4520 (1995); V. Antonelli, *Int. Work. on Hadronic Atoms and Positronium in the S.M.*, Dubna, 26-31 May 1998.
- [6] V. Antonelli, V. Ivanchenko, E. Kuraev, V. Laliena, Eur. Phys. J. **C5**, 535 (1998).
- [7] J. Pestieau, C. Smith and S. Trine, *hep-ph/0105034*, to be published in Int. J. Mod. Phys. **A**, and references therein.
- [8] J. Pestieau and C. Smith, *hep-ph/0111264*, Phys. Lett. **B524**, 395 (2002).

- [9] F. E. Low, Phys. Rev. **110**, 974 (1958); H. Chew, Phys. Rev. **123**, 377 (1961); J. Pestieau, Phys. Rev. **160**, 1555 (1967).
- [10] U. Jentschura, G. Soff, V. Ivanov and S. Karshenboim, Phys. Rev. **A56**, 4483 (1997).
- [11] M.I. Vysotsky, Yad.Fiz. **29**, 845 (1979).
- [12] R. Barbieri and E. Remiddi, Nucl. Phys. **B141**, 413 (1978).
- [13] A. Ore and J. L. Powell, Phys. Rev. **75**, 1696 (1949).
- [14] B. De Tollis, Nuovo Cim. **32**, 757 (1964), *ibid.* **35**, 1182 (1965); V. Constantini, B. De Tollis, G. Pistoni, Nuovo Cim. 2A, 733 (1971); M. Laursen, M. Samuel, G. Tupper, A. Sen, Phys. Rev. **D27**, 196 (1983).
- [15] W. Heisenberg, H. Euler, Z. Phys. **98**, 714 (1936); A. Dobado, A. Gomez-Nicola, A. Maroto, J. Pelaez, "Effective Lagrangians for the Standard Model", Springer, 1997.
- [16] L.G. Afanasev *et al*, Phys. Lett. **B236**, 116 (1990); Particle Data Group (D.E. Groom *et al.*), Eur.Phys.J. **C15**, 1 (2000).
- [17] B. Kniehl and A. Penin, Phys. Rev. Lett. **85**, 5094 (2000); K. Melnikov and A. Yelkhovsky, Phys.Rev.Lett. **86**, 1498 (2001).
- [18] R. Barbieri, P. Christillin, E. Remiddi, Phys. Rev. **A8**, 2266 (1973).
- [19] CLEO Collaboration (B. Nemati *et al.*), Phys. Rev. **D55**, 5273 (1997).