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Strong and compact relaxations in the original space using
a compact extended formulation

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Strong and compact relaxations in the original space using a compact extended formulation

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Abstract For certain integer programs, one way to obtain a strong dual bound is to use an extended formulation and then solve the associated linear programming relaxation. The classical way to obtain a bound of the same value in the original variable space is through the use of Benders' algorithm. Here, we propose an alternative approach based on a decomposition of the dual optimal solution of the extended formulation linear program. An example of the approach using the multi-commodity formulation of a two-level production/transportation problem is presented.

Introduction

We consider integer (or mixed integer) problems of the form

$$z = \min\{cx | Ax \geq b, Fx \geq g, x \in \mathbb{Z}^n\} \tag{1}$$

with the property that a tight and compact extended formulation

$$P = \{(x, w) | Bx + Dw \geq d\}$$

is known for the set $X = \{x \in \mathbb{Z}^n : Fx \geq g\}$. Here, tight means that $\text{proj}_x(P) = \text{conv}(X)$ and compact that P is of polynomial size relative to the description of

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X. Typically in such cases a description of $\mathrm{conv}(X)$ by valid inequalities in the original space $\mathbb{R}^n$ is of exponential size and is often only partially known. See Conforti et al. (2010) for a recent survey on extended formulations for combinatorial optimization problems and Vanderbeck and Wolsey (2010) for integer programmig problems.

Given P, a natural approach to solve problem (1) is to work with the formulation

$$z = \min\{cx | Ax \geq b, Bx + Dw \geq d, x \in \mathbb{Z}^n\} \tag{2}$$

for which the linear programming relaxation

$$z_{\mathrm{LP}} = \min\{cx | Ax \geq b, Bx + Dw \geq d, x \in \mathbb{R}^n\} \tag{3}$$

of value $z_{\mathrm{LP}} = \min\{cx | Ax \geq b, x \in \mathrm{conv}(X)\}$ should provide a good lower bound on z.

Here, we address the question of how to provide a linear program

$$z^*_{\mathrm{LP}} = \min\{cx | Ax \geq b, Cx \geq e\} \tag{4}$$

in the x-space such that

(i) $z_{\mathrm{LP}} = z^*_{\mathrm{LP}}$ and the optimal solutions of the linear programs (3) and (4) are essentially the same,
(ii) $Cx \geq e$ is efficiently computable, and in particular the number of inequalities in $Cx \geq e$ is not large, on the order of n,
(iii) the inequalities are strong.

Three reasons to address such a question are

(a) a desire to view and understand the valid inequalities that are implicitly being generated by the extended formulation for a given objective vector c. The possibility of obtaining such descriptions with software such as PORTA Christof and Loebel (1997) is limited to very small problem instances.
(b) though of polynomial size, the extended formulation P may be large in practice so that even though solving the linear program (3) may be feasible, solving the associated integer program (2) may be impossible. So one would like to work in $\mathbb{R}^n$ with the formulation

$$z = \min\{cx | Ax \geq b, Cx \geq e, x \in \mathbb{Z}^n\} \tag{5}$$

whose linear programming relaxation is strong using information derived from P.
(c) the possibility of providing an alternative to Benders' algorithm Benders (1962) when solving the linear programming relaxation at the top node of an IP or MIP.

In Sect. 2 of this note, we describe an algorithm to produce the desired set of inequalities $Cx \geq e$, and show that the resulting linear program (4) has the desired properties. In Sect. 3, we present an example involving a multi-commodity reformulation of a two-level production problem along with some remarks concerning the inequalities generated.

A decomposition algorithm

A first observation is that (i) and (ii) above are in general incompatible. Indeed, the set of optimal solutions of the linear program (3) can be a polyhedron with an exponential number of facets. In this case, it is simply impossible to give a compact LP description of this set in the original variable space.

The algorithm that we describe next is going to satisfy (i) but for a relaxation of (3). More specifically, we suppose that we have solved the linear program over the extended formulation and have dropped all the constraints for which the optimal dual variables are zero.

The algorithm

(i) Solve the linear program (3) or its dual.

(ii) Drop the primal constraints for which the optimal dual variables are zero, or equivalently, suppose that all constraints are binding at the optimal solution. Thus, the extended formulation

$$z_{\mathrm{LP}}(\mathrm{EF}) = \min\{cx + 0w : Ax \geq b,\ Bx + Dw \geq d,\ x \in \mathbb{R}^n,\ w \in \mathbb{R}^p\} \quad (6)$$

has $b \in R^{q_1}$, $d \in R^{q_2}$ and feasible region Q, where the optimal primal solution (x^*, w^*) satisfies all the constraints as equalities. Therefore, there is an optimal dual solution $(v^*, u^*) > 0$ satisfying

$$\begin{aligned} va + ud &= cx^* \\ vA + uB &= c \\ uD &= 0 \\ v, \quad u &\geq 0. \end{aligned}$$

(iii) Decompose u^* as

$$u^* = \sum_{t=1}^{r} \lambda_t u^t,$$

where $\{u^t\}_{t=1}^r$ are extreme rays of the dual cone $U = \{u : uD = 0, u \geq 0\}$ and $\lambda_t > 0$ for $t = 1, \ldots, r$.

(iv) The reformulated linear program is

$$\min\{cx | Ax \geq b, u^t Bx \geq u^t d,\ t = 1, \ldots, r\}.$$

Observation 1 *The decomposition in step (iii) can be implemented in polynomial time with an algorithm of* Schrijver (1986) *involving the solution of at most* q_2 *linear programs over the dual cone* U.

The simple algorithm is as follows:

Decomposition Algorithm
Initialization: $\rho^0 = u^* \neq 0,\ t = 1$.
Iteration t:

1. Given ρ^{t-1}, find the set $I^{t-1} = \{i : \rho_i^{t-1} = 0\}$ of dual constraints that are tight (i.e. find the face on which ρ^{t-1} lies).
2. Given some non-tight constraint (i.e. some $i \notin I^{t-1}$), choose that variable as objective function to be maximized and solve the linear program $\max\{u_i : u \in U, u_i = 0\ i \in I^{t-1}\}$ to obtain an extreme ray u^t of U.
3. Calculate $\lambda^t = \max\{\lambda : \rho^{t-1} - \lambda u^t \in U\}$.
4. Set $\rho^t = \rho^{t-1} - \lambda^t u^t$.
5. Stop if $\rho^t = 0$.
6. Increase t.

On termination at iteration T, we have $u^* = {}_{t=1}^{T} \lambda_t u^t$ with $\lambda_t > 0$ and u^t an extreme ray of U for all t.

Let $Y = \text{proj}_x(Q)$ and $Y^{\text{opt}} = \{x \in Y, cx = cx^*\}$ be the set of optimal solutions in the original variable space.

Proposition 2 *The set of optimal solutions in the original variable space is equal to* x^* *plus the projection of the lineality space of* Q*:*

$$Y^{\text{opt}} = \{x : x = x^* + \tilde{x}, A\tilde{x} = 0, B\tilde{x} + D\tilde{w} = 0\}$$

Proof The direction $\supseteq$ directly follows from the feasibility of x^*. For the other direction, suppose $(x, w) = (x^*, w^*) + (\tilde{x}, \tilde{w}) \in Y$ and $cx = cx^*$. As $(x, w) \in Y,\ Ax^* = a$ and $Bx^* + Dw^* = d$, it follows that $A\tilde{x} \geq 0$ and $B\tilde{x} + D\tilde{w} \geq 0$. Suppose that there exists such an optimal solution with $A^i\tilde{x} > 0$ or $B^i\tilde{x} + D^i\tilde{w} > 0$ for some row i. Then, $cx - cx^* = c\tilde{x} = v^*A\tilde{x} + u^*B\tilde{x} > v^*0 + u^*(0 - D\tilde{w}) = 0$, where the strict inequality follows from $(v^*, u^*) > 0$, contradicting the optimality of (x, w). The claim follows. $\square$

Consider now the dual cone $U = \{u \in \mathbb{R}_+^{q_2} : uD = 0\}$ of dimension r. As $u^* > 0,\ u^* \in \text{relint}(U)$, by Carathéodory's theorem u^* can be expressed in the form $u^* = {}_{j=}^{r} \lambda_j^* u^j$ with $\lambda^* > 0$ and $u^1, \ldots, u^r$ extreme rays of U spanning the linear subspace $L_U = \{u \in \mathbb{R}^{q_2} : uD = 0\}$.

Let

$$R = \{x \in \mathbb{R}^n : Ax \geq a, u^jBx \geq u^jd\ \ j = 1, \ldots, r\}, \text{ and}$$

let $R^{\text{opt}} = R \cap \{x : cx = cx^*\}$.

Claim 1. $\text{proj}_x Q \subseteq R$.
Claim 2. $u^j Bx^* = u^j d$ for $j = 1, \dots, r$.

Specifically one has $0 = u^*(d - Bx^* - Dw^*) = u^*(d - Bx^*) = \sum_{j=1}^{r} \lambda_j^* u^j (d - Bx^*)$. As $\lambda^* > 0$ and $u^j Bx^* \geq u^j d$ for $j = 1, \dots, r$, the claim follows.

Proposition 3

$$Y^{\text{opt}} = R^{\text{opt}}.$$

Proof As $Ax^* = a$ and $u^j Bx^* = u^j d$ for $j = 1, \dots, r$, $R = \{x : x = x^* + \tilde{x}, A\tilde{x} \geq 0, u^j B\tilde{x} \geq 0 \text{ for } j = 1, \dots, r\}$. As in the proof of Proposition 2, if $A^i\tilde{x} > 0$ or $u^j B\tilde{x} > 0$ for some i or j, then $c\tilde{x} = v^* A\tilde{x} + \sum_{j=1}^{r} \lambda_j^* u^j B\tilde{x} > 0$, contradicting optimality.

Thus,

$$\begin{aligned} R^{\text{opt}} &= \{x : x = x^* + \tilde{x}, A\tilde{x} = 0, u^j B\tilde{x} = 0\} \\ &= \{x : x = x^* + \tilde{x}, A\tilde{x} = 0, B\tilde{x} \perp L_D\} \\ &= \{x : x = x^* + \tilde{x}, A\tilde{x} = 0, B\tilde{x} = D\mu\} \\ &= Y^{\text{opt}}, \end{aligned}$$

where the second equality follows from the fact that $u^1, \dots, u^r$ span L_U. □

Thus, Proposition 3 shows that the family of inequalities generated by our decomposition algorithm is rich enough to satisfy (i) above. Looking more closely at the proof of the previous proposition, one crucial element is that $u^1, \dots, u^r$ span the linear subspace $L_U = \{u \in \mathbb{R}^{q_2} : uD = 0\}$. We show next how to weaken this condition to obtain a sufficient and necessary condition for $Y^{\text{opt}} = R^{\text{opt}}$.

Each u in the dual cone U gives rise to a valid inequality $uBx \geq ud$. However, two distinct u^1, u^2 might give rise to the same inequality, namely if $(u^1 - u^2)B = 0$, or equivalently if $(u^1 - u^2)$ is perpendicular to the image of the linear operator B (ImB, equivalently the space generated by the columns of B). This shows that projecting u orthogonally onto the subspace ImB will not modify the inequality obtained.

So let us define $L_V = \text{proj}_{ImB}(L_U)$, $V = \text{proj}_{ImB}(U)$ and $v^* = \text{proj}_{ImB}(u^*)$. We have the two following results.

Proposition 4 *$uBx \geq ud$ is a facet of $conv(X)$ if and only if $proj_{ImB}(u)$ is an extreme ray of the cone V.*

Proposition 5 *Let $\{v^t\}_{t=1,\dots,r}$ be points in V such that $v^* = \sum_{j=}^{r} \lambda_j v^j$ with $\lambda > 0$. Then $Q^{\text{opt}} = R^{\text{opt}}$ if and only if $v^1, \dots, v^r$ span the linear subspace L_V.*

To prove these two propositions, we need the following useful lemma. It essentially says that different rays of the cone V yield different valid inequalities for X.

Lemma 6 *If $v^1 B = v^2 B$ and $v_1, v_2 \in ImB$, then $v^1 = v^2$.*

Proof $v^1, v^2 \in ImB$ means $\exists w^1, w^2$ such that $v^1 = Bw^1$ and $v^2 = Bw^2$. Substituting for v^i yields $w^1 B^T B = w^2 B^T B$ or equivalently $(w^1 - w^2) B^T B = 0$. Multiplying by $(w^1 - w^2)$ yields $(w^1 - w^2) B^T B (w^1 - w^2) = 0$ or $[B(w^1 - w^2)]^T B(w^1 - w^2) = 0$ which is only possible if $B(w^1 - w^2) = 0$. Substituting back v^1 and v^2 gives the desired result. □

Proof of Proposition 4 $\Rightarrow$ Suppose that $v = \text{proj}_{ImB}(u)$ is not an extreme ray of the cone V so that there exist $v^1, v^2 \in V$ with $v^1 \neq v^2$ such that $v = \lambda v^1 + (1-\lambda) v^2$ with $0 < \lambda < 1$ and therefore $vB = \lambda v^1 B + (1-\lambda) v^2 B$. By Lemma 6, $v^1 B \neq v^2 B$ and therefore $uBx \geq ud$ cannot be a facet.

$\Leftarrow$ Suppose that $uBx \geq ud$ for $u \in U$ is not a facet of $conv(X)$ so that there exist u^1, u^2 in U such that $uB = \lambda u^1 B + (1-\lambda) u^2 B$ with $0 < \lambda < 1$ and $u^1 B \neq u^2 B$. Let $v = \text{proj}_{ImB}(u)$, $v^1 = \text{proj}_{ImB}(u^1)$ and $v^2 = \text{proj}_{ImB}(u^2)$. Clearly $vB = \lambda v^1 B + (1-\lambda) v^2 B = (\lambda v^1 + (1-\lambda) v^2) B$ and therefore by Lemma 6, $v = \lambda v^1 + (1-\lambda) v^2$. Since also $v^1 \neq v^2$ (otherwise we would have $u^1 B = u^2 B$), v is not an extreme ray of V. □

Proof of Proposition 5 Proposition 4 implies that there is a linear bijection between points in the cone V and valid inequalities of the form $uBx \geq ud$ for $u \in U$. The result follows. □

These two propositions characterize the individual strength of inequalities (being a facet or not) and collective strength of a family, respectively. However, they both rely on V which is defined as the projection of U onto the subspace ImB. Unfortunately, this projection step cannot be carried out efficiently from an algorithmic point of view.

The multi-commodity reformulation of a two-level production–transportation problem

We consider a single-item 2-level uncapacitated production/transportation problem. The time horizon is discretized in a finite number of periods, and for each period t and client c, a nonnegative demand d_t^c has to be met. Production is possible at several production sites, and products are then transported to the clients. Holding inventory is possible both at the production facilities and at the client sites. There are fixed costs associated with production and transportation. A standard formulation uses the following variables:

x_t^p production level at facility p in period t,
y_t^p binary setup variable associated with production level x_t^p,
s_t^p inventory level at the end of period t at facility p,
X_t^{pc} quantity transportated from facility p to client c in period t,
Y_t^{pc} binary setup variable associated with X_t^{pc},
S_t^c inventory level at the end of period t at client site c.
X^{2LS} is the feasible set of the problem defined as follows:

$$\min \sum_{p,k} b_k^p x_k^p + \sum_{p,c,k} B_k^{pc} X_k^{pc} + \sum_{p,k} q_k^p y_k^p + \sum_{p,c,k} Q_k^{pc} Y_k^{pc} \tag{7}$$

$$s^p_{t-1} + x^p_t = \sum_{c=1}^{\mathrm{NC}} X^{pc}_t + s^p_t \quad \forall p, t, \tag{8}$$

(2LS)

$$S^c_{t-1} + \sum_{p=1}^{\mathrm{NP}} X^{pc}_t = d^c_t + S^c_t \quad \forall c, t, \tag{9}$$

$$x^p_t \le M y^p_t \quad \forall p, t, \tag{10}$$

$$X^{pc}_t \le M Y^{pc}_t \quad \forall p, c, t, \tag{11}$$

$$y \in \{0,1\}^{\mathrm{NP}\times\mathrm{NT}}, \quad Y \in \{0,1\}_+^{\mathrm{NP}\times\mathrm{NC}\times\mathrm{NT}}. \tag{12}$$

We assume that initial stocks are zero in both the production sites and at the clients. Note that should inventory costs appear explicitly in the objective function, they can be eliminated by substituting a standard linear expression in the variables x, X and the demands d.

Multi-commodity formulation

We now present a multi-commodity reformulation for the set $X^{2\mathrm{LS}}$. For each pair c, t, the product used to satisfy the demand d^c_t is viewed as a distinct product. Thus, we introduce the following variables:

- w^{pc}_{tu} is the amount produced in period t at production site p to satisfy the demand of client c in period u,
- W^{pc}_{tu} is the amount transported from production site p to client c in period t to be used to satisfy the demand in period u of client c,
- σ^{pc}_{tu} is the amount stocked at production site p at the end of period t to be used to satisfy the demand in period u of client c,
- Σ^c_{tu} is the amount stocked at client c at the end of period t to satisfy the demand of period u.

The resulting multi-commodity formulation (MC) is as follows:

(MC)

$$\begin{aligned}
&\sigma^{pc}_{t-1,u} + w^{pc}_{tu} = W^{pc}_{tu} + \sigma^{pc}_{tu} \quad \forall\, p, c, t, u, \quad \text{with } t \le u,\\
&\Sigma^c_{t-1,u} + \sum_p W^{pc}_{tu} = d^c_u \delta_{tu} + \Sigma^c_{tu} \quad \forall\, c, t, u, \quad \text{with } t \le u,\\
&w^{pc}_{tu} \le d^c_u y^p_t \quad \forall\, p, c, t, u, \quad \text{with } t \le u,\\
&W^{pc}_{tu} \le d^c_u Y^{pc}_t \quad \forall\, p, c, t, u, \quad \text{with } t \le u,\\
&x^p_t = \sum_{c=1}^{NC} \sum_{u=t}^{NT} w^{pc}_{tu} \quad \forall\, p, t,\\
&z^{pc}_t = \sum_{u=t}^{\mathrm{NT}} W^{pc}_{tu} \quad \forall\, p, c, t,\\
&w^{pc}_{tu}, W^{pc}_{tu}, \Sigma^c_{tu}, \sigma^{pc}_{tu} \in \mathbb{R}^1_+ \quad \forall\, p, c, t, u, \quad \text{with } t \le u,\\
&y \in \{0,1\}^{\mathrm{NP}\times\mathrm{NT}}, \quad Y \in \mathbb{Z}_+^{\mathrm{NP}\times\mathrm{NC}\times\mathrm{NT}},
\end{aligned}$$

where $\delta_{tu} = 1$ if $t = u$ and $\delta_{tu} = 0$ otherwise. Here, the first two sets of constraints are flow balance constraints at production sites and client areas, the next two are the tightened variable upper bound constraints, and the last two link the original production and transportation variables to the corresponding multi-commodity variables.

Eliminating the stock variables, we obtain the equivalent formulation with which we will work:

$$
\begin{aligned}
\min \sum_{p,k} b_k^p x_k^p + \sum_{p,c,k} B_k^{pc} X_k^{pc} + \sum_{p,k} q_k^p y_k^p + \sum_{p,c,k} Q_k^{pc} Y_k^{pc} & \\
x_k^p - \sum_{c=1}^{\mathrm{NC}} \sum_{t=k}^{n} d_t^c w_{kt}^{pc} \geq 0 \quad \forall\, p, k, & \qquad (u_k^p) \\
X_k^{pc} - \sum_{t=k}^{n} d_t^c W_{kt}^{pc} \geq 0 \quad \forall\, p, c, k, & \qquad (U_k^{pc}) \\
y_k^p - w_{kt}^{pc} \geq 0 \quad \forall\, p, c, k, t, \quad \text{with } k \leq t, & \qquad (v_{kt}^{pc}) \\
Y_t^{pc} - W_{tu}^{pc} \geq 0 \quad \forall\, p, c, k, t \quad \text{with } k \leq t, & \qquad (V_{kt}^{pc}) \\
\sum_{k=1}^{\tau} w_{kt}^{pc} - \sum_{k=1}^{\tau} W_{kt}^{pc} \geq 0 \quad \forall\, p, c, \tau, t, \quad \text{with } \tau \leq t, & \qquad (\gamma_{\tau t}^{pc}) \\
\sum_{p} \sum_{k=1}^{t} w_{kt}^{1pc} \geq 1 \quad \forall\, c, t, & \qquad (\delta_t^c) \\
w_{kt}^{pc}, w_{kt}^{pc}, \in \mathbb{R}_+^1 \quad \forall\, p, c, t, u, \quad \text{with } k \leq t, & \\
y \in \mathbb{R}_+^{\mathrm{NP}\times\mathrm{NT}}, \quad Y \in \mathbb{R}_+^{\mathrm{NP}\times\mathrm{NC}\times\mathrm{NT}}, &
\end{aligned}
$$

The dual is

$$
\begin{aligned}
\max \sum_c \sum_t \delta_t^c & \\
u_k^p \leq b_k^p \quad \forall\, p, k & \\
U_k^{pc} \leq B_k^{pc} \quad \forall\, p, c, k & \\
\sum_c \sum_{t=k}^{n} v_{kt}^{pc} \leq q_k^p \quad \forall\, p, k & \\
\sum_{t=k}^{n} V_{kt}^{pc} \leq Q_k^{pc} \quad \forall\, p, c, k & \\
-d_t^c u_k^p - v_{kt}^{pc} + \sum_{\tau=k}^{t} \gamma_{kt}^{pc} \leq 0 \quad \forall\, p, c, k, t & \\
-d_t^c U_k^p - V_{kt}^{1pc} - \sum_{\tau=k}^{t} \gamma_{kt}^{pc} + \delta_t^c \leq 0 \quad \forall\, p, c, k, t & \\
u, U, v, V, \gamma, \delta \geq 0 &
\end{aligned}
$$

Now given an optimal solution $(u^*, U^*, v^*, V^*, \gamma^*, \delta^*)$, using the Decomposition Algorithm described above, we decompose it as a nonnegative combination of the extreme rays of the following cone:

$$-d_t^c u_k^p - v_{kt}^{pc} + \sum_{\tau=k}^{t} \gamma_{kt}^{pc} \leq 0 \quad \forall\, p, c, k, t$$

$$-d_t^c U_k^p - V_{kt}^{pc} - \sum_{\tau=k}^{t} \gamma_{kt}^{pc} + \delta_t^c \leq 0 \quad \forall\, p, c, k, t$$

$$u, U, v, V, \gamma, \delta \geq 0$$

On termination at iteration T,

$$(u^*, U^*, v^*, V^*, \gamma^*, \delta^*) = \sum_{t=1}^{T} \lambda_t \sigma^t$$

Taking $\sigma^t = (\bar{u}, \bar{U}, \bar{v}, \bar{V}, \bar{\gamma}, \bar{\delta})$, the corresponding valid inequality is

$$\sum_p \sum_t \bar{u}_t^p x_t^p + \sum_p \sum_c \sum_t \bar{U}_t^{pc} X_t^{pc} + \sum_p \sum_k \sum_c \sum_{t=k}^{n} \bar{v}_{kt}^{pc} \quad y_k^p$$
$$+ \sum_p \sum_c \sum_k \sum_{t=k}^{n} \bar{V}_{kt}^{pc} \quad Y_k^{pc} \geq \sum_c \sum_t \bar{\delta}_t^c.$$

Example 7 The instance involves one production site, two clients and three periods. $d^1 = (8, 3, 6)$ and $d^2 = (12, 7, 13)$. The formulation produced by the decomposition algorithm is

Minimize $x_1^1 + 0.5x_2^1 + 0.1x_3^1 + 0.5X_1^{11} + 0.2X_2^{11} + 0.9X_3^{11} + X_1^{12} + 0.4X_2^{12} + 0.2X_3^{12} + y_1^1 + 10y_2^1 + 16y_3^1 + Y_1^{11} + 5Y_2^{11} + 4Y_3^{11} + Y_1^{12} + 14Y_2^{12} + 3Y_3^{12}$

$$cut(1): \quad x_1^1 + x_3^1 + 22y_2^1 + 7Y_2^{12} \geq 49$$
$$cut(2): \quad x_1^1 + x_2^1 + X_3^{11} + 13y_3^1 \geq 49$$
$$cut(3): \quad x_1^1 + x_2^1 + 19y_3^1 \geq 49$$
$$cut(4): \quad x_1^1 + X_3^{11} + 16y_2^1 + 13y_3^1 + 6Y_2^{11} + 7Y_2^{12} \geq 49$$
$$cut(5): \quad x_1^1 + 22y_2^1 + 19y_3^1 + 7Y_2^{12} \geq 49$$
$$cut(6): \quad X_1^{12} + X_2^{12} + X_3^{12} \geq 32$$
$$cut(7): \quad X_1^{12} + X_2^{12} + 13Y_3^{12} \geq 32$$
$$cut(8): \quad X_1^{12} + 7Y_2^{12} \geq 19$$
$$cut(9): \quad X_1^{12} + 20Y_2^{12} + 13Y_3^{12} \geq 32$$
$$cut(10): \quad X_1^{11} + X_2^{11} + X_3^{11} \geq 17$$

$$cut(11): \quad X_1^{11} + X_3^{11} + 9Y_2^{11} \geq 17$$

$$cut(12): \quad y_1^1 \geq 1$$

$$cut(13): \quad Y_1^{12} \geq 1$$

$$cut(14): \quad Y_1^{11} \geq 1$$

In Rardin and Wolsey (1993), showed that the projection of the multi-commodity formulation for a single item, single source fixed charge network flow problem consists of "dicut" inequalities. Thus, each of the inequalities above has a simple explanation. For example, letting d_{tu}^c denote $\quad_{\ell} d_{\ell}^c$, cut(1) can be written as follows:

$$x_1^1 + d_2^1 Y_2^1 + (d_3^1 + d_{23}^2) y_2^1 + x_3^1 \geq d_{13}^1 + d_{13}^2,$$

where the demand d_2^2 is covered by the terms $x_1^1 + d_2^2 y_2^1$, the demand d_2^1 by the terms $x_1^1 + d_2^1 Y_2^1$, etc.

Final remarks

The projection procedure outlined above provides a tool to display a variety of inequalities in the original space. We have not yet succeeded in demonstrating the viability of our approach for our second objective, namely providing a practically efficient computational tool for the solution of instances for which the extended formulation is too large for branch-and-bound/cut. Here certain difficulties still need to be overcome. The first is that the number of extreme rays/cuts generated is very large and time consuming as a linear program has to be solved to generate each cut. This raises the question of whether it is possible to order the generation of the cuts, so that the most important cuts (for bound improvement) are generated first. More generally one can ask whether there are alternatives to the Schrijver algorithm for expressing a ray of a polyhedral cone as a positive combination of its extreme rays.

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