

Part II

Institutional Asset Liability Management and Mortality risk.

Chapter 4

A martingale approach applied to the management of pension liabilities.

In the present chapter and the followings, we adopt the point of view of an institutional investor managing pension liabilities and exposed to mortality risk. Our purpose is here to determine the optimal assets allocation of pure endowment insurance contracts, maximizing the expected utility of terminal surplus under a budget constraint. This category of contracts is in fact the building block of the most of pension liabilities (e.g. such as annuities). Under the assumption that the insurer's deflator is well determined, the optimal target wealth process is obtained by martingale approach. Given that the market resulting from the combination of insurance and financial products is incomplete owing to the mortality, the terminal target wealth may only be partly hedged. We explore here two ways to find the best replicating strategy. The first one relies on the Kunita Watanabe decomposition whereas the second approach is based upon dynamic programming. Finally, we illustrate this method for CARA and CRRA utility functions. The purpose of chapter 4 being rather methodological, we have voluntarily worked in a simplified model. The method developed will be next applied to more complex issues met by pension funds in chapter 5 and 6.

4.1 Introduction.

Due to the presence of mortality risk, which is not yet hedgeable by traditional financial tools, the combination of life insurance and financial markets is incomplete and standard utility optimization methods have to be used with care. The contribution of this chapter is precisely to show how the celebrated martingale approach, developed by Cox & Huang (1989) may be used to handle assets allocation problems in pension business.

Two classical ways are usually exploited to study the optimal assets allocation of pension liabilities. The first one is the martingale approach, already mentioned above. However, the existing literature based on this orientation, neglects the mortality risk. The interested reader may refer to Boulier et al. (2001), Deelstra et al. (2003, 2004) for examples of management and design of a pension fund. The second method relies on stochastic control and the resolution of the Hamilton Jacobi Bellman equation. This approach was successfully applied to the management and pricing of a wide variety of insurance contracts with exponential utility (CARA). Some results of this chapter will be compared with those obtained by Young and

Zariphopoulou (2002). Another application of stochastic control is the management of one life annuity studied by Menoncin et al. (2004).

In this chapter, the martingale method is applied in a simplified setting: the financial market is composed of two assets (cash, stocks) and the mortality, source of incompleteness, is modelled by a Poisson process. On the liability side, we have pure endowment policies of same maturity and guarantee. Under the assumption that the risk neutral measure of the global market (financial and actuarial) is equal to the product of the financial risk neutral measure and of the historical actuarial measure, the expected utility of a terminal surplus is maximized under a budget constraint, by the approach of Lagrange multipliers. The surplus is defined as the difference between a target terminal wealth and the sum of capitals paid out to alive affiliates. The budget constraint guarantees the actuarial equilibrium between the insurer's current wealth and the expected future benefits. The self financed investment strategy replicating at best the optimal wealth process is next obtained either by martingale decomposition of the wealth process or either by dynamic programming.

The last part of this work is devoted to examples. In the first application, we consider an exponential (CARA) utility. The value function and the optimal investment policy are compared with the results of Young and Zariphopoulou (2002) obtained by direct resolution of the Bellman equation. Next, the case of a power (CRRA) utility is addressed. At our knowledge, solutions for such kind of utility were not yet developed in the context of the management of pension liabilities. A comparison reveals that the choice of a power utility leads to an optimal assets allocation relatively well adapted to ALM purposes.

The outline of the chapter is as follows. The portfolio of life insurances and the financial market are described in section 2 and 3. The choice of the deflator is discussed in section 4. Next, the optimization problem and the dynamic of the fund are presented. In section 6, we establish a general solution and two ways to infer the optimal assets allocation. In section 7 and 8, particular cases of CARA and CRRA utilities are studied.

4.2 Liabilities.

We consider a portfolio of pure life endowments, that counts initially n_x affiliates of age x . The insurance company will deliver a fixed capital K to each individual who attains the age $x + T$. As in Møller (1998), remaining lifetimes are assumed independent and identically distributed exponential random variables, noted $T_1, T_2, \dots, T_{n_x}$ defined on a probability space $(\Omega^a, \mathcal{F}^a, P^a)$. At time t , the hazard rate of T_i , also called the mortality rate of the affiliates, is a deterministic function written $\mu(x + t)$. The total number of deaths at instant t is noted N_t and defined by:

$$N_t = \sum_{i=1}^{n_x} I(T_i \leq t)$$

Where $I(\cdot)$ is an indicator variable. The actuarial filtration $\mathcal{F}^a$ is the one generated by N_t . The stochastic intensity of N_t is formally described as follows:

$$\mathbb{E}(dN_t | \mathcal{F}_{t-}^a) = (n_x - N_{t-}) \cdot \mu(x + t) \cdot dt$$

At time t , the compensated process of N_t :

$$M_t = N_t - \int_0^t (n_x - N_{u-}) \cdot \mu(x+u) \cdot du \quad (4.2.1)$$

is a martingale under P^a . Remark that, at maturity, the total payment done by the fund is worth $(n_x - N_T) \cdot K$ and the expectation at time $t \leq T$ of this total cash flow is:

$$\begin{aligned} \mathbb{E}((n_x - N_T) \cdot K | \mathcal{F}_t^a) &= \mathbb{E} \left(\sum_{i=1}^{n_x} I(T_i > T) | \mathcal{F}_t^a \right) \cdot K \\ &= \sum_{T_i > t} \mathbb{E}(I(T_i > T) | \mathcal{F}_t^a) \cdot K \\ &= (n_x - N_t) \cdot \underbrace{\exp \left(- \int_t^T \mu(x+u) \cdot du \right)}_{T-t p_{x+t}} \cdot K \end{aligned}$$

$T-t p_{x+t}$ is the real probability that an individual of age $x+t$, survives till age $x+T$.

4.3 Assets.

We consider a financial market composed of one risky asset, S_t (stocks) and one risk-less asset, a cash account, which provides a constant return r . Stocks S_t are driven by a geometric Brownian motion:

$$\frac{dS_t}{S_t} = (r + \nu) \cdot dt + \sigma \cdot dW_t^{P^f}$$

Where $W_t^{P^f}$, ν , σ are respectively a Brownian motion, the constant risk premium and the constant volatility of stocks. The financial probability space is noted $(\Omega^f, \mathcal{F}^f, P^f)$ where $\mathcal{F}^f$ is the filtration generated by $W_t^{P^f}$. The financial market is complete and there exists an unique equivalent measure, the risk neutral measure written Q^f , such that the discounted asset prices are martingales under Q^f . Let $\lambda = \frac{\nu}{\sigma}$ be the cost of the risk of stocks. Under Q^f , the stocks obey to the following SDE:

$$\begin{aligned} \frac{dS_t}{S_t} &= r \cdot dt + \sigma \cdot \underbrace{(dW_t^{P^f} + \lambda \cdot dt)}_{dW_t^{Q^f}} \\ &= r \cdot dt + \sigma \cdot dW_t^{Q^f} \end{aligned}$$

Where $W_t^{Q^f}$ is a Brownian motion under the financial risk neutral measure.

4.4 Deflators.

Let $(\Omega, \mathcal{F}, P)$ be the probability space resulting from the combination of the insurance and financial markets.

$$\Omega = \Omega^a \times \Omega^f \quad \mathcal{F} = \mathcal{F}^a \otimes \mathcal{F}^f \vee \mathcal{N} \quad P = P^a \times P^f$$

Where the sigma algebra $\mathcal{N}$ is generated by all subsets of null sets from $\mathcal{F}^a \otimes \mathcal{F}^f$. This global market is incomplete owing to the presence of the mortality, a non hedgeable risk. Then, there exists multiple equivalent martingale measures under which the discounted prices of treadable assets are martingale. For the sake of simplicity, we assume that the global risk neutral measure, noted Q , is equal to the product of the financial risk neutral measure and of the historical actuarial measure P^a . Note that this measure is known in the financial literature as the minimal measure, introduced by Föllmer and Sondermann (1986). This assumption is commonly accepted by actuaries, who relies on diversification to hedge the mortality risk. The deflator associated to pension benefits is, in this context, equal to the financial deflator. At time t , for a claim occurring at T , we note it $H(t, T)$.

$$\begin{aligned} H(t, T) &= e^{-\int_t^T r \cdot du} \cdot \frac{\left(\frac{dQ^f}{dP^f}\right)_T}{\left(\frac{dQ^f}{dP^f}\right)_t} \\ &= \exp\left(-\int_t^T r \cdot du - \frac{1}{2} \cdot \int_t^T \lambda^2 \cdot du - \int_t^T \lambda \cdot dW_u^P\right) \end{aligned}$$

And the conditional expectation of the deflator is equal to the price of a zero coupon: $\mathbb{E}(H(t, T)|\mathcal{F}_t) = \exp(-r \cdot (T - t))$. In the next chapter, one will provide examples of deflator defining an actuarial measure different from the real measure.

4.5 The dynamic of the fund and the optimization problem.

The asset manager optimizes the investment policy so as to maximize the utility of the surplus at instant T . This surplus is the difference between a total target asset $\tilde{X}_T$ and the total of capitals paid out to survivors. In particular, the value function $V(t, x, n)$ at time t , for a wealth x and for n observed deceases is:

$$V(t, x, n) = \sup_{\tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(U \left(\tilde{X}_T - (n_x - N_T) \cdot K \right) \mid \underbrace{\tilde{X}_t = x, N_t = n}_{\mathcal{F}_t} \right) \quad (4.5.1)$$

The utility function is strictly increasing and concave. The optimal terminal wealth belongs to the set $\mathcal{A}_t(x)$ which is delimited by a feasibility condition stipulating that the expectation of the deflated terminal wealth is at most equal to the current wealth.

$$\mathcal{A}_t(x) = \left\{ \tilde{X}_T \text{ such that } \mathbb{E} \left(H(t, T) \cdot \tilde{X}_T \mid \mathcal{F}_t \right) \leq x \right\} \quad (4.5.2)$$

In the sequel, this constraint is called the budget constraint. This condition corresponds to a current actuarial practice which consists in imposing that the reserve of the company is at least equal to the expectation of future benefits. To avoid any confusion, we insist on the fact that $\tilde{X}_T$ is not necessary self financed (the optimal target wealth can depends on mortality which is not hedgeable).

Under the assumption that the fund is closed (no cash in or cash out), the optimal investment strategy, replicating at best the optimal target wealth, is obtained by projecting the optimal target wealth into the space of self financed processes. If the self financed wealth process and

the fraction of the fund invested in stocks are respectively noted X_t and π_t , X_t obeys the following SDE:

$$dX_t = (r + \pi_t \nu) \cdot X_t \cdot dt + \pi_t \sigma \cdot X_t \cdot dW_t^P$$

As explained in the next section, two methods of projection are available: one based on the Kunita Watanabe decomposition of $\tilde{X}_T$ and the other on dynamic programming.

We draw the attention of the reader on the fact that if the optimization is done on a domain of controls restricted to the set of replicable terminal wealth, $\mathcal{A}_t^\pi(x)$ rather than $\mathcal{A}_t(x)$

$$\mathcal{A}_t^\pi(x) = \left\{ X_T \mid \exists (\pi_t)_t \text{ } \mathcal{F}_t \text{ adapted} : e^{-r \cdot (T-t)} X_T = x + \int_t^T \frac{\pi_s \cdot X_s}{S_s} \cdot d \left(e^{-r \cdot (s-t)} \cdot S_s \right) \right\}$$

The expectation (4.7.7) is not always defined for a power utility, widely used for ALM purposes in the setting of complete markets. Intuitively, if the available asset, x , is insufficient to hedge the survival of all affiliates (that is nearly almost the case for life insurers), the probability of having a negative terminal surplus is not null and the the power utility of such negative surplus doesn't exist. Optimizing the utility on the enlarged set of controls $\mathcal{A}_t(x)$, allows us to avoid this drawback (results are developed in section 4.8).

4.6 A general solution.

Our method uses Lagrange multipliers. We refer to Karatzas & Shreve (1998) for a detailed presentation of this technique, developed by Cox Huang (1989) in complete markets. Let $y_t \in \mathbb{R}^+$ be the Lagrange multiplier associated to the budget constraint (4.5.2). The Lagrangian is defined by:

$$\begin{aligned} \mathcal{L}(t, x, n, \tilde{X}_T, y_t) = & \mathbb{E} \left(U \left(\tilde{X}_T - (n_x - N_T) \cdot K \right) \mid \mathcal{F}_t \right) \\ & + y_t \cdot \left(x - \mathbb{E} \left(H(t, T) \cdot \tilde{X}_T \mid \mathcal{F}_t \right) \right) \end{aligned} \quad (4.6.1)$$

The existence of the optimal terminal wealth, noted $\tilde{X}_T^*$, is guaranteed if we find an optimal Lagrange multiplier $y_t^* > 0$ such that $\tilde{X}_T^*$ is a saddle point of the Lagrangian (4.6.1). The value function verifies:

$$V(t, x, n) = \inf_{y \in \mathbb{R}^+} \left(\sup_{\tilde{X}_T} \mathcal{L}(t, x, n, \tilde{X}_T, y_t) \right)$$

Under the assumptions that the function $U(\cdot)$ is $\mathbb{C}^1$ strictly concave and increasing, the derivative of $U(\cdot)$ admits a continuous inverse function, noted $I(\cdot)$. It suffices to derive the expression (4.6.1) with respect to $\tilde{X}_T$, to obtain the optimal wealth process in function of the Lagrange multiplier

$$\tilde{X}_T^* = I(y_t^* \cdot H(t, T)) + (n_x - N_T) \cdot K \quad (4.6.2)$$

And the optimal Lagrange multiplier y_t^* is such that the budget constraint is binding i.e.

$$x = \mathbb{E} \left(H(t, T) \cdot (I(y_t^* \cdot H(t, T)) + (n_x - N_T) \cdot K) \mid \mathcal{F}_t \right)$$

Once the optimal Lagrange multiplier determined, the value function is also calculable:

$$V(t, x, n) = \mathbb{E} (U (I (y_t^*.H(t, T))) | \mathcal{F}_t) \quad (4.6.3)$$

The optimal terminal wealth (4.6.2), depends on the number of survivors at time T and clearly cannot be replicated by a portfolio of financial assets. However two possibilities are conceivable to infer the investment policy approaching at best $\tilde{X}_T^*$. The first one consists to decompose $(\tilde{X}_t^*)_t$ in a sum of an adapted process, of a Brownian martingale, and of a zero mean jump martingale. The second solution relies on dynamic programming.

4.6.1 Decomposition of $\tilde{X}_T^*$.

The following proposition is the key tool to establish the investment policy hedging partially $\tilde{X}_T^*$.

Proposition 4.6.1. *Let $\tilde{X}_T^*$ be an $\mathcal{F}_T^f \times \mathcal{F}_T^a$ measurable non negative random variable, such that*

$$\mathbb{E} \left(\tilde{X}_T^*.H(t, T) | \mathcal{F}_t \right) = x \quad (4.6.4)$$

Then there exists a predictable admissible portfolio π_t such that the associated final wealth X_T verifies

$$\mathbb{E} (X_T | \mathcal{F}_t) = \mathbb{E} \left(\tilde{X}_T^* | \mathcal{F}_t \right) \quad (4.6.5)$$

and minimizes the square of the spread between X_T and $\tilde{X}_T^$, under P :*

$$\pi_t \text{ minimizes } \mathbb{E} \left(\left(X_T - \tilde{X}_T^* \right)^2 | \mathcal{F}_t \right) \quad (4.6.6)$$

Proof. Under the probability measure Q , by the Tower property of conditional expectations, we know that

$$\begin{aligned} L_s &= \mathbb{E} \left(\tilde{X}_T^*.H(t, T) | \mathcal{F}_s \right) \\ &= \mathbb{E}^Q \left(\tilde{X}_T^*.e^{-\int_t^T r.du} | \mathcal{F}_s \right) \end{aligned}$$

with $s \geq t$, is a local martingale and hence a global supermartingale. Moreover, by assumption $\mathbb{E}^Q(L_T | \mathcal{F}_t) = L_t = x$ and L_s is in fact a martingale. Then, there exists a Kunita Watanabe decomposition of L_s (for a proof see Kunita Watanabe 1967). In particular there exist two progressively measurable processes φ_1 , φ_2 such that

$$L_s = L_t + \int_t^s \varphi_1(u).dW_u^{Q^f} + \int_t^s \varphi_2(u).dM_u$$

Where $\int_t^s \varphi_2(u).dM_u$ is a martingale orthogonal to the space of stochastic Brownian integral (M_u is the compensated process of N_u , see equation (4.2.1)). As $L_t = x$, we have that:

$$\tilde{X}_T^*.e^{-\int_t^T r.du} = x + \int_t^T \varphi_1(u).dW_u^{Q^f} + \int_t^T \varphi_2(u).dM_u$$

And, by comparison of this decomposition with the self financed wealth process,

$$X_T \cdot e^{-\int_t^T r \cdot du} = x + \int_t^T e^{-\int_t^u r \cdot dv} \cdot \pi_u \cdot X_u \cdot \sigma \cdot dW_u^{Q^f}$$

The strategy π_u replicating at best $\tilde{X}_T^*$ is:

$$\pi_u = \frac{\varphi_1(u)}{e^{-\int_t^u r \cdot dv} \cdot \sigma} \cdot \frac{1}{X_u} \quad \forall u \in [t, T] \quad (4.6.7)$$

By construction of π_t , assertion (4.6.5) is verified under P . For any other self financed strategy π'_u , such that

$$\pi'_u = \frac{\varphi'_1(u)}{e^{-\int_t^u r \cdot dv} \cdot \sigma} \cdot \frac{1}{X_u}$$

the expectation of the quadratic spread between X_T and Z under P becomes:

$$\begin{aligned} \mathbb{E} \left(\left(X_T - \tilde{X}_T^* \right)^2 \middle| \mathcal{F}_t \right) &= \\ &= e^{+2 \cdot \int_t^T r \cdot du} \cdot \mathbb{E} \left(\left(\int_t^T (\varphi'_1(u) - \varphi_1(u)) \cdot dW_u^{Q^f} + \int_t^T \varphi_2(u) \cdot dM_u \right)^2 \middle| \mathcal{F}_t \right) \\ &= e^{+2 \cdot \int_t^T r \cdot du} \cdot \underbrace{\mathbb{E} \left(\left(\int_t^T (\varphi'_1(u) - \varphi_1(u)) \cdot (dW_u^{P^f} + \lambda \cdot du) \right)^2 \middle| \mathcal{F}_t \right)}_{\geq 0} \\ &\quad + e^{+2 \cdot \int_t^T r \cdot du} \cdot \mathbb{E} \left(\left(\int_t^T \varphi_2(u) \cdot dM_u \right)^2 \middle| \mathcal{F}_t \right) \end{aligned} \quad (4.6.8)$$

As M_t the compensated process of N_t is identical under P and Q , the right hand term of this last expression is minimized when $\varphi'_1(u) = \varphi_1(u)$. $\square$

Remark 4.6.2. According to equation (4.6.8), the expectation $\mathbb{E} \left(\left(X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right)^2 \middle| \mathcal{F}_t \right)$ is minimized by the investment policy π_t , eq. 4.6.7, for any $\Delta t \in [0, T-t]$. If $t + \Delta t$ is the first exit time of $\left(X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right)$ from an interval $\mathcal{O}$ round $\left(X_t - \tilde{X}_t^* \right) = 0$, given that $f(x) = x^2$ is local lipschitz, we have that

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\left(X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right)^2 - \left(X_t - \tilde{X}_t^* \right)^2 \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E} \left[C^{\mathcal{O}} \cdot \left| \left(X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right) - \underbrace{\left(X_t - \tilde{X}_t^* \right)}_{=0} \right| \middle| \mathcal{F}_t \right] \end{aligned}$$

Where $C^{\mathcal{O}}$ is constant. Minimizing $\mathbb{E} \left[\left| X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right| \middle| \mathcal{F}_t \right]$ is therefore equivalent to minimizing $\mathbb{E} \left(\left(X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^* \right)^2 \middle| \mathcal{F}_t \right)$. This observation will be useful to prove the equivalence between the martingale decomposition and the dynamic programming approach.

4.6.2 Dynamic programming.

The second possibility to obtain the optimal investment policy, relies on dynamic programming (for an introduction, see Fleming and Rishel 1975). Given a small step of time, Δt , this principle states that

$$V(t, x, n) = \mathbb{E} \left(V(t + \Delta t, \tilde{X}_{t+\Delta t}^*, N_{t+\Delta t}) | \mathcal{F}_t \right)$$

Given that $(\tilde{X}_t^*)_t$ is the process maximizing the value function, any other process $(X_t)_t \neq (\tilde{X}_t^*)_t$ belonging to the set $\mathcal{A}_t(x)$, and in particular a replicable process, satisfies the inequality:

$$V(t, x, n) \geq \mathbb{E} (V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}) | \mathcal{F}_t) \quad (4.6.9)$$

And the closest process to $(\tilde{X}_t^*)_t$ is determined by an investment strategy maximizing the right hand term of (4.6.9). Indeed, the value function $V(t, x, n)$ is concave in x and then and then local Lipschitz in x :

$$\forall \mathcal{O} \subset \mathbb{R}, \exists C^{\mathcal{O}} \in \mathbb{R}^+ \mid x_1, x_2 \in \mathcal{O} \mid |V(t, x_1, n) - V(t, x_2, n)| \leq C^{\mathcal{O}} \cdot |x_1 - x_2|$$

And if $t + \Delta t$ is the first exit time of $\tilde{X}_{t+\Delta t}^*$ or $X_{t+\Delta t}$ from an interval $\mathcal{O}$ round x , the inequality (4.6.9) is bounded as follows:

$$\begin{aligned} 0 &\leq \mathbb{E} \left[V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}) - V(t + \Delta t, \tilde{X}_{t+\Delta t}^*, N_{t+\Delta t}) \mid \mathcal{F}_t \right] \\ &\leq \mathbb{E} \left[C^{\mathcal{O}} \cdot |X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^*| \mid \mathcal{F}_t \right] \end{aligned}$$

Where $C^{\mathcal{O}}$ is constant. Minimizing $\mathbb{E} [|X_{t+\Delta t} - \tilde{X}_{t+\Delta t}^*| \mid \mathcal{F}_t]$ is therefore equivalent to maximizing the right hand term of (4.6.9). Furthermore according remark 4.6.2, this demonstrates the equivalence with the martingale decomposition.

By application of the Itô's lemma for jump processes (see Øksendal and Sulem 2005, chapter one or Duffie 2001, annex F), the expectation of the value function at instant $t + \Delta t$ is given by:

$$\begin{aligned} \mathbb{E} (V(t + \Delta t, X_{t+\Delta t}, N_{t+\Delta t}) | \mathcal{F}_t) = & \\ & V(t, x, n) + \mathbb{E} \left(\int_t^{t+\Delta t} G^\pi(s, X_s, N_s) \cdot ds \mid \mathcal{F}_t \right) + \\ & \mathbb{E} \left(\int_t^{t+\Delta t} (V(s, X_s, N_s) - V(s, X_s, N_{s-})) \cdot dN_s \mid \mathcal{F}_t \right) \end{aligned} \quad (4.6.10)$$

Where $G^\pi(s, X_s, N_s)$ is the generator of the value function:

$$G^\pi(s, X_s, N_s) = V_s + (r + \pi_s \cdot \nu) \cdot X_s \cdot V_X + \frac{1}{2} \cdot \pi_s^2 \cdot \sigma^2 \cdot X_s^2 \cdot V_{XX} \quad (4.6.11)$$

V_s, V_X, V_{XX} are partial derivatives of the value function with respect to time and wealth. Deriving the generator with respect to π provides the investment strategy maximizing the right hand term of (4.6.9):

$$\pi_s = - \frac{\nu}{\sigma^2} \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_s} \quad (4.6.12)$$

As the value function is known, see equation (4.6.3), partial derivatives V_X and V_{XX} are easily calculable.

4.7 CARA utility.

We assume that the preferences of the asset manager are modelled by a CARA (constant absolute risk aversion) utility function with a risk aversion parameter noted α . The value function at time t is then rewritten:

$$V(t, x, n) = \sup_{\tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(-\frac{1}{\alpha} \exp \left(-\alpha \cdot \left(\tilde{X}_T - (n_x - N_T) \cdot K \right) \right) \middle| \mathcal{F}_t \right)$$

From equation (4.6.2), we infer the optimal terminal wealth:

$$\tilde{X}_T^* = -\frac{1}{\alpha} \cdot \ln(y_t^* \cdot H(t, T)) + (n_x - N_T) \cdot K \quad (4.7.1)$$

The value function at time t is easily calculable:

$$\begin{aligned} V(t, x, n) &= -\frac{1}{\alpha} \cdot y_t^* \cdot \mathbb{E}(H(t, T) | \mathcal{F}_t) \\ &= -\frac{1}{\alpha} \cdot y_t^* \cdot \exp(-r \cdot (T - t)) \end{aligned} \quad (4.7.2)$$

And the Lagrange multiplier is such that the budget constraint is saturated:

$$\begin{aligned} -\frac{1}{\alpha} \cdot \ln y_t^* &= \frac{1}{\mathbb{E}(H(t, T) | \mathcal{F}_t)} \cdot [(x - \mathbb{E}((n_x - N_T) \cdot H(t, T) \cdot K | \mathcal{F}_t)) \\ &\quad + \frac{1}{\alpha} \cdot \mathbb{E}(H(t, T) \cdot \ln H(t, T) | \mathcal{F}_t)] \end{aligned} \quad (4.7.3)$$

The independence between mortality and the financial market entails that:

$$\begin{aligned} \mathbb{E}((n_x - N_T) \cdot H(t, T) \cdot K | \mathcal{F}_t) &= K \cdot \mathbb{E}((n_x - N_T) | \mathcal{F}_t) \cdot \mathbb{E}(H(t, T) | \mathcal{F}_t) \\ &= K \cdot e^{-r \cdot (T-t)} \cdot (n_x - N_t) \cdot {}_{T-t}p_{x+t} \end{aligned}$$

After calculations (see appendix A for details), we get that:

$$\mathbb{E}(H(t, T) \cdot \ln H(t, T) | \mathcal{F}_t) = e^{-r \cdot (T-t)} \cdot \left[-\left(r - \frac{\lambda^2}{2} \right) \cdot (T - t) \right]$$

And inserting (4.7.3) in (4.7.2), leads to the following value function:

$$\begin{aligned} V(t, x, n) &= -\frac{1}{\alpha} \cdot \exp \left(-\alpha \cdot e^{r \cdot (T-t)} \cdot \underbrace{\left(x - e^{-r \cdot (T-t)} \cdot K \cdot (n_x - n) \cdot {}_{T-t}p_{x+t} \right)}_{\text{Equity}} \right) \\ &\quad \cdot \exp \left(-\frac{\nu^2}{2 \cdot \sigma^2} \cdot (T - t) \right) \end{aligned} \quad (4.7.4)$$

The value function is proportional to the difference between the current market of the fund and the discounted expected claims paid on time T . This amount may be seen as the equity of the fund. The optimal investment strategy is built either by decomposition of $\tilde{X}_T^*$ or either by dynamic programming.

4.7.1 Decomposition of $\tilde{X}_T^*$.

In this case the Kunita Watanabe decomposition is directly inferred from the form of $\tilde{X}_T^*$. The combination of expressions (4.7.1) and (4.7.3) gives that:

$$\tilde{X}_T^* = x \cdot \frac{1}{e^{-r(T-t)}} + \int_t^T \frac{1}{\alpha} \cdot \lambda \cdot dW_u^{Q^f} + K \cdot \underbrace{((n_x - N_T) - \mathbb{E}((n_x - N_T) | \mathcal{F}_t))}_{\int_t^T dM_t}$$

And the optimal investment policy is:

$$\pi_u = \frac{1}{\alpha} \cdot \frac{\lambda}{\sigma} \cdot \frac{e^{-r \cdot (T-t)}}{e^{-r \cdot (u-t)}} \cdot \frac{1}{X_u} \quad \forall u \in [t, T] \quad (4.7.5)$$

Contrary to the value function, the fraction of the fund invested in stocks at time t , is totally independent of the equity.

$$\pi_t = e^{-r \cdot (T-t)} \cdot \frac{1}{\alpha} \cdot \frac{\nu}{\sigma^2} \cdot \frac{1}{X_t} \quad (4.7.6)$$

4.7.2 Dynamic programming.

Deriving the value the value function with respect to x leads to:

$$\begin{aligned} \frac{\partial V(t, x, n)}{\partial x} &= -\alpha \cdot e^{r \cdot (T-t)} \cdot V(t, x, n) \\ \frac{\partial^2 V(t, x, n)}{\partial x^2} &= \left(\alpha \cdot e^{r \cdot (T-t)} \right)^2 \cdot V(t, x, n) \end{aligned}$$

The optimal assets allocation at time t is obtained by application of the formula (4.6.12) and is identical to the one calculated by the decomposition of $\tilde{X}_T^*$.

$$\begin{aligned} \pi_t &= -\frac{\nu}{\sigma^2} \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} \\ &= e^{-r \cdot (T-t)} \cdot \frac{1}{\alpha} \cdot \frac{\nu}{\sigma^2} \cdot \frac{1}{X_t} \end{aligned}$$

4.7.3 Relation with the Hamilton Jacobi Bellman equation.

V. Young and T. Zariphopoulou (2002) have already solved the problem of the management of one pure endowment insurance, for an exponential utility function, in a slightly different setting. More precisely, they restrict the set of admissible terminal wealth to the one of replicable processes. In this context, the value function, that we note $V^\pi(\cdot)$, is defined by:

$$V^\pi(t, x, n) = \sup_{X_T \in \mathcal{A}_t^\pi(x)} \mathbb{E}(U(X_T - (n_x - N_T) \cdot K) | \mathcal{F}_t) \quad (4.7.7)$$

Where $\mathcal{A}_t^\pi(x)$ is the set of attainable terminal wealth:

$$\mathcal{A}_t^\pi(x) = \left\{ X_T \mid \exists (\pi_t)_t \text{ } \mathcal{F}_t \text{ adapted} : e^{-r \cdot (T-t)} X_T = x + \int_t^T \pi_s \cdot X_s \cdot d \left(e^{-r \cdot (s-t)} \cdot S_s \right) \right\}$$

Restricting the set of admissible controls to $\mathcal{A}_t^\pi(x)$ entails that, if an optimal solution exists, the value function is solution of the HJB stochastic differential equation:

$$\begin{aligned} 0 = & V_s^\pi + \max_{\pi_s} \left((r + \pi_s \cdot \nu) \cdot X_s \cdot V_X^\pi + \frac{1}{2} \cdot \pi_s^2 \cdot \sigma^2 \cdot X_s^2 \cdot V_{XX}^\pi \right) \\ & + (n_x - N_s) \cdot \mu(x + s) \cdot \mathbb{E}(V^\pi(s, X_s, N_s + 1) - V^\pi(s, X_s, N_s)) \end{aligned} \quad (4.7.8)$$

With the following boundary condition:

$$V^\pi(T, x, n) = U(x - (n_x - n) \cdot K) \quad (4.7.9)$$

Whereas, as showed in paragraph 4.6.2, the value function $V(t, x, n)$ of the optimization problem on the enlarged set $\mathcal{A}_t(x)$ verifies only the stochastic differential inequality:

$$\begin{aligned} 0 \geq & V_s + \max_{\pi_s} \left((r + \pi_s \cdot \nu) \cdot X_s \cdot V_X + \frac{1}{2} \cdot \pi_s^2 \cdot \sigma^2 \cdot X_s^2 \cdot V_{XX} \right) \\ & + (n_x - N_s) \cdot \mu(x + s) \cdot \mathbb{E}(V(s, X_s, N_s + 1) - V(s, X_s, N_s)) \end{aligned} \quad (4.7.10)$$

When the portfolio counts one insurance policy, $n_x = 1$, the value function solution of the HJB equation found by V. Young and T. Zariphopoulou (2002) is slightly different from the one obtained by the martingale approach, (equation (4.7.4) with $n_x = 1$). If the affiliate is still alive at time t , their solution is:

$$\begin{aligned} V^\pi(t, x, 0) = & \\ & -\frac{1}{\alpha} \cdot \exp \left(-\alpha \cdot e^{r \cdot (T-t)} \cdot \left(x - e^{-r \cdot (T-t)} \cdot K \right) - \frac{\nu^2}{2 \cdot \sigma^2} \cdot (T-t) \right) \cdot {}_{T-t}p_{x+t} \\ & -\frac{1}{\alpha} \cdot \exp \left(-\alpha \cdot e^{r \cdot (T-t)} \cdot x - \frac{\nu^2}{2 \cdot \sigma^2} \cdot (T-t) \right) \cdot {}_{T-t}q_{x+t} \end{aligned}$$

If the individual deceases before t , the value function is the solution found by Merton (1969 and 1971), that maximizes the expected utility of terminal wealth.

$$V^\pi(t, x, 1) = -\frac{1}{\alpha} \cdot \exp \left(-\alpha \cdot e^{r \cdot (T-t)} \cdot x - \frac{\nu^2}{2 \cdot \sigma^2} \cdot (T-t) \right)$$

However, the optimal investment strategy is identical to the one obtained by maximization on the enlarged set $\mathcal{A}_t(x)$ (equation (4.7.6)).

To close this paragraph, we calculate the residue at time s , noted ϵ_s , of the Bellman equation for the solution found by martingale approach and check that it is negative. The combination of the optimal assets allocation (4.6.12) with the right hand term of (4.7.10), leads to the following definition of ϵ_s :

$$\begin{aligned} \epsilon_s = & V_s + r \cdot X_s \cdot V_X - \frac{1}{2} \cdot \frac{\nu^2}{\sigma^2} \cdot \frac{V_X^2}{V_{XX}} \\ & + (n_x - N_s) \cdot \mu(x + s) \cdot \mathbb{E}(V(s, X_s, N_s + 1) - V(s, X_s, N_s)) \end{aligned} \quad (4.7.11)$$

After simplification, we get that:

$$\begin{aligned} \epsilon_s = & \alpha \cdot K \cdot {}_{T-s}p_{x+s} \cdot (n_x - N_s) \cdot \mu(x + s) \cdot V(s, X_s, N_s) \\ & + (e^{-\alpha \cdot K \cdot {}_{T-s}p_{x+s}} - 1) \cdot (n_x - N_s) \cdot \mu(x + s) \cdot V(s, X_s, N_s) \end{aligned}$$

And by a development of Taylor of the exponential $e^{-\alpha.K.T-sp_{x+s}}$ round 0, we finally obtain a residue of the type:

$$\epsilon_s = e^{-\phi} \cdot (\alpha.K.T-sp_{x+s})^2 \cdot (n_x - N_s) \cdot \mu(x+s) \cdot V(s, X_s, N_s)$$

Where ϕ belongs to the interval $[-\alpha.K.T-sp_{x+s}, 0]$ and clearly the residue is negative.

4.8 CRRA utility.

In this section, preferences of the asset manager are reflected by a CRRA (constant relative risk aversion) of parameter $0 < \gamma < 1$. The optimization problem is only defined for positive terminal surplus.

$$V(t, x, n) = \sup_{\tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E} \left(\frac{1}{\gamma} \left(\tilde{X}_T - (n_x - N_T) \cdot K \right)^\gamma \middle| \mathcal{F}_t \right)$$

Again, we maximize the utility of the terminal surplus on the enlarged set of controls $\mathcal{A}_t(x)$ and project the optimal wealth process in the space of attainable wealth. We get that:

$$\tilde{X}_T^* = (y_t^* \cdot H(t, T))^{\frac{1}{\gamma-1}} + (n_x - N_T) \cdot K \quad (4.8.1)$$

The value function at time t is rewritten as a function of the optimal Lagrange multiplier:

$$V(t, x, n) = \frac{1}{\gamma} \cdot y_t^{*\frac{\gamma}{\gamma-1}} \cdot \mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right) \quad (4.8.2)$$

And the Lagrange multiplier is such that the budget constraint is saturated:

$$y_t^{*\frac{1}{\gamma-1}} = \frac{1}{\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right)} \cdot \left(x - K \cdot e^{-r \cdot (T-t)} \cdot (n_x - N_t) \cdot T-t p_{x+t} \right) \quad (4.8.3)$$

After calculations (see appendix B for details), we obtain that:

$$\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right) = e^{-\left(r \cdot \frac{\gamma}{\gamma-1} - \frac{1}{2} \cdot \lambda^2 \cdot \frac{\gamma}{(\gamma-1)^2} \right) \cdot (T-t)}$$

And inserting (4.8.3) in (4.8.2), leads to the following expression for the value function:

$$V(t, x, n) = \frac{1}{\gamma} \cdot \left(\underbrace{x - K \cdot e^{-r \cdot (T-t)} \cdot (n_x - n) \cdot T-t p_{x+t}}_{\text{equity}} \right)^\gamma \cdot \mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}_t \right)^{1-\gamma} \quad (4.8.4)$$

As in the CARA case, the value function depends on the equity of the fund. However, if the total asset is insufficient to cover the expected value of future claims, the value function is not defined and the optimization problem doesn't admit as solution. In the next two subsections, we build the optimal investment strategy by decomposition and dynamic programming.

4.8.1 Decomposition of $\tilde{X}_T^*$.

The Kunita Watanabe decomposition of $\tilde{X}_T^*$ requires more calculations than for a CARA utility. Firstly, we develop the discounted optimal terminal wealth at time t :

$$\begin{aligned} e^{-r.(T-t)}.\tilde{X}_T^* &= \left(x - K.e^{-r.(T-t)}.(n_x - N_t)._{T-t}p_{x+t} \right) \cdot \frac{e^{-r.(T-t)}.H(t, T)^{\frac{1}{\gamma-1}}}{\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}_t \right)} \\ &\quad + K.e^{-r.(T-t)}.(n_x - N_T) \end{aligned}$$

And after simplifications, we get that:

$$\begin{aligned} \frac{e^{-r.(T-t)}.H(t, T)^{\frac{1}{\gamma-1}}}{\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}_t \right)} &= \exp \left(-\frac{1}{2}.\lambda^2.\frac{1}{(\gamma-1)^2}.(T-t) - \lambda.\frac{1}{\gamma-1}.\int_t^T dW_u^{Q^f} \right) \\ &= \left(\frac{d\tilde{Q}}{dQ} \right)_T \bigg/ \left(\frac{d\tilde{Q}}{dQ} \right)_t \end{aligned}$$

Where $\frac{d\tilde{Q}}{dQ}$ defines a change of measure from the risk neutral measure Q to $\tilde{Q}$, a measure under which $dW_u^{\tilde{Q}} = dW_u^{Q^f} + \frac{\lambda}{\gamma-1}.du$ is a Brownian motion. The next step consists to derive the conditional expectation of the discounted terminal wealth $\tilde{X}_T^*$. This conditional expectation is noted L_s , for $s \geq t$:

$$\begin{aligned} L_s &= \mathbb{E}^Q \left(e^{-r.(T-t)}.\tilde{X}_T^* \mid \mathcal{F}_s \right) \\ &= \left(x - K.e^{-r.(T-t)}.(n_x - N_t)._{T-t}p_{x+t} \right) \\ &\quad \cdot \exp \left(-\frac{1}{2}.\lambda^2.\frac{1}{(\gamma-1)^2}.(s-t) - \lambda.\frac{1}{\gamma-1}.\int_t^s dW_u^{Q^f} \right) \\ &\quad + K.e^{-r.(T-t)}.\mathbb{E}^Q ((n_x - N_T) \mid \mathcal{F}_s) \end{aligned}$$

By use of the Itô's lemma for jumps processes, the dynamic of L_s is inferred:

$$\begin{aligned} dL_s &= - \left(x - K.e^{-r.(T-t)}.(n_x - N_t)._{T-t}p_{x+t} \right) \cdot \frac{\lambda}{\gamma-1} \cdot \frac{\left(\frac{d\tilde{Q}}{dQ} \right)_s}{\left(\frac{d\tilde{Q}}{dQ} \right)_t} \cdot dW_s^{Q^f} \\ &\quad -_{T-s}p_{x+s}.K.e^{-r.(T-t)}.dM_s \end{aligned}$$

And, by formula (4.6.7), the optimal investment strategy at time $s \geq t$ is:

$$\pi_s = -\frac{\lambda}{\sigma} \cdot \frac{1}{\gamma-1} \cdot \frac{\left(x - K.e^{-r.(T-t)}.(n_x - N_t)._{T-t}p_{x+t} \right)}{X_s} \cdot \frac{\left(\frac{d\tilde{Q}}{dQ} \right)_s}{\left(\frac{d\tilde{Q}}{dQ} \right)_t} \cdot \frac{1}{e^{-r.(s-t)}} \quad (4.8.5)$$

Contrary to the optimal assets allocation obtained with a CARA utility, the strategy is here function of the fund equity instead of the total assets.

4.8.2 Dynamic programming.

For the CRRA utility, to obtain the optimal assets allocation by dynamic programming is relatively easier than by decomposition. Indeed, it suffices to calculate the first and second order derivatives of the value function with respect to x .

$$\begin{aligned}\frac{\partial V(t, x, n)}{\partial x} &= \left(x - K.e^{-r.(T-t)}.(n_x - n)._{T-t}p_{x+t} \right)^{\gamma-1} \\ &\quad \cdot \mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right)^{1-\gamma} \\ \frac{\partial^2 V(t, x, n)}{\partial x^2} &= (\gamma - 1) \cdot \left(x - K.e^{-r.(T-t)}.(n_x - n)._{T-t}p_{x+t} \right)^{\gamma-2} \\ &\quad \cdot \mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_t \right)^{1-\gamma}\end{aligned}$$

The optimal assets allocation at time t is next obtained by application of the formula (4.6.12) and is identical to the one calculated by the decomposition of $\tilde{X}_T^*$.

$$\begin{aligned}\pi_t &= -\frac{\nu}{\sigma^2} \cdot \frac{V_X}{V_{XX}} \cdot \frac{1}{X_t} \\ &= -\frac{\nu}{\sigma^2} \cdot \frac{1}{\gamma - 1} \cdot \frac{\left(x - K.e^{-r.(T-t)}.(n_x - N_t)._{T-t}p_{x+t} \right)}{X_t}\end{aligned}$$

4.8.3 Residue of the Bellman equation.

At our knowledge, the Bellman equation corresponding to the CRRA utility has no explicit solution. But it is still possible to check that the residue of Bellman equation is well negative. After calculations, the combination of the residue (4.7.11) and of the value function (4.8.4) gives that:

$$\begin{aligned}\epsilon_s &= (n_x - N_s) \cdot \mu(x + s) \cdot \left(\mathbb{E} \left(H(s, T)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_s \right) \right)^{1-\gamma} \cdot \\ &\quad \left[- \left(x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s) \cdot_{T-s}p_{x+s} \right)^{\gamma-1} \cdot K \cdot e^{-r(T-s)} \cdot_{T-s}p_{x+s} \right. \\ &\quad \left. + \frac{1}{\gamma} \cdot \left(x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s - 1) \cdot_{T-s}p_{x+s} \right)^{\gamma} \right. \\ &\quad \left. - \frac{1}{\gamma} \cdot \left(x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s) \cdot_{T-s}p_{x+s} \right)^{\gamma} \right]\end{aligned}$$

And by a Taylor's development of $\left(x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s - 1) \cdot_{T-s}p_{x+s} \right)^{\gamma}$ round the value $\left(x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s) \cdot_{T-s}p_{x+s} \right)$, we obtain after simplification a residue of the form:

$$\begin{aligned}\epsilon_s &= (n_x - N_s) \cdot \mu(x + s) \cdot \left(\mathbb{E} \left(H(s, T)^{\frac{\gamma}{\gamma-1}} | \mathcal{F}_s \right) \right)^{1-\gamma} \\ &\quad \cdot \left(K \cdot e^{-r(T-s)} \cdot_{T-s}p_{x+s} \right)^2 \cdot (\gamma - 1) \cdot \phi^{\gamma-1}\end{aligned}$$

Where ϕ is a constant that belongs to the interval:

$$\begin{aligned}\phi &\in \left[x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s) \cdot_{T-s}p_{x+s} ; \right. \\ &\quad \left. x - e^{-r(T-s)} \cdot K \cdot (n_x - N_s - 1) \cdot_{T-s}p_{x+s} \right]\end{aligned}$$

It may be easily checked that the residue is strictly negative.

4.9 Conclusion.

The main contribution of this chapter is to show that the martingale method, widely used in the setting of complete financial markets, can be applied to the management of pension liabilities and in particular of pure endowments. We have addressed the maximization of the expected utility of terminal surplus, under a budget constraint, by an effective investment strategy. Our approach is based on two assumptions.

Firstly, the deflator of the insurer is well determined. This implies to choose a risk neutral measure amongst the set of equivalent measures, which is non empty owing to the incompleteness of the insurance market. In this work, the chosen risk neutral measure is equal to the product of the the financial risk neutral measure and of the actuarial historical measure. Such assumption is commonly accepted by actuaries to price insurance risks and the hedging of liabilities relies on diversification.

The second condition required to apply the martingale method is to maximize the utility on a set of admissible terminal wealths larger than the one of attainable wealths. More precisely, this set is delimited by a budget constraint which ensures that the current richness is at least equal to the deflated terminal wealth. The main consequence of enlarging the set of controls is that the corresponding value function is not anymore solution of the Bellman equation. This approach is mainly motivated by the hope of finding the optimal ALM policy coupled to the power utility of the surplus.

Once that the optimal terminal wealth is determined by the method of Lagrange multipliers, one projects it in the space of processes, replicable by an adapted investment policy. This operation is done either by Kunita Watanabe decomposition or either by dynamic programming. This second possibility is relatively simpler than the decomposition. It suffices indeed to calculate the first and second order derivatives of the value function, with respect to the wealth, to infer the optimal assets allocation.

Finally, results are presented when utilities are exponential (CARA) or power (CRRA). For an exponential utility, the value function found by martingale approach is compared with the solution of the Bellman equation. Those two approaches lead to the same optimal investment strategy, which is independent of liabilities. On the contrary, the optimal assets allocation found by the martingale approach for a power utility function, depends on the equity, defined as the difference between assets and expected discounted liabilities. The choice of a power utility function seems therefore more adapted to ALM purposes than the exponential.

Appendix A.

The calculation of $\mathbb{E}(H(t, T) \cdot \ln H(t, T) | \mathcal{F}_t)$ is relatively direct. Indeed, we have that

$$\begin{aligned} \mathbb{E}(H(t, T) \cdot \ln H(t, T) | \mathcal{F}_t) &= \mathbb{E}^Q \left(e^{-r(T-t)} \cdot \ln H(t, T) | \mathcal{F}_t \right) \\ &= e^{-r(T-t)} \cdot \mathbb{E}^Q (\ln H(t, T) | \mathcal{F}_t) \end{aligned}$$

And, by change of measure:

$$\begin{aligned}
\mathbb{E}^Q(\ln H(t, T) \mid \mathcal{F}_t) &= \mathbb{E}^Q \left(- \left(r + \frac{1}{2} \lambda^2 \right) \cdot (T - t) - \int_t^T \lambda dW_u \mid \mathcal{F}_t \right) \\
&= \mathbb{E}^Q \left(- \left(r - \frac{1}{2} \lambda^2 \right) \cdot (T - t) - \int_t^T \lambda dW_u^{Q^f} \mid \mathcal{F}_t \right) \\
&= - \left(r - \frac{1}{2} \lambda^2 \right) \cdot (T - t)
\end{aligned}$$

Appendix B.

The calculation of $\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}_t \right)$ is easily performed by an adapted change of measure:

$$\begin{aligned}
&\mathbb{E} \left(H(t, T)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}_t \right) \\
&= \mathbb{E} \left(\exp \left(\frac{\gamma}{\gamma-1} \cdot \left(- \int_t^T r \cdot du - \frac{1}{2} \int_t^T \lambda^2 \cdot du - \int_t^T \lambda dW_u \right) \right) \mid \mathcal{F}_t \right) \\
&= \exp \left(\frac{\gamma}{\gamma-1} \cdot \left(- \int_t^T r \cdot du + \frac{1}{2} \cdot \frac{1}{\gamma-1} \cdot \int_t^T \lambda^2 \cdot du \right) \right) \cdot \underbrace{\mathbb{E} \left(\frac{\left(\frac{d\tilde{P}}{dP} \right)_T}{\left(\frac{d\tilde{P}}{dP} \right)_t} \mid \mathcal{F}_t \right)}_{=1}
\end{aligned}$$

Where $\frac{d\tilde{P}}{dP}$ defines a change of measure from the real measure P to $\tilde{P}$, a measure under which $dW_u^{\tilde{P}} = dW_u + \frac{\lambda}{\gamma-1} \cdot du$ is a Brownian motion.

$$\frac{\left(\frac{d\tilde{P}}{dP} \right)_T}{\left(\frac{d\tilde{P}}{dP} \right)_t} = \exp \left(- \frac{1}{2} \int_t^T \left(\frac{\gamma}{\gamma-1} \lambda \right)^2 \cdot du - \int_t^T \frac{\gamma}{\gamma-1} \lambda dW_u \right)$$