Chapter 7

Timing synchronization

7.1 Introduction

The problem under consideration in this chapter is that of network timing synchronization at the level of the head-end. During the acquisition phase, the joint detector is optimized for the sampling phase at which the received signal is sampled at this time, and also for the K phases at which the chip-rate sequences are transmitted by the K users. The joint detector is assumed to be non-adaptive : its coefficients are constant. During the transmission, the system is in tracking phase : a synchronization mechanism is responsible to keep the sampling phase at the receiver and the K phases of the transmitters as close as possible to the nominal values for which the joint detector has been designed. The user-specific misalignment cannot be compensated by the common receiver clock: the timing error information has to be sent back to the user modems via the downlink.

In the present chapter we design a generic decision-directed (DD) multiuser timing estimator for the maximum likelihood (ML) criterion and compute the associated Cramèr-Rao bound (CRB). We also derive practical timing error detectors (TEDs) suited to multiuser FB-based transmission. Throughout this chapter, *the decision error rate is assumed to be very low*, so that the true symbols $I_p(n)$ are used rather than the decisions. The performance of the timing error detectors considered in this chapter has then to be considered as an upper bound on the true performance that should take the effect of wrong decisions into account.

7.2 Asynchronous FB-based multiple access

In Chapter 4, the FB-based multiple access model was presented in a perfectly synchronous context, i.e. with the assumption that the K transmitter clocks and the unique receiver clock were perfectly aligned during the whole duration of the transmission. In this section, we revisit the multiple access model by introducing $K+1$ delays τ_k^t and τ^r associated with the different clocks. The delay τ^r corresponds to the receiver clock while the remaining delays τ_1^t up to τ_K^t are related to the different transmitter clocks. As these clocks are physically distributed in different locations in the network, the associated delays are a priori independent.

The continuous-time signal transmitted by each user (4.23) becomes :

$$x_{k,\tau_k^t}(t) = \sum_{n=-\infty}^{+\infty} v_k(n)\, p(t - nT_c + \tau_k^t) \tag{7.1}$$

where $v_k(n)$ is the user-specific chip rate sequence obtained at the output of the FB (see (4.3)). The delays τ_k^t are assumed to be close to their nominal values which correspond to the desired user alignments.

After propagation over the multiuser channel, the received signal is sampled at a fractional rate M/T and with an additional delay τ^r representing the alignment of the receiver clock. The cumulated timing delays are denoted by $\tau_k = \tau_k^t + \tau^r$ and are gathered in the $K \times 1$ vector $\underline{\tau}$. In the nominal mode of operations, obtained at modem start-up, the delay vector is supposed to be $\underline{\tau}_0$. The channel timing error, which corresponds to a user misalignment and/or a receiver clock offset, is denoted by $\underline{\varepsilon} = \underline{\tau} - \underline{\tau}_0$. Without loss of generality, we assume in the sequel that the nominal delay vector $\underline{\tau}_0$ is all-zero. This is equivalent to assume that the desired user alignments are incorporated in the physical channel impulse responses $c_k(t)$. The M received polyphase sequences are now :

$$\begin{aligned} r_{\rho,\underline{\varepsilon}}(m) &= \bar{r}\left(mT + \rho T_s + \tau^r\right) && (7.2)\\ &= \sum_{n=-\infty}^{+\infty} \sum_{k=1}^{K} \sum_{p \in \mathcal{C}_k} h_{\rho,pk,\varepsilon_k}(m-n) I_p(n) + n_{\rho,\tau^r}(m) && (7.3) \end{aligned}$$

where the filtered continuous-time received signal $\bar{r}(t)$ is given by

$$\bar{r}(t) = \sum_{k=1}^{K} \left[x_{k,\tau_k^t}(t) \otimes \bar{c}_k(t) \right] + \bar{n}(t) \tag{7.4}$$

and the MN asynchronous channel sequences $h_{\rho,pk,\varepsilon_k}(m)$ are defined by

$$h_{\rho,pk,\varepsilon_k}(m) = \sum_{n=\infty}^{+\infty} s_p(n)\bar{c}_k(mT + \rho T_s - nT_c + \varepsilon_k). \tag{7.5}$$

With FIR channels, $L + 1$ asynchronous channel blocks $\underline{\underline{\mathrm{H}}}_{nk,\varepsilon_k}$ of size $M \times N_k$ can be defined for each user as in (4.45), on the basis of the asynchronous channel sequences $h_{\rho,pk,\varepsilon_k}(m)$ defined above. The corresponding asynchronous channel matrices $\underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}$ are obtained as in (4.46). The matrices of channel first and second derivatives $\dot{\underline{\underline{\mathrm{H}}}}_{k,\varepsilon_k}$ and $\ddot{\underline{\underline{\mathrm{H}}}}_{k,\varepsilon_k}$, as well as the corresponding channel blocks $\dot{\underline{\underline{\mathrm{h}}}}_{k,\varepsilon_k}(n)$ and $\ddot{\underline{\underline{\mathrm{h}}}}_{k,\varepsilon_k}$, are defined in the same fashion except that the polyphase components of $h_{pk}(t)$ are replaced by the polyphase components of $dh_{pk}(t)/dt$ and $d^2h_{pk}(t)/dt^2$, respectively.

The total symbol energy received from user k (see 4.32) is given by:

$$\mathcal{E}_k = \sum_{p\in\mathcal{C}_k} \mathcal{E}_{pk} = \sigma_I^2 \frac{T}{M} \mathrm{tr}\left(\underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}^T \underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}\right). \tag{7.6}$$

The mean square bandwidth of the signal received from user k is defined as follows:

$$\begin{aligned} W_k^2 &= \frac{\dfrac{\sigma_I^2}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \sum_{p\in\mathcal{C}_k} \omega^2 |H_{pk}(\omega)|^2 \, d\omega}{\dfrac{\sigma_I^2}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \sum_{p\in\mathcal{C}_k} |H_{pk}(\omega)|^2 \, d\omega} \\ &= -\frac{\mathrm{tr}\left(\ddot{\underline{\underline{\mathrm{H}}}}_{k,\varepsilon_k}^T \underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}\right)}{\mathrm{tr}\left(\underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}^T \underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}\right)} = \frac{\mathrm{tr}\left(\dot{\underline{\underline{\mathrm{H}}}}_{k,\varepsilon_k}^T \dot{\underline{\underline{\mathrm{H}}}}_{k,\varepsilon_k}\right)}{\mathrm{tr}\left(\underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}^T \underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}\right)} \end{aligned} \tag{7.7}$$

The above quantities $\mathcal{E}_k$ and W_k^2 do not depend on the timing delays ε_k even if they appear in the definitions.

The asynchronous FIR observation model relating the received samples within a length-N_fT observation window to the N transmitted symbol sequences is now

$$\left(\underline{\mathrm{r}}_{\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) = \sum_{k=1}^{K} \underline{\underline{\mathcal{H}}}_{N_f,k,\varepsilon_k} \left(\underline{\mathrm{I}}_k\right)_{n_2+L}^{n_1}(n) + \left(\underline{\mathrm{n}}_{\tau^r}\right)_{n_2}^{n_1}(n). \tag{7.8}$$

where the asynchronous global channel matrix $\underline{\underline{\mathcal{H}}}_{N_f,k,\varepsilon_k}$ is defined as in (4.47). The corresponding matrices of channel first and second derivatives are denoted by $\underline{\underline{\dot{\mathcal{H}}}}_{N_f,k,\varepsilon_k}$ and $\underline{\underline{\ddot{\mathcal{H}}}}_{N_f,k,\varepsilon_k}$, respectively.

One can be also interested in a FIR observation model relating a number N_f of consecutive outputs of the matched filter bank to the transmitted symbol sequences. The N_f consecutive outputs associated with the signatures allocated to the k-th user are given by

$$\left(\underline{\mathrm{y}}_{k,\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) = \underline{\underline{\mathrm{F}}}_0\, \underline{\underline{\mathcal{H}}}^T_{N_f+L,k,0}\, \left(\underline{\mathrm{r}}_{\underline{\varepsilon}}\right)_{n_2}^{n_1+L}(n) \qquad (7.9)$$

where the matrix $\underline{\underline{\mathrm{F}}}_0$ is defined by

$$\underline{\underline{\mathrm{F}}}_0 = \left(\ \underline{\underline{0}}_{N_f N\times LN}\ \middle|\ \underline{\underline{\mathrm{E}}}_{N_f N}\ \middle|\ \underline{\underline{0}}_{N_f N\times LN}\ \right). \qquad (7.10)$$

In an asynchronous mode of operation, the IIR observation model becomes :

$$\underline{\mathrm{R}}_{\underline{\varepsilon}}(z) = \sum_{k=1}^{K} \underline{\underline{\mathrm{H}}}_{k,\varepsilon_k}(z)\ \underline{\mathrm{I}}_k(z) + \underline{\mathrm{N}}_{\tau^r}(z). \qquad (7.11)$$

The N symbol-rate sequences obtained at the output of a bank of matched filters (5.12) are :

$$\underline{\mathrm{Y}}_{\underline{\varepsilon}}(z) = \underline{\underline{\mathrm{H}}}_{\underline{0}}^H(1/z^*)\,\underline{\mathrm{R}}_{\underline{\varepsilon}}(z) = \underline{\underline{\mathrm{X}}}_{\underline{\varepsilon}}(z)\,\underline{\mathrm{I}}(z) + \underline{\mathrm{W}}_{\tau^r}(z). \qquad (7.12)$$

where the global asynchronous interference matrix $\underline{\underline{\mathrm{X}}}_{\underline{\varepsilon}}(z)$ includes $K\times K$ blocks $\underline{\underline{\mathrm{X}}}_{kk',\varepsilon_{k'}}(z)$ of size $N_k \times N_{k'}$ defined as follows :

$$\underline{\underline{\mathrm{X}}}_{kk',\varepsilon_{k'}}(z) = \frac{T}{M}\,\underline{\underline{\mathrm{H}}}^H_{k,0}(1/z^*)\,\underline{\underline{\mathrm{H}}}_{k',\varepsilon_{k'}}(z). \qquad (7.13)$$

The noise cross-spectrum is $\underline{\underline{\gamma}}_w(z) = \frac{N_0}{2}\,\underline{\underline{\mathrm{X}}}_{\underline{0}}(z)$. Let $\underline{\underline{\mathrm{L}}}(z)$ be the MIMO spectral factor associated with the noise spectrum $\underline{\underline{\gamma}}_w(z)$, that is to say

$$\underline{\underline{\mathrm{X}}}_{\underline{0}}(z) = \underline{\underline{\mathrm{L}}}(z)\,\underline{\underline{\mathrm{L}}}^H(1/z). \qquad (7.14)$$

A MIMO noise whitening filter $\underline{\underline{\mathrm{L}}}^{-1}(z)$ can be placed at the output of the matched filter bank, providing N filtered sequences $\tilde{y}_{p,\underline{\varepsilon}}(n)$ as follows :

$$\tilde{\underline{\mathrm{Y}}}_{\underline{\varepsilon}}(z) = \underline{\underline{\mathrm{L}}}^{-1}(z)\ \underline{\mathrm{Y}}_{\underline{\varepsilon}}(z) = \tilde{\underline{\underline{\mathrm{X}}}}_{\underline{\varepsilon}}(z)\,\underline{\mathrm{I}}(z) + \tilde{\underline{\mathrm{W}}}_{\tau^r}(z) \qquad (7.15)$$

with $\underline{\underline{\tilde{X}}}_{\underline{\varepsilon}}(z) = \underline{\underline{L}}^{-1}(z)\, \underline{\underline{X}}_{\underline{\varepsilon}}(z)$ and $\underline{\underline{\tilde{X}}}_{\underline{0}}(z) = \underline{\underline{L}}^{H}(1/z)$. The noise-whitened matched filter outputs are also given by

$$\tilde{y}_{p,\underline{\varepsilon}}(n) = \sum_{m=-\infty}^{+\infty} \sum_{k=1}^{K} \sum_{p' \in \mathcal{C}_k} \tilde{x}_{pp',\varepsilon_k}(m)\, I_{p'}(n-m) + \tilde{\nu}_{p,\tau^r}(n). \tag{7.16}$$

7.3 ML-DD timing estimation and the Cramèr-Rao Lower Bound

7.3.1 Derivation of the ML-DD estimator

The log-likelihood function of the received signal is:

$$\Lambda_l\left(\underline{r}_{\underline{\varepsilon}} \,|\, \underline{I}, \hat{\underline{\varepsilon}}\right) = \kappa - \frac{1}{2\sigma_n^2} \left\| \left(\underline{r}_{\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) - \sum_{k=1}^{K} \left[\underline{\underline{\mathcal{H}}}_{N_f,k,\hat{\varepsilon}_k}\right] \left(\underline{I}_k\right)_{n_2+L}^{n_1}(n) \right\|^2 \tag{7.17}$$

where κ is a constant and $\|\underline{x}\|^2$ denotes the square norm of vector $\underline{x}$. The ML-DD estimate of the delays is the vector $\hat{\underline{\varepsilon}}^*$ that maximizes this expression. In other words, the first derivative of the log-likelihood function with respect to each timing parameter:

$$\begin{aligned}\frac{\partial \Lambda_l\left[\underline{r}_{\underline{\varepsilon}} \,|\, \underline{I}, \hat{\underline{\varepsilon}}\right]}{\partial \hat{\varepsilon}_k} = \frac{1}{\sigma_n^2} \Bigg[& \left(\underline{I}_k^T\right)_{n_2+L}^{n_1}(n) \left[\dot{\underline{\underline{\mathcal{H}}}}_{N_f,k,\hat{\varepsilon}_k}\right]^T \\ & \left(\left(\underline{r}_{\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) - \sum_{k'=1}^{K} \left[\underline{\underline{\mathcal{H}}}_{N_f,k',\hat{\varepsilon}_{k'}}\right] \left(\underline{I}_{k'}\right)_{n_2+L}^{n_1}(n) \right) \Bigg]\end{aligned} \tag{7.18}$$

is zero for $\hat{\varepsilon}_k = \hat{\varepsilon}_k^*$ and $k \in [1, K]$. This estimator is obviously unbiased as $\mathrm{E}_n\left\{\partial \Lambda_l / \partial \hat{\underline{\varepsilon}}\right\} = \underline{0}$ for $\hat{\underline{\varepsilon}} = \underline{\varepsilon}$. The timing estimation error is defined as $\underline{\Delta}_{\varepsilon} = \hat{\underline{\varepsilon}}^* - \underline{\varepsilon}$.

7.3.2 True Cramèr-Rao lower bound (DD-CRB)

The Fisher information matrix associated with this log-likelihood function is the $K \times K$ symmetric and positive definite matrix whose elements are defined by

$$\begin{aligned} J_{kk'}(\underline{\varepsilon}, \underline{I}) &= -\mathrm{E}_n \left\{ \frac{\partial^2 \Lambda_l\left[\underline{r}_{\underline{\varepsilon}} \,|\, \underline{I}, \hat{\underline{\varepsilon}}\right]}{\partial \hat{\varepsilon}_k \partial \hat{\varepsilon}_{k'}} \right\}_{\hat{\underline{\varepsilon}} = \underline{\varepsilon}} \\ &= \frac{1}{\sigma_n^2} \left(\left(\underline{I}_k^T\right)_{n_2+L}^{n_1}(n) \left[\underline{\underline{\Phi}}_{kk'}\right] \left(\underline{I}_{k'}\right)_{n_2+L}^{n_1}(n) \right) \end{aligned} \tag{7.19}$$

where $\underline{\underline{\Phi}}_{kk'} \triangleq \left[\underline{\underline{\dot{\mathcal{H}}}}_{N_f,k,\varepsilon_k}\right]^T \left[\underline{\underline{\dot{\mathcal{H}}}}_{N_f,k',\varepsilon_{k'}}\right]$. The expectation is taken with respect to the probability density function of the received signal, which is reduced here to the probability density function of the additive noise, as the symbols are assumed to be perfectly known. From the Cramèr-Rao theorem, we know that

$$\mathrm{E}_n \left\{ \underline{\underline{\Delta}}_\varepsilon \underline{\underline{\Delta}}_\varepsilon^T \right\} - \underline{\underline{\mathrm{J}}}(\underline{\varepsilon}, \underline{\mathrm{I}})^{-1} \geq \underline{\underline{0}} \tag{7.20}$$

where $\geq \underline{\underline{0}}$ is interpreted as meaning that the matrix is positive semi-definite. The lower bound on the timing error variance is dependent on the specific sequence of information symbols involved in the received signal segment [64]. In a tracking mode of operation, the timing parameters are continuously estimated and the lower bound on the average timing error variance becomes:

$$\mathrm{var}(\hat{\varepsilon}_k) \geq \mathrm{E}_I \left\{ \left[\underline{\underline{\mathrm{J}}}(\underline{\varepsilon}, \underline{\mathrm{I}})^{-1} \right]_{kk} \right\} \tag{7.21}$$

Equation (7.21) gives the true DD-CRB associated with the FB-based multiple access system.

7.3.3 Modified Cramèr-Rao lower bound (M-CRB)

The right-hand side of (7.21) is not easy to compute. Some simplifications are made possible by using the law of large numbers, if the length N_f of the observation interval (and the length of the corresponding sequences of symbols) is long. In that case, the diagonal elements of the Fisher information matrix are tightly distributed around a large mean, while the non-diagonal elements are tightly distributed around a zero mean:

$$\begin{aligned} J_{kk} &\sim \mathcal{N}\left[\left(N_f \frac{\sigma_I^2}{\sigma_n^2} \right) m_{kk}, \left(\sqrt{N_f} \frac{\sigma_I^2}{\sigma_n^2} \right) s_{kk} \right] \\ J_{kk'} &\sim \mathcal{N}\left[0, \left(\sqrt{N_f} \frac{\sigma_I^2}{\sigma_n^2} \right) s_{kk'} \right] \end{aligned} \tag{7.22}$$

with

$$\begin{aligned} m_{kk} &= \frac{1}{N_f} \mathrm{tr}\left(\underline{\underline{\Phi}}_{kk} \right) \\ s_{kk}^2 &= \frac{1}{N_f} \left[\sum_i \rho_i^2 \left(\underline{\underline{\Phi}}_{kk} \right)_{ii}^2 + \sum_i \sum_{j>i} \left(2\underline{\underline{\Phi}}_{kk} \right)_{ij}^2 \right] \\ s_{kk'}^2 &= \frac{1}{N_f} \left[\sum_i \sum_j \left(\underline{\underline{\Phi}}_{kk'} \right)_{ij}^2 \right]. \end{aligned} \tag{7.23}$$

The factor ρ_i^2 is defined by

$$\rho_i^2 = \mathrm{E}\left\{I^4\right\}/\sigma_I^4 - 1 < 4/5 \tag{7.24}$$

and depends on the PAM constellation used on the i-th symbol stream. The quantities defined in (7.23) are independent of the observation window size N_f (at least by neglecting the side-effect due to continuous transmission).

To get a tractable expression of the CRB, we have to observe two inequalities:

- The symmetric positive definite Fisher information matrix, has the property that:

$$\left[\underline{\underline{\mathrm{J}}}^{-1}\right]_{kk} \geq \frac{1}{J_{kk}} \quad \forall\, \underline{\mathrm{I}} \tag{7.25}$$

  Inequality (7.25) becomes an equality if and only if the Fisher information matrix is diagonal. In other words, considering it as an equality is equivalent to neglecting the performance degradation due to the joint estimation process. For a long observation window, the next approximation for the matrix inversion is valid:

$$\left[\underline{\underline{\mathrm{J}}}^{-1}\right]_{kk} \approx \frac{1}{J_{kk}}\left(1 + \sum_{k' \neq k} \frac{J_{kk'}^2}{J_{kk}\, J_{k'k'}}\right) \tag{7.26}$$

  and the average inverse of the Fisher information matrix becomes:

$$\mathrm{E}_I\left\{\left[\underline{\underline{\mathrm{J}}}^{-1}\right]_{kk}\right\} \approx \mathrm{E}_I\left\{\frac{1}{J_{kk}}\right\}\left(1 + \frac{1}{N_f}\sum_{k' \neq k} \frac{s_{kk'}^2}{m_{kk} m_{k'k'}}\right) \tag{7.27}$$

  The last factor appears as a correction term which decreases linearly with the size of the observation window.

- The application of Jensen's inequality[1] to the convex function $f(x) = 1/x$ gives:

$$\mathrm{E}_I\left\{\frac{1}{J_{kk}}\right\} \geq \frac{1}{\mathrm{E}_I\left\{J_{kk}\right\}} \tag{7.28}$$

[1] If $f(.)$ is a convex function and X is a random variable, then

$$\mathrm{E}\left\{f(X)\right\} \geq f\left(\mathrm{E}\left\{X\right\}\right)$$

As J_{kk} is concentrated near its mean, a second order Taylor expansion of its inverse can be used (see [65], pp. 113) to provide the following approximation:

$$\mathrm{E}_I\left\{\frac{1}{J_{kk}}\right\} \approx \frac{1}{\mathrm{E}_I\left\{J_{kk}\right\}}\left(1+\frac{1}{N_f}\frac{s_{kk}^2}{m_{kk}^2}\right) \tag{7.29}$$

The last factor appears again as a correction term which decreases linearly with the size of the observation window.

In the light of relations (7.25) and (7.28), we obtain a modified (looser) lower bound on the timing error variance:

$$\mathrm{var}(\hat{\varepsilon}_k) > \left[\mathrm{E}_I\left\{\underline{\underline{\mathrm{J}}}\left(\underline{\varepsilon},\underline{\mathrm{I}}\right)\right\}\right]_{kk}^{-1} = \frac{1}{\mathrm{E}_I\left\{J_{kk}\right\}} \tag{7.30}$$

From a careful examination of the $\left[\underline{\underline{\dot{\mathcal{H}}}}_{N_f,k,\varepsilon_k}\right]$ matrix structure shown in (4.47), we can check that

$$m_{kk} = \frac{1}{N_f}\mathrm{tr}\left(\underline{\underline{\Phi}}_{kk}\right) = \mathrm{tr}\left(\underline{\underline{\dot{\mathrm{H}}}}_{k,\varepsilon_k}^T\,\underline{\underline{\dot{\mathrm{H}}}}_{k,\varepsilon_k}\right) \tag{7.31}$$

Using the definitions (7.6) and (7.7), the proposed modified bound is finally given by :

$$\mathrm{var}(\hat{\varepsilon}_k) > \left[\frac{2\mathcal{E}_k}{N_0}\right]^{-1}\frac{1}{N_f\,W_k^2}. \tag{7.32}$$

Note that this modified bound does not depend on the true value to be estimated $\underline{\varepsilon}$.

The bound proposed in (7.30) corresponds to the modified Cramèr-Rao bound (M-CRB) introduced for vector parameter estimation by [66]. In that reference, which corresponds to a NDA scenario, the symbols are considered as nuisance parameters. In that scenario, the symbol-dependent (exponential) likelihood function has to be averaged over the symbols distribution before the logarithm is taken. This provides the following Fisher information matrix:

$$\bar{J}_{k,k'}(\underline{\varepsilon}) = -\mathrm{E}_n\left\{\frac{\partial^2 \ln\left[\mathrm{E}_I\left\{\Lambda\left(\underline{\mathrm{r}}_{\underline{\varepsilon}}\,|\,\underline{\mathrm{I}},\underline{\hat{\varepsilon}}\right)\right\}\right]}{\partial\hat{\varepsilon}_k\partial\hat{\varepsilon}_{k'}}\right\}_{\underline{\hat{\varepsilon}}=\underline{\varepsilon}} \tag{7.33}$$

The authors showed that the M-CRB is always lower than the true NDA-CRB. We finally have the following ordering between the bounds:

$$\underbrace{\left[\left(\mathrm{E}_I\left\{\underline{\underline{\mathrm{J}}}(\underline{\varepsilon},\underline{\mathrm{I}})\right\}\right)^{-1}\right]_{kk}}_{M-CRB} \;\leq\; \underbrace{\mathrm{E}_I\left\{\left[\underline{\underline{\mathrm{J}}}(\underline{\varepsilon},\underline{\mathrm{I}})^{-1}\right]_{kk}\right\}}_{DD-CRB} \;\leq\; \underbrace{\left[\underline{\underline{\bar{\mathrm{J}}}}(\underline{\varepsilon})^{-1}\right]_{kk}}_{NDA-CRB} \tag{7.34}$$

With PAM-distributed symbols $I_p(n)$, an exhaustive survey of all possible symbol combinations should be performed to obtain the expectation E(.) appearing in the expression of the true DD-CRB (7.21). The computation of the true DD-CRB has thus a complexity that increases exponentially with $(N_f + L)N$, i.e. the number of symbols used to obtain the timing estimate, and with the size of each PAM constellation. Let us consider a simple transmission scheme with a single user ($K = 1$), a single signature ($N = 1$) and an AWGN channel ($L = 1$). The ratio of the M-CRB with the true DD-CRB becomes :

$$\frac{\text{DD-CRB}}{\text{M-CRB}} = \left(\sigma_I^2 N_f\right) \mathrm{E}_I\left\{\frac{1}{\sum_{n=1}^{N_f} I_n^2}\right\} \tag{7.35}$$

$$\approx 1 + \frac{\rho^2}{N_f} \tag{7.36}$$

where ρ is defined in (7.24). Figure 7.1 illustrates the evolution of the ratio (7.35) with the length N_f of the observation window, for different values of the PAM constellation size. The solid lines give the true DD-CRB (obtained by an exhaustive computation), while the dashed lines give the asymptotic approximation (7.36). Obviously, the proposed asymptotic approximation is already tight with moderate values of N_f. Moreover, the M-CRB converges rapidly to the true DD-CRB. Extrapolating these results to the general multiuser FB scenario, we assume in the sequel that the M-CRB is a pertinent approximation of the true DD-CRB.

The M-CRB for a given user is the same as in the single user scenario, that is to say it only depends on the average matched filter bound (MFB) $2\mathcal{E}_k/N_0$ and the mean square bandwidth (MSB) W_k^2 of the considered user. However, both quantities are dependent on the individual channels and the waveforms allocated to the different users. In a FB system, the waveform allocation has thus a strong impact on the multiuser timing estimator performance: it should be matched to the different channels in such a way that the average MFB-MSB product is maximized for each user.

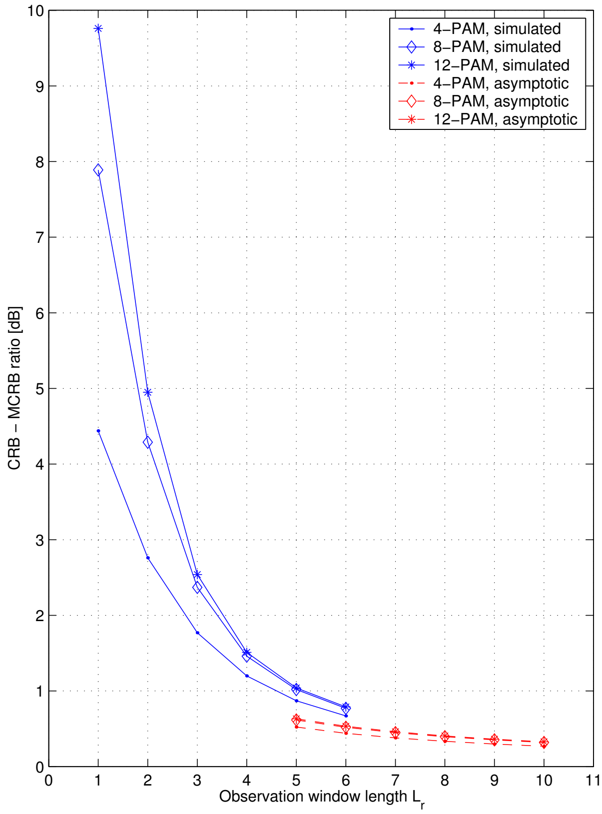

Figure 7.1: *Comparison of the true DD-CRB and the M-CRB*

In a single-user scenario, the M-CRB can be shown to be independent of the FB as long as it satisfies the paraunitary property :

$$\begin{aligned} \text{M-CRB} &= -\frac{1}{N_f}\left(\frac{\sigma_I^2}{N_0/2}\right)\left[\mathrm{tr}\left\{\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H\left(e^{j\Omega}\right)\ddot{\underline{\underline{\mathrm{G}}}}\left(e^{j\Omega}\right)\underline{\underline{\mathrm{S}}}\left(e^{j\Omega}\right)\frac{\mathrm{d}\Omega}{2\pi}\right\}\right]^{-1} \\ &= -\frac{1}{N_f}\left(\frac{\sigma_I^2}{N_0/2}\right)\sum_{p=1}^{N}\ddot{x}_{pp}(0) = -\frac{1}{N_f\,N}\left(\frac{\sigma_I^2}{N_0/2}\right)\ddot{g}(0) \qquad (7.37) \end{aligned}$$

where $g(t)$ is the channel correlation function.

7.4 Symbol-rate ML-DD timing estimation

The previous section was devoted to the performance computation of an optimal timing estimator placed at the channel output. This estimator has an access to the received signal samples at a fractional rate M/T, which represent a sufficient statistic of the continuous-time received signal.

The primary task of the receiver is to compute symbol estimates, by means of an appropriate joint detector. The JDs analyzed in Chapter 5 are firstly constituted by a bank of matched filters whose outputs are sampled at the symbol rate. The matched filter outputs are then combined through an appropriate MIMO filter (optionally with a feedback section) whose purpose is to decrease the residual interference. This MIMO filter is invertible, so there is no loss of information associated with it. The bank of matched filters, however, represent a non-invertible operation, in the sense that the channel output samples cannot be recovered from the matched filter outputs. This is not a problem regarding the process of symbol detection, as the matched filter outputs actually represent a sufficient statistic of the data symbols $I_p(n)$. But there is well a loss of information regarding the process of timing estimation.

A class of timing estimators, the so-called symbol-rate timing estimators, are intended to build estimates of the timing error on the basis of output samples produced at the symbol rate, that is to say at the output of the matched filter bank or that of the JD. The corresponding optimal performance is necessarily lower than that of a timing estimator working directly at the channel output. The purpose of this section is to derive the M-CRB associated with the symbol rate ML-DD timing estimation.

Following the observation model given by (7.9), the log-likelihood function of the received signal is now :

$$\Lambda_l\left(\underline{\mathrm{y}}_{\underline{\varepsilon}}\,|\,\underline{\mathrm{I}},\underline{\hat{\varepsilon}}\right) = \kappa - \frac{1}{2\sigma_n^2}\left\{\left[\left(\underline{\mathrm{y}}_{\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) - \left[\underline{\underline{\mathrm{F}}}_0\,\underline{\underline{\mathcal{H}}}_0^T\,\underline{\underline{\mathcal{H}}}_{\hat{\varepsilon}}\right]\,(\underline{\mathrm{I}}_k)_{n_2+L}^{n_1+L}(n)\right]^T\right.$$
$$\left.\left[\underline{\underline{\mathrm{F}}}_0\,\underline{\underline{\mathcal{H}}}_0^T\,\underline{\underline{\mathcal{H}}}_0\,\underline{\underline{\mathrm{F}}}_0^T\right]^{-1}\left[\left(\underline{\mathrm{y}}_{\underline{\varepsilon}}\right)_{n_2}^{n_1}(n) - \left[\underline{\underline{\mathrm{F}}}_0\,\underline{\underline{\mathcal{H}}}_0^T\,\underline{\underline{\mathcal{H}}}_{\hat{\varepsilon}}\right]\,(\underline{\mathrm{I}}_k)_{n_2+L}^{n_1+L}(n)\right]\right\} \quad (7.38)$$

The entries of the $K \times K$ Fisher information matrix associated with this log-likelihood function are now :

$$\begin{aligned} \overline{J}_{kk'}(\underline{\varepsilon}, \underline{\mathrm{I}}) &= -\mathrm{E}_n \left\{ \frac{\partial^2 \Lambda_l \left[\underline{\mathrm{y}}_{\underline{\varepsilon}} \,|\, \underline{\mathrm{I}}, \hat{\underline{\varepsilon}} \right]}{\partial \hat{\varepsilon}_k \partial \hat{\varepsilon}_{k'}} \right\}_{\hat{\underline{\varepsilon}} = \underline{\varepsilon}} \\ &= \frac{1}{\sigma_n^2} \left\{ \left(\underline{\mathrm{I}}_k^T \right)_{n_2+L}^{n_1+L}(n) \left[\overline{\underline{\underline{\Phi}}}_{kk'} \right] \left(\underline{\mathrm{I}}_{k'} \right)_{n_2+L}^{n_1+L}(n) \right\} \end{aligned} \tag{7.39}$$

where matrix $\overline{\underline{\underline{\Phi}}}_{kk'}$ is defined by

$$\overline{\underline{\underline{\Phi}}}_{kk'} = \underline{\underline{\dot{\mathcal{H}}}}_{k,\varepsilon_k}^T \left(\underline{\underline{\mathcal{H}}}_0 \underline{\underline{\mathrm{F}}}_0^T \right) \left(\underline{\underline{\mathrm{F}}}_0 \underline{\underline{\mathcal{H}}}_0^T \underline{\underline{\mathcal{H}}}_0 \underline{\underline{\mathrm{F}}}_0^T \right)^{-1} \left(\underline{\underline{\mathrm{F}}}_0 \underline{\underline{\mathcal{H}}}_0^T \right) \underline{\underline{\dot{\mathcal{H}}}}_{k',\varepsilon_{k'}} \tag{7.40}$$

The M-CRB on the symbol-rate timing estimator is expressed as follows :

$$\operatorname{var}(\hat{\varepsilon}_k) > \frac{1}{\mathrm{E}_I \left\{ \overline{J}_{kk} \right\}} = \left[\frac{\sigma_I^2}{\sigma_n^2} \operatorname{tr}\left(\overline{\underline{\underline{\Phi}}}_{kk} \right) \right]^{-1}. \tag{7.41}$$

From the examination of the channel matrix structure, it can be shown that for a long observation window $N_f \gg L$ the trace of $\overline{\underline{\underline{\Phi}}}_{kk'}$ can be approximated as follows :

$$\operatorname{tr}\left(\overline{\underline{\underline{\Phi}}}_{kk} \right) \approx N_f \operatorname{tr}\left\{ \int_{-\pi}^{\pi} \underline{\underline{\Psi}}_{kk}\left(e^{j\Omega} \right) \frac{\mathrm{d}\Omega}{2\pi} \right\} \tag{7.42}$$

with

$$\underline{\underline{\Psi}}_{kk}(z) = \begin{pmatrix} \underline{\underline{\dot{\mathrm{X}}}}_{1k,\varepsilon_k}^H(\frac{1}{z}) & \cdots & \underline{\underline{\dot{\mathrm{X}}}}_{Kk}^H(\frac{1}{z}) \end{pmatrix} \left[\underline{\underline{\mathrm{X}}}_0(z) \right]^{-1} \begin{pmatrix} \underline{\underline{\dot{\mathrm{X}}}}_{1k,\varepsilon_k}(z) \\ \vdots \\ \underline{\underline{\dot{\mathrm{X}}}}_{Kk,\varepsilon_k}(z) \end{pmatrix} \tag{7.43}$$

An equivalent expression in the time domain can be obtained by using Parseval's relation :

$$\operatorname{tr}\left(\overline{\underline{\underline{\Phi}}}_{kk} \right) \approx N_f \, \dot{\mathcal{X}}_k^2 \tag{7.44}$$

where

$$\dot{\mathcal{X}}_k^2 \triangleq \sum_{n=-\infty}^{+\infty} \sum_{p=1}^{N} \sum_{p' \in \mathcal{C}_k} \dot{\tilde{x}}_{pp',\varepsilon_k}^2(n) \tag{7.45}$$

The minimum error variance associated with a symbol-rate timing estimator is thus inversely proportional to the sum of the noise-whitened interference waveforms squared slopes, sampled at the symbol rate.

7.5 Joint timing error tracking structures

7.5.1 Feedforward structure

In a DD feedforward mode of operations, at time m, the synchronizer tries to estimate the vector of timing errors $\underline{\varepsilon}$ on the basis of a large set of measurement data, and the knowledge of the data symbols. The vector of measurement data is denoted by $\underline{\mathrm{x}}_{\underline{\varepsilon}}$: it can be made of channel outputs, matched filter outputs, or joint detector outputs.

A given DD feedforward timing error detector is characterized by a specific timing error vector function $\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\hat{\varepsilon}}\right)$ that should be close to the null vector for $\underline{\hat{\varepsilon}} = \underline{\varepsilon}$ in the absence of noise. In the ideal case, an exhaustive search on the trial vector $\underline{\hat{\varepsilon}}$ would be performed, and the proposed estimate would be $\underline{\hat{\varepsilon}}^*$ such that:

$$\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\hat{\varepsilon}}^*\right) = \underline{0}. \tag{7.46}$$

Because of the noise, the resulting estimate will differ from the true value $\underline{\varepsilon}$.

As suggested above, the TED output for a given candidate timing vector $\underline{\hat{\varepsilon}}$ depends on the known data symbols $\underline{\mathrm{I}}$ and the measurement data vector $\underline{\mathrm{x}}_{\underline{\varepsilon}}$, which is itself a function of the data symbols, the additive noise and the true timing error vector: $\underline{\mathrm{x}}_{\underline{\varepsilon}} = f\left(\underline{\mathrm{n}}, \underline{\mathrm{I}}, \underline{\varepsilon}\right)$. The TED output can usually be expressed as the sum of three terms as follows:

$$\begin{aligned} \underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\hat{\varepsilon}}\right) &= \underline{\mathrm{S}}_I\left(\underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) + \underline{\mathrm{N}}_n\left(\underline{\mathrm{n}}, \underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) \\ &= \underline{\mathrm{S}}_0\left(\underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) + \underline{\mathrm{N}}_s\left(\underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) + \underline{\mathrm{N}}_n\left(\underline{\mathrm{n}}, \underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) \end{aligned} \tag{7.47}$$

where

- The first term $\underline{\mathrm{S}}_0\left(\underline{\varepsilon}, \underline{\hat{\varepsilon}}\right)$ is the so-called 'S-hypercurve' of the TED. It is the expectation with respect to both data and additive noise of the TED output.

- The second term $\underline{\mathrm{N}}_s\left(\underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right)$ is the self-noise. It represents the deviation of the TED output from the S-hypercurve due to the specific pattern of data symbols contained in the received signal. In a given sense, the first two terms-sum $\underline{\mathrm{S}}_I\left(\underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right)$ gives the instantaneous pattern-dependent S-hypercurve.

- The third term $\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}}, \underline{\mathrm{I}}, \underline{\varepsilon}, \underline{\hat{\varepsilon}}\right)$ represents the contribution of the additive noise. In the DD scenario, its expectation with respect to the noise is supposed to be zero.

The S-hypercurve evaluated at the true timing error, that is to say $\underline{\mathrm{S}}_0\left(\underline{\varepsilon},\underline{\varepsilon}\right)$, gives the average bias of the TED. The term $\underline{\mathrm{S}}_I\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)$ is the instantaneous bias which depends on the data sequence.

To evaluate the mean square value of the estimation error, some approximations have to be done:

1. A first order expansion of the timing error vector function is performed:
$$\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\hat{\underline{\varepsilon}}\right) \approx \underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right) + \left[\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\right]\underline{\Delta}_{\varepsilon} \tag{7.48}$$
where $\left[\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\right]$ is the $K \times K$ TED Jacobian matrix with respect to the candidate timing vector, evaluated at $\hat{\underline{\varepsilon}} = \underline{\varepsilon}$. Using (7.46) and (7.48), the correlation matrix of the estimation error becomes:
$$\mathrm{E}_n\left\{\underline{\Delta}_{\varepsilon}\underline{\Delta}_{\varepsilon}^T\right\} = \mathrm{E}_n\left\{\left[\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\right]^{-1}\left[\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)^T\right]\right.$$
$$\left.\left[\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\right]^{-T}\right\} \tag{7.49}$$
This linearization is valid for small timing errors only.

2. At high signal-to-noise ratios, the noisy fluctuation of the Jacobian matrix with respect to its mean value is much smaller than this mean value. From (7.46) and (7.48), the timing error vector is thus approximated as:
$$\underline{\Delta}_{\varepsilon} \approx -\left[\mathrm{E}_n\left\{\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\right\}\right]^{-1}\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right) \tag{7.50}$$
$$= -\left[\frac{\partial \underline{\mathrm{S}}_I}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\right]^{-1}\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right) \tag{7.51}$$
The correlation matrix of the estimation error becomes:
$$\mathrm{E}_n\left\{\underline{\Delta}_{\varepsilon}\underline{\Delta}_{\varepsilon}^T\right\} = \left[\frac{\partial \underline{\mathrm{S}}_I}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\right]^{-1}\mathrm{E}_n\left\{\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)^T\right\}$$
$$\left[\frac{\partial \underline{\mathrm{S}}_I}{\partial \hat{\underline{\varepsilon}}}\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\right]^{-T} \tag{7.52}$$

with

$$\mathrm{E}_n\left\{\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)^T\right\}=\underline{\mathrm{S}}_I\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\underline{\mathrm{S}}_I\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)^T + \mathrm{E}_n\left\{\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}},\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}},\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)^T\right\} \quad (7.53)$$

The unbiased part of (7.52) has to be compared with the true CRB.

3. The correlation matrix obtained so far still depends on the specific symbol sequence that is contained in the signal. To obtain the mean performance of the timing detector, a statistical average has to be performed on the data symbols. To simplify the computations, we assume that the symbol-dependent fluctuation of the Jacobian matrix with respect to its mean value is much smaller than this mean value. In these conditions, the average correlation matrix of the estimation error simplifies into:

$$\mathrm{E}_{n,I}\left\{\underline{\Delta}_{\varepsilon}\underline{\Delta}_{\varepsilon}^T\right\}=\left[\frac{\partial\underline{\mathrm{S}}_0}{\partial\hat{\underline{\varepsilon}}}\left(\underline{\varepsilon},\underline{\varepsilon}\right)\right]^{-1}\left[\mathrm{E}_{n,I}\left\{\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)^T\right\}\right]\left[\frac{\partial\underline{\mathrm{S}}_0}{\partial\hat{\underline{\varepsilon}}}\left(\underline{\varepsilon},\underline{\varepsilon}\right)\right]^{-T} \quad (7.54)$$

with

$$\mathrm{E}_{n,I}\left\{\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\varepsilon}\right)^T\right\}=\underline{\mathrm{S}}_0\left(\underline{\varepsilon},\underline{\varepsilon}\right)\underline{\mathrm{S}}_0\left(\underline{\varepsilon},\underline{\varepsilon}\right)^T + \mathrm{E}_I\left\{\underline{\mathrm{N}}_s\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\underline{\mathrm{N}}_s\left(\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)^T\right\} + \mathrm{E}_{n,I}\left\{\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}},\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}},\underline{\mathrm{I}},\underline{\varepsilon},\underline{\varepsilon}\right)^T\right\} \quad (7.55)$$

The use of this approximation is equivalent to the use of the M-CRB instead of the true DD-CRB. The unbiased part of (7.54) has to be compared to the M-CRB. The mean square value obtained by this expression is lower than the true mean square value of the timing error estimate.

A remaining question is the following: does the performance of the estimator depend on the true value $\underline{\varepsilon}$ to be estimated? It is not the case in the single user scenario where the timing error function $S\left(\underline{\mathrm{x}}_{\varepsilon},\underline{\mathrm{I}},\hat{\varepsilon}\right)$ only depends on the difference $\varepsilon-\hat{\varepsilon}$. In the multiuser configuration, however, the timing error vector function $\underline{\mathrm{S}}\left(\underline{\mathrm{x}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\hat{\underline{\varepsilon}}\right)$ depends on all differences $\varepsilon_k-\hat{\varepsilon}_{k'}$ for every pair (k, k'). As a consequence, the performance of the multiuser timing estimator generally depends on the true value $\underline{\varepsilon}$ to be estimated. Intuitively, this can be explained by the idea that the inter-user offsets $\varepsilon_k - \varepsilon_{k'} \neq 0$ for $k \neq k'$ can have a positive or a negative impact on the joint timing estimation process.

Assuming in first-order approximation that the true value $\underline{\varepsilon}$ does not matter, the TED output only depends on the difference $\hat{\underline{\varepsilon}} - \underline{\varepsilon}$. The terms in the above expressions can then be computed for $(\hat{\underline{\varepsilon}}, \underline{\varepsilon}) = (\underline{0}, \underline{0})$ and the Jacobian matrix with respect to the candidate timing vector can be replaced by the Jacobian matrix with respect to the true timing vector:

$$\frac{\partial \underline{\mathrm{S}}}{\partial \hat{\underline{\varepsilon}}} \left(\underline{\mathrm{x}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\varepsilon} \right) = - \frac{\partial \underline{\mathrm{S}}}{\partial \underline{\varepsilon}} \left(\underline{\mathrm{x}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\varepsilon} \right) . \tag{7.56}$$

7.5.2 Feedback structure

In a feedback system, an instantaneous estimate of the timing error vector is computed at the symbol rate, and a timing correction is continuously applied by a feedback signal sent on the downlink to resynchronize the user transmitters.

At time n, a new block of N symbols is detected, and the TED output is computed with $\hat{\underline{\varepsilon}}(n) = \underline{0}$. In these conditions, the estimation error $\underline{\Delta}_{\varepsilon}(n)$ is equal to the true value itself, with the sign changed: $\underline{\Delta}_{\varepsilon}(n) = -\underline{\varepsilon}(n)$.

The TED output vector is then transformed into a vector of K timing corrections $\bar{\underline{\varepsilon}}(n)$ through the matrix gain $\underline{\underline{\mathrm{K}}}_c(n)$ which can be time-dependent. The timing corrections are finally filtered by means of a constant MIMO $K \times K$ loop filter $\underline{\underline{\mathrm{F}}}(z)$. The feedback loop equations are given by:

$$\underline{\varepsilon}(n+1) = \underline{\varepsilon}(n) - \bar{\underline{\varepsilon}}(n) \tag{7.57}$$

$$\bar{\underline{\varepsilon}}(n) = \underline{\underline{\mathrm{F}}}(z) \otimes \left[\underline{\underline{\mathrm{K}}}_c(n) \, \underline{\mathrm{S}} \left(\underline{\mathrm{x}}_{\underline{\varepsilon}}(n), \underline{\mathrm{I}}(n), \underline{0} \right) \right] \tag{7.58}$$

To evaluate the mean square value of the timing error $\underline{\varepsilon}$, some approximations have to be done:

1. A new Taylor expansion of the TED output is defined as follows:

$$\underline{\mathrm{S}} \left(\underline{\mathrm{x}}_{\underline{\varepsilon}}(n), \underline{\mathrm{I}}(n), \underline{0} \right) \approx \underline{\mathrm{S}} \left(\underline{\mathrm{x}}_{\underline{0}}(n), \underline{\mathrm{I}}(n), \underline{0} \right) + \left[\frac{\partial \underline{\mathrm{S}}}{\partial \underline{\varepsilon}} \left(\underline{\mathrm{x}}_{\underline{0}}(n), \underline{\mathrm{I}}(n), \underline{0} \right) \right] \underline{\varepsilon}(n) \tag{7.59}$$

The Taylor expansion proposed here is identical to the one previously proposed in (7.48), in the special cases where the TED output only depends on the difference $\underline{\hat{\varepsilon}} - \underline{\varepsilon}$, which implies (7.56).

2. The noisy fluctuation of the Jacobian matrix is supposed to be small as compared with its mean value:

$$\frac{\partial \underline{\mathrm{S}}}{\partial \underline{\varepsilon}} \left(\underline{\mathrm{x}}_{\underline{0}}(n), \underline{\mathrm{I}}(n), \underline{0}\right) \approx \frac{\partial \underline{\mathrm{S}}_I}{\partial \underline{\varepsilon}} \left(\underline{\mathrm{I}}(n), \underline{0}, \underline{0}\right) \triangleq \underline{\underline{\mathrm{K}}}_d(n) \tag{7.60}$$

This matrix provides the instantaneous (pattern-dependent) multiuser TED sensitivity.

3. The closed loop transfer function is time variant because the Jacobian matrix in the Taylor expansion depends on the specific sequence of symbols contained in the signal. To obtain a time-invariant system, and to ensure the decoupling between the timing errors, the matrix gain $\underline{\underline{\mathrm{K}}}_c(n)$ can be taken as the inverse of the Jacobian matrix:

$$\underline{\underline{\mathrm{K}}}_c(n) = \underline{\underline{\mathrm{k}}} \cdot \underline{\underline{\mathrm{K}}}_d(n)^{-1} \tag{7.61}$$

where $\underline{\underline{\mathrm{k}}}$ is a diagonal matrix of constant values. In these conditions, the feedback loop equation can be expressed in the z-domain as follows:

$$\underline{\varepsilon}(n) = \underline{\underline{\mathrm{H}}}_s(z)\, \underline{\mathrm{N}}_t(n) \tag{7.62}$$

where the loop transfer function $\underline{\underline{\mathrm{H}}}_s(z)$ and the equivalent noise $\underline{\mathrm{N}}_t(n)$ are given by:

$$\begin{aligned} \underline{\underline{\mathrm{H}}}_s(z) &\triangleq \left[(1-z)\, \underline{\underline{\mathrm{E}}}_K - \underline{\underline{\mathrm{F}}}(z)\, \underline{\underline{\mathrm{k}}}\right]^{-1} \underline{\underline{\mathrm{F}}}(z)\, \underline{\underline{\mathrm{k}}} & (7.63) \\ \underline{\mathrm{N}}_t(n) &\triangleq \underline{\underline{\mathrm{K}}}_d(n)^{-1} \left[\underline{\mathrm{S}}_0\left(\underline{0}, \underline{0}\right) + \underline{\mathrm{N}}_s\left(\underline{\mathrm{I}}(n), \underline{0}, \underline{0}\right)\right. \\ &+ \left.\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}}(n), \underline{\mathrm{I}}(n), \underline{0}, \underline{0}\right)\right] & (7.64) \end{aligned}$$

This equivalent noise is made up of three contributions (in volts): a constant bias, a pattern-dependent instantaneous bias ('self-noise'), and a contribution of the additive noise. These contributions are transformed into a vector of equivalent noise (in seconds) by the inverse of the time-variant TED sensitivity matrix $\underline{\underline{\mathrm{K}}}_d(n)$. The correlation matrix of the timing error vector is:

$$\mathrm{E}_{I,n}\left\{\underline{\varepsilon}(n)\underline{\varepsilon}^T(n)\right\} = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \underline{\underline{\mathrm{H}}}_s(e^{j\omega T})\, \underline{\underline{\gamma}}_{N_t}(e^{j\omega T})\, \underline{\underline{\mathrm{H}}}_s^H(e^{j\omega T})\, d\omega \tag{7.65}$$

where the cross-spectrum of the equivalent noise $\underline{\underline{\gamma}}_{N_t}(e^{j\omega T})$ is the z transform of the equivalent noise correlation:

$$\underline{\underline{\Gamma}}_{N_t}(k) = \mathrm{E}_{I,n}\left\{\underline{\mathrm{N}}_t(n)\,\underline{\mathrm{N}}_t^T(n+k)\right\} \tag{7.66}$$

4. A last approximation consists in neglecting the pattern-dependent nature of the TED sensitivity. The average value $\underline{\underline{\mathrm{K}}}_d = \mathrm{E}_I\left\{\underline{\underline{\mathrm{K}}}_d(n)\right\}$ is used in the computations, which provides a simple expression for the equivalent noise correlation:

$$\underline{\underline{\Gamma}}_{N_t}(k) \approx \underline{\underline{\mathrm{K}}}_d^{-1}\left[\underline{\underline{\Gamma}}_{S_0}(k) + \underline{\underline{\Gamma}}_{N_s}(k) + \underline{\underline{\Gamma}}_{N_n}(k)\right]\underline{\underline{\mathrm{K}}}_d^{-T} \tag{7.67}$$

with

$$\underline{\underline{\Gamma}}_{S_0}(k) = \underline{\mathrm{S}}_0\left(\underline{0},\underline{0}\right)\;\underline{\mathrm{S}}_0\left(\underline{0},\underline{0}\right)^T, \tag{7.68}$$

$$\underline{\underline{\Gamma}}_{N_s}(k) = \mathrm{E}_I\left\{\underline{\mathrm{N}}_s\left[\underline{\mathrm{I}}\left(n\right),\underline{0},\underline{0}\right]\;\underline{\mathrm{N}}_s\left[\underline{\mathrm{I}}\left(n+k\right),\underline{0},\underline{0}\right]^T\right\}, \tag{7.69}$$

and

$$\begin{aligned}\underline{\underline{\Gamma}}_{N_n}(k) = \mathrm{E}_{n,I}\{&\underline{\mathrm{N}}_n\left[\underline{\mathrm{n}}(n),\underline{\mathrm{I}}(n),\underline{0},\underline{0}\right]\\ &\underline{\mathrm{N}}_n\left(\underline{\mathrm{n}}(n+k),\underline{\mathrm{I}}(n+k),\underline{0},\underline{0}\right)^T\Big\}\end{aligned} \tag{7.70}$$

This approximation is also equivalent to the use of the M-CRB instead of the true DD-CRB.

Figure 7.2 illustrates the feedback joint timing synchronizer, and Figure 7.3 gives the associated linearized model.

7.6 Multiuser timing error detectors (TED)

7.6.1 Symbol rate multiuser TED

The optimal symbol-rate multiuser TED can be derived from the log-likelihood function in (7.38). Its input is made of N_f consecutive blocks of noise-whitened matched filter outputs $\tilde{y}_{p,\underline{\varepsilon}}(n)$. The optimal TED output is given by :

$$\underline{\mathrm{T}}\left(\underline{\tilde{\mathrm{y}}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\hat{\varepsilon}}\right) = \underline{\mathrm{S}}\left(\underline{\tilde{\mathrm{y}}}_{\underline{\varepsilon}},\underline{\mathrm{I}},\underline{\hat{\varepsilon}}\right) - \underline{\mathrm{S}}\left(\mathrm{E}_n\left\{\underline{\tilde{\mathrm{y}}}_{\underline{\hat{\varepsilon}}}\right\},\underline{\mathrm{I}},\underline{\hat{\varepsilon}}\right) \tag{7.71}$$

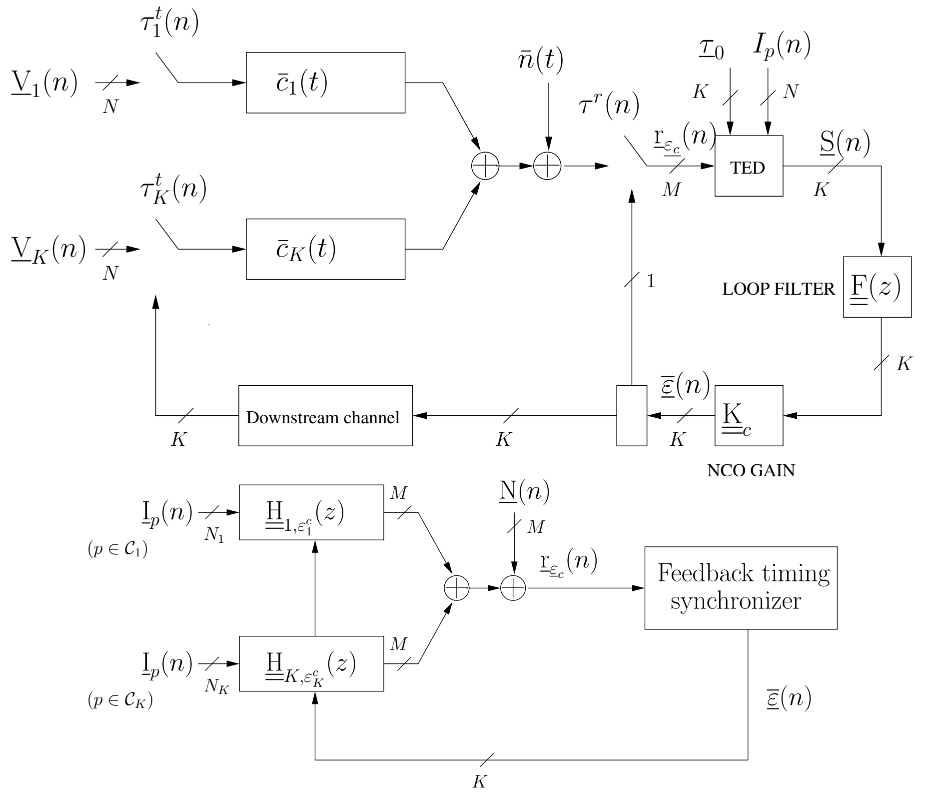

Figure 7.2: *Feedback joint timing synchronization*

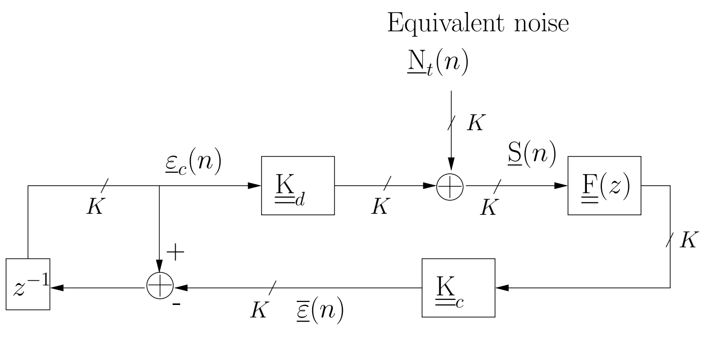

Figure 7.3: *Linearized model of the feedback joint timing synchronizer*

where the k-th element of the S-vector in last equation is given by

$$S_k\left(\underline{\tilde{\mathrm{y}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\hat{\varepsilon}}\right) = \sum_{n'=-n_1}^{n_2} \sum_{p=1}^{N} \tilde{y}_{p,\underline{\varepsilon}}\left(n+n'\right) \cdot \left[\sum_{m=-\infty}^{+\infty} \sum_{p'\in\mathcal{C}_k} \dot{\tilde{x}}_{pp',\hat{\varepsilon}_k}(m)\; I_{p'}\left(n+n'-m\right)\right] \tag{7.72}$$

The second term in (7.71) does not depend on the measurements $\underline{\tilde{\mathrm{y}}}_{\underline{\varepsilon}}$. This term is equivalent to the first term obtained when $\underline{\varepsilon} = \underline{\hat{\varepsilon}}$ and the additive noise is absent. It can be computed in the receiver on the basis of the known data symbols $I_p(n)$ and the known interference coefficients $x_{pp',\hat{\varepsilon}_k}(n)$ and $\dot{x}_{pp',\hat{\varepsilon}_k}(n)$, and subtracted from the first term. This correction term actually guarantees that the proposed timing estimate is unbiased for any combination of data symbols. In other words, it guarantees that the S-hypercurve is all-zero for $\underline{\varepsilon} = \underline{\hat{\varepsilon}}$, and it makes the self-noise equal to zero.

The S-hypercurve of the detector is given by

$$\begin{aligned}\left[\underline{\mathrm{S}}_0\left(\underline{\varepsilon}, \underline{\hat{\varepsilon}}\right)\right]_k &= \mathrm{E}_{n,I}\left\{S_k\left(\underline{\tilde{\mathrm{y}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{\hat{\varepsilon}}\right)\right\} \\ &= N_f\, \sigma_I^2 \sum_{m=-\infty}^{+\infty} \sum_{p=1}^{N} \sum_{p'\in\mathcal{C}_k} \tilde{x}_{pp',\varepsilon_k}(m)\, \dot{\tilde{x}}_{pp',\hat{\varepsilon}_k}(m)\end{aligned} \tag{7.73}$$

The Jacobian matrix associated with the S-hypercurve is diagonal :

$$\frac{\partial \underline{\mathrm{S}}_0}{\partial \underline{\hat{\varepsilon}}}\left(\underline{\varepsilon}, \underline{\hat{\varepsilon}}\right) = N_f\, \sigma_I^2\, \mathrm{diag}\left\{\dot{\mathcal{X}}_k^2\right\}_{1\le k\le K}. \tag{7.74}$$

where $\dot{\mathcal{X}}_k^2$ is defined in (7.45).

The following simplifications can be considered to obtain a practical low-complexity symbol-rate multiuser TED :

1. **Self-noise and bias mitigation.** Even if the implementation of the self-noise mitigation term (second term in (7.71)) is theoretically possible in the receiver, it increases the complexity of the TED. For this reason, it is usually discarded in practical implementations. In that case, a special attention should be paid to the possible bias associated with the S-hypercurve. In a single-user scenario, the bias and self-noise are generally equal to zero when the sequences of interference

coefficients $\tilde{x}_{pp'}(n)$ satisfy a number of symmetry properties :

$$\begin{aligned} \tilde{x}_{pp,0}(-n) &= \tilde{x}_{pp,0}(n) & \dot{\tilde{x}}_{pp,0}(0) &= 0 \\ \tilde{x}_{pp',0}(-n) &= \tilde{x}_{p'p,0}(n) & \dot{\tilde{x}}_{pp',0}(-n) &= -\dot{\tilde{x}}_{p'p,0}(n). \end{aligned} \tag{7.75}$$

These properties are indeed satisfied by the sequences of interference coefficients associated with the noise-whitened matched filter bank receiver, but also by those associated with the simple matched filter bank receiver and the IIR linear MMSE joint detector. The symmetry is lost in the case of a DF joint detector : in that case, a self-noise mitigation term is required to maintain a good level of performance (see a discussion in [67], for example).

In a multiuser scenario, bias and self-noise are present even in the case of symmetric sequences of interference coefficients. To show this, we rewrite the S-hypercurve (7.73) of the multiuser TED as follows :

$$\begin{aligned} [\underline{\mathrm{S}}_0\left(\underline{0},\underline{0}\right)]_k = N_f\,\sigma_I^2 \Bigg\{ &\sum_{p'\in\mathcal{C}_k} \tilde{x}_{p'p',0}(0)\,\dot{\tilde{x}}_{p'p',0}(0) \\ &+ \sum_{p'\in\mathcal{C}_k}\sum_{p\in\mathcal{C}_k,p>p'} \left[\tilde{x}_{pp',0}(0)\,\dot{\tilde{x}}_{pp',0}(0) + \tilde{x}_{p'p,0}(0)\,\dot{\tilde{x}}_{p'p,0}(0)\right] \\ &+ \sum_{m=1}^{+\infty}\sum_{p'\in\mathcal{C}_k}\sum_{p\in\mathcal{C}_k} \left[\tilde{x}_{pp',0}(m)\,\dot{\tilde{x}}_{pp',0}(m) + \tilde{x}_{p'p,0}(-m)\,\dot{\tilde{x}}_{p'p,0}(-m)\right] \\ &+ \sum_{m=-\infty}^{+\infty}\sum_{p\notin\mathcal{C}_k}\sum_{p'\in\mathcal{C}_k} \tilde{x}_{pp',0}(m)\,\dot{\tilde{x}}_{pp',0}(m) \Bigg\}. \end{aligned} \tag{7.76}$$

The first three terms in (7.76) are indeed zero when the symmetry properties in (7.75) are satisfied. The fourth term in (7.76) is not zero, however. This term is specific to the multiuser scenario. It can be shown in the same fashion that the self-noise level is increased in a multiuser scenario. Actually, the symmetry properties in (7.75) ensure that

$$\sum_{k=1}^{K}\left[\mathrm{E}_n\left\{S_k\left(\underline{\tilde{\mathrm{y}}}_0,\underline{\mathrm{I}},\underline{0}\right)\right\}\right] = 0, \tag{7.77}$$

that is to say the sum of self-noises is always zero, but the individual self-noise is non-zero. For this reason, we believe that the self-noise mitigation term should not be discarded in a multiuser configuration, in order to keep a good level of performance.

2. **Noise whitening and scaling.** The noise-whitening filter is not really implemented, as it is not useful for the primary task of the receiver, which is the detection of the information symbol sequences. So, it is preferable that the TED builds its estimate of the timing error on the basis of simple matched filter outputs, for example. The corresponding timing error variance is higher, of course. Actually the role of the noise-whitening filter is twofold :

   (a) It suppresses the correlation between successive noise samples at the output of a given matched filter branch, as well as the mutual correlation between noise samples at the output of different matched filter branches.

   (b) It normalizes the noise variance in each branch.

   While the first effect (whitening) is definitely lost when the noise-whitening filter is not implemented, the second effect (scaling) can be easily recovered by an appropriate modification of the weights associated with each symbol $I_{p'}(n+n'-m)$ in (7.72). The optimal weights can be computed easily. They are inversely proportional to the noise variance $\sigma^2_{w_p}$ at the p-th matched filter output. The modified TED output is

$$S_k\left(\underline{\mathrm{y}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \hat{\underline{\varepsilon}}\right) = \sum_{n'=-n_1}^{n_2} \sum_{p=1}^{N} y_{p,\underline{\varepsilon}}\left(n+n'\right) \cdot \left[\sum_{m=-\infty}^{+\infty} \sum_{p' \in \mathcal{C}_k} \frac{\dot{x}_{pp',\hat{\varepsilon}_k}(m)}{\sigma^2_{w_p}} I_{p'}\left(n+n'-m\right)\right] \quad (7.78)$$

3. **Sequence truncature and selection.** The detector proposed in (7.78) correlates the received signal $y_{p,\underline{\varepsilon}}(n)$ with the N_k symbol sequences $I_{p'}(n)$ with $p' \in \mathcal{C}_k$, using appropriate weights. A further simplification can be proposed by selecting a subset of sequences $I_{p'}(n)$ and/or by truncating these sequences.

   In a single-user ($K=1$) and single-carrier ($N=1$) scenario, this truncature leads to the so-called Mueller and Müller algorithm [68]. The unique sequence of interference coefficient first derivatives $\dot{x}_0(m)$ is there approximated by its 3 central coefficients as follows :

$$\dot{x}_0(m) \approx \{\cdots, 0, +\mathbf{1}, \mathbf{0}, -\mathbf{1}, 0, \cdots\}. \quad (7.79)$$

This provides the well-known Mueller and Müller TED :

$$S\left(\underline{\mathrm{y}}_{\varepsilon}, \underline{\mathrm{I}}, \hat{\varepsilon}\right) = \sum_{n'=-n_1}^{n_2} y_{\varepsilon}\left(n+n'\right) \left[I(n+n'+1) - I(n+n'-1)\right]. \tag{7.80}$$

This approximation is effective in the single-carrier scenario as the largest part of the available energy $\sum_n \dot{x}_0^2(n)$ is indeed located in the coefficients in position ± 1.

The extension of the Mueller and Müller algorithm to the case of FB-based transmission ($N > 1$) is not trivial. Let us consider a single-user system ($K = 1$) and a paraunitary FB $\underline{\underline{\mathrm{S}}}(z)$ of size N. The polyphase matrix $\underline{\underline{\dot{\mathrm{X}}}}_0(z)$ associated with the N^2 sequences of interference coefficient first derivatives $\dot{x}_{pp',0}(n)$ is given by

$$\underline{\underline{\dot{\mathrm{X}}}}_0(z) = \underline{\underline{\mathrm{S}}}^H(z)\, \underline{\underline{\dot{\mathrm{G}}}}_0(z)\, \underline{\underline{\mathrm{S}}}(z) \tag{7.81}$$

where $\underline{\underline{\dot{\mathrm{G}}}}_0(z)$ is the Toeplitz polyphase matrix associated with $\dot{g}(nT_c)$, the channel correlation first derivative, sampled at the chip rate. The total energy associated with the sequences $\dot{x}_{pp',0}(n)$ is

$$\begin{aligned}
\dot{\mathcal{X}}^2 &= \sum_{n=-\infty}^{+\infty} \sum_{p=1}^{N} \sum_{p'=1}^{N} \dot{x}_{pp',0}^2(n) && (7.82)\\
&= \int_{-\pi}^{\pi} \mathrm{tr}\left\{ \underline{\underline{\dot{\mathrm{X}}}}_0^H(e^{j\Omega})\, \underline{\underline{\dot{\mathrm{X}}}}_0(e^{j\Omega}) \right\} \frac{\mathrm{d}\Omega}{2\pi} && (7.83)\\
&= \int_{-\pi}^{\pi} \mathrm{tr}\left\{ \underline{\underline{\dot{\mathrm{G}}}}_0^H(e^{j\Omega})\, \underline{\underline{\dot{\mathrm{G}}}}_0(e^{j\Omega}) \right\} \frac{\mathrm{d}\Omega}{2\pi} && (7.84)\\
&= N \sum_{n=-\infty}^{+\infty} \dot{g}(nT_c)^2 && (7.85)
\end{aligned}$$

where (7.84) comes from the paraunitary property of the FB. So, the quantity $\dot{\mathcal{X}}^2$ is independent of the selected FB. From (7.85), it actually depends on N, but this only reflects that the observation window associated with a burst of N_f blocks has a duration of $N_f N T_c$ which obviously increases proportionally to N. While the sequence $\dot{g}(nT_c)$ itself can be effectively restricted to the central coefficients $\dot{g}(\pm T_c)$, it is a priori unclear how the available energy $\sum_n \dot{g}^2(nT_c)$ is distributed among the various sequences $\dot{x}_{pp',0}(n)$ through Equation (7.81). This redistribution of the energy depends on the selected FB.

A determining factor is the length of the signatures $s_p(n)$ constituting the FB. Let $N' = (n_b + 1)\, N$ be the signature length, which means that each signature extends over $n_b + 1$ successive blocks of symbols. This parameter n_b is equal to 0 for the basic FBs such as Walsh-Hadamard codes or DMT basis functions (see Section 4.2.3). The FB polyphase matrix is given by

$$\underline{\underline{\mathrm{S}}}(z) = \sum_{n=0}^{n_b} \underline{\underline{\mathrm{S}}}_n\, z^{-n}. \tag{7.86}$$

Introducing (7.86) into (7.81), we get

$$\underline{\underline{\dot{\mathrm{X}}}}_0(z) = \sum_{m=-n_b}^{n_b} \left[\sum_{n=0}^{n_b} \underline{\underline{\mathrm{S}}}_n^H\, \underline{\underline{\dot{\mathrm{G}}}}_0(z)\, \underline{\underline{\mathrm{S}}}_{n-m} \right] z^{-m} \tag{7.87}$$

For large values of N, the Toeplitz polyphase matrix $\underline{\underline{\dot{\mathrm{G}}}}_0(z)$ can be approximated by the zero-order term in its z-transform expansion :

$$\underline{\underline{\dot{\mathrm{G}}}}_0(z) = \sum_{n=-\infty}^{+\infty} \underline{\underline{\dot{\mathrm{G}}}}_{0,n}\, z^{-n} \approx \underline{\underline{\dot{\mathrm{G}}}}_{0,0}. \tag{7.88}$$

It comes that

$$\underline{\underline{\dot{\mathrm{X}}}}_0(z) \approx \sum_{m=-n_b}^{n_b} \underline{\underline{\dot{\mathrm{X}}}}_{0,n}\, z^{-n} \quad \text{with} \quad \underline{\underline{\dot{\mathrm{X}}}}_{0,n} = \sum_{n=0}^{n_b} \underline{\underline{\mathrm{S}}}_n^H\, \underline{\underline{\dot{\mathrm{G}}}}_{0,0}\, \underline{\underline{\mathrm{S}}}_{n-m}. \tag{7.89}$$

The sequences $\dot{x}_{pp',0}(n)$ can thus be truncated to the interval $n \in \{-n_b, n_b\}$ without a significant loss of performance. The performance loss decreases with N. An important special case is that of FBs extending over a single block, i.e. $n_b = 0$. For these FBs, the sequences $\dot{x}_{pp',0}(n)$ can be restricted to their central coefficient $n = 0$. The corresponding energy is

$$\begin{aligned}
\sum_{p=1}^{N} \sum_{p'=1}^{N} \dot{x}_{pp',0}^2(n) &= \operatorname{tr}\left\{ \left[\underline{\underline{\mathrm{S}}}^H\, \underline{\underline{\dot{\mathrm{G}}}}_{0,0}\, \underline{\underline{\mathrm{S}}} \right]^H \left[\underline{\underline{\mathrm{S}}}^H\, \underline{\underline{\dot{\mathrm{G}}}}_{0,0}\, \underline{\underline{\mathrm{S}}} \right] \right\} && (7.90)\\
&= \operatorname{tr}\left\{ \underline{\underline{\dot{\mathrm{G}}}}_{0,0}^H\, \underline{\underline{\dot{\mathrm{G}}}}_{0,0} \right\} && (7.91)\\
&= N \left[\dot{g}^2(0) + 2 \sum_{n=1}^{N-1} \frac{N-k}{N}\, \dot{g}^2(nT_c) \right], && (7.92)
\end{aligned}$$

which rapidly converges to (7.85) for large values of N. The simplified multiuser TED incorporating the proposed truncature is given by

$$S_k\left(\underline{\mathrm{y}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \hat{\underline{\varepsilon}}\right) = \sum_{n'=-n_1}^{n_2} \sum_{p=1}^{N} y_{p,\underline{\varepsilon}}\left(n+n'\right) \cdot \left[\sum_{p' \in \mathcal{C}_k} \frac{\dot{x}_{pp',\hat{\varepsilon}_k}(0)}{\sigma_{w_p}^2} I_{p'}\left(n+n'\right)\right] \quad (7.93)$$

for FBs with a single-block extension. A further approximation can be investigated by replacing the MF outputs $\underline{\mathrm{y}}_{\underline{\varepsilon}}$ with MMSE JD outputs $\hat{\underline{\mathrm{I}}}_{\underline{\varepsilon}}$ in the last equation, and by updating the weighting coefficients accordingly.

7.6.2 Early-late multiuser TED

The optimal multiuser TED at the channel output can be derived from the log-likelihood function in (7.17). It requires the implementation of a bank of derivated matched filters $\dot{h}_{pk}(-t)$ in the receiver. Its input is made of N_f consecutive blocks of derivated matched filter outputs $\dot{y}_{p,\underline{\varepsilon}}(n)$. The optimal TED output is given by :

$$\underline{\mathrm{T}}\left(\dot{\underline{\mathrm{y}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{0}\right) = \underline{\mathrm{S}}\left(\dot{\underline{\mathrm{y}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{0}\right) - \underline{\mathrm{S}}\left(\mathrm{E}_n\left\{\dot{\underline{\mathrm{y}}}_{\underline{0}}\right\}, \underline{\mathrm{I}}, \underline{0}\right) \quad (7.94)$$

where the k-th element of the S-vector in last equation is given by

$$S_k\left(\dot{\underline{\mathrm{y}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{0}\right) = \sum_{n'=-n_1}^{n_2} \sum_{p \in \mathcal{C}_k} \dot{y}_{p,\underline{\varepsilon}}\left(n+n'\right)\ I_p\left(n+n'\right). \quad (7.95)$$

The S-hypercurve of the detector is given by

$$\left[\underline{\mathrm{S}}_0\left(\underline{\varepsilon}, \underline{0}\right)\right]_k = N_f\, \sigma_I^2 \sum_{p \in \mathcal{C}_k} \dot{x}_{pp,\varepsilon_k}(0). \quad (7.96)$$

The Jacobian matrix associated with the S-hypercurve is diagonal :

$$\frac{\partial \underline{\mathrm{S}}_0}{\partial \underline{\varepsilon}}\left(\underline{\varepsilon}, \underline{0}\right) = N_f\, \sigma_I^2\, \mathrm{diag}\left\{\sum_{p \in \mathcal{C}_k} \ddot{x}_{pp,\varepsilon_k}(0)\right\}_{1 \le k \le K}. \quad (7.97)$$

The following simplifications can be considered to obtain a practical low-complexity symbol-rate multiuser TED :

1. **Self-noise and bias mitigation.** Like for the optimal symbol-rate multiuser TED, the second term in (7.94) is a self-noise mitigation term. Even with perfectly symmetrical interference waveforms, there would be a significant self-noise if that term was suppressed, due to the multiuser configuration.

2. **MMSE joint detector and scaling.** The bank of matched filters can be replaced by a joint detector designed with an MMSE criterion. The optimal linear MMSE JD is actually obtained by cascading the matched filter bank with a MIMO filter whose purpose is to reduce the residual interference. Let $w_p(t)$ be the filter associated with the MMSE-JD p-th branch. The continuous-time interference waveforms are given by $\overline{x}_{pp',\underline{\varepsilon}}(n) = \left[\bar{h}_{p'k}(t) \otimes w_p(t)\right](nT+\varepsilon)$. The idea here is thus to replace the bank of derivated matched filters $\dot{h}_{pk}(-t)$ by a bank of derivated MMSE filters $\dot{w}_p(t)$. The TED based on joint detector outputs $\hat{I}_p(n)$ is given by

$$S_k\left(\dot{\hat{\underline{\mathrm{I}}}}_{\underline{\varepsilon}}, \underline{\mathrm{I}}, \underline{0}\right) = \sum_{n'=-n_1}^{n_2} \sum_{p \in \mathcal{C}_k} \lambda_p^2 \, \dot{\hat{I}}_{p,\underline{\varepsilon}}\left(n+n'\right) \, I_p\left(n+n'\right) \qquad (7.98)$$

where the weights λ_p^2 associated with each JD branch have to be optimized. These weights are necessary as the SNR of each subchannel should be taken into account. Indeed, while the amplitudes of the matched filter outputs depend on the channel attenuation associated with each subchannel, that of the JD output are normalized. The optimal weights λ_p^2 in (7.98) can be shown to be

$$\lambda_p^2 = \frac{\ddot{\overline{x}}_{pp}(0)}{\int_{-\infty}^{+\infty} \dot{w}_p^2(t)\,\mathrm{d}t}. \qquad (7.99)$$

The optimal weights in (7.99) are effectively equal to 1 in the case where the detector filters $w_p(t)$ are identical to the matched filters $h_{pk}(-t)$.

3. **Early-late implementation.** The ultimate approximation of the optimal TED is obtained by replacing the derivative in (7.98) by a finite difference :

$$\dot{\hat{I}}_{p,\underline{\varepsilon}}\left(n\right) \approx \hat{I}_{p,\underline{\varepsilon}^+}\left(n\right) - \hat{I}_{p,\underline{\varepsilon}^-}\left(n\right) \qquad (7.100)$$

where $\varepsilon_k^+ = \varepsilon_k + T_s$ and $\varepsilon_k^- = \varepsilon_k - T_s$ for all k. The $2N$ sequences $\hat{I}_{p,\underline{\varepsilon}^+}\left(n\right)$ and $\hat{I}_{p,\underline{\varepsilon}^-}\left(n\right)$ can be obtained by feeding the bank of MMSE filters with adequately delayed or anticipated versions of the received fractional-rate sequence. The resulting early-late multiuser TED is finally

$$S_k\left(\left\{\hat{\underline{\mathrm{I}}}_{\underline{\varepsilon}^+},\hat{\underline{\mathrm{I}}}_{\underline{\varepsilon}^-}\right\},\underline{\mathrm{I}},\underline{0}\right)=\sum_{n'=-n_1}^{n_2}\sum_{p\in\mathcal{C}_k}\lambda_p^2\left[\hat{I}_{p,\underline{\varepsilon}^+}\left(n+n'\right)\right.$$
$$\left.-\hat{I}_{p,\underline{\varepsilon}^-}\left(n+n'\right)\right]I_p\left(n+n'\right)\quad(7.101)$$

The fundamental drawback associated with the early-late TED is the need to compute $3N$ different JD outputs for each block of received samples.

7.7 Asymptotic performance for an AWGN channel

In this section, the relative performance of the various TEDs are compared in a simplified scenario with a single user ($K = 1$) and a frequency-flat channel ($c(t) = \delta(t-\tau)$). The channel correlation $g(t)$ (including the shaping filter in the transmitter and the matched filter) corresponds to a full raised cosine filter with roll-off α. The variance of the timing estimate is given by

$$\sigma_\varepsilon^2=\frac{1}{N_f\,N}\left(\frac{\sigma_I^2}{N_0/2}\right)^{-1}F\left(\alpha\right).\qquad(7.102)$$

The first factor in (7.102) represents the length of the observation window, expressed in number of chip durations T_c. The second factor gives the effect of the signal-to-noise ratio on the timing detection. The last factor $F\left(\alpha\right)$, that only depends on the roll-off, is specific of the selected TED.

Table 7.1 gives the expression of the inverse factor $F\left(\alpha\right)^{-1}$ for the different TEDs presented in this chapter. The second column corresponds to a single-carrier system, while the third column is for a paraunitary FB-based transmission with N subchannels. The timing error variance associated with the first three TEDs is known to be independent of the FB and independent of N. The common factor $F\left(\alpha\right)$ can thus be computed analytically in the single-carrier case as a function of α. The last line corresponds to the truncated symbol-rate TED proposed in (7.93). The associated timing error variance only depends on N and α, and can be computed numerically by using (7.92). The computed functions $F\left(\alpha\right)$ are displayed in Figure 7.4.

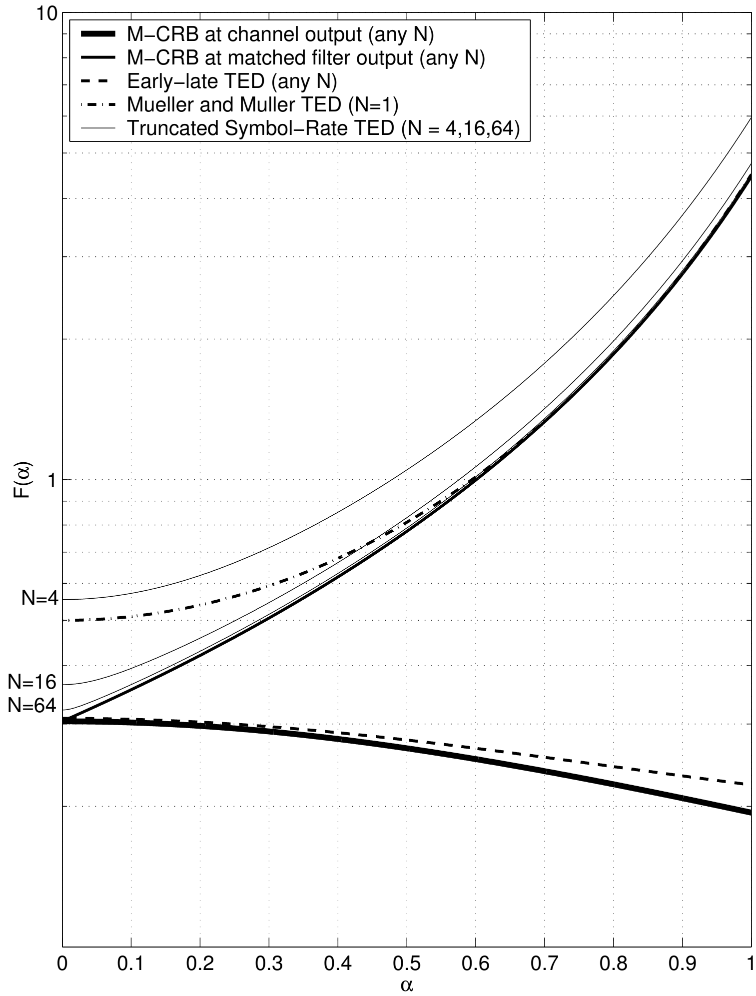

Figure 7.4: *Relative performance of various single-user TEDs for a frequency-flat channel*

Table 7.1: $F(\alpha)^{-1}$ *of various timing error detectors*

TED	$N=1$	$N>1$
ML (channel output)	$-\ddot{g}(0)$	$-\frac{1}{N}\sum_{p=1}^{N}\ddot{x}_{pp}(0)$
Early-late	$\frac{2\dot{g}^2(T_s)}{1-g(2T_s)}$	$\frac{2\left[\frac{1}{N}\sum_{p=1}^{N}\dot{x}_{pp}(T_s)\right]^2}{1-\frac{1}{N}\sum_{p=1}^{N}x_{pp}(2T_s)}$
ML (MF output)	$\sum_{n=-\infty}^{+\infty}\dot{g}^2(nT_c)$	$\frac{1}{N}\sum_{p=1}^{N}\sum_{p'=1}^{N}\sum_{n=-\infty}^{+\infty}\dot{x}^2_{pp'}(n)$
Mueller and Müller	$\sum_{n=-1}^{1}\dot{g}^2(nT_c)$	
Truncated symbol-rate		$\frac{1}{N}\sum_{p=1}^{N}\sum_{p'=1}^{N}\dot{x}^2_{pp'}(0)$

Table 7.2: *M-CRB for an uplink 5-user powerline channel with CAP-OFDMA and $N = 128$ tones*

	User 2	User 5	User 10	User 15	User 20
$\frac{2E_k}{N_0}$ [dB]	94.08	89.42	77.52	73.13	73.05
$\frac{1}{2\pi} W_k$ [MHz]	4.49	4.90	2.79	1.94	1.12
$N_f \frac{\sigma^2_{\varepsilon_k}}{T'^2_c}$ [dB]	-103.99	-100,09	-83,32	-75,74	-70,91

This figure provides a basic understanding of the performance loss that can be expected from :

- The implementation of a ML symbol-rate TED rather than an optimal ML TED using channel outputs.
- The early-late approximation.
- The truncature loss associated with the Mueller and Müller TED ($N = 1$).
- The truncature loss associated with the symbol rate multiuser TED proposed in this work ($N > 1$).

7.8 Applications to the powerline channel

Let us now illustrate the performance of the proposed multiuser timing error detectors with a practical example. We consider the 5-user uplink PLC system introduced in Section 5.8. The multiple access method is CAP-OFDMA with $N = 128$ tones. The power spectral density of the transmitted signals resulting from the signature allocation algorithm is presented in Figure 5.11.

Table 7.2 gives the M-CRB on the timing error variance, according to equation (7.32). The first row gives the user-specific global SNR $\frac{2\mathcal{E}_k}{N_0}$ (in dB), obtained by combining the received energy in all subchannels allocated to each user. The second row gives the mean bandwidth $\frac{W_k}{2\pi}$ (in MHz) of the received signals. Finally, the third row gives the minimum timing error variance, normalized with respect to the squared chip duration $T_c^{'2}$.

The performance computation of practical TEDs requires the analysis of the continuous-time interference waveforms $x_{pp'}(t)$ obtained by combining the equivalent channel $h_{p'k}(t)$ with the p-th branch of the receiver :

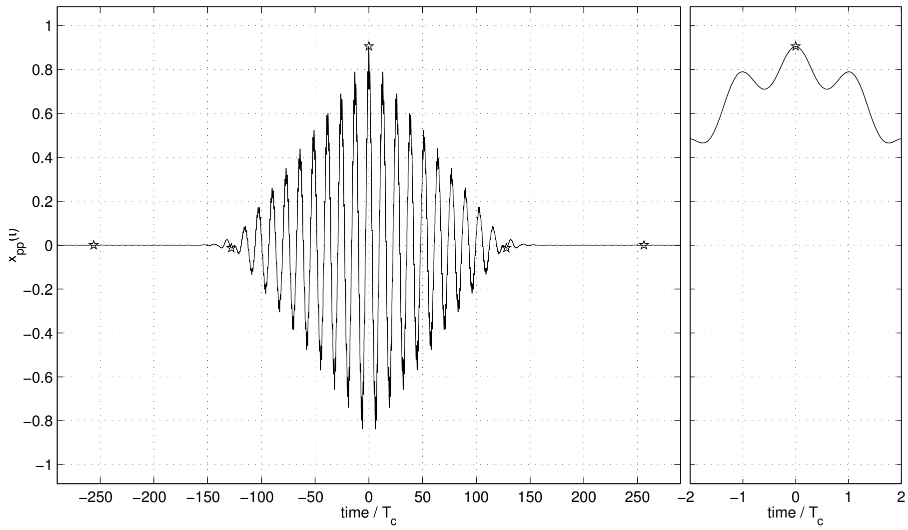

Figure 7.5: *Continuous-time interference waveform* $x_{10,10}(t)$ *(CAP-OFDMA,* $N = 128$*).*

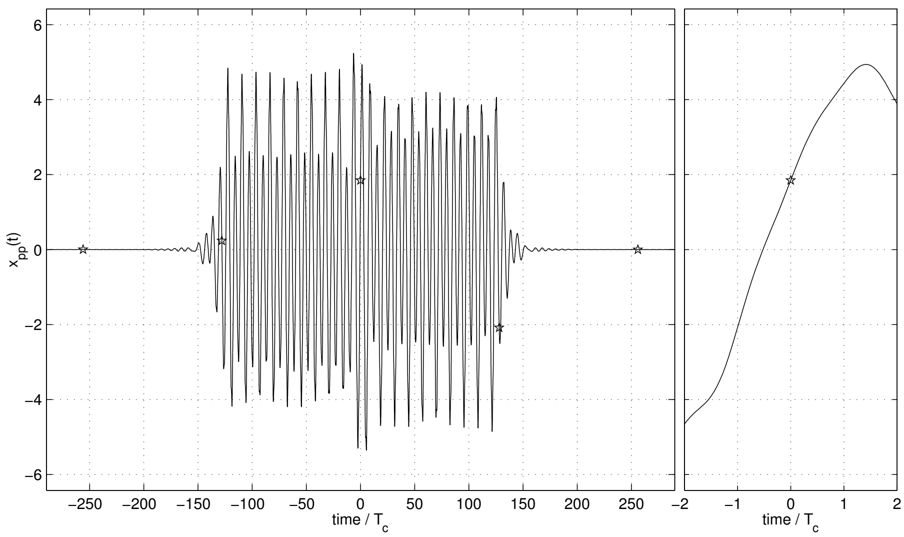

Figure 7.6: *Continuous-time interference waveform* $x_{40,20}(t)$ *(CAP-OFDMA,* $N = 128$*).*

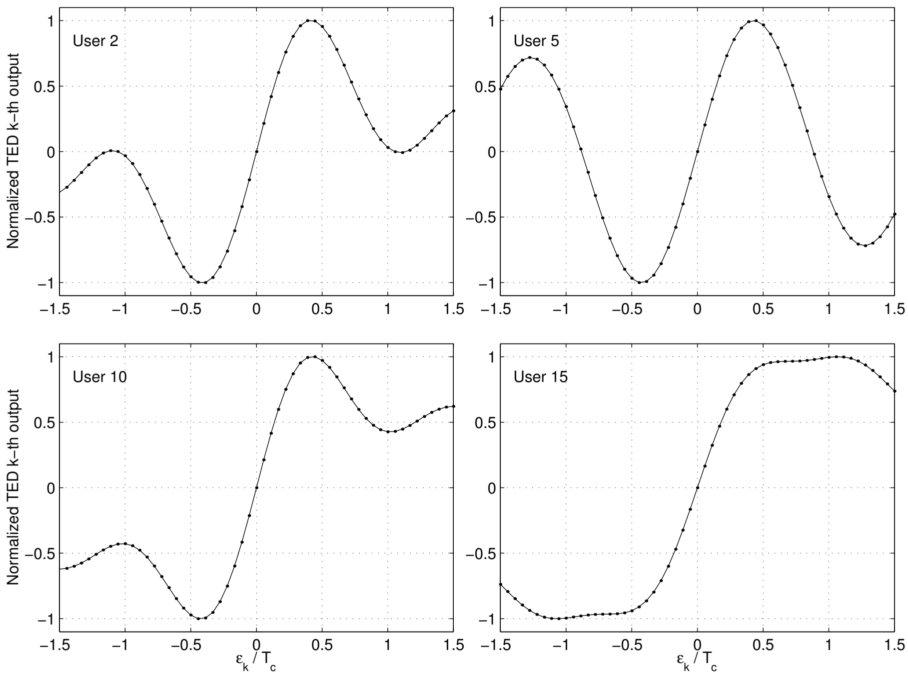

Figure 7.7: *Normalized S-curves for the Early-Late multiuser TED (CAP-OFDMA with a matched filter bank receiver).*

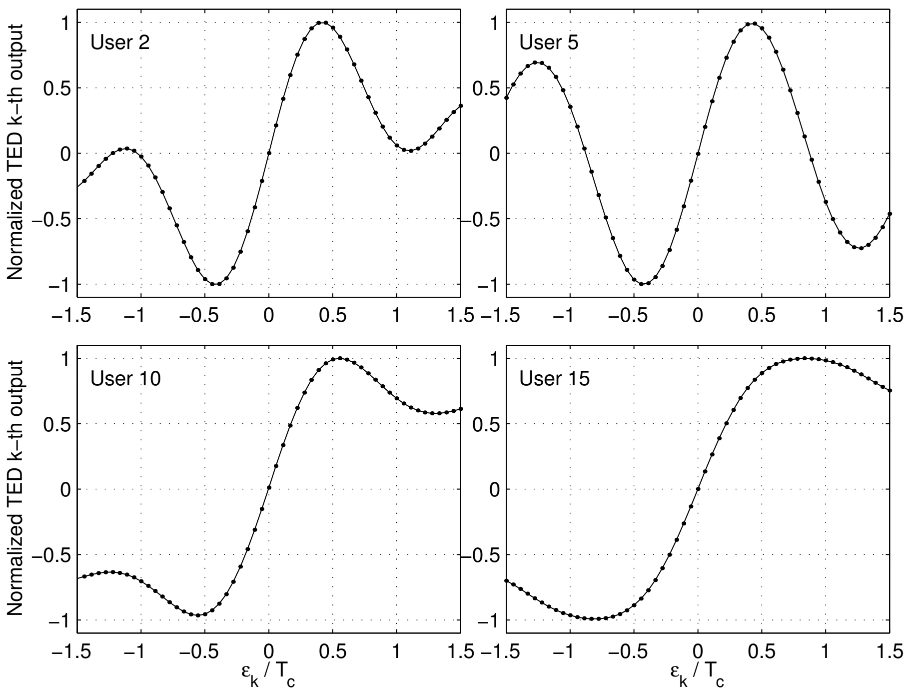

Figure 7.8: *Normalized S-curves for the Symbol-rate multiuser TED (CAP-OFDMA with a matched filter bank receiver).*

Table 7.3: *Timing error variance of the Early-Late multiuser TED vs. M-CRB [dB]*

Detector	User 2	User 5	User 10	User 15	User 20	weights
MF bank	0.08	0.01	0.20	0.09	0.21	λ_p^2
	0.15	0.04	0.29	0.45	0.38	1
MMSE-JD	**2.86**	**3.37**	**3.28**	**5.44**	**11.54**	λ_p^2
	4.15	6.77	4.47	8.14	24.52	1

- Figure 7.5 illustrates the $x_{10,10}(t)$ waveform obtained with a matched filter receiver. Signature 10 being allocated to user 15, this waveform corresponds to $h_{10,15}(t) \otimes h_{10,15}(-t)$. The stars represent the nominal interference coefficients obtained by sampling the continuous-time interference waveform at the symbol rate $1/T$, with a sampling phase equal to zero. The concavity of the waveform at the origin $\ddot{x}_{pp}(0)$, which can be evaluated on the right part of Figure 7.5, determines the contribution of the p-th subchannel to the Early-Late estimation of the k-th timing error ε_k.

- Figure 7.6 illustrates the $x_{40,20}(t)$ waveform obtained with a matched filter receiver. Signatures 20 and 40 are both allocated to user 10, so this waveform corresponds to $h_{20,10}(t) \otimes h_{40,10}(-t)$. The slope of the waveform at the origin $\dot{x}_{pp'}(0)$, which can be evaluated on the right part of Figure 7.6, determines the contribution of the p-th receiver branch and the p'-th subchannel to the Symbol-Rate estimation of the k-th timing error ε_k.

Figures 7.7 and 7.8 give the normalized S-curves for the Early-Late and Symbol-Rate multiuser TEDs, respectively, placed at the output of a matched filter bank receiver. The combination of these individual S-curves generates the global S-hypercurve. The slope of the S-curve at the origin determines the performance of the TED.

Finally, Tables 7.3 and 7.4 give the relative performance (in dB) of various Early-Late and Symbol-Rate timing error detectors as compared to the M-CRB. The timing error variance is computed according to Equation (7.54). Self-noise compensation is assumed in the computations, so the timing error variance is basically equal to the variance of the additive noise term, divided by the squared slope of the TED S-curve. In each case, two kinds of receivers are considered : a matched filter bank (which is ideal for the

Table 7.4: *Timing error variance of the Symbol-Rate multiuser TED vs. M-CRB [dB]*

Detector	User 2	User 5	User 10	User 15	User 20	weights
MF Bank	0.86	0.52	2.35	1.72	1.64	$\dot{x}_{pp'}/\sigma^2_{w_p}$
	0.84	1.78	8.93	7.36	3.28	$\dot{x}_{pp'}$
MMSE-JD	**3.89**	**3.91**	**4.29**	**5.95**	**6.51**	$\dot{x}_{pp'}/\sigma^2_{w_p}$
	26.48	22.49	5.56	6.27	8.00	$\dot{x}_{pp'}$

timing estimation but not for the symbol estimation), or a linear MMSE-JD. The performance of the proposed TEDs are found to be very close to the M-CRB as long as a matched filter bank is implemented in the receiver. The performance loss is important if the MF bank is replaced by a linear MMSE-JD, especially for the users with highly dispersive channels.

7.9 Conclusion

In this Chapter, the issue of Decision-Directed multiuser timing synchronization has been addressed, assuming perfect decisions. The FB-based multiple access system has been redefined to include user-specific timing errors ε_k.

The Cramèr-Rao bound (CRB) on the joint timing error estimator has been derived. The bound actually varies with the symbols used for transmission. The average bound (i.e. the expectation with respect to the transmitted symbols) is not easy to compute. For long observation windows, it has been shown to converge to the so-called modified CRB. The M-CRB, given by Equation (7.32), is easy to compute: it only depends on the total SNR $\frac{2\mathcal{E}_k}{N_0}$, the mean square bandwidth W_k^2, and the length of the observation window. Moreover, the timing errors related to different users are uncorrelated. In a single-user system, the M-CRB is independent of the selected FB. In a multiuser system, however, the signature allocation algorithm has a strong impact on the distribution of the user-specific timing error variances.

Another M-CRB can be computed by considering the problem of timing estimation at the symbol rate, i.e. at the output of a matched filter bank. The symbol-rate M-CRB has been derived. It is higher than the M-CRB computed at the channel output.

Feedforward and feedback *multiuser* timing error tracking structures have been analyzed in detail. A number of approximations have been identified, providing a tractable expression (7.54) for the correlation matrix of the joint timing estimation error.

Two practical multiuser TEDs have been proposed. The first one (7.93) is derived from the maximum-likelihood symbol-rate multiuser TED (7.71). Two major approximations are proposed : the noise-whitening filter is discarded and replaced by a well-chosen set of weights, and the sequences of interference waveform slopes are truncated to their central coefficient. The second one (7.101) is derived from the maximum-likelihood multiuser TED at channel output (7.94). The proposed approximations are the use of an MMSE joint detector (with an appropriate set of weights) rather than the matched filter bank, and the early-late implementation of the first-order derivative.

The performance of the proposed TEDs has been tested with the 5-user PLC channel. The performance loss with respect to the M-CRB is acceptable, and comparable for both systems.