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$\mathbb{Z}S_n$-MODULES AND POLYNOMIAL IDENTITIES WITH INTEGER COEFFICIENTS

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We show that, like in the case of algebras over fields, the study of multilinear polynomial identities of unitary rings can be reduced to the study of proper polynomial identities. In particular, the factors of series of $\mathbb{Z}S_n$-submodules in the $\mathbb{Z}S_n$-modules of multilinear polynomial functions can be derived by the analog of Young's (or Pieri's) rule from the factors of series in the corresponding $\mathbb{Z}S_n$-modules of proper polynomial functions. As an application, we calculate the codimensions and a basis of multilinear polynomial identities of unitary rings of upper triangular 2×2 matrices and infinitely generated Grassmann algebras over unitary rings. In addition, we calculate the factors of series of $\mathbb{Z}S_n$-submodules for these algebras. Also we establish relations between codimensions of rings and codimensions of algebras and show that the analog of Amitsur's conjecture holds in all torsion-free rings, and all torsion-free rings with 1 satisfy the analog of Regev's conjecture.

Keywords: Associative ring; polynomial identity; integral representation; upper triangular matrix; Grassmann algebra; codimension; Young diagram; symmetric group; group ring.

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Polynomial identities and their numeric and representational characteristics are well studied in the case of algebras over fields of characteristic zero (see e.g. [3, 5]). However, polynomial identities in rings also play an important role [12]. The systematic study of multilinear polynomial identities started in 1950 by Mal'cev [9] and Specht [14]. In his paper, Specht considered polynomial identities with integer coefficients. This paper is devoted to numeric and representational characteristics of polynomial identities with integer coefficients.

In Propositions 2.1–2.4, we prove basic facts about codimensions of polynomial identities with integer coefficients in the case when the ring is an algebra over a field. In Theorem 2.5, we consider algebras obtained from rings by the tensor product of the ring and a field. As a consequence of Theorem 2.5, we derive the analog of Amitsur's conjecture for all torsion-free rings and the analog of Regev's conjecture for all torsion-free rings with 1. In Theorem 3.1, we show that the ordinary codimensions of rings with 1 can be calculated using their proper codimensions by the same formula as in the case of algebras over fields [3, Theorem 4.3.12(ii)]. In Theorem 3.4, we prove an analog of Drensky's theorem [3, Theorem 12.5.4] that establishes a relation between $\mathbb{Z}S_n$-modules corresponding to proper and ordinary polynomial identities of unitary rings. In order to apply Theorem 3.4, we show that the analog of Young's (or Pieri's) rule holds for $\mathbb{Z}S_n$-modules too (Theorem 4.4).

In the case of algebras over fields, few examples are known where the numeric and representational characteristics can be precisely evaluated. Among those are the algebras of upper triangular matrices [7, 10, 11, 13] and the Grassmann algebra [8]. In Secs. 5 and 6, we apply the results obtained in the preceding sections, to evaluate the basis of multilinear polynomial identities with integer coefficients and calculate their numeric and representational characteristics for unitary rings of upper triangular 2×2 matrices and infinitely generated Grassmann algebras over unitary rings.

1. Introduction

Let R be a ring. If R has a unit element 1_R, then the number

$$\operatorname{char} R := \min\{n \in \mathbb{N} \mid n1_R = 0\} = \min\{n \in \mathbb{N} \mid na = 0 \text{ for all } a \in R\}$$

is called the *characteristic* of R. (As usual, if $n1_R \neq 0$ for all $n \in \mathbb{N}$, then $\operatorname{char} R := 0$.)

Let $\mathbb{Z}\langle X\rangle$ be the free associative ring without 1 on the countable set $X = \{x_1, x_2, \ldots\}$, i.e. the ring of polynomials in non-commuting variables from X without a constant term.

Let I be an ideal of a ring R. We say that I is a *T-ideal* of R if $\varphi(I) \subseteq I$ for all $\varphi \in \operatorname{End}(R)$. We say that $f \in \mathbb{Z}\langle X\rangle$ is a *polynomial identity* of R *with integer coefficients* if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in R$. In other words, f is a polynomial identity if $\psi(f) = 0$ for all $\psi \in \operatorname{Hom}(\mathbb{Z}\langle X\rangle, R)$. Note that the set $\operatorname{Id}(R, \mathbb{Z})$ of polynomial identities of R with integer coefficients is a T-ideal of $\mathbb{Z}\langle X\rangle$.

Let $P_n(\mathbb{Z})$ be the additive subgroup of $\mathbb{Z}\langle X\rangle$ generated by $x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$, $\sigma \in S_n$. (Here S_n is the nth symmetric group, $n \in \mathbb{N}$.) Then $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ is a finitely generated Abelian group which is the direct sum of free and primary cyclic groups:

$$\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R, \mathbb{Z})} \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{c_n(R,0)} \oplus \bigoplus_{\substack{p \text{ is a prime} \\ \text{number}}} \bigoplus_{k \in \mathbb{N}} \Big(\underbrace{\mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k}}_{c_n(R,p^k)} \Big).$$

We call the numbers $c_n(R,q)$ the *codimensions* of polynomial identities of R with integer coefficients.

Note that the symmetric group S_n is acting on $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ by permutations of variables, i.e. $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is a $\mathbb{Z}S_n$-module. We refer to $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ as the *$\mathbb{Z}S_n$-module of ordinary multilinear polynomial functions* on R.

Denote by $\Gamma_n(\mathbb{Z})$ the subgroup of $P_n(\mathbb{Z})$ that consists of *proper polynomials*, i.e. linear combinations of products of long commutators. (All long commutators in the paper are left normed, e.g. $[x,y,z,t] := [[[x,y],z],t]$.) Then $\Gamma_n(\mathbb{Z})$ is a $\mathbb{Z}S_n$-submodule of $P_n(\mathbb{Z})$. Obviously, $\Gamma_1(\mathbb{Z}) = 0$.

Analogously, we define the *codimensions* $\gamma_n(R,q)$ of proper polynomial identities of R:

$$\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \cong \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{\gamma_n(R,0)} \oplus \bigoplus_{\substack{p \text{ is a prime} \\ \text{number}}} \bigoplus_{k\in\mathbb{N}} \Big(\underbrace{\mathbb{Z}_{p^k}\oplus\cdots\oplus\mathbb{Z}_{p^k}}_{\gamma_n(R,p^k)} \Big).$$

If R has a unit element 1_R, then, by the definition, $\gamma_0(R,q)$ is the number of $\mathbb{Z}_q$ in the decomposition of the cyclic additive subgroup of R generated by 1_R. We refer to $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ as the *$\mathbb{Z}S_n$-module of proper multilinear polynomial functions* on R.

If A is an algebra over a field F, then we can consider codimensions $c_n(A,F) := \dim \frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A,F)}$ of polynomial identities of A with coefficients from F. (See [5, Definition 4.1.1].) Here, $\mathrm{Id}(A,F) \subset F\langle X\rangle$ is the set of polynomial identities of A with coefficients from F, and $P_n(F)$ is the subspace of $F\langle X\rangle$ generated by $x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$, $\sigma\in S_n$. The subspace of $P_n(F)$ consisting of proper polynomials, is denoted by $\Gamma_n(F)$.

We say that $\lambda = (\lambda_1,\dots,\lambda_s)$ is a (*proper or ordered*) *partition* of n and write $\lambda \vdash n$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, $\lambda_i \in \mathbb{N}$, and $\sum_{i=1}^s \lambda_i = n$. In this case, we write $\lambda \vdash n$. For our convenience, we assume $\lambda_i = 0$ for all $i > s$.

We say that $\mu = (\mu_1,\dots,\mu_s)$ is an *unordered partition* of n if $\mu_i \in \mathbb{N}$ and $\sum_{i=1}^s \mu_i = n$. In this case, we write $\mu \vDash n$. Again, for our convenience, we assume $\mu_i = 0$ for all $i > s$.

For every ordered or unordered partition λ one can assign the *Young diagram* D_λ which contains λ_k boxes in the kth row. If λ is unordered, then D_λ is called *generalized*. A Young diagram filled with numbers is called a *Young tableau*. A tableau corresponding to λ is denoted by T_λ.

In the representation theory of symmetric groups, partitions and their Young diagrams are widely used. (See [3, 5] for applications to PI-algebras.) Let $a_{T_\lambda} = \sum_{\pi\in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma\in C_{T_\lambda}} (\mathrm{sign}\,\sigma)\sigma$ be symmetrizers corresponding to a Young tableau T_λ, $\lambda \vdash n$. Then $S(\lambda) := (\mathbb{Z}S_n)b_{T_\lambda}a_{T_\lambda}$ is the corresponding Specht module. Moreover, modules $S(\lambda)$ that correspond to different T_λ but the same λ, are isomorphic too. (The proof is analogous to the case of fields.) Though $S(\lambda)$ are not irreducible over $\mathbb{Z}$ and even contain no irreducible $\mathbb{Z}S_n$-submodules (it is sufficient

to consider the submodule $0 \neq 2M \subsetneqq M$ for any submodule $M \subseteq S(\lambda)$), we will use them in order to describe the structure of $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$.

2. Codimensions of Algebras Over Fields

Every algebra over a field can be treated as a ring. Therefore, we have to deal with two different types of codimensions. Here, we establish a relation between them.

Proposition 2.1. *Let A be an algebra over a field F. Then $c_n(A,q) = 0$ for all $n \in \mathbb{N}$ and $q \neq \operatorname{char} F$.*

Proof. Note that $(\operatorname{char} F)f \in \mathrm{Id}(R,\mathbb{Z})$ for all $f \in \mathbb{Z}\langle X\rangle$. Hence $\operatorname{char} F > 0$ implies $c_n(A,q) = 0$ for all $n \in \mathbb{N}$ and $q \neq \operatorname{char} F$. If $\operatorname{char} F = 0$, then every $q = p^k \neq 0$ is invertible and $c_n(A,q) = 0$ for all $n \in \mathbb{N}$ and $q \neq 0$ too. □

Proposition 2.2. *Let A be an algebra over a field F, $\operatorname{char} F = 0$. Then $c_n(A,F) \leq c_n(A,0)$ for all $n \in \mathbb{N}$. Moreover, $c_n(A,\mathbb{Q}) = c_n(A,0)$ for all $n \in \mathbb{N}$.*

Proof. By Proposition 2.1, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(A,\mathbb{Z})}$ is a free Abelian group. Let $f_1,\dots,f_s$ be the preimages of its free generators in $P_n(\mathbb{Z})$. Note that $P_n(\mathbb{Z}) \subset P_n(\mathbb{Q}) \subseteq P_n(F)$ and for every $\sigma \in S_n$ the monomial $x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$ can be expressed as a linear combination with integer coefficients of $f_1,\dots,f_s$ and an element of $P_n(\mathbb{Z}) \cap \mathrm{Id}(A,\mathbb{Z})$. Hence the images of $f_1,\dots,f_s$ generate $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A,F)}$ and $c_n(A,F) \leq c_n(A,0) = s$.

Suppose $f_1,\dots,f_s$ are linearly dependent modulo $\mathrm{Id}(A,\mathbb{Q})$. In this case $\frac{r_1}{q_1}f_1 + \dots + \frac{r_1}{q_1}f_s \in \mathrm{Id}(A,\mathbb{Q})$ for some $q_i \in \mathbb{N}$, $r_i \in \mathbb{Z}$. Thus

$$f := r_1\left(\prod_{i=2}^{s} q_i\right)f_1 + r_1q_1\left(\prod_{i=3}^{s} q_i\right)f_2 + \dots + r_s\left(\prod_{i=1}^{s-1} q_i\right)f_s \in \mathrm{Id}(A,\mathbb{Q}).$$

However, $f \in \mathbb{Z}\langle X\rangle$. Hence $f \in \mathrm{Id}(A,\mathbb{Z})$ and all $r_i = 0$ since f_i are linearly independent modulo $\mathrm{Id}(A,\mathbb{Z})$. Therefore, the images of $f_1,\dots,f_s$ form a basis of $\frac{P_n(\mathbb{Q})}{P_n(\mathbb{Q})\cap \mathrm{Id}(A,\mathbb{Q})}$ and $c_n(A,\mathbb{Q}) = c_n(A,0) = s$. □

The next example shows that in the case $F \supsetneqq \mathbb{Q}$ we could have $c_n(A,F) < c_n(A,\mathbb{Q}) = c_n(A,0)$.

Example 2.3. Note that $P_3(\mathbb{Q}) \cong \mathbb{Q}S_3 \cong S^{\mathbb{Q}}(3) \oplus S^{\mathbb{Q}}(2,1) \oplus S^{\mathbb{Q}}(2,1) \oplus S^{\mathbb{Q}}(1^3)$. Let $a \in \mathbb{Q}S_3$ such that $S^{\mathbb{Q}}(2,1) = \mathbb{Q}S_3a$. Denote by f_1 and f_2 the polynomials that correspond to a in the copies of $S^{\mathbb{Q}}(2,1)$ in $P_3(\mathbb{Q})$. Let $F = \mathbb{Q}(\sqrt{2})$. Consider the T-ideal I of $F\langle X\rangle$ generated by $(f_1 + \sqrt{2}f_2)$. We claim that $c_3(F\langle X\rangle/I,F) = 4 < c_3(F\langle X\rangle/I,\mathbb{Q}) = 6$.

Proof. First we notice that $P_3(F) \cap \mathrm{Id}(F\langle X\rangle/I,F) = FS_3\cdot(f_1 + \sqrt{2}f_2) \cong S^F(2,1)$. Hence by the hook formula, $c_3(F\langle X\rangle/I,F) = 6 - 2 = 4$. However,

$P_3(\mathbb{Q}) \cap \mathrm{Id}(F\langle X\rangle/I, \mathbb{Q}) = P_3(\mathbb{Q}) \cap FS_3(f_1 + \sqrt{2}f_2) = 0$. Indeed, suppose $f = b(f_1+\sqrt{2}f_2) \in P_3(\mathbb{Q})$ for some $b \in FS_3$. Note that $b = b_1+\sqrt{2}b_2$ where $b_1, b_2 \in \mathbb{Q}S_3$. Therefore, $f = (b_1 + \sqrt{2}b_2)(f_1 + \sqrt{2}f_2) = (b_1f_1 + 2b_2f_2) + \sqrt{2}(b_1f_2 + b_2f_1)$ and $f \in P_3(\mathbb{Q})$ implies $b_1f_2 + b_2f_1 = 0$. Recall that $\mathbb{Q}S_3f_1 \oplus \mathbb{Q}S_3f_2$ is the direct sum of $\mathbb{Q}S_3$-submodules. Hence $b_1f_2 = b_2f_1 = 0$. However, $\mathbb{Q}S_3f_1 \cong \mathbb{Q}S_3f_2$. Thus $b_1f_1 = b_2f_2 = 0$ too, $f = 0$, $P_3(\mathbb{Q}) \cap \mathrm{Id}(F\langle X\rangle/I, \mathbb{Q}) = 0$ and $c_3(F\langle X\rangle/I, \mathbb{Q}) = 6$. □

The result, analogous to Proposition 2.2, holds in a positive characteristic.

Proposition 2.4. *Let A be an algebra over a field F, $\mathrm{char}\, F = p$. Then $c_n(A, F) \le c_n(A, p)$ for all $n \in \mathbb{N}$. Moreover, $c_n(A, \mathbb{Z}_p) = c_n(A, p)$ for all $n \in \mathbb{N}$.*

Proof. By Proposition 2.1, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(A,\mathbb{Z})}$ is the direct sum of copies of $\mathbb{Z}_p$. Let $f_1, \dots, f_s$ be the preimages of their standard generators in $P_n(\mathbb{Z})$. Note that $P_n(\mathbb{Z}_p)$ is an image of $P_n(\mathbb{Z})$ under the natural homomorphism, $P_n(\mathbb{Z}_p) \subseteq P_n(F)$ and for every $\sigma \in S_n$ the monomial $x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)}$ can be expressed as a linear combination with integer coefficients of $f_1, \dots, f_s$ and an element of $P_n(\mathbb{Z}) \cap \mathrm{Id}(A, \mathbb{Z})$. Hence the images of $f_1, \dots, f_s$ generate $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A,F)}$ and $c_n(A, F) \le c_n(A, p)$.

Suppose $f_1, \dots, f_s$ are linearly dependent modulo $\mathrm{Id}(A, \mathbb{Z}_p)$. In this case $\bar{m}_1f_1 + \dots + \bar{m}_sf_s \in \mathrm{Id}(A, \mathbb{Z}_p)$ for some $m_i \in \mathbb{Z}$. Thus $m_1f_1 + \dots + m_sf_s \in \mathrm{Id}(A, \mathbb{Z})$ and all $m_i \in p\mathbb{Z}$ since f_i generate modulo $\mathrm{Id}(A, \mathbb{Z})$ the direct sum of copies of $\mathbb{Z}_p$. Therefore, the images of $f_1, \dots, f_s$ form a basis of $\frac{P_n(\mathbb{Z}_p)}{P_n(\mathbb{Z}_p)\cap \mathrm{Id}(A,\mathbb{Z}_p)}$ and $c_n(A, \mathbb{Z}_p) = c_n(A, p) = s$. □

The next result is concerned with the extension of a ring to an algebra over a field.

Theorem 2.5. *Let R be a ring and let F be a field. Then,*

$$c_n(R \otimes_{\mathbb{Z}} F, F) = \begin{cases} c_n(R/\mathrm{Tor}\, R, 0) & \text{if } \mathrm{char}\, F = 0, \\ c_n(R/pR, p) & \text{if } \mathrm{char}\, F = p, \end{cases}$$

where $\mathrm{Tor}\, R := \{r \in R \mid mr = 0 \text{ for some } m \in \mathbb{N}\}$ is the torsion of R.

First, we prove the following lemma

Lemma 2.6. *Let R be a ring and let F be a field. Then,*

$$R \otimes 1_F \cong \begin{cases} R/\mathrm{Tor}\, R & \text{if } F = \mathbb{Q}, \\ R/pR & \text{if } F = \mathbb{Z}_p, \end{cases}$$

where $R \otimes 1_F \subseteq R \otimes_{\mathbb{Z}} F$ is a subring.

Proof. Consider the natural homomorphism $\varphi\colon R \to R \otimes 1_F$ where $\varphi(a) = a \otimes 1_F$, $a \in R$.

Suppose $F = \mathbb{Q}$. If $ma = 0$ for some $m \in \mathbb{N}$ and $a \in R$, then $\varphi(a) = a \otimes 1_{\mathbb{Q}} = ma \otimes \frac{1_{\mathbb{Q}}}{m} = 0$. Hence $\mathrm{Tor}\, R \subseteq \ker \varphi$.We claim that $\ker \varphi = \mathrm{Tor}\, R$.

Let $a \in \ker\varphi$, i.e. $a \otimes 1_{\mathbb{Q}} = 0$. By one of the definitions of the tensor product,

$$\begin{aligned}(a, 1_{\mathbb{Q}}) = &\sum_i \ell_i((a_i + b_i, q_i) - (a_i, q_i) - (b_i, q_i)) \\ &+ \sum_i m_i((c_i, s_i + t_i) - (c_i, s_i) - (c_i, t_i)) \\ &+ \sum_i n_i((k_i d_i, u_i) - (d_i, k_i u_i))\end{aligned}$$

holds for some $a_i, b_i, c_i, d_i \in R$, $k_i, \ell_i, m_i, n_i \in \mathbb{Z}$ and $q_i, s_i, t_i, u_i \in \mathbb{Q}$ in the free $\mathbb{Z}$-module $H_{R\times\mathbb{Q}}$ with the basis $R \times \mathbb{Q}$. We can find such $m \in \mathbb{N}$ that all $mq_i, ms_i, mt_i, mu_i \in \mathbb{Z}$. Then

$$\begin{aligned}(a, m) = &\sum_i \ell_i((a_i + b_i, mq_i) - (a_i, mq_i) - (b_i, mq_i)) \\ &+ \sum_i m_i((c_i, ms_i + mt_i) - (c_i, ms_i) - (c_i, mt_i)) \\ &+ \sum_i n_i((k_i d_i, mu_i) - (d_i, k_i mu_i))\end{aligned}$$

holds in the free $\mathbb{Z}$-module $H_{R\times\mathbb{Z}}$ with the basis $R \times \mathbb{Z}$. Note that in the right-hand side of the latter equality we have a relation in $R \otimes_{\mathbb{Z}} \mathbb{Z}$. Hence $a \otimes m = 0$ in $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ and $ma = 0$. Thus $a \in \operatorname{Tor} R$. Therefore, $\ker\varphi = \operatorname{Tor} R$ and $R \otimes 1_{\mathbb{Q}} \cong R/\operatorname{Tor} R$.

Suppose $F = \mathbb{Z}_p$. Then $\varphi(pR) = R \otimes p1_{\mathbb{Z}_p} = 0$ and $pR \subseteq \ker\varphi$. Let $a \in \ker\varphi$, i.e. $a \otimes 1_{\mathbb{Z}_p} = 0$. Then,

$$\begin{aligned}(a, 1_{\mathbb{Z}_p}) = &\sum_i q_i((a_i + b_i, \bar{\ell}_i) - (a_i, \bar{\ell}_i) - (b_i, \bar{\ell}_i)) \\ &+ \sum_i s_i((c_i, \bar{m}_i + \bar{n}_i) - (c_i, \bar{m}_i) - (c_i, \bar{n}_i)) \\ &+ \sum_i t_i((k_i d_i, \bar{u}_i) - (d_i, k_i \bar{u}_i))\end{aligned}$$

holds for some $a_i, b_i, c_i, d_i \in R$ and $k_i, \ell_i, m_i, n_i, q_i, s_i, t_i, u_i \in \mathbb{Z}$ in the free $\mathbb{Z}$-module $H_{R\times\mathbb{Z}_p}$ with the basis $R \times \mathbb{Z}_p$. Note that $H_{R\times\mathbb{Z}_p}$ is the factor module of $H_{R\times\mathbb{Z}}$ by the subgroup $\langle (a, m) - (a, m + p) \,|\, a \in R,\ m \in \mathbb{Z}\rangle_{\mathbb{Z}}$. Hence

$$\begin{aligned}(a, 1_{\mathbb{Z}}) = &\sum_i q_i((a_i + b_i, \ell_i) - (a_i, \ell_i) - (b_i, \ell_i)) \\ &+ \sum_i s_i((c_i, m_i + n_i) - (c_i, m_i) - (c_i, n_i)) \\ &+ \sum_i t_i((k_i d_i, u_i) - (d_i, k_i u_i)) + \sum_i \alpha_i((r_i, \beta_i) - (r_i, \beta_i + p))\end{aligned}$$

holds in $H_{R\times\mathbb{Z}}$ for some $r_i \in R$ and $\alpha_i, \beta_i \in \mathbb{Z}$. Thus $a \otimes 1_{\mathbb{Z}} = \sum_i \alpha_i r_i \otimes p$. Now, we use the isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ and get $a = \sum_i \alpha_i r_i p \in pR$. Therefore, $\ker\varphi = pR$ and $R \otimes 1_{\mathbb{Z}_p} \cong R/pR$. □

Proof of Theorem 2.5. Recall that $R \otimes 1_F$ is a subring of $R \otimes_{\mathbb{Z}} F$. Hence $P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} F, \mathbb{Z}) \subseteq P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes 1_F, \mathbb{Z})$. Conversely, $P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} F, \mathbb{Z}) \supseteq P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes 1_F, \mathbb{Z})$ since $R \otimes 1_F$ generates $R \otimes_{\mathbb{Z}} F$ as an F-vector space. Therefore, $c_n(R \otimes 1_F, \mathrm{char}\, F) = c_n(R \otimes_{\mathbb{Z}} F, \mathrm{char}\, F)$ and we get Theorem 2.5 for $F = \mathbb{Q}$ and $F = \mathbb{Z}_p$ from Lemma 2.6 and Propositions 2.2, 2.4. The general case follows from the fact that $(R \otimes_{\mathbb{Z}} F) \otimes_F K \cong R \otimes_{\mathbb{Z}} K$ (as a K-algebra) for any field extension $K \supseteq F$ and, by [5, Theorem 4.1.9],

$$c_n(R \otimes_{\mathbb{Z}} K, K) = c_n((R \otimes_{\mathbb{Z}} F) \otimes_F K, K) = c_n(R \otimes_{\mathbb{Z}} F, F). \qquad \square$$

Corollary 2.7. *Let R be a torsion-free ring satisfying a non-trivial polynomial identity. Then*

(1) *either $c_n(R,0) = 0$ for all $n \geq n_0$, $n_0 \in \mathbb{N}$, or there exist $d \in \mathbb{N}$, $C_1, C_2 > 0$, $q_1, q_2 \in \mathbb{R}$ such that $C_1 n^{q_1} d^n \leq c_n(R,0) \leq C_2 n^{q_2} d^n$ for all $n \in \mathbb{N}$; in particular, polynomial identities of R satisfy the analog of Amitsur's conjecture, i.e. there exists $\lim_{n\to\infty} \sqrt[n]{c_n(R,0)} \in \mathbb{Z}_+$;*
(2) *if R contains 1, then there exist $C > 0$ and $q \in \mathbb{Z}$ such that $c_n(R,0) \sim C n^{\frac{q}{2}} d^n$ as $n \to \infty$, i.e. the analog of Regev's conjecture holds in R. (We write $f \sim g$ if $\lim \frac{f}{g} = 1$.)*

Proof. By Theorem 2.5, $c_n(R,0) = c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$. Now, we apply [5, Theorem 6.5.2] and [2, Theorem 4.2.2]. □

Remark 2.8. If R is a torsion-free ring, then $c_n(R,q) = 0$ for all $q \neq 0$ since $f \in \mathrm{Id}(R,\mathbb{Z})$ for all $f \in \mathbb{Z}\langle X\rangle$ such that $mf \in \mathrm{Id}(R,\mathbb{Z})$ for some $m \in \mathbb{N}$.

We conclude the section with an example.

Example 2.9. Let $R = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{2^k}$. Then $c_n(R,0) = 1$ and $c_n(R,q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. Although $mR \neq 0$ for all $m \in \mathbb{N}$, $R \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$.

Proof. The ring R is commutative. Hence all monomials from $P_n(\mathbb{Z})$ are proportional to $x_1 x_2 \dots x_n$ modulo $\mathrm{Id}(R,\mathbb{Z})$. However, $m x_1 x_2 \dots x_n \notin \mathrm{Id}(R,\mathbb{Z})$ for all $m \in \mathbb{N}$. (It is sufficient to substitute $x_1 = x_2 = \dots = x_n = \overline{1}_{\mathbb{Z}_{2^k}}$ for $2^k > m$.) Thus $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})} \cong \mathbb{Z}$ and $c_n(R,0) = 1$ and $c_n(R,q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. However, $a \otimes q = 2^k a \otimes \frac{q}{2^k}$ for all $a \in R$, $q \in \mathbb{Q}$, and $k \in \mathbb{N}$. Choosing k sufficiently large, we get $a \otimes q = 2^k a \otimes \frac{q}{2^k} = 0$. Thus $R \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$. □

3. Relation between $\mathbb{Z}S_n$-Modules of Proper and Ordinary Polynomial Functions

First, we describe the relation between proper and ordinary codimensions.

Theorem 3.1. *Let R be a unitary ring. Then $c_n(R,q) = \sum_{j=0}^{n} \binom{n}{j}\gamma_j(R,q)$ for every $n \in \mathbb{N}$ and $q \in \{p^k \mid p,k \in \mathbb{N}, p \text{ is prime}\} \cup \{0\}$.*

Proof. First, we notice that

$$P_n(\mathbb{Z}) = \bigoplus_{k=0}^{n} \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}\sigma_{i_1,\dots,i_k}\Gamma_{n-k}(\mathbb{Z}) \tag{3.1}$$

is the direct sum of $\mathbb{Z}$-modules where $\Gamma_0(\mathbb{Z}) := \mathbb{Z}$ and $\sigma_{i_1,\dots,i_k} \in S_n$ is any permutation such that $\sigma((n-k)+j) = i_j$ for all $1 \le j \le k$.

One way to prove (3.1) is to use the Poincaré–Birkhoff–Witt theorem for Lie algebras over rings [1, Theorem 2.5.3].

Another way is to show this explicitly in the spirit of Specht [14]. Using the equalities $yx = [y,x] + xy$ and $[\dots,\dots]x = x[\dots,\dots] + [[\dots,\dots],x]$, we can present every polynomial from $P_n(\mathbb{Z})$ as a linear combination of polynomials $x_{i_1}x_{i_2}\dots x_{i_k}f$ where $1 \le i_1 < i_2 < \dots < i_k \le n$ and f is a proper multilinear polynomial of degree $(n-k)$ in the variables from the set $\{x_1,x_2,\dots,x_n\}\backslash\{x_{i_1},x_{i_2},\dots,x_{i_k}\}$. In other words, $f \in \sigma_{i_1,\dots,i_k}\Gamma_{n-k}(\mathbb{Z})$. In order to check that the sum in (3.1) is direct, we consider a linear combination of $x_{i_1}x_{i_2}\dots x_{i_k}\sigma_{i_1,\dots,i_k}f$ where $f \in \Gamma_{n-k}(\mathbb{Z})$, for different k and i_j and choose the term $g := x_{i_1}x_{i_2}\dots x_{i_k}\sigma_{i_1,\dots,i_k}f$ with the greatest k among the terms with a nonzero coefficient. Then, we substitute $x_{i_1} = x_{i_2} = \dots = x_{i_k} = 1$ and $x_j = x_j$ for the rest of the variables. (We assume that we are working in the free ring with 1 on the set $X = \{x_1,x_2,\dots\}$.) All the other terms vanish and we get $f = 0$. Therefore, the sum is direct and (3.1) holds.

Substituting $x_{i_1} = x_{i_2} = \dots = x_{i_k} = 1_R$ and arbitrary elements of R for the other x_j, we obtain

$$\begin{aligned} P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z}) = {} & (\mathrm{char}\, R)\mathbb{Z}x_1x_2\dots x_n \\ & \oplus \bigoplus_{k=0}^{n-2} \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k} \\ & \times \sigma_{i_1,\dots,i_k}(\mathrm{Id}(R,\mathbb{Z}) \cap \Gamma_{n-k}(\mathbb{Z})). \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})} \cong \bigoplus_{k=0}^{n} \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} \frac{\Gamma_{n-k}(\mathbb{Z})}{\Gamma_{n-k}(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$$

(direct sum of $\mathbb{Z}$-modules) for an arbitrary ring R with the unit 1_R. (We define $\frac{\Gamma_0(\mathbb{Z})}{\Gamma_0(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} := \langle 1_R\rangle_{\mathbb{Z}} \subseteq R$.) Calculating the number of the components, we obtain Theorem 3.1. ∎

Corollary 3.2. *Let R be a unitary ring. Then all multilinear polynomial identities of R are consequences of proper multilinear polynomial identities of R and the identity* $(\operatorname{char} R)x \equiv 0$.

Proof. This follows from (3.2). □

Corollary 3.3. *Let R be a unitary ring and let the sequence $(c_n(R,q))_{n=1}^{\infty}$ be polynomially bounded for some q. Then $c_n(R,q)$ is a polynomial in $n \in \mathbb{N}$.*

Proof. If the sequence $(c_n(R,q))_{n=1}^{\infty}$ is polynomially bounded, then by Theorem 3.1 there exists $j_0 \in \mathbb{N}$ such that $\gamma_j(R,q) = 0$ for all $j \geq j_0$. Now, we apply Theorem 3.1 once again. □

If H is a subgroup of a group G and M is a left $\mathbb{Z}H$-module, then $M \uparrow G := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$. The G-action on $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$ is induced as follows: $g_0(g \otimes a) := g_0 g \otimes a$ for $a \in M$, $g, g_0 \in G$.

Now, we prove an analog of Drensky's theorem [3, Theorem 12.5.4]:

Theorem 3.4. *Let R be a unitary ring,* $\operatorname{char} R = \ell$, $\ell \in \mathbb{Z}_+$. *Consider for every $n \in \mathbb{N}$ the series of $\mathbb{Z}S_n$-submodules*

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \supsetneqq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})},$$

where each M_k is the image of $\bigoplus_{t=k}^{n} \mathbb{Z}S_n(x_1 \dots x_{n-t}\Gamma_t(\mathbb{Z}))$ and $M_{n+1} := 0$. Then $M_0/M_2 \cong \mathbb{Z}_\ell$ (trivial S_n-action),

$$\begin{aligned} M_t/M_{t+1} &\cong \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \uparrow S_n \\ &:= \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \end{aligned}$$

for all $2 \leq t \leq n$ where S_{n-t} is permuting $x_{t+1}, \dots, x_n$ and $\mathbb{Z}$ is a trivial $\mathbb{Z}S_{n-t}$-module.

Proof. First we notice that M_0/M_2 is generated by the image of $x_1 x_2 \dots x_n$. Suppose the image of $k x_1 x_2 \dots x_n$ belongs to M_2 for some $k \in \mathbb{N}$. All the polynomials in M_2 vanish under the substitution $x_1 = \cdots = x_n = 1_R$ since each of them contain at least one commutator. Hence, we get $k1_R = 0$, $\ell \mid k$ and $M_0/M_2 \cong \mathbb{Z}_\ell$.

Note that $\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ where S_{n-t} acts trivially. Consider the bilinear map

$$\varphi\colon \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \to M_t/M_{t+1}$$

defined by $\varphi(\sigma, f) = x_{\sigma(t+1)} x_{\sigma(t+2)} \dots x_{\sigma(n)}(\sigma f)$ for $\sigma \in S_n$, $f \in \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$. Note that $\varphi(\sigma\pi, f) = \varphi(\sigma, \pi f)$ for all $\pi \in S_t \times S_{n-t}$ and M_t/M_{t+1} is generated by all $\varphi(\sigma, f)$ for $\sigma \in S_n$ and $f \in \Gamma_t(\mathbb{Z})$.

Suppose L is an Abelian group and $\psi\colon \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \to L$ is a $\mathbb{Z}$-bilinear map and $\psi(\sigma\pi, f) = \psi(\sigma, \pi f)$ for all $\pi \in S_t \times S_{n-t}$. First we define $\bar{\psi}\colon M_t \to L$ on the elements that generate M_t modulo M_{t+1}:

$$\bar{\psi}(x_{i_1}x_{i_2}\dots x_{i_{n-t}}f) = \psi(\sigma, \sigma^{-1}f),$$

where $\sigma \in S_n$ and $\sigma^{-1}f \in \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ (e.g. we can take $\sigma(k) = i_k$ for $1 \le k \le n-t$). Clearly, $\bar{\psi}(x_{i_1}x_{i_2}\dots x_{i_{n-t}}f)$ does not depend on the choice of σ. Suppose the image $\bar{f}_0$ of a polynomial

$$f_0 = \sum_{i_1<\dots<i_{n-t}} x_{i_1}x_{i_2}\dots x_{i_{n-t}} f_{i_1,\dots,i_{n-t}}$$

belongs to M_{t+1} for some $f_{i_1,\dots,i_{n-t}} \in \Gamma_t(\mathbb{Z})$. Substituting

$$x_{i_1} = x_{i_2} = \dots = x_{i_{n-t}} = 1_R$$

and arbitrary values for the other x_j, we get zero for every $i_1 < \dots < i_{n-t}$. Hence $f_{i_1,\dots,i_{n-t}} \in \mathrm{Id}(R,\mathbb{Z})$ and $\bar{\psi}(\bar{f}_0) = 0$. Thus, we can define $\bar{\psi}$ to be zero on M_{t+1} and we may assume that $\bar{\psi}\colon M_t/M_{t+1} \to L$.

Note that $\bar{\psi}\varphi = \psi$. Hence $M_t/M_{t+1} \cong \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t\times S_{n-t})} \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right)$ (isomorphism of Abelian groups) where $\varphi(\sigma, f) \mapsto \sigma \otimes f$. Therefore, this is an isomorphism of $\mathbb{Z}S_n$-modules too. □

4. A Particular Case of the Littlewood–Richardson Rule

Let $\mu \vDash n$, $\lambda \vdash n'$, $n' \le n$. Suppose $\lambda_i \le \mu_i$ for all $i \in \mathbb{N}$. Denote by $M(\mu)$ the free Abelian group generated by all μ-tabloids. Now we treat D_λ as a Young subdiagram in D_μ. Later on we always assume that in a pair $(\lambda;\mu)$ we have $\lambda_1 = \mu_1$.

Following [6, Definition 17.4], we define a $\mathbb{Z}S_n$-submodule $S(\lambda,\mu) \subseteq M(\mu)$ where

$$S(\lambda;\mu) := \langle e^{\lambda,\mu}_{T_\mu} \mid T_\mu \text{ is a tableau of the shape } \mu\rangle_{\mathbb{Z}}$$

and $e^{\lambda,\mu}_{T_\mu} := \sum_{\sigma\in C_{T_\lambda}} (\operatorname{sign}\sigma)\sigma[T_\mu]$. Here T_λ is the subtableau of T_μ defined by the partition λ and $C_{T_\lambda} \subseteq S_n$ is the subgroup that leaves the numbers out of T_λ invariant and puts every number from each column of T_λ to the same column. By $[T_\mu]$ we denote the tabloid corresponding to T_μ. We assume $S(0;0) = 0$ for the zero partitions $0 \vdash 0$. Note that $S(\lambda;\lambda) \cong S(\lambda)$. (The proof is completely analogous to the case when the coefficients are taken from a field.)

Let F be a field and let $M^F(\mu)$ be the vector space over F with the formal basis consisting of all μ-tabloids. In other words, $M^F(\mu) = M(\mu) \otimes_{\mathbb{Z}} F$. We define $S^F(\lambda;\mu)$ as the subspace in $M^F(\mu)$ generated by $S(\lambda;\mu) \otimes 1$.

Lemma 4.1. *Let $\mu \vDash n$, $\lambda \vdash n'$, $n' \le n$. Suppose $\lambda_i \le \mu_i$ for all $i \in \mathbb{N}$. Then $M(\mu)/S(\lambda;\mu)$ has no torsion.*

Proof. Recall that $M(\mu)$ is a finitely generated free Abelian group and $S(\lambda;\mu)$ is its subgroup. Hence, we can choose a basis $a_1, a_2, \dots, a_t$ in $M(\mu)$ such that

$m_1a_1, m_2a_2, \ldots, m_ka_k$ is a basis of $S(\lambda;\mu)$ for some $m_i \in \mathbb{N}$. We claim that all $m_i = 1$. First, we notice that $a_1 \otimes 1, a_2 \otimes 1, \ldots, a_t \otimes 1$ form a basis of $M^F(\mu)$ and $m_1a_1 \otimes 1, m_2a_2 \otimes 1, \ldots, m_ka_k \otimes 1$ generate $S^F(\lambda;\mu)$ for any field F. Thus $\dim_F S^F(\lambda;\mu) = k$ for $\operatorname{char} F = 0$ and $\dim_F S^F(\lambda;\mu) < k$ if $\operatorname{char} F \mid m_i$ for at least one m_i. However, by [6, Theorem 17.13(III)], $\dim_F S^F(\lambda;\mu)$ does not depend on the field F. Therefore, all $m_i = 1$ and $M(\mu)/S(\lambda;\mu)$ is a free Abelian group. □

Let $c \geq 2$ be a natural number satisfying the following conditions: $\mu_{c-1} = \lambda_{c-1}$ and $\mu_c > \lambda_c$. Then, we define the operators A_c ("adding") and R_c ("raising") in the following way:

(1) if $\lambda_c = \lambda_{c-1}$, then $A_c(\lambda;\mu) = (0;0)$ where $0 \vdash 0$ is a zero partition, otherwise $A_c(\lambda;\mu) = (\tilde{\lambda};\mu)$ where $\tilde{\lambda}_i = \lambda_i$ for $i \neq c$ and $\tilde{\lambda}_c = \lambda_c + 1$;
(2) $R_c(\lambda;\mu) = (\tilde{\lambda};\tilde{\mu})$ where $\tilde{\mu}_i = \mu_i$ for $i \neq c-1, c$; $\tilde{\mu}_c = \lambda_c$, $\tilde{\mu}_{c-1} = \mu_{c-1} + (\mu_c - \lambda_c)$, $\tilde{\lambda}_1 = \tilde{\mu}_1$ and $\tilde{\lambda}_i = \lambda_i$ for $i > 1$.

Fix $i \in \mathbb{N}$ and $0 \leq v \leq \mu_{i+1}$. Let $\nu \vDash n$, $\nu_j = \mu_j$ for $j \neq i, i+1$, $\nu_i = \mu_i + \mu_{i+1} - v$, $\nu_{i+1} = v$. Then, we define $\psi_{i,v} \in \operatorname{Hom}_{\mathbb{Z}S_n}(M(\mu), M(\nu))$ in the following way: $\psi_{i,v}[T_\mu] = \sum[T_\nu]$ where the summation runs over the set of all tabloids $[T_\nu]$ such that $[T_\nu]$ agrees with $[T_\mu]$ in all the rows except the ith and the $(i+1)$th, and the $(i+1)$th is a subset of size v of the $(i+1)$th row in $[T_\mu]$. Analogously, we define $\psi^F_{i,v} \in \operatorname{Hom}_{FS_n}(M^F(\mu), M^F(\nu))$ for any field F.

Lemma 4.2. (1) $\psi_{c-1,\lambda_c} S(\lambda;\mu) = S(R_c(\lambda;\mu))$;
(2) $\ker \psi_{c-1,\lambda_c} \cap S(\lambda;\mu) = S(A_c(\lambda;\mu))$.

Proof. The proof of the first part of the lemma and of the embedding $\ker \psi_{c-1,\lambda_c} \supseteq S(A_c(\lambda;\mu))$ is completely analogous to [6, Lemma 17.12]. Now, we notice that there exists a natural embedding $M(\lambda) \otimes 1 \subset M^{\mathbb{Q}}(\lambda)$. By [6, Theorem 17.13], $\ker \psi^{\mathbb{Q}}_{c-1,\lambda_c} \cap S^{\mathbb{Q}}(\lambda;\mu) = S^{\mathbb{Q}}(A_c(\lambda;\mu))$. Thus if $\psi_{c-1,\lambda_c} a = 0$ for some $a \in S(\lambda;\mu)$, then $ma \in S(A_c(\lambda;\mu))$ for some $m \in \mathbb{N}$ and $a \in S(A_c(\lambda;\mu))$ since $M(\mu)/S(A_c(\lambda;\mu))$ is torsion-free by Lemma 4.1. □

Lemma 4.3. *Let $n \in \mathbb{N}$, $\lambda \vdash n'$, $\mu \vDash n$, $n' \leq n$, $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Then $S(\lambda;\mu)$ has a chain of submodules*

$$S(\lambda;\mu) = M_0 \supsetneqq M_1 \supsetneqq M_2 \supsetneqq \cdots \supsetneqq M_t = 0$$

with factors M_i/M_{i+1} isomorphic to Specht modules. Moreover, $S(\lambda;\mu)/M_i$ is torsion-free for any i.

Proof. If $\mu = \lambda$, then $S(\lambda;\mu) = S(\lambda)$ and there is nothing to prove. If $\mu \neq \lambda$, then we find $c \in \mathbb{N}$ such that $\lambda_i = \mu_i$ for all $1 \leq i \leq c-1$ and $\lambda_c < \mu_c$. Since we always assume $\lambda_1 = \mu_1$, we have $c \geq 2$. Now, we apply Lemma 4.2. Note that $\tilde{\lambda}_c > \lambda_c$ where $A_c(\lambda;\mu) = (\tilde{\lambda};\tilde{\mu})$ and R_c moves the boxes of D_μ upper. Applying Lemma 4.2 many times, we get the first part of Lemma 4.3 by induction.

Suppose $S(\lambda;\mu)/M_i$ is not torsion-free and $ma \in M_i$ for some $a \in S(\lambda;\mu)$, $a \notin M_i$ and $m \in \mathbb{N}$. Then, we can find an index $0 \leq k < i$ such that $a \in M_k$, $a \notin M_{k+1}$. However, $ma \in M_i \subseteq M_{k+1}$, i.e. the Specht module M_k/M_{k+1} is not torsion-free either. We get a contradiction since all Specht modules are subgroups in finitely generated free Abelian groups. □

Now we can prove the $\mathbb{Z}$-analog of the particular case of the Littlewood–Richardson rule that sometimes is referred to as Young's rule [5, Theorem 2.3.3], [3, Theorem 12.5.2] and sometimes as Pieri's formula [4, (A.7)].

Theorem 4.4. *Let $t, n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $t < n$, and $\lambda \vdash t$ and let $\mathbb{Z}$ be the trivial $\mathbb{Z}S_{n-t}$-module. Then*

$$(S(\lambda)/mS(\lambda)) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} ((S(\lambda)/mS(\lambda)) \otimes_{\mathbb{Z}} \mathbb{Z})$$

has a series of submodules with factors $S(\nu)/mS(\nu)$ where ν runs over the set of all partitions $\nu \vdash n$ such that

$$\lambda_n \leq \nu_n \leq \lambda_{n-1} \leq \nu_{n-1} \leq \cdots \leq \lambda_2 \leq \nu_2 \leq \lambda_1 \leq \nu_1.$$

(*Each factor occurs exactly once.*)

Proof. Suppose $\lambda = (\lambda_1, \ldots, \lambda_s)$, $\lambda_s > 0$. Then $S(\lambda) \uparrow S_n \cong S(\lambda;\mu)$ where $\mu = (\lambda_1, \ldots, \lambda_s, n-t)$. Now, Lemma 4.3 implies the theorem for $m = 0$.

Suppose $m > 0$. Then $(S(\lambda)/mS(\lambda)) \uparrow S_n \cong (S(\lambda) \uparrow S_n)/(m(S(\lambda) \uparrow S_n))$. Let

$$S(\lambda) \uparrow S_n = M_0 \supsetneqq M_1 \supsetneqq M_2 \supsetneqq \cdots \supsetneqq M_t = 0,$$

where $M_{i-1}/M_i \cong S(\lambda^{(i)})$, $\lambda^{(i)} \vdash n$, $1 \leq i \leq t$.

Hence

$$(S(\lambda) \uparrow S_n)/(m(S(\lambda) \uparrow S_n)) = \overline{M_0} \supsetneqq \overline{M_1} \supsetneqq \overline{M_2} \supsetneqq \cdots \supsetneqq \overline{M_t} = 0,$$

where $\overline{M_i} \cong (M_i + m(S(\lambda) \uparrow S_n))/m(S(\lambda) \uparrow S_n)$ and

$$\begin{aligned}
\overline{M_{i-1}}/\overline{M_i} &\cong (M_{i-1} + m(S(\lambda) \uparrow S_n))/(M_i + m(S(\lambda) \uparrow S_n)) \\
&\cong M_{i-1}/M_{i-1} \cap (M_i + m(S(\lambda) \uparrow S_n)) \\
&= M_{i-1}/(M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n)) \\
&\cong (M_{i-1}/M_i)/((M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n))/M_i).
\end{aligned}$$

By Lemma 4.3, $(S(\lambda) \uparrow S_n)/M_{i-1}$ is torsion-free. Hence $M_{i-1} \cap m(S(\lambda) \uparrow S_n) = mM_{i-1}$ and

$$\begin{aligned}
\overline{M_{i-1}}/\overline{M_i} &\cong (M_{i-1}/M_i)/((M_i + mM_{i-1})/M_i) \\
&= (M_{i-1}/M_i)/(m(M_{i-1}/M_i)) \cong S(\lambda^{(i)})/mS(\lambda^{(i)}).
\end{aligned}$$

The description of $\lambda^{(i)}$ is obtained from the proof of Lemma 4.3. □

5. Algebras of Upper Triangular Matrices

5.1. *Codimensions and multilinear identities*

Let M be an (R_1, R_2)-bimodule for commutative rings R_1, R_2 with 1 and let $R = \left(\begin{smallmatrix} R_1 & M \\ 0 & R_2 \end{smallmatrix}\right)$.

In this section, we calculate $c_n(R, q)$ for all $q = p^k$ and $q = 0$, describe the structure of the $\mathbb{Z}S_n$-module $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ and find such multilinear polynomials that elements of $\mathrm{Id}(R, \mathbb{Z}) \cap P_n(\mathbb{Z})$ are consequences of them.

Remark 5.1. If F is a field of characteristic 0 and $A = \mathrm{UT}_2(F) := \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, then $c_n(A, F)$ and generators of $\mathrm{Id}(A, F)$ as a T-ideal can be found, e.g. in [5, Theorem 4.1.5]. The structure of the FS_n-module $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A,F)}$ can be determined using proper cocharacters [**3**, Theorem 12.5.4].

Theorem 5.2. *All polynomials from $P_n(\mathbb{Z}) \cap \mathrm{Id}(R, \mathbb{Z})$, $n \in \mathbb{N}$, are consequences of the left-hand sides of the following polynomial identities in R:*

$$[x, y][z, t] \equiv 0, \tag{5.1}$$

$$\ell x \equiv 0, \tag{5.2}$$

$$m[x, y] = 0, \tag{5.3}$$

where $[x, y] := xy - yx$,

$$\ell := \min\{n \in \mathbb{N} \,|\, na = 0 \text{ for all } a \in R_1 \cup R_2\},$$
$$m := \min\{n \in \mathbb{N} \,|\, na = 0 \text{ for all } a \in M\}.$$

(If one of the corresponding sets is empty, we define $\ell = 0$ or $m = 0$, respectively. Note that $m \,|\, \ell$.)

Moreover, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \cong \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^{(n-2)2^{n-1}+1}$ where $\mathbb{Z}_0 := \mathbb{Z}$.

Remark 5.3. Now $c_n(R, q)$ can be easily computed. If $R_1 = R_2 = M$ and $R_1 = R_2$ is a field, we obtain the same numbers as in [5, Theorem 4.1.5].

Proof of Theorem 5.2. Denote by e_{ij} the matrix units. Then $R = R_1e_{11} \oplus R_2e_{22} \oplus Me_{12}$ (direct sum of subspaces), $[R, R] \subseteq Me_{12}$ and (5.1)–(5.3) are indeed polynomial identities of R.

Now, we consider an arbitrary mononomial from $P_n(\mathbb{Z})$ and find the first inversion among the indices of its variables. We replace the corresponding pair of variables with the sum of their commutator and their product in the right order. Note that $[x, y]u[z, t] = [x, y][z, t]u + [x, y][u, [z, t]] \equiv 0$ is a consequence of (5.1). Therefore, we may assume that all the variables to the right of the commutator have

increasing indices. For example:

$$\begin{aligned} x_3x_1x_4x_2 &= x_1x_3x_4x_2 + [x_3,x_1]x_4x_2 \\ &\overset{(5.1)}{\equiv} x_1x_3x_2x_4 + x_1x_3[x_4,x_2] + [x_3,x_1]x_2x_4 \\ &= x_1x_2x_3x_4 + x_1[x_3,x_2]x_4 + x_1x_3[x_4,x_2] + [x_3,x_1]x_2x_4. \end{aligned}$$

Continuing this procedure, we present any element of $P_n(\mathbb{Z})$ modulo the consequences of (5.1) as a linear combination of polynomials $f_0 := x_1x_2\dots x_n$ and

$$x_{i_1}\dots x_{i_k}[x_s,x_r]x_{j_1}\dots x_{j_{n-k-2}} \quad \text{for } i_1<\dots<i_k<s,r<s, j_1<\dots<j_{n-k-2}. \tag{5.4}$$

Denote the set of polynomials (5.4) by Ξ.

Consider the free Abelian group $\mathbb{Z}(\Xi\cup\{f_0\})$ with the basis $\Xi\cup\{f_0\}$. Now, we have the surjective homomorphism $\varphi\colon \mathbb{Z}(\Xi\cup\{f_0\})\to \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ where $\varphi(f)$ is the image of $f\in\Xi\cup\{f_0\}$ in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$. We claim that $\ker\varphi$ is generated by ℓf_0 and all mf where $f\in\Xi$.

Suppose that a linear combination f_1 of f_0 and elements from Ξ is a polynomial identity, however f_1 is not a linear combination of ℓf_0 and mf, $f\in\Xi$. If we substitute

$$x_1 = x_2 = \dots = x_n = 1_{R_i}e_{ii} \quad \text{where } i\in\{1,2\},$$

all $f\in\Xi$ vanish. Therefore, the coefficient of f_0 is a multiple of ℓ. Now, we find $f_2 := x_{i_1}\dots x_{i_k}[x_s,x_r]x_{j_1}\dots x_{j_{n-k-2}}\in\Xi$ with the greatest k such that the coefficient β of f_2 in f_1 is not a multiple of m. Then we substitute $x_{i_1}=\dots=x_{i_k}=x_s=1_{R_1}e_{11}$, $x_r = ae_{12}$, $x_{j_1}=\dots=x_{j_{n-k-2}} = 1_{R_1}e_{11}+1_{R_2}e_{22}=1_R$ where $a\in M$ and $\beta a\neq 0$. Our choice of f_2 implies that f_2 is the only summand in f_1 that could be nonzero under this substitution. Hence f_1 does not vanish and we get a contradiction. Therefore, $\ker\varphi$ is generated by ℓf_0 and mf, $f\in\Xi$. In particular, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}\cong \mathbb{Z}_\ell\oplus(\mathbb{Z}_m)^{|\Xi|}$ and every multilinear polynomial identity of R is a consequence of (5.1)–(5.3).

Note that

$$\begin{aligned} |\Xi| &= \sum_{k=2}^{n}(k-1)\binom{n}{k} = \sum_{k=2}^{n}\frac{n!}{(k-1)!(n-k)!} - \sum_{k=2}^{n}\binom{n}{k} \\ &= n\sum_{k=1}^{n-1}\frac{(n-1)!}{k!(n-k-1)!} - (2^n-n-1) \\ &= n(2^{n-1}-1) - (2^n-n-1) = (n-2)2^{n-1}+1 \end{aligned}$$

and the theorem follows. □

Corollary 5.4. *Multilinear polynomial identities of* $\mathrm{UT}_2(\mathbb{Q})$ *as a ring are generated by* (5.1).

5.2. $\mathbb{Z}S_n$-modules

Note that the Jacobi identity and (5.1) imply that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is generated as a $\mathbb{Z}$-module by $[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$ where $1 \le i \le n-1$.

Lemma 5.5. *Let R be the ring from Sec.* 5.1 *and* $T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$. *Then*

$$b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv n(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \pmod{P_n(\mathbb{Z}) \cap \mathrm{Id}(R)}. \tag{5.5}$$

Proof. Indeed,

$$\begin{aligned} & b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \\ & \quad \equiv b_{T_\lambda}(n-2)! \sum_{i=1}^{n-1} [x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}] \\ & \quad = (n-2)! \sum_{i=2}^{n-1} ([x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}] \\ & \qquad - [x_i, x_1, x_n, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]) + 2(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \\ & \quad \equiv n(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \end{aligned}$$

since, by the Jacobi identity, $[x_i, x_1, x_n] = [x_i, x_n, x_1] + [x_n, x_1, x_i]$. □

First, we determine the structure of $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ for $R = \mathrm{UT}_2(\mathbb{Q})$.

Lemma 5.6. *Let* $T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$. *Then,* $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})} \cong (\mathbb{Z}S_n) b_{T_\lambda} a_{T_\lambda}$.

Proof. We claim that if $u b_{T_\lambda} a_{T_\lambda} = 0$ for some $u \in \mathbb{Z}S_n$, then

$$u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \in \Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z}).$$

Indeed, by (5.5),

$$n(n-2)! u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv u b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] = 0.$$

Since $\mathrm{UT}_2(\mathbb{Q})$ has no torsion, $u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv 0$ is a polynomial identity of $\mathrm{UT}_2(\mathbb{Q})$.

Thus, we can define the surjective homomorphism $\varphi : (\mathbb{Z}S_n) b_{T_\lambda} a_{T_\lambda} \to \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})}$ by $\varphi(\sigma b_{T_\lambda} a_{T_\lambda}) = \sigma[x_1, x_n, x_2, x_3, \dots, x_{n-1}]$ for $\sigma \in S_n$.

Analogously, we can define the surjective homomorphism

$$\varphi_0 : (\mathbb{Q}S_n) b_{T_\lambda} a_{T_\lambda} \to \frac{\Gamma_n(\mathbb{Q})}{\Gamma_n(\mathbb{Q}) \cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Q})}$$

by $\varphi(\sigma b_{T_\lambda} a_{T_\lambda}) = \sigma[x_1, x_n, x_2, x_3, \dots, x_{n-1}]$ for $\sigma \in S_n$. Since $(\mathbb{Q}S_n) b_{T_\lambda} a_{T_\lambda}$ is an irreducible $\mathbb{Q}S_n$-module, φ_0 is an isomorphism of $\mathbb{Q}S_n$-modules. We claim that φ is an isomorphism of $\mathbb{Z}S_n$-modules.

Indeed, suppose

$$u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \in \Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z})$$

for some $u \in \mathbb{Z}S_n$. Then $\varphi_0(ub_{T_\lambda}a_{T_\lambda}) = u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \in \Gamma_n(\mathbb{Q}) \cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z})$ and $ub_{T_\lambda}a_{T_\lambda} = 0$. Hence φ is an isomorphism and the lemma is proven. □

Theorem 5.7. *Let R and m be, respectively, the ring and the number from Sec.* 5.1. *Then, $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \cong S(\lambda)/mS(\lambda)$ where $\lambda = (n-1, 1)$, for all $n \geq 2$.*

Proof. Recall that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is generated as a $\mathbb{Z}$-module by $[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$ where $1 \leq i \leq n-1$. We exploit the same trick as in the proof of Theorem 5.2. Using the substitution $x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_n = 1_{R_1}e_{11}$, $x_i = ae_{12}$ where $a \in M$, we obtain that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is the direct sum of $n-1$ cyclic groups isomorphic to $\mathbb{Z}_m$ and generated by $[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$ where $1 \leq i \leq n-1$.

By Theorem 5.2 and its corollary, we have the natural surjective homomorphism $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})} \to \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$. The remarks above imply that the kernel equals $m\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})}$. Now the theorem follows from Lemma 5.6. □

Applying Theorems 3.4, 4.4 and 5.7, we immediately get the following theorem.

Theorem 5.8. *Let R, ℓ, and m be, respectively, the ring and the numbers from Sec.* 5.1. *Then there exists a chain of $\mathbb{Z}S_n$-submodules in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ with the set of factors that consists of one copy of $\mathbb{Z}_\ell$ and $(\lambda_1 - \lambda_2 + 1)$ copies of $S(\lambda_1, \lambda_2, \lambda_3)/mS(\lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3) \vdash n$, $\lambda_2 \geq 1$, $\lambda_3 \in \{0, 1\}$.*

6. Grassmann Algebras

Let R be a commutative ring with a unit element 1_R, $\mathrm{char}\, R = \ell$ where either ℓ is an odd natural number or $\ell = 0$. We define the Grassmann algebra G_R over a ring R as the R-algebra with a unit, generated by the countable set of generators e_i, $i \in \mathbb{N}$, and the anti-commutative relations $e_ie_j = -e_je_i$, $i, j \in \mathbb{N}$. Here, we consider the same questions as for the upper triangular matrices.

6.1. *Codimensions and polynomial identities*

This lemma is known but we provide its proof for the reader's convenience.

Lemma 6.1. *The polynomial identity $[y, x][z, t] + [y, z][x, t] \equiv 0$ is a consequence of $[x_1, x_2, x_3] \equiv 0$. In particular, $[x, y]u[z, t] + [x, t]u[z, y] \equiv 0$ for all $u \in \mathbb{Z}\langle X\rangle$.*

Proof. Note that

$$\begin{aligned}[x, yt, z] &= [[x, y]t, z] + [y[x, t], z] \\ &= [x, y, z]t + [x, y][t, z] + [y, z][x, t] + y[x, t, z] \\ &\equiv [x, y][t, z] + [y, z][x, t] \\ &= [y, x][z, t] + [y, z][x, t] \qquad (6.1)\end{aligned}$$

modulo $[x_1, x_2, x_3] \equiv 0$. (Here, we have used Jacobi's identity too.) Hence

$$\begin{aligned}[x, y]u[z, t] + [x, t]u[z, y] &= [x, y][u, [z, t]] + [x, y][z, t]u \\ &\quad + [x, t][u, [z, y]] + [x, t][z, y]u \\ &\equiv [x, y][z, t]u + [x, t][z, y]u \\ &\equiv 0. \qquad (6.2)\end{aligned}$$

□

Theorem 6.2. *All polynomials from $P_n(\mathbb{Z}) \cap \mathrm{Id}(G_R, \mathbb{Z})$, $n \in \mathbb{N}$, are consequences of the left-hand sides of the following polynomial identities in R:*

$$[x, y, z] \equiv 0, \qquad (6.3)$$

$$\ell x \equiv 0. \qquad (6.4)$$

Moreover, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(G_R, \mathbb{Z})} \cong (\mathbb{Z}_\ell)^{2^{n-1}}$.

Proof. Define $G_R^{(0)} = \langle e_{i_1} e_{i_2} \dots e_{i_{2k}} \,|\, k \in \mathbb{Z}_+ \rangle_R$ and $G_R^{(1)} = \langle e_{i_1} e_{i_2} \dots e_{i_{2k+1}} \,|\, k \in \mathbb{Z}_+ \rangle_R$. Clearly, $G_R = G_R^{(0)} \oplus G_R^{(1)}$ (direct sum of R-submodules), $[G_R, G_R] \subseteq G_R^{(0)}$, $G_R^{(0)} = Z(G_R)$. Hence $[x_1, x_2, x_3] \equiv 0$ is a polynomial identity. Obviously, (6.4) is a polynomial identity too.

Let

$$\begin{aligned}\Xi = \{x_{i_1} \dots x_{i_k} [x_{j_1}, x_{j_2}] \dots [x_{j_{2m-1}}, x_{j_{2m}}] \,|\, & i_1 < \dots < i_k, \\ & j_1 < \dots < j_{2m}, k + 2m = n, k, m \in \mathbb{Z}_+\} \subset P_n(\mathbb{Z}).\end{aligned}$$

By Lemma 6.1, every polynomial from $P_n(\mathbb{Z})$ can be presented modulo (6.3) as a linear combination of polynomials from Ξ. For example,

$$\begin{aligned}x_3 x_2 x_4 x_1 &= -[x_2, x_3] x_4 x_1 + x_2 x_3 x_4 x_1 \\ &= ([x_2, x_3][x_1, x_4] - [x_2, x_3] x_1 x_4) + (x_2 x_3 x_1 x_4 - x_2 x_3 [x_1, x_4]) \\ &\equiv -[x_2, x_1][x_3, x_4] - x_1 x_4 [x_2, x_3] + x_2 x_1 x_3 x_4 \\ &\quad - x_2 [x_1, x_3] x_4 - x_2 x_3 [x_1, x_4] \\ &\equiv [x_1, x_2][x_3, x_4] - x_1 x_4 [x_2, x_3] + x_1 x_2 x_3 x_4 \\ &\quad - [x_1, x_2] x_3 x_4 - x_2 x_4 [x_1, x_3] - x_2 x_3 [x_1, x_4]\end{aligned}$$

$$\equiv [x_1,x_2][x_3,x_4] - x_1x_4[x_2,x_3] + x_1x_2x_3x_4$$
$$- x_3x_4[x_1,x_2] - x_2x_4[x_1,x_3] - x_2x_3[x_1,x_4].$$

Consider the free Abelian group $\mathbb{Z}\Xi$ with the basis Ξ. Now, we have the surjective homomorphism $\varphi\colon \mathbb{Z}\Xi \to \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$ where $\varphi(f)$ is the image of $f \in \Xi$ in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$. We claim that $\ker\varphi$ is generated by ℓf where $f \in \Xi$.

Suppose that a linear combination f_1 of elements from Ξ is a polynomial identity, however f_1 is not a linear combination of ℓf, $f \in \Xi$. Now, we find

$$f_2 := x_{i_1}\dots x_{i_k}[x_{j_1},x_{j_2}]\dots[x_{j_{2m-1}},x_{j_{2m}}] \in \Xi$$

with the greatest k such that the coefficient β of f_2 in f_1 is not a multiple of ℓ. Then we substitute $x_{i_1} = \dots = x_{i_k} = 1_{G_R}$, $x_{j_i} = e_i$, $1 \le i \le 2m$. Our choice of f_2 implies that f_2 is the only summand in f_1 that could be nonzero under this substitution. Hence, the value of f_1 equals $(2^m\beta 1_R)e_1e_2\dots e_m = 0$. However, G_R is a free R-module and $e_1e_2\dots e_m$ is one of its basis elements. Therefore $2^m\beta 1_R = 0$, $\ell \,|\, (2^m\beta)$ and $\ell \,|\, \beta$ since $2 \nmid \ell$. We get a contradiction.

Thus $\ker\varphi$ is generated by ℓf, $f \in \Xi$. In particular, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong (\mathbb{Z}_\ell)^{|\Xi|}$ and every multilinear polynomial identity of G_R is a consequence of (6.3) and (6.4).

We now calculate $|\Xi|$. The number of these polynomials equals the number of choices of $x_{i_1},\dots,x_{i_k}$. If n is odd, this number equals $\binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{n}$. If n is even, the number equals $\binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{n}$. But the both are equal to 2^{n-1}. Indeed, denote $s_0 = \sum_{i\,\text{even}} \binom{n}{i}$ and $s_1 = \sum_{i\,\text{odd}} \binom{n}{i}$. Then $2^n = (1+1)^n = s_0 + s_1$ and $0 = (1-1)^n = s_0 - s_1$. So $|\Xi| = s_0 = s_1 = 2^{n-1}$. □

6.2. $\mathbb{Z}S_n$-modules

First, we determine the structure of $\mathbb{Z}S_n$-modules of proper polynomial functions.

Theorem 6.3. *Let G_R be the Grassmann algebra over R. Let $\lambda = (1^{2m})$ and $T_\lambda = \begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline 2m\\\hline\end{array}$. Then $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong S(\lambda)/\ell S(\lambda)$ for all $m \in \mathbb{N}$, where $\ell = \operatorname{char} R$, and $\frac{\Gamma_{2m+1}(\mathbb{Z})}{\Gamma_{2m+1}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} = 0$ for all $m \in \mathbb{Z}_+$.*

Proof. Note $S(\lambda)$ is a free cyclic group generated by $b_{T_\lambda}[T_\lambda]$ and $\sigma b_{T_\lambda}[T_\lambda] = (\operatorname{sign}\sigma) b_{T_\lambda}[T_\lambda]$ for all $\sigma \in S_n$.

The proof of Theorem 6.2 implies that $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong \mathbb{Z}_\ell$ is a cyclic group generated by $[x_1,x_2]\dots[x_{2m-1},x_{2m}]$. By Lemma 6.1,

$$\sigma[x_1,x_2]\dots[x_{2m-1},x_{2m}] = (\operatorname{sign}\sigma)[x_1,x_2]\dots[x_{2m-1},x_{2m}] \quad \text{for all } \sigma \in S_n.$$

Hence $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong S(\lambda)/\ell S(\lambda)$. The first assertion is proved. The second assertion is evident since every long commutator of length greater than 2 is a polynomial identity of G_R. □

Theorem 6.4. *Let G_R be the Grassmann algebra over the R. Then there exists a chain of $\mathbb{Z}S_n$-submodules in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$ with factors $S(n-k,1^k)/\ell S(n-k,1^k)$ for each $0 \le k \le n-1$ (each factor occurs exactly once) where $\ell = \operatorname{char} R$.*

Proof. Now, we apply Theorems 3.4, 4.4 and 6.3. By Theorem 4.4, a diagram consisting of a single column can generate only diagrams $D_{(n-k,1^k)}$. Since we have diagrams of an even length only, each factor occurs only once. □

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