

Vacuum structure for scalar cosmological perturbations in Modified Gravity Models

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We have found for the general class of Modified Gravity Models $f(R, \mathcal{G})$ a new instability which can arise in vacuum for the scalar modes of the cosmological perturbations if the background is not de Sitter. In particular, the short-wavelength modes, if stable, in general have a group velocity which depends linearly in k , the wave number. Therefore these modes will be in general superluminal. We have also discussed the condition for which in general these scalar modes will be ghost-like. There is a subclass of these models, defined out of properties of the function $f(R, \mathcal{G})$ and to which the $f(R)$ and $f(\mathcal{G})$ models belong, which however do not have this feature.

I. INTRODUCTION

Modified Gravity Models (MGM) have been introduced as an alternative to the quintessence picture for dark energy [1, 2, 3, 4, 5, 6, 7, 8, 9]. The idea is speculative but attractive: maybe the acceleration of the universe is due to a modification of the behaviour of gravity at large scales. Gravitation seems to be described by an action which is not renormalizable, and the symmetry of the theory, does not restrict enough the possibility for building up an action for the $g_{\mu\nu}$ variables, the metric tensor components, as already pointed out in 1917 [10, 11]. However, to build a sensible gravitational theory is not an easy task. Some tried to change it to have a renormalizable theory at 1-loop [12, 13, 14], but in general this leads to fourth-order gravity theories and spurious degrees of freedom [15, 16, 17, 18, 19, 20, 21, 22].

Lovelock [23] found and studied a special class of scalars, the so called Lovelock scalars, built out of the Riemann and metric tensors only. In four dimensions the only such scalars which are not identically zero are three: a constant, the Ricci scalar, and the Gauss-Bonnet combination $\mathcal{G}$. This last scalar is made of quadratic combinations of the Riemann tensor, which in turn depends on second derivatives of $g_{\mu\nu}$. Naively, one might expect that this term in a Lagrangian would automatically lead to four derivatives for $g_{\mu\nu}$. However, all the Lovelock scalars have the property (which makes them special among all possible scalar invariants) that the equations of motion still remain of second order. This is still true even if these same curvature invariants are coupled to some other matter scalar field. Another property of the Gauss-Bonnet term is that, in four dimensions, it can be written as a total derivative [24], therefore the only way for it to give a contribution to the equations of motion is to couple it to some other field. Indeed, string theory seems to predict the existence of such couplings in the low-energy effective action [25, 26, 27, 28, 29].

In the aim of building up the action for gravity, since general covariance is not strong enough, we can take a bottom-up approach and look for the most general action which can be built out of the Riemann and the metric tensor. Therefore recently models have been introduced which use one or both these scalars. These are the so called $f(R)$ [2, 3, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48], $f(\mathcal{G})$ [4, 5, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60], and $f(R, \mathcal{G})$ theories [6, 20, 22, 61, 62, 63, 64]. All the Lagrangians written through these scalars share the property that they can be rewritten as scalar fields coupled to R and $\mathcal{G}$ respectively [64]. Therefore introducing functions of R and $\mathcal{G}$ adds scalar degrees of freedom only to the theory, eliminating the spurious spin-2 extra degrees of freedom which are typically ghost-like [20, 64, 65, 66].

It is well known that $f(R)$ theories can be written as a scalar tensor theory which, through a conformal transformation, takes in the Einstein frame the same form of the Lagrangian for a scalar field minimally coupled to gravity [67, 68]. However, the same technique does not prove it useful for the $f(\mathcal{G})$ theories. The vacuum structure of these theories has been deeply studied for maximally symmetric spaces [20, 62]. As for the behaviour of the same theories on a Friedmann-Lemaître-Robertson-Walker (FLRW) background some papers appeared [65, 66, 69], but, as we shall see later, they only treated a subset of the whole class of $f(R, \mathcal{G})$ theories.

In this paper we will give a general study of cosmological perturbations in vacuum for a general $f(R, \mathcal{G})$ Lagrangian. Some papers claimed that they studied the most general MGM [66, 69]. However, we think that only a subset of these theories was considered (among these, the $f(R)$ and the $f(\mathcal{G})$ theories). We present here a new result for the most general MGM, that is the dispersion relation for the scalar perturbations changes, and typically scalar modes,

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if stable, will be superluminal with a speed proportional to the wave number k . This change for the dispersion relation is always present except for a special subclass, which actually contains all the models for which cosmological perturbation theory was discussed [66, 69]. For such models, we recover the same results found before.

This new feature is essential to study the vacuum structure for these theories and an additional bound (the speed of the short wavelength modes) should be added to the previous one found in the literature of these models. We want to stress out that this is a physical property of the general class of these theories, and not a feature of a bad gauge choice, or a spurious feature which can be removed by a suitable field redefinition. Although the study of this action in the presence of matter fields is very important, the analysis of the vacuum case is the first thing to address. We will come back to this point in the discussion section.

The paper is structured as follows. In section II we introduce the model together with the equations of motion for a FLRW background. In section III, we perform a perturbation analysis for the scalar modes and derive a master equation. In section IV, we analyze the master equation and discuss properties of the solution. In section V, we consider a toy model of MGM to demonstrate how our findings can actually set bounds on MGM. Section VI is devoted to the study of special cases where the structure of the master equation for high k modes differs from the general case. In section VII, based on our findings obtained from the study of the scalar perturbations, we discuss various implications which must be taken into account for building up a sensible MGM. We also comment on the past studies where the scalar perturbations for the MGM were studied. Finally, in section VIII we present our conclusions.

II. MODIFIED GRAVITY MODELS

We consider the modified gravity action given by

$$S = \frac{M_P^2}{16\pi} \int d^4x \sqrt{-g} f(R, \mathcal{G}), \quad (1)$$

where R is the Ricci scalar and $\mathcal{G}$ is the so-called Gauss-Bonnet term defined by

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \quad (2)$$

Since we want to study the general theory of cosmological perturbations for the modified gravity action (1), we do not assume any particular functional form of $f(R, \mathcal{G})$.

Although (1) is the basic action we consider, for convenience of the actual analysis, we mainly use a different action, equivalent to (1), which is given by

$$\begin{aligned} S &= \frac{M_P^2}{16\pi} \int d^4x \sqrt{-g} \left[f(\lambda, \sigma) + (R - \lambda)F(\lambda, \sigma) + (\mathcal{G} - \sigma)\xi(\lambda, \sigma) \right] \\ &= \frac{M_P^2}{16\pi} \int d^4x \sqrt{-g} \left[RF(\lambda, \sigma) + \mathcal{G}\xi(\lambda, \sigma) - V(\lambda, \sigma) \right], \end{aligned} \quad (3)$$

where λ and σ are auxiliary fields and

$$F(\lambda, \sigma) \equiv \frac{\partial f}{\partial \lambda}, \quad \xi(\lambda, \sigma) \equiv \frac{\partial f}{\partial \sigma}, \quad (4)$$

$$V(\lambda, \sigma) \equiv \lambda F(\lambda, \sigma) + \sigma \xi(\lambda, \sigma) - f(\lambda, \sigma). \quad (5)$$

By the following way, we can verify that the action (3) is equivalent to (1). From the variation of S with respect to λ and σ , we have equations for λ and σ which are given by

$$(R - \lambda)F_\lambda + (\mathcal{G} - \sigma)F_\sigma = 0, \quad (6)$$

$$(R - \lambda)F_\sigma + (\mathcal{G} - \sigma)\xi_\sigma = 0, \quad (7)$$

where $F_\lambda = \partial F / \partial \lambda$, $F_\sigma = \partial F / \partial \sigma$ and $\xi_\sigma = \partial \xi / \partial \sigma$. If a combination $F_\lambda \xi_\sigma - F_\sigma^2$ does not vanish, the two equations are independent and λ and σ are given by

$$\lambda = R, \quad (8)$$

$$\sigma = \mathcal{G}. \quad (9)$$

Eliminating λ and σ in the original action by using these results, we find S in (3) reduces to (1): the equivalence of these two actions also holds for the equations of motion. If the combination $F_\lambda \xi_\sigma - F_\sigma^2$ vanishes, (6) and (7) are not independent from each other. This does not mean we can not eliminate λ and σ from the original action. Because (6) is the condition that the action remains the same under the variation, all the possible pairs of λ and σ that satisfy (6) give the same action. We can put any (λ, σ) we want into S as long as (λ, σ) are the solutions of (6). Obviously, $\lambda = R$ and $\sigma = \mathcal{G}$ are the solution and we can put them into (3) to find that it recovers the original action (1).

A. Equations of motion

The equations of motion for $g_{\mu\nu}$, in the presence also of a matter component with stress-energy tensor $T_{\mu\nu}$, are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Sigma_{\mu\nu} = \frac{8\pi}{M_P^2 F} T_{\mu\nu}, \quad (10)$$

where $\Sigma_{\mu\nu}$ is the effective energy momentum tensor defined by

$$\begin{aligned} \Sigma_{\mu\nu} = & \frac{1}{F} (\nabla_\mu \nabla_\nu F - g_{\mu\nu} \square F + 2R \nabla_\mu \nabla_\nu \xi - 2g_{\mu\nu} R \square \xi - 4R_\mu{}^\lambda \nabla_\lambda \nabla_\nu \xi \\ & - 4R_\nu{}^\lambda \nabla_\lambda \nabla_\mu \xi + 4R_{\mu\nu} \square \xi + 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta \xi + 4R_{\mu\alpha\beta\nu} \nabla^\alpha \nabla^\beta \xi - \frac{1}{2} g_{\mu\nu} V). \end{aligned} \quad (11)$$

The equations of motions for λ and σ are instead given by $\lambda = R$ and $\sigma = \mathcal{G}$.

B. Background dynamics

We assume that the background spacetime is a flat FLRW universe whose metric is given by

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i, \quad (12)$$

where the indices $i, j, \dots$ are raised by δ_{ij} . Then the background components are

$$G_0^0 = -3H^2, \quad (13)$$

$$G_j^i = - \left(H^2 + 2\frac{\ddot{a}}{a} \right) \delta_j^i, \quad (14)$$

$$\Sigma_0^0 = \frac{1}{F} \left[3H\dot{F} + 12H^3\dot{\xi} - \frac{1}{2}V \right], \quad (15)$$

$$\Sigma_j^i = \frac{1}{F} \left[2H\dot{F} + 8H\frac{\ddot{a}}{a}\dot{\xi} + 4H^2\ddot{\xi} + \ddot{F} - \frac{1}{2}V \right] \delta_j^i, \quad (16)$$

$$R = 6(2H^2 + \dot{H}), \quad (17)$$

$$\mathcal{G} = 24H^2(H^2 + \dot{H}), \quad (18)$$

from which we have, in vacuum

$$3H^2 = \frac{1}{F} \left[\frac{1}{2}V - 3H\dot{F} - 12H^3\dot{\xi} \right], \quad (19)$$

$$\dot{H} = \frac{1}{2F + 8H\dot{\xi}} \left[-\ddot{F} + H\dot{F} - 4H^2(\ddot{\xi} - H\dot{\xi}) \right], \quad (20)$$

$$\lambda = 6(2H^2 + \dot{H}), \quad (21)$$

$$\sigma = 24H^2(H^2 + \dot{H}). \quad (22)$$

III. SCALAR PERTURBATION

A. Perturbed metric

We consider scalar perturbations around the metric Eq. (12). We write the perturbed metric as

$$ds^2 = -(1 + 2\alpha) dt^2 - 2a(t)\partial_i\beta dt dx^i + a^2(t)(\delta_{ij} + 2\phi\delta_{ij} + 2\partial_i\partial_j\gamma) dx^i dx^j. \quad (23)$$

For later convenience, we define χ by the following equation,

$$\chi \equiv a(\beta + a\dot{\gamma}). \quad (24)$$

This represents the shear potential of the unit vector normal to Σ_t , where Σ_t is a time-like hypersurface of constant t .

B. Gauge transformation

There are degrees of freedom of choosing Σ_t . Changing from Σ_t to $\tilde{\Sigma}_t$ corresponds to a time coordinate transformation: $t \rightarrow t + T(t, x^i)$. Under this transformation, the perturbation variables transform as

$$\tilde{\alpha} = \alpha - \dot{T}, \quad (25)$$

$$\tilde{\phi} = \phi - HT, \quad (26)$$

$$\tilde{\chi} = \chi - T, \quad (27)$$

$$\tilde{\delta F} = \delta F - \dot{F}T, \quad (28)$$

$$\tilde{\delta \xi} = \delta \xi - \dot{\xi}T. \quad (29)$$

C. Perturbation equations in general gauge

The perturbations of the Einstein tensor can be written in Fourier space as

$$\delta G_0^0 = 2 \left(3H^2 \alpha - \frac{k^2}{a^2} \phi - 3H\dot{\phi} + \frac{k^2}{a^2} H\chi \right), \quad (30)$$

$$\delta G_i^0 = 2\partial_i(\dot{\phi} - H\alpha), \quad (31)$$

$$\delta G_j^i - \frac{1}{3}\delta_j^i \delta G_\ell^\ell = a^{-2} \left(\partial^i \partial_j - \frac{1}{3}\delta_j^i \Delta \right) (\dot{\chi} + H\chi - \phi - \alpha). \quad (32)$$

It can be shown that under the gauge transformation $t \rightarrow t + T(t, x^i)$ the Einstein tensor transforms as

$$\delta \tilde{G}_0^0 = \delta G_0^0 + 6H\dot{H}T, \quad (33)$$

$$\delta \tilde{G}_i^0 = \delta G_i^0 - 2\dot{H}\partial_i T, \quad (34)$$

$$\delta \tilde{G}_j^i - \frac{1}{3}\delta_j^i \delta \tilde{G}_\ell^\ell = \delta G_j^i - \frac{1}{3}\delta_j^i \delta G_\ell^\ell. \quad (35)$$

We do not consider the trace part of G_j^i because the perturbation equations for the trace part can be derived by the combinations of other equations and do not bring new informations. We find that the traceless part of δG_j^i is gauge invariant.

The perturbations of Σ_ν^μ given in Eq. (11) can be written as

$$\begin{aligned} \delta \Sigma_0^0 = \frac{1}{F} & \left[3(\dot{F} + 12H^2\dot{\xi})\dot{\phi} + 8\frac{k^2}{a^2}H\dot{\xi}\phi - 6H(\dot{F} + 8H^2\dot{\xi})\alpha - \frac{k^2}{a^2}(\dot{F} + 12H^2\dot{\xi})\chi \right. \\ & \left. + 3H(\delta\dot{F} + 4H^2\delta\dot{\xi}) + \left(-3(H^2 + \dot{H}) + \frac{k^2}{a^2} \right) (\delta F + 4H^2\delta\xi) \right], \end{aligned} \quad (36)$$

$$\delta \Sigma_i^0 = \frac{1}{F} \partial_i [-\delta\dot{F} + H\delta F - 4H^2\delta\dot{\xi} + 4H^3\delta\xi + (\dot{F} + 12H^2\dot{\xi})\alpha - 8H\dot{\xi}\phi], \quad (37)$$

$$\begin{aligned} \delta \Sigma_j^i - \frac{1}{3}\delta_j^i \delta \Sigma_\ell^\ell = \frac{1}{a^2 F} & \left(\partial^i \partial_j - \frac{1}{3}\delta_j^i \Delta \right) \left\{ 4H\dot{\xi}\alpha + 4\dot{\xi}\phi - [\dot{F} + 4(H^2 + \dot{H})\dot{\xi} + 4H\dot{\xi}] \chi \right. \\ & \left. - 4H\dot{\xi}\dot{\chi} + \delta F + 4(H^2 + \dot{H})\delta\xi \right\}. \end{aligned} \quad (38)$$

With the help of the background equations, we find that under the gauge transformation $t \rightarrow t + T(t, x^i)$, Σ_ν^μ transform as

$$\delta \tilde{\Sigma}_0^0 = \delta \Sigma_0^0 + 6H\dot{H}T, \quad (39)$$

$$\delta \tilde{\Sigma}_i^0 = \delta \Sigma_i^0 - 2\dot{H}\partial_i T, \quad (40)$$

$$\delta \tilde{\Sigma}_j^i - \frac{1}{3}\delta_j^i \delta \tilde{\Sigma}_\ell^\ell = \delta \Sigma_j^i - \frac{1}{3}\delta_j^i \delta \Sigma_\ell^\ell. \quad (41)$$

The explicit verification that both G_ν^μ and Σ_ν^μ transform in the same way under the gauge transformation supports that the derived perturbation equations are indeed correct.

Collecting these results, the perturbation equations, in Fourier space, are given by

$$3H^2\alpha - \frac{k^2}{a^2}\phi - 3H\dot{\phi} + \frac{k^2}{a^2}H\chi = \frac{1}{2F}\left[3(\dot{F} + 12H^2\dot{\xi})\dot{\phi} + 8\frac{k^2}{a^2}H\dot{\xi}\phi - 6H(\dot{F} + 8H^2\dot{\xi})\alpha - \frac{k^2}{a^2}(\dot{F} + 12H^2\dot{\xi})\chi + 3H(\delta\dot{F} + 4H^2\delta\dot{\xi}) + \left(-3(H^2 + \dot{H}) + \frac{k^2}{a^2}\right)(\delta F + 4H^2\delta\xi)\right], \quad (42)$$

$$\dot{\phi} - H\alpha = \frac{1}{2F}\left[H\delta F - \delta\dot{F} - 4H^2\delta\dot{\xi} + 4H^3\delta\xi + (\dot{F} + 12H^2\dot{\xi})\alpha - 8H\dot{\xi}\dot{\phi}\right], \quad (43)$$

$$\dot{\chi} + H\chi - \phi - \alpha = \frac{1}{F}\left[4H\dot{\xi}\alpha + 4\ddot{\xi}\phi - (\dot{F} + 4(H^2 + \dot{H})\dot{\xi} + 4H\ddot{\xi})\chi - 4H\dot{\xi}\dot{\chi} + \delta F + 4(H^2 + \dot{H})\delta\xi\right], \quad (44)$$

$$\delta\lambda = -2\left[6(2H^2 + \dot{H})\alpha + \frac{k^2}{a^2}(\dot{\chi} + 2H\chi) - 2\frac{k^2}{a^2}\phi - \frac{k^2}{a^2}\alpha - 3\ddot{\phi} - 12H\dot{\phi} + 3H\dot{\alpha}\right], \quad (45)$$

$$\delta\sigma = -8\left[12(H^2 + \dot{H})H^2\alpha + 2\frac{k^2}{a^2}H(H^2 + \dot{H})\chi + \frac{k^2}{a^2}H^2\dot{\chi} - 3H^2\ddot{\phi} + 3H^3\dot{\alpha} - 6H(2H^2 + \dot{H})\dot{\phi} - 2\frac{k^2}{a^2}(H^2 + \dot{H})\phi - \frac{k^2}{a^2}H^2\alpha\right]. \quad (46)$$

D. Gauge invariant variables and the master equation

It is convenient to analyze the perturbation equations in terms of gauge invariant variables. Therefore, let us define the following gauge independent combinations of fields

$$\delta\Gamma = \dot{F}\delta\xi - \dot{\xi}\delta F, \quad (47)$$

$$\Phi = \phi - \frac{H(\delta F + 4H^2\delta\xi)}{\dot{F} + 4H^2\dot{\xi}}, \quad (48)$$

$$\Psi = \phi + \frac{\delta F + 4H^2\delta\xi}{2(F + 4H\dot{\xi})} + \frac{1}{2}\chi \left[\frac{FH - \dot{F}}{F + 4H\dot{\xi}} - 3H \right]. \quad (49)$$

We will see that these three fields are enough to understand the behaviour of the scalar perturbations in the most general theory of gravity $f(R, \mathcal{G})$. This field definition cannot be applied on maximally symmetric backgrounds, where $H = H_0$, and $\dot{F} = \dot{\xi} = 0$. These backgrounds must be studied as a special case, which will be done later. In general, one can solve equation (43) for α and substitute it into equation (42). Afterwards one can use the fields introduced before, to change (42) into

$$\frac{k^2}{a^2}\Psi = A_1(t)\dot{\Phi} + A_2(t)\delta\Gamma, \quad (50)$$

where $A_{1,2}$ are functions of the background only and are defined in the appendix. It must be noticed that, in order to write down this equation, we made use of the equations of motion in order to replace $\ddot{F}$ in terms of the other backgrounds variables. Along the same lines one can rewrite equation (44) into the following form

$$\dot{\Psi} = A_3(t)\delta\Gamma + A_4(t)\Psi + A_5(t)\Phi. \quad (51)$$

We need at least another equation in order to make the equations closed. For this aim, we use then the definition of the field $\delta\Gamma$ and write the following formula

$$\delta\Gamma = (\dot{F}\xi_\sigma - \dot{\xi}F_\sigma)\delta\sigma + (\dot{F}\xi_\lambda - \dot{\xi}F_\lambda)\delta\lambda = (F_\lambda\xi_\sigma - F_\sigma^2)(\dot{\lambda}\delta\sigma - \dot{\sigma}\delta\lambda). \quad (52)$$

It is possible to rewrite both F_λ and ξ_σ in terms of $\dot{F}$, $\dot{\xi}$, and F_σ as follows

$$F_\lambda = \frac{\dot{F} - F_\sigma \dot{\sigma}}{\dot{\lambda}}, \quad (53)$$

$$\xi_\sigma = \frac{\dot{\xi} - F_\sigma \dot{\lambda}}{\dot{\sigma}}. \quad (54)$$

Therefore we can put equations (45) and (46) into equation (52) to write another equation for $\delta\Gamma$ and the other fields. The equation that one obtains has the following form

$$p_1(t) \delta\dot{\Gamma} + p_2(t) \dot{\Phi} + p_3(t) \delta\Gamma + \frac{k^2}{a^2} [p_4(t) \delta\Gamma + \dot{\Psi} + p_5(t) \Psi + p_6(t) \Phi] + p_7(t) \ddot{\Phi} = 0, \quad (55)$$

where the p_i are all functions of time only. This complicated equation can be simplified as follows. From equation (51) one can solve for $\dot{\Psi} = \dot{\Psi}(\Psi, \Phi, \delta\Gamma)$. By using equation (50) one can find $\dot{\Phi} = \dot{\Phi}(\Psi, \delta\Gamma)$. By differentiating once this last equation and by replacing $\dot{\Psi}$ and $\ddot{F}$ (this last one, by using the background equations) one can also find $\ddot{\Phi} = \ddot{\Phi}(\Phi, \Psi, \delta\Gamma, \delta\dot{\Gamma})$. Inserting this last equation into (55) to eliminate $\ddot{\Phi}$ and also replacing $\dot{\Phi}$ and $\dot{\Psi}$ with expressions that do not contain time derivatives of the fields, we have an equation given by

$$\delta\Gamma = \frac{k^2}{a^2} [A_6(t) \Phi + A_7(t) \Psi], \quad (56)$$

where $A_{6,7}$ are given in the appendix. We find that $\delta\dot{\Gamma}$ automatically disappears from the equation, which is crucial to derive the closed second order differential equation. As we will see later on, some gravity models identically yield $\delta\Gamma = 0$. In these models, we can verify that A_6 and A_7 also identically vanish and (56) does not give any information.

Now, it is possible to find a closed second order differential equation in time for the Fourier fields Φ , or Ψ , or $\delta\Gamma$. For example, let us use equation (56) into equations (51) and (50) obtaining

$$\frac{k^2}{a^2} [1 - A_2(t) A_7(t)] \Psi = A_1(t) \dot{\Phi} + A_2(t) A_6(t) \frac{k^2}{a^2} \Phi, \quad (57)$$

$$\dot{\Psi} = \frac{k^2}{a^2} A_3(t) A_6(t) \Phi + \frac{k^2}{a^2} A_3(t) A_7(t) \Psi + A_4(t) \Psi + A_5(t) \Phi. \quad (58)$$

Then let us solve (57) for Ψ , finding $\Psi = \Psi(\Phi, \dot{\Phi})$. Then we can use this result into equation (58) to find an equation of the form

$$\frac{1}{a^3 Q(t)} \partial_t [a^3 Q(t) \dot{\Phi}] + B_1(t) \frac{k^2}{a^2} \Phi + B_2(t) \frac{k^4}{a^4} \Phi = 0, \quad (59)$$

where Q , B_1 , and B_2 are defined in the appendix. The field Φ indeed seems to be a “good” field to study in the sense that its equation of motion has a relatively simple dependence in time and k .

Eq. (59) is the main result of this paper. The scalar perturbations have only two independent degrees of freedom, which are $\Phi(t_0)$ and $\dot{\Phi}(t_0)$. All the informations of the perturbation behavior are completely determined by (59). To see this, looking at equation (57) and (56), one can write both Ψ and $\delta\Gamma$ as linear combinations of Φ and $\dot{\Phi}$. As for the metric perturbation variables themselves such as α , χ and ϕ , because of their gauge dependence, they cannot be written in terms of Φ , Ψ and $\delta\Gamma$ which are gauge invariant. But this does not mean that we need further knowledges which are not contained in Φ , Ψ and $\delta\Gamma$. It merely means that only the gauge invariant combinations out of $\{\alpha, \chi, \phi\}$, or equivalently $\{\alpha, \chi, \phi\}$ in a specific gauge, are expressed by Φ , Ψ and $\delta\Gamma$. As an illustration, let us consider a gauge where $\delta F = -4H^2 \delta\xi$, which we call the MGM Gauge (MGMG). There are no remaining gauge degrees of freedom in this gauge. In this gauge, we have $\phi = \Phi$ from (48). By using the remaining two equations (47) and (49), $\delta\xi$ and χ are uniquely expressed in terms of Φ , Ψ and $\delta\Gamma$. After that, α is given by (44) and finally all the metric perturbation variables are determined. Once the metric perturbations are determined in a specific gauge, those in other gauges are simply obtained by gauge transformation.

Therefore, the knowledge of Φ and $\dot{\Phi}$ is enough to understand the behavior of the metric perturbations.

The master equation (59) contains a term proportional to k^4 , or equivalently a term of fourth order spatial derivative in real space. This term does not vanish in generic $f(R, \mathcal{G})$ models, nor is a spurious result due to a bad choice of fields/gauge: the field Φ is gauge invariant, and this k^4 behaviour would still be there by studying either $\delta\Gamma$ or Ψ . Furthermore in the MGMG Φ has a simple meaning which can be directly related to experimental data. Therefore, for the general theory, we don't have a standard wave equation and we expect that the perturbations propagate in space in non-trivial ways.

It should be noted, however, that $B_2(t)$ identically vanishes if $F_\lambda \xi_\sigma - F_\sigma^2 = 0$. Interestingly, most of the modified gravity models concerned by the past literature, such as $f(R)$ gravity and $R + f(\mathcal{G})$ gravity models, belong to these special cases. In these special cases, the propagation properties of the perturbations deeply differ from those of the generic cases. We defer to study the special cases in section VI. In this section, we consider the generic cases where $B_2(t) \neq 0$.

The extreme complexity of $B_1(t)$ and $B_2(t)$ requires numerical calculations to solve exactly the differential equation and to see how the perturbations evolve. However, studies for limiting cases still allow us to obtain important informations such as instability of the perturbations, which will be done in section IV.

E. Action of the perturbations

By expanding the action (3) up to second order in the perturbation variables and eliminating all the auxiliary fields using the equations of motion, we find that the action in terms of Φ can be written as

$$S = \frac{M_P^2}{\pi} \int dt d^3x Q a^3 \left[\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \frac{B_1}{a^2} (\vec{\nabla} \Phi)^2 - \frac{1}{2} \frac{B_2}{a^4} (\vec{\nabla}^2 \Phi)^2 \right]. \quad (60)$$

From this action, we correctly recover the master equation (59). We find that the sign of the kinetic term in the action is equal to that of Q . We will then call that mode for which its kinetic energy has a negative sign, that is $Q < 0$, a ghost. Typically, if a ghost is present and coupled with other normal fields, we can expect that the vacuum decays into metric and normal fields, because the energy conservation allows such a process to happen. Therefore, one would expect strong bounds from the vacuum decay [70]. Here we give the condition $Q > 0$ as a necessary one in order to have a theory without ghosts degrees of freedom. However, we are not interested in the details of the quantization procedure which is outlined in [69]. Although the quantization mentioned in [69] only treats the case $B_2 = 0$, the k^4 term will only change the dispersion relation $\omega_k(t)$ for each decoupled mode. Therefore, this change will only lead to an explicit expression for the solution $\bar{\Phi}_k(t)$ which is, in general, different from the $B_2(t) = 0$ case. Anyhow, we will leave this point in detail for a future project, when matter fields will be introduced. In summary, the first bound we will consider is the one coming from not having ghosts in the theory, that is requiring a positive Q .

IV. STUDY OF THE MASTER EQUATION

A. Long wavelength limit

Let us first study a case where the wavelength of the mode we consider is much larger than any typical length scale. In this case, neglecting terms that are quadratic and quartic in k/a in (59) yields

$$\frac{1}{a^3 Q(t)} \partial_t [a^3 Q(t) \dot{\Phi}] = 0. \quad (61)$$

We can immediately integrate this differential equation and the result is

$$\Phi = C_k + D_k \int_0^t \frac{dt'}{a^3(t') Q(t')}, \quad (62)$$

where C_k, D_k are independent of t , being integration constants for each k .

Ψ can be determined by substituting the above solution into (57). To the leading order, Ψ is given by

$$\Psi \approx \frac{1}{1 - A_2 A_7} \left(A_1 \frac{D_k}{a^3 Q} \frac{a^2}{k^2} + A_2 A_6 C_k \right). \quad (63)$$

Depending on the leading power of k for both C_k and D_k , each term on the right hand side can be dominant.

B. Short wavelength limit

Let us next study a case where the wavelength of the mode we consider is much smaller than any typical length scale. In the large k limit, the time scale for the change of Φ is much smaller than that for the change of the background

quantities. Therefore, we can use WKB approximation to obtain the solutions. The solutions of (59), under the WKB approximation, can be given by

$$\begin{aligned} \Psi(t) \approx & c_+ \exp \left[i \int^t dt' \left(\omega_k^{(+)} + \frac{i\dot{\omega}_k^{(+)}}{2\omega_k^{(+)} - i(a^3 Q)'/(a^3 Q)} \right) \right] \\ & + c_- \exp \left[-i \int^t dt' \left(\omega_k^{(-)} - \frac{i\dot{\omega}_k^{(-)}}{2\omega_k^{(-)} - i(a^3 Q)'/(a^3 Q)} \right) \right], \end{aligned} \quad (64)$$

where $c_{\pm}$ are constants and $\omega_k^{(\pm)}$ are the roots of the following quadratic equation

$$\omega^2 - i \frac{(a^3 Q)'}{a^3 Q} \omega - \frac{k^4}{a^4} B_2 - \frac{k^2}{a^2} B_1 = 0. \quad (65)$$

In the large k limit, $\omega_k^{(\pm)}$ are given by

$$\omega_k^{(\pm)} = \pm \sqrt{B_2} \frac{k^2}{a^2} \pm \frac{B_1}{2\sqrt{B_2}} + \frac{i}{2} \frac{(a^3 Q)'}{a^3 Q} + \mathcal{O}(k^{-2}). \quad (66)$$

We see that if B_2 is negative, then $\omega_k^{(\pm)}$ are pure imaginary. Therefore, Φ grows exponentially in time. The growth rate increases in proportion to k^2 . The smaller the wavelength of the mode is, the smaller the time scale of the instability is. One may guess this instability is due to a bad choice of perturbation variables and there might exist a special gauge where all the metric perturbation variables do not exhibit such an instability, i.e. where the deviations from the FLRW universe remain small. If this were possible, Φ , written in terms of the metric perturbation variables in that gauge, should not show an exponential growth, which is inconsistent with the exponential growth of Φ we have just found. Therefore this instability is not a due to a bad choice for the perturbation variables and the FLRW universe is indeed unstable on small scales. Due to this instability, the perturbations will grow until they become $\mathcal{O}(1)$ where the linear perturbation theory no longer works. We do not know what happens if the system goes into the non-linear regime and will not consider it furthermore.

If B_2 is positive, then the leading term of $\omega_k^{(\pm)}$ is real and the perturbations propagate. In this case, the group velocity is given by

$$v_g(k) = a \frac{\partial |\omega_k^{(\pm)}|}{\partial k} \approx 2\sqrt{B_2} \frac{k}{a}. \quad (67)$$

At any time t_0 , this velocity exceeds the speed of light for those modes above a critical wavenumber k_c which is given by $k_c = a(t_0)/[2\sqrt{B_2(t_0)}]$. The propagation of the short wavelength modes inevitably becomes superluminal up to the cutoff of the theory $v_{g,\max} < 2\sqrt{B_2}\Lambda_{\text{cutoff}}$. The smaller the wavelength of the mode is, the larger the propagation speed. For a given k , the speed of propagation is background dependent. Because of this, if v_g reduces quickly with time, for some backgrounds, it may be possible that the friction term (the third one in the right hand side of Eq. 66) gives the strongest contribution to the stability of each mode, at least for $t \rightarrow \infty$. It still remains a lack of consensus among researchers if the propagation speed larger than the speed of light causes crucial problems that immediately make us give up considering such models as a possible modification for a consistent theory of gravity. More details are given in the discussion section.

We will also impose B_1 to be positive as there is always some intermediate range of k 's, for which the $B_1 k^2$ term will be dominating over the $B_2 k^4$ one.

To summarize, for the general $f(R, \mathcal{G})$ MGM models, except for those special cases where $F_\lambda \xi_\sigma - F_\sigma^2 = 0$, the small wavelength modes inevitably either suffer from strong instability or acquire superluminal propagation.

V. APPLICATION TO A TOY MODEL

Just for the purpose of illustrating how the general formalism we have developed in the previous section can be applied to an explicit MGM, let us consider the following toy model, introduced in ([6]), whose $f(R, \mathcal{G})$ is given by,

$$f(R, \mathcal{G}) = R + \frac{M^6}{R^2 + \alpha \mathcal{G}}, \quad (68)$$

where M is a constant of mass dimension and α is a dimensionless constant. In this model, we have that

$$\frac{\partial^2 f}{\partial R^2} \frac{\partial^2 f}{\partial \mathcal{G}^2} - \left(\frac{\partial^2 f}{\partial R \partial \mathcal{G}} \right)^2 = - \frac{4M^{12}\alpha^2}{(R^2 + \alpha\mathcal{G})^5}, \quad (69)$$

which does not vanish in general. Therefore, this model does not belong to the special class of MGM, and its master equation will have the k^4 -term.

It is known that the background equations can have power-law solution such as $a(t) \propto t^p$ for four different values of p . They are given by

$$p_1 = \frac{51 + 10\alpha + y(\alpha)}{4(6 + \alpha)}, \quad (70)$$

$$p_2 = \frac{51 + 10\alpha - y(\alpha)}{4(6 + \alpha)}, \quad (71)$$

$$p_3 = \frac{6 + \alpha + \sqrt{6\alpha + \alpha^2}}{2(6 + \alpha)}, \quad (72)$$

$$p_4 = \frac{6 + \alpha - \sqrt{6\alpha + \alpha^2}}{2(6 + \alpha)}, \quad (73)$$

where $y = y(\alpha) = \sqrt{1521 + 840\alpha + 100\alpha^2}$. If the radicand becomes negative, which indeed happens for some values of α , the corresponding power-law solution will not exist.

From now on, we only consider the case $p = p_1$. Analysis for other p_i can be done in exactly the same way. From the background equation, we find that the power-law solution is an attractor if the following inequality

$$\frac{273 + 50\alpha + 3y(\alpha)}{6 + \alpha} > 0, \quad (74)$$

is satisfied. If not, the universe cannot settle down to this power-law solution. From the definition of $y(\alpha)$, we find that $y(\alpha)$ is real for $\alpha \leq -3(14 + 3\sqrt{3})/10 \approx -5.76$ and for $\alpha \geq 3(-14 + 3\sqrt{3})/10 \approx -2.64$. For such ranges of α , the power-law solution becomes an attractor for $\alpha \leq -507/80 \approx -6.34$ and for $\alpha \geq -2.64$. Moreover, such an attractor is an accelerating one for $\alpha \geq -2.64$, whereas it is a decelerating one for $\alpha \leq -6.34$.

Using the formulae given in the appendix, the expressions for B_1 , B_2 and Q at late time ($t \rightarrow \infty$) are given by

$$B_1 = \frac{135(26364 - 391y) + 16\alpha\{60(3660 - 49y) + \alpha[70716 - 631y + 10\alpha(907 + 40\alpha - 4y)]\}}{45(15 + 8\alpha)(57 + 8\alpha)y}, \quad (75)$$

$$H^2 B_2 = \frac{-16\alpha^2(6 + \alpha)(51 + 10\alpha + y)}{3[8\alpha(15 + 2\alpha) + 3(39 + y)]\{351(39 + y) + 2\alpha[5301 + 87y + 10\alpha(129 + 10\alpha + y)]\}}, \quad (76)$$

$$H^6 Q = \frac{10125(15 + 8\alpha)(57 + 8\alpha)y}{4s(\alpha)}, \quad (77)$$

$$s(\alpha) = -4108137345(39 + y) - 2\alpha\{-9477(-18610644 + 44279y) + 10\alpha[-297(-46531983 + 545000y) + 320\alpha(27(579515 - 6608y) + 5\alpha[581040 - 4563y + 10\alpha(5403 + 200\alpha - 20y)])]\}. \quad (78)$$

Here we have multiplied B_2 and Q by H^2 and H^6 to make them dimensionless. Graphs of these quantities as functions of α are shown in Fig. 1.

Here we find that the region of α where all of the quantities become positive is $-2.64 < \alpha < -2.25$. This range is outside the allowed range coming from different constraints regarding the fit to Supernovae Type Ia data and the viability of the cosmological evolution from radiation domination up to today [63, 71]. Therefore, according to our results, for these models, in the α -interval allowed by theory/experiment, the background attractor p_1 will be in general unstable. In fact, outside this region, at least one of $\{B_1, B_2, Q\}$ becomes negative. The asymptotic value of $H^2 B_2$ for $\alpha \rightarrow -\infty$ is $-1/9$. At first glance, this is incompatible because the action (68) in the limit $\alpha \rightarrow \pm\infty$ reduces to a form $f(R, \mathcal{G}) \rightarrow R + M^6/(\alpha\mathcal{G})$ which belong to the special case. The origin of this gap comes from the naive expectation that R^2 in the denominator of (68) will be much smaller than $\alpha\mathcal{G}$ for $|\alpha| \gg 1$ and can be neglected. This expectation is true for positively large α . For negatively large α , p_1 becomes $9/(4\alpha)$ and R^2 in the denominator of (68) remains of the same order of magnitude as $\alpha\mathcal{G}$. Therefore, as long as this background solution is concerned, the perturbation behaviors in the limit $-\alpha \gg 1$ do not reduce to the special case.

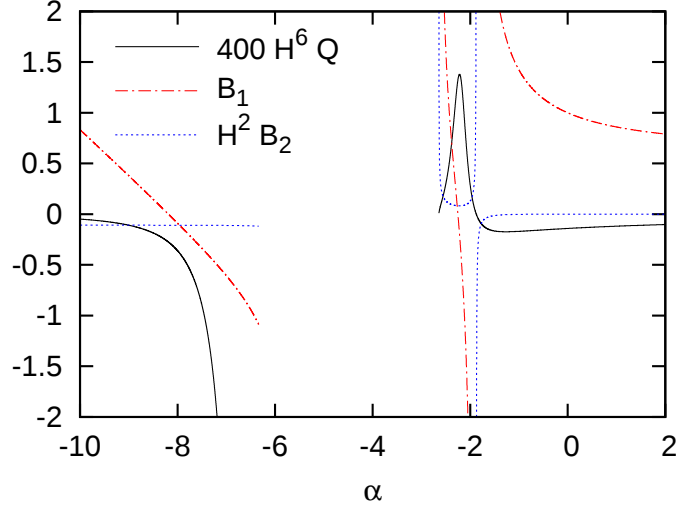


FIG. 1: Plot of Q, B_1, B_2 for one attractor of the toy model considered. B_1 becomes positive for three regions of α : (i) $\alpha < -8.19$, (ii) $-2.64 < \alpha < -2.25$, (iii) $\alpha > -1.875$. $H^2 B_2$ becomes positive only for $-2.64 < \alpha < -1.875$. $H^6 Q$ becomes positive only for $-2.64 < \alpha < -1.875$.

VI. SPECIAL CASES

In the previous section, we found that the perturbations propagate in non-trivial ways on small scales due to the k^4 -term in the master equation (59). However, there are special cases where B_2 , the coefficient in front of $k^4 \Phi$, vanishes identically. In these cases, we do not have the exotic behavior such as a strong instability or a superluminal propagation observed in the generic models due to the k^4 -term. Instead of the k^4 -term, the k^2 -term becomes important. In this section, we study these special cases.

From the explicit expression of B_2 given in the appendix, we see that the special case happens if either $\dot{H} = 0$ or $F_\lambda \xi_\sigma - F_\sigma^2 = 0$. Therefore, de Sitter space for any $f(R, \mathcal{G})$ MGM belongs to this case. We can easily verify that both $R + f(\mathcal{G})$ and $f(R)$ models for any background solutions also belong to this case. As we will explain later, there are other infinite numbers of $f(R, \mathcal{G})$ models that, irrespective of the background solutions, yield $B_2 = 0$. We will provide a systematic method to find those special gravity models and will list some explicit forms of $f(R, \mathcal{G})$.

Let us comment that vanishing B_2 is equivalent to $\delta\Gamma = 0$. To see this, let us first assume $B_2 = 0$. This implies that $\dot{H} = 0$ or $F_\lambda \xi_\sigma - F_\sigma^2 = 0$. If $\dot{H} = 0$, then $\dot{\lambda}$ and $\dot{\sigma}$ are also zero. From (52), we immediately find that $\delta\Gamma$ is also zero. If $F_\lambda \xi_\sigma - F_\sigma^2 = 0$, (52) tells us again that $\delta\Gamma = 0$. Therefore, $B_2 = 0$ implies $\delta\Gamma = 0$. Next let us assume that $\delta\Gamma = 0$. This implies that either of $F_\lambda \xi_\sigma - F_\sigma^2 = 0$ or $\dot{\lambda} \delta\sigma - \dot{\sigma} \delta\lambda = 0$. In the first case, it is obvious from the expression of B_2 given in the appendix that B_2 vanishes. In the second case, both $\dot{\lambda}$ and $\dot{\sigma}$ must vanish, which implies $\dot{H} = 0$. This again gives $B_2 = 0$. Therefore, $B_2 = 0$ is equivalent to $\delta\Gamma = 0$.

Since there are two independent conditions for the models to belong to this special case, we will study them separately.

A. Special cases I: $\dot{H} = 0$

If the background space-time is de Sitter, we have $\dot{\lambda} = \dot{\sigma} = 0$. Since F and ξ are both functions of λ and σ only, we also have that $\dot{F} = \dot{\xi} = 0$. Except for the scale factor, which varies like $a(t) \propto e^{Ht}$, all the other background quantities appearing in the perturbed equations are constants. Keeping this in mind, the perturbed equations (42),

(43), (44), (45) and (46) become

$$3H^2\alpha - \frac{k^2}{a^2}\phi - 3H\dot{\phi} + \frac{k^2}{a^2}H\chi = \frac{1}{2F} \left[3H(\delta\dot{F} + 4H^2\delta\xi) + \left(-3H^2 + \frac{k^2}{a^2} \right) (\delta F + 4H^2\delta\xi) \right], \quad (79)$$

$$\dot{\phi} - H\alpha = \frac{1}{2F} [H\delta F - \delta\dot{F} - 4H^2\delta\xi + 4H^3\delta\xi], \quad (80)$$

$$\dot{\chi} + H\chi - \phi - \alpha = \frac{1}{F} [\delta F + 4H^2\delta\xi], \quad (81)$$

$$\delta\lambda = -2 \left[12H^2\alpha + \frac{k^2}{a^2}(\dot{\chi} + 2H\chi) - 2\frac{k^2}{a^2}\phi - \frac{k^2}{a^2}\alpha - 3\ddot{\phi} - 12H\dot{\phi} + 3H\dot{\alpha} \right], \quad (82)$$

$$\delta\sigma = -8 \left[12H^4\alpha + 2\frac{k^2}{a^2}H^3\chi + \frac{k^2}{a^2}H^2\dot{\chi} - 3H^2\ddot{\phi} + 3H^3\dot{\alpha} - 12H^3\dot{\phi} - 2\frac{k^2}{a^2}H^2\phi - \frac{k^2}{a^2}H^2\alpha \right]. \quad (83)$$

These equations can be rewritten in terms of gauge independent fields. We can adopt the same field Ψ as in the general case (but evaluated on the de Sitter background), however Φ is not well defined for a de Sitter background. We will choose then another field Φ_{DS} defined as

$$\Phi_{\text{DS}} = -\frac{\delta F + 4H^2\delta\xi}{2F}. \quad (84)$$

This combination is gauge invariant on de Sitter and reduces to ϕ in the Newtonian gauge ($\chi = 0$). Therefore the equations of motion can be rewritten as

$$\Psi = 0, \quad (85)$$

$$\ddot{\Phi}_{\text{DS}} = -3H\dot{\Phi}_{\text{DS}} - \left[\frac{k^2}{a^2} - 4H^2 + \frac{F}{3(F_\lambda + 8H^2F_\sigma + 16H^4\xi_\sigma)} \right] \Phi_{\text{DS}}, \quad (86)$$

where δF and $\delta\xi$ are not independent from each other as, on this background, we have $\delta\sigma = 4H^2\delta\lambda$. Eq. (86) is equal to the Klein-Gordon equation in de Sitter space-time with an effective mass

$$m_{\text{eff}}^2 = -4H^2 + \frac{F}{3(F_\lambda + 8H^2F_\sigma + 16H^4\xi_\sigma)}. \quad (87)$$

If m_{eff}^2 is negative, then the long wavelength modes are unstable (tachyonic instability). For any $f(R, \mathcal{G})$ gravity models, the short wavelength modes are stable and the sound velocity is unity. The last quantity we need to find for this special case, is the expression for Q for Φ_{DS} . Its action can be written as

$$S = \frac{M_P^2}{\pi} \int dt d^3x Q a^3 \left[\frac{1}{2} \dot{\Phi}_{\text{DS}}^2 - \frac{1}{2} \frac{1}{a^2} (\vec{\nabla} \Phi_{\text{DS}})^2 - \frac{1}{2} m_{\text{eff}}^2 \Phi_{\text{DS}}^2 \right], \quad (88)$$

where

$$Q = \frac{3}{4} F. \quad (89)$$

Therefore F must be positive in order to avoid ghost-like degrees of freedom to propagate.

B. Special cases II: $F_\lambda \xi_\sigma - F_\sigma^2 = 0$

Before we study the perturbation equation, let us try to find what kinds of models belong to this case. Because F and ξ are functions of λ and σ , we have

$$\delta F = F_\lambda \delta\lambda + F_\sigma \delta\sigma, \quad (90)$$

$$\delta\xi = F_\sigma \delta\lambda + \xi_\sigma \delta\sigma. \quad (91)$$

Here we have used the identity $\xi_\lambda = F_\sigma$ in the equation for $d\xi$. Because $F_\lambda \xi_\sigma - F_\sigma^2$ is the determinant of the matrix of the linear transformation above, the condition $F_\lambda \xi_\sigma - F_\sigma^2 = 0$ tells us that δF and $\delta\xi$ are not independent. This

implies that F is a function of ξ . In other words, F and ξ are related to each other by a single variable φ like $F = F(\varphi)$, $\xi = \xi(\varphi)$. Such modified gravity models are equivalent to the action

$$S = \frac{M_P^2}{16\pi} \int d^4x \sqrt{-g} [\tilde{F}(\varphi)R + \tilde{\xi}(\varphi)\mathcal{G} - \tilde{U}(\varphi)], \quad (92)$$

where $\tilde{F}$, $\tilde{\xi}$ and $\tilde{U}$ are arbitrary functions. To see that this action is equivalent to the $f(R, \mathcal{G})$ action for the special case, let us first take the variation of (92) with respect to φ . After the variation, we obtain an equation given by

$$R \frac{d\tilde{F}}{d\varphi} + \mathcal{G} \frac{d\tilde{\xi}}{d\varphi} - \frac{d\tilde{U}}{d\varphi} = 0. \quad (93)$$

By solving this algebraic equation with respect to φ , we can write φ as a function of R and $\mathcal{G}$, i.e. $\varphi = \varphi(R, \mathcal{G})$. Substituting this solution into (92), we find that it reduces to the $f(R, \mathcal{G})$ action. Then we can calculate F and ξ as

$$F = \frac{\partial f}{\partial R} = \tilde{F}(\varphi(R, \mathcal{G})), \quad (94)$$

$$\xi = \frac{\partial f}{\partial \mathcal{G}} = \tilde{\xi}(\varphi(R, \mathcal{G})). \quad (95)$$

Therefore, $\tilde{F}$ and $\tilde{\xi}$ are equal to F and ξ . From these equations, we can easily verify that this model belongs to the special case, since

$$\frac{\partial F}{\partial R} \frac{\partial \xi}{\partial \mathcal{G}} - \frac{\partial F}{\partial \mathcal{G}} \frac{\partial \xi}{\partial R} = 0. \quad (96)$$

In this way, we can construct $f(R, \mathcal{G})$ models that belongs to the special case. Though finding analytic solution of (93) is difficult in general, we can in principle find as many models as we want by doing the above procedures for various $\tilde{F}$, $\tilde{\xi}$ and $\tilde{U}$. In this paper, we provide only a few simple $f(R, \mathcal{G})$ models by using the above method. The first example is a case where $\tilde{\xi}(\varphi) = c$ (c is a dimensionless constant). In this case, (93) tells us that φ depends only on R . Therefore, the corresponding modified gravity model is $f(R, \mathcal{G}) = f(R) + c\mathcal{G}$. Note that the term $c\mathcal{G}$ only adds a total derivative in the action and does not contribute to the equation of motion. Therefore, this model is equivalent to the $f(R)$ gravity model. The second example is the opposite of the first case, i.e. $\tilde{F}(\varphi) = c$. The corresponding modified gravity model is $f(R, \mathcal{G}) = cR + f(\mathcal{G})$. The third example is a case where $\tilde{F}(\varphi) = M^2 \tilde{\xi}(\varphi)$ (M is a constant of mass dimension). In this case, (93) tells us that φ depends only on the combination $R + \mathcal{G}/M^2$. The corresponding modified gravity model is $f(R, \mathcal{G}) = f(R + \mathcal{G}/M^2)$. We list these models in Table I. For the special models, the perturbation equation (59) can now be written as

$$\frac{1}{a^3 Q(t)} \partial_t [a^3 Q(t) \dot{\Phi}] + B_1(t) \frac{k^2}{a^2} \Phi = 0, \quad (97)$$

where B_1 in these cases is significantly reduced to a much simpler expression than the general case and is given by

$$B_1(t) = \frac{16H\dot{\xi}[\dot{F} + 4(H^2 + \dot{H})\dot{\xi}] + F[3\dot{F} + 4(3H^2 + 4\dot{H})\dot{\xi}] - 4(\dot{F} + 4H^2\dot{\xi})\ddot{\xi}}{3(F + 4H\dot{\xi})(\dot{F} + 4H^2\dot{\xi})}. \quad (98)$$

In particular, for $f(R)$ gravity models, it reduces to

$$B_1(t) = 1. \quad (99)$$

Therefore, we correctly recover the well-known result that the propagation speed is equal to the velocity of light in $f(R)$ models.

For $cR + f(\mathcal{G})$ gravity models, the propagation speed was calculated in [69]. In this case, according to our formula (98), $B_1(t)$ reduces to

$$B_1(t) = 1 + \frac{2\dot{H}}{H^2}. \quad (100)$$

If the universe, in vacuum, accelerates more slowly than $a(t) \propto t^2$, B_1 becomes negative and the FLRW universe is unstable on small scales. Therefore for these theories, acceleration (faster than t^2) is a condition for vacuum stability.

Special $f(R, \mathcal{G})$ models	remark
$f(R) + \mathcal{G}/M^2$	$\tilde{\xi}(\varphi) = 1/M^2$, M :Constant of mass dimension
$cR + f(\mathcal{G})$	$\tilde{F}(\varphi) = c$, c :Dimensionless constant
$f(R + \mathcal{G}/M^2)$	$\tilde{F}(\varphi) = M^2 \tilde{\xi}(\varphi)$

TABLE I: We list some of the special $f(R, \mathcal{G})$ models for which $B_2(t)$ identically vanishes. In the first case, different values of M give the same modified gravity model because a linear term in $\mathcal{G}$ in the action is a total derivative and does not contribute to the equations of motion.

On the other hand, if the universe undergoes super-acceleration $\dot{H} > 0$, the propagation of the perturbations becomes superluminal.

For $f(R + \mathcal{G}/M^2)$ gravity models, $B_1(t)$ reduces to

$$B_1(t) = 1 + \frac{8\dot{H}}{M^2 + 4H^2}. \quad (101)$$

Therefore, similarly to the $cR + f(\mathcal{G})$ models, the propagation becomes superluminal if the universe undergoes super-acceleration.

VII. DISCUSSION

So far we have considered the scalar modes. In the literature [66, 69], it has been claimed that the cosmological scalar perturbations for a general Lagrangian of the kind $f(\phi, R) + \xi(\phi)\mathcal{G} - U(\phi)$, were studied¹. However, we think that the scalar perturbations of this Lagrangian were not studied there. In fact, the authors considered only the following Lagrangian $F(\phi)R + \xi(\phi)\mathcal{G} - U(\phi)$, which actually reduces to the special case discussed above. By doing so, in the literature, the k^4 -term has always been neglected, and the most general MGM was never then fully studied. In fact, the novelty of our analysis resides in the study of the general case where the modes do get a non-trivial modification for the dispersion relation.

As for the vector modes, they are not important since, as already stated in [65], they do not propagate. We have explicitly verified that this remains true even for the general cases for which $F_\lambda \xi_\sigma - F_\sigma^2 \neq 0$ and $\dot{H} \neq 0$. The tensor modes instead, do propagate, but, differently from the scalars, we checked that there is no difference between general and special cases, and that their evolution equation coincides with the one given in [69]. More in detail their equations are

$$\frac{1}{a^3 Q_{TT}} \partial_t (a^3 Q_{TT} \dot{C}_{\mu\nu}) - \frac{c_{TT}^2}{a^2} \vec{\nabla}^2 C_{\mu\nu} = 0, \quad (102)$$

where

$$Q_{TT} = \frac{1}{2} F + 2H\dot{\xi}, \quad (103)$$

$$c_{TT}^2 = \frac{F + 4\ddot{\xi}}{F + 4H\dot{\xi}}. \quad (104)$$

However, they still give an important contribution in order to set constraints for these theories. To summarize, we can then set the following points in order to understand the vacuum structures for the MGM.

- In the general case one must set the following conditions for the tensor modes $Q_{TT} > 0$, $c_{TT}^2 > 0$. The scalar modes instead have a more complicated structure as they will acquire a non-trivial dispersion relation. We found, at least for the long and short wavelength modes, that $B_1 > 0$ and $B_2 > 0$ are the necessary conditions to avoid instability. Furthermore, in order to avoid ghosts, one needs to set the condition $Q > 0$.

¹ Eliminating the auxiliary field ϕ by using the equation of motion, we find that this model is equivalent to the general $f(R, \mathcal{G})$ gravity models.

- The backgrounds for the general MGM, which are not de Sitter, if all the previous conditions are satisfied, at least in the scalar sector, will always have modes with superluminal propagations. In fact, for the short-wavelength ones, independently of the background, their propagation speed grows linearly with their wave number k/a , up to the cutoff of the theory. The presence of superluminal modes might be present also for the tensor modes, but this is only a background dependent feature, i.e. the propagation speed is independent of k . Recently, in the literature, superluminal modes in cosmology have been discussed in the context of different theories, such as k-essence models. Whether or not these modes represent a real issue on a background which explicitly brakes Lorentz invariance is still matter of discussion and a clear and definite opinion shared by everybody on this issue is still not achieved [72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82]. Some problems may arise from having a superluminal modes interacting with matter fields, either indirectly (by looking for example at the evolution of the clustering $\delta\rho/\rho$) or directly (such as the production of a kind of Cerenkov radiation), but this would imply to go beyond linear perturbation theory.
- The behaviour of the scalar modes can be used in order to distinguish among different MGM. In fact, we have shown that the modified gravity models which possesses second order differential equations can be divided into two categories
 1. General Gravity Models, for which $F_\sigma^2 - F_\lambda \xi_\sigma \neq 0$. The scalar modes have two degrees of freedom, but their equation of motion is non-trivial in the sense that it has also a $\nabla^4 \Phi$ term. This terms, at least for the short-wavelength modes, changes the dispersion relation such that the group velocity of these modes becomes proportional to k , i.e $v_g = 2\sqrt{B_2}k/a$ (if $B_2 > 0$). Therefore the theory, unless the vacuum is de Sitter, will always have superluminal modes. This is a new feature of these model which, up to our knowledge, was not considered before in the literature. It should be noticed that this feature is not a spurious one, in the sense that it can be gauged away. In fact, the analysis has been done through gauge invariant fields. Besides, no matter which gauge invariant field is used to study the large k behaviour, exactly the same B_2 coefficient appears, as the parameter in front of the k^4 -term.
 2. Special Gravity Models, for which $F_\sigma^2 - F_\lambda \xi_\sigma = 0$. The $f(R)$ and $R + f(\mathcal{G})$ theories belong to this class (however general $f_1(R) + f_2(\mathcal{G})$ does not), but they are not the only ones. In fact we have found that there is a larger group of theories which all share the same feature $B_2 = 0$, which distinguishes them from the General case. This means that the equation of motion for the modes is similar to a standard wave equation, where speed of propagation is background dependent (except for the $f(R)$ theories for which $c^2 = 1$). As far as we know, these are the models mostly studied in the literature, but actually they represent a special case of the whole class of MGM.
 3. The de Sitter background is a “good” background for all modified gravity theories, provided that it is stable, that is if the scalar field has an effective positive squared mass, and provided that $F > 0$, in order to remove ghosts degrees of freedom.

In this paper we have only considered the vacuum case. Although the presence of matter is important, the vacuum itself has presented a rich structure due to the presence of the k^4 -term. Thinking on how matter could change the whole picture, it is difficult to imagine that this k^4 -term discontinuously disappears if we add standard matter fields (which, in the action, are expected to contribute only to the k^2 -terms) into the theory, because this k^4 dispersion relation is solely due to the modification of gravity and not due to the nature of matter. In this sense, the essence of the k^4 propagation, already present in vacuum case, clearly supports the importance of studying the vacuum.

The next obvious and important extension of this work is to add matter fields into these MGM, that we will study in another project of ours. Although conclusive results will be given in our forthcoming paper, we can naively expect the following effects on the propagation of modes due to the existence of matter. First, the master equation for the scalar perturbation of gravity has time dependent coefficients determined by the background dynamics. Since the background dynamics changes if matter fields are present, one obvious effect, by including the matter, is that those time dependent coefficients in the master equation will change. Second, we are adding new degrees of freedom. Therefore, we expect that one variable is not enough to specify the perturbation behavior. We will obtain at least two coupled evolution equations for the perturbations.

VIII. CONCLUSIONS

We have studied the structure of the scalar cosmological perturbation for the most general classes of MGM, $f(R, \mathcal{G})$ which do not have spurious spin-2 degrees of freedom. Indeed we have discovered, for the general case, a non-trivial propagation for the modes that has not been studied before. In fact, in the past literature [66, 69], claims were made

about studying the general class of MGM. However, we believe that this study was never actually performed, as the authors only discussed a special subcase of the general MGM.

This new propagation affects primarily the short-wavelength modes on non-de Sitter backgrounds, for which their group velocity becomes proportional to their wave number, $v_g = 2\sqrt{B_2(t)}k/a$, where B_2 is a background dependent quantity. This amounts to have a non-standard dispersion relation. In turn, this implies that these models will in general have superluminal modes. Besides we have also given the conditions that the same modes may be stable, i.e. they do not acquire a negative squared speed, or they do not become ghosts degrees of freedom. Although there is still discussion about the presence in cosmology of superluminal modes, we have here a new necessary constraint to impose for these MGM, that is $0 \leq B_2 < +\infty$.

However, not all MGM share this feature. Indeed there is a subclass (to which the common $f(R)$ and $f(\mathcal{G})$ theories belong) which does not have any longer these modified dispersion relation.

We strongly believe, that in the process to find (through a phenomenological bottom-up approach) the ultimate low-energy effective theory of gravity (and possibly to find at last that it coincides with General Relativity, albeit a cosmological constant), this work can be considered fundamental in order to set constraints to the cosmology of these models.

The search for the gravitational action requires indeed the study of local gravity constraints, the study of cosmological solutions, but, also in the light of new experiments, more and more the analysis of cosmological perturbations. It is necessary to understand the behaviour of these same models in the presence of matter fields, but we leave this issue for future work.

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APPENDIX A: EXPRESSIONS OF $A_i(t)$, ($i = 1, \dots, 7$), B_1, B_2 AND Q

We give explicit forms for $A_i(t)$.

$$A_1(t) = -\frac{3(\dot{F} + 4H^2\dot{\xi})^2}{2(F + 4H\dot{\xi})[\dot{F} + 2H(F + 6H\dot{\xi})]} \quad (A1)$$

$$A_2(t) = -\frac{12H^2\dot{H}}{(F + 4H\dot{\xi})[\dot{F} + 2H(F + 6H\dot{\xi})]} \quad (A2)$$

$$A_3(t) = -\frac{2\dot{H}}{(F + 4H\dot{\xi})^2} \quad (A3)$$

$$A_4(t) = -\frac{2\{(\dot{F} + 4H^2\dot{\xi})[\dot{F} + 4(\dot{H} + H^2)\dot{\xi}] + FH(\dot{F} + 8H^2\dot{\xi}) + 4H\ddot{\xi}[\dot{F} + H(F + 8H\dot{\xi})] + F^2H^2\}}{(F + 4H\dot{\xi})[\dot{F} + 2H(F + 6H\dot{\xi})]} \quad (A4)$$

$$A_5(t) = \frac{16H\dot{\xi}[\dot{F} + 4(\dot{H} + H^2)\dot{\xi}](\dot{F} + 4H^2\dot{\xi}) + F[3\dot{F} + 4(4\dot{H} + 3H^2)\dot{\xi}](\dot{F} + 4H^2\dot{\xi}) - 4\ddot{\xi}(\dot{F} + 4H^2\dot{\xi})^2}{2(F + 4H\dot{\xi})^2[\dot{F} + 2H(F + 6H\dot{\xi})]} \quad (A5)$$

$$A_6(t) = -\frac{2\dot{H}(\dot{F} + 4H^2\dot{\xi})^2\{6F_\sigma[\ddot{H}(\dot{F} + 4H^2\dot{\xi}) + 4H\dot{H}[\dot{F} + 2(\dot{H} + 2H^2)\dot{\xi}]] - \dot{F}\dot{\xi}\}}{3H[\dot{F} + 2H(F + 6H\dot{\xi})][2\dot{H}(\dot{F}\dot{H} + 2H^2\dot{F} - 48F_\sigma H\dot{H}^3 + 8H^4\dot{\xi}) + H\ddot{H}(\dot{F} + 4H^2\dot{\xi})]}, \quad (A6)$$

$$A_7(t) = \frac{2\dot{H}(32FH^3\dot{\xi} + 4F^2H^2 + 64H^4\dot{\xi}^2)\{6F_\sigma[\ddot{H}(\dot{F} + 4H^2\dot{\xi}) + 4H\dot{H}[\dot{F} + 2(\dot{H} + 2H^2)\dot{\xi}]] - \dot{F}\dot{\xi}\}}{3H[\dot{F} + 2H(F + 6H\dot{\xi})][2\dot{H}(\dot{F}\dot{H} + 2H^2\dot{F} - 48F_\sigma H\dot{H}^3 + 8H^4\dot{\xi}) + H\ddot{H}(\dot{F} + 4H^2\dot{\xi})]}. \quad (A7)$$

A useful relation among these quantities is the following $A_2A_6 = A_1A_3A_7$, moreover both A_6 and A_7 are proportional to $F_\sigma^2 - F_\lambda\xi_\sigma$, so that they identically vanish for the special cases. We also have

$$Q = \frac{A_1(F + 4H\dot{\xi})^2}{2(A_2A_7 - 1)[\dot{F} + 2H(F + 6H\dot{\xi})]}. \quad (A8)$$

Also we have defined

$$B_1 = \frac{-A_6\dot{A}_2 - A_2^2(A_4A_6A_7 - A_7\dot{A}_6 + A_6\dot{A}_7) + A_2(A_4A_6 - \dot{A}_6) + A_5(A_2A_7 - 1)^2}{A_1(A_2A_7 - 1)}, \quad (\text{A9})$$

$$B_2 = -\frac{A_2A_6^2}{A_1^2A_7} = \frac{64\dot{H}^2(F_\sigma^2 - F_\lambda\xi_\sigma)}{3[F_\lambda + 8H^2(F_\sigma + 2H^2\xi_\sigma)]\{F + 24H[\dot{H}(F_\sigma + 4H^2\xi_\sigma) + 4H\dot{H}[F_\sigma + 2\xi_\sigma(\dot{H} + 2H^2)]]\}}. \quad (\text{A10})$$

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