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Hybrid life insurance valuation based on a new standard deviation premium principle in a stochastic interest rate framework

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Abstract

In a complete arbitrage-free financial market, financial products are valued with the risk-neutral measure and these products are completely hedgeable. In life insurance, the approach is different as the valuation is based on an insurance premium principle which includes a safety loading. The insurer reduces the risk by pooling a vast number of independent risks. In our framework, we suggest valuations of a class of products that are dependent on both mortality and financial risk, namely hybrid life products.

The main contribution of this paper is to present a generalized standard deviation premium principle in a stochastic interest rate framework, and to integrate it in different valuation operators suggested in the literature. We illustrate our methods with a classical application, namely a Pure Endowment with profit. Several numerical results are presented, and an extensive sensitivity analysis is included.

Keywords:

Financial risk, actuarial risk, hybrid products, fair valuation, risk decomposition.

1 Introduction

The fair value of a liability is generally understood to be the amount for which it could be transferred between knowledgeable willing parties in an arm's length transaction. In a complete financial market, we use the risk-neutral measure to determine the fair value of a purely financial product. This theoretical risk-neutral probability differs from the real-world probability, which is sometimes also referred to as the physical probability. The methodology of this valuation is justified by the hedging technique; any hedgeable risk can be valuated at its hedge price. In practice, financial markets are rarely complete. Therefore we face the problem of valuating liabilities in incomplete financial markets. Several ways of valuating unhedgeable claims have been considered in the literature, see e.g. the risk-neutral Esscher measure in [9] and the minimal entropy martingale measure in [8]. On the other hand, for purely insurance products, insurance premium principles are applied. The insurer justifies his approach by the independent aspect of his risks and therefore applies the law of large numbers; although there are risks which are not quite diversifiable such as natural catastrophes or longevity. Securitization methods (e.g. CAT-BOND) is one of the possibilities for the insurer to deal with these systematic risks.

Defining a fair valuation of hybrid life contracts means defining a valuation that allows us to strike a balance between the standard financial approach and the actuarial approach, which is not a simple task. In order to be consistent with the financial market, the concept of market-consistency was used in the literature, see e.g. [13] or [10]; whilst to be consistent with the actuarial market, the concept of actuarial-consistency has been used in the literature, see e.g. [4].

Several attempts have been made in the literature in order to study the valuation of hybrid products. Following the axiomatization of [3], we will first conduct a profound study of the axioms that a valuation operator should verify in the presence of stochastic interest rates and their relation to several alternative insurance premium principles. Then, we will tackle the market-consistency property and the actuarial-consistency property. Furthermore, we will clearly define the concept of a fair valuation. In our study, we concentrate upon the Two-step method of [13], the Conditional standard deviation principle presented in [12] and the Three-step method of [5]. The primary contribution of this paper is to propose a new formulation of the classical standard deviation premium principle in a stochastic interest rate framework and to integrate it in the early mentioned valuations methods.

We organize the remainder of this article as follows. In section 2, we introduce the assumptions considered in this study and the new standard deviation premium principle. In section 3, we present our generalization of the Two-step method and the Conditional standard deviation principle. In section 4 we discuss our generalization of the Three-step method. In section 5 we study a Pure Endowment product with terminal bonus and finally in section 6 we present numerical results. Section 7 concludes the paper.

2 New standard deviation premium principle

2.1 Assumptions

In this subsection we start by specifying the assumptions which are assumed to hold in the following: Absence of arbitrage opportunities, complete financial market and independence between the actuarial and financial market.
Furthermore, our study will be done in a static time interval $[0, T]$, where T is the maturity. We will consider a portfolio of n insured individuals belonging to the same cohort, all aged x years at time 0 with remaining lifetimes $(\tau_1, \tau_2, ..., \tau_n)$ following the same distribution $(\tau_i)_{1 \leq i \leq n} \sim \tau_1$.
We will consider the following probability space

$$(\Omega^a \times \Omega^f, \mathcal{F}^a \vee \mathcal{F}^f, \mathbb{P}^a \times \mathbb{P}^f),$$

where
-$\mathbb{P}^a$, $\mathbb{P}^f$ denote the real world actuarial and financial probability, respectively. On the other hand, $\mathbb{Q}^f$ denotes the risk-neutral financial probability.
-$\mathcal{F}^f$ is the financial filtration.
-$\mathcal{F}^a$ is the actuarial filtration. It is dependent on the filtration $(\mathfrak{S}_t)_t$ generated by the systematic mortality risk factors and the filtration $\vee_{i=1}^n \Im^i$ holding the information of the unsystematic mortality risk. In respect of each individual $1 \leq i \leq n$, we define at time $t < T$, $\Im_t^i = \sigma(1_{\tau_i \leq u} : u \leq t)$.
Therefore, we have that

$$\mathcal{F}^a = \mathfrak{S} \vee (\vee_{i=1}^n \Im^i).$$

In particular, we denote $\mathcal{F} \triangleq \mathcal{F}^a \vee \mathcal{F}^f$.
The conditional expectation at time t $\mathbb{E}[.|\mathcal{F}_t]$ and the conditional variance at time t $\mathbb{V}[.|\mathcal{F}_t]$ will be denoted by $\mathbb{E}_{\mathcal{F}_t}[.]$ and $\mathbb{V}_{\mathcal{F}_t}[.]$, respectively.
It will be assumed that the n individuals have lifetimes, which are independent conditional on knowing the systematic mortality risk information at the maturity. In a more mathematical sense, this means for any $t \in [0, T]$

$$\mathbb{P}^a(\tau_1 > t, \tau_2 > t, ..., \tau_n > t | \mathfrak{S}_T) = \prod_{i=1}^{n} \mathbb{P}^a(\tau_i > t | \mathfrak{S}_T). \tag{1}$$

For financial instruments, we will consider a single stochastic investment fund $(S_t)_{0\leq t\leq T}$ and stochastic interest rates $(r_t)_t$. The price of a zero-coupon bond of maturity T at a time $t<T$ can be derived as follows

$$P(t,T) \triangleq \mathbb{E}^{\mathbb{P}^a\times\mathbb{Q}^f}_{\mathcal{F}_t}[e^{-\int_t^T r_s \mathrm{d}s}]. \tag{2}$$

For the actuarial instruments, stochastic mortality rates $(\lambda_t)_{0\leq t\leq T}$, adapted to the filtration $\mathfrak{G}$ will be used. Thus, the probability of an individual at time $t<T$ to survive until maturity T is

$${}_{T-t}p_{x+t} \triangleq \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_t}[e^{-\int_t^T \lambda_s \mathrm{d}s}]. \tag{3}$$

We assume that the n insureds have chosen the same payoff. Following [5], we are going to present the payoff of hybrid life contracts H^n_T as the following arithmetic average of payoffs of each insured

$$H^n_T \triangleq \frac{\sum_{j=1}^n g(\tau_j)h(S_T)}{n}, \tag{4}$$

where $g(\tau_j):(0,\infty[\to[0,\infty[$ and $h(S_T):(0,\infty[\to[0,\infty[$ are two functions $\mathcal{F}^a_T$-measurable and $\mathcal{F}^f_T$-measurable, respectively; the former represents the actuarial payoff part while the latter denotes the financial payoff part.
We assume that $\mathbb{E}^{\mathbb{P}^a\times\mathbb{Q}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s}h(S_T)g(\tau_j)]$, $\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s}h(S_T)g(\tau_j)]$, $\mathbb{V}^{\mathbb{P}^a\times\mathbb{Q}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s}h(S_T)g(\tau_j)]$ and $\mathbb{V}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s}h(S_T)g(\tau_j)]$ are finite for all $j=1,...,n$.
The class of hybrid products will be denoted by

$$\mathcal{H}_T \triangleq \left\{\frac{\sum_{j=1}^n g(\tau_j)h(S_T)}{n}\right\}. \tag{5}$$

The sub-class of financial investment funds (i.e. no actuarial payoff) is denoted by

$$\mathcal{H}^f_T \triangleq \{h(S_T)\} \subset \mathcal{H}_T, \tag{6}$$

whereas the sub-class of actuarial assets (i.e. no financial payoff) is indicated by

$$\mathcal{H}^a_T \triangleq \left\{\frac{\sum_{j=1}^n g(\tau_j)}{n}\right\} \subset \mathcal{H}_T. \tag{7}$$

Furthermore, the class of financial assets is indicated by

$$\tilde{\mathcal{H}}^f_T \triangleq \left\{h(S_T)e^{-\int_0^T r_s \mathrm{d}s}\right\}.$$

2.2 Axiomatization and standard deviation premium principle

In our study, we focus on the generic meaning of fair valuation in an insurance context as opposed to a particular technical meaning mandated by a particular regulation or legislation. According to [3], we follow some axioms that a valuation operator must verify in the following.

Definition 1 (Valuation operator).

Let $\pi_0 : \mathcal{H}_T \to \mathbb{R}$ *a function which represents the value at time* 0 *of the hybrid payoff* $H_T^n \in \mathcal{H}_T$ *defined in* (4)*. We say that* π_0 *is a valuation operator if it verifies the following axioms*

-***Normalization*** $\pi_0(0|\mathcal{F}_0) = 0.$

-***Translation invariance*** $\forall x \in \mathbb{R}, \forall H_T^n \in \mathcal{H}_T\ \pi_0(H_T^n + x|\mathcal{F}_0) = \pi_0(H_T^n|\mathcal{F}_0) + P(0,T)x.$

-***Positive homogeneity*** $\forall a > 0, \forall H_T^n \in \mathcal{H}_T\ \pi_0(aH_T^n|\mathcal{F}_0) = a\pi_0(H_T^n|\mathcal{F}_0).$

-***Sub-additivity*** $\forall H_T^n \in \mathcal{H}_T, \forall G_T^n \in \mathcal{H}_T\ \pi_0(H_T^n+G_T^n|\mathcal{F}_0) \leq \pi_0(H_T^n|\mathcal{F}_0)+\pi_0(G_T^n|\mathcal{F}_0).$

The normalization property entails that when a payoff of a product equals zero, it automatically has a zero value. In a stochastic interest rate framework, it is a natural choice that a constant value x at time T, is discounted by a zero-coupon bond $P(0,T)$ defined in (2), in order to calculate its value at time 0. Therefore, the translation invariance property uses the price of a zero-coupon to discount. The positive homogeneity states that a-times a given payoff leads to a-times the value of that payoff, and finally the sub-additivity ensures the diversification. We underline that the axiom of constant payoff $\pi_0(x|\mathcal{F}_0) = P(0,T)x$ can be directly deducted from the two first axioms (normalization and translation invariance). Our goal is to study and compare various valuation operators satisfying these four axioms and able to cope in a stochastic interest rate framework with hybrid financial and actuarial products.

First, we can develop operators directly inspired by finance. In that context, we intend to make an extension of the general asset pricing method; the valuation of any financial product $h(S_T) \in \mathcal{H}_T^f$ is determined by $\mathbb{E}_{\mathcal{F}_t^f}^{\mathbb{Q}^f}[e^{-\int_t^T r_s \mathrm{d}s} h(S_T)]$. Based on [1], we adapt this valuation in the context of hybrid payoffs $H_T^n \in \mathcal{H}_T$ and propose the following financial valuation operator π_0^f

$$\pi_0^f(H_T^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n]. \tag{8}$$

On the other hand, we can look at actuarial, derived from some standard actuarial approach. Typically, the insurer evaluates the product using the best estimate of futures cash-flows, augmented by an appropriate security load and ensures the diversification of his overall portfolio.

In the framework of this paper, we propose the following new hybrid standard deviation premium principle in a stochastic interest rate framework.

Definition 2 (**Generalized standard deviation premium principle**).
The generalized standard deviation premium principle in a stochastic interest rate framework is the valuation operator defined by

$$\pi_0^a(H_T^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[H_T^n]} \times P(0,T), \tag{9}$$

for $H_T^n \in \mathcal{H}_T$.

It is composed of two terms, where the first term represents the best estimate. Inspired by the standard deviation premium principle, the second part is the safety loading under the real probability measure, discounted by the zero-coupon price. For a purely actuarial payoff $g_T^n \in \mathcal{H}_T^a$ we obtain indeed a discounted standard deviation premium principle

$$\pi_0^a(g_T^n|\mathcal{F}_0) = P(0,T) \times (\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[g_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g_T^n]}). \tag{10}$$

Remark 1.
One might ponder whether another natural standard deviation premium principle in a stochastic interest rate framework might be formulated, namely

$$\pi_0^{\tilde{a}}(H_T^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n]}.$$

However, since this does not satisfy the axiom of translation invariance introduced above, it is not a valuation operator in our setting. Furthermore, it does not satisfy the standard deviation premium principle (10)*. Indeed, for a purely actuarial payoff* $g_T^n \in \mathcal{H}_T^a$

$$\pi_0^{\tilde{a}}(g_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s}] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g_T^n] + \frac{\alpha}{2}\Big(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g_T^{n\,2}] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T 2r_s \mathrm{d}s}] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g_T^n]^2 \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s}]^2\Big)^{\frac{1}{2}}.$$

Remark 2.
We also ignored the following valuation operator

$$\pi_0^{a^*}(H_T^n|\mathcal{F}_0) = P(0,T)\big(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[H_T^n]}\big),$$

since it assumes a total independence between interest rates and the investment fund; whereas, in the case of π_0^a *in* (9)*, we assume independence only in the variance part, and thus in the safety loading.*

Therefore the operator π_0^a in (9) seems to be the most appropriate generalization. We will now introduce the market-consistent property and the actuarial-consistent property.

Definition 3 (**Market-consistent property**).

A valuation operator π_0 is market-consistent if and only if

$$\forall h(S_T) \in \mathcal{H}_T^f, \forall H_T^n \in \mathcal{H}_T : \pi_0(H_T^n + h(S_T)|\mathcal{F}_0) = \pi_0(H_T^n|\mathcal{F}_0) + \pi_0^f(h(S_T)|\mathcal{F}_0).$$

As a consequence, one can directly see that

$$\forall h(S_T) \in \mathcal{H}_T^f, \forall H_T^n \in \mathcal{H}_T : \pi_0(H_T^n|\mathcal{F}_0) = \pi_0(H_T^n - h(S_T)|\mathcal{F}_0) + \pi_0^f(h(S_T)|\mathcal{F}_0).$$

Consequently, valuing H_T^n is equivalent to valuing $H_T^n - h(S_T)$ separately and hedging the other part $h(S_T)$ in the financial market. In other words, a market-consistent valuation is a valuation in agreement with the financial market. The financial operator π_0^f introduced in (8) satisfies the market-consistent property.

Definition 4 (**Actuarial-consistent property**).

A valuation operator satisfies the actuarial-consistent property if and only if $\forall g_T^n \in \mathcal{H}_T^a$

$$\pi_0(g_T^n|\mathcal{F}_0) = \pi_0^a(g_T^n|\mathcal{F}_0).$$

This signifies, that the operator is equal to the actuarial valuation π_0^a for actuarial payoffs. The suggested valuation operator π_0^a in (9) is actuarial-consistent.

Remark 3.

As discussed in [4], one might ask whether it would be more appropriate to define the actuarial-consistent property as the following

$$\forall H_T^n \in \mathcal{H}_T, \forall g_T^n \in \mathcal{H}_T^a : \pi_0(H_T^n + g_T^n|\mathcal{F}_0) = \pi_0(H_T^n|\mathcal{F}_0) + \pi_0^a(g_T^n|\mathcal{F}_0). \tag{11}$$

The disadvantage of this is the fact that this would lead to a contradiction for standard deviation principles (see appendix A for details).

Definition 5 (**Fair valuation**).

A fair valuation is a valuation that is both market-consistent and actuarial-consistent.

Proposition 1.

The financial operator π_0^f defined by (8) *and the actuarial operator π_0^a defined by* (9) *are no fair valuations.*

Proof.

π_0^f is not actuarial-consistent since for $g_T^n \in \mathcal{H}_T^a$

$$\pi_0^f(g_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} g_T^n] \neq \pi_0^a(g_T^n|\mathcal{F}_0).$$

π_0^a is not market-consistent since for $h(S_T) \in \mathcal{H}_T^f$ and $H_T^n \in \mathcal{H}_T$

$$\pi_0^a(H_T^n + h(S_T)|\mathcal{F}_0) = \pi_0^f(h(S_T)|\mathcal{F}_0) + \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[H_T^n + h(S_T)]} \times P(0,T),$$

and therefore $\pi_0^a(H_T^n + h(S_T)|\mathcal{F}_0) \neq \pi_0^f(h(S_T)|\mathcal{F}_0) + \pi_0^a(H_T^n|\mathcal{F}_0)$. $\square$

Therefore we will concentrate upon other valuation methodologies for the fair valuation of hybrid financial - actuarial products.

3 Generalization of the Two-step method and the Conditional standard deviation principle

In this section, we concentrate upon operator valuations from the literature and generalize them to fair valuation methods according to our axiomatization while using the generalized standard deviation premium principle in a stochastic interest rate framework defined by (9).

3.1 Adaptation of the Two-step method

We start by a valuation that is very close to, but slightly different from, the Two-step valuation proposed by [13]. Inspired by [13], we introduce the definition of a $\mathcal{F}_T^f$-conditional valuation in a stochastic interest rate framework.

Definition 6 ($\mathcal{F}_T^f$**-conditional valuation).**
A $\mathcal{F}_T^f$-conditional valuation is a $\pi_0(.|\mathcal{F}_0 \vee \mathcal{F}_T^f) : \mathcal{H}_T \to \tilde{\mathcal{H}}_T^f$ function that fulfills the following properties
-***Normalization*** $\pi_0(0|\mathcal{F}_0 \vee \mathcal{F}_T^f) = 0$.
-***Translation invariance*** $\forall H_T^n \in \mathcal{H}_T$, $\forall h(S_T) \in \mathcal{H}_T^f$

$$\pi_0(H_T^n + h(S_T)|\mathcal{F}_0 \vee \mathcal{F}_T^f) = \pi_0(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f) + h(S_T)e^{-\int_0^T r_s \mathrm{d}s}.$$

-***Positive homogeneity*** $\forall H_T^n \in \mathcal{H}_T$, $\forall h(S_T) \in \mathcal{H}_T^f$

$$\pi_0(H_T^n \times h(S_T)|\mathcal{F}_0 \vee \mathcal{F}_T^f) = h(S_T) \times \pi_0(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f).$$

According to [13], the Two-step method is defined as the mathematical expectation of a $\mathcal{F}_T^f$-conditional valuation i.e.

$\forall H_T^n \in \mathcal{H}_T$, $\forall \pi_0$ $\mathcal{F}_T^f$-conditional valuation

$$Two - step^{\pi_0} \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[\pi_0(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f)].$$

Applying this Two-step methodology using π_0^a defined in (9), we obtain the following definition

Definition 7 (The Two-step method).

$\forall H_T^n \in \mathcal{H}_T$, the Two-step method $\pi_0^{(2a)}$ is specified by

$$\pi_0^{(2a)}(H_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[\pi_0^a(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f)], \tag{12}$$

with $\pi_0^a(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f) = e^{-\int_0^T r_s \mathrm{d}s} \times (\mathbb{E}_{\mathcal{F}_0 \vee \mathcal{F}_T^f}^{\mathbb{P}^a \times \mathbb{Q}^f}[H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0 \vee \mathcal{F}_T^f}^{\mathbb{P}^a \times \mathbb{P}^f}[H_T^n]})$ *is a* $\mathcal{F}_T^f$-*conditional valuation.*

It is easy to check that the Two-step method $\pi_0^{(2a)}$ is a fair valuation.

Remark 4.

It is pertinent to note that the Two-step standard-deviation principle proposed by [13] (see example 3.4)

$$\pi_0^{(2p)}(H_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[\pi_0^p(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f)],$$

$$\pi_0^p(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f) = e^{-\int_0^T r_s \mathrm{d}s} \times (\mathbb{E}_{\mathcal{F}_0 \vee \mathcal{F}_T^f}^{\mathbb{P}^a \times \mathbb{P}^f}[H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0 \vee \mathcal{F}_T^f}^{\mathbb{P}^a \times \mathbb{P}^f}[H_T^n]}).$$

coincides with our proposal in (12) *inasmuch as our study is conducted on the set of hybrid life products* $\mathcal{H}_T$ *as defined in* (5). *This correspondence does not maintain for more general hybrid life payoffs; typically for products where the financial payoff depends on mortality outcomes. For instance, a 5-year coupon bond issued for hedging against the risk of abnormally high mortality.*

In the following lemma we derive a formula which will be useful for applications.

Lemma 1.

Let $H_T^n \in \mathcal{H}_T$, then

$$\pi_0^{(2a)}(H_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\frac{\sum_{j=1}^n g(\tau_j)}{n}]}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} h(S_T)]. \tag{13}$$

Proof.
As a first step, recalling the form of the hybrid payoff, namely $H_T^n = \frac{\sum_{j=1}^n g(\tau_j)h(S_T)}{n}$, we easily obtain

$$\pi_0^a(H_T^n|\mathcal{F}_0 \vee \mathcal{F}_T^f) = e^{-\int_0^T r_s \mathrm{d}s} h(S_T) \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[\frac{\sum_{j=1}^n g(\tau_j)}{n}] + \Big(\frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\frac{\sum_{j=1}^n g(\tau_j)}{n}]} \times h(S_T) e^{-\int_0^T r_s \mathrm{d}s}\Big).$$

As a second step, we use the assumption of independence between the financial and actuarial market to obtain the result. □

3.2 Generalized Conditional standard deviation principle

Based on [12] and the generalized standard deviation premium principle in a stochastic interest rate framework defined in (9), we introduce the following generalization of the Conditional standard deviation principle denoted by $\pi_0^{(2b)}$.

Definition 8 (**Generalized Conditional standard deviation principle**).
The generalized Conditional standard deviation principle $\pi_0^{(2b)}$ is defined by

$$\pi_0^{(2b)}(H_T^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0 \vee \mathcal{F}_T^f}^{\mathbb{P}^a \times \mathbb{P}^f}[H_T^n]]} \times P(0,T). \quad (14)$$

It can be easily checked that $\pi_0^{(2b)}$ is a fair valuation. In the following lemma, we derive the formula of the generalized Conditional standard deviation principle $\pi_0^{(2b)}$ for $H_T^n \in \mathcal{H}_T$.

Lemma 2.
Let $H_T^n \in \mathcal{H}_T$, then

$$\pi_0^{(2b)}(H_T^n|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \Big(\frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[h(S_T)^2]}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\frac{\sum_{j=1}^n g(\tau_j)}{n}]} \times P(0,T)\Big). \quad (15)$$

Proof.
It is sufficient to specify the general form of the hybrid payoff within the variance.
□

Remark 5.

One could raise the question whether a natural generalization of the Conditional standard deviation principle as presented in [12] could be

$$\pi_0^{(2b)^*}(H_T^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathcal{F}_T^f}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s} H_T^n]]}.$$

However, $\pi_0^{(2b)^*}$ does not represent a fair valuation since it is not actuarial-consistent according to our axiomatization; i.e. it is not consistent with the discounted standard deviation premium principle in (10)*. Indeed, for a purely actuarial risk $g_T^n \in \mathcal{H}_T^a$*

$$\begin{aligned}\pi_0^{(2b)^*}(g_T^n|\mathcal{F}_0) &= P(0,T)\times\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[g_T^n] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g_T^n]\times\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T 2r_s \mathrm{d}s}]}\\ &\neq \pi_0^a(g_T^n).\end{aligned}$$

4 Generalization of the Three-step method

The insurer's valuation is based on the mean-loading method, where he applies a mutualization of a large number of independent risks. In finance, another approach is adopted which is mainly based on the risk-neutral measure. This approach is justified by hedging strategies to replicate portfolios and therefore eliminate the risk. The payoffs studied in our context rely on financial and actuarial risks, which proves to be quite complicated for the insurer who seeks to have a clear idea on the nature and the origin of the risk that he incurs. Thus, we aim to understand how the insurer can hedge and reduce these risks, and to do so, we would like to have a simple decomposition of our hybrid payoff $H_T^n = \frac{\sum_{j=1}^n g(\tau_j)h(S_T)}{n}$.
Following the lines of [5], we consider the following decomposition for the actuarial part

$$g(\tau_j) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)] + (g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]) + (\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]).$$

Hence, the hybrid payoff (4) becomes

$$\begin{aligned}H_T^n = &\frac{\sum_{j=1}^n \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]h(S_T)}{n} + \frac{\sum_{j=1}^n (g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n} +\\ &\frac{\sum_{j=1}^n (\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}.\end{aligned}$$

Let us now denote

$$ {}_nH_T^1 \triangleq \frac{\sum_{j=1}^n \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]h(S_T)}{n}. \tag{16} $$

$$ {}_nH_T^2 \triangleq \frac{\sum_{j=1}^n (g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}. \tag{17} $$

$$ {}_nH_T^3 \triangleq \frac{\sum_{j=1}^n (\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}. \tag{18} $$

Definition 9 (The Three-step method).
The Three-step method π_0^3 is defined by

$$ \pi_0^3(H_T^n|\mathcal{F}_0) = \pi_0^f({}_nH_T^1|\mathcal{F}_0) + \pi_0^a({}_nH_T^2|\mathcal{F}_0) + \pi_0^n({}_nH_T^3|\mathcal{F}_0), $$

where π_0^f is given by (8), π_0^a is given by (9) and the valuation π_0^n is a valuation operator for systematic cash flows neither hedgeable nor diversifiable (a concrete proposition of valuation for this last part will be given afterwards).

It is easy to check that the Three-step method π_0^3 is a fair valuation. Let us detail in a stochastic interest rate framework, the three components of this Three step method.

4.1 Valuation of the financial hedgeable part ${}_nH_T^1$

At first view, we can already resume that ${}_nH_T^1$ in (16) is purely financial ($\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)] \in \mathbb{R}$); this part can be hedged in the financial market, and therefore the valuation π_0^f in (8) is a direct consequence of the market consistency. We present in lemma 3 the derivation formula of $\pi_0^f({}_nH_T^1|\mathcal{F}_0)$.

Lemma 3.
Let $H_T^n \in \mathcal{H}_T$. Then

$$ \pi_0^f({}_nH_T^1|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} h(S_T)] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]. \tag{19} $$

Proof.
By the definition of π_0^f in (8) and ${}_nH_T^1$ in (16), we obtain

$$ \pi_0^f({}_nH_T^1|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} \frac{\sum_{j=1}^n \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]h(S_T)}{n}], $$

since $\frac{\sum_{j=1}^n \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]}{n}$ is $\mathcal{F}_0$-measurable and $(\tau_j)_{1\le j\le n}$ identically distributed we obtain the result.

□

4.2 Valuation of the diversifiable part ${}_nH_T^2$

Concerning ${}_nH_T^2$ in (17), it has been shown in [5] that this part is diversifiable in a deterministic interest rate framework. Therefore, this part will be evaluated on the basis of insurance principles. We present in lemma 4 a derivation formula for $\pi_0^a({}_nH_T^2|\mathcal{F}_0)$ and we prove that this reaches asymptotically a zero value by n tending to $+\infty$.

Lemma 4.

Let $H_T^n \in \mathcal{H}_T$. Then

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[h(S_T)^2]} \times \sqrt{\frac{1}{n}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{G}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]]} \times P(0,T), \quad (20)$$

and as a consequence $\pi_0^a({}_nH_T^2|\mathcal{F}_0) \underset{n\to+\infty}{\longrightarrow} 0.$

Proof.

See Appendix B. □

4.3 Valuation of the systematic part ${}_nH_T^3$

It now remains to examine the third part ${}_nH_T^3$ in (18) of the decomposition. This part contains the risk that we can neither eliminate nor reduce, and we can clearly see that it depends on the insurance market as well as on the financial market. To be more concrete, it reflects what is meant by the "systematic" risk, for instance longevity risk. In order to price it, we suggest an Esscher transform as below

$$\pi_0^n({}_nH_T^3|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{M}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} \times_n H_T^3], \quad (21)$$

where

$$\mathbb{E}_{\mathcal{F}_0}^{\mathbb{M}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s} \times_n H_T^3] = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\int_0^T r_s \mathrm{d}s} \times_n H_T^3 \times \frac{e^{-\theta^{(1)}X_T^{(1)}}}{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[e^{-\theta^{(1)}X_T^{(1)}}]} \times \frac{d\mathbb{Q}^f}{d\mathbb{P}^f}].$$

$X_T^{(1)}$ represents the systematic actuarial risk of ${}_nH_T^3$ and $\theta^{(1)}$ a parameter which will be calibrated. The explicit derivation formula of $\pi_0^n({}_nH_T^3|\mathcal{F}_0)$ will be addressed in the application part.

5 Application to a Pure Endowment contract with terminal bonus

In this section, we proceed to the application part with the aim of illustrating the different valuations presented previously and comparing them. Pure Endowment contracts with terminal bonus are among the most classic hybrid products. The insurer pays the insured a basic amount C in the case of life after T years and in addition, if the insurer makes an outstanding gain, he shares a percentage of this profit with the insured. We conduct the study for a benefit of one euro and by the axiom of positive homogeneity, we will multiply by the amount C at the end to get the final value.

5.1 General assumptions

In the following, we assume that $g(\tau_j) = 1_{\tau_j > T}$ expressing whether the insured will survive or not after T years. The premium P is determined by ${}_TE_x = (\frac{1}{1+i})^T \times {}_T\tilde{p}_x$ where i is the technical rate and ${}_T\tilde{p}_x$ is the prudent survival probability. The (first order basis) premium is fully invested in a financial investment fund $(S_t)_{t\geq 0}$, therefore $S_0 = P$. It will be presumed that the insurer will share the gain with its insured in the event that the financial investment fund exceeds the premium P capitalized with the insurer's own technical rate i, i.e. $h(S_T) = 1 + \beta(S_T - P(1+i)^T)^+$, where β is the rate of the profit-sharing. With this definition, profit sharing is at maturity and only financial, we do not intend to add the gain that comes from mortality. Hence the hybrid life payoff is

$$H_T^n = \frac{\sum_{j=1}^{n} 1_{\tau_j > T}(1 + \beta(S_T - P(1+i)^T)^+)}{n}.$$

We now present the dynamics of the models which describe the financial and actuarial risks. For the interest rates, we use an Hull and White model. Then under $\mathbb{P}^f$

$$dr_t = (b(t) - ar_t)dt + \sigma_r d\widetilde{W}_r(t),$$

where $(\tilde{W}_r)_{t\geq 0}$ is a Brownian motion, $b(t)$ is a deterministic function, a is the constant speed of mean reversion and σ_r is the constant volatility.
Regarding the financial investment fund $(S_t)_{t\geq 0}$, we follow the model presented in [6] and we introduce under $\mathbb{P}^f$

$$dS_t = \mu_S(t)S_t dt + S_t\sigma_S(\rho d\widetilde{W}_r(t) + \sqrt{1-\rho^2}d\widetilde{W}_S(t)),$$

where $(\tilde{W}_S)_{t\geq 0}$ is a Brownian motion independent from $(\tilde{W}_r)_{t\geq 0}$, $\mu_S(t)$ is the instantaneous return of $(S_t)_{t\geq 0}$, σ_S is the volatility of the investment fund, and ρ

defines the correlation between the investment fund and interest rates.
In order to switch to the risk-neutral probability, we use Girsanov's Theorem to define the following Brownian motions $(W_r(t))_t$ and $(W_S(t))_t$

$$W_r(t) \triangleq \widetilde{W}_r(t) - \int_0^t \nu_u du.$$

$$W_S(t) \triangleq \widetilde{W}_S(t) + \int_0^t \eta(u)\, du.$$

ν_u and $\eta(u)$ are deterministic functions of u, which represent market prices of risk. We state that $\eta(u) = (\frac{\mu_S(u)-r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu_u \rho}{\sqrt{1-\rho^2}})$, thereby the instantaneous return of the investment fund $\mu_S(u)$ is stochastic. Moreover, the Radon-Nikodym derivative of $\mathbb{Q}^f$ with respect to $\mathbb{P}^f$ is

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}^f} = e^{\int_0^t \nu_u d\widetilde{W}_r(u) - \int_0^t (\frac{\mu_S(u)-r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu_u \rho}{\sqrt{1-\rho^2}})\, d\widetilde{W}_S(u) - \frac{1}{2}\int_0^t (\nu_u^2 + (\frac{\mu_S(u)-r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu_u \rho}{\sqrt{1-\rho^2}})^2)\, du}.$$

Consequently, we obtain the following dynamics under $\mathbb{Q}^f$

$$dS_t = r_t S_t dt + S_t \sigma_S(\rho dW_r(t) + \sqrt{1-\rho^2} dW_S(t)).$$

$$dr_t = (\theta(t) - a r_t)dt + \sigma_r dW_r(t),$$

with

$$\theta(t) = b(t) + \sigma_r \nu_t$$

a deterministic function chosen so as to exactly fit the term structure of interest rates currently observed in the market. Since the prices of the zero-coupon bonds given by Hull and White model in $t = 0$ coincide with those of the market, we obtain the following

$$P(0,T) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_0^T r_s \mathrm{d}s}] = P^{Market}(0,T).$$

Remark 6.

The constant interest rate case studied in [5] is included in our framework, e.g. by setting $\sigma_r = 0$, $a = 1$, $\theta(t) = r_0$, $\rho = 0$, $\mu_S(t) = \mu_S(0)$, $\eta(t) = \eta(0)$ and $\nu_t = 0$ for all t.

On the actuarial side, we model the stochastic mortality rates $(\lambda_t)_{t\geq 0}$ under $\mathbb{P}^a$ by an Ornstein–Uhlenbeck intensity process see [11]

$$d\lambda_t = \mu_m \lambda_t dt + \sigma_m dW_m(t),$$

where $\mu_m > 0$ and $\sigma_m > 0$. By applying Itô's lemma, we arrive at

$$\lambda_t = \lambda_0 e^{\mu_m t} + e^{\mu_m t}\sigma_m \int_0^t e^{-\mu_m u} dW_m(u).$$

After integrating from 0 to t and using Fubini's theorem, we reach

$$\int_0^t \lambda_u du = \lambda_0 \frac{e^{\mu_m t} - 1}{\mu_m} - \int_0^t \frac{\sigma_m}{\mu_m}[1 - e^{\mu_m (t-u)}] dW_m(u).$$

One can then notice that $\int_0^t \lambda_u du$ has a normal distribution, and that therefore $e^{-\int_0^t \lambda_u du}$ has a log-normal distribution. Based on the latter, we obtain the probability of survival until maturity T (see also [5])

$$_T p_x = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\int_0^T \lambda_u du}] = e^{-m_{0,T} + \frac{1}{2}s_{0,T}^2}, \tag{22}$$

where

$$m_{0,T} = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\int_0^T \lambda_u \mathrm{d}u] = \lambda_0 \frac{e^{\mu_m T} - 1}{\mu_m},$$

and

$$s_{0,T}^2 = \mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[\int_0^T \lambda_u \mathrm{d}u] = \frac{\sigma_m^2}{2\mu_m^3}(1 - e^{\mu_m T})^2 + \frac{\sigma_m^2}{\mu_m^2}(T - \frac{e^{\mu_m T} - 1}{\mu_m}).$$

The main advantage of this model is that we keep the Gaussian character of $X_T^{(1)} := -\int_0^T \lambda_s \mathrm{d}s$.

Regarding the calibration of its parameter $\theta^{(1)}$ used in the Esscher valuation of the part $_n H_T^3$ defined in (18), one can calibrate it using the Solvency 2 Cost-of-Capital approach [14]. Indeed, the fair valuation π_{CoC} of the risk $X_T^{(1)}$ using the Solvency 2 Cost of Capital approach is based on the Value at risk VaR

$$\pi_{CoC}(X_T^{(1)}) = -m_{0,T} + 6\%(VaR_{99,5\%}[X_T^{(1)}] + m_{0,T}).$$

Since $X_T^{(1)}$ has a normal distribution, then $VaR_{99,5\%}[X_T^{(1)}] = -m_{0,T} + 2,58 s_{0,T}$. In contrast, the valuation π_{Essch} of $X_T^{(1)}$ under an Esscher transform equals

$$\begin{aligned}\pi_{Essch}(X_T^{(1)}) &= \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[X_T^{(1)} \times \frac{e^{-\theta^{(1)} X_T^{(1)}}}{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[e^{-\theta^{(1)} X_T^{(1)}}]}] \\ &= -m_{0,T} - \theta^{(1)} s_{0,T}^2.\end{aligned}$$

By identification $\pi_{CoC}(X_T^{(1)}) = \pi_{Essch}(X_T^{(1)})$, we obtain the following estimation of the parameter $\theta^{(1)}$

$$\theta^{(1)} = \frac{-0.16}{s_{0,T}}.$$

In the following subsections, we derive the evaluation formula of the Pure Endowment product with terminal bonus for each of the valuation methods defined in the previous sections.

5.2 Valuation in the Two-step method

The valuation is given by

Proposition 2.
The Two-step valuation $\pi_0^{(2a)}$ in (12) of the Pure Endowment product with terminal bonus is determined by

$$\pi_0^{(2a)}(H_T^n|\mathcal{F}_0) \quad = (P(0,T) + \beta P(\Phi(d) - P(0,T)G\Phi(d - n(0,T)))) \times [{}_Tp_x + \tfrac{\alpha}{2} L_0^M], \tag{23}$$

where

$$d = \frac{\frac{1}{2}n(0,T)^2 - ln(G) - ln(P(0,T))}{n(0,T)},$$

$$n(0,T)^2 = \sigma_S^2 T + \frac{\sigma_r^2}{a^2}T + \frac{\sigma_r^2}{a^3}(\frac{-3}{2} - \frac{1}{2}e^{-2aT} + 2e^{-aT}) + \frac{2\rho\sigma_S\sigma_r T}{a} - \frac{2\rho\sigma_S\sigma_r(1 - e^{-aT})}{a^2},$$

$$G = (1+i)^T,$$

$$L_0^M = \sqrt{(e^{s_{0,T}^2} - 1)e^{-2m_{0,T}+s_{0,T}^2} + \frac{1}{n}(e^{-m_{0,T}+\frac{1}{2}s_{0,T}^2} - e^{-2m_{0,T}+2s_{0,T}^2})}.$$

Proof.
Starting from lemma 1, using the price of a call option in a stochastic interest rate framework with a strike equal to PG and the fact that $\sum_{j=1}^n 1_{\tau_j > T|\mathcal{F}_0 \vee \mathfrak{S}_T}$ follows a binomial distribution $Bin(n, e^{-\int_0^T \lambda_s \mathrm{d}s})$, one easily obtains formula (23). □

We can rewrite this formula (23) as follows

$$\pi_0^{(2a)}(H_T^n|\mathcal{F}_0) = (P(0,T) + \beta P(\Phi(d) - P(0,T)G\Phi(d - n(0,T)))) \times {}_Tp_x \times (1 + \frac{\alpha}{2}\overline{L_0^M}),$$

where $\overline{L_0^M} = \frac{L_0^M}{{}_Tp_x} = \sqrt{(e^{s_{0,T}^2} - 1) + \frac{1}{n}(e^{m_{0,T} - \frac{1}{2}s_{0,T}^2} - e^{s_{0,T}^2})}$.
Based on this form, the price can be interpreted as a zero coupon bond plus the portion of the profit sharing, taking into account the survival probability of the insured during T years, and inflated by a proportional loading factor equal to $\frac{\alpha}{2}\overline{L_0^M}$.

5.3 Valuation in the generalized Conditional standard deviation principle

The valuation is given by

Proposition 3.

The generalized Conditional standard deviation valuation $\pi_0^{(2b)}$ in (14) of the Pure Endowment product with terminal bonus is determined by

$$\pi_0^{(2b)}(H_T^n|\mathcal{F}_0) = \big(P(0,T) + \beta P(\Phi(d) - P(0,T)G\Phi(d - n(0,T)))\big)_T p_x + P(0,T)\tfrac{\alpha}{2} L_0^F L_0^M, \tag{24}$$

where

$$L_0^F = \Big[1 + 2\beta P(e^{\sigma_S \int_0^T \tilde{\lambda}(u)du + A_T + \frac{1}{2}V(0,T) + \sigma_S\sigma_r\rho\frac{T-B(0,T)}{a}}\Phi(f) - G\Phi(f-\Sigma)) + \beta^2\big(P^2$$
$$e^{2(\sigma_S \int_0^T \tilde{\lambda}(u)du + A_T + V(0,T)) + \sigma_S^2 T + 4\sigma_r\sigma_S\rho\frac{T-B(0,T)}{a}}\Phi(f+\Sigma) + (PG)^2\Phi(f-\Sigma)$$
$$- 2P^2 G e^{\sigma_S \int_0^T \tilde{\lambda}(u)du + A_T + \frac{1}{2}V(0,T) + \sigma_S\sigma_r\rho\frac{T-B(0,T)}{a}}\Phi(f)\big)\Big]^{\frac{1}{2}},$$
$$V(0,T) = \frac{-\sigma_r^2}{2a^3}(1-e^{-aT})^2 + \frac{\sigma_r^2}{a^2}(T - \frac{1-e^{-aT}}{a}),$$
$$B(0,T) = \frac{1-e^{-aT}}{a},$$
$$A_T = -ln(P(0,T)) + \frac{1}{2}V(0,T) - \sigma_r\int_0^T \nu_x \frac{1-e^{-a(T-x)}}{a}dx,$$
$$f = \frac{\sigma_S \int_0^T \tilde{\lambda}(u)du + \frac{1}{2}\sigma_S^2 T + A_T - ln(G) + 2\sigma_r\sigma_S\rho\frac{T-B(0,T)}{a} + V(0,T)}{\Sigma},$$
$$\tilde{\lambda}(u) = \sqrt{1-\rho^2}\eta(u) - \nu_u\rho,$$
$$\Sigma^2 = V(0,T) + 2\sigma_S\sigma_r\rho\frac{T-B(0,T)}{a} + \sigma_S^2 T,$$

and where we use the notation introduced in proposition 2.

Proof.

Through lemma 2 and based on the law of the financial investment fund under $\mathbb{P}^f$, namely

$$S_T \sim LN\Big(ln(P) + \sigma_S\int_0^T \tilde{\lambda}(u)du - \frac{1}{2}\sigma_S^2 T + A_T, V(0,T) + 2\sigma_S\sigma_r\rho\frac{T-B(0,T)}{a} + \sigma_S^2 T\Big), \tag{25}$$

the formula (24) is easily obtained. □

5.4 Valuation in the Three step method

The valuation of the three components ${}_nH_T^1$, ${}_nH_T^2$ and ${}_nH_T^3$ of the Pure Endowment contract with terminal bonus are respectively given by the following propositions 4, 5 and 6.

Proposition 4.

The financial hedgeable part ${}_nH_T^1$ in the Three-step method is evaluated by

$$\pi_0^f({}_nH_T^1|\mathcal{F}_0) = \big(P(0,T) + \beta P(\Phi(d) - P(0,T)G\Phi(d - n(0,T)))\big)_T p_x. \tag{26}$$

Proof.

Starting from lemma 3 and the price of a call option in a stochastic interest rate framework with a strike equal to PG, we find the desired formula (26). □

Proposition 5.

The diversifiable part ${}_nH_T^2$ in the Three-step method is evaluated by

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \tfrac{\alpha}{2} \times P(0,T) \times L_0^F \times L_1^M, \tag{27}$$

where

$$L_1^M = \sqrt{\frac{1}{n}(e^{-m_{0,T}+\frac{1}{2}s_{0,T}^2} - e^{-2m_{0,T}+2s_{0,T}^2})}.$$

Proof.

Starting from lemma 4, the law of the financial investment fund under $\mathbb{P}^f$ determined in (25), and given that $1_{\tau_1>T|\mathcal{F}_0\vee\mathfrak{S}_T}$ follows a Bernoulli distribution $Bern(e^{-\int_0^T \lambda_s \mathrm{d}s})$, we obtain the desired formula (27). □

Proposition 6.

The fair value of the systematic risk ${}_nH_T^3$ in the Three-step method is determined by

$$\begin{aligned}\pi_0^n({}_nH_T^3|\mathcal{F}_0) =_T p_x \times (e^{-s_{0,T}^2\times\theta^{(1)}} - 1)\times \\ \Big[P(0,T) + \beta P \times \Big(\Phi(d) - P(0,T)G\Phi(d - n(0,T))\Big)\Big].\end{aligned} \tag{28}$$

Proof.

In the case of the Pure Endowment product with terminal bonus, ${}_nH^3_T$ is given by

$$ {}_nH^3_T = (e^{-\int_0^T \lambda_s \mathrm{d}s} - e^{-m_{0,T}+\frac{1}{2}s^2_{0,T}})h(S_T), $$

and therefore

$$ \pi^n_0({}_nH^3_T|\mathcal{F}_0) = \frac{1}{\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{\int_0^T \theta^{(1)}\lambda_s \mathrm{d}s}]} \times \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s} \times (e^{-\int_0^T \lambda_s \mathrm{d}s} - e^{-m_{0,T}+\frac{1}{2}s^2_{0,T}})h(S_T)\times $$
$$ e^{\int_0^T \theta^{(1)}\lambda_s \mathrm{d}s} \times \frac{d\mathbb{Q}^f}{d\mathbb{P}^f}]. $$

We use the independence between the actuarial and financial markets to obtain

$$ \pi^n_0({}_nH^3_T|\mathcal{F}_0) = L^M \times L^F. $$

Where

$$ L^M = \Big(\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{\int_0^T (\theta^{(1)}-1)\lambda_s \mathrm{d}s}] - e^{-m_{0,T}+\frac{1}{2}s^2_{0,T}} \times \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{\int_0^T \theta^{(1)}\lambda_s \mathrm{d}s}]\Big) \times \frac{1}{\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{\int_0^T \theta^{(1)}\lambda_s \mathrm{d}s}]}, $$

and

$$ L^F = \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s} \times h(S_T) \times \frac{d\mathbb{Q}^f}{d\mathbb{P}^f}]. $$

Using the fact that $e^{\int_0^T \theta^{(1)}\lambda_s \mathrm{d}s}$ and $e^{\int_0^T (\theta^{(1)}-1)\lambda_s \mathrm{d}s}$ follows a log-normal distribution, we obtain after an easy calculation

$$ L^M = {}_Tp_x \times (e^{-s^2_{0,T}\times\theta^{(1)}} - 1). $$

With regard to L^F, we switch to the risk-neutral measure $\mathbb{Q}^f$ in order to evaluate it, then

$$ L^F = \mathbb{E}^{\mathbb{P}^a\times\mathbb{Q}^f}_{\mathcal{F}_0}[e^{-\int_0^T r_s \mathrm{d}s} \times h(S_T)] = P(0,T) + \beta P \times \Big(\Phi(d) - P(0,T)G\Phi(d-n(0,T))\Big), $$

and this concludes the proof. □

Here, we can interpret the Esscher valuation as a zero-coupon bond plus the portion of the profit sharing, taking into account the probability of survival of the insured during T years, and inflated by a loading factor controlled by the parameter $\theta^{(1)}$, which can be calibrated in a consistent way by the cost of capital approach.

6 Numerical results

In this section, we aim to present numerical values for the pricing of the Pure Endowment product with terminal bonus. For this purpose, we will use Black-Scholes-Vasicek specification, which is a particular case of the Black-Scholes-Hull-and-White settings described in the previous section. In particular, the dynamics of the model under $\mathbb{P}^f$ reduce to

$$dr_t = (b - ar_t)dt + \sigma_r d\widetilde{W}_r(t),$$

$$dS_t = \mu_S S_t dt + S_t \sigma_S(\rho d\widetilde{W}_r(t) + \sqrt{1-\rho^2} d\widetilde{W}_S(t)).$$

In order to switch to $\mathbb{Q}^f$, we notice that

$$W_r(t) = \widetilde{W}_r(t) - \nu t,$$

$$W_S(t) = \widetilde{W}_S(t) + \int_0^t \Big(\frac{\mu_S - r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu\rho}{\sqrt{1-\rho^2}}\Big)\, du,$$

and therefore, the dynamics of the Black-Scholes-Vasicek model under $\mathbb{Q}^f$ are

$$dS_t = r_t S_t dt + S_t \sigma_S(\rho dW_r(t) + \sqrt{1-\rho^2} dW_S(t)),$$

$$dr_t = (\theta - ar_t)dt + \sigma_r dW_r(t),$$

with

$$\theta = b + \sigma_r \nu.$$

It is well known that in the Vasicek model

$$P(0,T) = e^{-\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[\int_0^T r_s \mathrm{d}s] + \frac{1}{2}\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[\int_0^T r_s \mathrm{d}s]}. \tag{29}$$

Where

$$\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[\int_0^T r_s \mathrm{d}s] = \frac{1-e^{-aT}}{a}(r_0 - \frac{\theta}{a}) + \frac{\theta}{a}T,$$

and

$$\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[\int_0^T r_s \mathrm{d}s] = \frac{-\sigma_r^2}{2a^3}(1-e^{-aT})^2 + \frac{\sigma_r^2}{a^2}(T - \frac{1-e^{-aT}}{a}).$$

We choose portfolios of either $n = 500$, $n = 2000$ or $n = 5000$ pure endowment insurance contracts with profit that have a time to maturity of $T = 30$ for a cohort of 35-year-old females. We choose the technical interest rate $i = 0.9\%$ and the amount $C = 100000$. For the standard deviation principle, we choose a risk

aversion parameter $\alpha = 0.3$.
We compute the actuarial premium P (for a basic benefit of one euro)

$$P =_{30} E_{35} = (\frac{1}{1+i})^{30} \times_{30} \tilde{p}_{35} = 0.7116739,$$

where the survival probability ${}_{30}\tilde{p}_{35}$ is calculated by the Makeham's formula [1]

$${}_{30}\tilde{p}_{35} = s^{30} g^{c^{65}-c^{35}}.$$

It now remains to specify the numerical values that we will use for our various models.
The parameters a, σ_r, r_0, θ are taken from [7], while σ_S and ρ are taken from the numerical example of the paper [6]. Finally, μ_S is chosen as in [2] and the rate of the profit-sharing β is chosen as in [5]. We are left to specify the actuarial parameters. In order to calibrate μ_m and σ_m, we minimize the least square errors on the survival probabilities computed between the observed survival probabilities and the predicted mortality by the stochastic model, mathematically

$$(\mu_m, \sigma_m) = argmin \sum_{T=1}^{40} \left(e^{-m_{0,T}+\frac{1}{2}s^2_{0,T}} -_T p_x^{Obs} \right)^2,$$

where $({}_T p_x^{Obs})_{1 \leq T \leq 40}$ are the observed survival probabilities taken from the Human Mortality Database. We choose the 1965 female cohort and we set $\lambda_0 \triangleq -ln({}_1 p_x^{Obs})$, according to [15]. We summarize all the numerical values of the parameters in Table 1.

We now present the valuation of our contract with respect to the Two-step method, the generalized Conditional standard deviation principle and the Three-step method. We notice that the formulas (23), (26) and (28) simplify to the Black-Scholes-Vasicek specification by setting $P(0,T)$ as derived in (29).

- The generalized Conditional standard deviation principle in this setting equals
  $$\pi_0^{(2b)}(H_T^n|\mathcal{F}_0) = \big(P(0,T)+\beta P(\Phi(d)-P(0,T)G\Phi(d-n(0,T)))\big)_T p_x + P(0,T)\frac{\alpha}{2}L_1^F L_0^M,$$
  where $L_1^F = \Big[1 + 2\beta P(e^{T\mu_S}\Phi(l) - G\Phi(l - \sigma_S\sqrt{T})) + \beta^2\big(P^2 e^{2T(\mu_S+\frac{1}{2}\sigma_S^2)}\Phi(l + \sigma_S\sqrt{T}) + (PG)^2\Phi(l - \sigma_S\sqrt{T}) - 2P^2 G e^{\mu_S T}\Phi(l)\big)\Big]^{\frac{1}{2}}$,
  $l = \frac{T(\mu_S+\frac{1}{2}\sigma_S^2)-ln(G)}{\sigma_S\sqrt{T}}$.

[1] We choose the survival specification (FR) where the Belgian Royal Decree parameters are $s = 0.999669730966$, $g = 0.999951440172$ and $c = 1.116792453830$.

Parameters	Numerical Values
r_0	1.6%
a	40%
θ	0.8%
σ_r	0.5%
σ_S	15%
ρ	20%
μ_S	5%
β	40.19%
λ_0	0.001080584
μ_m	0.0745323698
σ_m	0.0002248628
$\theta^{(1)}$	$\theta^{(1)} = \frac{-0.16}{s_{0,T}} = -3.373053$

Table 1: Parameters of the model.

- The diversifiable part ${}_nH_T^2$ in the Three-step method is evaluated by

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \frac{\alpha}{2} \times P(0,T) \times L_1^F \times L_1^M.$$

The valuation of the contract according to the Two-step method and the generalized Conditional standard deviation principle are presented in the following Table 2.

Portfolio size n	$\pi_0^{(2a)}(H_T^n|\mathcal{F}_0)$
500	60903.55
2000	60889.75
5000	60886.95
∞	60885.07

Portfolio size n	$\pi_0^{(2b)}(H_T^n|\mathcal{F}_0)$
500	61388.13
2000	61362.29
5000	61357.04
∞	61353.53

Table 2: The valuation of the contract with respect to $\pi_0^{(2a)}(H_T^n|\mathcal{F}_0)$ and $\pi_0^{(2b)}(H_T^n|\mathcal{F}_0)$.

We observe that the Two-step valuations decrease with the size of the portfolio n with a limit value of 60885.07. Similarly, the generalized Conditional standard deviation principle has a limit value of 61353.53. Importantly, the two valuations are close. The results of the Three-step method are summarized in Table 3. We

can see from the outset that the valuation of the financial part takes the largest space in the Three-step method. We clearly see the diversifiable aspect of the actuarial part. We can nearly eliminate the actuarial risk with a large portfolio. In order to compare the Two-step method, the generalized Conditional standard deviation principle, and the Three-step method, we present Figure 1. We get that $\pi_0^{(2a)}$ is the least expensive, and $\pi_0^{(2b)}$ is cheaper than $\pi_0^{(3)}$ for a portfolio size less than approximately 302.
Next, we look at the sensitivity of our valuations regarding the volatility of interest rates σ_r, in comparison to the constant interest rate case studied in [5]. To this end, we present Figure 2 where one observes a rather surprising price increase. This trend reflects the influence of the stochasticity of interest rates on hybrid product valuations. In order to confirm the impact of the correlation ρ between interest rates and the financial investment fund on valuations, we present Figure 3. Noticeably, the valuations increase with respect to ρ. This finding is not surprising, because, as the correlation increases, the product becomes riskier and, therefore, more expensive. The impact of volatility changes, however, is more important than changes in this correlation coefficient.

Portfolio size n	$\pi_0^f({}_nH_T^1\vert\mathcal{F}_0)$	$\pi_0^a({}_nH_T^2\vert\mathcal{F}_0)$	$\pi_0^n({}_nH_T^3\vert\mathcal{F}_0)$	$\pi_0^3(H_T^n\vert\mathcal{F}_0)$
500	60347.86	266.08	719.9	61333.84
2000	60347.86	133.04	719.9	61200.8
5000	60347.86	84.14	719.9	61151.9
∞	60347.86	0	719.9	61067.76

Table 3: The valuation of the contract with respect to the Three-step method.

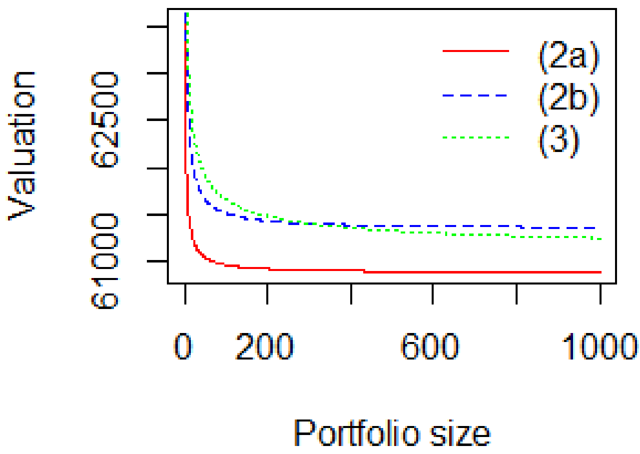

Figure 1: Comparison of $\pi_0^{(2a)}(H_T^n|\mathcal{F}_0)$, $\pi_0^{(2b)}(H_T^n|\mathcal{F}_0)$ and $\pi_0^{(3)}(H_T^n|\mathcal{F}_0)$.

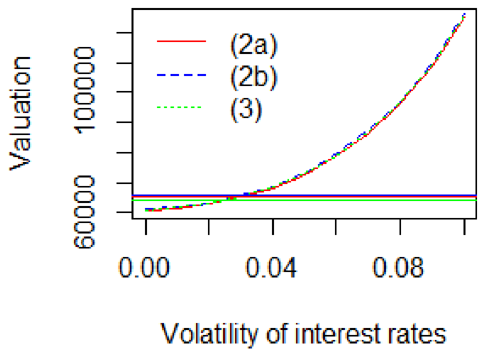

Figure 2: $\pi_0^{(2a)}(H_T^n|\mathcal{F}_0)$, $\pi_0^{(2b)}(H_T^n|\mathcal{F}_0)$ and $\pi_0^{(3)}(H_T^n|\mathcal{F}_0)$ with respect to σ_r. Constant interest rate case vs stochastic interest rate case. ($n = 5000$).

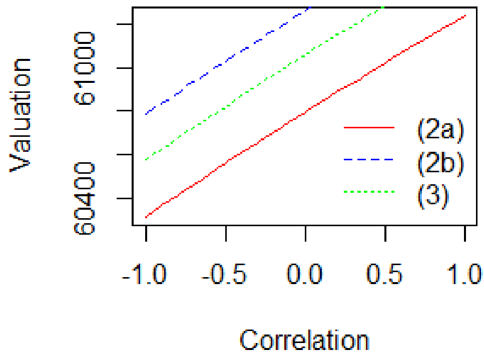

Figure 3: $\pi_0^{(2a)}(H_T^n|\mathcal{F}_0)$, $\pi_0^{(2b)}(H_T^n|\mathcal{F}_0)$ and $\pi_0^{(3)}(H_T^n|\mathcal{F}_0)$ with respect to ρ. $(n = 5000)$

7 Conclusion

In this paper, we have studied the valuation of hybrid life products, combining the financial and actuarial approaches.
From the literature, we followed an axiomatization of valuation operators in the presence of stochastic interest rates and subsequently studied their relation to several alternative insurance premium principles. In this setting, we introduced a new formulation of the classical standard deviation premium principle in a stochastic interest rate framework. We further defined the market-consistent and actuarial-consistent properties, thus developing the concept of fair valuation. Using this modelling framework, we generalized the Two-step method, the Conditional standard deviation principle and the Three step-method. We illustrated these valuations for a Pure Endowment product with financial terminal bonus using the Black-Scholes-Hull-and-White settings. Moreover, we obtained numerical values with the Black-Scholes-Vasicek specification. In our numerical experiment, the Two-step valuation method led to lower values than those obtained using the generalized Conditional standard deviation principle and the Three-step method. With an increasing portfolio size, the generalized Conditional standard deviation principle values started falling below the Three-step method values, until reaching a certain point, from which they become larger. In the Three-step method, the financial part represented the main part contributing to the total value and the systematic risk value was rather small, whereas the values of the actuarial part reduced when the size of the portfolio increased. Further, the values increased

according to the volatility of interest rates in comparison to the constant interest rate case, and also according to the correlation between interest rates and the financial investment fund.
In this study, we supposed independence between the actuarial and financial markets. Nevertheless, in the context of the health crisis recently experienced by the world, the increase in mortality has impacted financial markets. Therefore, relaxing this hypothesis in future research, would be desirable.

APPENDIX

Appendix A: Details about Remark 3.
Let us prove that (11) implies the linearity of π_0^a on the set $\mathcal{H}_T^a$.

Proof.

Let g_T^n, $\tilde{g_T}^n \in \mathcal{H}_T^a$. Then by (11), we write

$$\pi_0(g_T^n + \tilde{g_T}^n|\mathcal{F}_0) = \pi_0(g_T^n|\mathcal{F}_0) + \pi_0^a(\tilde{g_T}^n|\mathcal{F}_0).$$

Using the axiom of normalization, we obtain $\pi_0(g_T^n|\mathcal{F}_0) = \pi_0^a(g_T^n|\mathcal{F}_0)$.
Since $g_T^n + \tilde{g_T}^n \in \mathcal{H}_T^a$, we get $\pi_0(g_T^n + \tilde{g_T}^n|\mathcal{F}_0) = \pi_0^a(g_T^n + \tilde{g_T}^n|\mathcal{F}_0)$.
Therefore

$$\pi_0^a(g_T^n + \tilde{g_T}^n|\mathcal{F}_0) = \pi_0^a(g_T^n|\mathcal{F}_0) + \pi_0^a(\tilde{g_T}^n|\mathcal{F}_0).$$

□

Appendix B: Proof of lemma 4.

Proof.

Starting from the definition of π_0^a in (9), one easily notices that

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[{}_nH_T^2 e^{-\int_0^T r_s\mathrm{d}s}] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[{}_nH_T^2]} \times P(0,T).$$

Using the independence between mortality and finance as well as the tower rule, we obtain for the first term

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[{}_nH_T^2 e^{-\int_0^T r_s\mathrm{d}s}] &= \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s\mathrm{d}s}\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}] \\
&= \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s\mathrm{d}s}h(S_T)]\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}] \\
&= \frac{1}{n}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[e^{-\int_0^T r_s\mathrm{d}s}h(S_T)]\sum_{j=1}^n(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}[g(\tau_j)] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)]) \\
&= 0.
\end{aligned}$$

Consequently

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[{}_nH_T^2]} \times P(0,T)$$
$$= \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}]} \times P(0,T).$$

Let us denote $\tilde{A} \triangleq \frac{\sum_{j=1}^n(g(\tau_j)-\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}$, then it follows from the calculation above that $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\tilde{A}] = 0$. Moreover,

$$\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])h(S_T)}{n}] = \mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\tilde{A}] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[h(S_T)^2],$$

and therefore

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[h(S_T)^2]} \times \sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}]} \times P(0,T).$$

We further notice that

$$\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j)-\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}] = \frac{1}{n^2}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[\sum_{j=1}^n g(\tau_j) - n\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]]] + \frac{1}{n^2}\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[\sum_{j=1}^n g(\tau_j) - n\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]]].$$

Hence

$$\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j)-\mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}] = \frac{1}{n^2}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[\sum_{j=1}^n g(\tau_j)]] + \frac{1}{n^2} \times 0.$$

Since the remaining lifetimes $(\tau_j)_{1\leq j\leq n}$ are independent and identically distributed conditional on the filtration of systematic mortality $\mathfrak{S}_T$ (see (1)), this variance can be rewritten as

$$\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\frac{\sum_{j=1}^n(g(\tau_j) - \mathbb{E}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_j)])}{n}] = \frac{1}{n^2}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[n \times \mathbb{V}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]]$$
$$= \frac{1}{n}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]].$$

Finally

$$\pi_0^a({}_nH_T^2|\mathcal{F}_0) = \frac{\alpha}{2}\sqrt{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[h(S_T)^2]} \times \sqrt{\frac{1}{n}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{P}^f}[\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{S}_T}^{\mathbb{P}^a\times\mathbb{P}^f}[g(\tau_1)]]} \times P(0,T). \qquad \square$$

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