

ON THE ALMOST-CIRCULAR SYMPLECTIC INDUCED GINIBRE ENSEMBLE

SUNG-SOO BYUN* AND CHRISTOPHE CHARLIER†

ABSTRACT. We consider the symplectic induced Ginibre process, which is a Pfaffian point process on the plane. Let N be the number of points. We focus on the almost-circular regime where most of the points lie in a thin annulus $\mathcal{S}_N$ of width $O(\frac{1}{N})$ as $N \rightarrow \infty$. Our main results are the bulk scaling limits of all correlation functions near the real axis, and also away from the real axis. Near the real axis, the limiting correlation functions are Pfaffians with a new correlation kernel, which interpolates the limiting kernels in the bulk of the symplectic Ginibre ensemble and of the anti-symmetric Gaussian Hermitian ensemble of odd size. Away from the real axis, the limiting correlation functions are determinants, and the kernel is the same as the one appearing in the bulk limit of almost-Hermitian random matrices. Furthermore, we obtain precise large N asymptotics for the probability that no points lie outside $\mathcal{S}_N$, as well as of several other “semi-large” gap probabilities.

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1. INTRODUCTION AND MAIN RESULTS

The symplectic induced Ginibre ensemble is the Pfaffian point process for N points $\zeta = \{\zeta_j\}_{j=1}^N$ on the plane whose joint probability distribution $\mathbf{P}_N$ is given by

$$(1.1) \quad d\mathbf{P}_N(\zeta) = \frac{1}{N!Z_N} \prod_{1 \leq j < k \leq N} |\zeta_j - \zeta_k|^2 |\zeta_j - \bar{\zeta}_k|^2 \prod_{j=1}^N |\zeta_j - \bar{\zeta}_j|^2 e^{-NQ_N(\zeta_j)} dA(\zeta_j),$$

where $dA(\zeta) := d^2\zeta/\pi$, and Z_N is the normalisation constant. Here the potential Q_N depends strongly on N and is given by

$$(1.2) \quad Q_N(\zeta) := a_N |\zeta|^2 - 2b_N \log |\zeta|, \quad a_N = \frac{N}{\rho^2}, \quad b_N = \frac{N}{\rho^2} - 1,$$

where $\rho \in (0, \infty)$ is independent of N . If b_N is replaced by 0, then (1.1) is the eigenvalue distribution of a symplectic Ginibre matrix, i.e. a Gaussian random matrix with quaternion entries [33]. More generally, for integer values of b_N , (1.1) is the eigenvalue distribution of a symplectic Ginibre matrix conditioned to have b_N zero eigenvalues [4], see also [28, eq (2.40)] (such matrices are called symplectic induced Ginibre matrices [24]).

The ensemble (1.1) also admits a statistical mechanics interpretation [39, 28] as a two-dimensional Coulomb gas with an additional complex conjugation symmetry. From this point of view, the parameters a_N and b_N play different roles in the statistics of (1.1): a_N is a confining parameter, whereas b_N corresponds to the repulsion between the points and the origin. Since a_N and b_N are of order N , this confinement and this repulsion from 0 are much stronger than the point-point repulsion. As a

*Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea. e-mail: sungsoobyun@kias.re.kr. SB acknowledges support from the National Research Foundation of Korea, Grant NRF-2019R1A5A1028324, Samsung Science and Technology Foundation, Grant SSTF-BA1401-51, and KIAS Individual via the Center for Mathematical Challenges at Korea Institute for Advanced Study, Grant SP083201.

†Centre for Mathematical Sciences, Lund University, 22100 Lund, Sweden. e-mail: christophe.charlier@math.lu.se. CC acknowledges support from the Novo Nordisk Fonden Project, Grant 0064428, the Swedish Research Council, Grant No. 2021-04626, and the Ruth and Nils-Erik Stenbäck Foundation.

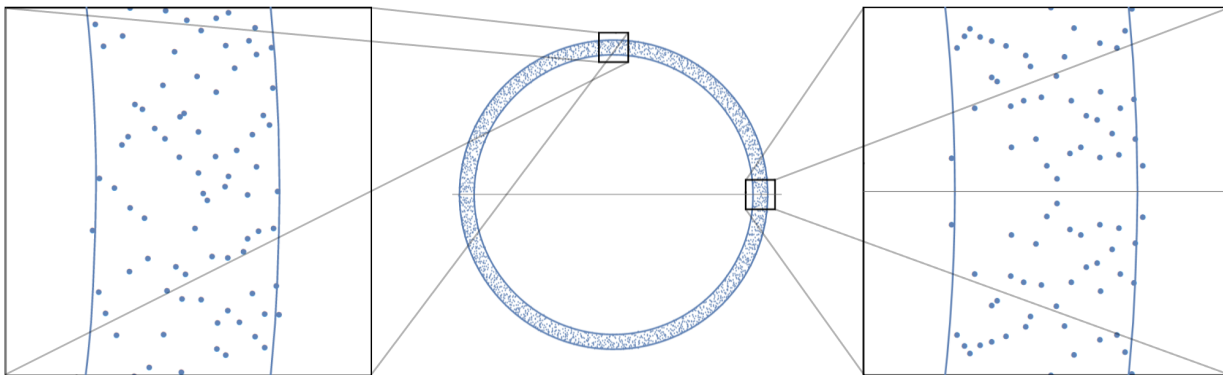


FIGURE 1. Illustration of $(\zeta, \bar{\zeta})$, where ζ represents the symplectic induced Ginibre ensemble in the almost-circular regime, with $N = 1000$ and $\rho = 10$. The zooms are taken near 1 and i .

consequence, one expects by [47, Section IV.6] and [17, Theorem 3.1]¹ that for large N the points are confined in a thin annulus with high probability. Let $\mathcal{S}_N$ be the support of the equilibrium measure associated with (1.1). $\mathcal{S}_N$ is called the droplet, and by [47, Section IV.6] it is given by

$$(1.3) \quad \mathcal{S}_N := \{\zeta \in \mathbb{C} : r_1 \leq |\zeta| \leq r_2\}, \quad r_1 = \sqrt{\frac{b_N}{a_N}}, \quad r_2 = \sqrt{\frac{2 + b_N}{a_N}}.$$

With the choice (1.2) of a_N and b_N , it follows that

$$(1.4) \quad r_1 = 1 - \frac{\rho^2}{2N} + O\left(\frac{1}{N^2}\right), \quad r_2 = 1 + \frac{\rho^2}{2N} + O\left(\frac{1}{N^2}\right), \quad \text{as } N \rightarrow \infty,$$

and therefore $\mathcal{S}_N$ is a thin annulus of width $\frac{\rho^2}{N} + O(N^{-2})$ as $N \rightarrow \infty$, see Figure 1. For this reason, we will refer to the choice of parameters (1.2)—which is the focus of this paper—as the *almost-circular regime*.

In this work N is large and ρ is fixed. As an interesting aside, we mention that in the other regime where $\rho \rightarrow 0$ while N is kept fixed, the points also lie in a thin annulus, and by taking formally $\rho = 0$, (1.1) becomes the eigenvalue distribution on the unit circle of a symplectic unitary random matrix, see [26, Section 2.6].

Planar ensembles with thin droplets were introduced in the works [31, 29, 30] on the complex elliptic Ginibre ensemble and were recently studied in [14, 22] for more general random normal matrix models. The work [14] treats universality questions for general bandlimited point processes using Ward's equation, while [22] deals with almost-circular ensembles associated with general radially symmetric potentials. In the earlier works [31, 29, 30, 14, 22], the considered point processes are determinantal. A major difference with our case is that (1.1) is Pfaffian.

We also mention that if Q_N is replaced by $2(1 + \frac{L+1}{N}) \log(1 + |\zeta|^2) - \frac{4L}{N} \log|\zeta|$, then (1.1) is called the symplectic induced spherical ensemble and was studied in [44, 21].

In this work, we study scaling limits and gap probabilities of (1.1) as $N \rightarrow \infty$ while ρ is kept fixed. Our results can be summarized as follows.

¹The result [47, Section IV.6] also holds for varying potentials as in our case, but [17, Theorem 3.1] only holds for fixed potentials.

- (i) **(Scaling limits)** One might expect from (1.1) that different limiting correlation structures will emerge as $N \rightarrow \infty$ depending on whether we look at the statistics near the real line or not. Our results confirm this expectation. Theorem 1.1 (a) deals with the limiting correlation structure of (1.1) near a point p on the unit circle, $p \neq -1, 1$, while Theorem 1.1 (b) deals with the other cases $p = \pm 1$. For $p = \pm 1$, the Pfaffian structure is preserved in the limit, and it involves a new skew pre-kernel $\kappa^{\mathbb{R}}$ (see (1.14) below) which interpolates between the limiting pre-kernels in the bulk of the symplectic Ginibre ensemble and of the anti-symmetric Gaussian Hermitian ensemble of odd size (see Remark 1.4 and Proposition 1.5). For the other case $|p| = 1$, $p \neq -1, 1$, we find that the Pfaffian structure of (1.1) simplifies in the limit and becomes determinantal, see (1.9)–(1.10). The limiting correlation kernel $K^{\mathbb{C}}$ is given by (1.11) and already appeared in [31, 29, 30, 7, 16, 14, 22], but the way it arises in this work, namely as the large N limit of the kernel of a Pfaffian point process, is new to our knowledge. (See also [18, Theorem 11] for a limiting determinantal structure arising in another Pfaffian point process.) In Theorem 1.1 (c), we study a transition between the two families of correlation functions associated with $\kappa^{\mathbb{R}}$ and $K^{\mathbb{C}}$.
- (ii) **(Gap probabilities)** For typical configurations, most of the points of (1.1) lie in $\mathcal{S}_N$. In fact, by analogy with e.g. [42, Eq.(70)], we expect about $\sim \sqrt{N}$ of the ζ_j 's to lie slightly outside $\mathcal{S}_N$. In Theorem 1.6 below, we obtain precise asymptotics for the probability $\mathbb{P}_N^{12}$ that *all* ζ_j lie in $\mathcal{S}_N$, up and including the term of order 1. We also obtain similar results for the probability $\mathbb{P}_N^1$ that $\min_{j \in \{1, \dots, N\}} |\zeta_j| \geq r_1$, and for the probability $\mathbb{P}_N^2$ that $\max_{j \in \{1, \dots, N\}} |\zeta_j| \leq r_2$.

In the following subsections, we give more background and state our main results.

1.1. Scaling limits. An interesting feature of (1.1) is that it is integrable and provides one of the few known examples of Pfaffian point processes in the plane [37]. The k -point correlation function of (1.1) is defined by

$$(1.5) \quad \mathbf{R}_{N,k}(\zeta_1, \dots, \zeta_k) := \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \mathbf{P}_N(\zeta) \prod_{j=k+1}^N dA(\zeta_j).$$

Since (1.1) is a Pfaffian point process, all correlation functions can be expressed as Pfaffians involving a (skew) pre-kernel $\mathbf{x}_N$. More precisely, we have

$$(1.6) \quad \mathbf{R}_{N,k}(\zeta_1, \dots, \zeta_k) = \text{Pf} \left[e^{-\frac{N}{2}(Q_N(\zeta_j) + Q_N(\bar{\zeta}_l))} \begin{pmatrix} \mathbf{x}_N(\zeta_j, \zeta_l) & \mathbf{x}_N(\zeta_j, \bar{\zeta}_l) \\ \mathbf{x}_N(\bar{\zeta}_j, \zeta_l) & \mathbf{x}_N(\bar{\zeta}_j, \bar{\zeta}_l) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k (\bar{\zeta}_j - \zeta_j),$$

where

$$(1.7) \quad \mathbf{x}_N(\zeta, \eta) := \mathbf{G}_N(\zeta, \eta) - \mathbf{G}_N(\eta, \zeta),$$

and $\mathbf{G}_N$ is given in terms of the standard Γ function by

$$(1.8) \quad \mathbf{G}_N(\zeta, \eta) := \sqrt{\pi} \left(\frac{a_N N}{2} \right)^{b_N N + \frac{3}{2}} \sum_{k=0}^{N-1} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{(\sqrt{\frac{a_N N}{2}} \eta)^{2l}}{\Gamma(l + \frac{b_N N}{2} + 1)}.$$

We prove (1.6)–(1.8) in Lemma 2.2 below. In this subsection, we obtain scaling limits of all correlation functions $\{\mathbf{R}_{N,k}\}_{k=1}^{\infty}$ as $N \rightarrow \infty$ at different points of the droplet.

The problem of deriving scaling limits for correlation functions of point processes is a classical topic in random matrix theory in connection with the local universality conjecture. This conjecture asserts, roughly speaking, that the limiting correlations between the points should only depend on the symmetry class of the random matrix model and on the points around which these correlations are studied, see e.g. [40] for a survey.

For planar Pfaffian point processes, such problems began to be addressed in the pioneering works of Mehta [45] and Kanzieper [37], who studied the limiting local statistics of the symplectic Ginibre ensemble at the origin. In recent years, there has been a growing number of works in that direction. To name a few, the limiting local statistics of the symplectic Ginibre ensemble at the real bulk/edge have been studied in [6, 43, 10]. These results have been extended to the elliptic Ginibre ensemble, see [8] for the limiting kernel at the origin, and [19] for the limiting kernel anywhere on the real line. For the elliptic Ginibre ensemble in the almost-Hermitian regime, the scaling limits for the kernel at the origin and at the real edge were discovered in [37] and [11] respectively. These scaling limits have then been extended to the entire real bulk/edge in [20]. Similar problems were studied in [38] for the truncated symplectic ensemble and in [20] for the Ginibre ensemble with boundary confinements. We also refer to [6, 4, 5, 35] for various results on scaling limits of correlation kernels in the context of Mittag-Leffler ensembles, Laguerre ensembles, and products of Ginibre matrices.

For the point process (1.1), it is natural to expect important differences between the limiting statistics near the real line and those away from the real line. Indeed, the factor $|\zeta_j - \bar{\zeta}_j|^2$ implies that the ζ_j 's repel from the real line. Furthermore, when the local statistics around the real line are considered, both interaction terms $|\zeta_j - \zeta_k|^2$ and $|\zeta_j - \bar{\zeta}_k|^2$ in (1.1) should contribute as $N \rightarrow \infty$. One therefore expects a limiting Pfaffian structure to emerge in the large N limit. On the other hand, when the local statistics are considered away from the real line, only one of the terms $|\zeta_j - \zeta_k|^2$ and $|\zeta_j - \bar{\zeta}_k|^2$ makes a non-trivial contribution; the other one behaves like a constant background charge which leads to a trivial contribution in the large N limit. Therefore, away from the real axis, the limiting local statistics are expected to be determinantal and to be the same as the ones appearing in the random normal matrix model. We now state our first main results, which confirm these expectations.

Theorem 1.1. *Let $\rho \in (0, \infty)$ and $k \in \mathbb{N}_{>0}$ be fixed, let Q_N be as in (1.2), and let $\gamma_N := \sqrt{2}\rho/N$.*

(a) **Scaling limit of $\mathbf{R}_{N,k}$ away from the real line.**

Let $p := e^{i\theta}$, where $\theta \in [0, 2\pi) \setminus \{0, \pi\}$. As $N \rightarrow \infty$, we have

$$(1.9) \quad \gamma_N^{2k} \mathbf{R}_{N,k} \left(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k \right) = R_k^{\mathbb{C}}(z_1, \dots, z_k) + o(1),$$

uniformly for $z_1, \dots, z_k$ in compact subsets of $\mathbb{C}$, where

$$(1.10) \quad R_k^{\mathbb{C}}(z_1, \dots, z_k) := \det \left[e^{-|z_j|^2 - |z_l|^2} K^{\mathbb{C}}(z_j, z_l) \right]_{j,l=1}^k,$$

$$(1.11) \quad K^{\mathbb{C}}(z, w) := \frac{e^{2z\bar{w}}}{2} \left(\operatorname{erfc}(z + \bar{w} - 2a) - \operatorname{erfc}(z + \bar{w} + 2a) \right), \quad a := \frac{\rho}{2\sqrt{2}}.$$

(b) **Scaling limit of $\mathbf{R}_{N,k}$ near the real line.**

Let $p = p_N := e^{i\theta_N}$, where $\theta_N = \frac{\sqrt{2}\rho}{N}t$ and $t \in \mathbb{R}$ is fixed. As $N \rightarrow \infty$, we have

$$(1.12) \quad \gamma_N^{2k} \mathbf{R}_{N,k} \left(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k \right) = R_k^{\mathbb{R}}(z_1 + it, \dots, z_k + it) + o(1),$$

uniformly for $z_1, \dots, z_k$ in compact subsets of $\mathbb{C}$, where

$$(1.13) \quad R_k^{\mathbb{R}}(z_1, \dots, z_k) := \operatorname{Pf} \left[e^{-|z_j|^2 - |z_l|^2} \begin{pmatrix} \kappa^{\mathbb{R}}(z_j, z_l) & \kappa^{\mathbb{R}}(z_j, \bar{z}_l) \\ \kappa^{\mathbb{R}}(\bar{z}_j, z_l) & \kappa^{\mathbb{R}}(\bar{z}_j, \bar{z}_l) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k (\bar{z}_j - z_j),$$

$$(1.14) \quad \kappa^{\mathbb{R}}(z, w) := \sqrt{\pi} e^{z^2 + w^2} \left(\int_{-a}^a W(f_w, f_z)(u) du + f_w(a) f_z(-a) - f_z(a) f_w(-a) \right),$$

$W(f, g) := fg' - gf'$ is the Wronskian, $a := \frac{\rho}{2\sqrt{2}}$ and

$$(1.15) \quad f_z(u) := \frac{1}{2} \operatorname{erfc}(\sqrt{2}(z - u)).$$

Statement (b) above also holds with $p = p_N := -e^{i\theta_N}$.

(c) **The $R_k^{\mathbb{R}}$ -to- $R_k^{\mathbb{C}}$ transition**
As $t \rightarrow \infty$, we have

$$(1.16) \quad R_k^{\mathbb{R}}(z_1 + it, \dots, z_k + it) = R_k^{\mathbb{C}}(z_1, \dots, z_k) + o(1),$$

uniformly for $z_1, \dots, z_k$ in compact subsets of $\mathbb{C}$.

Remark 1.2. The 1-point functions

$$R_1^{\mathbb{C}}(z) = \frac{1}{2} \left(\operatorname{erfc}(z + \bar{z} - 2a) - \operatorname{erfc}(z + \bar{z} + 2a) \right),$$

$$R_1^{\mathbb{R}}(z) = \sqrt{\pi}(\bar{z} - z) e^{(z - \bar{z})^2} \left(\int_{-a}^a W(f_{\bar{z}}, f_z)(u) du + f_{\bar{z}}(a) f_z(-a) - f_z(a) f_{\bar{z}}(-a) \right),$$

are represented in Figure 2 for several choices of a .

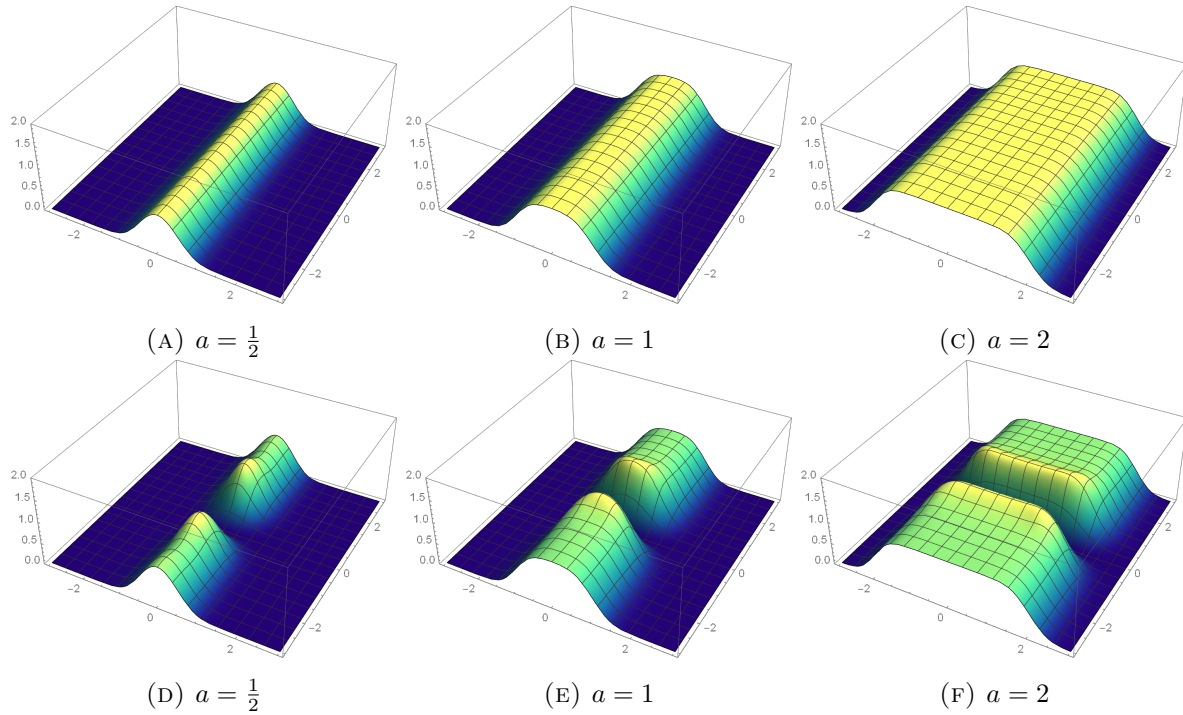


FIGURE 2. The functions $(x, y) \mapsto R_1^{\mathbb{C}}(x + iy)$ (first row) and $(x, y) \mapsto R_1^{\mathbb{R}}(x + iy)$ (second row) for the indicated values of a .

Remark 1.3. The limiting kernel $K^{\mathbb{C}}$ in Theorem 1.1 (a) already appeared in the study of several determinantal processes [31, 29, 30, 7, 16, 14, 22]. The way $K^{\mathbb{C}}$ arises in Theorem 1.1 (a) is novel in that (1.1) is Pfaffian.

As is known [7, Remark 4.(c)], $K^{\mathbb{C}}$ interpolates between two other known limiting kernels: as $\rho \rightarrow 0$, $K^{\mathbb{C}}$ converges (after a proper rescaling) to the sine kernel $K^{\sin}(x, y) := \frac{\sin(\pi(x-y))}{\pi(x-y)}$, while as $\rho \rightarrow \infty$ it converges to the limiting kernel in the bulk of the complex Ginibre ensemble, namely $K^{\exp}(z, w) := \exp \left[z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2) \right]$.

Remark 1.4. The skew pre-kernel $\kappa^{\mathbb{R}}$ is new to the best of our knowledge. Here we look at some particular limits involving $\kappa^{\mathbb{R}}$ and $R_k^{\mathbb{R}}$ when $\rho \rightarrow 0$ and $\rho \rightarrow \infty$ (although we emphasize that Theorem 1.1 is valid only for ρ fixed).

(1) **Matching with the bulk limit of the symplectic Ginibre ensemble when $\rho \rightarrow \infty$.**

Recall that the (classical) symplectic Ginibre ensemble is the planar Pfaffian point process (1.1) with $a_N = 1$ and $b_N = 0$. The limiting pre-kernel κ^W in the bulk of the symplectic Ginibre ensemble is known (see [6, Remark 2.3 (ii)]) and given by

$$(1.17) \quad \kappa^W(z, w) := \sqrt{\pi} e^{z^2 + w^2} \int_{-\infty}^{\infty} W(f_w, f_z)(u) du,$$

where f_z is given in (1.15) and we recall that $W(f, g) := fg' - gf'$ is the Wronskian. Since ρ is proportional to the width of the droplet (see (1.4)), it is natural to expect $\kappa^{\mathbb{R}}$ to tend to κ^W as $\rho \rightarrow +\infty$. A direct analysis shows that this is the case. Indeed, for any fixed $z, w \in \mathbb{C}$, using that $\lim_{a \rightarrow +\infty} f_z(-a) = 0$ and that $W(f_w, f_z)(u)$ has fast decay as $u \rightarrow \pm\infty$, we obtain

$$\lim_{\rho \rightarrow \infty} \kappa^{\mathbb{R}}(z, w) = \kappa^W(z, w).$$

(2) **Matching with the bulk limit of a chiral Gaussian unitary ensemble when $\rho \rightarrow 0$.**

The chiral Gaussian unitary ensemble with parameter $\nu = \frac{1}{2}$ [48] is the determinantal point process for N points $\boldsymbol{\xi} = \{\xi_j\}_{j=1}^N$ on $\mathbb{R}_+$ whose joint probability distribution $\mathbf{P}_N^{\chi^{\text{GUE}}}$ is defined by

$$(1.18) \quad d\mathbf{P}_N^{\chi^{\text{GUE}}}(\boldsymbol{\xi}) = \frac{1}{N! Z_N^{\chi^{\text{GUE}}}} \prod_{1 \leq j < k \leq N} |\xi_j^2 - \xi_k^2|^2 \prod_{j=1}^N \xi_j^2 e^{-N\xi_j^2} d\xi_j,$$

and whose limiting correlation kernel at the origin is given by

$$(1.19) \quad K^{\chi^{\text{sin}}}(y_1, y_2) := \frac{\sin(4(y_1 - y_2))}{2(y_1 - y_2)} - \frac{\sin(4(y_1 + y_2))}{2(y_1 + y_2)}, \quad y_1, y_2 \in \mathbb{R}_+,$$

see [26, Section 7.2]. We mention that (1.18) is also the ensemble of anti-symmetric Hermitian matrices when their size is odd [45, Section 13.1]. The kernel (1.19) is also equivalent to the well-known Bessel kernel in squared variables with parameter $\nu = \frac{1}{2}$ (see e.g. [26, Eq.(7.54)]²). When taking $\rho \rightarrow 0$, one heuristically expects the correlation functions $R_k^{\mathbb{R}}$ in (1.13) to “become determinantal” and to be related to $K^{\chi^{\text{sin}}}$. Indeed, by denoting $\zeta_j = 1 + i\xi_j$ ($\xi_j \in \mathbb{R}$) and replacing $dA(\zeta_j)$ by $d\xi_j$ in (1.1), we formally obtain

$$(1.20) \quad d\mathbf{P}_N(\boldsymbol{\zeta}) \sim \prod_{1 \leq j < k \leq N} |\xi_j^2 - \xi_k^2|^2 \prod_{j=1}^N \xi_j^2 e^{-NQ_N(1+i\xi_j)} d\xi_j.$$

The main difference between (1.18) and (1.20) lies in the potential, but by universality heuristics one expects this difference to be irrelevant for the computation of the limiting kernel. We make this heuristic more precise in Proposition 1.5 below.

Proposition 1.5. Recall that $a = \frac{\rho}{2\sqrt{2}}$ and that $R_k^{\mathbb{R}}$ is defined in (1.13). Let $k \in \mathbb{N}_{>0}$ and $z_1, \dots, z_k \in \mathbb{C}$ be fixed. As $\rho \rightarrow 0$, we have

$$(1.21) \quad \frac{1}{a^{2k}} R_k^{\mathbb{R}}\left(\frac{z_1}{a}, \dots, \frac{z_k}{a}\right) \xrightarrow{d} \det \left[K^{\chi^{\text{sin}}}(y_j, y_l) \right]_{j,l=1}^k, \quad (y_k = \text{Im } z_k),$$

²There is a typo in [26, Eq.(7.54)]: the $\pm$ sign on the right-hand side should instead be $\mp$.

where $K^{\chi^{\sin}}$ is given by (1.19), and where “ $\xrightarrow{d}$ ” in (1.21) means that for any bounded and continuous function $f : \mathbb{C}^k \rightarrow \mathbb{R}$ with compact support, we have

$$\lim_{\rho \rightarrow 0} \int_{\mathbb{C}^k} f(z_1, \dots, z_k) \frac{1}{a^{2k}} R_k^{\mathbb{R}}\left(\frac{z_1}{a}, \dots, \frac{z_k}{a}\right) \prod_{j=1}^k dA(z_j) = \int_{\mathbb{R}^k} f(y_1, \dots, y_k) \det \left[K^{\chi^{\sin}}(y_j, y_l) \right]_{j,l=1}^k \prod_{j=1}^k dy_j.$$

Our results are summarized in Figure 3, where to lighten the notation we use $y_j := \text{Im } z_j$ and

$$K_{2 \times 2}^W(z, w) := e^{-|z|^2 - |w|^2} \begin{pmatrix} \kappa^W(z, w) & \kappa^W(z, \bar{w}) \\ \kappa^W(\bar{z}, w) & \kappa^W(\bar{z}, \bar{w}) \end{pmatrix},$$

where κ^W is given by (1.17).

$$\begin{array}{c} \gamma_N^{2k} \mathbf{R}_{N,k} \left(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k \right), \quad p = \pm e^{i \frac{\sqrt{2}\rho}{N} t} \\ \text{Pfaffian} \\ \downarrow N \rightarrow \infty \quad \text{Thm 1.1(b)} \\ \det \left[K^{\chi^{\sin}}(y_j, y_l) \right]_{j,l=1}^k \xleftarrow[\text{Prop 1.5}]{\rho \rightarrow 0} R_k^{\mathbb{R}}(z_1 + it, \dots, z_k + it) \xrightarrow[\text{Rmk 1.4(1)}]{\rho \rightarrow \infty} \text{Pf} \left[K_{2 \times 2}^W(z_j, z_l) \right]_{j,l=1}^k \prod_{j=1}^k \frac{2y_j}{i} \\ \text{Pfaffian} \\ \downarrow t \rightarrow \infty \quad \text{Thm 1.1(c)} \\ \det \left[K^{\sin}(y_j, y_l) \right]_{j,l=1}^k \xleftarrow[\text{Rmk 1.3}]{\rho \rightarrow 0} R_k^{\mathbb{C}}(z_1, \dots, z_k) \xrightarrow[\text{Rmk 1.3}]{\rho \rightarrow \infty} \det \left[K^{\exp}(z_j, z_l) \right]_{j,l=1}^k \\ \text{Determinant} \\ \uparrow N \rightarrow \infty \quad \text{Thm 1.1(a)} \\ \gamma_N^{2k} \mathbf{R}_{N,k} \left(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k \right), \quad p = e^{i\theta}, \quad p \neq -1, 1 \\ \text{Pfaffian} \end{array}$$

FIGURE 3. The second column and the second row summarize our main findings. The third row was already known [31, 7] and is included in this diagram to place our results in their overall context.

1.2. Gap probabilities. Deriving asymptotic formulas of gap probabilities is a classical problem in random matrix theory with a rich history [27, 32]. The problem is usually considered challenging when the hole region is large. For a point process with N points, a “large” hole region refers to a region that contains, with high probability, a number of points proportional to N . For the disk hole region in the complex Ginibre point process, this problem was investigated by several authors [34, 25, 36]. In recent years, these results have been extended to other potentials, hole regions and models in [12, 13, 9, 2, 1, 41], and have been improved to higher precision in [23]. For more results on large gap asymptotics of two-dimensional point processes, we refer to [32, 23]. To our knowledge, the only paper prior to this work on large gap probabilities of planar Pfaffian point processes is [12].

In this subsection, we obtain large N asymptotics, up to and including the term of order 1, for the following three gap probabilities:

$$(1.22) \quad \mathbb{P}_N^1 := \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [0, r_1]\right\} = 0\right) = \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [r_1, \infty)\right\} = N\right),$$

$$(1.23) \quad \mathbb{P}_N^2 := \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [r_2, \infty)\right\} = 0\right) = \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [0, r_2]\right\} = N\right),$$

$$(1.24) \quad \mathbb{P}_N^{12} := \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [0, r_1] \cup [r_2, \infty)\right\} = 0\right) = \mathbb{P}\left(\#\left\{\zeta_j : |\zeta_j| \in [r_1, r_2]\right\} = N\right),$$

where the ζ_j are distributed according to (1.1) with Q_N as in (1.2), and where we recall that r_1 and r_2 are given by (1.3). Exact expressions for these probabilities in terms of the incomplete gamma function are given in (4.2), (4.3) and (4.4) respectively.

As argued in item (ii) above, for typical configurations the hole regions associated with $\mathbb{P}_N^1$, $\mathbb{P}_N^2$ and $\mathbb{P}_N^{12}$ contain about $\sim \sqrt{N}$ points. Hence, the problems of determining the large N asymptotics of $\mathbb{P}_N^1$, $\mathbb{P}_N^2$, $\mathbb{P}_N^{12}$ can be seen as “semi-large” gap problems (and are simpler than “large” gap problems such as the ones considered in [12]).

Theorem 1.6. Asymptotics of semi-large gap probabilities

Let $\rho \in (0, \infty)$ be fixed. As $N \rightarrow \infty$, we have

$$(1.25) \quad \begin{aligned} \mathbb{P}_N^1 &= \exp\left(NC_1 - C_0 + O(N^{-1})\right), & \mathbb{P}_N^2 &= \exp\left(NC_1 + C_0 + O(N^{-1})\right), \\ \mathbb{P}_N^{12} &= \exp\left(N\tilde{C}_1 + O(N^{-1})\right), \end{aligned}$$

where

$$(1.26) \quad C_1 = \int_0^1 \log\left(1 - \frac{1}{2} \operatorname{erfc}(\sqrt{2}\rho x)\right) dx,$$

$$(1.27) \quad \tilde{C}_1 = \int_0^1 \log\left(\frac{1}{2} \operatorname{erfc}(\sqrt{2}\rho(x-1)) - \frac{1}{2} \operatorname{erfc}(\sqrt{2}\rho x)\right) dx,$$

$$(1.28) \quad C_0 = \frac{\log(2 - \operatorname{erfc}(\sqrt{2}\rho))}{2} - \frac{\rho}{3\sqrt{2\pi}} \int_0^1 \frac{e^{-2\rho^2 x^2} (5 + 3\rho^2 x - 2\rho^2 x^2)}{1 - \frac{1}{2} \operatorname{erfc}(\sqrt{2}\rho x)} dx.$$

Theorem 1.6 has been verified numerically, see Figure 4.

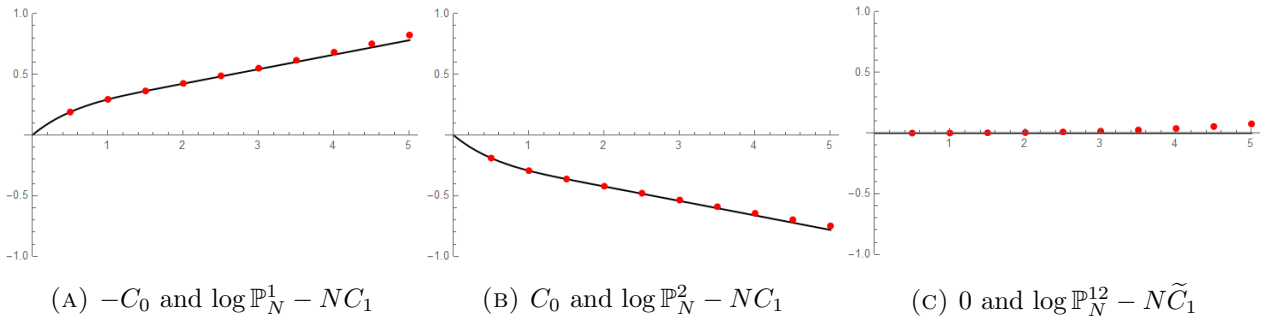


FIGURE 4. The functions $\rho \mapsto -C_0, C_0, 0$ (black lines) versus $\log \mathbb{P}_N^1 - NC_1$, $\log \mathbb{P}_N^2 - NC_1$ and $\log \mathbb{P}_N^{12} - N\tilde{C}_1$ with $N = 100$ and ten equispaced values of ρ from 0.5 to 5 (red dots).

Remark 1.7. Inequalities among $\mathbb{P}_N^1$, $\mathbb{P}_N^2$, and $\mathbb{P}_N^{12}$ for large N

It readily follows from the definitions (1.22), (1.23) and (1.24) that $\mathbb{P}_N^1, \mathbb{P}_N^2 > \mathbb{P}_N^{12}$. Theorem 1.6 allows to obtain more quantitative comparisons between these three probabilities. Indeed, it follows from (1.26), (1.27) and $\operatorname{erfc}(x) < 2$ ($x \in \mathbb{R}$) that $C_1 > \tilde{C}_1$ for all $\rho \in (0, \infty)$. Moreover, by (1.28), $C_0 < 0$ for all $\rho \in (0, \infty)$ (see also Figure 4 (B)). Thus we have

$$(1.29) \quad \log \mathbb{P}_N^1 \underbrace{>}_{O(1)} \log \mathbb{P}_N^2 \underbrace{\gg}_{O(N)} \log \mathbb{P}_N^{12}, \quad \text{as } N \rightarrow \infty.$$

In (1.29), the notation below $>$ and $\gg$ indicates the order of the difference between the left- and right-hand sides.

1.3. Outline. Different strategies will be used to prove Theorem 1.1 (a) and (b). However, for both, the starting point is the same: using skew-orthogonal polynomial techniques, we first express $\gamma_N^{2k} \mathbf{R}_{N,k}(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k)$ and the associated normalized skew pre-kernel $\hat{\mathbf{x}}_N$ in terms of the incomplete Gamma function (see (2.7), (2.8) and Lemma 2.2). We then prove a so-called generalised Christoffel-Darboux formula for $\hat{\mathbf{x}}_N(\zeta, \eta) := e^{-a_N N \zeta \eta} \hat{\mathbf{x}}_N(\zeta, \eta)$, which in our case is a differential identity expressing $\partial_\zeta \hat{\mathbf{x}}_N$ in terms of $\hat{\mathbf{x}}_N$ and the incomplete Gamma function (see Proposition 2.3).

In Section 3 we prove Theorem 1.1. Parts (a), (b) and (c) of this theorem are proven in Subsections 3.1, 3.2 and 3.3, respectively. In short, part (a) is proved via a first order Riemann sum approximation, part (b) is proved using our generalised Christoffel-Darboux formula, while part (c) follows from a direct analysis of $R_k^{\mathbb{R}}$. Proposition 1.5 is proved in Subsection 3.4.

Theorem 1.6 is proved in Section 4. Using again skew-orthogonal polynomials, we express $\mathbb{P}_N^1, \mathbb{P}_N^2, \mathbb{P}_N^{12}$ in terms of the incomplete gamma function γ . We then use some known asymptotics expansion of γ (which we recall in Appendix A) together with some precise Riemann sum approximations to complete the proof of Theorem 1.6.

1.4. Notation. Throughout this paper, $p = e^{i\theta}$ denotes the base/zooming point on the unit circle around which the local statistics of (1.1) are considered. Given $\mathbf{z} = \{z_j\}_{j=1}^N$, we define $\boldsymbol{\zeta} = \{\zeta_j\}_{j=1}^N \subset \mathbb{C}$ by

$$(1.30) \quad \zeta_j := p + p\gamma_N z_j, \quad \gamma_N = \sqrt{\frac{2}{a_N N}}, \quad j = 1, \dots, N,$$

and to shorten the notation we also define

$$(1.31) \quad R_{N,k}(z_1, \dots, z_k) := \gamma_N^{2k} \mathbf{R}_{N,k}(p + p\gamma_N z_1, \dots, p + p\gamma_N z_k) = \gamma_N^{2k} \mathbf{R}_{N,k}(\zeta_1, \dots, \zeta_k),$$

where we recall that $\mathbf{R}_{N,k}$ is given by (1.5).

2. SKEW-ORTHOGONAL POLYNOMIALS AND CHRISTOFFEL-DARBOUX FORMULA

In this section we first use the skew-orthogonal formalism for planar symplectic ensembles introduced by Kanzieper [37] to express $\hat{\mathbf{x}}_N$ as in (1.7)–(1.8) in terms of Γ . We then obtain a generalised Christoffel-Darboux formula for a normalized skew pre-kernel (Proposition 2.3).

Skew-orthogonal polynomials. Consider the following skew-symmetric form $\langle \cdot, \cdot \rangle_s$

$$\langle f, g \rangle_s := \int_{\mathbb{C}} \left(f(\zeta) g(\bar{\zeta}) - g(\zeta) f(\bar{\zeta}) \right) (\zeta - \bar{\zeta}) e^{-N Q_N(\zeta)} dA(\zeta).$$

A family $\{q_m\}_{m \geq 0}$ of monic polynomials q_m of degree m is said to be a family of skew-orthogonal polynomials if the following skew-orthogonality conditions hold: for all $k, l \in \mathbb{N}$

$$(2.1) \quad \langle q_{2k}, q_{2l} \rangle_s = \langle q_{2k+1}, q_{2l+1} \rangle_s = 0, \quad \langle q_{2k}, q_{2l+1} \rangle_s = -\langle q_{2l+1}, q_{2k} \rangle_s = r_k \delta_{k,l},$$

where r_k is a positive constant called the k -th skew-norm and $\delta_{k,l}$ is the Kronecker delta. The existence of $\{q_m\}_{m \geq 0}$ follows from a Gram-Schmidt skew-orthogonalisation procedure [8, Theorem 2.4], and a sufficient condition to ensure uniqueness is to set the coefficient z^{2k} in $q_{2k+1}(z)$ to 0 for each $k \in \mathbb{N}$ [8, Lemma 2.2]. It follows from the general theory [37] that (1.6) holds with

$$(2.2) \quad \mathfrak{x}_N(\zeta, \eta) = \sum_{k=0}^{N-1} \frac{q_{2k+1}(\zeta)q_{2k}(\eta) - q_{2k}(\zeta)q_{2k+1}(\eta)}{r_k}.$$

(In fact the above theory from [37] applies in a much broader setting, for example for smooth potentials $Q : \mathbb{C} \rightarrow \mathbb{R}$ of sufficient increase near ∞ (not necessarily rotation-invariant), but this will not be needed for us.)

Remark 2.1. *Different pre-kernels can yield the same correlation functions $\mathbf{R}_{N,k}$. For instance, if $\{g_N : \mathbb{C} \rightarrow \mathbb{C}\}_{N=0}^{+\infty}$ is a sequence satisfying $g_N(\bar{\zeta}) = 1/g_N(\zeta)$ for each $\zeta \in \mathbb{C}$ and $N \in \mathbb{N}$, then replacing $\mathfrak{x}_N(\zeta, \eta)$ in (1.6) by $g_N(\zeta)g_N(\eta)\mathfrak{x}_N(\zeta, \eta)$ does not modify $\mathbf{R}_{N,k}$; in other words, the two pre-kernels $\mathfrak{x}_N(\zeta, \eta)$ and $g_N(\zeta)g_N(\eta)\mathfrak{x}_N(\zeta, \eta)$ give rise to same point process.*

Since Q_N is rotation-invariant, by e.g. [35, p.7] and [8, Corollary 3.3] the following holds: the family $\{q_k\}_{k=0}^{+\infty}$ defined by

$$(2.3) \quad q_{2k+1}(\zeta) = \zeta^{2k+1}, \quad q_{2k}(\zeta) = \zeta^{2k} + \sum_{l=0}^{k-1} \zeta^{2l} \prod_{j=0}^{k-l-1} \frac{h_{2l+2j+2}}{h_{2l+2j+1}}, \quad k = 0, 1, \dots$$

where h_k is given by

$$(2.4) \quad h_k := \int_{\mathbb{C}} |\zeta|^{2k} e^{-NQ_N(\zeta)} dA(\zeta),$$

satisfies (2.1) with

$$(2.5) \quad r_k = 2h_{2k+1}.$$

For definiteness, from now we let $\{q_k\}_{k=0}^{+\infty}$ be the family of skew-orthogonal polynomials defined in (2.3), and we define $\mathfrak{x}_N$ as in (2.2) in terms of those $\{q_k\}_{k=0}^{+\infty}$. It follows from (1.6), the rescaling (1.31) and the definition (1.2) of Q_N that

$$(2.6) \quad R_{N,k}(z_1, \dots, z_k) = \text{Pf} \left[e^{-\frac{a_N N}{2}(|\zeta_j|^2 + |\zeta_l|^2)} \begin{pmatrix} \widehat{\mathfrak{x}}_N(\zeta_j, \zeta_l) & \widehat{\mathfrak{x}}_N(\zeta_j, \bar{\zeta}_l) \\ \widehat{\mathfrak{x}}_N(\bar{\zeta}_j, \zeta_l) & \widehat{\mathfrak{x}}_N(\bar{\zeta}_j, \bar{\zeta}_l) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k \frac{\bar{\zeta}_j - \zeta_j}{\gamma_N},$$

where

$$(2.7) \quad \widehat{\mathfrak{x}}_N(\zeta, \eta) = \gamma_N^3 \zeta^{b_N N} \eta^{b_N N} \mathfrak{x}_N(\zeta, \eta),$$

and the principal branches are used for $\zeta^{b_N N}$ and $\eta^{b_N N}$ (i.e. $\zeta^{b_N N} = e^{b_N N \log \zeta}$, $\log \zeta := \log |\zeta| + i \arg \zeta$ and $\arg \zeta \in (-\pi, \pi]$). To obtain (2.7), we have also used Remark 2.1 with $g_N(\zeta) = \zeta^{b_N N}/|\zeta|^{b_N N}$ to replace $|\zeta \eta|^{b_N N}$ in the calculations by $\zeta^{b_N N} \eta^{b_N N}$.

Recall that the incomplete gamma functions $\Gamma(a, \zeta)$, $\gamma(a, \zeta)$ and the regularised gamma function $Q(a, \zeta)$ are given by

$$(2.8) \quad Q(a, \zeta) := \frac{\Gamma(a, \zeta)}{\Gamma(a)}, \quad \Gamma(a, \zeta) := \int_{\zeta}^{\infty} t^{a-1} e^{-t} dt, \quad \gamma(a, \zeta) := \int_0^{\zeta} t^{a-1} e^{-t} dt = \Gamma(a) - \Gamma(a, \zeta),$$

see also e.g. [46, Chapter 8].

Lemma 2.2. *Let $\zeta, \eta \in \mathbb{C}$. We have $\widehat{\mathfrak{X}}_N(\zeta, \eta) = \widehat{\mathbf{G}}_N(\zeta, \eta) - \widehat{\mathbf{G}}_N(\eta, \zeta)$, where*

$$(2.9) \quad \begin{aligned} \widehat{\mathbf{G}}_N(\zeta, \eta) &:= \sqrt{\pi} \sum_{k=0}^{N-1} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{(\sqrt{\frac{a_N N}{2}} \eta)^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} \\ &= \sqrt{\pi} \sum_{k=0}^{N-1} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} e^{\frac{a_N N}{2} \eta^2} \left(\mathbf{Q}\left(k + \frac{b_N N}{2} + 1, \frac{a_N N}{2} \eta^2\right) - \mathbf{Q}\left(\frac{b_N N}{2}, \frac{a_N N}{2} \eta^2\right) \right). \end{aligned}$$

Proof. Recall that Q_N is defined in (1.2). Hence, by (2.4) we have

$$(2.10) \quad h_k = 2 \int_0^\infty r^{2k+1} e^{-NQ(r)} dr = 2 \int_0^\infty r^{2k+1+2b_N N} e^{-a_N N r^2} dr = \frac{\Gamma(1+k+b_N N)}{(a_N N)^{1+k+b_N N}},$$

and thus

$$\begin{aligned} \prod_{j=0}^{k-l-1} \frac{h_{2l+2j+2}}{h_{2l+2j+1}} &= \prod_{j=0}^{k-l-1} \frac{\Gamma(2l+2j+3+b_N N)}{(a_N N)^{2l+2j+3+b_N N}} \frac{(a_N N)^{2l+2j+2+b_N N}}{\Gamma(2l+2j+2+b_N N)} \\ &= \prod_{j=0}^{k-l-1} \frac{l+j+1+b_N N/2}{a_N N/2} = \left(\frac{2}{a_N N}\right)^{k-l} \frac{\Gamma(k + \frac{b_N N}{2} + 1)}{\Gamma(l + \frac{b_N N}{2} + 1)}, \quad 0 \leq l < k. \end{aligned}$$

Substituting the above in the definition (2.3) of q_{2k} yields

$$q_{2k}(\zeta) = \sum_{l=0}^k \left(\frac{2}{a_N N}\right)^{k-l} \frac{\Gamma(k + \frac{b_N N}{2} + 1)}{\Gamma(l + \frac{b_N N}{2} + 1)} \zeta^{2l}, \quad k \geq 0.$$

Using (2.5), (2.10), and the duplication formula of the gamma function

$$(2.11) \quad \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}),$$

we obtain

$$r_k = 2 \frac{\Gamma(2+2k+b_N N)}{(a_N N)^{2+2k+b_N N}} = \frac{1}{\sqrt{\pi}} \left(\frac{2}{a_N N}\right)^{2+2k+b_N N} \Gamma(k + \frac{b_N N}{2} + 1) \Gamma(k + \frac{b_N N}{2} + \frac{3}{2}).$$

Now, by (2.2), we get (1.7) with

$$\begin{aligned} \mathbf{G}_N(\zeta, \eta) &= \sum_{k=0}^{N-1} \frac{q_{2k+1}(\zeta) q_{2k}(\eta)}{r_k} = \sum_{k=0}^{N-1} \frac{(a_N N)^{2k+2+b_N N}}{2 \Gamma(2k+2+b_N N)} \zeta^{2k+1} \sum_{l=0}^k \left(\frac{2}{a_N N}\right)^{k-l} \frac{\Gamma(k + \frac{b_N N}{2} + 1)}{\Gamma(l + \frac{b_N N}{2} + 1)} \eta^{2l} \\ &= \sqrt{\pi} \left(\frac{a_N N}{2}\right)^{b_N N + \frac{3}{2}} \sum_{k=0}^{N-1} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{(\sqrt{\frac{a_N N}{2}} \eta)^{2l}}{\Gamma(l + \frac{b_N N}{2} + 1)}. \end{aligned}$$

It then follows from (2.7) that $\widehat{\mathfrak{X}}_N(\zeta, \eta) = \widehat{\mathbf{G}}_N(\zeta, \eta) - \widehat{\mathbf{G}}_N(\eta, \zeta)$, where $\widehat{\mathbf{G}}_N(\zeta, \eta) := \gamma_N^3 \zeta^{b_N N} \eta^{b_N N} \mathbf{G}_N(\zeta, \eta)$. This proves the first equation in (2.9). The second expression in (2.9) follows from the recurrence relation of the incomplete Gamma function (see e.g. [46, Eq.(8.8.9)]), namely

$$(2.12) \quad \sum_{k=0}^{N-1} \frac{z^{k+c}}{\Gamma(k+c+1)} = e^z \left(\mathbf{Q}(N+c, z) - \mathbf{Q}(c, z) \right).$$

□

We now obtain an identity for $\partial_\zeta \widehat{\mathfrak{X}}_N(\zeta, \eta)$ in terms of $\widehat{\mathfrak{X}}_N(\zeta, \eta)$ and $\mathbf{Q}$. As mentioned earlier, such identities are typically called generalised Christoffel-Darboux formulas (see e.g. [42, Proposition 2.3]).

Proposition 2.3. (*Christoffel-Darboux formula*) For $\theta \in [0, 2\pi)$, and $\zeta, \eta \in \mathbb{C}$, let

$$(2.13) \quad \tilde{\mathfrak{X}}_N(\zeta, \eta) := e^{-a_N N \zeta \eta} \hat{\mathfrak{X}}_N(\zeta, \eta) = e^{-2\mu\nu} \hat{\mathfrak{X}}_N(\zeta, \eta),$$

where

$$(2.14) \quad \mu := \sqrt{\frac{a_N N}{2}} \zeta, \quad \nu := \sqrt{\frac{a_N N}{2}} \eta.$$

Then we have

$$(2.15) \quad \begin{aligned} \sqrt{\frac{2}{a_N N}} \partial_\zeta \tilde{\mathfrak{X}}_N(\zeta, \eta) &= 2(\mu - \nu) \tilde{\mathfrak{X}}_N(\zeta, \eta) + 2 \left(Q(2N + b_N N, 2\mu\nu) - Q(b_N N, 2\mu\nu) \right) \\ &\quad - 2\sqrt{\pi} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} e^{\nu^2 - 2\mu\nu} \left(Q(N + \frac{b_N N}{2}, \nu^2) - Q(\frac{b_N N}{2}, \nu^2) \right) \\ &\quad - 2\sqrt{\pi} \frac{\mu^{b_N N-1}}{\Gamma(\frac{b_N N}{2})} e^{\nu^2 - 2\mu\nu} \left(Q(N + \frac{b_N N+1}{2}, \nu^2) - Q(\frac{b_N N+1}{2}, \nu^2) \right). \end{aligned}$$

Remark 2.4. Curiously, the term $Q(2N + b_N N, 2\mu\nu) - Q(b_N N, 2\mu\nu)$ appearing in the right-hand side of (2.15) is equal to the kernel of the complex induced Ginibre ensemble times $e^{\frac{N}{2}(Q_N(\nu) + Q_N(\mu))}$. Similar identities featuring possible relations between orthogonal and skew-orthogonal polynomial kernels have already appeared in the literature, see [19, 6, 4, 44] for two-dimensional point processes and [3, 51] for one-dimensional point processes.

Proof of Proposition 2.3. By (2.9), we have

$$\hat{\mathbf{G}}_N(\zeta, \eta) := \sqrt{\pi} \sum_{k=0}^{N-1} \frac{\mu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)},$$

where μ and ν are given by (2.14). Differentiating $\hat{\mathbf{G}}_N(\zeta, \eta)$ with respect to the ζ -variable yields

$$\begin{aligned} \partial_\zeta \hat{\mathbf{G}}_N(\zeta, \eta) &= \sqrt{\pi} \frac{a_N N}{2} \sum_{k=0}^{N-1} \frac{(2k+1+b_N N) \mu^{2k+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} \\ &= \sqrt{\pi 2 a_N N} \sum_{k=0}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)}. \end{aligned}$$

Rearranging the summation, we have

$$\partial_\zeta \hat{\mathbf{G}}_N(\zeta, \eta) = \sqrt{\pi 2 a_N N} \left[\sum_{k=1}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \frac{\mu^{b_N N}}{\Gamma(\frac{1}{2} + \frac{b_N N}{2})} \frac{\nu^{b_N N}}{\Gamma(\frac{b_N N}{2} + 1)} \right].$$

Since

$$\begin{aligned} \sum_{k=1}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} &= \mu \sum_{k=0}^{N-2} \frac{\mu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^{k+1} \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} \\ &= \mu \sum_{k=0}^{N-2} \frac{\mu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \sum_{k=1}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \frac{\nu^{2k+b_N N}}{\Gamma(k + \frac{b_N N}{2} + 1)}, \end{aligned}$$

we have

$$\begin{aligned}
& \sqrt{\pi} \left[\sum_{k=1}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \frac{\mu^{b_N N}}{\Gamma(\frac{1}{2} + \frac{b_N N}{2})} \frac{\nu^{b_N N}}{\Gamma(\frac{b_N N}{2} + 1)} \right] \\
&= \mu \sqrt{\pi} \sum_{k=0}^{N-2} \frac{\mu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \sum_{l=0}^k \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \sqrt{\pi} \sum_{k=0}^{N-1} \frac{\mu^{2k+b_N N}}{\Gamma(k + \frac{1}{2} + \frac{b_N N}{2})} \frac{\nu^{2k+b_N N}}{\Gamma(k + \frac{b_N N}{2} + 1)} \\
&= \mu \widehat{\mathbf{G}}_N(\zeta, \eta) - \sqrt{\pi} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^{N-1} \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \sum_{k=0}^{N-1} \frac{(2\mu\nu)^{2k+b_N N}}{\Gamma(2k+1+b_N N)},
\end{aligned}$$

where we have used (2.11) for the last line. We have just shown that

$$\frac{\partial_{\zeta} \widehat{\mathbf{G}}_N(\zeta, \eta)}{\sqrt{2a_N N}} = \mu \widehat{\mathbf{G}}_N(\zeta, \eta) - \sqrt{\pi} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^{N-1} \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} + \sum_{k=0}^{N-1} \frac{(2\mu)^{2k+b_N N} \nu^{2k+b_N N}}{\Gamma(2k+1+b_N N)}.$$

In a similar way, we obtain

$$\frac{\partial_{\zeta} \widehat{\mathbf{G}}_N(\eta, \zeta)}{\sqrt{2a_N N}} = \mu \widehat{\mathbf{G}}_N(\eta, \zeta) - \sum_{k=0}^{N-1} \frac{(2\mu)^{2k+1+b_N N} \nu^{2k+1+b_N N}}{\Gamma(2k+2+b_N N)} + \sqrt{\pi} \frac{\mu^{b_N N-1}}{\Gamma(\frac{b_N N}{2})} \sum_{k=0}^{N-1} \frac{\nu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})}.$$

Combining above equations, we have

$$\begin{aligned}
\frac{\partial_{\zeta} \widehat{\mathbf{X}}_N(\zeta, \eta)}{\sqrt{2a_N N}} &= \mu \widehat{\mathbf{X}}_N(\zeta, \eta) + \sum_{k=0}^{2N-1} \frac{(2\mu)^{k+b_N N} \nu^{k+b_N N}}{\Gamma(k+1+b_N N)} \\
&\quad - \sqrt{\pi} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} \sum_{l=0}^{N-1} \frac{\nu^{2l+b_N N}}{\Gamma(l + \frac{b_N N}{2} + 1)} - \sqrt{\pi} \frac{\mu^{b_N N-1}}{\Gamma(\frac{b_N N}{2})} \sum_{k=0}^{N-1} \frac{\nu^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})}.
\end{aligned}$$

Substituting (2.12) in the above expression, we obtain

$$\begin{aligned}
\sqrt{\frac{2}{a_N N}} \partial_{\zeta} \widehat{\mathbf{X}}_N(\zeta, \eta) &= 2\mu \widehat{\mathbf{X}}_N(\zeta, \eta) + 2e^{2\mu\nu} \left(Q(2N + b_N N, 2\mu\nu) - Q(b_N N, 2\mu\nu) \right) \\
&\quad - 2\sqrt{\pi} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} e^{\nu^2} \left(Q(N + \frac{b_N N}{2}, \nu^2) - Q(\frac{b_N N}{2}, \nu^2) \right) \\
&\quad - 2\sqrt{\pi} \frac{\mu^{b_N N-1}}{\Gamma(\frac{b_N N}{2})} e^{\nu^2} \left(Q(N + \frac{b_N N+1}{2}, \nu^2) - Q(\frac{b_N N+1}{2}, \nu^2) \right).
\end{aligned}$$

Using then (2.13), we get the desired identity (2.15). $\square$

3. PROOFS OF THEOREM 1.1 AND PROPOSITION 1.5

Parts (a), (b) and (c) of Theorem 1.1 are proven in Subsections 3.1, 3.2 and 3.3, respectively, and Proposition 1.5 is proved in Subsection 3.4.

Recall that $p = e^{i\theta}$ is the zooming point around which we rescale the correlation functions, see (1.31), and that z_j and ζ_j are related as (1.30). Recall also that Theorem 1.1 (a) deals with $\theta \in [0, 2\pi) \setminus \{0, \pi\}$ and that Theorem 1.1 (b) deals with $\theta \approx 0, \pi$. We start this section with a general lemma valid for all θ .

Lemma 3.1. *Let $\theta \in [0, 2\pi)$. The following identity holds*

$$R_{N,k}(z_1, \dots, z_k) = \text{Pf} \left[\begin{pmatrix} \tilde{\mathfrak{X}}_N(\zeta_j, \zeta_l) e^{-(|z_j|^2 + |z_l|^2 - 2e^{2i\theta} z_j z_l + (1 - e^{2i\theta}) a_N N + \sqrt{2a_N N}(z_j + z_l)(1 - e^{2i\theta}))} & & \\ & \tilde{\mathfrak{X}}_N(\bar{\zeta}_j, \zeta_l) e^{-(|z_j|^2 + |z_l|^2 - 2\bar{z}_j z_l)} & \\ & & \ddots \\ & & & \tilde{\mathfrak{X}}_N(\zeta_j, \bar{\zeta}_l) e^{-(|z_j|^2 + |z_l|^2 - 2z_j \bar{z}_l)} & \\ & & & & \ddots \\ & & & & & \tilde{\mathfrak{X}}_N(\bar{\zeta}_j, \bar{\zeta}_l) e^{-(|z_j|^2 + |z_l|^2 - 2e^{-2i\theta} \bar{z}_j \bar{z}_l + (1 - e^{-2i\theta}) a_N N + \sqrt{2a_N N}(\bar{z}_j + \bar{z}_l)(1 - e^{-2i\theta}))} \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k \frac{\bar{\zeta}_j - \zeta_j}{\gamma_N}.$$

Proof. Using (2.6) and (2.13), we first rewrite $R_{N,k}$ as

(3.1)

$$R_{N,k}(z_1, \dots, z_k) = \text{Pf} \left[e^{-\frac{a_N N}{2}(|\zeta_j|^2 + |\zeta_l|^2)} \begin{pmatrix} \tilde{\mathfrak{X}}_N(\zeta_j, \zeta_l) e^{a_N N \zeta_j \zeta_l} & \tilde{\mathfrak{X}}_N(\zeta_j, \bar{\zeta}_l) e^{a_N N \zeta_j \bar{\zeta}_l} \\ \tilde{\mathfrak{X}}_N(\bar{\zeta}_j, \zeta_l) e^{a_N N \bar{\zeta}_j \zeta_l} & \tilde{\mathfrak{X}}_N(\bar{\zeta}_j, \bar{\zeta}_l) e^{a_N N \bar{\zeta}_j \bar{\zeta}_l} \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k \frac{\bar{\zeta}_j - \zeta_j}{\gamma_N}.$$

In the spirit of (1.30), given $z, w \in \mathbb{C}$, let us define ζ and η by

$$(3.2) \quad \zeta = e^{i\theta} \left(1 + \sqrt{\frac{2}{a_N N}} z \right), \quad \eta = e^{i\theta} \left(1 + \sqrt{\frac{2}{a_N N}} w \right).$$

A direct computation shows that

$$(3.3) \quad \frac{a_N N}{2}(|\zeta|^2 + |\eta|^2 - 2\zeta\eta) = |z|^2 + |w|^2 - 2e^{2i\theta} zw + (1 - e^{2i\theta}) a_N N + \sqrt{2a_N N} \left((z + w)(1 - e^{2i\theta}) - i \text{Im}(z + w) \right),$$

$$(3.4) \quad \frac{a_N N}{2}(|\zeta|^2 + |\eta|^2 - 2\zeta\bar{\eta}) = |z|^2 + |w|^2 - 2z\bar{w} - \sqrt{2a_N N} i \text{Im}(z + \bar{w}).$$

Furthermore, when using (3.3) and (3.4) in (3.1), the terms $i \text{Im}(z + w)$ and $i \text{Im}(z + \bar{w})$ cancel out when computing the Pfaffian in (3.1). The claim follows. $\square$

3.1. Proof of Theorem 1.1 (a). As in the statement of Theorem 1.1 (a), here $\theta \in [0, 2\pi) \setminus \{0, \pi\}$. Let us write

$$(3.5) \quad K_N(z, w) := \frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} \tilde{\mathfrak{X}}_N(\zeta, \bar{\eta}),$$

$$(3.6) \quad e_N(z, w) := \frac{\sqrt{2a_N N} \sin \theta}{i} e^{(e^{2i\theta} - 1)(a_N N + \sqrt{2a_N N}(z + w)) + 2e^{2i\theta} zw} \tilde{\mathfrak{X}}_N(\zeta, \eta),$$

with ζ and η as in (3.2).

Proof of Theorem 1.1 (a). Combining Lemma 3.1 with (3.3), (3.4), (3.5), (3.6), and

$$\frac{\bar{\zeta}_j - \zeta_j}{\gamma_N} = \frac{2 \sin \theta}{i \gamma_N} + e^{-i\theta} \bar{z}_j - e^{i\theta} z_j = \frac{2 \sin \theta}{i \gamma_N} + O(1), \quad \text{as } N \rightarrow \infty,$$

we obtain after a computation that

$$(3.7) \quad R_{N,k}(z_1, \dots, z_k) = \text{Pf} \left[e^{-|z_j|^2 - |z_l|^2} \begin{pmatrix} e_N(z_j, z_l) & K_N(z_j, z_l) \\ -K_N(z_l, z_j) & -e_N(z_j, z_l) \end{pmatrix} \right]_{j,l=1}^k + o(1),$$

where the $o(1)$ -term is uniform on compact subsets of $\mathbb{C}$. It turns out that

$$(3.8) \quad K_N(z, w) = K^{\mathbb{C}}(z, w) + o(1), \quad e_N(z, w) = o(1), \quad \text{as } N \rightarrow \infty,$$

uniformly for z, w in compact subsets of $\mathbb{C}$, where $K^{\mathbb{C}}$ is defined in (1.11). We postpone the proof of (3.8) to Lemma 3.2 below. Using (3.7) and (3.8), we obtain

$$R_{N,k}(z_1, \dots, z_k) = \text{Pf} \left[e^{-|z_j|^2 - |z_l|^2} \begin{pmatrix} 0 & K^{\mathbb{C}}(z_j, z_l) \\ -K^{\mathbb{C}}(z_l, z_j) & 0 \end{pmatrix} \right]_{j,l=1}^k + o(1), \quad \text{as } N \rightarrow \infty$$

uniformly for $z_1, \dots, z_k$ in compact subsets of $\mathbb{C}$. Using row and column operations, we get

$$R_{N,k}(z_1, \dots, z_k) = (-1)^{\frac{k(k-1)}{2}} \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} + o(1), \quad M := \left(e^{-|z_j|^2 - |z_l|^2} K^{\mathbb{C}}(z_j, z_l) \right)_{j,l=1}^k.$$

The claim now follows from the algebraic identity

$$(3.9) \quad (-1)^{\frac{k(k-1)}{2}} \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = \det(M).$$

□

We now prove (3.8).

Lemma 3.2. *Let $\theta \in [0, 2\pi) \setminus \{0, \pi\}$. As $N \rightarrow \infty$,*

$$(3.10) \quad K_N(z, w) = K^{\mathbb{C}}(z, w) + o(1), \quad e_N(z, w) = o(1),$$

uniformly for z, w in compact subsets of $\mathbb{C}$, where $K^{\mathbb{C}}$ is defined in (1.11).

Proof. Recall that $K_N(z, w)$ is defined in (3.5) in terms of $\tilde{\mathbf{x}}_N(\zeta, \bar{\eta})$, where ζ and η are given by (3.2). Recall also from (2.13) that

$$(3.11) \quad \tilde{\mathbf{x}}_N(\zeta, \bar{\eta}) := e^{-a_N N \zeta \bar{\eta}} \widehat{\mathbf{x}}_N(\zeta, \bar{\eta}) = e^{-a_N N \zeta \bar{\eta}} \left(\widehat{\mathbf{G}}_N(\zeta, \bar{\eta}) - \widehat{\mathbf{G}}_N(\bar{\eta}, \zeta) \right),$$

where $\widehat{\mathbf{x}}_N$ and $\widehat{\mathbf{G}}_N$ are defined in the statement of Lemma 2.2. The first step of the proof consists in obtaining the large N asymptotics of the summand in the second expression of (2.9) that is valid uniformly for $k \in \{0, \dots, N-1\}$. For this, we use (1.2), (3.2) and Stirling's formula, and obtain after a direct computation that

$$(3.12) \quad \begin{aligned} \sqrt{\pi} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} &= \frac{\rho}{N} e^{i\theta(2k+1+b_N N)} \exp \left(\frac{N^2}{2\rho^2} + \frac{\sqrt{2}z}{\rho} N - z^2 - \sqrt{2}\rho z - \frac{\rho^2}{4} \right) \\ &\times \exp \left(\frac{(2\sqrt{2}\rho z + \rho^2)k}{N} - \rho^2 \left(\frac{k}{N} \right)^2 \right) (1 + O(N^{-1})), \quad \text{as } N \rightarrow \infty \end{aligned}$$

uniformly for $k \in \{0, \dots, N-1\}$. Applying (A.2), we have

$$(3.13) \quad \begin{aligned} e^{\frac{a_N N}{2} \bar{\eta}^2} Q \left(k + \frac{b_N N}{2} + 1, \frac{a_N N}{2} \bar{\eta}^2 \right) &= \frac{1}{\sqrt{\pi}} \frac{\rho}{N} \frac{e^{-i\theta(2k+2+b_N N)}}{e^{-2i\theta} - 1} \exp \left(\frac{(2\sqrt{2}\rho \bar{w} + \rho^2)k}{N} - \rho^2 \left(\frac{k}{N} \right)^2 \right) \\ &\times \exp \left(\frac{N^2}{2\rho^2} + \frac{\sqrt{2}\bar{w}}{\rho} N - \bar{w}^2 - \sqrt{2}\rho \bar{w} - \frac{\rho^2}{4} \right) (1 + O(N^{-1})) \end{aligned}$$

as $N \rightarrow \infty$ uniformly for $k \in \{-1, 0, \dots, N-1\}$. Combining the above equations with

$$a_N N \zeta \bar{\eta} = \frac{N^2}{\rho^2} + \frac{\sqrt{2}}{\rho} (z + \bar{w}) N + 2z\bar{w}, \quad \frac{e^{-i\theta}}{e^{-2i\theta} - 1} = \frac{i}{2 \sin \theta},$$

we obtain

$$\begin{aligned}
s_{k,1} &:= \frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} e^{-a_N N \zeta \bar{\eta}} \sqrt{\pi} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \exp\left(\frac{a_N N}{2} \bar{\eta}^2\right) Q\left(k + \frac{b_N N}{2} + 1, \frac{a_N N}{2} \bar{\eta}^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\rho}{N} \exp\left(-z^2 - \bar{w}^2 - \sqrt{2}\rho(z + \bar{w}) - \frac{\rho^2}{2} + \frac{(2\sqrt{2}(z + \bar{w})\rho + 2\rho^2)k}{N} - 2\rho^2\left(\frac{k}{N}\right)^2\right) (1 + O(N^{-1})). \\
s_{k,2} &:= \frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} e^{-a_N N \zeta \bar{\eta}} \sqrt{\pi} \frac{(\sqrt{\frac{a_N N}{2}} \zeta)^{2k+1+b_N N}}{\Gamma(k + \frac{3}{2} + \frac{b_N N}{2})} \exp\left(\frac{a_N N}{2} \bar{\eta}^2\right) Q\left(\frac{b_N N}{2}, \frac{a_N N}{2} \bar{\eta}^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\rho}{N} e^{i\theta(2k+2)} \exp\left(-z^2 - \bar{w}^2 - \sqrt{2}\rho(z + \bar{w}) - \frac{\rho^2}{2}\right) (1 + O(N^{-1})),
\end{aligned}$$

as $N \rightarrow \infty$ uniformly for $k \in \{0, \dots, N-1\}$. Since $\sum_{k=0}^{N-1} e^{i\theta(2k+2)} = O(1)$ as $N \rightarrow \infty$, we have $\sum_{k=0}^{N-1} s_{k,2} = O(N^{-1})$ as $N \rightarrow \infty$. Hence, using a first order Riemann sum approximation, we obtain

$$\begin{aligned}
\frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} e^{-a_N N \zeta \bar{\eta}} \widehat{\mathbf{G}}_N(\zeta, \bar{\eta}) &= \sum_{k=0}^{N-1} (s_{k,1} - s_{k,2}) = O(N^{-1}) + \sum_{k=0}^{N-1} s_{k,1} \\
&= \frac{1}{\sqrt{2\pi}} \rho \exp\left(-z^2 - \bar{w}^2 - \sqrt{2}\rho(z + \bar{w}) - \frac{\rho^2}{2}\right) \int_0^1 \exp\left((2\sqrt{2}(z + \bar{w})\rho + 2\rho^2)x - 2\rho^2 x^2\right) dx + O(N^{-1}).
\end{aligned}$$

The above integral can be evaluated explicitly,

$$\int_0^1 \exp\left((2\sqrt{2}(z + \bar{w})\rho + 2\rho^2)x - 2\rho^2 x^2\right) dx = \sqrt{\frac{\pi}{2}} \frac{1}{\rho} \exp\left(z^2 + \bar{w}^2 + \sqrt{2}\rho(z + \bar{w}) + \frac{\rho^2}{2}\right) K^{\mathbb{C}}(z, w),$$

where $K^{\mathbb{C}}$ is defined in (1.11), and therefore

$$(3.14) \quad \frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} e^{-a_N N \zeta \bar{\eta}} \widehat{\mathbf{G}}_N(\zeta, \bar{\eta}) = \frac{1}{2} K^{\mathbb{C}}(z, w) + O(N^{-1}).$$

By interchanging ζ and $\bar{\eta}$ in the above computations and using $e^{i\theta}/(e^{2i\theta} - 1) = -i/(2 \sin \theta)$, we also obtain

$$(3.15) \quad \frac{\sqrt{2a_N N} \sin \theta}{i} e^{2z\bar{w}} e^{-a_N N \zeta \bar{\eta}} \widehat{\mathbf{G}}_N(\bar{\eta}, \zeta) = -\frac{1}{2} K^{\mathbb{C}}(z, w) + O(N^{-1}).$$

Then by combining (3.5), (3.11), (3.14) and (3.15), we obtain the first asymptotics of (3.10). To prove the second part of (3.10), we use the formula (3.6) for e_N together with (3.11) (with $\bar{\eta}$ replaced by η) to write

$$(3.16) \quad e_N(z, w) := \frac{\sqrt{2a_N N} \sin \theta}{i} e^{(e^{2i\theta}-1)(a_N N + \sqrt{2a_N N}(z+w)) + 2e^{2i\theta}zw} e^{-a_N N \zeta \eta} \left(\widehat{\mathbf{G}}_N(\zeta, \eta) - \widehat{\mathbf{G}}_N(\eta, \zeta) \right).$$

The asymptotics of $\widehat{\mathbf{G}}_N$ can be obtained using (3.12) and (3.13) (with $\bar{\eta}$, $\bar{w}$ and $-\theta$ replaced by η , w and θ , respectively). The above exponential $e^{(e^{2i\theta}-1)(a_N N + \sqrt{2a_N N}(z+w)) + 2e^{2i\theta}zw}$ gets perfectly cancelled

in the asymptotics; indeed, using $a_N N \zeta \eta = e^{2i\theta} \left(\frac{N^2}{\rho^2} + \frac{\sqrt{2}}{\rho} (z+w)N + 2zw \right)$, we obtain

$$\begin{aligned} & \frac{\sqrt{2a_N N} \sin \theta}{i} e^{(e^{2i\theta}-1)(a_N N + \sqrt{2a_N N}(z+w)) + 2e^{2i\theta} zw} e^{-a_N N \zeta \eta} \widehat{\mathbf{G}}_N(\zeta, \eta) \\ &= \frac{1}{N} \frac{\sqrt{2} \rho \sin \theta}{\sqrt{\pi} i} \sum_{k=0}^{N-1} e^{i\theta(2k+1+b_N N)} \exp \left(-z^2 - w^2 - \sqrt{2} \rho (z+w) - \frac{\rho^2}{2} \right) \\ & \quad \times \left[\exp \left(\frac{(2\sqrt{2} \rho (z+w) + 2\rho^2)k}{N} - 2 \left(\frac{\rho k}{N} \right)^2 \right) \frac{e^{i\theta(2k+2+b_N N)}}{e^{2i\theta} - 1} - \frac{e^{i\theta b_N N}}{e^{2i\theta} - 1} \right] \left(1 + O(N^{-1}) \right), \end{aligned}$$

as $N \rightarrow +\infty$ uniformly for $k \in \{0, \dots, N-1\}$. By (3.16), we thus have

$$e_N(z, w) = \frac{1}{N} O \left(\sum_{k=0}^{N-1} N^{-1} \right) = O(N^{-1}), \quad \text{as } N \rightarrow +\infty.$$

□

3.2. Proof of Theorem 1.1 (b). Recall that $t \in \mathbb{R}$ and $\theta_N := \gamma_N t = \frac{\sqrt{2}\rho}{N} t$. The two cases $p = e^{i\theta_N}$ and $p = -e^{i\theta_N}$ are similar, so to avoid repetition we will only consider the case $p := e^{i\theta_N}$. Recall that a_N and b_N are defined by (1.2). In this subsection, given $z, w \in \mathbb{C}$, we define ζ and η as in (3.2) with $\theta = \theta_N$, namely

$$(3.17) \quad \zeta = e^{i\theta_N} \left(1 + \sqrt{\frac{2}{a_N N}} z \right), \quad \eta = e^{i\theta_N} \left(1 + \sqrt{\frac{2}{a_N N}} w \right),$$

so that μ and ν in (2.14) become

$$\mu := \sqrt{\frac{a_N N}{2}} \zeta = e^{i\theta_N} \left(z + \sqrt{\frac{a_N N}{2}} \right), \quad \nu := \sqrt{\frac{a_N N}{2}} \eta = e^{i\theta_N} \left(w + \sqrt{\frac{a_N N}{2}} \right).$$

In particular, ζ, η, μ, ν always depend on N in this subsection, although this is not indicated in the notation. The following lemma is a rather direct consequence of Proposition 2.3 and Lemma 3.1.

Lemma 3.3. *Let $t \in \mathbb{R}$ and $\rho > 0$ be fixed, and $p := e^{i\theta_N}$. As $N \rightarrow \infty$*

$$(3.18) \quad \begin{aligned} R_{N,k}(z_1, \dots, z_k) &= \text{Pf} \left[e^{-|z_j+it|^2 - |z_l+it|^2} \begin{pmatrix} \widetilde{\mathfrak{X}}_N(\zeta_j, \zeta_l) e^{2(z_j+it)(z_l+it)} & \widetilde{\mathfrak{X}}_N(\zeta_j, \bar{\zeta}_l) e^{2(z_j+it)(\bar{z}_l-it)} \\ \widetilde{\mathfrak{X}}_N(\bar{\zeta}_j, \zeta_l) e^{2(\bar{z}_j-it)(z_l+it)} & \widetilde{\mathfrak{X}}_N(\bar{\zeta}_j, \bar{\zeta}_l) e^{2(\bar{z}_j-it)(\bar{z}_l-it)} \end{pmatrix} \right]_{j,l=1}^k \\ &\quad \times \prod_{j=1}^k (\bar{z}_j - z_j - 2it) + o(1), \end{aligned}$$

uniformly for $z_1, \dots, z_k$ in compact subsets of $\mathbb{C}$, where $\zeta_j := p + p\gamma_N z_j$ for $j = 1, \dots, k$. Furthermore, for any $z, w \in \mathbb{C}$, we have

$$(3.19) \quad \begin{aligned} e^{-i\theta_N} \frac{d}{dz} \widetilde{\mathfrak{X}}_N(\zeta, \eta) &= 2e^{i\theta_N} (z-w) \widetilde{\mathfrak{X}}_N(\zeta, \eta) + 2 \left(Q(2N + b_N N, 2\mu\nu) - Q(b_N N, 2\mu\nu) \right) \\ &\quad - 2\sqrt{\pi} e^{2i\theta_N(z-w)^2} \frac{\mu^{2N+b_N N}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} e^{-\mu^2} \left(Q(N + \frac{b_N N}{2}, \nu^2) - Q(\frac{b_N N}{2}, \nu^2) \right) \\ &\quad - 2\sqrt{\pi} e^{2i\theta_N(z-w)^2} \frac{\mu^{b_N N-1}}{\Gamma(\frac{b_N N}{2})} e^{-\mu^2} \left(Q(N + \frac{b_N N+1}{2}, \nu^2) - Q(\frac{b_N N+1}{2}, \nu^2) \right), \end{aligned}$$

where $\zeta = \zeta(z)$ and $\eta = \eta(w)$ are as in (3.17).

Proof. The expansion (3.18) directly follows from Lemma 3.1 with $\theta = \theta_N$. To obtain the exponentials inside the Pfaffian in (3.18), we have used the following large N expansion

$$\begin{aligned} & |z_j|^2 + |z_l|^2 - 2e^{2i\theta_N} z_j z_l + (1 - e^{2i\theta_N}) a_N N + \sqrt{2a_N N} (z_j + z_l) (1 - e^{2i\theta_N}) \\ &= |z_j + it|^2 + |z_l + it|^2 - 2(z_j + it)(z_l + it) - 2it \operatorname{Re}(z_j + z_l) - \frac{2\sqrt{2tN}}{\rho} i + o(1), \end{aligned}$$

the identity $|z_j|^2 + |z_l|^2 - 2z_j \bar{z}_l = |z_j + it|^2 + |z_l + it|^2 - 2(z_j + it)(\bar{z}_l - it) - 2it \operatorname{Re}(z_j - z_l)$, and the fact that the terms containing $\operatorname{Re}(z_j + z_l)$, $\operatorname{Re}(z_j - z_l)$ and $\frac{2\sqrt{2tN}}{\rho} i$ cancel out when computing the Pfaffian. The differential identity (3.19) immediately follows from (3.17), $\partial_\zeta = e^{-i\theta_N} \sqrt{\frac{a_N N}{2}} \partial_z$ and Proposition 2.3. $\square$

We now derive the large N asymptotics of $\partial_z \tilde{\mathfrak{X}}_N(\zeta, \eta)$ using the right-hand side of (3.19).

Lemma 3.4. *Let $t \in \mathbb{R}$ and $\rho > 0$ be fixed. Let $z_t := z + it$ and $w_t := w + it$. As $N \rightarrow \infty$, we have*

$$\begin{aligned} (3.20) \quad \partial_z \tilde{\mathfrak{X}}_N(\zeta, \eta) &= 2(z_t - w_t) \tilde{\mathfrak{X}}_N(\zeta, \eta) + \operatorname{erfc}(z_t + w_t - \frac{\rho}{\sqrt{2}}) - \operatorname{erfc}(z_t + w_t + \frac{\rho}{\sqrt{2}}) \\ &\quad - \frac{e^{(z_t - w_t)^2}}{\sqrt{2}} \left(e^{-(\sqrt{2}z_t - \frac{\rho}{2})^2} + e^{-(\sqrt{2}z_t + \frac{\rho}{2})^2} \right) \left(\operatorname{erfc}(\sqrt{2}w_t - \frac{\rho}{2}) - \operatorname{erfc}(\sqrt{2}w_t + \frac{\rho}{2}) \right) + o(1), \end{aligned}$$

uniformly for z, w in compact subsets of $\mathbb{C}$, where ζ and η are as in (3.17).

Proof of Lemma 3.4. By (1.2) and (2.14), we have

$$\begin{aligned} 2N + b_N N &= \frac{N^2}{\rho^2} + N, & b_N N &= \frac{N^2}{\rho^2} - N, \\ 2\mu\nu &= \frac{N^2}{\rho^2} + \frac{\sqrt{2}}{\rho} (z_t + w_t) N + 2z_t w_t + o(1), & \nu^2 &= \frac{N^2}{2\rho^2} + \frac{\sqrt{2}}{\rho} w_t N + w_t^2 + o(1) \end{aligned}$$

as $N \rightarrow \infty$ uniformly for z, w in compact subsets of $\mathbb{C}$. By [46, Eq.(8.11.10)],

$$(3.21) \quad Q(s+1, s + \sqrt{2}sz) = \frac{1}{2} \operatorname{erfc}(z) + \frac{1}{3} \sqrt{\frac{2}{\pi}} (1 + z^2) e^{-z^2} \frac{1}{\sqrt{s}} + O(1/s), \quad s \rightarrow +\infty$$

uniformly for z in compact subsets of $\mathbb{C}$. It readily follows from (3.21) that

$$(3.22) \quad Q(2N + b_N N, 2\mu\nu) - Q(b_N N, 2\mu\nu) = \frac{1}{2} \left(\operatorname{erfc}(z_t + w_t - \frac{\rho}{\sqrt{2}}) - \operatorname{erfc}(z_t + w_t + \frac{\rho}{\sqrt{2}}) \right) + o(1),$$

$$(3.23) \quad Q(N + \frac{b_N N}{2}, \nu^2) - Q(\frac{b_N N}{2}, \nu^2) = \frac{1}{2} \left(\operatorname{erfc}(\sqrt{2}w_t - \frac{\rho}{2}) - \operatorname{erfc}(\sqrt{2}w_t + \frac{\rho}{2}) \right) + o(1),$$

$$(3.24) \quad Q(N + \frac{b_N N + 1}{2}, \nu^2) - Q(\frac{b_N N + 1}{2}, \nu^2) = \frac{1}{2} \left(\operatorname{erfc}(\sqrt{2}w_t - \frac{\rho}{2}) - \operatorname{erfc}(\sqrt{2}w_t + \frac{\rho}{2}) \right) + o(1),$$

as $N \rightarrow \infty$ uniformly for z, w in compact subsets of $\mathbb{C}$. Also, by Stirling's formula, we have

$$(3.25) \quad \frac{\mu^{2N + b_N N} e^{-\mu^2}}{\Gamma(N + \frac{1}{2} + \frac{b_N N}{2})} = \frac{e^{-(\sqrt{2}z_t - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + o(1), \quad \frac{\mu^{b_N N - 1} e^{-\mu^2}}{\Gamma(\frac{b_N N}{2})} = \frac{e^{-(\sqrt{2}z_t + \frac{\rho}{2})^2}}{\sqrt{2\pi}} + o(1),$$

as $N \rightarrow \infty$ uniformly for z in compact subsets of $\mathbb{C}$. Combining (3.22), (3.23), (3.24) and (3.25), we obtain (3.20). $\square$

We now finish the proof of Theorem 1.1 (b).

Proof of Theorem 1.1 (b). The proof can be summarized as follows: we first use Lemma 3.4 to obtain large N asymptotics for $\tilde{\mathfrak{X}}_N(\zeta, \eta)$. We then substitute these asymptotics in (3.18) to obtain the leading order large N behavior of $R_{N,k}(z_1, \dots, z_k)$.

In Lemma 3.4, we have derived (3.20), which can be seen as a family of ODE (indexed by N) of the form $\partial_z \tilde{\mathfrak{X}}_N = c_0(z) \tilde{\mathfrak{X}}_N + c_1(z) + \mathcal{E}_N(z)$ where $\mathcal{E}_N(z) \rightarrow 0$ as $N \rightarrow +\infty$ uniformly for z in compact subsets of $\mathbb{C}$. By [19, Lemma 3.10], the limit

$$\tilde{\kappa}(z, w) := \lim_{N \rightarrow +\infty} \tilde{\mathfrak{X}}_N(\zeta, \eta) = \lim_{N \rightarrow +\infty} \tilde{\mathfrak{X}}_N(e^{i\theta_N}(1 + \gamma_N z), e^{i\theta_N}(1 + \gamma_N w))$$

exists for all $z \in \mathbb{C}$, is analytic and satisfies $\partial_z \tilde{\kappa} = c_0(z) \tilde{\kappa} + c_1(z)$ and $\partial_z \tilde{\kappa}|_{z=w} = 0$. More precisely, we have

$$(3.26) \quad \begin{aligned} \partial_z \tilde{\kappa}(z, w) &= 2(z_t - w_t) \tilde{\kappa}(z, w) + \operatorname{erfc}(z_t + w_t - \frac{\rho}{\sqrt{2}}) - \operatorname{erfc}(z_t + w_t + \frac{\rho}{\sqrt{2}}) \\ &\quad - \frac{e^{(z_t - w_t)^2}}{\sqrt{2}} \left(e^{-(\sqrt{2}z_t - \frac{\rho}{2})^2} + e^{-(\sqrt{2}z_t + \frac{\rho}{2})^2} \right) \left(\operatorname{erfc}(\sqrt{2}w_t - \frac{\rho}{2}) - \operatorname{erfc}(\sqrt{2}w_t + \frac{\rho}{2}) \right). \end{aligned}$$

For a given w , we view (3.26) as a first order ODE in z with the initial condition $\tilde{\kappa}(w, w) = 0$. Since $c_0(z)$ and $c_1(z)$ are analytic, uniqueness of the solution to this ODE follows from standard theory.

To obtain the solution of (3.26), we first rewrite it as

$$(3.27) \quad \partial_z \tilde{\kappa}(z, w) = 2(z_t - w_t) \tilde{\kappa}(z, w) + e^{(z-w)^2} (\partial_z F_1(z_t, w_t) + \partial_z F_2(z_t, w_t)),$$

where

$$\begin{aligned} F_1(z, w) &:= \frac{1}{\sqrt{2}} \int_{-a}^a (e^{-2(z-u)^2} \operatorname{erfc}(\sqrt{2}(w-u)) - e^{-2(w-u)^2} \operatorname{erfc}(\sqrt{2}(z-u))) du, \\ F_2(z, w) &:= \frac{\sqrt{\pi}}{4} \left[\operatorname{erfc}(\sqrt{2}(z+a)) \operatorname{erfc}(\sqrt{2}(w-a)) - \operatorname{erfc}(\sqrt{2}(z-a)) \operatorname{erfc}(\sqrt{2}(w+a)) \right]. \end{aligned}$$

Indeed, using integration by parts, we obtain

$$\begin{aligned} \partial_z F_1(z, w) &= e^{-(z-w)^2} \left(\operatorname{erfc}(z+w-2a) - \operatorname{erfc}(z+w+2a) \right) \\ &\quad - \frac{1}{\sqrt{2}} \left(e^{-2(z-a)^2} \operatorname{erfc}(\sqrt{2}(w-a)) - e^{-2(z+a)^2} \operatorname{erfc}(\sqrt{2}(w+a)) \right), \\ \partial_z F_2(z, w) &= \frac{1}{\sqrt{2}} \left(e^{-2(z-a)^2} \operatorname{erfc}(\sqrt{2}(w+a)) - e^{-2(z+a)^2} \operatorname{erfc}(\sqrt{2}(w-a)) \right). \end{aligned}$$

Note that $F_1(w, w) = F_2(w, w) = 0$. It is now readily checked that the unique solution of (3.27) satisfying $\tilde{\kappa}(w, w) = 0$ is given by

$$\tilde{\kappa}(z, w) := e^{(z-w)^2} \left[F_1(z_t, w_t) + F_2(z_t, w_t) \right] = e^{-2z_t w_t} \kappa^{\mathbb{R}}(z_t, w_t),$$

where $\kappa^{\mathbb{R}}(z, w)$ is given by (1.14). □

3.3. Proof of Theorem 1.1 (c). We first obtain the large t asymptotics of

$$e^{-|z+it|^2 - |w+it|^2} \kappa^{\mathbb{R}}(z + it, \bar{w} - it) \quad \text{and} \quad e^{-|z+it|^2 - |w+it|^2} \kappa^{\mathbb{R}}(z + it, w + it),$$

where $\kappa^{\mathbb{R}}$ is defined in (1.14). Using the well-known $z \rightarrow +\infty$ asymptotics of $\operatorname{erfc}(z)$ (see e.g. [46, Eq.(7.12.1)]), we obtain

$$\sqrt{\pi} e^{-|z+it|^2 - |w+it|^2} e^{(z+it)^2 + (\bar{w}-it)^2} \int_{-a}^a W(f_{\bar{w}-it}, f_{z+it})(u) du = \frac{i}{2t} e^{-|z|^2 - |w|^2} K^{\mathbb{C}}(z, w) c(z, w) + O(t^{-2}),$$

as $t \rightarrow \infty$ uniformly for z and w in compact subsets of $\mathbb{C}$, where $c(z, w) := e^{-(z+\bar{z})it+(w+\bar{w})it}$ satisfies $c(z, w) = 1/c(w, z)$ is a therefore an unimportant cocycle. We also have

$$\sqrt{\pi} e^{-|z+it|^2-|w+it|^2} e^{(z+it)^2+(\bar{w}-it)^2} \left(f_w(a) f_z(-a) - f_z(a) f_w(-a) \right) = O(t^{-2}), \quad \text{as } t \rightarrow \infty.$$

Then by (1.14), we have

$$e^{-|z+it|^2-|w+it|^2} \kappa^{\mathbb{R}}(z+it, \bar{w}-it) = \frac{i}{2t} e^{-|z|^2-|w|^2} K^{\mathbb{C}}(z, w) c(z, w) + O(t^{-2}), \quad \text{as } t \rightarrow \infty,$$

uniformly for z and w in compact subsets of $\mathbb{C}$. Similarly, we have

$$e^{-|z+it|^2-|w+it|^2} \kappa^{\mathbb{R}}(z+it, w+it) = O(t^{-2}), \quad \text{as } t \rightarrow \infty,$$

uniformly for z and w in compact subsets of $\mathbb{C}$. Let us write

$$M := \left(c(z_j, z_l) e^{-|z_j|^2-|z_l|^2} K^{\mathbb{C}}(z_j, z_l) \right)_{j,l=1}^k.$$

Combining the above expansions with (1.12) and (1.11), and performing elementary row and column operations, we obtain that as $t \rightarrow \infty$,

$$\begin{aligned} R_k^{\mathbb{R}}(z_1+it, \dots, z_k+it) &= (-1)^{\frac{k(k-1)}{2}} \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} + o(1), \\ &= \det(M) + o(1) = \det \left(e^{-|z_j|^2-|z_l|^2} K^{\mathbb{C}}(z_j, z_l) \right)_{j,l=1}^k + o(1), \end{aligned}$$

which gives (1.16). Here, the second identity follows from (3.9), whereas the third one follows from the fact that the cocycle c cancels out when forming the determinant. The proof is complete.

3.4. Proof of Proposition 1.5. We start with an auxiliary lemma.

Lemma 3.5. *For any function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfying $f(x, y) = f(y, x)$, any $x_1, \dots, x_k \in \mathbb{C}$ and any $y_1, \dots, y_k \in \mathbb{C} \setminus \{0\}$, we have*

$$(3.28) \quad \text{Pf} \left[f(x_j, x_l) \begin{pmatrix} \frac{y_j-y_l}{2y_j y_l} & \frac{y_j+y_l}{2y_j y_l} \\ -\frac{y_j+y_l}{2y_j y_l} & -\frac{y_j-y_l}{2y_j y_l} \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k y_j = \det \left[f(x_j, x_l) \right]_{j,l=1}^k.$$

To be concrete, for $k = 1, 2$ formula (3.28) reads as follows: letting $f_{jk} := f(x_j, x_k)$,

$$\text{Pf} \begin{pmatrix} 0 & f_{11} \frac{1}{y_1} \\ -f_{11} \frac{1}{y_1} & 0 \end{pmatrix} y_1 = f_{11}, \quad \text{Pf} \begin{pmatrix} 0 & f_{11} \frac{1}{y_1} & f_{12} \frac{y_1-y_2}{2y_1 y_2} & f_{12} \frac{y_1+y_2}{2y_1 y_2} \\ -f_{11} \frac{1}{y_1} & 0 & -f_{12} \frac{y_1+y_2}{2y_1 y_2} & -f_{12} \frac{y_1-y_2}{2y_1 y_2} \\ -f_{12} \frac{y_1-y_2}{2y_1 y_2} & f_{12} \frac{y_1+y_2}{2y_1 y_2} & 0 & f_{22} \frac{1}{y_2} \\ -f_{12} \frac{y_1+y_2}{2y_1 y_2} & f_{12} \frac{y_1-y_2}{2y_1 y_2} & -f_{22} \frac{1}{y_2} & 0 \end{pmatrix} y_1 y_2 = f_{11} f_{22} - f_{12}^2.$$

Proof of Lemma 3.5. Let

$$F := \left[f_{jl} \begin{pmatrix} \frac{y_j-y_l}{2y_j y_l} & \frac{y_j+y_l}{2y_j y_l} \\ -\frac{y_j+y_l}{2y_j y_l} & -\frac{y_j-y_l}{2y_j y_l} \end{pmatrix} \right]_{j,l=1}^k = \left[\frac{f_{jl}}{2} \begin{pmatrix} \frac{1}{y_l} - \frac{1}{y_j} & \frac{1}{y_l} + \frac{1}{y_j} \\ -\frac{1}{y_l} - \frac{1}{y_j} & \frac{1}{y_j} - \frac{1}{y_l} \end{pmatrix} \right]_{j,l=1}^k.$$

We first show that

$$(3.29) \quad \text{Pf}(F) \prod_{j=1}^k y_j = \text{Pf} \left[f_{jl} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]_{j,l=1}^k.$$

For this, recall that for any $k \times k$ matrix $A = (a_{j,l})_{j,l=1}^k$ and any $2k \times 2k$ skew-symmetric matrix B ,

$$(3.30) \quad \det(A) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^k a_{j,\sigma(j)}, \quad \text{Pf}(B)^2 = \det(B),$$

where S_k is the symmetric group of all permutations of size k . Using row and column operations, we observe that

$$\begin{aligned} \det(F) &= \det \left[\frac{f_{jl}}{2} \begin{pmatrix} y_l^{-1} - y_j^{-1} & y_l^{-1} + y_j^{-1} \\ -y_l^{-1} - y_j^{-1} & y_j^{-1} - y_l^{-1} \end{pmatrix} \right]_{j,l=1}^k = \det \left[\frac{f_{jl}}{2} \begin{pmatrix} -2y_j^{-1} & 2y_j^{-1} \\ -y_l^{-1} - y_j^{-1} & y_j^{-1} - y_l^{-1} \end{pmatrix} \right]_{j,l=1}^k \\ &= \det \left[\frac{f_{jl}}{2} \begin{pmatrix} 0 & 2y_j^{-1} \\ -2y_l^{-1} & y_j^{-1} - y_l^{-1} \end{pmatrix} \right]_{j,l=1}^k = \det \left[\frac{f_{jl}}{2} \begin{pmatrix} 0 & 2y_j^{-1} \\ -2y_l^{-1} & -y_l^{-1} \end{pmatrix} \right]_{j,l=1}^k \\ &= \det \left[f_{jl} \begin{pmatrix} 0 & y_j^{-1} \\ -y_l^{-1} & 0 \end{pmatrix} \right]_{j,l=1}^k = \det \left[f_{jl} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k y_j^{-2}. \end{aligned}$$

Combining the above with the second identity in (3.30) yields (3.29). Finally, using again row and column operations, we obtain

$$(3.31) \quad \text{Pf} \left[f_{jl} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]_{j,l=1}^k = (-1)^{\frac{k(k-1)}{2}} \text{Pf} \left[\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} \right]_{j,l=1}^k, \quad M := (f_{jl})_{j,l=1}^k.$$

The desired identity (3.28) follows from (3.29), (3.31) and (3.9). $\square$

We are now ready to prove Proposition 1.5.

Proof of Proposition 1.5. Throughout the proof, let $x_j := \text{Re } z_j$ and $y_j = \text{Im } z_j$ ($j = 1, \dots, k$). Using the definition of $R_k^{\mathbb{R}}$, the left-hand side of (1.21) can be rewritten as

$$(3.32) \quad \tilde{R}_k(z_1, \dots, z_k) := \frac{1}{a^{2k}} R_k^{\mathbb{R}}\left(\frac{z_1}{a}, \dots, \frac{z_k}{a}\right) = \frac{1}{a^{3k}} \text{Pf} \left[e^{-\frac{|z_j|^2 + |z_l|^2}{a^2}} \begin{pmatrix} \kappa^{\mathbb{R}}\left(\frac{z_j}{a}, \frac{z_l}{a}\right) & \kappa^{\mathbb{R}}\left(\frac{z_j}{a}, \frac{\bar{z}_l}{a}\right) \\ \kappa^{\mathbb{R}}\left(\frac{\bar{z}_j}{a}, \frac{z_l}{a}\right) & \kappa^{\mathbb{R}}\left(\frac{\bar{z}_j}{a}, \frac{\bar{z}_l}{a}\right) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k (-2i y_j).$$

Let us first compute the leading order behavior of $\frac{1}{a^3} e^{-\frac{|z|^2 + |w|^2}{a^2}} \kappa^{\mathbb{R}}\left(\frac{z}{a}, \frac{w}{a}\right)$ as $a \rightarrow 0$. Using the well-known $z \rightarrow +\infty$ asymptotics of $\text{erfc}(z)$ (see e.g. [46, Eq.(7.12.1)]), we find

$$f_{z/a}(u) = \frac{1}{2} \text{erfc}(\sqrt{2}(\frac{z}{a} - u)) = \frac{1}{2\sqrt{2\pi}} \frac{a}{z} e^{-2(\frac{z}{a} - u)^2} (1 + o(1)), \quad \text{as } a \rightarrow 0,$$

uniformly for u in compact subsets of $\mathbb{C}$. Thus we have

$$(3.33) \quad \begin{aligned} & \frac{1}{a^3} \sqrt{\pi} e^{-\frac{|z|^2 + |w|^2 - z^2 - w^2}{a^2}} \left(f_{w/a}(a) f_{z/a}(-a) - f_{z/a}(a) f_{w/a}(-a) \right) \\ &= -\sqrt{\pi} e^{-\frac{|z|^2 + |w|^2 + z^2 + w^2}{a^2}} \frac{1}{8\pi a z w} \left(e^{4(z-w)} - e^{-4(z-w)} \right) (1 + o(1)), \quad \text{as } a \rightarrow 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} W(f_{w/a}, f_{z/a})(u) &= -\sqrt{\frac{2}{\pi}} \left(e^{-2(\frac{w}{a} - u)^2} f_{z/a}(u) - e^{-2(\frac{z}{a} - u)^2} f_{w/a}(u) \right) \\ &= -\frac{a}{2\pi} \left(\frac{1}{z} - \frac{1}{w} \right) e^{-2(\frac{z}{a} - u)^2 - 2(\frac{w}{a} - u)^2} (1 + o(1)), \quad \text{as } a \rightarrow 0 \end{aligned}$$

uniformly for u in compact subsets of $\mathbb{C}$. Thus we have

$$\frac{1}{a^3} \sqrt{\pi} e^{-\frac{|z|^2+|w|^2-z^2-w^2}{a^2}} W(f_{w/a}, f_{z/a})(au) = -\frac{1}{2a^2 \sqrt{\pi}} \left(\frac{1}{z} - \frac{1}{w} \right) e^{-\frac{|z|^2+|w|^2+z^2+w^2}{a^2}} e^{4u(z+w)} (1 + o(1)).$$

This gives

$$\begin{aligned} & \frac{1}{a^3} \sqrt{\pi} e^{-\frac{|z|^2+|w|^2-z^2-w^2}{a^2}} \int_{-a}^a W(f_{w/a}, f_{z/a})(u) du = \frac{\sqrt{\pi}}{a^2} e^{-\frac{|z|^2+|w|^2-z^2-w^2}{a^2}} \int_{-1}^1 W(f_{w/a}, f_{z/a})(au) du \\ & = -e^{-\frac{|z|^2+|w|^2+z^2+w^2}{a^2}} \frac{1}{2a\sqrt{\pi}} \left(\frac{1}{z} - \frac{1}{w} \right) \int_{-1}^1 e^{4u(z+w)} du (1 + o(1)) \\ & = -e^{-\frac{|z|^2+|w|^2+z^2+w^2}{a^2}} \frac{1}{8a\sqrt{\pi}} \frac{1}{zw} \frac{w-z}{z+w} \left(e^{4(z+w)} - e^{-4(z+w)} \right) (1 + o(1)) \quad \text{as } a \rightarrow 0. \end{aligned}$$

Summing the last asymptotic formula with (3.33) (and using (1.14)) gives

$$\begin{aligned} \tilde{\kappa}(z, w) &:= \frac{1}{a^3} e^{-\frac{|z|^2+|w|^2}{a^2}} \kappa^{\mathbb{R}}\left(\frac{z}{a}, \frac{w}{a}\right) \\ (3.34) \quad &= \frac{2}{a\sqrt{\pi}} e^{-\frac{|z|^2+|w|^2+z^2+w^2}{a^2}} \frac{z-w}{4zw} \left(\frac{\sinh(4(z+w))}{2(z+w)} - \frac{\sinh(4(z-w))}{2(z-w)} + o(1) \right), \quad \text{as } a \rightarrow 0. \end{aligned}$$

We now use the Gaussian approximation of the Dirac delta: for any continuous function $f : \mathbb{R} \rightarrow \mathbb{C}$ with compact support and any fixed $\lambda > 0$, we have

$$(3.35) \quad \int_{-\infty}^{+\infty} \frac{1}{a\sqrt{\pi}} e^{-\lambda(x/a)^2} f(x) dx = \frac{f(0)}{\sqrt{\lambda}} + o(1) \quad \text{as } a \rightarrow 0.$$

For short, in what follows we will denote the above as $\frac{1}{a\sqrt{\pi}} e^{-\lambda(x/a)^2} \xrightarrow{d} \frac{\delta(x)}{\sqrt{\lambda}}$ as $a \rightarrow 0$.

Before considering the general case $k \in \mathbb{N}_{>0}$, it is instructive to first look at the simpler cases $k = 1, 2$. For $k = 1$, using (3.32), (3.34), (3.35), and the identities $\tilde{\kappa}(z_1, z_1) = 0$, $\tilde{\kappa}(z_1, \bar{z}_1) = -\tilde{\kappa}(\bar{z}_1, z_1)$, we obtain

$$\begin{aligned} \tilde{R}_1(z_1) &= \tilde{\kappa}(z_1, \bar{z}_1)(-2iy_1) = \frac{1}{a\sqrt{\pi}} e^{-4(x_1/a)^2} \frac{y_1^2}{x_1^2 + y_1^2} \left(\frac{\sinh(8x_1)}{2x_1} - \frac{\sin(8y_1)}{2y_1} + o(1) \right) \\ &\xrightarrow{d} \frac{\delta(x_1)}{2} \frac{y_1^2}{x_1^2 + y_1^2} \left(\frac{\sinh(8x_1)}{2x_1} - \frac{\sin(8y_1)}{2y_1} + o(1) \right) = \delta(x_1) K^{\chi \sin}(y_1, y_1), \quad \text{as } a \rightarrow 0. \end{aligned}$$

For $k = 2$, using an exact computation of the Pfaffian in (3.32), we get

$$\tilde{R}_2(z_1, z_2) = \left(\tilde{\kappa}(z_1, \bar{z}_1) \tilde{\kappa}(z_2, \bar{z}_2) - |\tilde{\kappa}(z_1, z_2)|^2 + |\tilde{\kappa}(z_1, \bar{z}_2)|^2 \right) 4y_1 y_2.$$

By (3.34), as $a \rightarrow 0$, the terms in the above right-hand side have the same exponential factor $\frac{4}{a^2\pi} e^{-4(x_1/a)^2 - 4(x_2/a)^2}$ in their asymptotics. Using (3.35), we obtain after some computation that

$$\tilde{R}_2(z_1, z_2) \xrightarrow{d} \delta(x_1) \delta(x_2) \left(K^{\chi \sin}(y_1, y_1) K^{\chi \sin}(y_2, y_2) - K^{\chi \sin}(y_1, y_2)^2 \right), \quad \text{as } a \rightarrow 0.$$

We now turn to the general case. For $k \in \mathbb{N}_{>0}$, using (3.34) and

$$\prod_{j=1}^k \frac{2}{a\sqrt{\pi}} e^{-4(x_j/a)^2} \xrightarrow{d} \prod_{j=1}^k \delta(x_j),$$

we obtain

$$\begin{aligned} \tilde{R}_k(z_1, \dots, z_k) &= \text{Pf} \left[\begin{pmatrix} \tilde{\kappa}(z_j, z_l) & \tilde{\kappa}(z_j, \bar{z}_l) \\ \tilde{\kappa}(\bar{z}_j, z_l) & \tilde{\kappa}(\bar{z}_j, \bar{z}_l) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k (-2i y_j) \\ &\xrightarrow{d} \text{Pf} \left[\begin{pmatrix} \frac{y_j - y_l}{2y_j y_l} K^{\chi \sin}(y_j, y_l) & \frac{y_j + y_l}{2y_j y_l} K^{\chi \sin}(y_j, y_l) \\ -\frac{y_j + y_l}{2y_j y_l} K^{\chi \sin}(y_j, y_l) & -\frac{y_j - y_l}{2y_j y_l} K^{\chi \sin}(y_j, y_l) \end{pmatrix} \right]_{j,l=1}^k \prod_{j=1}^k y_j \delta(x_j), \quad \text{as } a \rightarrow 0. \end{aligned}$$

Now Lemma 3.5 completes the proof. $\square$

4. PROOF OF THEOREM 1.6: SEMI-LARGE GAP PROBABILITIES

The first step in proving Theorem 1.6 is to obtain exact identities for $\mathbb{P}_N^1$, $\mathbb{P}_N^2$ and $\mathbb{P}_N^{12}$. For this, we will rely on the following well-known formula for partition functions of planar symplectic ensembles.

Lemma 4.1. (See e.g. [8, Remark 2.5 and Corollary 3.3])

Let $\mathbf{w}$ be a rotation-invariant weight with sufficient decay at ∞ ,

$$\int_0^{+\infty} r^j \mathbf{w}(r) dr < +\infty, \quad \text{for all } j \geq 0.$$

Then the partition function

$$(4.1) \quad \tilde{Z}_N := \frac{1}{N!} \int_{\mathbb{C}} \dots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq N} |\zeta_j - \zeta_k|^2 |\zeta_j - \bar{\zeta}_k|^2 \prod_{j=1}^N |\zeta_j - \bar{\zeta}_j|^2 \mathbf{w}(\zeta_j) dA(\zeta_j)$$

can be rewritten as

$$\tilde{Z}_N = \prod_{k=0}^{N-1} \tilde{r}_k, \quad \text{where} \quad \tilde{r}_k := 2 \int_{\mathbb{C}} |\zeta|^{4k+2} \mathbf{w}(\zeta) dA(\zeta) = 4 \int_0^{+\infty} r^{4k+3} \mathbf{w}(r) dr.$$

Lemma 4.2. For any $\rho > 0$ and $N \in \mathbb{N}_{>0}$, the following identities hold:

$$(4.2) \quad \log \mathbb{P}_N^1 = \sum_{j=0}^{N-1} \log \left(1 - \frac{\gamma(2+2j+b_N N, b_N N)}{\Gamma(2+2j+b_N N)} \right),$$

$$(4.3) \quad \log \mathbb{P}_N^2 = \sum_{j=0}^{N-1} \log \left(\frac{\gamma(2+2j+b_N N, b_N N + 2N)}{\Gamma(2+2j+b_N N)} \right),$$

$$(4.4) \quad \log \mathbb{P}_N^{12} = \sum_{j=0}^{N-1} \log \left(\frac{\gamma(2+2j+b_N N, b_N N + 2N)}{\Gamma(2+2j+b_N N)} - \frac{\gamma(2+2j+b_N N, b_N N)}{\Gamma(2+2j+b_N N)} \right),$$

where γ is defined in (2.8).

Proof. Let

$$\begin{aligned} \mathbf{w}_1(\zeta) &:= e^{-NQ_N(\zeta)} \begin{cases} 0, & \text{if } |\zeta| \leq r_1, \\ 1, & \text{otherwise,} \end{cases} & \mathbf{w}_2(\zeta) &:= e^{-NQ_N(\zeta)} \begin{cases} 0, & \text{if } |\zeta| \geq r_2, \\ 1, & \text{otherwise,} \end{cases} \\ \mathbf{w}_{12}(\zeta) &:= e^{-NQ_N(\zeta)} \begin{cases} 0, & \text{if } |\zeta| \in [0, r_1] \cup [r_2, \infty], \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

By the definitions (1.22)–(1.24), we have $\mathbb{P}_N^1 = Z_N^1/Z_N$, $\mathbb{P}_N^2 = Z_N^2/Z_N$ and $\mathbb{P}_N^{12} = Z_N^{12}/Z_N$, where Z_N , Z_N^1 , Z_N^2 , Z_N^{12} are given by the right-hand side of (4.1) with $\mathbf{w}$ replaced by e^{-NQ_N} , $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_{12}$, respectively. Combining Lemma 4.1 with

$$\begin{aligned} 2 \int_{r_1}^{\infty} r^{4k+3+2b_N N} e^{-a_N N r^2} dr &= \frac{\Gamma(2+2k+b_N N, a_N N r_1^2)}{(a_N N)^{2+2k+b_N N}}, \\ 2 \int_0^{r_2} r^{4k+3+2b_N N} e^{-a_N N r^2} dr &= \frac{\gamma(2+2k+b_N N, a_N N r_2^2)}{(a_N N)^{2+2k+b_N N}}, \\ 2 \int_{r_1}^{r_2} r^{4k+3+2b_N N} e^{-a_N N r^2} dr &= \frac{\gamma(2+2k+b_N N, a_N N r_2^2) - \gamma(2+2k+b_N N, a_N N r_1^2)}{(a_N N)^{2+2k+b_N N}}, \end{aligned}$$

we obtain

$$\begin{aligned} Z_N &= \frac{2^N}{(a_N N)^{(b_N+1)N^2+N}} \prod_{k=0}^{N-1} \Gamma(2+2k+b_N N), \\ Z_N^1 &= \frac{2^N}{(a_N N)^{(b_N+1)N^2+N}} \prod_{k=0}^{N-1} \Gamma(2+2k+b_N N, b_N N), \\ Z_N^2 &= \frac{2^N}{(a_N N)^{(b_N+1)N^2+N}} \prod_{k=0}^{N-1} \gamma(2+2k+b_N N, b_N N + 2N), \\ Z_N^{12} &= \frac{2^N}{(a_N N)^{(b_N+1)N^2+N}} \prod_{k=0}^{N-1} \left(\gamma(2+2k+b_N N, b_N N + 2N) - \gamma(2+2k+b_N N, b_N N) \right), \end{aligned}$$

and the claim follows. $\square$

Proof of Theorem 1.6. By Lemmas 4.2 and A.2, we have

$$\begin{aligned} \log \mathbb{P}_N^1 &= \sum_{j=0}^{N-1} \log \left(1 - \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,1} \sqrt{\tilde{a}_j/2} \right) + R_{\tilde{a}_j}(\eta_{j,1}) \right), \\ \log \mathbb{P}_N^2 &= \sum_{j=0}^{N-1} \log \left(\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,2} \sqrt{\tilde{a}_j/2} \right) - R_{\tilde{a}_j}(\eta_{j,2}) \right), \\ \log \mathbb{P}_N^{12} &= \sum_{j=0}^{N-1} \log \left(\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,2} \sqrt{\tilde{a}_j/2} \right) - R_{\tilde{a}_j}(\eta_{j,2}) - \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,1} \sqrt{\tilde{a}_j/2} \right) + R_{\tilde{a}_j}(\eta_{j,1}) \right), \end{aligned}$$

where

$$(4.5) \quad \tilde{a}_j = 2 + 2j + b_N N, \quad \lambda_{j,1} = \frac{b_N N}{\tilde{a}_j}, \quad \lambda_{j,2} = \frac{b_N N + 2N}{\tilde{a}_j},$$

$$(4.6) \quad \eta_{j,k} = (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \log \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}, \quad k = 1, 2.$$

We complete the proof and give full details only for $\mathbb{P}_N^{12}(\rho)$. The proofs for $\mathbb{P}_N^1(\rho)$ and $\mathbb{P}_N^2(\rho)$ are similar (and simpler), so we omit them. Using (1.2), (4.5) and (4.6), we find

$$\begin{aligned}\lambda_{j,1} &= 1 - \frac{2\rho^2(1+j)}{N^2} + O(N^{-3}), & \lambda_{j,2} &= 1 + \frac{2\rho^2}{N} - \frac{2\rho^2(1+j-\rho^2)}{N^2} + O(N^{-3}), \\ \eta_{j,1} &= -\frac{2(1+j)\rho^2}{N^2} + O(N^{-3}), & \eta_{j,2} &= \frac{2\rho^2}{N} - \frac{2\rho^2(3+3j+\rho^2)}{3N^2} + O(N^{-3}),\end{aligned}$$

as $N \rightarrow \infty$. It follows that

$$\begin{aligned}-\eta_{j,1}\sqrt{\tilde{a}_j/2} &= \sqrt{2}\rho\frac{1+j}{N} + \frac{(1+j)\rho^3}{\sqrt{2}N^2} + O(N^{-3}), \\ -\eta_{j,2}\sqrt{\tilde{a}_j/2} &= -\sqrt{2}\rho\left(1 - \frac{j}{N}\right) + \sqrt{2}\rho\frac{1+\rho^2/6}{N} + \frac{\rho^3(12+12j-11\rho^2)}{36\sqrt{2}N^2} + O(N^{-3}),\end{aligned}$$

as $N \rightarrow \infty$. Using the above and (A.3), we then get

$$\begin{aligned}&\log\left(\frac{1}{2}\operatorname{erfc}\left(-\eta_{j,2}\sqrt{\tilde{a}_j/2}\right) - R_{\tilde{a}_j}(\eta_{j,2}) - \frac{1}{2}\operatorname{erfc}\left(-\eta_{j,1}\sqrt{\tilde{a}_j/2}\right) + R_{\tilde{a}_j}(\eta_{j,1})\right) \\ &= f_1(j/N) + \frac{f_2(j/N)}{N} + O(N^{-2}),\end{aligned}$$

uniformly for $j \in \{0, 1, \dots, N-1\}$, where

$$\begin{aligned}f_1(x) &:= \log\left(\frac{1}{2}\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \frac{1}{2}\operatorname{erfc}(\sqrt{2}\rho x)\right), \\ f_2(x) &:= \frac{-\rho}{3\sqrt{2}\pi} \frac{e^{-2\rho^2x^2}(2x^2\rho^2 - 3\rho^2x - 5) + e^{-2\rho^2(1-x)^2}(5 + \rho^2(1+x-2x^2))}{\frac{1}{2}\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \frac{1}{2}\operatorname{erfc}(\sqrt{2}\rho x)}.\end{aligned}$$

Since $\{f_1^{(\ell)}, f_2^{(\ell)}\}_{\ell=0,1,2}$ are continuous and bounded on $[0, 1]$, it follows from [23, Lemma 3.4] (with $A = a_0 = 0$, $B = 1$ and $b_0 = -1$) that as $N \rightarrow +\infty$,

$$\begin{aligned}\sum_{j=0}^{N-1} f_1(j/N) &= N \int_0^1 f_1(x) dx + \frac{f_1(0) - f_1(1)}{2} + O(N^{-1}), \\ \frac{1}{N} \sum_{j=0}^{N-1} f_2(j/N) &= \int_0^1 f_2(x) dx + O(N^{-1}).\end{aligned}$$

This yields

$$\begin{aligned}\log \mathbb{P}_N^{12}(\rho) &= N \int_0^1 \log\left(\frac{1}{2}\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \frac{1}{2}\operatorname{erfc}(\sqrt{2}\rho x)\right) dx + \frac{1}{2} \log\left(\frac{\operatorname{erfc}(-\sqrt{2}\rho) - 1}{1 - \operatorname{erfc}(\sqrt{2}\rho)}\right) \\ &\quad + \frac{2\rho}{3\sqrt{2}\pi} \int_0^1 \frac{e^{-2\rho^2x^2}(5 + 3\rho^2x - 2\rho^2x^2) - e^{-2\rho^2(1-x)^2}(5 + (1+x-2x^2)\rho^2)}{\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \operatorname{erfc}(\sqrt{2}\rho x)} dx + O(N^{-1})\end{aligned}$$

as $N \rightarrow +\infty$. Since $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$,

$$\log\left(\frac{\operatorname{erfc}(-\sqrt{2}\rho) - 1}{1 - \operatorname{erfc}(\sqrt{2}\rho)}\right) = 0.$$

Using also

$$\int_0^1 \frac{e^{-2\rho^2x^2}(5 + 3\rho^2x - 2\rho^2x^2)}{\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \operatorname{erfc}(\sqrt{2}\rho x)} dx = \int_0^1 \frac{e^{-2\rho^2(1-x)^2}(5 + (1+x-2x^2)\rho^2)}{\operatorname{erfc}(-\sqrt{2}\rho(1-x)) - \operatorname{erfc}(\sqrt{2}\rho x)} dx.$$

we obtain (1.25). $\square$

APPENDIX A. UNIFORM ASYMPTOTICS OF THE INCOMPLETE GAMMA FUNCTION

In this appendix, we collect some known asymptotics of the incomplete gamma function.

Lemma A.1. (Taken from [46, Section 8.11.7], [50] and [15, (1.32) and below]). Let E_{sz} be the exterior region of the Szegő curve $\{z \in \mathbb{C} : |z| \leq 1, |ze^{1-z}| = 1\}$. Note that $\{z : |\arg(z-1)| < \frac{3\pi}{4}\} \subset E_{sz}$. As $\tilde{a} \rightarrow +\infty$,

$$(A.1) \quad Q(\tilde{a}, \tilde{a}z) \sim \frac{\tilde{a}^{\tilde{a}-1}}{\Gamma(\tilde{a})} e^{-\tilde{a}z} \frac{z^{\tilde{a}}}{z-1} \sum_{k=0}^{+\infty} \frac{b_k(z)}{(z-1)^{2k+1}} \frac{(-1)^k}{\tilde{a}^k}$$

uniformly for z in compact subsets of E_{sz} . The coefficients $b_k(z)$ are defined recursively by

$$b_0(z) = 1, \quad b_k(z) = z(1-z)b'_{k-1}(z) + (2k-1)zb_{k-1}(z), \quad (k = 1, 2, \dots).$$

In particular, as $\tilde{a} \rightarrow +\infty$ we have

$$(A.2) \quad Q(\tilde{a}, \tilde{a}z) = \frac{1}{\sqrt{2\pi\tilde{a}}} e^{\tilde{a}-\tilde{a}z} \frac{z^{\tilde{a}}}{z-1} \left(1 - \left(\frac{1}{12} + \frac{z}{(z-1)^2} \right) \frac{1}{\tilde{a}} + \left(\frac{1}{288} + \frac{z(2z+1)}{(z-1)^4} \right) \frac{1}{\tilde{a}^2} + O\left(\frac{1}{\tilde{a}^3}\right) \right),$$

uniformly for z in compact subsets of E_{sz} .

The asymptotic expansion (A.1) is stated in [50] for $|\arg(z-1)| < \frac{3\pi}{4}$ and extended in [15, (1.32) and below] for $z \in E_{sz}$. Lemma A.1 is used in Subsection 3.1, see (3.13). In fact, to prove Theorem 1.1 (a) for a given $\theta \in [0, 2\pi) \setminus \{0, \pi\}$, we need (A.2) with z close to $e^{2i\theta}$. Hence, to handle the case $\theta \notin [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]$, the results from [50] are enough for us; however for the other case $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]$ we rely on [15].

Lemma A.2. (Taken from [49, Section 11.2.4]). For $\tilde{a} > 0$ and $z > 0$, we have

$$\frac{\gamma(\tilde{a}, z)}{\Gamma(\tilde{a})} = \frac{1}{2} \operatorname{erfc}(-\eta\sqrt{\tilde{a}/2}) - R_{\tilde{a}}(\eta), \quad R_{\tilde{a}}(\eta) = \frac{e^{-\frac{1}{2}\tilde{a}\eta^2}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{a}u^2} g(u) du,$$

where $g(u) = \frac{dt}{du} \frac{1}{\lambda-t} + \frac{1}{u+i\eta}$,

$$\lambda = \frac{z}{\tilde{a}}, \quad \eta = (\lambda-1) \sqrt{\frac{2(\lambda-1-\log \lambda)}{(\lambda-1)^2}}, \quad u = -i(t-1) \sqrt{\frac{2(t-1-\log t)}{(t-1)^2}},$$

where $\operatorname{sign}(\eta) = \operatorname{sign}(\lambda-1)$, and $\operatorname{sign}(u) = \operatorname{sign}(\operatorname{Im} t)$ with $t \in \mathcal{L} := \{\frac{\theta}{\sin \theta} e^{i\theta} : -\pi < \theta < \pi\}$ and $u \in \mathbb{R}$ (in particular $u = -i(t-1) + \mathcal{O}((t-1)^2)$ as $t \rightarrow 1$). Furthermore,

$$(A.3) \quad R_{\tilde{a}}(\eta) \sim \frac{e^{-\frac{1}{2}\tilde{a}\eta^2}}{\sqrt{2\pi\tilde{a}}} \sum_{j=0}^{+\infty} \frac{c_j(\eta)}{\tilde{a}^j} \quad \text{as } \tilde{a} \rightarrow +\infty$$

uniformly for $z \in [0, \infty)$, where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in [0, \infty)$) and given by

$$c_0 = \frac{1}{\lambda-1} - \frac{1}{\eta}, \quad c_j = \frac{1}{\eta} \frac{d}{d\eta} c_{j-1}(\eta) + \frac{\gamma_j}{\lambda-1}, \quad j \geq 1,$$

where the γ_j are the Stirling coefficients

$$\gamma_j = \frac{(-1)^j}{2^j j!} \left[\frac{d^{2j}}{dx^{2j}} \left(\frac{1}{2} \frac{x^2}{x - \log(1+x)} \right)^{j+\frac{1}{2}} \right]_{x=0}.$$

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REFERENCES

- [1] K. Adhikari. Hole probabilities for β -ensembles and determinantal point processes in the complex plane. *Electron. J. Probab.*, 23:Paper No. 48, 21, 2018.
- [2] K. Adhikari and N. K. Reddy. Hole probabilities for finite and infinite Ginibre ensembles. *Int. Math. Res. Not. IMRN*, (21):6694–6730, 2017.
- [3] M. Adler, P. J. Forrester, T. Nagao, and P. van Moerbeke. Classical skew orthogonal polynomials and random matrices. *J. Statist. Phys.*, 99(1-2):141–170, 2000.
- [4] G. Akemann. The complex Laguerre symplectic ensemble of non-Hermitian matrices. *Nuclear Phys. B*, 730(3):253–299, 2005.
- [5] G. Akemann and F. Basile. Massive partition functions and complex eigenvalue correlations in matrix models with symplectic symmetry. *Nuclear Phys. B*, 766(1-3):150–177, 2007.
- [6] G. Akemann, S.-S. Byun, and N.-G. Kang. Scaling limits of planar symplectic ensembles. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 18:Paper No. 007, 40, 2022.
- [7] G. Akemann, M. Cikovic, and M. Venker. Universality at weak and strong non-Hermiticity beyond the elliptic Ginibre ensemble. *Comm. Math. Phys.*, 10.1007/s00220-018-3201-1, 2018.
- [8] G. Akemann, M. Ebke, and I. Parra. Skew-orthogonal polynomials in the complex plane and their Bergman-like kernels. *Comm. Math. Phys.*, 389:621–659, 2022.
- [9] G. Akemann, J. R. Ipsen, and E. Strahov. Permanent processes from products of complex and quaternionic induced Ginibre ensembles. *Random Matrices Theory Appl.*, 3(4):1450014, 54, 2014.
- [10] G. Akemann, M. Kieburg, A. Mielke, and T. Prosen. Universal signature from integrability to chaos in dissipative open quantum systems. *Phys. Rev. Lett.*, 123(25):254101, 2019.
- [11] G. Akemann and M. J. Phillips. The interpolating Airy kernels for the $\beta = 1$ and $\beta = 4$ elliptic Ginibre ensembles. *J. Stat. Phys.*, 155(3):421–465, 2014.
- [12] G. Akemann, M. J. Phillips, and L. Shifrin. Gap probabilities in non-Hermitian random matrix theory. *J. Math. Phys.*, 50(6):063504, 32, 2009.
- [13] G. Akemann and E. Strahov. Hole probabilities and overcrowding estimates for products of complex Gaussian matrices. *J. Stat. Phys.*, 151(6):987–1003, 2013.
- [14] Y. Ameur and S.-S. Byun. Almost-Hermitian random matrices and bandlimited point processes. *arXiv:2101.03832*.
- [15] Y. Ameur and J. Cronvall. Szegő type asymptotics for the reproducing kernel in spaces of full-plane weighted polynomials. *arXiv:2107.11148*.
- [16] Y. Ameur, N.-G. Kang, N. Makarov, and A. Wennman. Scaling limits of random normal matrix processes at singular boundary points. *J. Funct. Anal.*, 278(3):108340, 2020.
- [17] F. Benaych-Georges and F. Chapon. Random right eigenvalues of Gaussian quaternionic matrices. *Random Matrices Theory Appl.*, 1(2):1150009, 18, 2012.
- [18] A. Borodin and C.D. Sinclair. The Ginibre ensemble of real random matrices and its scaling limits. *Comm. Math. Phys.* 291 (2009), no. 1, 177–224.
- [19] S.-S. Byun and M. Ebke. Universal scaling limits of the symplectic elliptic Ginibre ensembles. *arXiv:2108.05541*.
- [20] S.-S. Byun, M. Ebke, and S.-M. Seo. Wronskian structures of planar symplectic ensembles. *arXiv:2110.12196*.
- [21] S.-S. Byun and P. J. Forrester. Spherical induced ensembles with symplectic symmetry. *arXiv:2209.01934*.
- [22] S.-S. Byun and S.-M. Seo. Random normal matrices in the almost-circular regime. *Bernoulli (to appear)*, *arXiv:2112.11353*.
- [23] C. Charlier. Large gap asymptotics on annuli in the random normal matrix model. *arXiv:2110.06908*.
- [24] J. Fischmann, W. Bruzda, B. A. Khoruzhenko, H.-J. Sommers, and K. Życzkowski. Induced Ginibre ensemble of random matrices and quantum operations. *J. Phys. A*, 45(7):075203, 31, 2012.
- [25] P. J. Forrester. Some statistical properties of the eigenvalues of complex random matrices. *Phys. Lett. A*, 169(1-2):21–24, 1992.
- [26] P. J. Forrester. *Log-gases and Random Matrices (LMS-34)*. Princeton University Press, Princeton, 2010.
- [27] P. J. Forrester. Asymptotics of spacing distributions 50 years later. *Random matrix theory, interacting particle systems, and integrable systems, Math. Sci. Res. Inst. Publ.*, 65:199–222, 2014.
- [28] P. J. Forrester. Analogies between random matrix ensembles and the one-component plasma in two-dimensions. *Nucl. Phys. B*, 904:253–281, 2016.
- [29] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers. Almost Hermitian random matrices: crossover from Wigner-Dyson to Ginibre eigenvalue statistics. *Phys. Rev. Lett.*, 79(4):557–560, 1997.
- [30] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers. Almost-Hermitian random matrices: eigenvalue density in the complex plane. *Phys. Lett. A*, 226(1-2):46–52, 1997.

- [31] Y. V. Fyodorov, H.-J. Sommers, and B. A. Khoruzhenko. Universality in the random matrix spectra in the regime of weak non-Hermiticity. *Ann. Inst. H. Poincaré Phys. Théor.*, 68(4):449–489, 1998.
- [32] S. Ghosh and A. Nishry. Point processes, hole events, and large deviations: random complex zeros and Coulomb gases. *Constr. Approx.*, 48(1):101–136, 2018.
- [33] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.*, 6(3):440–449, 1965.
- [34] R. Grobe, F. Haake, and H.-J. Sommers. Quantum distinction of regular and chaotic dissipative motion. *Phys. Rev. Lett.*, 61(17):1899, 1988.
- [35] J. R. Ipsen. Products of independent quaternion Ginibre matrices and their correlation functions. *J. Phys. A*, 46(26):265201, 16, 2013.
- [36] B. Jancovici, J. L. Lebowitz, and G. Manificat. Large charge fluctuations in classical Coulomb systems. *J. Statist. Phys.*, 72(3-4):773–787, 1993.
- [37] E. Kanzieper. Eigenvalue correlations in non-Hermitian symplectic random matrices. *J. Phys. A*, 35(31):6631–6644, 2002.
- [38] B. A. Khoruzhenko and S. Lysychnkin. Truncations of random symplectic unitary matrices. *arXiv:2111.02381*.
- [39] M. K.-H. Kiessling and H. Spohn. A note on the eigenvalue density of random matrices. *Comm. Math. Phys.*, 199(3):683–695, 1999.
- [40] A. B. J. Kuijlaars. Universality. *Chapter 6 in: Oxford Handbook of Random Matrix Theory*, (G. Akemann, J. Baik, and P. Di Francesco, eds.), Oxford University Press, 2011 (*arXiv:1103.5922*).
- [41] B. Lacroix-A-Chéz-Toine, J. A. M. Garzón, C. S. H. Calva, I. P. Castillo, A. Kundu, S. N. Majumdar, and G. Schehr. Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble. *Phys. Rev. E*, 100(1):012137, 2019.
- [42] S.-Y. Lee and R. Riser. Fine asymptotic behavior for eigenvalues of random normal matrices: ellipse case. *J. Math. Phys.*, 57(2):023302, 29, 2016.
- [43] S. Lysychnkin. Complex eigenvalues of high dimensional quaternion random matrices. *PhD Thesis, Queen Mary University of London United Kingdom*, 2021.
- [44] A. Mays and A. Ponsaing. An induced real quaternion spherical ensemble of random matrices. *Random Matrices Theory Appl.*, 6(1):1750001, 29, 2017.
- [45] M. L. Mehta. *Random matrices*. Academic Press, Inc., Boston, MA, second edition, 1991.
- [46] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Editors). *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge, 2010.
- [47] E. B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.
- [48] E. V. Shuryak and J. Verbaarschot. Random matrix theory and spectral sum rules for the Dirac operator in QCD. *Nucl. Phys. A*, 560(1):306–320, 1993.
- [49] N. Temme. Special functions: An introduction to the classical functions of mathematical physics. *John Wiley & Sons*, 1996.
- [50] F. G. Tricomi. Asymptotische Eigenschaften der unvollständigen Gammafunktion. *Math. Z.*, 53:136–148, 1950.
- [51] H. Widom. On the relation between orthogonal, symplectic and unitary matrix ensembles. *J. Statist. Phys.*, 94(3-4):347–363, 1999.