

Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles

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Abstract

We study the moment generating function of the disk counting statistics of a two-dimensional determinantal point process which generalizes the complex Ginibre point process. This moment generating function involves an $n \times n$ determinant whose weight is supported on the whole complex plane, is rotation-invariant, and has discontinuities along circles centered at 0. These discontinuities can be thought of as a two-dimensional analogue of jump-type Fisher-Hartwig singularities. In this paper, we obtain large n asymptotics for this determinant, up to and including the term of order $n^{-\frac{1}{2}}$. We allow for any finite number of discontinuities in the bulk, one discontinuity at the edge, and any finite number of discontinuities bounded away from the bulk. As an application, we obtain the large n asymptotics of all the cumulants of the disk counting function up to and including the term of order $n^{-\frac{1}{2}}$, both in the bulk and at the edge. This improves on the best known results for the complex Ginibre point process, and for general values of our parameters these results are completely new. Our proof makes a novel use of the uniform asymptotics of the incomplete gamma function.

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1 Introduction and statement of results

Consider the determinant

$$D_n := \frac{1}{n!} \int_{\mathbb{C}} \dots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j = \det \left(\int_{\mathbb{C}} z^j \bar{z}^k w(z) d^2 z \right)_{j,k=0}^{n-1},$$

where the weight w is given by

$$w(z) := |z|^{2\alpha} e^{-n|z|^{2b}} \omega(|z|), \quad b > 0, \alpha > -1, \quad (1.1)$$

and the function ω is defined by

$$\omega(x) = \prod_{\ell=1}^p \begin{cases} e^{u_\ell}, & \text{if } x < r_\ell, \\ 1, & \text{if } x \geq r_\ell, \end{cases} \quad (1.2)$$

for some $p \in \mathbb{N}_{>0} := \{1, 2, \dots\}$, $u_1, \dots, u_p \in \mathbb{R}$, and $0 < r_1 < \dots < r_p < +\infty$. Thus w is supported on the whole complex plane, is rotation-invariant (i.e. $w(z) = w(|z|)$), has a root-type singularity at 0, and has p discontinuities along circles centered at 0. We are interested in the asymptotics of D_n as $n \rightarrow +\infty$.

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A large number of problems arising in statistical mechanics, integrable operators, orthogonal polynomials, random matrix theory and the theory of Gaussian multiplicative chaos can be expressed in terms of structured determinants associated with a weight having root-type and/or jump-type singularities (the so-called Fisher-Hartwig singularities) [11, 21, 49, 3]. Of particular interest is the asymptotic behavior of these determinants as the size of the underlying matrix gets large. In the case where the weight is supported on a one-dimensional set, these asymptotics have already been widely studied and have a long history. Early works include [41] by Lenard, [51] by Widom, [6] by Basor, [12] by Böttcher and Silbermann, and [24] by Ehrhardt; see also [21] for more historical background. More recent results have been obtained in e.g. [20, 22] for Toeplitz determinants, [10] for Fredholm determinants, [33, 31, 8, 15, 17] for Hankel determinants, [20, 7] for Toeplitz+Hankel determinants and [16] for Muttalib-Borodin determinants. Much less is known when the weight is supported on a two-dimensional set. The first result in this direction is due to Webb and Wong [50], who obtained the large n asymptotics of $\det(\int_{\mathbb{C}} z^j \bar{z}^k |z - z_0|^{2\alpha} e^{-n|z|^2} d^2 z)_{j,k=0}^{n-1}$ with $|z_0| < 1$ fixed and $\operatorname{Re} \alpha > -1$, i.e. they considered the case of a Gaussian weight perturbed with a planar root-type singularity located in the bulk. Deaño and Simm in [19] then investigated the “edge regime” when $n \rightarrow +\infty$ and simultaneously $|z_0| \rightarrow 1$ at a critical speed. The case of two merging planar root-type singularities in the bulk was also studied in [19], among other things. We also mention that the related topic of planar orthogonal polynomials associated with a Gaussian weight having root-type singularities has been studied in [4, 5, 38, 9, 39, 40], see also [1].

Only limited results are available on asymptotics of determinants with planar discontinuities. In [14], second order asymptotics were obtained for Ginibre-type weights with discontinuities along smooth curves. In the rotation-invariant setting, but for more general potentials, second order asymptotics were obtained in [34]. More refined asymptotics, including the third term of order 1, were then obtained in [25] for Ginibre-type weights. We discuss these works in more detail in Remarks 1.3, 1.4 and 1.7 below. The purpose of this paper is to develop a systematic approach to obtain precise asymptotics of determinants with a discontinuous rotation-invariant weight.

When the weight is supported on a one-dimensional set, it is now well-understood that the asymptotics analysis of determinants with jump-type Fisher-Hartwig singularities involves hypergeometric functions, see e.g. [31]. Our situation presents an obvious but important difference with earlier works such as [31], namely that in [31] the discontinuities are located at several isolated points, while in our two-dimensional setting the discontinuities take place along circles. Here we find that the asymptotic analysis of D_n involves the uniform asymptotics of the incomplete γ function, see Lemma 1.12 below.

To motivate the study of D_n , let us consider the probability density function

$$\frac{1}{n!Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n |z_j|^{2\alpha} e^{-nV(z_j)}, \quad V(z) = |z|^{2b}, \quad b > 0, \quad \alpha > -1, \quad (1.3)$$

where Z_n is the normalization constant and $z_1, \dots, z_n \in \mathbb{C}$. This is the determinantal point process which characterizes the log-potential Coulomb gas with n particles in the external field $V(z) - \frac{2\alpha}{n} \log |z|$ at the inverse temperature $\beta = 2$ [27]. The density (1.3) is also the law of the eigenvalues of an $n \times n$ normal matrix M taken with respect to the probability measure [43]

$$\frac{1}{Z_n} |\det(M)|^{2\alpha} e^{-n \operatorname{tr}((MM^*)^b)} dM,$$

where Z_n is the normalization constant, M^* is the conjugate transpose of M , “tr” stands for “trace” and dM denotes the measure on the set of normal $n \times n$ matrices that is induced by the flat Euclidian metric of $\mathbb{C}^{n \times n}$. The special case $b = 1$ and $\alpha = 0$ of (1.3), known as the complex Ginibre point process [29], is also the joint eigenvalue density of an $n \times n$ random matrix whose entries are

independent complex Gaussian random variables with mean 0 and variance $\frac{1}{n}$. For more background on two-dimensional determinantal point processes such as (1.3), we refer to [30].

Given a Borel set $A \subset \mathbb{C}$, we denote $N(A) := \#\{z_j \in A\}$, i.e. $N(A)$ is the random variable that counts the number of points that fall into A . Let $p \in \mathbb{N}_{>0} := \{1, 2, \dots\}$, let $0 < r_1 < \dots < r_p$, and for $r > 0$, let $D_r := \{z \in \mathbb{C} : |z| < r\}$. We are interested in the joint moment generating function of $N(D_{r_1}), \dots, N(D_{r_p})$, which is given by

$$\mathbb{E} \left[\prod_{\ell=1}^p e^{u_\ell N(D_{r_\ell})} \right] = \frac{1}{n! Z_n} \int_{\mathbb{C}} \dots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j = \frac{D_n}{Z_n}, \quad (1.4)$$

where $u_1, \dots, u_p \in \mathbb{R}$ and the weight w was defined in (1.1). We mention that Z_n can be easily expanded as $n \rightarrow +\infty$ (see Remark 1.10 below). Thus the two problems of obtaining the large n asymptotics of D_n and of $\mathbb{E}[\prod_{\ell=1}^p e^{u_\ell N(D_{r_\ell})}]$ are essentially equivalent.

The main result of this paper is an explicit formula for the large n asymptotics of $\mathbb{E}[\prod_{\ell=1}^p e^{u_\ell N(D_{r_\ell})}]$, up to and including the term of order $n^{-\frac{1}{2}}$. These asymptotics depend very much on whether the r_ℓ 's are smaller, equal, or bigger than the critical value $b^{-\frac{1}{2b}}$. This is because the normalized empirical distribution $\frac{1}{n} \sum_{j=1}^n \delta_{z_j}$ of (1.3) converges as $n \rightarrow +\infty$ weakly almost surely (see e.g. [32, 13]) to an equilibrium measure μ [47],

$$d\mu(z) = \frac{1}{4\pi} \Delta V(z) d^2 z = \frac{b^2}{\pi} |z|^{2b-2} d^2 z, \quad (1.5)$$

which is supported on the closed disk centered at 0 of radius $b^{-\frac{1}{2b}}$. In Theorem 1.1 below, we treat the general situation where

$$0 < r_1 < \dots < r_m < r_{m+1} = b^{-\frac{1}{2b}} \left(1 + \sqrt{2b} \frac{\mathfrak{s}}{\sqrt{n}} \right)^{\frac{1}{2b}} < r_{m+2} < \dots < r_p < +\infty, \quad \mathfrak{s} \in \mathbb{R},$$

i.e. w has m discontinuities strictly inside the support of μ (the bulk), one discontinuity close to the edge, and $p - m - 1$ discontinuities outside the support of μ .

The large n asymptotics of $\mathbb{E}[\prod_{\ell=1}^p e^{u_\ell N(D_{r_\ell})}]$ are naturally described in terms of the two functions

$$\mathcal{F}(t, s) := \log \left(1 + \frac{s-1}{2} \operatorname{erfc}(t) \right), \quad \mathcal{G}(t, s) := \frac{1-s}{1 + \frac{s-1}{2} \operatorname{erfc}(t)} \frac{e^{-t^2}}{\sqrt{\pi}} = \frac{d}{dt} \mathcal{F}(t, s), \quad (1.6)$$

where $t \in \mathbb{R}$, $s \in \mathbb{C} \setminus (-\infty, 0]$, the principal branch is chosen for the log, and erfc is the complementary error function defined by

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx. \quad (1.7)$$

We now state our main result.

Theorem 1.1. *Let $p \in \mathbb{N}_{>0}$, $m \in \{0, 1, \dots, p-1\}$, $\mathfrak{s} \in \mathbb{R}$, and*

$$\alpha > -1, \quad b > 0, \quad 0 < r_1 < \dots < r_m < b^{-\frac{1}{2b}} < r_{m+2} < \dots < r_p < +\infty,$$

be fixed parameters, and for $n \in \mathbb{N}_{>0}$, define $r_{m+1} := b^{-\frac{1}{2b}} \left(1 + \sqrt{2b} \frac{\mathfrak{s}}{\sqrt{n}} \right)^{\frac{1}{2b}}$. For any fixed $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E} \left[\prod_{j=1}^p e^{u_j N(D_{r_j})} \right] = \exp \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O} \left(\frac{(\log n)^2}{n} \right) \right), \quad \text{as } n \rightarrow +\infty \quad (1.8)$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$\begin{aligned}
C_1 &= \sum_{j=1}^m b r_j^{2b} u_j + \sum_{j=m+1}^p u_j, \\
C_2 &= \sum_{j=1}^m \sqrt{2b} r_j^b \int_0^{+\infty} \left(\mathcal{F}(t, e^{u_j}) + \mathcal{F}(t, e^{-u_j}) \right) dt + \sqrt{2b} \int_0^{+\infty} \mathcal{F}(t, e^{-u_{m+1}}) dt \\
&\quad + \sqrt{2b} \mathfrak{s} u_{m+1} + \sqrt{2b} \int_0^{-\mathfrak{s}} \mathcal{F}(t, e^{u_{m+1}}) dt, \\
C_3 &= -\left(\frac{1}{2} + \alpha\right) \sum_{j=1}^m u_j + \left(\frac{1}{2} + \alpha\right) \mathcal{F}(\mathfrak{s}, e^{-u_{m+1}}) + 4b \sum_{j=1}^m \int_0^{+\infty} t \left(\mathcal{F}(t, e^{u_j}) - \mathcal{F}(t, e^{-u_j}) \right) dt \\
&\quad - 2b \int_0^{+\infty} (2t - \mathfrak{s}) \mathcal{F}(t, e^{-u_{m+1}}) dt + 2b \int_0^{-\mathfrak{s}} (2t + \mathfrak{s}) \mathcal{F}(t, e^{u_{m+1}}) dt \\
&\quad + b \sum_{j=1}^m \int_{-\infty}^{+\infty} \mathcal{G}(t, e^{u_j}) \frac{5t^2 - 1}{3} dt + b \int_{-\infty}^{-\mathfrak{s}} \mathcal{G}(t, e^{u_{m+1}}) \frac{5t^2 + 3\mathfrak{s}t - 1}{3} dt, \\
C_4 &= \sum_{j=1}^m \frac{6\sqrt{2}b}{r_j^b} \int_0^{+\infty} t^2 \left(\mathcal{F}(t, e^{u_j}) + \mathcal{F}(t, e^{-u_j}) \right) dt + (2b)^{3/2} \int_0^{+\infty} (3t^2 - 2\mathfrak{s}t) \mathcal{F}(t, e^{-u_{m+1}}) dt \\
&\quad + (2b)^{3/2} \int_0^{-\mathfrak{s}} (3t^2 + 2\mathfrak{s}t) \mathcal{F}(t, e^{u_{m+1}}) dt + \sum_{j=1}^m \frac{-b}{\sqrt{2} r_j^b} \int_{-\infty}^{+\infty} \mathcal{G}(t, e^{u_j}) \frac{21t - 193t^3 + 50t^5}{18} dt \\
&\quad - \frac{b^{3/2}}{\sqrt{2}} \int_{-\infty}^{-\mathfrak{s}} \mathcal{G}(t, e^{u_{m+1}}) \frac{21t - 193t^3 + 50t^5 + 6\mathfrak{s}(1 - 29t^2 + 10t^4) - 9\mathfrak{s}^2(3t - 2t^3)}{18} dt \\
&\quad - \sum_{j=1}^m \frac{b}{2\sqrt{2} r_j^b} \int_{-\infty}^{+\infty} \left(\mathcal{G}(t, e^{u_j}) \frac{5t^2 - 1}{3} \right)^2 dt - \frac{b^{3/2}}{2\sqrt{2}} \int_{-\infty}^{-\mathfrak{s}} \left(\mathcal{G}(t, e^{u_{m+1}}) \frac{5t^2 + 3\mathfrak{s}t - 1}{3} \right)^2 dt \\
&\quad + \left(\left(\frac{1}{2} + \alpha\right) \frac{2\mathfrak{s}^2 - 1}{3\sqrt{2}} \sqrt{b} + \frac{1 + 6\alpha + 6\alpha^2}{12\sqrt{2b}} \right) \mathcal{G}(-\mathfrak{s}, e^{u_{m+1}}).
\end{aligned}$$

In particular, since $\mathbb{E}[\prod_{j=1}^p e^{u_j N(D_{r_j})}]$ is analytic for $u_1, \dots, u_p \in \mathbb{C}$ and positive for $u_1, \dots, u_p \in \mathbb{R}$, Cauchy's formula combined with (1.8) implies that for any $k_1, \dots, k_p \in \mathbb{N} := \{0, 1, \dots\}$, $k_1 + \dots + k_p \geq 1$, and $u_1, \dots, u_p \in \mathbb{R}$, we have

$$\partial_{u_1}^{k_1} \dots \partial_{u_p}^{k_p} \left\{ \log \mathbb{E} \left[\prod_{j=1}^p e^{u_j N(D_{r_j})} \right] - \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} \right) \right\} = \mathcal{O} \left(\frac{(\log n)^2}{n} \right), \quad \text{as } n \rightarrow +\infty. \tag{1.9}$$

Remark 1.2. With more efforts, we expect that the estimates $\mathcal{O}(\frac{(\log n)^2}{n})$ in (1.8) and (1.9) can actually be shown to be $\mathcal{O}(n^{-1})$.

Remark 1.3. For $b = 1$, $\alpha = 0$, $u_2 = \dots = u_p = 0$ and $r_1 < 1$ (the bulk regime of the complex Ginibre point process), the first two leading coefficients C_1 and C_2 were previously obtained in [34, eq (28)], and C_3 was obtained in [25, Proposition 2.1]. For $b = 1$, $\alpha = 0$, $u_2 = \dots = u_p = 0$ and $r_1 = (1 + \sqrt{2} \frac{\mathfrak{s}}{\sqrt{n}})^{\frac{1}{2}}$ (the edge regime of the complex Ginibre point process), C_1, C_2, C_3 were also obtained in [25, Proposition 2.1].

Remark 1.4. Consider the probability measure (1.3) with a general radial potential $V(z) = V(|z|)$ satisfying $V(|z|)/(2 \log |z|) \rightarrow +\infty$ as $|z| \rightarrow +\infty$, and assume that the equilibrium measure is supported on D_{r_*} for a certain $r_* > 0$. In this general setting, it was argued in [34] that for $r < r_*$ fixed, we have $\mathbb{E}[e^{uN(D_r)}] = \exp(C_1 n + C_2 \sqrt{n} + o(\sqrt{n}))$ as $n \rightarrow +\infty$ for certain coefficients C_1 and C_2 . It was also conjectured in [34, eq (34)] that C_2 can be written in the form $C_2 = c_V \tilde{C}_2(u)$, where c_V is independent of u , and $\tilde{C}_2(u)$ is a universal quantity independent of V . Theorem 1.1 establishes this conjecture for $V(z) = |z|^{2b}$. More generally, we note from Theorem 1.1 that the coefficients C_2, C_3 and C_4 appearing in the large n asymptotics of $\mathbb{E}[\prod_{j=1}^m e^{u_j N(D_{r_j})}]$ are of the forms

$$C_2 = \sum_{j=1}^m b r_j^b \tilde{C}_2(u_j), \quad C_3 = -\left(\frac{1}{2} + \alpha\right) \sum_{j=1}^m u_j + \sum_{j=1}^m b \tilde{C}_3(u_j), \quad C_4 = \sum_{j=1}^m b r_j^{-b} \tilde{C}_4(u_j),$$

for some explicit $\{\tilde{C}_k(u)\}_{k=2}^4$ that are independent of α, b and $r_1, \dots, r_m$. We also find it remarkable that C_3 is completely independent of $r_1, \dots, r_m$.

Remark 1.5. For one-dimensional log-correlated point processes, asymptotic formulas for moment generating functions of bulk counting statistics are typically of the form $\exp(D_1 n + D_2 \log n + D_3 + o(1))$, see e.g. [15], and thus differ drastically from the asymptotics (1.8).

Recall that the cumulants $\{\kappa_j = \kappa_j(n, r, b, \alpha)\}_{j \in \mathbb{N}_{>0}}$ of the random variable $N(D_r)$ are defined through the expansion

$$\log \mathbb{E}[e^{uN(D_r)}] = \kappa_1 u + \frac{\kappa_2 u^2}{2!} + \frac{\kappa_3 u^3}{3!} + \frac{\kappa_4 u^4}{4!} + \dots, \quad \text{as } u \rightarrow 0,$$

or equivalently by

$$\kappa_j = \partial_u^j \log \mathbb{E}[e^{uN(D_r)}] \Big|_{u=0}. \quad (1.10)$$

More generally, the joint cumulants of $N(D_{r_1}), \dots, N(D_{r_p})$ are defined by

$$\kappa_{j_1, \dots, j_p} := \partial_{u_1}^{j_1} \dots \partial_{u_p}^{j_p} \log \mathbb{E}[e^{u_1 N(D_{r_1}) + \dots + u_p N(D_{r_p})}] \Big|_{u_1 = \dots = u_p = 0}, \quad j_1, \dots, j_p \in \mathbb{N}_{>0}. \quad (1.11)$$

We can deduce from Theorem 1.1 the following results.

Corollary 1.6.

(a) (Asymptotics for the cumulants in the bulk regime)

Let $j \in \mathbb{N}_{>0}$, $\alpha > -1$, $b > 0$ and $r \in (0, b^{-\frac{1}{2b}})$ be fixed. As $n \rightarrow +\infty$, we have

$$\kappa_j = \begin{cases} b r^{2b} n + d_j + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j = 1, \\ d_j + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j \text{ is odd and } j \neq 1, \\ c_j \sqrt{n} + e_j n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j \text{ is even,} \end{cases} \quad (1.12)$$

where

$$c_j = \sqrt{2} b r^b \int_0^{+\infty} \partial_u^j \left(\mathcal{F}(t, e^u) + \mathcal{F}(t, e^{-u}) \right) \Big|_{u=0} dt, \\ d_j = \begin{cases} -\frac{1}{2} - \alpha, & \text{if } j = 1 \\ 0, & \text{if } j \geq 2 \end{cases} + 4b \int_0^{+\infty} t \partial_u^j \left(\mathcal{F}(t, e^u) - \mathcal{F}(t, e^{-u}) \right) \Big|_{u=0} dt$$

$$\begin{aligned}
& + b \int_{-\infty}^{+\infty} \partial_u^j \mathcal{G}(t, e^u) \Big|_{u=0} \frac{5t^2 - 1}{3} dt, \\
e_j = & \frac{6\sqrt{2}b}{r^b} \int_0^{+\infty} t^2 \partial_u^j \left(\mathcal{F}(t, e^u) + \mathcal{F}(t, e^{-u}) \right) \Big|_{u=0} dt - \frac{b}{r^b} \int_{-\infty}^{+\infty} \partial_u^j \mathcal{G}(t, e^u) \Big|_{u=0} \frac{21t - 193t^3 + 50t^5}{18\sqrt{2}} dt \\
& - \frac{b}{2\sqrt{2}r^b} \int_{-\infty}^{+\infty} \partial_u^j [\mathcal{G}(t, e^u)^2] \Big|_{u=0} \left(\frac{5t^2 - 1}{3} \right)^2 dt.
\end{aligned} \tag{1.13}$$

For $j = 1$ and $j = 2$, these integrals can be simplified further, and we obtain

$$\kappa_1 = \mathbb{E}[N(D_r)] = br^{2b}n + \frac{b - 1 - 2\alpha}{2} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), \tag{1.14}$$

$$\kappa_2 = \text{Var}[N(D_r)] = \frac{br^b}{\sqrt{\pi}}\sqrt{n} - \frac{b}{16\sqrt{\pi}r^b}\frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), \tag{1.15}$$

as $n \rightarrow +\infty$. As one would expect, the leading term of $\mathbb{E}[N(D_r)]$ in (1.14) can be rewritten as $br^{2b} = \int_{D_r} d\mu$, where μ is the equilibrium measure defined in (1.5).

(b) (Asymptotics for the cumulants in the edge regime)

Let $j \in \mathbb{N}_{>0}$, $\alpha > -1$, $b > 0$ and $\mathfrak{s} \in \mathbb{R}$ be fixed, and for $n \in \mathbb{N}_{>0}$, let $r := b^{-\frac{1}{2b}}(1 + \sqrt{2b}\frac{\mathfrak{s}}{\sqrt{n}})^{\frac{1}{2b}}$. As $n \rightarrow +\infty$, we have

$$\kappa_j = \begin{cases} n + c_j\sqrt{n} + d_j + e_j n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j = 1, \\ c_j\sqrt{n} + d_j + e_j n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j \geq 2, \end{cases} \tag{1.16}$$

where

$$\begin{aligned}
c_j = & \begin{cases} \sqrt{2b}\mathfrak{s}, & \text{if } j = 1 \\ 0, & \text{if } j \geq 2 \end{cases} + \sqrt{2b} \int_0^{+\infty} \partial_u^j \mathcal{F}(t, e^{-u}) \Big|_{u=0} dt + \sqrt{2b} \int_0^{-\mathfrak{s}} \partial_u^j \mathcal{F}(t, e^u) \Big|_{u=0} dt, \\
d_j = & \left(\frac{1}{2} + \alpha \right) \partial_u^j \mathcal{F}(\mathfrak{s}, e^{-u}) \Big|_{u=0} - 2b \int_0^{+\infty} (2t - \mathfrak{s}) \partial_u^j \mathcal{F}(t, e^{-u}) \Big|_{u=0} dt \\
& + 2b \int_0^{-\mathfrak{s}} (2t + \mathfrak{s}) \partial_u^j \mathcal{F}(t, e^u) \Big|_{u=0} dt + b \int_{-\infty}^{-\mathfrak{s}} \partial_u^j \mathcal{G}(t, e^u) \Big|_{u=0} \frac{5t^2 + 3\mathfrak{s}t - 1}{3} dt, \\
e_j = & (2b)^{3/2} \int_0^{+\infty} (3t^2 - 2\mathfrak{s}t) \partial_u^j \mathcal{F}(t, e^{-u}) \Big|_{u=0} dt + (2b)^{3/2} \int_0^{-\mathfrak{s}} (3t^2 + 2\mathfrak{s}t) \partial_u^j \mathcal{F}(t, e^u) \Big|_{u=0} dt \\
& - \frac{b^{3/2}}{\sqrt{2}} \int_{-\infty}^{-\mathfrak{s}} \partial_u^j \mathcal{G}(t, e^u) \Big|_{u=0} \frac{21t - 193t^3 + 50t^5 + 6\mathfrak{s}(1 - 29t^2 + 10t^4) - 9\mathfrak{s}^2(3t - 2t^3)}{18} dt \\
& - \frac{b^{3/2}}{2\sqrt{2}} \int_{-\infty}^{-\mathfrak{s}} \partial_u^j [\mathcal{G}(t, e^u)^2] \Big|_{u=0} \left(\frac{5t^2 + 3\mathfrak{s}t - 1}{3} \right)^2 dt \\
& + \left(\left(\frac{1}{2} + \alpha \right) \frac{2\mathfrak{s}^2 - 1}{3\sqrt{2}} \sqrt{b} + \frac{1 + 6\alpha + 6\alpha^2}{12\sqrt{2}b} \right) \partial_u^j \mathcal{G}(-\mathfrak{s}, e^u) \Big|_{u=0}.
\end{aligned}$$

For $j = 1$ and $j = 2$, the coefficients c_j , d_j and e_j can be evaluated explicitly (in terms of erfc) using integration by parts, and we obtain the following:

$$c_1 = \frac{\sqrt{b}\mathfrak{s}}{\sqrt{2}} \text{erfc}(\mathfrak{s}) - \frac{\sqrt{b}}{\sqrt{2\pi}} e^{-\mathfrak{s}^2},$$

$$\begin{aligned}
d_1 &= -\frac{1}{2} \left(\frac{1}{2} + \alpha - \frac{b}{2} \right) \operatorname{erfc}(\mathfrak{s}) - \frac{b\mathfrak{s}}{3\sqrt{\pi}} e^{-\mathfrak{s}^2}, \\
e_1 &= \frac{e^{-\mathfrak{s}^2}}{\sqrt{2\pi}} \left(\frac{b(2+4\alpha) - 1 - 6\alpha - 6\alpha^2}{12\sqrt{b}} + \frac{(3b-2-4\alpha)\mathfrak{s}^2}{6} \sqrt{b} - \frac{2\mathfrak{s}^4}{9} b^{3/2} \right), \\
c_2 &= \frac{\sqrt{b}}{2\sqrt{\pi}} \operatorname{erfc}(\sqrt{2}\mathfrak{s}) + \sqrt{b} \frac{e^{-\mathfrak{s}^2}}{\sqrt{2\pi}} (1 - \operatorname{erfc}(\mathfrak{s})) + \frac{\sqrt{b}\mathfrak{s}}{\sqrt{2}} \operatorname{erfc}(\mathfrak{s}) \left(\frac{1}{2} \operatorname{erfc}(\mathfrak{s}) - 1 \right), \\
d_2 &= -\frac{b}{12\pi} e^{-2\mathfrak{s}^2} + \frac{b\mathfrak{s}}{2\sqrt{2\pi}} \operatorname{erfc}(\sqrt{2}\mathfrak{s}) + \frac{b\mathfrak{s}}{3\sqrt{\pi}} e^{-\mathfrak{s}^2} (1 - \operatorname{erfc}(\mathfrak{s})) \\
&\quad + \frac{b-1-2\alpha}{4} \operatorname{erfc}(\mathfrak{s}) \left(\frac{1}{2} \operatorname{erfc}(\mathfrak{s}) - 1 \right), \\
e_2 &= \frac{e^{-\mathfrak{s}^2}}{12\sqrt{2\pi}b} \left(1 - 2b + 6\alpha - 4b\alpha + 6\alpha^2 + 2(2-3b+4\alpha)b\mathfrak{s}^2 + \frac{8b^2}{3}\mathfrak{s}^4 \right) (1 - \operatorname{erfc}(\mathfrak{s})) \\
&\quad - \frac{b^{3/2}\mathfrak{s}}{72\sqrt{2}\pi} e^{-2\mathfrak{s}^2} - \frac{b^{3/2}(1+4\mathfrak{s}^2)}{32\sqrt{\pi}} \operatorname{erfc}(\sqrt{2}\mathfrak{s}). \tag{1.17}
\end{aligned}$$

In particular, for $r = b^{-\frac{1}{2b}}$ (thus $D_r = \operatorname{supp} \mu$ and $\mathfrak{s} = 0$), as $n \rightarrow +\infty$ we have

$$\begin{aligned}
\mathbb{E}[N(D_r)] &= n - \frac{\sqrt{b}}{\sqrt{2\pi}} \sqrt{n} + \frac{b-1-2\alpha}{4} + \frac{b(2+4\alpha) - 1 - 6\alpha - 6\alpha^2}{12\sqrt{2\pi}b\sqrt{n}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), \\
\operatorname{Var}[N(D_r)] &= \frac{\sqrt{b}}{2\sqrt{\pi}} \sqrt{n} + \frac{1+2\alpha-b}{8} - \frac{b}{12\pi} - \frac{b^{3/2}}{32\sqrt{\pi}} \frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right).
\end{aligned}$$

(c) (Asymptotics for the cumulants in the regime bounded away from the bulk)

Let $\alpha > -1$, $b > 0$ and $r > b^{-\frac{1}{2b}}$ be fixed. As $n \rightarrow +\infty$, we have

$$\kappa_j = \begin{cases} n + \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j = 1, \\ \mathcal{O}\left(\frac{(\log n)^2}{n}\right), & \text{if } j \geq 2. \end{cases} \tag{1.18}$$

(d) (Asymptotics for the joint cumulants)

Let $p \in \mathbb{N}$, $p \geq 2$, $m \in \{0, 1, \dots, p-1\}$, $j_1, \dots, j_p \in \mathbb{N}$, $\alpha > -1$, $b > 0$, and

$$0 < r_1 < r_2 < \dots < r_m < b^{-\frac{1}{2b}} < r_{m+2} < \dots < r_p < +\infty, \quad \mathfrak{s} \in \mathbb{R},$$

be fixed parameters, and for $n \in \mathbb{N}_{>0}$, define $r_{m+1} = b^{-\frac{1}{2b}} (1 + \sqrt{2b} \frac{\mathfrak{s}}{\sqrt{n}})^{\frac{1}{2b}}$. If at least two j_ℓ 's are positive, then as $n \rightarrow +\infty$ we have

$$\kappa_{j_1, \dots, j_p} = \mathcal{O}\left(\frac{(\log n)^2}{n}\right). \tag{1.19}$$

(e) (joint Gaussian fluctuations)

Let $\alpha > -1$, $b > 0$, $\mathfrak{s} \in \mathbb{R}$ and $0 < r_1 < r_2 < \dots < r_m < b^{-\frac{1}{2b}}$ be fixed, and for $n \in \mathbb{N}_{>0}$, define $r_{m+1} := b^{-\frac{1}{2b}} (1 + \sqrt{2b} \frac{\mathfrak{s}}{\sqrt{n}})^{\frac{1}{2b}}$. Consider the random variables

$$N_j := \pi^{1/4} \frac{N(D_{r_j}) - br_j^{2b}n}{\sqrt{br_j^b} n^{1/4}}, \quad j = 1, \dots, m, \tag{1.20}$$

$$N_{m+1} := \frac{N(D_{r_{m+1}}) - (n + c_1\sqrt{n})}{\sqrt{c_2} n^{1/4}}, \quad (1.21)$$

where c_1, c_2 are as in (1.17). As $n \rightarrow +\infty$, $(N_1, \dots, N_{m+1})$ converges in distribution to a multivariate normal random variable of mean $(0, \dots, 0)$ and covariance matrix I_{m+1} , where I_{m+1} is the $(m+1) \times (m+1)$ identity matrix.

Remark 1.7. Some parts of Corollary 1.6 were already known:

- In [45], Rider obtained various results for the variance and covariance of radial and angular statistics in the complex Ginibre point process (which corresponds to $b = 1$ and $\alpha = 0$ in our setting). In particular, for $(b, \alpha) = (1, 0)$, the leading coefficient $c_2 = br^b/\sqrt{\pi}$ of (1.15) was determined in [45, Theorem 1.6].
- Let N_{Ell} be the number of points lying outside the droplet of the Elliptic Ginibre ensemble. Fine asymptotics for $\mathbb{E}[N_{\text{Ell}}]$, including the term of order $n^{-\frac{1}{2}}$, were obtained in [37, eq (70)]. In particular, for the edge regime, the coefficients $c_1|_{(\mathfrak{s}, b, \alpha)=(0, 1, 0)}$, $d_1|_{(\mathfrak{s}, b, \alpha)=(0, 1, 0)}$ and $e_1|_{(\mathfrak{s}, b, \alpha)=(0, 1, 0)}$ were previously found in [37].
- Let $N_A := \#\{z_j : z_j \in A\}$ be the number of points of the complex Ginibre process lying in a given Borel set A . The following was proved in [14, Theorem 1.6]: when A is in the bulk, has smooth boundary, and is independent of n , the cumulants $\{\kappa_j(A)\}_{j=1}^{+\infty}$ satisfy

$$\kappa_j(A) = \begin{cases} \alpha_{j,0}n + \sum_{\ell=1}^N \alpha_{j,\ell}n^{1-\ell} + \mathcal{O}(n^{-N}), & \text{if } j = 1, \\ \sum_{\ell=1}^N \alpha_{j,\ell}n^{1-\ell} + \mathcal{O}(n^{-N}), & \text{if } j \text{ is odd and } j \geq 3, \\ \beta_{j,0}n^{\frac{1}{2}} + \sum_{\ell=1}^N \beta_{j,\ell}n^{\frac{1}{2}-\ell} + \mathcal{O}(n^{-N-\frac{1}{2}}), & \text{if } j \text{ is even,} \end{cases} \quad (1.22)$$

for any $N \in \mathbb{N}$ and some $\alpha_{j,\ell}, \beta_{j,\ell} \in \mathbb{R}$. Moreover, the constants $\alpha_{1,0}, \beta_{j,0}$ were also determined explicitly in [14, Theorem 1.6]. Our asymptotics (1.6) are consistent with (1.22). (Actually, our results also suggest that the cumulants $\kappa_j(D_r)$ of the Mittag-Leffler ensemble with general b and α also satisfy an all-order expansion of the form (1.22).)

- The coefficients $\{c_j\}_{j=1}^{+\infty}$ were obtained for both the bulk and the edge regimes of the complex Ginibre point process in [35, eqs (55)–(67)] (the analysis of [35] is done in the context of fermions in a rotating trap, and this model is equivalent to the complex Ginibre point process [35], see also [42]). The coefficients $\{d_j|_{(b,\alpha)=(1,0)}\}_{j=1}^{+\infty}$ were then obtained in [25, Remark 4] for the bulk regime.
- Corollary 1.6 (e), when specialized to $(b, \alpha) = (1, 0)$, was already known from [45] for $m = 1$, and from [25, Proposition 2.2] for general $m \in \mathbb{N}_{>0}$. Note that disk counting statistics are linear statistics with indicator type test functions (thus non-smooth). We mention in passing that for smooth linear statistics of non-Hermitian random matrices, some Gaussian fluctuation formulas were already obtained by Forrester in [26], then proved in [46] for Ginibre matrices, and then in more generality in [2, 36].

Remark 1.8. As a sanity check, note that c_1 and c_2 in (1.17) satisfy $c_1 < 0$ and $c_2 > 0$ for all $\mathfrak{s} \in \mathbb{R}$, which is consistent with $\kappa_1 = \mathbb{E}[N(D_r)] \leq n$ and $\kappa_2 = \text{Var}[N(D_r)] > 0$. Note also that c_2 in (1.17) decays exponentially fast as $\mathfrak{s} \rightarrow +\infty$, which suggests that for any $\epsilon > 0$, $\text{Var}[N(D_r)]$ decays very fast as $n \rightarrow +\infty$ and simultaneously $n^{-\epsilon}\mathfrak{s} \rightarrow +\infty$ (but this is only a heuristic since the error terms in (1.16) are proved for fixed $\mathfrak{s}$). The error terms in (1.18) and (1.19) are far from optimal and could be easily improved if needed, but we do not pursue that here (see also [45, Theorem 1.7] where an exponentially small error term was obtained for $\text{Cov}[N(D_{r_1}), N(D_{r_2})]$ in the Ginibre case).

Proof of Corollary 1.6. Proof of parts (a), (b), (c) and (d): the asymptotics (1.12), (1.16), (1.18) and (1.19) are obtained by combining (1.10)–(1.11) with Theorem 1.1 (using in particular (1.9)). To obtain (1.12), one needs to further note that

$$\begin{aligned} \partial_u^j \left(\mathcal{F}(t, e^u) + \mathcal{F}(t, e^{-u}) \right) &= 0 \quad \text{for } j \text{ odd}, & \partial_u^j \left(\mathcal{F}(t, e^u) - \mathcal{F}(t, e^{-u}) \right) &= 0 \quad \text{for } j \text{ even}, \\ t \mapsto \mathcal{G}_j(t) &:= \partial_u^j \mathcal{G}(t, e^u) \Big|_{u=0} \quad \text{satisfies } \mathcal{G}_j(t) = \mathcal{G}_j(-t) \text{ for } j \text{ odd and } \mathcal{G}_j(t) = -\mathcal{G}_j(-t) \text{ for } j \text{ even}, \end{aligned}$$

from which it easily follows that the coefficients c_j, d_j, e_j of (1.13) satisfy $c_j = e_j = 0$ for j odd and $d_j = 0$ for j even. The simplified formulas (1.14), (1.15), (1.17) are then obtained using integration by parts. We now turn to the proof of part (e). For this, recall that there exists $\delta > 0$ such that (1.8) holds uniformly for $u_1, \dots, u_{m+1} \in \{z \in \mathbb{C} : |z| \leq \delta\}$. Hence, using Theorem 1.1 with $p = m + 1$ and

$$u_j = \pi^{1/4} \frac{t_j}{\sqrt{br_j^b} n^{1/4}}, \quad j = 1, \dots, m, \quad u_{m+1} := \frac{t_{m+1}}{\sqrt{c_2} n^{1/4}},$$

where $t_1, \dots, t_{m+1} \in \mathbb{R}$ are arbitrary but fixed and c_2 is as in (1.17), we obtain

$$\mathbb{E} \left[\prod_{j=1}^{m+1} e^{t_j N_j} \right] = \exp \left(\sum_{j=1}^{m+1} \frac{t_j^2}{2} + \mathcal{O}(n^{-\frac{1}{4}}) \right), \quad \text{as } n \rightarrow +\infty,$$

which implies the claim. $\square$

Outline of the proof of Theorem 1.1.

Since w is rotation-invariant, D_n can be identically expressed in terms of one-fold integrals (albeit not being a Selberg integral). This fact is well-known and has already been used in different contexts, see e.g. [45, 28, 19, 23]. For convenience, we also give a proof of this result here.

Lemma 1.9. *Let w be a rotation invariant weight satisfying*

$$\int_0^{+\infty} u^j w(u) du < +\infty, \quad \text{for all } j \geq 0.$$

Then

$$\frac{1}{n!} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j = (2\pi)^n \prod_{j=0}^{n-1} \int_0^{+\infty} u^{2j+1} w(u) du.$$

Proof. It follows from e.g. [50, Lemma 2.1] that

$$\frac{1}{n!} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j = \det \left(\int_{\mathbb{C}} z^j \bar{z}^k w(z) d^2 z \right)_{j,k=0}^{n-1}. \quad (1.23)$$

Since w is rotation-invariant,

$$\int_{\mathbb{C}} z^j \bar{z}^k w(z) d^2 z = \begin{cases} 0, & \text{if } j \neq k, \\ 2\pi \int_0^{+\infty} u^{2j+1} w(u) du, & \text{if } j = k, \end{cases}$$

and the claim follows. $\square$

Remark 1.10. Recall that Z_n is the normalization constant of (1.3). Applying Lemma 1.9 to $w(z) = |z|^{2\alpha} e^{-n|z|^{2b}}$, we obtain

$$Z_n := \frac{1}{n!} \int_{\mathbb{C}} \dots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n |z_j|^{2\alpha} e^{-n|z_j|^{2b}} d^2 z_j = n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^n \Gamma\left(\frac{j+\alpha}{b}\right). \quad (1.24)$$

The above right-hand side can be easily expanded as $n \rightarrow +\infty$ using [44, formula 5.11.1]

$$\log \Gamma(z) = z \log z - z - \frac{\log z}{2} + \frac{\log 2\pi}{2} + \frac{1}{12z} + \mathcal{O}(z^{-3}), \quad \text{as } z \rightarrow +\infty,$$

and we obtain

$$\begin{aligned} \log Z_n = & -\frac{3+2\log b}{4b} n^2 - \frac{1}{2} n \log n + \left(\frac{\log(2\pi)}{2} + \frac{b-2\alpha-1}{2b} (1 + \log b) + \log \frac{\pi}{b} \right) n \\ & + \frac{1-3b+b^2+6\alpha-6b\alpha+6\alpha^2}{12b} \log n + \mathfrak{g}(b, \alpha) + \mathcal{O}(n^{-2}), \quad \text{as } n \rightarrow +\infty, \end{aligned} \quad (1.25)$$

for a certain constant $\mathfrak{g}(b, \alpha)$. As noticed in [18, Proposition 1.4], if $b = \frac{n_1}{n_2}$ for some $n_1, n_2 \in \mathbb{N}_{>0}$, then $\mathfrak{g}(b, \alpha)$ is explicitly given by¹

$$\begin{aligned} \mathfrak{g}(b, \alpha) = & n_1 n_2 \zeta'(-1) + \frac{b(n_2 - n_1) + 2n_1 \alpha}{4b} \log(2\pi) \\ & - \frac{1-3b+b^2+6\alpha-6b\alpha+6\alpha^2}{12b} \log n_1 - \sum_{j=1}^{n_2} \sum_{k=1}^{n_1} \log G\left(\frac{j + \frac{\alpha}{b} - 1}{n_2} + \frac{k}{n_1}\right), \end{aligned}$$

where G is Barnes' G -function. Thus one can obtain the large n asymptotics of D_n by combining (1.4), (1.8) and (1.25).

For convenience, let us write ω (which was defined in (1.2)) as

$$\omega(x) = \sum_{\ell=1}^{p+1} \omega_\ell \mathbf{1}_{[0, r_\ell)}(x), \quad \omega_\ell := \begin{cases} e^{u_\ell + \dots + u_p} - e^{u_{\ell+1} + \dots + u_p}, & \text{if } \ell < p, \\ e^{u_p} - 1, & \text{if } \ell = p, \\ 1, & \text{if } \ell = p+1, \end{cases} \quad (1.26)$$

where $r_{p+1} := +\infty$. Applying Lemma 1.9 to $w = w$, we immediately get the following exact identity for D_n :

$$\begin{aligned} D_n = & (2\pi)^n \prod_{j=0}^{n-1} \int_0^{+\infty} u^{2j+1+2\alpha} e^{-nu^{2b}} \omega(u) du \\ = & n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^n \left(\sum_{\ell=1}^p \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) + \Gamma\left(\frac{j+\alpha}{b}\right) \right), \end{aligned} \quad (1.27)$$

where $\gamma(a, z)$ is the incomplete gamma function

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt.$$

As can be seen from (1.27), to obtain the large n asymptotics of D_n , we need the asymptotics of $\gamma(a, z)$ as $z \rightarrow +\infty$ uniformly for $a \in [\frac{1+\alpha}{b}, \frac{z}{br_1^{2b}} + \frac{\alpha}{b}]$. These asymptotics are already known and are stated in the following lemmas.

¹ $\mathfrak{g}(b, \alpha)$ here corresponds to $d(\frac{1}{b}, \frac{\alpha}{b} - 1)$ in [18].

Lemma 1.11. (Taken from [44, formula 8.11.2]). Let $a > 0$ be fixed. As $z \rightarrow +\infty$,

$$\gamma(a, z) = \Gamma(a) + \mathcal{O}(e^{-\frac{z}{a}}).$$

Lemma 1.12. (Taken from [48, Section 11.2.4]). We have

$$\frac{\gamma(a, z)}{\Gamma(a)} = \frac{1}{2} \operatorname{erfc}(-\eta \sqrt{a/2}) - R_a(\eta), \quad R_a(\eta) = \frac{e^{-\frac{1}{2}a\eta^2}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} g(u) du,$$

where erfc is defined in (1.7), $\lambda := \frac{z}{a}$, $g(u) := \frac{dt}{du} \frac{1}{\lambda - t} + \frac{1}{u + i\eta}$,

$$\eta = (\lambda - 1) \sqrt{\frac{2(\lambda - 1 - \log \lambda)}{(\lambda - 1)^2}}, \quad u = -i(t - 1) \sqrt{\frac{2(t - 1 - \log t)}{(t - 1)^2}}, \quad (1.28)$$

where the principal branch is used for the roots. In particular, $\eta > 0$ for $\lambda > 1$, $\eta < 0$ for $\lambda < 1$, and $u \in \mathbb{R}$ for $t \in \mathcal{L} := \{\frac{\theta}{\sin \theta} e^{i\theta} : -\pi < \theta < \pi\}$. Furthermore, as $a \rightarrow +\infty$, uniformly for $z \in [0, \infty)$,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \sum_{j=0}^{\infty} \frac{c_j(\eta)}{a^j}, \quad (1.29)$$

where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in [0, \infty)$). The first two coefficients are given by (see [48, p. 312])

$$c_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \quad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda - 1)^3} - \frac{1}{(\lambda - 1)^2} - \frac{1}{12(\lambda - 1)}.$$

In particular, the following hold:

(i) Let $\delta > 1$ be fixed, and let $z = \lambda a$. As $a \rightarrow +\infty$, uniformly for $\lambda \geq 1 + \delta$,

$$\gamma(a, z) = \Gamma(a) (1 + \mathcal{O}(e^{-\frac{a\eta^2}{2}})).$$

(ii) Let $z = \lambda a$. As $a \rightarrow +\infty$, uniformly for λ in compact subsets of $(0, 1)$,

$$\gamma(a, z) = \Gamma(a) \mathcal{O}(e^{-\frac{a\eta^2}{2}}).$$

2 Proof of Theorem 1.1

In this section, $\log$ always denotes the principal branch of the logarithm, and c and C denote positive constants which may change within a computation.

Let M' be a large integer independent of n , let $\epsilon > 0$ be a small constant independent of n , and let $M := M' \sqrt{\log n}$. Define

$$j_{\ell,-} := \lceil \frac{bnr_{\ell}^{2b}}{1+\epsilon} - \alpha \rceil, \quad j_{\ell,+} := \lfloor \frac{bnr_{\ell}^{2b}}{1-\epsilon} - \alpha \rfloor, \quad \ell = 1, \dots, m, \\ j_{m+1,-} := \lceil \frac{n}{1+\epsilon} - \alpha \rceil, \quad j_{m+1,+} := n,$$

$j_{0,-} := 1$, $j_{0,+} := M'$. We take ϵ sufficiently small such that

$$\frac{br_{\ell}^{2b}}{1-\epsilon} < \frac{br_{\ell+1}^{2b}}{1+\epsilon}, \quad \text{for all } \ell \in \{1, \dots, m\}.$$

Using (1.27), we split $\log D_n$ into $2m + 4$ parts

$$\log D_n = S_{-1} + S_0 + \sum_{k=1}^m (S_{2k-1} + S_{2k}) + S_{2m+1} + S_{2m+2}, \quad (2.1)$$

with

$$S_{-1} = -\frac{1}{2b}n^2 \log n - \frac{1+2\alpha}{2b}n \log n + n \log \frac{\pi}{b}, \quad S_0 = \sum_{j=1}^{M'} \log \left(\sum_{\ell=1}^{p+1} \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) \right), \quad (2.2)$$

$$S_{2k-1} = \sum_{j=j_{k-1,+}+1}^{j_{k,-}-1} \log \left(\sum_{\ell=1}^{p+1} \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) \right), \quad S_{2k} = \sum_{j=j_{k,-}}^{j_{k,+}} \log \left(\sum_{\ell=1}^{p+1} \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) \right), \quad (2.3)$$

$$S_{2m+1} = \sum_{j=j_{m,+}+1}^{j_{m+1,-}-1} \log \left(\sum_{\ell=1}^{p+1} \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) \right), \quad S_{2m+2} = \sum_{j=j_{m+1,-}}^n \log \left(\sum_{\ell=1}^{p+1} \omega_\ell \gamma\left(\frac{j+\alpha}{b}, nr_\ell^{2b}\right) \right). \quad (2.4)$$

For convenience, we also define

$$\Omega_\ell = \sum_{j=\ell}^{p+1} \omega_j = \begin{cases} e^{u_\ell + \dots + u_p}, & \text{if } \ell \leq p, \\ 1 & \text{if } \ell = p+1, \end{cases} \quad (2.5)$$

so that ω can be rewritten as

$$\omega(x) = \sum_{\ell=1}^{p+1} \omega_\ell \mathbf{1}_{[0, r_\ell)}(x) = \sum_{\ell=1}^{p+1} \Omega_\ell \mathbf{1}_{[r_{\ell-1}, r_\ell)}(x).$$

Lemma 2.1. *For any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_0 = M' \log \Omega_1 + \sum_{j=1}^{M'} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \mathcal{O}(e^{-cn}), \quad \text{as } n \rightarrow +\infty, \quad (2.6)$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. By (2.2) and Lemma 1.11, as $n \rightarrow +\infty$ we have

$$S_0 = \sum_{j=1}^{M'} \log \left(\sum_{\ell=1}^{p+1} \omega_\ell [\Gamma\left(\frac{j+\alpha}{b}\right) + \mathcal{O}(e^{-cn})] \right) = \sum_{j=1}^{M'} \log (\Omega_1 \Gamma\left(\frac{j+\alpha}{b}\right)) + \mathcal{O}(e^{-cn}),$$

where the first error term in the above expression is independent of $u_1, \dots, u_p$. This clearly implies the claim. $\square$

Lemma 2.2. *Let $k \in \{1, \dots, m+1\}$. For any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_{2k-1} = (j_{k,-} - j_{k-1,+} - 1) \log \Omega_k + \sum_{j=j_{k-1,+}+1}^{j_{k,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \mathcal{O}(e^{-cn})$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. Recall that S_{2k-1} is defined in (2.3) and (2.4), and define $a_j := \frac{j+\alpha}{b}$, $\lambda_{j,\ell} := \frac{bnr_\ell^{2b}}{j+\alpha}$ and

$$\eta_{j,\ell} := (\lambda_{j,\ell} - 1) \sqrt{\frac{2(\lambda_{j,\ell} - 1 - \ln \lambda_{j,\ell})}{(\lambda_{j,\ell} - 1)^2}}. \quad (2.7)$$

Let us treat the case $k \geq 2$ first. We use Lemma 1.12 (i)–(ii) with a and λ replaced by a_j and $\lambda_{j,\ell}$ respectively, where $j \in \{j_{k-1,+} + 1, \dots, j_{k,-} - 1\}$ and $\ell \in \{1, \dots, p\}$. This gives

$$S_{2k-1} = \sum_{j=j_{k-1,+}+1}^{j_{k,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=j_{k-1,+}+1}^{j_{k,-}-1} \log \left(\sum_{\ell=1}^{k-1} \omega_\ell \mathcal{O}(e^{-\frac{a_j \eta_{j,\ell}^2}{2}}) + \sum_{\ell=k}^p \omega_\ell (1 + \mathcal{O}(e^{-\frac{a_j \eta_{j,\ell}^2}{2}})) + 1 \right), \quad (2.8)$$

as $n \rightarrow +\infty$, where the above error terms are independent of $\omega_1, \dots, \omega_p$. Note that for each $k \in \{2, \dots, m+1\}$, there exist positive constants $\{c_j, c'_j\}_{j=1}^3$ such that $c_1 n \leq a_j \leq c'_1 n$, $c_2 \leq |\lambda_{j,\ell} - 1| \leq c'_2$ and $c_3 \leq \eta_{j,\ell}^2 \leq c'_3$ hold for all n sufficiently large, for all $j \in \{j_{k-1,+} + 1, \dots, j_{k,-} - 1\}$ and for all $\ell \in \{1, \dots, p\}$. This proves the claim for $k = 2, \dots, m+1$ with $c = \frac{c_1 c_3}{2}$. The proof for $k = 1$ is only slightly different. By Lemma 1.12 (i), for any $\epsilon' > 0$ there exist $A = A(\epsilon')$, $C = C(\epsilon') > 0$ such that $|\frac{\gamma(a, z)}{\Gamma(a)} - 1| \leq C e^{-\frac{a \eta^2}{2}}$ for all $a \geq A$, for all $\lambda = \frac{z}{a} \in [1 + \epsilon', +\infty]$, and where η is defined by (1.28). Let us take $\epsilon' = \frac{\epsilon}{2}$ and choose M' large enough so that $a_j = \frac{j+\alpha}{b} \geq A(\frac{\epsilon}{2})$ for all $j \in \{M'+1, \dots, j_{1,-} - 1\}$. Hence, as in (2.8) we find

$$S_1 = \sum_{j=M'+1}^{j_{1,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=M'+1}^{j_{1,-}-1} \log \left(\sum_{\ell=1}^p \omega_\ell (1 + \mathcal{O}(e^{-\frac{a_j \eta_{j,\ell}^2}{2}})) + 1 \right), \quad \text{as } n \rightarrow +\infty.$$

It is easy to check that for each $\ell \in \{1, \dots, p\}$, the quantity $a_j \eta_{j,\ell}^2$ decreases as j increases from $M' + 1$ to $j_{1,-} - 1$. Hence

$$\frac{a_j \eta_{j,\ell}^2}{2} \geq \frac{a_{j_{1,-}-1} \eta_{j_{1,-}-1,\ell}^2}{2} \geq cn, \quad \text{for all } j \in \{M' + 1, \dots, j_{1,-} - 1\}, \ell \in \{1, \dots, p\},$$

for a sufficiently small $c > 0$. This proves the claim for $k = 1$. $\square$

We now turn our attention to the sums S_{2k} , $k = 1, \dots, m+1$.

Lemma 2.3. *For any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_{2k} = S_{2k}^{(1)} + S_{2k}^{(2)} + S_{2k}^{(3)} + \mathcal{O}(e^{-cn}), \quad k = 1, 2, \dots, m,$$

as $n \rightarrow \infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$S_{2k}^{(v)} = \sum_{j: \lambda_{j,k} \in I_v} \log \left(\omega_k \gamma(a_j, z_k) + \Omega_{k+1} \Gamma\left(\frac{j+\alpha}{b}\right) \right), \quad v = 1, 2, 3, \quad k = 1, \dots, m \quad (2.9)$$

with

$$a_j := \frac{j+\alpha}{b}, \quad z_k := nr_k^{2b}, \quad \lambda_{j,k} := \frac{z_k}{a_j} = \frac{bnr_k^{2b}}{j+\alpha},$$

and where

$$I_1 = [1 - \epsilon, 1 - \frac{M}{\sqrt{n}}), \quad I_2 = [1 - \frac{M}{\sqrt{n}}, 1 + \frac{M}{\sqrt{n}}], \quad I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon].$$

Similarly, for any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that

$$S_{2m+2} = S_{2m+2}^{(2)} + S_{2m+2}^{(3)} + \mathcal{O}(e^{-cn}),$$

as $n \rightarrow \infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$S_{2m+2}^{(v)} = \sum_{j: \lambda_{j,m+1} \in I_v} \log \left(\omega_{m+1} \gamma(a_j, z_{m+1}) + \Omega_{m+2} \Gamma\left(\frac{j+\alpha}{b}\right) \right), \quad v = 2', 3, \quad (2.10)$$

with

$$a_j := \frac{j+\alpha}{b}, \quad z_{m+1} := nr_{m+1}^{2b}, \quad \lambda_{j,m+1} := \frac{z_{m+1}}{a_j} = \frac{bnr_{m+1}^{2b}}{j+\alpha},$$

and where

$$I_{2'} = \left[\frac{1+\frac{\mathcal{R}}{\sqrt{n}}}{1+\frac{\alpha}{n}}, 1 + \frac{M}{\sqrt{n}} \right], \quad I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon], \quad \mathcal{R} := \sqrt{2b} \mathfrak{s}.$$

Proof. By (2.3), (2.4) and Lemma 1.12 (i)–(ii), we have

$$S_{2k} = \sum_{j=j_{k,-}}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=j_{k,-}}^{j_{k,+}} \log \left(\sum_{\ell=1}^{k-1} \omega_\ell \mathcal{O}(e^{-cn}) + \omega_k \frac{\gamma(\frac{j+\alpha}{b}, nr_k^{2b})}{\Gamma(\frac{j+\alpha}{b})} + \sum_{\ell=k+1}^{p+1} \omega_\ell (1 + \mathcal{O}(e^{-cn})) \right), \quad (2.11)$$

as $n \rightarrow +\infty$, where the above error terms are independent of $\omega_1, \dots, \omega_p$. Let us choose $\delta > 0$ sufficiently small such that

$$\omega_k \frac{\gamma(\frac{j+\alpha}{b}, nr_k^{2b})}{\Gamma(\frac{j+\alpha}{b})} + \Omega_{k+1}$$

remains bounded away from the interval $(-\infty, 0]$ as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$ and for $j \in \{j_{k,-}, \dots, j_{k,+}\}$. By (2.11),

$$S_{2k} = \sum_{j=j_{k,-}}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=j_{k,-}}^{j_{k,+}} \log \left(\Omega_{k+1} + \omega_k \frac{\gamma(\frac{j+\alpha}{b}, nr_k^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right) + \mathcal{O}(e^{-cn}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$. Note that $\lambda_{j,k} \in [1 - \epsilon, 1 + \epsilon]$ for each $k \in \{1, \dots, m\}$ and $j \in \{j_{k,-}, \dots, j_{k,+}\}$, while $\lambda_{j,m+1} \in [\frac{1+\frac{\mathcal{R}}{\sqrt{n}}}{1+\frac{\alpha}{n}}, 1 + \epsilon]$ for each $j \in \{j_{m+1,-}, \dots, j_{m+1,+}\}$. The claim now follows directly by splitting S_{2k} , $k = 1, \dots, m$ into three parts and S_{2m+2} into two parts. $\square$

For $k = 1, \dots, m$, define $g_{k,-} := \lceil \frac{bnr_k^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \rceil$, $g_{k,+} := \lfloor \frac{bnr_k^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \rfloor$, and

$$\begin{aligned} \theta_{k,-}^{(n,M)} &:= g_{k,-} - \left(\frac{bnr_k^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \right) = \left\lceil \frac{bnr_k^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \right\rceil - \left(\frac{bnr_k^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \right), \\ \theta_{k,+}^{(n,M)} &:= \left(\frac{bnr_k^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \right) - g_{k,+} = \left(\frac{bnr_k^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \right) - \left\lfloor \frac{bnr_k^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \right\rfloor. \end{aligned}$$

Clearly, $\theta_{k,-}^{(n,M)}, \theta_{k,+}^{(n,M)} \in [0, 1]$, and formally we can write

$$\sum_{j: \lambda_{j,k} \in I_3} = \sum_{j=j_{k,-}}^{g_{k,-}-1}, \quad \sum_{j: \lambda_{j,k} \in I_2} = \sum_{j=g_{k,-}}^{g_{k,+}}, \quad \sum_{j: \lambda_{j,k} \in I_1} = \sum_{j=g_{k,+}+1}^{j_{k,+}}. \quad (2.12)$$

Define also $g_{m+1,-} := \left\lceil \frac{bnr_{m+1}^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right\rceil$ and

$$\theta_{m+1,-}^{(n,M)} := g_{m+1,-} - \left(\frac{bnr_{m+1}^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) = \left\lceil \frac{bnr_{m+1}^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right\rceil - \left(\frac{bnr_{m+1}^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) \in [0, 1).$$

Formally, we have

$$\sum_{j: \lambda_{j,m+1} \in I_3} = \sum_{j=j_{m+1,-}}^{g_{m+1,-}-1}, \quad \sum_{j: \lambda_{j,m+1} \in I_2} = \sum_{j=g_{m+1,-}}^n. \quad (2.13)$$

We collect in the following lemma some straightforward but useful asymptotic formulas.

Lemma 2.4. *As $n \rightarrow +\infty$, we have*

$$\begin{aligned} \sum_{j=g_{k,+}+1}^{j_{k,+}} 1 &= j_{k,+} - g_{k,+} = j_{k,+} - \left(\frac{bnr_k^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) + \theta_{k,+}^{(n,M)} \\ &= j_{k,+} - br_k^{2b}n - bMr_k^{2b}\sqrt{n} - bM^2r_k^{2b} + \alpha + \theta_{k,+}^{(n,M)} - bM^3r_k^{2b}n^{-\frac{1}{2}} + \mathcal{O}(M^4n^{-1}), \end{aligned} \quad (2.14)$$

$$\begin{aligned} \sum_{j=j_{k,-}}^{g_{k,-}-1} 1 &= g_{k,-} - j_{k,-} = \left(\frac{bnr_k^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) + \theta_{k,-}^{(n,M)} - j_{k,-} \\ &= br_k^{2b}n - j_{k,-} - bMr_k^{2b}\sqrt{n} + bM^2r_k^{2b} - \alpha + \theta_{k,-}^{(n,M)} - bM^3r_k^{2b}n^{-\frac{1}{2}} + \mathcal{O}(M^4n^{-1}), \end{aligned} \quad (2.15)$$

where (2.14) holds for all $k \in \{1, \dots, m\}$ and (2.15) holds for all $k \in \{1, \dots, m+1\}$. For $k = m+1$, (2.15) can be further expanded using $r_{m+1} = b^{-\frac{1}{2b}}(1 + \frac{\mathcal{R}}{\sqrt{n}})^{\frac{1}{2b}}$, and this gives

$$\begin{aligned} \sum_{j=j_{m+1,-}}^{g_{m+1,-}-1} 1 &= g_{m+1,-} - j_{m+1,-} = \left(\frac{bnr_{m+1}^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) + \theta_{m+1,-}^{(n,M)} - j_{m+1,-} = n - j_{m+1,-} \\ &+ (\mathcal{R} - M)\sqrt{n} + M(M - \mathcal{R}) - \alpha + \theta_{m+1,-}^{(n,M)} - M^2(M - \mathcal{R})n^{-\frac{1}{2}} + \mathcal{O}(M^4n^{-1}) \end{aligned} \quad (2.16)$$

as $n \rightarrow +\infty$.

Proof. The proof is a short computation. $\square$

Our next task is to evaluate $\{S_{2k}^{(v)}\}_{k=1,\dots,m}^{v=1,2,3}$ and $\{S_{2m+2}^{(v)}\}_{v=2,3}$. These sums are rather delicate to analyze and involve the asymptotics of $\gamma(a, z)$ in the regime $a \rightarrow +\infty$, $z \rightarrow +\infty$ with $\lambda = \frac{z}{a} \in [1 - \epsilon, 1 + \epsilon]$.

Lemma 2.5. *For any $k \in \{1, \dots, m\}$ and any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_{2k}^{(1)} &= \sum_{j=g_{k,+}+1}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \left(j_{k,+} - br_k^{2b}n - bMr_k^{2b}\sqrt{n} - bM^2r_k^{2b} \right. \\ &\quad \left. + \alpha + \theta_{k,+}^{(n,M)} - bM^3r_k^{2b}n^{-\frac{1}{2}} \right) \log \Omega_{k+1} + \mathcal{O}(M^4n^{-1}), \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. Since $I_1 = [1 - \epsilon, 1 - \frac{M}{\sqrt{n}}]$, by (2.9) and Lemma 1.12 we have

$$\begin{aligned} S_{2k}^{(1)} &= \sum_{j: \lambda_{j,k} \in I_1} \log \left(\omega_k \gamma(a_j, z_k) + \Omega_{k+1} \Gamma\left(\frac{j+\alpha}{b}\right) \right) \\ &= \sum_{j: \lambda_{j,k} \in I_1} \left[\log \Gamma\left(\frac{j+\alpha}{b}\right) + \log \left(\omega_k \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,k} \sqrt{a_j/2} \right) - R_{a_j}(\eta_{j,k}) \right] + \Omega_{k+1} \right) \right], \end{aligned}$$

where $\eta_{j,k} = (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}$. Since

$$\eta_{j,k} = \lambda_{j,k} - 1 + \mathcal{O}((\lambda_{j,k} - 1)^2) \leq -\frac{M}{\sqrt{n}} + \mathcal{O}\left(\frac{M^2}{n}\right), \quad \text{as } n \rightarrow \infty, \quad (2.17)$$

$$-\eta_{j,k} \sqrt{a_j/2} \geq \frac{Mr_k^b}{\sqrt{2}} + \mathcal{O}\left(\frac{M^2}{\sqrt{n}}\right), \quad \text{as } n \rightarrow \infty, \quad (2.18)$$

uniformly for $j \in \{j : \lambda_{j,k} \in I_1\}$, by choosing M' large enough we have

$$R_{a_j}(\eta_{j,k}) = \mathcal{O}(e^{-\frac{r_k^{2b} M^2}{4}}) = \mathcal{O}(n^{-10}), \quad \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,k} \sqrt{a_j/2} \right) = \mathcal{O}(e^{-\frac{r_k^{2b} M^2}{4}}) = \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$ uniformly for $j \in \{j : \lambda_{j,k} \in I_1\}$, and thus (using also (2.12))

$$\begin{aligned} S_{2k}^{(1)} &= \sum_{j=g_{k,+}+1}^{j_{k,+}} \left[\log \Gamma\left(\frac{j+\alpha}{b}\right) + \log \Omega_{k+1} \right] + \mathcal{O}(n^{-9}) \\ &= \sum_{j=g_{k,+}+1}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + (j_{k,+} - g_{k,+}) \log \Omega_{k+1} + \mathcal{O}(n^{-9}), \end{aligned} \quad (2.19)$$

as $n \rightarrow +\infty$. Since the error terms in (2.17) and (2.18) are clearly independent of $\omega_1, \dots, \omega_p$, the asymptotics (2.19) are uniform for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$. The claim now follows after inserting (2.14) in (2.19). $\square$

Lemma 2.6. *For any $k \in \{1, \dots, m+1\}$ and any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_{2k}^{(3)} &= \sum_{j=j_{k,-}}^{g_{k,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \left(br_k^{2b} n - j_{k,-} - bMr_k^{2b} \sqrt{n} + bM^2 r_k^{2b} \right. \\ &\quad \left. - \alpha + \theta_{k,-}^{(n,M)} - bM^3 r_k^{2b} n^{-\frac{1}{2}} \right) \log \Omega_k + \mathcal{O}(M^4 n^{-1}), \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$. For $k = m+1$, the above formula can be further expanded using $r_{m+1} = b^{-\frac{1}{2b}} (1 + \frac{\mathcal{R}}{\sqrt{n}})^{\frac{1}{2b}}$, and this gives

$$\begin{aligned} S_{2m+2}^{(3)} &= \sum_{j=j_{m+1,-}}^{g_{m+1,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \left(n - j_{m+1,-} + (\mathcal{R} - M) \sqrt{n} + M(M - \mathcal{R}) \right. \\ &\quad \left. - \alpha + \theta_{m+1,-}^{(n,M)} - M^2(M - \mathcal{R}) n^{-\frac{1}{2}} \right) \log \Omega_{m+1} + \mathcal{O}(M^4 n^{-1}), \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. The proof is similar to Lemma 2.5. Recall that $I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon]$. Thus by (2.9), (2.10) and Lemma 1.12, we have

$$S_{2k}^{(3)} = \sum_{j: \lambda_{j,k} \in I_3} \left[\log \Gamma\left(\frac{j+\alpha}{b}\right) + \log \left(\omega_k \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,k} \sqrt{a_j/2} \right) - R_{a_j}(\eta_{j,k}) \right] + \Omega_{k+1} \right) \right],$$

where $\eta_{j,k} = (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}$. Using the fact that

$$\begin{aligned} \eta_{j,k} &= \lambda_{j,k} - 1 + \mathcal{O}((\lambda_{j,k} - 1)^2) \geq \frac{M}{\sqrt{n}} + \mathcal{O}\left(\frac{M^2}{n}\right), & \text{as } n \rightarrow \infty, \\ -\eta_{j,k} \sqrt{a_j/2} &\leq -\frac{Mr_k^b}{\sqrt{2}} + \mathcal{O}\left(\frac{M^2}{\sqrt{n}}\right), & \text{as } n \rightarrow \infty, \end{aligned}$$

uniformly for $j \in \{j : \lambda_{j,k} \in I_3\}$, by choosing M' large enough we have

$$R_{a_j}(\eta_{j,k}) = \mathcal{O}(e^{-\frac{r_k^{2b} M^2}{4}}) = \mathcal{O}(n^{-10}), \quad \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,k} \sqrt{a_j/2} \right) = 1 - \mathcal{O}(e^{-\frac{r_k^{2b} M^2}{4}}) = 1 - \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$ uniformly for $j \in \{j : \lambda_{j,k} \in I_3\}$, and thus

$$\begin{aligned} S_{2k}^{(3)} &= \sum_{j=j_{k,-}}^{g_{k,-}-1} \left[\log \Gamma\left(\frac{j+\alpha}{b}\right) + \log \Omega_k \right] + \mathcal{O}(n^{-9}) \\ &= \sum_{j=j_{k,-}}^{g_{k,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + (g_{k,-} - j_{k,-}) \log \Omega_k + \mathcal{O}(n^{-9}), \end{aligned}$$

where we have also used (2.12) and (2.13) for the first equality. The claim now follows directly from (2.15) and (2.16). $\square$

Our next goal is to obtain the large n asymptotics of the sums $\{S_{2k}^{(2)}\}_{k=1, \dots, m+1}$. This is the most technical part of the proof of Theorem 1.1. Let us define

$$\begin{aligned} M_{j,k} &:= \sqrt{n}(\lambda_{j,k} - 1), & \text{for all } k \in \{1, \dots, m\} \text{ and } j \in \{j : \lambda_{j,k} \in I_2\} = \{g_{k,-}, \dots, g_{k,+}\}, \\ M_{j,m+1} &:= \sqrt{n}(\lambda_{j,m+1} - 1), & \text{for all } j \in \{j : \lambda_{j,m+1} \in I_2\} = \{g_{m+1,-}, \dots, n\}. \end{aligned}$$

By definition of I_2 , as $n \rightarrow +\infty$ the points $M_{g_{k,-},k}, \dots, M_{g_{k,+},k}$ tend to spread all over the interval $[-M, M]$ for each $k \in \{1, \dots, m\}$. Similarly, by definition of I_2' , as $n \rightarrow +\infty$ the points $M_{g_{m+1,-},m+1}, \dots, M_{n,m+1}$ tend to spread all over the interval $[\frac{\mathcal{R}n - \alpha\sqrt{n}}{n+\alpha}, M]$. Note also that $M_{j,k}$ decreases as j increases. The following lemma will be useful to obtain the large n asymptotics of $\{S_{2k}^{(2)}\}_{k=1, \dots, m+1}$.

Lemma 2.7. *Let $f \in C^3(\mathbb{R})$ be a function such that $|f|, |f'|, |f''|, |f'''|$ are bounded. For any $k \in \{1, \dots, m\}$, as $n \rightarrow +\infty$ we have*

$$\begin{aligned} \sum_{j=g_{k,-}}^{g_{k,+}} f(M_{j,k}) &= br_k^{2b} \int_{-M}^M f(t) dt \sqrt{n} - 2br_k^{2b} \int_{-M}^M t f(t) dt + \left(\frac{1}{2} - \theta_{k,-}^{(n,M)} \right) f(M) + \left(\frac{1}{2} - \theta_{k,+}^{(n,M)} \right) f(-M) \\ &+ \frac{1}{\sqrt{n}} \left[3br_k^{2b} \int_{-M}^M t^2 f(t) dt + \left(\frac{1}{12} + \frac{\theta_{k,-}^{(n,M)}(\theta_{k,-}^{(n,M)} - 1)}{2} \right) \frac{f'(M)}{br_k^{2b}} - \left(\frac{1}{12} + \frac{\theta_{k,+}^{(n,M)}(\theta_{k,+}^{(n,M)} - 1)}{2} \right) \frac{f'(-M)}{br_k^{2b}} \right] \\ &+ \mathcal{O}(M^4 n^{-1}). \end{aligned} \tag{2.20}$$

Also, as $n \rightarrow +\infty$, we have

$$\begin{aligned} \sum_{j=g_{m+1,-}}^n f(M_{j,m+1}) &= \int_{\mathcal{R}}^M f(t)dt (\sqrt{n} + \mathcal{R}) - 2 \int_{\mathcal{R}}^M t f(t)dt + \left(\frac{1}{2} - \theta_{m+1,-}^{(n,M)}\right) f(M) + \left(\frac{1}{2} + \alpha\right) f(\mathcal{R}) \\ &+ \frac{1}{\sqrt{n}} \left[-2\mathcal{R} \int_{\mathcal{R}}^M t f(t)dt + 3 \int_{\mathcal{R}}^M t^2 f(t)dt + \left(\frac{1}{12} + \frac{\theta_{m+1,-}^{(n,M)}(\theta_{m+1,-}^{(n,M)} - 1)}{2}\right) f'(M) \right. \\ &\quad \left. - \frac{1 + 6\alpha + 6\alpha^2}{12} f'(\mathcal{R}) \right] + \mathcal{O}(M^4 n^{-1}). \end{aligned} \quad (2.21)$$

Proof. Let $k \in \{1, \dots, m\}$. For conciseness, we will write M_j instead of $M_{j,k}$. Since $f \in C^3(\mathbb{R})$ is bounded and has bounded derivatives, we have

$$\begin{aligned} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} f(t)dt &= \sum_{j=g_{k,-}+1}^{g_{k,+}} \int_{M_j}^{M_{j-1}} f(t)dt = \sum_{j=g_{k,-}+1}^{g_{k,+}} \left\{ f(M_j)(M_{j-1} - M_j) \right. \\ &\quad \left. + f'(M_j) \frac{(M_{j-1} - M_j)^2}{2} + f''(M_j) \frac{(M_{j-1} - M_j)^3}{6} \right\} + \sum_{j=g_{k,-}+1}^{g_{k,+}} \mathcal{O}((M_{j-1} - M_j)^4), \end{aligned} \quad (2.22)$$

as $n \rightarrow +\infty$. Note that

$$M_{j-1} - M_j = \frac{bn^{3/2}r_k^{2b}}{(j+\alpha)(j-1+\alpha)} = \frac{1}{br_k^{2b}n^{1/2}} + \frac{2M_j}{br_k^{2b}n} + \left(\frac{1}{b^2r_k^{4b}} + \frac{M_j^2}{br_k^{2b}}\right) \frac{1}{n^{3/2}} + \mathcal{O}\left(\frac{1+|M_j|}{n^2}\right), \quad (2.23)$$

as $n \rightarrow +\infty$ uniformly for $j \in \{g_{k,-}+1, \dots, g_{k,+}\}$. Substituting (2.23) in (2.22) and rearranging the terms, we get

$$\begin{aligned} \sum_{j=g_{k,-}+1}^{g_{k,+}} f(M_j) &= br_k^{2b} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} f(t)dt \sqrt{n} - \frac{1}{\sqrt{n}} \sum_{j=g_{k,-}+1}^{g_{k,+}} \left(2M_j f(M_j) + \frac{1}{2br_k^{2b}} f'(M_j) \right) \\ &- \frac{1}{n} \sum_{j=g_{k,-}+1}^{g_{k,+}} \left(\frac{1}{br_k^{2b}} f(M_j) + M_j^2 f(M_j) + \frac{2}{br_k^{2b}} M_j f'(M_j) + \frac{1}{6b^2r_k^{4b}} f''(M_j) \right) + \mathcal{O}(M^4 n^{-1}) \end{aligned} \quad (2.24)$$

as $n \rightarrow +\infty$. In the same way as for (2.24), by replacing $f(t)$ above by $tf(t)$, $f'(t)$, $t^2 f(t)$ and $f''(t)$, we obtain respectively

$$\begin{aligned} \sum_{j=g_{k,-}+1}^{g_{k,+}} M_j f(M_j) &= br_k^{2b} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} t f(t)dt \sqrt{n} \\ &- \frac{1}{\sqrt{n}} \sum_{j=g_{k,-}+1}^{g_{k,+}} \left(2M_j^2 f(M_j) + \frac{1}{2br_k^{2b}} f(M_j) + \frac{1}{2br_k^{2b}} M_j f'(M_j) \right) + \mathcal{O}(M^4 n^{-\frac{1}{2}}), \end{aligned} \quad (2.25a)$$

$$\begin{aligned} \sum_{j=g_{k,-}+1}^{g_{k,+}} f'(M_j) &= br_k^{2b} (f(M_{g_{k,-}}) - f(M_{g_{k,+}})) \sqrt{n} \\ &- \frac{1}{\sqrt{n}} \sum_{j=g_{k,-}+1}^{g_{k,+}} \left(2M_j f'(M_j) + \frac{1}{2br_k^{2b}} f''(M_j) \right) + \mathcal{O}(M^3 n^{-\frac{1}{2}}), \end{aligned} \quad (2.25b)$$

$$\sum_{j=g_{k,-}+1}^{g_{k,+}} M_j^2 f(M_j) = br_k^{2b} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} t^2 f(t) dt \sqrt{n} + \mathcal{O}(M^4), \quad (2.25c)$$

$$\sum_{j=g_{k,-}+1}^{g_{k,+}} f''(M_j) = br_k^{2b} \left(f'(M_{g_{k,-}}) - f'(M_{g_{k,+}}) \right) \sqrt{n} + \mathcal{O}(M^2), \quad (2.25d)$$

as $n \rightarrow +\infty$. Substituting (2.25) in (2.24) yields

$$\begin{aligned} \sum_{j=g_{k,-}}^{g_{k,+}} f(M_j) &= br_k^{2b} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} f(t) dt \sqrt{n} - 2br_k^{2b} \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} t f(t) dt + \frac{f(M_{g_{k,-}}) + f(M_{g_{k,+}})}{2} \\ &+ \frac{br_k^{2b}}{\sqrt{n}} \left(3 \int_{M_{g_{k,+}}}^{M_{g_{k,-}}} t^2 f(t) dt + \frac{f'(M_{g_{k,-}}) - f'(M_{g_{k,+}})}{12b^2 r_k^{4b}} \right) + \mathcal{O}(M^4 n^{-\frac{1}{2}}), \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (2.26)$$

Note that the sum on the left-hand side of (2.26) starts at $j = g_{k,-}$. The integrals on the right-hand side can be expanded using

$$\begin{aligned} M_{g_{k,-}} &= M - \frac{\theta_{k,-}^{(n,M)}}{br_k^{2b} \sqrt{n}} - \frac{2M\theta_{k,-}^{(n,M)}}{br_k^{2b} n} + \mathcal{O}(M^2 n^{-\frac{3}{2}}), & \text{as } n \rightarrow +\infty, \\ M_{g_{k,+}} &= -M + \frac{\theta_{k,+}^{(n,M)}}{br_k^{2b} \sqrt{n}} - \frac{2M\theta_{k,+}^{(n,M)}}{br_k^{2b} n} + \mathcal{O}(M^2 n^{-\frac{3}{2}}), & \text{as } n \rightarrow +\infty. \end{aligned}$$

We then find (2.20) after a computation. We now turn to the proof of (2.21). Again, for conciseness we will write M_j instead of $M_{j,m+1}$. In the same way as for (2.26), we have

$$\begin{aligned} \sum_{j=g_{m+1,-}}^n f(M_j) &= \int_{M_n}^{M_{g_{m+1,-}}} f(t) dt (\sqrt{n} + \mathcal{R}) - 2 \int_{M_n}^{M_{g_{m+1,-}}} t f(t) dt + \frac{f(M_{g_{m+1,-}}) + f(M_n)}{2} \\ &+ \frac{1}{\sqrt{n}} \left(-2\mathcal{R} \int_{M_n}^{M_{g_{m+1,-}}} t f(t) dt + 3 \int_{M_n}^{M_{g_{m+1,-}}} t^2 f(t) dt + \frac{f'(M_{g_{m+1,-}}) - f'(M_n)}{12} \right) + \mathcal{O}(M^4 n^{-\frac{1}{2}}) \end{aligned} \quad (2.27)$$

as $n \rightarrow +\infty$, where we have used that $br_{m+1}^{2b} = 1 + \frac{\mathcal{R}}{\sqrt{n}}$ and $M_n := \frac{\mathcal{R}n - \alpha\sqrt{n}}{n + \alpha}$. We then obtain (2.21) from a direct computation using the expansions

$$\begin{aligned} M_{g_{m+1,-}} &= M - \frac{\theta_{m+1,-}^{(n,M)}}{\sqrt{n}} - \frac{(2M - \mathcal{R})\theta_{m+1,-}^{(n,M)}}{n} + \mathcal{O}(M^2 n^{-\frac{3}{2}}), & \text{as } n \rightarrow +\infty, \\ M_n &= \mathcal{R} - \frac{\alpha}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{2}}), & \text{as } n \rightarrow +\infty. \end{aligned}$$

□

Lemma 2.8. *For any $k \in \{1, \dots, m\}$ and any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_{2k}^{(2)} &= \sum_{j=g_{k,-}}^{g_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \tilde{C}_{2,k}^{(M)} \sqrt{n} + \tilde{C}_{3,k}^{(n,M)} + \frac{1}{\sqrt{n}} \tilde{C}_{4,k}^{(n,M)} + \mathcal{O}(M^4 n^{-1}), \\ \tilde{C}_{2,k}^{(M)} &= br_k^{2b} \int_{-M}^M \log \left(\frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right) + \Omega_{k+1} \right) dt, \end{aligned}$$

$$\begin{aligned}
\tilde{C}_{3,k}^{(n,M)} &= 2br_k^{2b} \int_{-M}^M t \log \left(\frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right) + \Omega_{k+1} \right) dt \\
&+ \left(\frac{1}{2} - \theta_{k,-}^{(n,M)} \right) \log \left(\frac{\omega_k}{2} \operatorname{erfc} \left(-\frac{M r_k^b}{\sqrt{2}} \right) + \Omega_{k+1} \right) + \left(\frac{1}{2} - \theta_{k,+}^{(n,M)} \right) \log \left(\frac{\omega_k}{2} \operatorname{erfc} \left(\frac{M r_k^b}{\sqrt{2}} \right) + \Omega_{k+1} \right) \\
&+ br_k^{2b} \int_{-M}^M \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right)} \left[\frac{1}{3r_k^b} - \frac{5t^2 r_k^b}{6} \right] \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} dt, \\
\tilde{C}_{4,k}^{(n,M)} &= 3br_k^{2b} \int_{-M}^M t^2 \log \left(\frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right) + \Omega_{k+1} \right) dt \\
&+ br_k^{2b} \int_{-M}^M \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right)} \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \frac{t(42 - 193r_k^{2b}t^2 + 25r_k^{4b}t^4)}{72r_k^b} dt \\
&- \frac{br_k^{2b}}{2} \int_{-M}^M \left[\frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{t r_k^b}{\sqrt{2}} \right)} \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \left(\frac{1}{3r_k^b} - \frac{5t^2 r_k^b}{6} \right) \right]^2 dt \\
&+ \left[\left(\frac{1}{2} - \theta_{k,-}^{(n,M)} \right) \left(\frac{1}{3r_k^b} - \frac{5M^2 r_k^b}{6} \right) + \left(\frac{1}{12} + \frac{\theta_{k,-}^{(n,M)}(\theta_{k,-}^{(n,M)} - 1)}{2} \right) \frac{1}{br_k^b} \right] \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(-\frac{M r_k^b}{\sqrt{2}} \right)} \frac{e^{-\frac{M^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \\
&+ \left[\left(\frac{1}{2} - \theta_{k,+}^{(n,M)} \right) \left(\frac{1}{3r_k^b} - \frac{5M^2 r_k^b}{6} \right) - \left(\frac{1}{12} + \frac{\theta_{k,+}^{(n,M)}(\theta_{k,+}^{(n,M)} - 1)}{2} \right) \frac{1}{br_k^b} \right] \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{M r_k^b}{\sqrt{2}} \right)} \frac{e^{-\frac{M^2 r_k^{2b}}{2}}}{\sqrt{2\pi}},
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. For convenience, we will write M_j and λ_j instead of $M_{j,k}$ and $\lambda_{j,k}$. By (2.9) and Lemma 1.12, we have

$$S_{2k}^{(2)} = \sum_{j: \lambda_j \in I_2} \log \Gamma \left(\frac{j+\alpha}{b} \right) + \sum_{j: \lambda_j \in I_2} \log \left(\omega_k \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_j \sqrt{a_j/2} \right) - R_{a_j}(\eta_j) \right] + \Omega_{k+1} \right),$$

where $\eta_j = (\lambda_j - 1) \sqrt{\frac{2(\lambda_j - 1 - \ln \lambda_j)}{(\lambda_j - 1)^2}}$. Recall that for all $j \in \{j : \lambda_j \in I_2\}$, we have $1 - \frac{M}{\sqrt{n}} \leq \lambda_j = \frac{bnr_k^{2b}}{j+\alpha} \leq 1 + \frac{M}{\sqrt{n}}$, and that $-M \leq M_j \leq M$. Since

$$\begin{aligned}
\eta_j &= (\lambda_j - 1) \left(1 - \frac{\lambda_j - 1}{3} + \frac{7}{36}(\lambda_j - 1)^2 + \mathcal{O}((\lambda_j - 1)^3) \right) = \frac{M_j}{\sqrt{n}} - \frac{M_j^2}{3n} + \frac{7M_j^3}{36n^{3/2}} + \mathcal{O}\left(\frac{M^4}{n^2}\right), \\
-\eta_j \sqrt{a_j/2} &= -\frac{M_j r_k^b}{\sqrt{2}} + \frac{5M_j^2 r_k^b}{6\sqrt{2}\sqrt{n}} - \frac{53M_j^3 r_k^b}{72\sqrt{2}n} + \mathcal{O}(M^4 n^{-\frac{3}{2}}),
\end{aligned} \tag{2.28}$$

as $n \rightarrow +\infty$ uniformly in $j \in \{j : \lambda_j \in I_2\}$, using (1.29) we obtain

$$\begin{aligned}
R_{a_j}(\eta_j) &= \frac{e^{-\frac{M_j^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \left(\frac{-1}{3r_k^b \sqrt{n}} - \left[\frac{M_j}{12r_k^b} + \frac{5M_j^3 r_k^b}{18} \right] \frac{1}{n} + \mathcal{O}((1 + M_j^6)n^{-\frac{3}{2}}) \right), \\
\frac{1}{2} \operatorname{erfc} \left(-\eta_j \sqrt{a_j/2} \right) &= \frac{1}{2} \operatorname{erfc} \left(-\frac{M_j r_k^b}{\sqrt{2}} \right) + \frac{1}{2} \operatorname{erfc}' \left(-\frac{M_j r_k^b}{\sqrt{2}} \right) \frac{5M_j^2 r_k^b}{6\sqrt{2}\sqrt{n}} \\
&+ \frac{M_j^3}{288n} \left(25M_j r_k^{2b} \operatorname{erfc}'' \left(-\frac{M_j r_k^b}{\sqrt{2}} \right) - 53\sqrt{2}r_k^b \operatorname{erfc}' \left(-\frac{M_j r_k^b}{\sqrt{2}} \right) \right) + \mathcal{O} \left(e^{-\frac{M_j^2 r_k^{2b}}{2}} (1 + M_j^8) n^{-\frac{3}{2}} \right),
\end{aligned}$$

as $n \rightarrow +\infty$. Thus we have

$$\begin{aligned}
S_{2k}^{(2)} &= \sum_{j: \lambda_j \in I_2} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=g_{k,-}}^{g_{k,+}} \log \left\{ \omega_k \left[\frac{1}{2} \operatorname{erfc}\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) + \frac{1}{2} \operatorname{erfc}'\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) \frac{5M_j^2 r_k^b}{6\sqrt{2}\sqrt{n}} \right. \right. \\
&\quad \left. \left. + \frac{M_j^3}{288n} \left(25M_j r_k^{2b} \operatorname{erfc}''\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) - 53\sqrt{2} r_k^b \operatorname{erfc}'\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) \right) \right] \right. \\
&\quad \left. + \frac{e^{-\frac{M_j^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \left(\frac{1}{3r_k^b \sqrt{n}} + \left[\frac{M_j}{12r_k^b} + \frac{5M_j^3 r_k^b}{18} \right] \frac{1}{n} \right) \right\} + \Omega_{k+1} \Big) + \mathcal{O}(n^{-1}) \\
&= \Sigma_1^{(n)} + \frac{1}{\sqrt{n}} \Sigma_2^{(n)} + \frac{1}{n} \Sigma_3^{(n)} + \mathcal{O}(n^{-1}), \tag{2.29}
\end{aligned}$$

as $n \rightarrow +\infty$, where

$$\begin{aligned}
\Sigma_1^{(n)} &= \sum_{j=g_{k,-}}^{g_{k,+}} \log \left(\frac{\omega_k}{2} \operatorname{erfc}\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) + \Omega_{k+1} \right), \\
\Sigma_2^{(n)} &= \sum_{j=g_{k,-}}^{g_{k,+}} \frac{\omega_k}{\frac{\omega_k}{2} \operatorname{erfc}\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) + \Omega_{k+1}} \left(\frac{1}{2} \operatorname{erfc}'\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) \frac{5M_j^2 r_k^b}{6\sqrt{2}} + \frac{e^{-\frac{M_j^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \frac{1}{3r_k^b} \right), \\
\Sigma_3^{(n)} &= \sum_{j=g_{k,-}}^{g_{k,+}} \left\{ \frac{\omega_k}{\frac{\omega_k}{2} \operatorname{erfc}\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) + \Omega_{k+1}} \left(\frac{M_j^3}{288} \left[25M_j r_k^{2b} \operatorname{erfc}''\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) - 53\sqrt{2} r_k^b \operatorname{erfc}'\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) \right] \right. \right. \\
&\quad \left. \left. + \frac{e^{-\frac{M_j^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \left[\frac{M_j}{12r_k^b} + \frac{5M_j^3 r_k^b}{18} \right] \right) - \frac{1}{2} \left[\frac{\omega_k}{\frac{\omega_k}{2} \operatorname{erfc}\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) + \Omega_{k+1}} \left(\frac{1}{2} \operatorname{erfc}'\left(-\frac{M_j r_k^b}{\sqrt{2}}\right) \frac{5M_j^2 r_k^b}{6\sqrt{2}} + \frac{e^{-\frac{M_j^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \frac{1}{3r_k^b} \right) \right]^2 \right\}.
\end{aligned}$$

By (2.20), the large n asymptotics of $\Sigma_1^{(n)}$, $\Sigma_2^{(n)}$ and $\Sigma_3^{(n)}$ are of the form

$$\begin{aligned}
\Sigma_1^{(n)} &= \Sigma_{1,2} \sqrt{n} + \Sigma_{1,3} + \frac{1}{\sqrt{n}} \Sigma_{1,4} + \mathcal{O}(M^4 n^{-1}), \\
\frac{1}{\sqrt{n}} \Sigma_2^{(n)} &= \Sigma_{2,3} + \frac{1}{\sqrt{n}} \Sigma_{2,4} + \mathcal{O}(n^{-1}), \quad \frac{1}{n} \Sigma_3^{(n)} = \frac{1}{\sqrt{n}} \Sigma_{3,4} + \mathcal{O}(n^{-1}),
\end{aligned}$$

where the coefficients $\Sigma_{1,2}, \Sigma_{1,3}, \Sigma_{1,4}, \Sigma_{2,3}, \Sigma_{2,4}, \Sigma_{3,4}$ are explicit (but we do not write them down).

After simplification (using for example $\operatorname{erfc}'(x) = -\frac{2e^{-x^2}}{\sqrt{\pi}}$), we obtain

$$\Sigma_{1,2} = \tilde{C}_{2,k}^{(M)}, \quad \Sigma_{1,3} + \Sigma_{2,3} = \tilde{C}_{3,k}^{(n,M)}, \quad \Sigma_{1,4} + \Sigma_{2,4} + \Sigma_{3,4} = \tilde{C}_{4,k}^{(n,M)}.$$

□

We are now in a position to compute the large n asymptotics of S_{2k} for $k \in \{1, \dots, m\}$.

Lemma 2.9. *For any $k \in \{1, \dots, m\}$ and any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned}
S_{2k} &= \sum_{j=j_{k,-}}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \left(j_{k,+} - br_k^{2b} n \right) \log \Omega_{k+1} \\
&\quad + \left(br_k^{2b} n - j_{k,-} \right) \log \Omega_k + C_{2,k} \sqrt{n} + C_{3,k} + \frac{1}{\sqrt{n}} C_{4,k} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$\begin{aligned}
C_{2,k} &= br_k^{2b} \left[\int_0^\infty \log \left(1 + \frac{\omega_k}{2\Omega_{k+1}} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt + \int_0^\infty \log \left(1 - \frac{\omega_k}{2\Omega_k} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt \right], \\
C_{3,k} &= \left(\frac{1}{2} + \alpha \right) \log \Omega_{k+1} + \left(\frac{1}{2} - \alpha \right) \log \Omega_k \\
&+ 2br_k^{2b} \left[\int_0^\infty t \log \left(1 + \frac{\omega_k}{2\Omega_{k+1}} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt - \int_0^\infty t \log \left(1 - \frac{\omega_k}{2\Omega_k} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt \right] \\
&+ br_k^{2b} \int_{-\infty}^\infty \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{tr_k^b}{\sqrt{2}} \right)} \left[\frac{1}{3r_k^b} - \frac{5t^2 r_k^b}{6} \right] \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} dt, \\
C_{4,k} &= 3br_k^{2b} \left[\int_0^\infty t^2 \log \left(1 + \frac{\omega_k}{2\Omega_{k+1}} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt + \int_0^\infty t^2 \log \left(1 - \frac{\omega_k}{2\Omega_k} \operatorname{erfc} \frac{tr_k^b}{\sqrt{2}} \right) dt \right] \\
&+ br_k^b \int_{-\infty}^\infty \frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{tr_k^b}{\sqrt{2}} \right)} \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \frac{t(42 - 193r_k^{2b}t^2 + 25r_k^{4b}t^4)}{72} dt \\
&- \frac{1}{2} br_k^{2b} \int_{-\infty}^\infty \left(\frac{\omega_k}{\Omega_{k+1} + \frac{\omega_k}{2} \operatorname{erfc} \left(\frac{tr_k^b}{\sqrt{2}} \right)} \left[\frac{1}{3r_k^b} - \frac{5t^2 r_k^b}{6} \right] \frac{e^{-\frac{t^2 r_k^{2b}}{2}}}{\sqrt{2\pi}} \right)^2 dt.
\end{aligned}$$

Proof. By combining Lemmas 2.5, 2.6 and 2.8, we have

$$\begin{aligned}
S_{2k} &= \sum_{j=j_{k,-}}^{j_{k,+}} \log \Gamma \left(\frac{j+\alpha}{b} \right) + \left(j_{k,+} - br_k^{2b}n \right) \log \Omega_{k+1} + \left(br_k^{2b}n - j_{k,-} \right) \log \Omega_k \\
&+ C_{2,k}^{(M)} \sqrt{n} + C_{3,k}^{(n,M)} + \frac{1}{\sqrt{n}} C_{4,k}^{(n,M)} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$\begin{aligned}
C_{2,k}^{(M)} &:= \tilde{C}_{2,k}^{(M)} - bMr_k^{2b} \left[\log \Omega_{k+1} + \log \Omega_k \right], \\
C_{3,k}^{(n,M)} &:= \tilde{C}_{3,k}^{(n,M)} + \left(-bM^2 r_k^{2b} + \alpha + \theta_{k,+}^{(n,M)} \right) \log \Omega_{k+1} + \left(bM^2 r_k^{2b} - \alpha + \theta_{k,-}^{(n,M)} \right) \log \Omega_k, \\
C_{4,k}^{(n,M)} &:= \tilde{C}_{4,k}^{(n,M)} - bM^3 r_k^{2b} \left[\log \Omega_{k+1} + \log \Omega_k \right].
\end{aligned}$$

By choosing M' sufficiently large, we obtain

$$C_{2,k}^{(M)} = C_{2,k} + \mathcal{O}(n^{-10}), \quad C_{3,k}^{(n,M)} = C_{3,k} + \mathcal{O}(n^{-10}), \quad C_{4,k}^{(n,M)} = C_{4,k} + \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$, which finishes the proof. $\square$

We now turn our attention to the asymptotic analysis of $S_{2m+2}^{(2)}$ and then of S_{2m+2} .

Lemma 2.10. *For any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_{2m+2}^{(2)} = \sum_{j=g_{m+1,-}}^n \log \Gamma \left(\frac{j+\alpha}{b} \right) + \tilde{C}_{2,m+1}^{(M)} \sqrt{n} + \tilde{C}_{3,m+1}^{(n,M)} + \frac{1}{\sqrt{n}} \tilde{C}_{4,m+1}^{(n,M)} + \mathcal{O}(M^4 n^{-1}),$$

$$\begin{aligned}
\tilde{C}_{2,m+1}^{(M)} &= \int_{-M}^{-\mathcal{R}} \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right) + \Omega_{m+2} \right) dt, \\
\tilde{C}_{3,m+1}^{(n,M)} &= \int_{-M}^{-\mathcal{R}} (2t + \mathcal{R}) \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right) + \Omega_{m+2} \right) dt \\
&\quad + \left(\frac{1}{2} - \theta_{m+1,-}^{(n,M)} \right) \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{M}{\sqrt{2b}} \right) + \Omega_{m+2} \right) + \left(\frac{1}{2} + \alpha \right) \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{-\mathcal{R}}{\sqrt{2b}} \right) + \Omega_{m+2} \right) \\
&\quad + \int_{-M}^{-\mathcal{R}} \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right)} \frac{2b - 3\mathcal{R}t - 5t^2}{6\sqrt{b}} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} dt, \\
\tilde{C}_{4,m+1}^{(n,M)} &= \int_{-M}^{-\mathcal{R}} t(3t + 2\mathcal{R}) \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right) + \Omega_{m+2} \right) dt \\
&\quad + \int_{-M}^{-\mathcal{R}} \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right)} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} \frac{t(42b^2 - 193bt^2 + 25t^4) + 6\mathcal{R}(2b^2 - 29bt^2 + 5t^4) - 9\mathcal{R}^2(3bt - t^3)}{72b^{3/2}} dt \\
&\quad - \frac{1}{2} \int_{-M}^{-\mathcal{R}} \left[\frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \left(\frac{t}{\sqrt{2b}} \right)} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} \frac{2b - 3\mathcal{R}t - 5t^2}{6\sqrt{b}} \right]^2 dt \\
&\quad + \left[\frac{1}{12} + \frac{\theta_{m+1,-}^{(n,M)}(\theta_{m+1,-}^{(n,M)} - 1)}{2} + \left(\frac{1}{2} - \theta_{m+1,-}^{(n,M)} \right) \frac{2b + 3\mathcal{R}M - 5M^2}{6} \right] \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{M}{\sqrt{2b}} \right)} \frac{e^{-\frac{M^2}{2b}}}{\sqrt{2\pi b}} \\
&\quad + \left(\left(\frac{1}{2} + \alpha \right) \frac{b - \mathcal{R}^2}{3} - \frac{1 + 6\alpha + 6\alpha^2}{12} \right) \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{\mathcal{R}}{\sqrt{2b}} \right)} \frac{e^{-\frac{\mathcal{R}^2}{2b}}}{\sqrt{2\pi b}},
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$.

Proof. The first part of the proof is identical to the beginning of the proof of Lemma 2.8. Namely, in the same way as for (2.29), we have

$$\begin{aligned}
S_{2m+2}^{(2)} &= \sum_{j: \lambda_j \in I_{2'}} \log \Gamma \left(\frac{j+\alpha}{b} \right) + \sum_{j=g_{m+1},-}^n \log \left(\omega_{m+1} \left[\frac{1}{2} \operatorname{erfc} \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) + \frac{1}{2} \operatorname{erfc}' \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) \right] \frac{5M_j^2 r_{m+1}}{6\sqrt{2}\sqrt{n}} \right. \\
&\quad \left. + \frac{M_j^3}{288n} \left(25M_j r_{m+1}^{2b} \operatorname{erfc}'' \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) - 53\sqrt{2} r_{m+1}^b \operatorname{erfc}' \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) \right) \right. \\
&\quad \left. + \frac{e^{-\frac{M_j^2 r_{m+1}^{2b}}{2}}}{\sqrt{2\pi}} \left(\frac{1}{3r_{m+1}^b \sqrt{n}} - \left[\frac{M_j}{12r_{m+1}^b} + \frac{5M_j^3 r_{m+1}^b}{18} \right] \frac{1}{n} \right) + \Omega_{m+2} \right) + \mathcal{O}(n^{-1}) \\
&= \sum_{j: \lambda_j \in I_{2'}} \log \Gamma \left(\frac{j+\alpha}{b} \right) + \Sigma_1^{(n)} + \frac{1}{\sqrt{n}} \Sigma_2^{(n)} + \frac{1}{n} \Sigma_3^{(n)} + \mathcal{O}(n^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1^{(n)} &= \sum_{j=g_{m+1},-}^n \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) + \Omega_{m+2} \right), \\
\Sigma_2^{(n)} &= \sum_{j=g_{m+1},-}^n \frac{\omega_{m+1}}{\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) + \Omega_{m+2}} \left(\frac{1}{2} \operatorname{erfc}' \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) \frac{5M_j^2 r_{m+1}^b}{6\sqrt{2}} + \frac{e^{-\frac{M_j^2 r_{m+1}^{2b}}{2}}}{\sqrt{2\pi}} \frac{1}{3r_{m+1}^b} \right), \\
\Sigma_3^{(n)} &= \sum_{j=g_{m+1},-}^n \left\{ \frac{\omega_{m+1}}{\frac{\omega_{m+1}}{2} \operatorname{erfc} \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) + \Omega_{m+2}} \left(\frac{M_j^3}{288} \left[25M_j r_{m+1}^{2b} \operatorname{erfc}'' \left(-\frac{M_j r_{m+1}^b}{\sqrt{2}} \right) \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -53\sqrt{2}r_{m+1}^b \operatorname{erfc}'\left(-\frac{M_j r_{m+1}^b}{\sqrt{2}}\right) \Bigg] + \frac{e^{-\frac{M_j^2 r_{m+1}^{2b}}{2}}}{\sqrt{2\pi}} \left[\frac{M_j}{12r_{m+1}^b} + \frac{5M_j^3 r_{m+1}^b}{18} \right] \Bigg) \\
& - \frac{1}{2} \left[\frac{\omega_{m+1}}{\frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{M_j r_{m+1}^b}{\sqrt{2}}\right) + \Omega_{m+2}} \left(\frac{1}{2} \operatorname{erfc}'\left(-\frac{M_j r_{m+1}^b}{\sqrt{2}}\right) \frac{5M_j^2 r_{m+1}^b}{6\sqrt{2}} + \frac{e^{-\frac{M_j^2 r_{m+1}^{2b}}{2}}}{\sqrt{2\pi}} \frac{1}{3r_{m+1}^b} \right) \right]^2 \Bigg\},
\end{aligned}$$

where, for conciseness, we have written M_j instead of $M_{j,m+1}$. Note that one cannot yet apply Lemma 2.7 because r_{m+1} depends on n (if $\mathcal{R} \neq 0$). Using $\operatorname{erfc}'(x) = -\frac{2e^{-x^2}}{\sqrt{\pi}}$, $\operatorname{erfc}''(x) = \frac{4xe^{-x^2}}{\sqrt{\pi}}$ and

$$r_{m+1}^b = \frac{1}{\sqrt{b}} + \frac{\mathcal{R}}{2\sqrt{b}} \frac{1}{\sqrt{n}} - \frac{\mathcal{R}^2}{8\sqrt{b}n} + \mathcal{O}(n^{-\frac{3}{2}}), \quad \text{as } n \rightarrow +\infty,$$

we obtain

$$\Sigma_1^{(n)} + \frac{1}{\sqrt{n}} \Sigma_2^{(n)} + \frac{1}{n} \Sigma_3^{(n)} = \tilde{\Sigma}_1^{(n)} + \frac{1}{\sqrt{n}} \tilde{\Sigma}_2^{(n)} + \frac{1}{n} \tilde{\Sigma}_3^{(n)} + \mathcal{O}(n^{-1}), \quad \text{as } n \rightarrow +\infty,$$

where

$$\begin{aligned}
\tilde{\Sigma}_1^{(n)} &= \sum_{j=g_{m+1,-}}^n \log \left(\frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{M_j}{\sqrt{2b}}\right) + \Omega_{m+2} \right), \\
\tilde{\Sigma}_2^{(n)} &= \sum_{j=g_{m+1,-}}^n \frac{\omega_{m+1}}{\frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{M_j}{\sqrt{2b}}\right) + \Omega_{m+2}} \frac{1}{2\sqrt{b}} \left(\frac{2b - 5M_j^2}{3} + \mathcal{R}M_j \right) \frac{e^{-\frac{M_j^2}{2b}}}{\sqrt{2\pi}}, \\
\tilde{\Sigma}_3^{(n)} &= \sum_{j=g_{m+1,-}}^n \left\{ \frac{\omega_{m+1}}{\frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{M_j}{\sqrt{2b}}\right) + \Omega_{m+2}} \left(\frac{6b^2 M_j + 73b M_j^3 - 25M_j^5 - (12b^2 + 42b M_j^2 - 30M_j^4) \mathcal{R}}{72b^{3/2}} \right. \right. \\
& \quad \left. \left. - \frac{9(bM_j + M_j^3) \mathcal{R}^2}{72b^{3/2}} \right) \frac{e^{-\frac{M_j^2}{2b}}}{\sqrt{2\pi}} - \frac{\omega_{m+1}^2}{\left(\frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{M_j}{\sqrt{2b}}\right) + \Omega_{m+2}\right)^2} \frac{(2b + 3M_j \mathcal{R} - 5M_j^2)^2}{72b} \frac{e^{-\frac{M_j^2}{2b}}}{2\pi} \right\}.
\end{aligned}$$

The large n asymptotics of $\tilde{\Sigma}_1^{(n)}$, $\tilde{\Sigma}_2^{(n)}$ and $\tilde{\Sigma}_3^{(n)}$ can now be obtained using (2.21), and by choosing M' sufficiently large, we obtain the claim after a computation and rearranging the terms. $\square$

Lemma 2.11. *For any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned}
S_{2m+2} &= \sum_{j=j_{m+1,-}}^n \log \Gamma\left(\frac{j+\alpha}{b}\right) + (n - j_{m+1,-}) \log \Omega_{m+1} \\
&+ C_{2,m+1} \sqrt{n} + C_{3,m+1} + \frac{1}{\sqrt{n}} C_{4,m+1} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$\begin{aligned}
C_{2,m+1} &= \int_0^\infty \log \left(1 - \frac{\omega_{m+1}}{2\Omega_{m+1}} \operatorname{erfc} \frac{t}{\sqrt{2b}} \right) dt + \mathcal{R} \log \frac{\Omega_{m+1}}{\Omega_{m+2}} + \int_0^{-\mathcal{R}} \log \left(1 + \frac{\omega_{m+1}}{2\Omega_{m+2}} \operatorname{erfc} \frac{t}{\sqrt{2b}} \right) dt, \\
C_{3,m+1} &= \left(\frac{1}{2} + \alpha \right) \log \left(\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc} \frac{-\mathcal{R}}{\sqrt{2b}} \right) + \left(\frac{1}{2} - \alpha \right) \log \Omega_{m+1} \\
&- \int_0^\infty (2t - \mathcal{R}) \log \left(1 - \frac{\omega_{m+1}}{2\Omega_{m+1}} \operatorname{erfc} \frac{t}{\sqrt{2b}} \right) dt + \int_0^{-\mathcal{R}} (2t + \mathcal{R}) \log \left(1 + \frac{\omega_{m+1}}{2\Omega_{m+2}} \operatorname{erfc} \frac{t}{\sqrt{2b}} \right) dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{-\infty}^{-\mathcal{R}} \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc}\left(\frac{t}{\sqrt{2b}}\right)} \frac{2b - 3\mathcal{R}t - 5t^2}{6\sqrt{b}} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} dt, \\
C_{4,m+1} &= \int_0^\infty (3t^2 - 2\mathcal{R}t) \log \left(1 - \frac{\omega_{m+1}}{2\Omega_{m+1}} \operatorname{erfc} \frac{t}{\sqrt{2b}}\right) dt + \int_0^{-\mathcal{R}} (3t^2 + 2\mathcal{R}t) \log \left(1 + \frac{\omega_{m+1}}{2\Omega_{m+2}} \operatorname{erfc} \frac{t}{\sqrt{2b}}\right) dt \\
& + \int_{-\infty}^{-\mathcal{R}} \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc}\left(\frac{t}{\sqrt{2b}}\right)} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} \frac{t(42b^2 - 193bt^2 + 25t^4) + 6\mathcal{R}(2b^2 - 29bt^2 + 5t^4) - 9\mathcal{R}^2(3bt - t^3)}{72b^{3/2}} dt \\
& - \frac{1}{2} \int_{-\infty}^{-\mathcal{R}} \left(\frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc}\left(\frac{t}{\sqrt{2b}}\right)} \frac{2b - 3\mathcal{R}t - 5t^2}{6\sqrt{b}} \frac{e^{-\frac{t^2}{2b}}}{\sqrt{2\pi}} \right)^2 dt \\
& + \left(\left(\frac{1}{2} + \alpha \right) \frac{b - \mathcal{R}^2}{3} - \frac{1 + 6\alpha + 6\alpha^2}{12} \right) \frac{\omega_{m+1}}{\Omega_{m+2} + \frac{\omega_{m+1}}{2} \operatorname{erfc}\left(-\frac{\mathcal{R}}{\sqrt{2b}}\right)} \frac{e^{-\frac{\mathcal{R}^2}{2b}}}{\sqrt{2\pi b}}.
\end{aligned}$$

Proof. By combining Lemmas 2.6 and 2.10, we have

$$\begin{aligned}
S_{2m+2} &= \sum_{j=j_{m+1,-}}^n \log \Gamma\left(\frac{j+\alpha}{b}\right) + (n - j_{m+1,-}) \log \Omega_{m+1} \\
&+ C_{2,m+1}^{(M)} \sqrt{n} + C_{3,m+1}^{(n,M)} + \frac{1}{\sqrt{n}} C_{4,m+1}^{(n,M)} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$, where

$$\begin{aligned}
C_{2,m+1}^{(M)} &= \tilde{C}_{2,m+1}^{(M)} + (\mathcal{R} - M) \log \Omega_{m+1}, \\
C_{3,m+1}^{(n,M)} &= \tilde{C}_{3,m+1}^{(n,M)} + (M^2 - M\mathcal{R} - \alpha + \theta_{k,-}^{(n,M)}) \log \Omega_{m+1}, \\
C_{4,m+1}^{(n,M)} &= \tilde{C}_{4,m+1}^{(n,M)} + (-M^3 + M^2\mathcal{R}) \log \Omega_{m+1}.
\end{aligned}$$

Note that S_{2m+2} is independent of M . By choosing M' sufficiently large, we obtain

$$C_{2,m+1}^{(M)} = C_{2,m+1} + \mathcal{O}(n^{-10}), \quad C_{3,m+1}^{(n,M)} = C_{3,m+1} + \mathcal{O}(n^{-10}), \quad C_{4,m+1}^{(n,M)} = C_{4,m+1} + \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$, which concludes the proof. $\square$

We now finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining (2.1) with Lemmas 2.1, 2.2, 2.9 and 2.11, we infer that for any $x_1, \dots, x_p \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned}
\log D_n &= S_{-1} + S_0 + \sum_{k=1}^m (S_{2k-1} + S_{2k}) + S_{2m+1} + S_{2m+2} \\
&= -\frac{1}{2b} n^2 \log n - \frac{1+2\alpha}{2b} n \log n + n \log \frac{\pi}{b} + M' \log \Omega_1 + \sum_{j=1}^{M'} \log \Gamma\left(\frac{j+\alpha}{b}\right) \\
&+ \sum_{k=1}^m \left\{ (j_{k,-} - j_{k-1,+} - 1) \log \Omega_k + \sum_{j=j_{k-1,+}+1}^{j_{k,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=j_{k,-}}^{j_{k,+}} \log \Gamma\left(\frac{j+\alpha}{b}\right) \right. \\
&\left. + (j_{k,+} - br_k^{2b} n) \log \Omega_{k+1} + (br_k^{2b} n - j_{k,-}) \log \Omega_k + C_{2,k} \sqrt{n} + C_{3,k} + \frac{1}{\sqrt{n}} C_{4,k} \right\}
\end{aligned}$$

$$\begin{aligned}
& + (j_{m+1,-} - j_{m,+} - 1) \log \Omega_{m+1} + \sum_{j=j_{m,+}+1}^{j_{m+1,-}-1} \log \Gamma\left(\frac{j+\alpha}{b}\right) + \sum_{j=j_{m+1,-}}^n \log \Gamma\left(\frac{j+\alpha}{b}\right) \\
& + \left(n - j_{m+1,-}\right) \log \Omega_{m+1} + C_{2,m+1} \sqrt{n} + C_{3,m+1} + \frac{1}{\sqrt{n}} C_{4,m+1} + \mathcal{O}(M^4 n^{-1})
\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_p \in \{z \in \mathbb{C} : |z - x_p| \leq \delta\}$. Recall that Z_n is given by (1.24). Hence, simplifying the above yields

$$\begin{aligned}
\log D_n &= \log Z_n + M \log \Omega_1 + \sum_{k=1}^m \left\{ \left(j_{k,+} - br_k^{2b} n \right) \log \Omega_{k+1} + \left(br_k^{2b} n - j_{k-1,+} - 1 \right) \log \Omega_k + C_{2,k} \sqrt{n} \right. \\
&+ \left. C_{3,k} + \frac{1}{\sqrt{n}} C_{4,k} \right\} + (n - j_{m,+} - 1) \log \Omega_{m+1} + C_{2,m+1} \sqrt{n} + C_{3,m+1} + \frac{1}{\sqrt{n}} C_{4,m+1} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

as $n \rightarrow +\infty$. Using (1.4), we then find

$$\begin{aligned}
\log \mathbb{E} \left[\prod_{j=1}^p e^{u_j N(D_{r_j})} \right] &= (br_1^{2b} n - 1) \log \Omega_1 + \sum_{k=2}^m (br_k^{2b} n - br_{k-1}^{2b} n - 1) \log \Omega_k \\
&+ (n - br_m^{2b} n - 1) \log \Omega_{m+1} + \sum_{k=1}^{m+1} \left\{ C_{2,k} \sqrt{n} + C_{3,k} + \frac{1}{\sqrt{n}} C_{4,k} \right\} + \mathcal{O}(M^4 n^{-1}),
\end{aligned}$$

which can be rewritten as (1.8) using (1.6), (1.26) and (2.5). This finishes the proof of Theorem 1.1. $\square$

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