# IDEAL EFFORTS AND CONSENSUS IN A MULTI-LAYER NETWORK GAME

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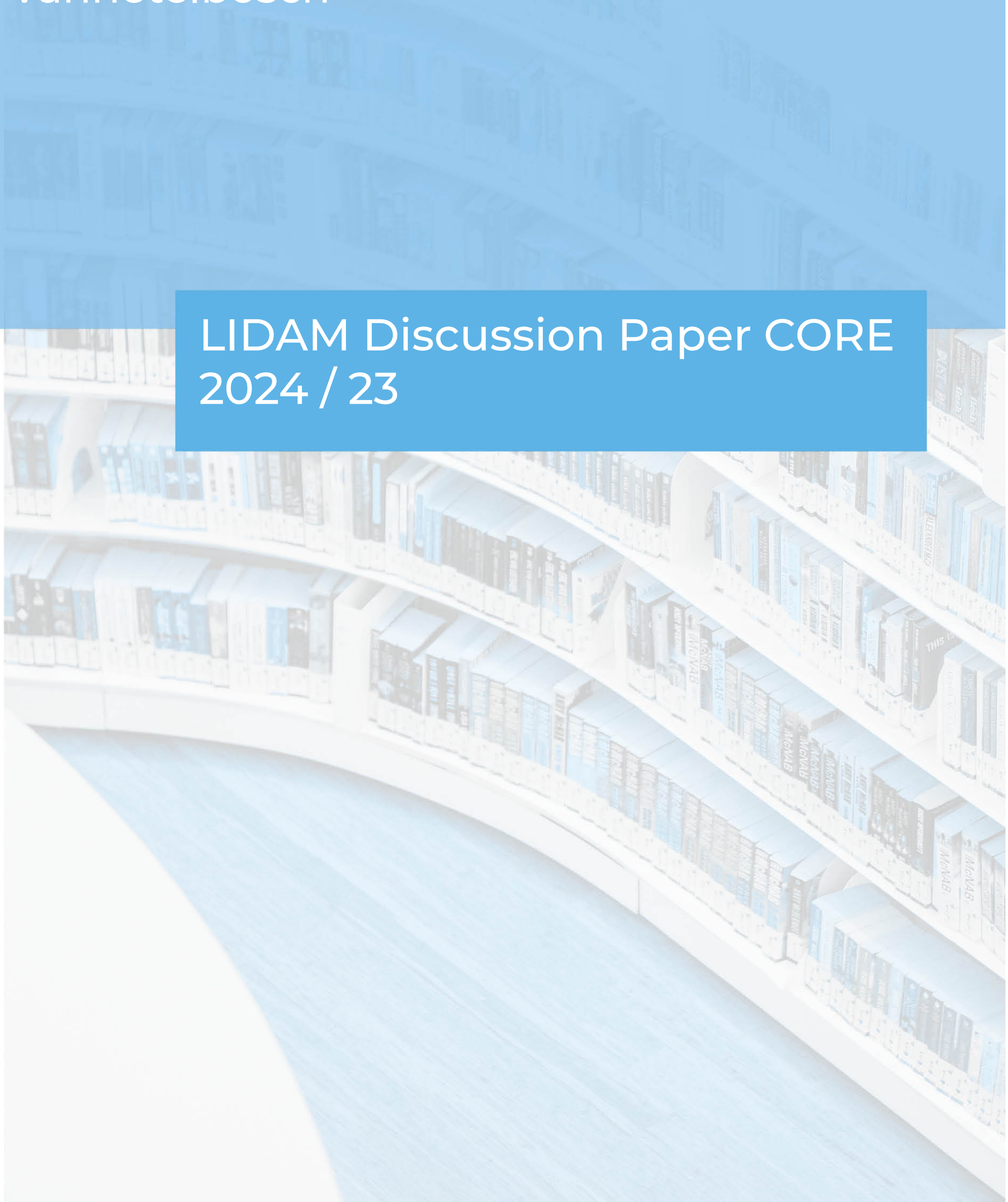

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# Ideal efforts and consensus in a multi-layer network game

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**Abstract**

We study a network game on a fixed multi-layer network of two types of relationships. The social interactions in the first layer carries a pressure to conform with the social norm within the layer. The second layer provides additional strategic complementarities from players' interaction. Players are endowed with personal ideal efforts and are heterogeneous in their ideal efforts and productivity. Each player repeatedly chooses her effort level in the network game and updates her ideal effort based on the new effort choice. Each player suffers disutility when her effort differs from her neighbors' efforts or is inconsistent with her ideal effort. We find the pure Nash equilibrium of the game in each period and provide conditions for the convergence of efforts and ideals to a steady state. Furthermore, we provide conditions for emerging long-run consensus about ideals in groups of players and the entire network.

**Keywords**: multi-layer networks · network games · personal norms · social norms · strategic complementarities

**JEL Classifications**: A14 · C72 · D85

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# 1 Introduction

Individual choices are often affected by the social environment.[1] The behaviors and the attitudes of social contacts affect their well-being, driving their choices and leading them to adapt or adopt their opinions leading to conformity.[2] However, individual beliefs on many issues vary significantly in society and individuals differ in their willingness to change those beliefs.[3] Despite the importance of a consensus in problems where coordination matters, the presence of diverse opinions about the issue is an important part in the evolution of the beliefs and the dynamics of individual choices.

Recent research has shown that studying the social and economic relations in the context of network interactions can provide valuable insights about group outcomes and individual choices.[4] The literature on network games provides a game-theoretic framework for analyzing the behavior of players embedded in a network in a variety of economic settings. The structure of the network, players' characteristics and economic incentives affect the individuals' choices.[5] The topology of the network impacts the efforts dynamics and learning in the population (see e.g Golub and Jackson (2010) or Azemoglu and Ozdaglar (2011)). Most of the network games commonly studied assume that individuals are embedded in a network structure representing one type of relationships.[6] However, very often individuals are engaged in multiple types of relationships that influence individual choices regarding a single issue. Individuals can belong to both social and professional networks, firms can be involved in several R&D collaborative relationships, and countries can sign multiple economic and/or military alliances. This raises an important question: When agents interact in several dimensions of a multi-layer network, how individuals'

[1] Behavioral spillovers and peer effects through social interaction are present in education, labor markets, environmental choices, crime, etc. For surveys on theory and empirical evidence see Durlauf (2004), Ioannides and Datcher Loury (2004), Ioannides and Ioannides (2012).

[2] See the pioneering work by Asch (1955) on opinions and social pressure.

[3] Bednar and Page (2007) define *behavioral stickiness* as one of the elements of cultural behavior and show that the individuals may not alter their behavior despite changes in incentives.

[4] There is a wide range of applications of network theory in social and economic problems, such as in labor markets (e.g. Calvó-Armengol (2004), Calvó-Armengol and Jackson (2004)), financial markets (e.g. Gale and Kariv (2007), Elliott, Golub and Jackson (2014)), R&D collaborations (e.g. Goyal and Moraga-Gonzalez (2001), Dawid and Hellmann (2014), Mauleon, Sempere-Monerris and Vannetelbosch (2023)), as well as peer effects in networks (see the recent survey by Bramoullé, Djebbari and Fortin (2020)), opinion diffusion (e.g. Golub and Jackson (2010), Jackson and Yariv (2011)), adoption of environmentally friendly behavior (Currarini, Marchiori and Tavoni (2016)) and technology adoption (e.g. Beaman, BenYishay, Magruder and Mobarak (2021)).

[5] See Jackson and Zenou (2015) for a comprehensive introduction to network games.

[6] Few exceptions are the papers of Belhaj and Deroïan (2014) and Chen, Zenou and Zhou (2018) that capture multiple activities on a single-layer network, and Walsh (2019) who studies a public good game on a two-layer network where players choose to invest in one of the layers.

choices are affected by the presence of spillovers, social norms and peer pressure in the network? In this paper we study, in the presence of a multi-layer network of interactions, the consequences of social norms and conformity on the long run consensus of ideals in the society and on individuals' efforts.

We analyze a network game with social and personal norms regarding the effort choice among a population of players embedded in a two-layer network and exerting costly efforts. The two layers of the network influence differently the individual's choice of effort. The presence of social interactions in the first layer carries a pressure to conform with the social norm within the layer. The second layer represents a network where the interaction with neighbors generates strategic complementarity of efforts. Players have personal norms related to the effort choice, or *ideal efforts*,[7] and they are heterogeneous in their ideal efforts and productivity. Players repeatedly participate in a network game and suffer disutility when their efforts are different from those exerted by their neighbors. This disutility can be interpreted as a conflict cost for non-compliance with the social norm.[8] In addition, players incur disutility when there is a discrepancy between their beliefs about the ideal effort and their actual efforts. This disutility can be described as cognitive dissonance or inner conflict.[9] The novelty of this paper is that players suffer from these two different types of disutilities in a two-layer network where the interaction with neighbors generates strategic complementarity of efforts in the second layer.

Apart from individual efforts influencing the decisions of other players through spillovers, the efforts of all players in a network are additionally affected by their idiosyncratic productivities and ideal efforts, their position in the network and the costs from miscoordination in the first-layer network and from the inconsistencies with their ideal efforts. However, players are myopic in the sense that they do not consider how their effort choice affects those of other players in the network. We derive the unique Nash equilibrium of the network game. We find that the equilibrium efforts of the players are proportional to their weighted Katz-Bonacich centrality in the combined network of two-layers,[10] that is,

[7]The notion of ideal efforts in network games is used in recent works by Olcina, Panebianco and Zenou (2024) and Galeotti, Golub, Goyal and Rao (2021).

[8]That is, individuals are penalized if they deviate from the effort level of their network in the first network layer. See e.g. Ballester, Calvó-Armengol and Zenou (2010), Patacchini and Zenou (2012), Liu, Patacchini and Zenou (2014), Boucher (2016), Atay, Mauleon, Schopohl and Vannetelbosch (2023) about peer effects and conformism in social networks.

[9]The *Theory of Cognitive Dissonance* introduced by social psychologists dates back to Festinger (1957). It suggests that people strive for internal consistency continuously aligning the cognitions with their actions in order to minimize the cognitive dissonance. Experimental work by Elliot and Devine (1994) proves this result. Akerlof and Dickens (1982) and Rabin (1994) incorporate the psychological theory of cognitive dissonance into theoretical economic models of rational choice. They show that such approach provides a better explanation of the welfare consequences of some economic phenomena.

[10]This network centrality measure, introduced first by Katz (1953) and reestablished later by Bonacich

the network centrality measure indicating the weighted number of paths in the network stemming from a given player, additionally weighted by idiosyncratic productivities and ideal efforts of each player on the paths.[11]

We assume that the norms, both social and personal, tend to change. After each period network game, players update their ideal efforts by adapting those to their actual efforts in the network game. The ideal effort is updated as a linear combination of the ideal effort in the previous period and the actual effort. This form of updating implies that the players adjust their ideals to justify their actions and to manage their cognitive dissonance.[12] Such linear updating method is most similar to the well-known DeGroot updating introduced by DeGroot (1974) and widely used in opinion formation and social learning models. Compared to DeGroot updating where the beliefs are revised according to the social norm, in our model the ideal effort is updated according to the actual effort that is, in turn, a linear combination of social norm, the current ideal effort of the player, her productivity, and effort complementarities from the additional network layer. A similar approach to DeGroot updating has been adopted by DeMarzo, Vayanos and Zwiebel (2003) who define the notion of persuasion bias as a failure to adjust for possible repetitions of information they receive and instead treat all information as new. Correctly adjusting for repetitions of information would become extremely complicated when individuals interact repeatedly on complex social networks.[13] Such bounded rationality of players is assumed in most non-Bayesian learning models.

Similarly to subjective beliefs, we assume that ideal efforts are a subjective measure idiosyncratic to the players. We study the dynamics of ideal efforts given the multi-layer network structure, the taste for conformity, the sensitivity to cognitive dissonance in the network, and players' productivity coefficients. We show that under certain conditions on the network layer of strategic complementarities, the steady-state exists. Moreover, in presence of structural equivalence of players in the given two-layer network (i.e., players similar in their relational patterns and neighborhoods), the ideals of players converge to a consensus. Referring to ideal efforts as to an individual subjective matter, we refrain from studying the correctness of it. Thus, the consensus over the ideals in this setting

(1987), captures the power or the status as a relative degree of influence of the player in the network.

[11]Following the seminal work by Ballester, Calvó-Armengol and Zenou (2006), the link between the Katz-Bonacich centrality of a player and her effort is an established result in the literature on network games.

[12]An early experimental study by Brehm (1956) shows that people attempt to increase the desirability of their made choices to stay consistent with their decisions. In psychology literature such ex-post rationalization of decisions and choice-supportive bias is present in experimental results (see e.g. Mather and Johnson (2000), Mather, Shafir and Johnson (2000)).

[13]See DeMarzo, Vayanos and Zwiebel (2003) for a detailed discussion of this assumption and Choi, Gale and Kariv (2012) for experimental evidence.

represents the possibility of coordination in the network, rather than the "correctness" of learning and convergence to true action. A number of works have studied the ability of correct aggregation of information and conditions to converge to the truth. Golub and Jackson (2010) characterize the population and the learning process as wise if, given the existence of consensus, the influence of the most influential player vanishes as the network grows. Bala and Goyal (1998) study a non-Bayesian model of social learning in the context of technology adoption and show that the structure of the network has important implications in the adoption of new technologies, and their diffusion in society. Foerster, Mauleon and Vannetelbosch (2016) show that manipulation can connect a segregated society and thus lead to mutual consensus in a model of boundedly rational opinion dynamics. Acemoglu and Ozdaglar (2011) provide an overview of works on opinion dynamics and social learning. In this paper, we study the dynamics of ideal efforts and we show that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a consensus and the steady state. The sensitivity to cognitive dissonance, that captures the behavioral stickiness of the players, affects the speed of convergence to a steady-state but does not change the long-run efforts. The taste for conformity, on the other hand, affects the steady-state efforts' levels.

Finally, this paper is closely related to the recent work by Olcina, Panebianco and Zenou (2024) on the long-run integration behavior of ethnic minorities. They study a network game in which there is a conflict between an individual's integration choice and that of her friends and between an individual's assimilation choice and that of her family and community. They show that the key determinant of the different integration choices is their position in the network and the initial preferences of the persons they are connected to.[14] Similarly, we study a network game where the economic incentives, together with the ideal efforts and the social norms, affect the effort choice of the players. The novelty of this paper is that we allow players to be embedded in a two-layer network.[15] We show that the layer of network complementarities affects the structural similarities of players in the network providing conditions for the consensus in the ideal efforts of the players.

The paper is organized as follows. In Section 2 we describe the network spillover game with conformism and private dissonance and we characterize the unique Nash equilibrium efforts of the network game. In Section 3 we characterize the convergence to the steady-state and the consensus. Some numerical examples are used to study the dynamics of the ideal efforts. In Section 4 we conclude the paper. The proofs not given in the main text

[14]Recently, Luo, Mauleon and Vannetelbosch (2024) analyze how the presence of farsighted individuals can affect the long-run integration outcome and under which circumstances this can destabilize segregation in a friendship network formation model with two different communities.

[15]Joshi, Mahmud and Sarangi (2020) and Billand, Bravard, Joshi, Mahmud and Sarangi (2023) propose two different models on the endogenous formation of multilayer networks.

are provided in the appendix.

# 2 Model

## 2.1 The network game

We have a set $N = \{1, 2, ..., n\}$ of players embedded in a multi-layer network of social influences. The network consists of two layers, where one layer represents the network $\boldsymbol{g}$ of social connections, and the other is the network of complementarities $\boldsymbol{l}$. The network $\boldsymbol{g}$ is a directed weighted network described by the weighted adjacency matrix $\mathbf{G}^*$ with elements $g^*_{ij}$ for any neighbors $i, j \in N$ in the network $\boldsymbol{g}$. The weight $g^*_{ij}$ on the link $i \to j$ is normalized by the degree of player $i$ on $\boldsymbol{g}$, such that $\sum\limits_{j \in N} g^*_{ij} = 1$ for all $i \in N$. The network $\boldsymbol{l}$ is represented by the matrix $\mathbf{\Lambda}$ with elements $\lambda_{ij} > 0$ as coefficients of effort complementarity from player $j$ given the existence of a directed link $i \to j$ in the network $\boldsymbol{l}$ for all $i, j \in N$. By convention, we assume there are no self-loops in the network, i.e. $g^*_{ii} = 0$ and $\lambda_{ii} = 0$ for all $i \in N$.

Players are initially endowed with *personal ideals*, or *personal norms*, $y^0_i$.[16] Each period $t$ they choose their efforts $x^t_i$ based on the network game on the layers $\boldsymbol{g}$ and $\boldsymbol{l}$. The choice of effort for each player $i$ is based on her idiosyncratic productivity $\theta_i$, complementarities from their neighbors in the network $\boldsymbol{l}$, the cost from miscoordination due to the difference between her effort and those of each of her neighbors in the network $\boldsymbol{g}$,[17] as well as the cost from the discrepancy between her actual effort $x^t_i$ and her *ideal behavior* $y^t_i$.[18] After each period network game, players update their ideal efforts as a linear combination of their ideal efforts and their actual efforts, with a speed of updating $\xi \in [0; 1]$.

The timing of the model is the following for each period $t$.

1. *Network game and effort selection.* Players $i \in N$ choose the best response efforts $x^t_i$ in the two-layer network game by maximizing the following utility:

$$u^t_i(x) = \theta_i x^t_i + x^t_i \sum_{j \in N} \lambda_{ij} x^t_j - \underbrace{\frac{\omega_1}{2} \sum_{j \in N} g^*_{ij} (x^t_i - x^t_j)^2}_{\text{Conformism or conflict}} - \frac{\omega_2}{2} (x^t_i)^2 - \underbrace{\frac{\omega_3}{2} (x^t_i - y^t_i)^2}_{\substack{\text{Consistency or} \\ \text{cognitive dissonance}}}, \quad (1)$$

$$x^t_i = BR_i(\mathbf{x}^t_{-i}; \mathbf{G}^*, \mathbf{\Lambda}, y^t_i, \theta_i, \omega_1, \omega_2, \omega_3),$$

[16] The ideal efforts can be perceived as a personal norm of a player, personal standard, an attitude towards the specific issue, feeling of moral obligation, or self-expectations (Ajzen and Fishbein, 1970; Schwartz, 1973; Schwartz and Howard, 1980).

[17] Minimization of this cost of conflict leads to conformism similarly to conventional local-average models with descriptive social norms.

[18] In network games literature, the notion of ideal efforts and consistency is present in works by Olcina, Panebianco and Zenou (2024) and Galeotti, Golub, Goyal and Rao (2021).

where $\mathbf{x}_{-i}^t$ is the list of efforts of players $N \setminus i$.

2. *Updating ideal efforts.* Players update their ideal efforts based on the actual effort $x_i^t$ and ideal effort $y_i^t$ of the current period:

$$y_i^{t+1} = \xi x_i^t + (1-\xi) y_i^t. \tag{2}$$

The coefficients $\omega_1$, $\omega_2$ and $\omega_3$ in the utility of player $i$ in (1) are the cost coefficients of the model. The difference in efforts of neighbors in the network $\boldsymbol{g}$ is amplified by the coefficient $\omega_1$, that is, the cost of conflict. The cost of conflict may also be regarded as a taste for conformity. The cost of unit effort is given by the coefficient $\frac{\omega_2}{2}$. In addition, $\omega_3$ is the cost of moral conflict or sensitivity to cognitive dissonance. When the cost of conflict $\omega_1$ is high, the disutility from miscoordination in the network $\boldsymbol{g}$ is high, forcing the neighbors to coordinate around social norm faster, resulting in conformity. If the sensitivity to cognitive dissonance $\omega_3$ is high, the players remain consistent with their ideal efforts longer.[19]

## 2.2 Equilibrium

Solving for the first-order conditions on the utility (1) of player $i$, we find her best response effort in period $t$:

$$x_i^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left( \theta_i + \omega_3 y_i^t + \sum_{j \in N} \lambda_{ij} x_j^t + \omega_1 \sum_{j \in N} g_{ij}^* x_j^t \right).$$

One can see that the best response of the player is the linear combination of her productivity $\theta_i$, the ideal effort $y_i^t$, complementarities $\sum\limits_{j \in N} \lambda_{ij} x_j^t$ from the network $\boldsymbol{l}$, and the social norm $\sum\limits_{j \in N} g_{ij}^* x_j^t$ in the network $\boldsymbol{g}$.

Let $\mathbf{x}^t$, $\mathbf{y}^t$ be the vectors of efforts and ideal efforts in period $t$ respectively, and let $\boldsymbol{\theta}$ denote the vector of idiosyncratic elements $\theta_i$, for all $i \in N$. Then, the best response in matrix form will be given by:

$$\mathbf{x}^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left( \boldsymbol{\theta} + \omega_3 \mathbf{y}^t + \boldsymbol{\Lambda} \mathbf{x}^t + \omega_1 \mathbf{G}^* \mathbf{x}^t \right). \tag{3}$$

We characterize the conditions for the existence of unique Nash equilibrium in the following proposition.

[19]The cost of private dissonance indicates the level of flexibility of the players. This leaves room for extending the model by introducing heterogeneity in players' types. Particularly, dividing the players into stubborn (or persistent) and flexible agents, with the first group having higher coefficient $\omega_3$ compared to the flexible agents, and thus *sticky* ideals.

**Proposition 1.** *Assuming the spectral radius of* $\mathbf{\Lambda}+\omega_1\mathbf{G}^*$ *is smaller than* $(\omega_1+\omega_2+\omega_3)$, *the unique Nash equilibrium in pure strategies is then given by:* [20]

$$\mathbf{x}^t = \frac{1}{\omega_1+\omega_2+\omega_3}\left(\mathbf{I} - \frac{1}{\omega_1+\omega_2+\omega_3}(\mathbf{\Lambda}+\omega_1\mathbf{G}^*)\right)^{-1}(\boldsymbol{\theta}+\omega_3\mathbf{y}^t).$$

One can see that the unique Nash equilibrium of the game is proportional to the weighted Katz-Bonacich centrality of the players on a network represented by the matrix $(\mathbf{\Lambda}+\omega_1\mathbf{G}^*)$ and weighted by the linear combination of productivites and ideal efforts at time $(\boldsymbol{\theta}+\omega_3\mathbf{y}^t)$.[21] The matrix $(\mathbf{\Lambda}+\omega_1\mathbf{G}^*)$ represents a network formed by projecting the links of the layer $\boldsymbol{l}$ on $\boldsymbol{g}$, where the adjacency matrix of the layer $\boldsymbol{g}$ is weighted by the coefficient of conflict cost.

To be able to characterize the dynamics of ideal efforts by the multi-layer network representation, we follow Olcina, Panebianco and Zenou (2024) and we define the following vectors of length $2n$.

$$\hat{\mathbf{x}}^t := \left[\begin{array}{c}\boldsymbol{\theta}\\ \hline \mathbf{x}^t\end{array}\right] \qquad \hat{\mathbf{y}}^t := \left[\begin{array}{c}\boldsymbol{\theta}\\ \hline \mathbf{y}^t\end{array}\right]$$

The vectors $\hat{\mathbf{x}}^t$ and $\hat{\mathbf{y}}^t$ are constructed using the vector $\boldsymbol{\theta}$ as first $n$ elements, and vectors $\mathbf{x}^t$ and $\mathbf{y}^t$ as the second half of the respective vector. Additionally, we define the following matrix.

$$\bar{\mathbf{G}} := \frac{1}{\omega_1+\omega_2}\left[\begin{array}{c|c}(\omega_1+\omega_2)\mathbf{I} & \mathbf{O}\\ \hline \mathbf{I} & (\mathbf{\Lambda}+\omega_1\mathbf{G}^*)\end{array}\right] \tag{4}$$

Using the definitions above and the equation (3) we can write the augmented vector of efforts $\hat{\mathbf{x}}^t$ as the weighted average of $\hat{\mathbf{y}}^t$ and the social norm in the network described by the augmented adjacency matrix $\bar{\mathbf{G}}$.

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1+\omega_2+\omega_3}\hat{\mathbf{y}}^t + \frac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}\bar{\mathbf{G}}\hat{\mathbf{x}}^t$$

One can notice, that the first $n$ elements of the vector $\hat{\mathbf{x}}^t$, the vector $\boldsymbol{\theta}$, does not change over time and does not carry any economic meaning. While the elements from $n+1$ to $2n$ correspond to (3). This allows us to rewrite the equilibrium efforts and characterize new conditions for equilibrium existence.

[20]The spectral radius of a matrix is its largest absolute eigenvalue. Debreu and Herstein (1953) show that for a square matrix $A$, $(sI-A)^{-1}$ is well-defined and non-negative if and only if the spectral radius of $A$ is smaller than $s$. The link between the equilibrium efforts of the network game and the Katz-Bonacich centralities has been first established in the seminal work by Ballester, Calvó-Armengol and Zenou (2006).

[21]E.g. see Remark 1 in Ballester, Calvó-Armengol and Zenou (2006).

**Proposition 2.** *Let $\delta_1$ be the spectral radius of the matrix $\bar{\mathbf{G}}$. If $\delta_1 < \dfrac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$, then the unique Nash equilibrium in pure strategies is given by:*

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1+\omega_2+\omega_3}\left(\mathbf{I} - \frac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}\bar{\mathbf{G}}\right)^{-1}\hat{\mathbf{y}}^t.$$

One can see that 1 is always an eigenvalue of $\bar{\mathbf{G}}$ by construction, whereas for the parameter values satisfying $\omega_2 \geq 1 + \max\limits_i \sum\limits_{j\in N} \lambda_{ij}$ it is also the spectral radius.

Lemma 1 provides conditions on parameter values and combinations that ensure the equilibrium existence in Proposition 2.

Let's introduce the following notations:

$$\overline{\omega}_2 \equiv \max_i \sum_{j\in N} \lambda_{ij},$$
$$\overline{\overline{\omega}}_2 \equiv \max_j \sum_{i\in N\backslash j} (\lambda_{ij} + \omega_1 g^*_{ij}) - \omega_1 - 1.$$

**Lemma 1.** *Let $\delta_1$ be the spectral radius of the matrix $\bar{\mathbf{G}}$. The condition $\delta_1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ is satisfied if one of the conditions below holds:*

1. *Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\overline{\omega}}_2$ and $\omega_3 > \overline{\overline{\omega}}_2 + 1 - \omega_2$;*
2. *Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2)$ and $\omega_3 > 1$;*
3. *Given $\overline{\omega}_2 \leq \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\overline{\omega}}_2$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;*
4. *$\omega_2 \in [\overline{\omega}_2; 1 + \overline{\omega}_2)$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;*
5. *$\omega_2 \geq 1 + \overline{\omega}_2$ and $\omega_3 > 0$.*

Figure 1 illustrates the conditions provided in Lemma 1. The combination of parameter values of $\omega_2$ and $\omega_3$ in the shaded area above the black line in Figure 1 ensures the existence of equilibrium given that $\max\limits_i \sum\limits_{j\in N} \lambda_{ij} > \max\limits_j \sum\limits_{i\in N\backslash j} (\lambda_{ij} + \omega_1 g_{ij}) - \omega_1 - 1$. The shaded area above the green line in the figure displays the satisfying parameter combinations, when the opposite is true.

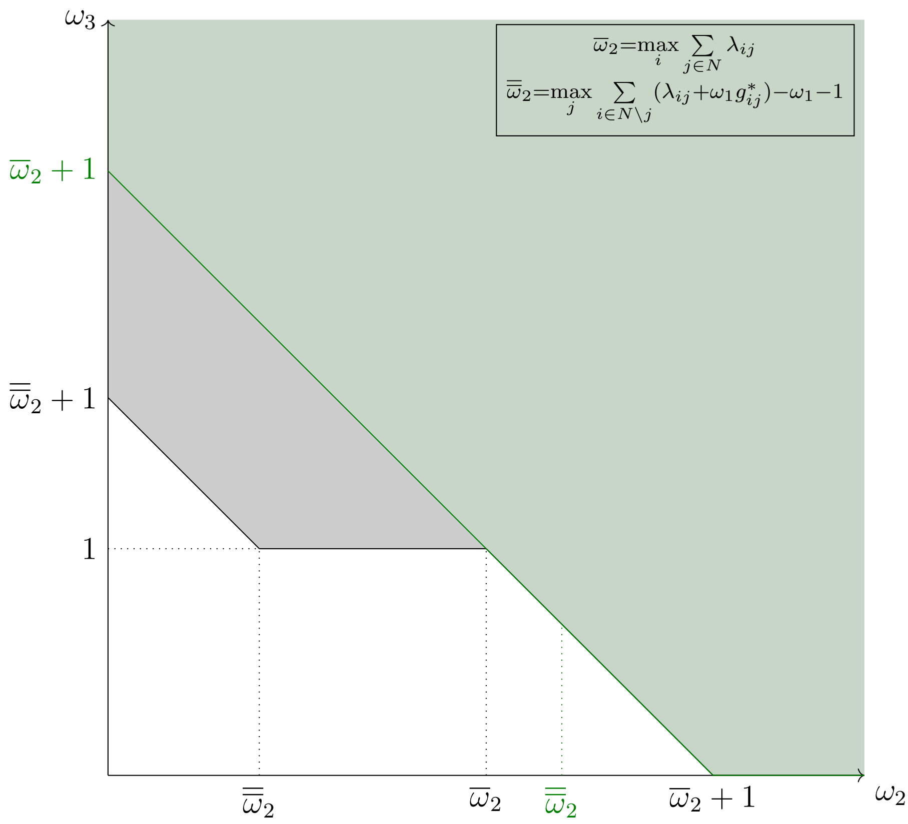



Figure 1: The shaded area covers the combination of parameter values according to the conditions on $\bar{\mathbf{G}}$ provided by Lemma 1.

When the coefficients of cost and cognitive dissonance are too low relative to complementarities in the network, the positive feedback loops in players' efforts escalate infinitely, so the Nash equilibrium does not exist. Thus, the cost coefficients have to be high enough to curb the complementarities in the network measured by the spectral radius of the matrix $\bar{\mathbf{G}}$. The combinations of parameters in Figure 1 ensure this requirement and the existence of Nash equilibria.

# 3 Dynamics and Convergence of Ideal Efforts

## 3.1 Updating Ideal Efforts

Let $\phi = \dfrac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3}$. We can then rewrite the augmented vector of equilibrium efforts in Proposition 2 as follows:

$$\hat{\mathbf{x}}^t = (1 - \phi) \left(\mathbf{I} - \phi \bar{\mathbf{G}}\right)^{-1} \hat{\mathbf{y}}^t. \tag{5}$$

Similarly to $\hat{\mathbf{x}}^t$, in the stage of updating the ideal efforts, we can apply the updating rule on the augmented vector $\hat{\mathbf{y}}^t$. The first $n$ elements of $\hat{\mathbf{y}}^t$ remain unchanged and equal to $\boldsymbol{\theta}$ over time, while the elements $n+1$ to $n$ correspond to ideal efforts $\mathbf{y}^t$ in time period $t$.

$$\hat{\mathbf{y}}^{t+1} = \xi\hat{\mathbf{x}}^t + (1-\xi)\hat{\mathbf{y}}^t \tag{6}$$

Plugging $\hat{\mathbf{x}}^t$ in (5) into equation (6) we have:

$$\hat{\mathbf{y}}^{t+1} =[\xi(1-\phi)\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1} + (1-\xi)\mathbf{I}]\hat{\mathbf{y}}^t.$$

The transition to a new ideal effort depends on player's current ideal effort and the equilibrium effort. The latter is determined by her location in the network, that is, her weighted Katz-Bonacich centrality.

### 3.2 Convergence and Consensus

Let us define the following matrices:

$$\mathbf{M} := \left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1} \tag{7}$$

$$\mathbf{T} := \xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I} \tag{8}$$

We can rewrite the ideal efforts using the notations in (8).

$$\hat{\mathbf{y}}^{t+1} = \mathbf{T}\hat{\mathbf{y}}^t = \mathbf{T}^{t+1}\hat{\mathbf{y}}^0 \tag{9}$$

Given the initial ideal efforts $\hat{\mathbf{y}}^0$ and the matrix $\mathbf{T}$, we can find the ideal efforts at any time period $t$, as well as the limit beliefs $\hat{\mathbf{y}}^\infty$. Proposition 4 provides the conditions for existence of the limit ideal efforts. Moreover, it shows that in order to find the ideal and equilibrium efforts in the limit it is sufficient to use the augmented matrix of weights $\bar{\mathbf{G}}$ instead of $\mathbf{T}$. To do so, we first define the relationship between the corresponding eigenvalues of the two matrices.

**Proposition 3.** *Consider the distinct and ordered sets of eigenvalues $\delta_i$ of $\bar{\mathbf{G}}$, and $\tau_i$ of $\mathbf{T}$. Then for all $i \in [1,n]$:*

$$\tau_i = \xi\frac{1-\phi}{1-\phi\delta_i} + (1-\xi).$$

One can notice that if $\sum\limits_{j\in N}\lambda_{ij} = \omega_2 - 1$ for all $i \in N$, then the matrix of weights $\bar{\mathbf{G}}$, and thus, the matrix $(1-\phi)\mathbf{M}$ are row-normalized matrices.[22] As a result, the matrix $\mathbf{T}$ is row-normalized as a convex combination of a row-stochastic matrix and an identity. With this setting, the dynamics described in (9) is a time-homogeneous Markov process with transition matrix $\mathbf{T}$.

[22]For a row-normalized $\bar{\mathbf{G}}$ and $\phi < 1$, the matrix $\mathbf{M} = \sum\limits_{k=0}^{\infty}\phi^k\bar{\mathbf{G}}^k$ is converging Neumann series with sum of elements in each row equal to $\frac{1}{1-\phi}$. Therefore, the matrix $(1-\phi)\mathbf{M}$ is row-normalized.

**Proposition 4.** *For a given network with layers* $\boldsymbol{g}$ *and* $\boldsymbol{l}$ *such that* $\max_i \sum_{j \in N} \lambda_{ij} \leq \omega_2 - 1$, *and* $\bar{\mathbf{G}}$ *is diagonalizable, there exist* $\bar{\mathbf{G}}^\infty = \lim_{t \to \infty} \bar{\mathbf{G}}^t$ *and* $\mathbf{T}^\infty = \lim_{t \to \infty} \mathbf{T}^t$ *such that:*

$$\hat{\mathbf{y}}^\infty = \mathbf{T}^\infty \hat{\mathbf{y}}^0 = \bar{\mathbf{G}}^\infty \hat{\mathbf{y}}^0$$

*and*

$$\hat{\mathbf{x}}^\infty = (1 - \phi) \left(\mathbf{I} - \phi \bar{\mathbf{G}}\right)^{-1} \bar{\mathbf{G}}^\infty \hat{\mathbf{y}}^0.$$

A special case fulfilling the conditions Proposition 4 are the networks with undirected layers $\boldsymbol{g}$ and $\boldsymbol{l}$ described by symmetric matrices $\mathbf{G}^*$ and $\Lambda$. This makes the augmented matrix $\bar{\mathbf{G}}$ symmetric and thus diagonalizable. Therefore, for symmetric matrices $\mathbf{G}^*$ and $\Lambda$, the existence of limit ideal efforts and actions depends on the maximum row-sum norm of the matrix of weights $\Lambda$.

It is easy to show that matrices $\bar{\mathbf{G}}$ and $\mathbf{T}$ are commuting matrices, that is $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.[23] It is known that commuting matrices share the same set of eigenvectors. Using this fact, together with the assumption on diagonalizability of $\bar{\mathbf{G}}$, we can rewrite the two matrices as follows.

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \qquad \text{and} \qquad \mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1},$$

where $\mathbf{E}$ is the matrix of eigenvectors of $\bar{\mathbf{G}}$ and $\mathbf{T}$ as columns, $\boldsymbol{\Delta}$ is the diagonal matrix of eigenvalues of $\bar{\mathbf{G}}$, and $\boldsymbol{\Sigma}$ is the diagonal matrix of eigenvalues of $\mathbf{T}$. Thus, there exists a transformation $P(.)$ such that $P(\boldsymbol{\Delta}) = \boldsymbol{\Sigma}$ [24] given which, we can find $\hat{\mathbf{y}}^\infty$:

$$\boldsymbol{\Sigma}^\infty = \lim_{t \to \infty} \boldsymbol{\Sigma}^t = \lim_{t \to \infty} P(\boldsymbol{\Delta})^t$$

$$\hat{\mathbf{y}}^\infty = \mathbf{E}\boldsymbol{\Sigma}^\infty \mathbf{E}^{-1} \hat{\mathbf{y}}^0.$$

Given the convergence of $\bar{\mathbf{G}}$ in Proposition 4 we can find the steady-state equilibrium efforts of the players.

$$\mathbf{x}^\infty = \mathbf{y}^\infty = \frac{1}{\omega_1 + \omega_2} \left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2} (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\right)^{-1} \boldsymbol{\theta} \tag{10}$$

Indeed, when $\hat{\mathbf{y}}^{t+1} = \hat{\mathbf{y}}^t = \hat{\mathbf{y}}^\infty$, the equilibrium efforts are identical to the ideals, $\hat{\mathbf{y}}^\infty = \hat{\mathbf{x}}^\infty$. From equilibrium efforts in (5) in the steady state we have

$$\hat{\mathbf{x}}^\infty = (1 - \phi) \left(\mathbf{I} - \phi \bar{\mathbf{G}}\right)^{-1} \hat{\mathbf{x}}^\infty \qquad \Longrightarrow \qquad \hat{\mathbf{x}}^\infty = \bar{\mathbf{G}}\hat{\mathbf{x}}^\infty.$$

[23] See the proof of Step 1 in the proof of Proposition 4 in the Appendix.

[24] E.g. see Zhang (2011), Chapter 3, Theorem 3.1.

Recalling the definition of $\bar{\mathbf{G}}$ in (4) we find the steady-state equilibrium efforts of players.

$$\mathbf{x}^{\infty} = \frac{1}{\omega_1 + \omega_2} \left( \boldsymbol{\theta} + \left( \boldsymbol{\Lambda} + \omega_1 \mathbf{G}^* \right) \mathbf{x}^{\infty} \right)$$

Assume there exists a row vector $\mathbf{e}^{\intercal}$, such that $y_i^{\infty} = \mathbf{e}^{\intercal}\mathbf{y}^{\infty}$ for all players $i \in N$. Denote $y^{\infty} := \mathbf{e}^{\intercal}\mathbf{y}^{\infty}$. This implies, that in the steady state the ideal efforts of the players converge to a consensus $y^{\infty}$, in which case we have:

$$\mathbf{y}^{\infty} = \frac{1}{\omega_1 + \omega_2} \left( \boldsymbol{\theta} + \left( \boldsymbol{\Lambda} + \omega_1 \right) \mathbf{y}^{\infty} \right),$$

$$y_i^{\infty} = \frac{\theta_i}{\omega_2 - \sum\limits_{j \in N} \lambda_{ij}} \quad \text{s.t.} \quad y_i^{\infty} = y^{\infty} \quad \text{for all} \quad i \in N. \tag{11}$$

A consensus over ideal efforts occurs among *structurally equivalent players* (Lorrain and White, 1971), that is, the players that share the same neighborhood and similar relationship patterns in the network, and have the same productivity. Such strict form of players' equivalence ensures the consensus. A more general definition of equivalent structure of player is the *regular equivalence* (Sailer, 1978; White and Reitz, 1983).

**Definition 1.** Let $C \subset N$ be a subset of players in the network. Given the layers $\boldsymbol{g}$ and $\boldsymbol{l}$, and productivity coefficients $\boldsymbol{\theta}$, we say that the players in $C$ are regularly equivalent if their neighborhoods are equivalent to each other, and for all $i \in C$ there exists a consensus such that $y_i^{\infty} = y_C^{\infty}$.

Unlike structurally equivalent players, regularly equivalent players do not necessarily share the same neighbors in the network, but they have similar connection patterns and are connected to similar neighbors. The neighbors, in turn, are also regularly equivalent. The consensus ideal effort of regularly equivalent players is given by:

$$y_C^{\infty} = \frac{\theta_i + \sum\limits_{j \in N \setminus C} (\lambda_{ij} + \omega_1 g_{ij}^*) y_j^{\infty}}{\omega_2 - \sum\limits_{j \in C} \lambda_{ij} + \omega_1 (1 - \sum\limits_{j \in C} g_{ij}^*)} \tag{12}$$

Using the definition of ideal efforts in the steady state in (10) and assuming there exists consensus among structurally equivalent players in $C$, such that $y_i^{\infty} = y_C^{\infty} =$ for all $i \in C$, we can find the consensus ideal for players in $C$.

$$\left( \mathbf{I} - \frac{1}{\omega_1 + \omega_2} (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*) \right) \mathbf{y}^{\infty} = \frac{1}{\omega_1 + \omega_2} \boldsymbol{\theta}$$

For player $i \in C$ with $y_i^\infty = y_C^\infty$ we have:

$$y_C^\infty - \frac{1}{\omega_1+\omega_2}\left(\sum_{j\in N\setminus C}\lambda_{ij}y_j^\infty + \underbrace{\sum_{j\in C}\lambda_{ij}y_j^\infty}_{y_C^\infty\sum_{j\in C}\lambda_{ij}} + \omega_1\sum_{j\in N\setminus C}g_{ij}^* y_j^\infty + \omega_1\underbrace{\sum_{j\in C}g_{ij}^* y_j^\infty}_{y_C^\infty\sum_{j\in C}g_{ij}^*}\right) = \frac{1}{\omega_1+\omega_2}\theta_i$$

$$y_C^\infty - \frac{1}{\omega_1+\omega_2}\left(y_C^\infty(\sum_{j\in C}\lambda_{ij} + \omega_1\sum_{j\in C}g_{ij}^*) + \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^\infty\right) = \frac{1}{\omega_1+\omega_2}\theta_i$$

$$y_C^\infty(\omega_1+\omega_2) - y_C^\infty(\sum_{j\in C}\lambda_{ij} + \omega_1\sum_{j\in C}g_{ij}^*) - \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^\infty = \theta_i$$

$$y_C^\infty\left(\omega_2 - \sum_{j\in C}\lambda_{ij} + \omega_1(1-\sum_{j\in C}g_{ij}^*)\right) = \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^\infty + \theta_i$$

Thus, the consensus effort of regularly equivalent group of players is given by (12).

The regular equivalence is associated with similarities in networks characteristics of the players, such as the neighborhood and the weights on all layers of the network, the idiosyncratic productivity parameters of the neighbors, as well as the productivity of the player herself.

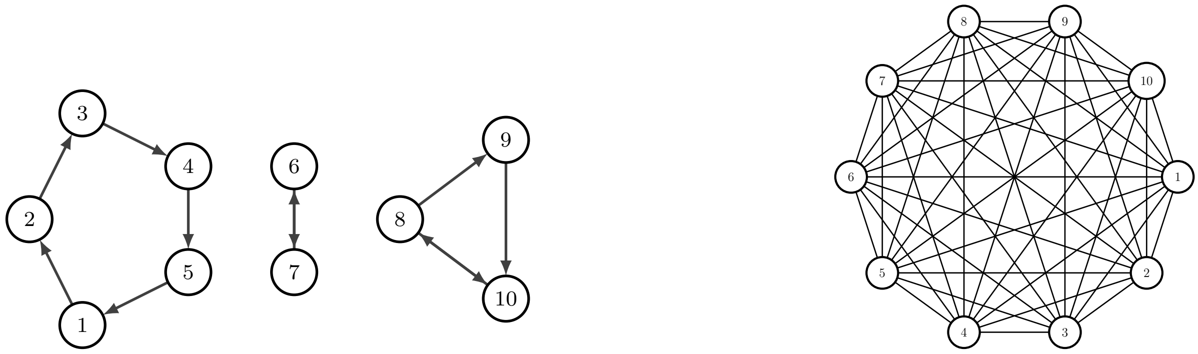
## 3.3 Numerical Examples

Consider a network of $n = 10$ players connected through a social network $\boldsymbol{g}$ (Figure 2) with normalized equal weights on the directed links, and a network of additional complementarities $\boldsymbol{l}$ (Figure 3). In each period the players engage in the two-layer network game in (1) where the productivity of effort is $\theta_i = 1$ for all players, and the effort cost parameter is $\omega_2 = 2$. In the initial period 0, the players are endowed with ideal efforts $y_i = 0.2i$:

$$\mathbf{y}^{0\intercal} = [0.2\ 0.4\ 0.6\ 0.8\ 1\ 1.2\ 1.4\ 1.6\ 1.8\ 2].$$

After each period game the players update their ideal efforts according to equation (2) with the speed $\xi = 0.2$. We consider an arbitrary set of combinations for parameters of conflict cost and sensitivity to cognitive dissonance with $\omega_1, \omega_3 \in \{2, 8, 15\}$.

Consider the network of social interactions in Figure 2a. In the absence of complementarity layer $\boldsymbol{l}$, the ideal efforts in all three components of the network $\boldsymbol{g}$ converge to the consensus $y^\infty = 0.5$ by expression (11). The layer of complementarities in the network affects the steady-state level of efforts and ideals, as well as the existence of the consensus.



(a) (b)

Figure 2: Social network $\boldsymbol{g}$

Let us now add the layer $\boldsymbol{l}$ in Figure 3a described by the following adjacency matrix.

$$\boldsymbol{\Lambda}_{(a)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

For this and further examples, we distinguish the following sets of players that form disconnected components in both networks in Figures 2a and 3a:

$$C_1 = \{1, 2, 3, 4, 5\}, \quad C_2 = \{6, 7\}, \quad C_3 = \{8, 9, 10\}.$$

Moreover, we assume that the players in each set $C_i$ are homogeneous in their aggregate complementarity coefficients such that

$$\sum_{j \in N} \lambda_{ij} = 0.5 \ \forall i \in C_1, \quad \sum_{j \in N} \lambda_{ij} = 0.7 \ \forall i \in C_2 \quad \text{and} \quad \sum_{j \in N} \lambda_{ij} = 1 \ \forall i \in C_3. \tag{13}$$

It is easy to see that the players within each component $C_{i, i \in \{1,2\}}$ are regularly equivalent. The players in $C_3$ are regularly equivalent since player 8 receives the same aggregate complementarity from the regularly equivalent players 9 and 10. Moreover, the components are disconnected in both layers of the network, thus the steady-state efforts and ideals can be found using the equation (11). The following vector shows the ideal efforts of players in the steady state:

$$\mathbf{y}^{\infty\intercal} = [0.667 \ 0.667 \ 0.667 \ 0.667 \ 0.667 \ 0.769 \ 0.769 \ 1 \ 1 \ 1].$$

Unsurprisingly, we can see that the complementarity of efforts from $\boldsymbol{l}$ increase the ideal efforts of players in the steady state. Figure 4 shows the dynamics of the ideal efforts of the players. One can observe that with a higher cost of conflict $\omega_1$ the ideal efforts within each component $C_i$ converge faster. While the increase in the cost of cognitive dissonance $\omega_3$ delays the convergence to the steady-state.

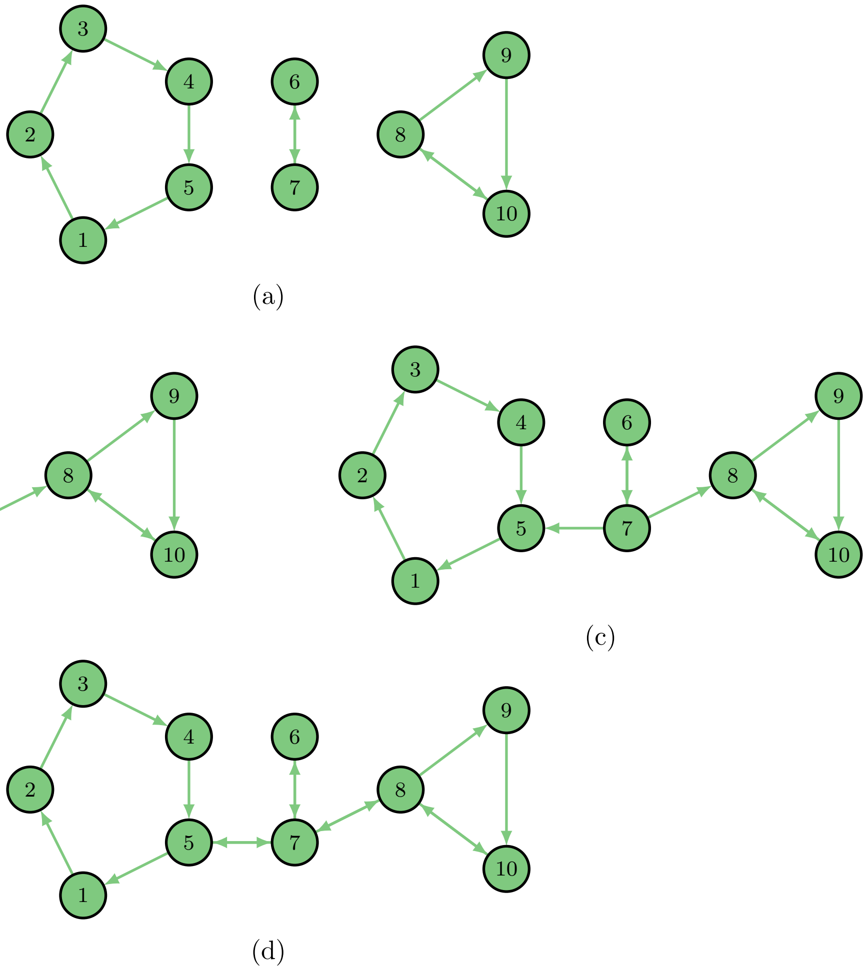



Figure 3: Network of complementarities $\boldsymbol{l}$

Assume now a complete network of social interactions $\boldsymbol{g}$ in Figure 2b. The regular equivalence of players within each set $C_i$ remains. Thus, the consensus over ideal efforts among players in $C_i$ satisfies the expression (12). E.g. for any player $i \in C_3$, given $\omega_1 = 15$, we find:

$$y^{\infty}_{C_3} = \frac{\theta_i + \omega_1(\frac{5}{9}y^{\infty}_{C_1} + \frac{2}{9}y^{\infty}_{C_2})}{\omega_2 - \sum\limits_{j \in C_3} \lambda_{ij} + \omega_1 \frac{7}{9}} = 0.778.$$

The ideal efforts of players in the steady states, given the cost of conflict $\omega_1 = 15$, are the following:

$$\mathbf{y}^{\infty\intercal} = [0.756\ 0.756\ 0.756\ 0.756\ 0.756\ 0.765\ 0.765\ 0.778\ 0.778\ 0.778].$$

The dynamics of ideal effort with the complete network $\boldsymbol{g}$ are displayed in Figure 5. Comparing the two examples above and the dynamics of ideal efforts in Figures 4 and 5, we can see that the variance between the steady-state ideals of the components is lower in the network with densely connected layer $\boldsymbol{g}$.

Let us further consider only the social network $\boldsymbol{g}$ depicted in Figure 2a for the rest of the examples below. Further, let us consider the layer of effort complementarities $\boldsymbol{l}$ in

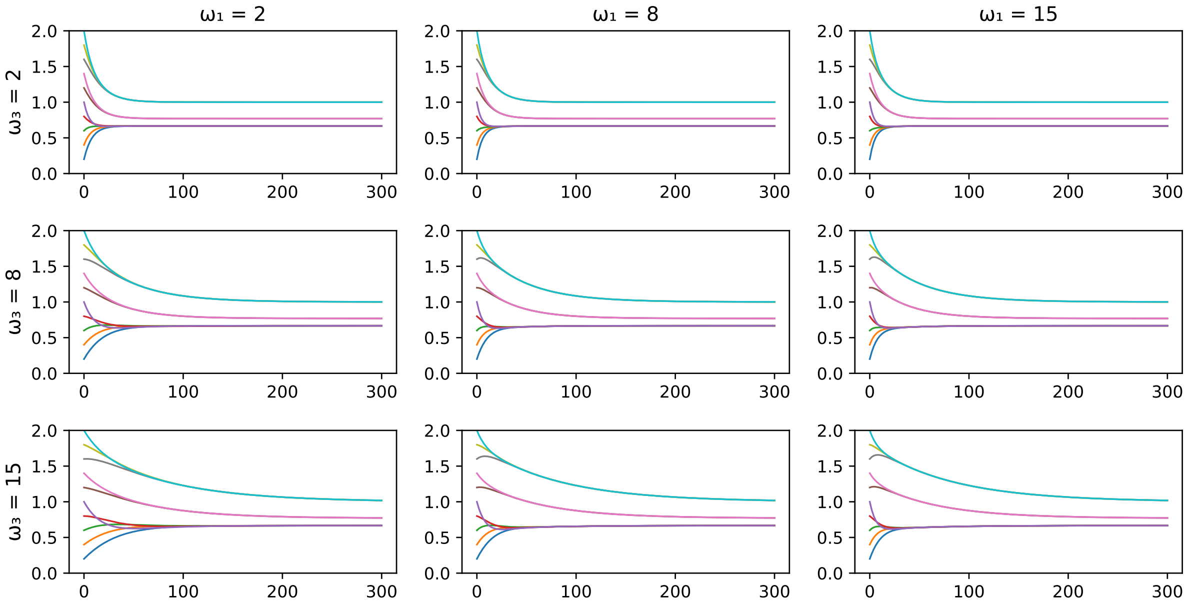



Figure 4: Dynamics of ideal efforts in the network of complementarities in Figure 3a and the network of social interactions in Figure 2a

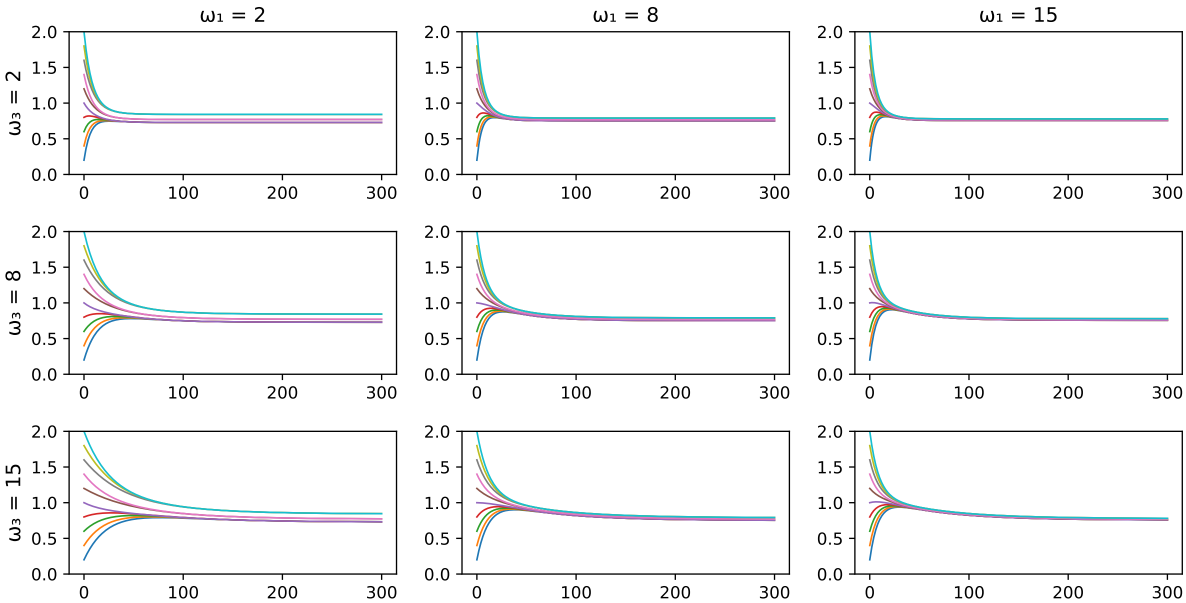



Figure 5: Dynamics of ideal efforts in the network of complementarities in Figure 3a and the complete network of social interactions in Figure 2b.

Figure 3b. We assume that the aggregate of the complementarity coefficients of players do not change by adding a new link. Thus, by adding a connection in the network layer $\boldsymbol{l}$, we redistribute the complementarity coefficients of outgoing links maintaining the characteristics described in (13). By doing so, we can analyze the effect of the change in the structure of the layer $\boldsymbol{l}$ without increasing the complementarity levels in the network. Assume, the layer $\boldsymbol{l}$ is described by the following matrix of complementarity coefficients.

$$\boldsymbol{\Lambda}_{(b)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The additional link $7 \to 8$ in $\boldsymbol{l}$ makes the player 7 influenced by the component $C_3$ driving the ideal efforts of players 6 and 7 closer to $y^{\infty}_{C_3}$. Unlike the case with $\boldsymbol{l}$ in Figure 3a, players 6 and 7 are not equivalent anymore, so the consensus over ideal effort in $C_2$ is not achieved.

Further, we add a complementarity link connecting player 7 to player 5 as in Figure 3c, the component $C_2$ is then additionally influenced by the efforts in the component $C_1$. Assume the link $7 \to 5$ with $\lambda_{75} = 0.2$ in the matrix of weights $\boldsymbol{\Lambda}_{(c)}$ describing the layer in Figure 3c.

$$\boldsymbol{\Lambda}_{(c)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

As shown in Table 1, the steady-state consensus effort in the component $C_1$ is 0.667 and is lower than that of player 6 and component $C_3$ in the previous example. Receiving complementarity from the lower-effort component $C_1$, the ideal efforts of player 7 decreases. At the same time, the pressure to conform in layer $\boldsymbol{g}$ forces player 6 to adjust her effort according to the new social norm.

Consider now the layer $\boldsymbol{l}$ in Figure 3d where players 5 and 8 reciprocate player 7 with complementarity links. The matrix of weights is the following.

$$\boldsymbol{\Lambda}_{(d)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The new links make the network connected. There is no regular equivalence between players anymore, which results in all players having different steady-state efforts. The ideal efforts of the players in $C_1$ are positively affected by the efforts in the rest of the network. This positive effect is weaker for players that are further from player 5.

The steady-state ideal efforts in previous examples are summarized in Table 1.

Table 1

| | | $y_1^\infty$ | $y_2^\infty$ | $y_3^\infty$ | $y_4^\infty$ | $y_5^\infty$ | $y_6^\infty$ | $y_7^\infty$ | $y_8^\infty$ | $y_9^\infty$ | $y_{10}^\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mathbf{\Lambda}_{(a)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.769 | 0.769 | 1 | 1 | 1 |
| $\omega_1 = 2$ | $\mathbf{\Lambda}_{(b)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.783 | 0.79 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(c)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.777 | 0.78 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(d)}$ | 0.668 | 0.669 | 0.67 | 0.671 | 0.673 | 0.776 | 0.779 | 0.98 | 0.989 | 0.985 |
| $\omega_1 = 8$ | $\mathbf{\Lambda}_{(b)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.785 | 0.787 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(c)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.778 | 0.779 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(d)}$ | 0.6687 | 0.669 | 0.6694 | 0.667 | 0.671 | 0.777 | 0.778 | 0.983 | 0.986 | 0.984 |
| $\omega_1 = 15$ | $\mathbf{\Lambda}_{(b)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.786 | 0.787 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(c)}$ | 0.667 | 0.667 | 0.667 | 0.667 | 0.667 | 0.778 | 0.779 | 1 | 1 | 1 |
| | $\mathbf{\Lambda}_{(d)}$ | 0.669 | 0.6692 | 0.6695 | 0.6698 | 0.67 | 0.777 | 0.778 | 0.983 | 0.985 | 0.984 |

One can also observe that while the sensitivity to cognitive dissonance $\omega_3$ does not affect the steady-state levels of ideals efforts, increasing the said parameter slows down the emergence of the consensus among players and the steady state. The higher conflict cost $\omega_1$, on the other hand, accelerates the agreement between players connected in the layer of social influences $\boldsymbol{g}$.

# 4 Conclusion

In this paper, we have proposed a network game model with multidimensionality of network relations, where people repeatedly choose their efforts and update their ideal beliefs based on their current choices. Specifically, we study a model with a two-layer network where the first layer dictates the social norm, and the second is a network of interactions with strategic complementarities in efforts. The players in the network are initially endowed with heterogeneous ideal efforts and differ in their productivity. In line with the network game literature, we find that the pure Nash equilibrium of the game is proportional to Katz-Bonacich centrality in the combined network of the two layers. Additionally, we find the steady-state equilibrium efforts and ideals. We derive the conditions for the existence of consensus about ideal efforts in the whole population and among structurally and regularly equivalent players.

The dynamics of ideal efforts in our model show that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a consensus and the steady state. The sensitivity to cognitive dissonance, that captures the behavioral stickiness of the players, affects the speed of convergence to a steady-state but does not change the long-run efforts. The taste for conformity, on the other hand, affects the steady-state efforts' levels. Consideration of possible heterogeneity in the cost of the cognitive dissonance, as well as the taste for conformity, is left for future research.

We show that the multidimensionality of network interactions may change the overall effort levels and the beliefs in society. While focusing on the network layer with strategic complementarities we leave the consideration of other types of relations for future research. In particular, the direct extension of the model provided in this paper is the consideration of strategic substitutability along with complementarities on the same network layer. In the paper, we have assumed a given two-layer network. Endogenizing the formation of the network layer with complementarities is left for future research.[25]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# Appendix

*Proof of Proposition 1.* Given $x_i^*$ is the optimal effort choice for $i$

$$\frac{\partial u_i^t}{\partial x_i^t}(x_i^{t*}) \equiv 0$$

$$\begin{aligned}
\frac{\partial u_i^t(x^t)}{\partial x_i^t} &= \theta_i + \sum_{j \in N \setminus i} \lambda_{ij} x_j^t - \omega_1 \sum_{j \in N} g_{ij}^* (x_i^t - x_j^t) - \omega_2 x_i^t - \omega_3 (x_i^t - y_i^t) = \\
&= \theta_i + \sum_{j \in N \setminus i} \lambda_{ij} x_j^t - \omega_1 x_i^t \underbrace{\sum_{j \in N} g_{ij}^*}_{=1} + \omega_1 \sum_{j \in N} g_{ij}^* x_j^t - \omega_2 x_i^t - \omega_3 x_i^t + \omega_3 y_i^t = \\
&= (\theta_i + \omega_3 y_i^t) + \sum_{j \in N \setminus i} \lambda_{ij} x_j^t + \omega_1 \sum_{j \in N} g_{ij}^* x_j^t - (\omega_1 + \omega_2 + \omega_3) x_i^t \\
&= (\theta_i + \omega_3 y_i^t) + \sum_{j \in N \setminus i} (\lambda_{ij} + \omega_1 g_{ij}^*) x_j^t - (\omega_1 + \omega_2 + \omega_3) x_i^t
\end{aligned}$$

[25] Mauleon and Vannetelbosch (2016) provide a comprehensive introduction to network formation games.

Thus the equilibrium effort choice of $i$ is:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1+\omega_2+\omega_3)} + \sum_{j\in N\setminus i} \frac{\lambda_{ij} + \omega_1 g_{ij}^*}{(\omega_1+\omega_2+\omega_3)} x_j^t,$$

for all $i \in N$, or:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1+\omega_2+\omega_3)} + \frac{1}{(\omega_1+\omega_2+\omega_3)} \sum_{j\in N\setminus i} \lambda_{ij} x_j^t + \frac{\omega_1}{(\omega_1+\omega_2+\omega_3)} \sum_{j\in N} g_{ij}^* x_j^t,$$

We can rewrite it in a matrix form:

$$\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}\boldsymbol{\theta} + \frac{\omega_3}{(\omega_1+\omega_2+\omega_3)}\mathbf{y}^t + \frac{1}{(\omega_1+\omega_2+\omega_3)}\boldsymbol{\Lambda}\mathbf{x}^t + \frac{\omega_1}{(\omega_1+\omega_2+\omega_3)}\mathbf{G}^*\mathbf{x}^t,$$

So the, matrix form solution of the equilibrium efforts is the following:

$$\left(\mathbf{I} - \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\Lambda} + \omega_1\mathbf{G}^*)\right)\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\theta} + \omega_3\mathbf{y}^t)$$

And thus the vector of equilibrium efforts is:

$$\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}\left(\mathbf{I} - \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\Lambda} + \omega_1\mathbf{G}^*)\right)^{-1}(\boldsymbol{\theta} + \omega_3\mathbf{y}^t)$$

□

*Proof of Lemma 1.* In order to ensure the condition of Proposition 2 on the spectral radius $\delta_1$ of the matrix $\bar{\mathbf{G}}$ in (4), we can use maximum row-sum and column-sum norms of the matrix, $|||\bar{\mathbf{G}}|||_\infty$ and $|||\bar{\mathbf{G}}|||_1$, to find an upper bound for the spectral radius[26].
One can see that the row-sum for each of the first $n$ rows of the matrix $\bar{\mathbf{G}}$ is equal to 1. While for the consecutive $n$ rows, the sum of row elements of $(n+i)$th row is equal to $\frac{1+\omega_1+\sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$, for each $i \in N$. As a result, the maximum row-sum norm of the matrix is the maximum of given values:

$$|||\bar{\mathbf{G}}|||_\infty = \max\left\{1, \frac{1}{\omega_1+\omega_2}(1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}\right\}$$

Moreover, Gershgorin circle theorem allows us to see from the first $n$ rows of the matrix $\bar{\mathbf{G}}$ that 1 is one of its' eigenvalues.
Similarly, the sum for each of the first $n$ column elements of the matrix $\bar{\mathbf{G}}$ is $\frac{1+\omega_1+\omega_2}{\omega_1+\omega_2}$.

[26]This follows common linear algebra results, e.g one can see the Theorem 5.6.9 in Horn and Johnson (2012)

And the sum of column elements for columns $n+j$ for each $j \in N$ is $\frac{\sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2}$. Thus the maximum column-sum norm of the matrix is the following:

$$|||\bar{\mathbf{G}}|||_1 = \max\left\{1+\frac{1}{\omega_1+\omega_2}, \frac{1}{\omega_1+\omega_2}\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})\right\}$$

Now let's find the better proxy for the upper bound $UB(\delta_1)$ of the spectral radius by finding $\min\{|||\bar{\mathbf{G}}|||_\infty, |||\bar{\mathbf{G}}|||_1\}$ for all parameter values that satisfy $\omega_2 \geq \max_i \sum_{j\in N} \lambda_{ij}$. And $|||\bar{\mathbf{G}}|||_1$ can refine the upper bound for $\delta_1$ when $\omega_2 < \max_i \sum_{j\in N} \lambda_{ij}$.

Case 1: Assume $|||\bar{\mathbf{G}}|||_\infty = 1$. Thus $1 \geq \frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$, which follows by $\omega_2 \geq 1 + \max_i \sum_{j\in N} \lambda_{ij}$.

(a) If $|||\bar{\mathbf{G}}|||_1 = 1+\frac{1}{\omega_1+\omega_2} > 1$.

(b) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2} \geq 1+\frac{1}{\omega_1+\omega_2} > 1$.

Case 2: Assume $|||\bar{\mathbf{G}}|||_\infty = \frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$. This means that $\frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2} > 1$. Then we have $\omega_2 < 1 + \max_i \sum_{j\in N} \lambda_{ij}$.

(a) If $|||\bar{\mathbf{G}}|||_1 = 1+\frac{1}{\omega_1+\omega_2}$. Then $\omega_2 \geq \max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij}) - \omega_1 - 1$

And $1+\frac{1}{\omega_1+\omega_2} \geq \frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$ only when $\omega_2 \geq \max_i \sum_{j\in N}\lambda_{ij}$.

(b) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2} \geq 1+\frac{1}{\omega_1+\omega_2}$. Then $\omega_2 < \max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij}) - \omega_1 - 1$.

Thus $|||\bar{\mathbf{G}}|||_\infty$ is the upper bound for the spectral radius if $\frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2} \geq \frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$.

Let's introduce the following notations:

$$\overline{\omega}_2 \equiv \max_i \sum_{j\in N}\lambda_{ij}$$
$$\overline{\overline{\omega}}_2 \equiv \max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g^*_{ij}) - \omega_1 - 1$$

To sum up

$$\begin{array}{llllll}
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = 1 & \text{when} & \omega_2 \geq \overline{\omega}_2 + 1. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\omega}_2; \overline{\omega}_2 + 1). \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = 1 + \frac{1}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2) \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 \geq \overline{\omega}_2.
\end{array}$$

We can now say that for $\delta_1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ it is sufficient to have $UB(\delta_1) < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$. Thus,

- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = 1$, then $1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 0$.
- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$, then $\frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1+\overline{\omega}_2-\omega_2$.
- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2}$, then $1 + \frac{1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1$.
- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2}$, then $\frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > \overline{\overline{\omega}}_2+1-\omega_2$.

These results are illustrated in Figure 1. □

*Proof of Proposition 3.* Notice the following:

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \implies \boldsymbol{\Delta} = \mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}$$
$$\mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1} \implies \boldsymbol{\Sigma} = \mathbf{E}^{-1}\mathbf{T}\mathbf{E}$$

multiplying $\mathbf{T}$ in (8) by $\mathbf{E}$ from the left and by $\mathbf{E}^{-1}$ from the right we get the following:

$$\begin{aligned}
\mathbf{E}^{-1}\mathbf{T}\mathbf{E} &= \mathbf{E}^{-1}\left(\xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I}\right)\mathbf{E} \\
\boldsymbol{\Sigma} &= \xi(1-\phi)\mathbf{E}^{-1}\mathbf{M}\mathbf{E} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\mathbf{E}^{-1}\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1}\mathbf{E} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{E}^{-1}(\mathbf{I} - \phi\bar{\mathbf{G}})\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{I} - \phi\mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{I} - \phi\boldsymbol{\Delta}\right)^{-1} + (1-\xi)\mathbf{I}
\end{aligned}$$

So we have:

$$\boldsymbol{\Sigma} = \xi(1-\phi)\left(\mathbf{I} - \phi\boldsymbol{\Delta}\right)^{-1} + (1-\xi)\mathbf{I}$$

We can find the vector of eigenvalues $\boldsymbol{\tau}$ of $\mathbf{T}$ by multiplying the above expression by a vector of ones, $\mathbf{1}$, such that $\boldsymbol{\tau} = P(\boldsymbol{\delta})$.

Notice that:

$$(\mathbf{I} - \phi\boldsymbol{\Delta})^{-1} = \sum_{t=0}^{\infty} \phi^t \boldsymbol{\Delta}^t$$

$$\begin{aligned}\boldsymbol{\Sigma}\mathbf{1} &= \xi(1-\phi)\left(\mathbf{I} - \phi\boldsymbol{\Delta}\right)^{-1}\mathbf{1} + (1-\xi)\mathbf{I}\mathbf{1} \\ \boldsymbol{\tau} &= \xi(1-\phi)\sum_{t=0}^{\infty} \phi^t \boldsymbol{\Delta}^t \mathbf{1} + (1-\xi)\mathbf{1}\end{aligned}$$

And for each distinct eigenvalue we have:

$$\begin{aligned}\tau_i &= \xi(1-\phi)\sum_{t=0}^{\infty} \phi^t \delta_i^t + (1-\xi) \\ &= \xi\frac{1-\phi}{1-\phi\delta_i} + (1-\xi)\end{aligned}$$

Recall that $\phi = \dfrac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3}$ and from the condition in Proposition 2 we have that $\delta_1 < \dfrac{\omega_1 + \omega_2 + \omega_3}{\omega_1 + \omega_2}$ which implies that $\phi\delta_i < 1$. One can notice that when $\delta_i < 1$ then $\tau_i < 1$, as a convex combination of 1 and a value below 1, for all $i \in [1, n]$. Additionally, this is always true when $\omega_3 = 0$.

So when $\boldsymbol{\Delta}^{\infty}$ converge $\boldsymbol{\Sigma}^{\infty}$ converge too.

□

*Proof of Proposition 4.* Similarly to Proposition 1 in Olcina, Panebianco and Zenou (2024) we prove the proposition by following two steps.

Step 1. $\bar{\mathbf{G}}$ and $\mathbf{T}$ are commuting matrices: $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.

Recalling the notations in (8) we have that

$$\bar{\mathbf{G}}\mathbf{T} = \xi(1-\phi)\bar{\mathbf{G}}\mathbf{M} + (1-\xi)\bar{\mathbf{G}}.$$

To show that $\bar{\mathbf{G}}$ and $\mathbf{M}$ are commuting matrices we first show that $\bar{\mathbf{G}}$ and $\mathbf{M}^{-1}$ commute.

$$\bar{\mathbf{G}}\mathbf{M}^{-1} = \bar{\mathbf{G}}\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right) = \left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)\bar{\mathbf{G}} = \mathbf{M}^{-1}\bar{\mathbf{G}}.$$

Multiplying the above equality by $\mathbf{M}$ from left and right we have that $\bar{\mathbf{G}}\mathbf{M} = \mathbf{M}\bar{\mathbf{G}}$. From which it directly follows that $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.

Step 2. From diagonalizability of $\bar{\mathbf{G}}$ we can infer that $\mathbf{T}$ is also diagonalizable given that $\mathbf{M}$ is a polynomial of $\bar{\mathbf{G}}$, and $\mathbf{T}$ is a linear convex combination of $\mathbf{M}$ and $\mathbf{I}$.

As commuting matrices, $\bar{\mathbf{G}}$ and $\mathbf{T}$ share the same set of eigenvectors, thus also the eigenvector associated with the eigenvalue equal to 1. We can rewrite the two matrices as

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \qquad \text{and} \qquad \mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1},$$

where $\mathbf{E}$ is the matrix of eigenvectors of $\bar{\mathbf{G}}$ and $\mathbf{T}$ as columns. $\boldsymbol{\Delta}$ is the diagonal matrix of eigenvalues of $\bar{\mathbf{G}}$, and $\boldsymbol{\Sigma}$ is the diagonal matrix of eigenvalues of $\mathbf{T}$.[27] The eigenvalues in $\boldsymbol{\Delta}$ and $\boldsymbol{\Sigma}$ are ordered in the decreasing manner, with the order of eigenvectors in $\mathbf{E}$ corresponding to associated eigenvalues. From the above-mentioned it directly follows that:

$$\bar{\mathbf{G}}^t = \mathbf{E}\boldsymbol{\Delta}^t\mathbf{E}^{-1} \qquad \text{thus} \qquad \bar{\mathbf{G}}^\infty = \mathbf{E}\boldsymbol{\Delta}^\infty\mathbf{E}^{-1} \qquad \text{with} \qquad \boldsymbol{\Delta}^\infty = \lim_{t\to\infty} \boldsymbol{\Delta}^t$$

and

$$\mathbf{T}^t = \mathbf{E}\boldsymbol{\Sigma}^t\mathbf{E}^{-1} \qquad \text{thus} \qquad \mathbf{T}^\infty = \mathbf{E}\boldsymbol{\Sigma}^\infty\mathbf{E}^{-1} \qquad \text{with} \qquad \boldsymbol{\Sigma}^\infty = \lim_{t\to\infty} \boldsymbol{\Sigma}^t$$

The condition $\max_i \sum_{j\in N} \lambda_{ij} \leq \omega_2 - 1$ ensures that $\delta_1 = 1$.[28] Which, as it is easy to see from Proposition 3, implies that $\tau_1 = 1$. While for any other eigenvalue $\delta_i < 1$ of $\bar{\mathbf{G}}$, we have that $\tau_i < 1$. This follows that

$$\boldsymbol{\Delta}^\infty = \boldsymbol{\Sigma}^\infty \qquad \text{and} \qquad \bar{\mathbf{G}}^\infty = \mathbf{T}^\infty.$$

Using the results above in (9) we find the $\hat{\mathbf{y}}^\infty$. And plugging it into the equilibrium efforts in equation (5) we prove the second part of the proposition. □

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[27] Notice that given the diagonalizability of $\bar{\mathbf{G}}$, it has $2n$ eigenvectors.

[28] Recall that, by construction of the matrix in (4), 1 is always an eigenvalue of $\bar{\mathbf{G}}$.

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