

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 1/36

GENESIS PHASE B1

Degree-1 gravitational potential, geocenter and application to GENESIS

ROB/AWS/UCLouvain/ULiège (Belgium)



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 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 2/36

CHANGE RECORDS

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V1.1(1) draft	29/03/2025	New document created by V. Dehant and M. Folgueira	Following question on geocenter sensitivity of GENESIS
V1.1(2)	24/04/2025	Revision by G. López Rosson	Improvements on understanding and interpretation of the results
V1.1(3)	09/06/2025	Revision by A. Couhert	Notes on orbit inclination
V1.1(4)	25/06/2025	Modifications by G. López Rosson	Improvements for readability, addition of new tables for $F(i)$, $G(e)$.
V1.1(5)	11/07/2025	Modifications by G. López Rosson	Improvements for readability, new graphics.

¹ vx.y ==> x = version number, y = issue number

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 3/36

TABLE OF CONTENTS

1. GENERAL INFORMATION	4
1.1 Scope of the Document	4
1.2 Applicable documents	4
1.3 Reference documents	4
2. EXISTING EXPRESSION FOR THE GRAVITATIONAL FIELD IN TERMS OF THE ORBITAL ELEMENTS.....	5
2.1 Unperturbed satellite orbit – two-body problem – Keplerian orbit	5
2.1.1 Kepler law	5
2.1.2 Anomalies and their relationships	5
2.1.3 Position and velocity of the satellite	8
2.1.4 Orbital elements	8
2.1.5 Link between position and velocity of the satellite, and orbital elements	9
2.1.6 Expressions relating r , E , v as well as their cosine and sine in terms of Bessel functions	10
2.2 Satellite orbit perturbed by a gravitational field.....	13
2.2.1 Gravitational potential in spherical harmonics	13
2.2.2 Force acting on the satellite	14
2.2.3 Expression of the perturbing function in V in terms of osculating orbital elements	16
2.2.3.1 Expression of the perturbing function in V in terms of inclination – $F(i)$ function	17
2.2.3.2 Expression of the perturbing function in V in terms of eccentricity – $G(e)$ function	22
2.2.3.2.1 Kaula's approach	22
2.2.3.2.2 Balmino's approach	24
2.2.4 Degree 1	28
2.2.4.1 F-functions	28
2.2.4.2 G-functions	29
2.2.4.3 Altogether, gravitational potential of degree 1	30
2.2.4.4 Conclusion	33
2.3 Recommendation of the WG on Orbit inclination	35
3. REFERENCES	36

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 4/36

1. GENERAL INFORMATION

1.1 Scope of the Document

The document aims at finding an expression for the gravity field of degree 1 (important for geocenter) as a function of the orbital elements of the spacecraft.

1.2 Applicable documents

<i>Title</i>	<i>Reference</i>
GENESIS-ROB-AA-0001-Abbreviations-Acronyms_v1.2(1) provides the abbreviations and acronyms used in this context.	GENESIS-ROB-AA-0001

1.3 Reference documents

<i>Title</i>	<i>Reference</i>

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 5/36

2. EXISTING EXPRESSION FOR THE GRAVITATIONAL FIELD IN TERMS OF THE ORBITAL ELEMENTS

We begin our study reviewing the paper by Balmino (1992), which builds upon the foundational work in Kaula's book *Theory of Satellite Geodesy* (1966). We also consulted the books by Capderou (2005), Montenbruck (2005), and Vallado (2013), as well as lecture notes by Duriez (2007).

2.1 Unperturbed satellite orbit – two-body problem – Keplerian orbit

When the Earth is modeled as a point mass and a satellite orbits this point, the resulting trajectory is an ellipse with the Earth located at one of its foci. In this case, the gravitational potential follows the classical Newtonian formulation. However, when the Earth is modeled as an extended body with varying internal densities, the satellite's orbit evolves over time due to perturbations. In such scenarios, the gravitational force acting on the satellite depends on its orbital elements, which will be discussed in Section 2.2, dedicated to the perturbed case.

To prepare for this analysis, we first present the expressions for the satellite's position in terms of its orbital elements. These expressions are valid in the unperturbed case, which is why they are introduced here.

2.1.1 Kepler first law

Assuming the Earth is a perfectly spherical, homogeneous body with geocenter O , Kepler's First Law states that: “a satellite follows an elliptical orbit with the Earth located at one of the foci” (see Figure 1).

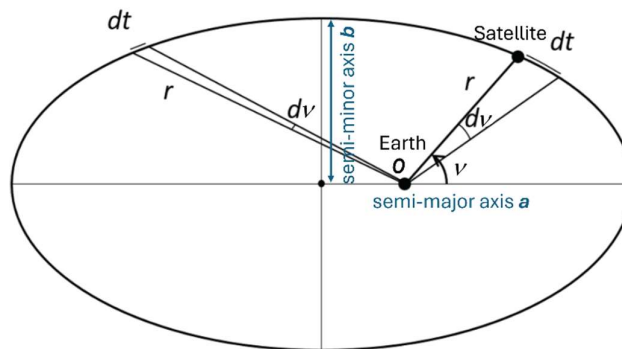


Figure 1: Kepler's orbit diagram.

2.1.2 Anomalies and their relationships

Anomalies are specific angles that describe the satellite's position along its orbit around a central body (like the Earth) at a given time.

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 6/36

The position of a satellite in an Earth-centered reference frame can be described using its distance from Earth (r) and the true anomaly (v), or alternatively using the mean anomaly (M) and the eccentric anomaly (E).

The mean anomaly M is defined based on the satellite's mean angular velocity and is given by:

$$M = n(t - \tau) \quad (1)$$

with:

$n = \frac{2\pi}{T}$: the mean motion,

T : the orbital period,

t : current time, and,

τ : the time of passage at perigee (A).

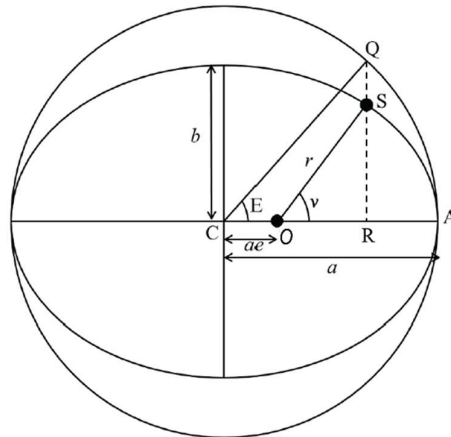


Figure 2: Geometric representation of the anomalies.

These anomalies are interconnected through well-established mathematical relationships:

1. Relationship between eccentric anomaly and true anomaly

Considering the geometry shown in Figure 2 and using trigonometric relationships, it is easy to see that the cosine and sine of the true anomaly v can be expressed in terms of the eccentric anomaly E , the eccentricity e , and the semi-major axis a of the orbit:

$$\begin{aligned} r \cos v &= a \cos E - ae \\ r \sin v &= a\sqrt{1 - e^2} \sin E \end{aligned}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 7/36

By squaring and summing these two equations and then taking the square root, we obtain that the distance from Earth r is:

$$r = \frac{a(1 - e^2)}{1 + e \cos v} = a (1 - e \cos E) \quad (2)$$

Through further manipulation, the following relationship is derived:

$$\tan^2 \frac{v}{2} = \frac{1 + e}{1 - e} \tan^2 \frac{E}{2} \quad (3)$$

2. Relationship between mean anomaly and eccentric anomaly

From geometric considerations involving triangles OSA and OSR (see **Error! Reference source not found.**), we derive the classical **Kepler equation**:

$$M = E - e \sin E \quad (4)$$

3. Series expansion: true anomaly as a function of eccentric anomaly

Using the development $y = x + q \sin 2x + \frac{1}{2}q^2 \sin 4x + \frac{1}{3}q^3 \sin 6x + \dots$ which results from solving $\tan y = p \tan x$ for $q = \frac{p-1}{p+1} < 1$, and applying this to Eq. (3) $y = \frac{v}{2}, x = \frac{E}{2}$, $p = \sqrt{\frac{1+e}{1-e}}$, we obtain:

$$v = E + e \sin E + \frac{e^2}{4} \sin 2E + \dots \quad (3)'$$

4. Series expansion: eccentric anomaly as a function of mean anomaly

Starting from Eq. (4), and treating eccentricity e as a function of E , i.e.,

$$e = f(E) = \frac{E-M}{\sin E},$$

we apply the following expansion formula:

$$x = x_0 + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!} \left[\frac{d^{k-1}}{dx^{k-1}} \left(\frac{x-x_0}{f(x)-y_0} \right)^k \right]_{x=x_0} \quad \text{for } y = e, y_0 = 0, x = E, x_0 = M,$$

and we obtain the series:

$$E = M + e \sin M + \frac{e^2}{2} \sin 2M + \dots \quad (4)'$$

In summary, we have derived series expansions for both the true anomaly (Eq. (3)') and the eccentric anomaly (Eq. (4)') in terms of the eccentricity and mean anomaly. In Section 2.1.5, we will present the general forms of these expressions using Bessel functions.

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 8/36

2.1.3 Position and velocity of the satellite

The satellite position $\mathbf{r}$ in the orbital plane is given by a vector of which the components are:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} r \cos v \\ r \sin v \\ 0 \end{pmatrix} = \begin{pmatrix} a (\cos E - e) \\ a \sqrt{1 - e^2} \sin E \\ 0 \end{pmatrix} \quad (5)$$

Using the Kepler equation and its derivative $\frac{dE}{dt} = \frac{n}{1 - e \cos E}$ (n being the mean motion $n = \sqrt{\frac{\mu}{a^3}}$), the velocity, $\dot{\mathbf{r}}$ is:

$$\dot{\mathbf{r}} = \frac{n a}{1 - e \cos E} \begin{pmatrix} -\sin E \\ \sqrt{1 - e^2} \cos E \\ 0 \end{pmatrix} \quad (6)$$

2.1.4 Orbital elements

The orbit lies in space, within a plane of whose orientation is determined by the initial conditions at the time of orbit insertion. This orbital plane, and the satellite's trajectory within it, can be fully described by six orbital elements, or Keplerian elements.

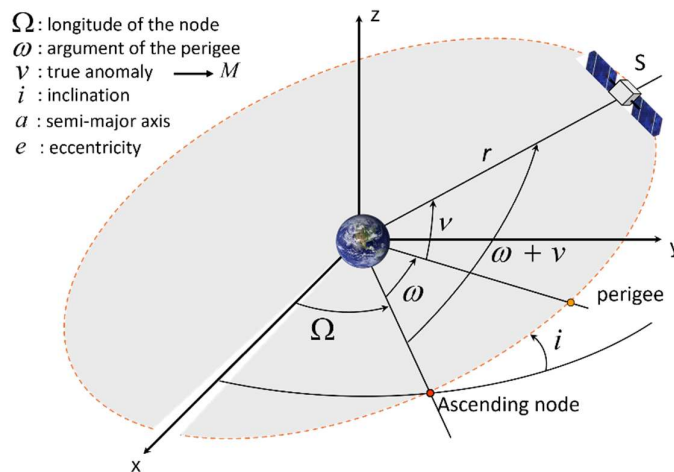


Figure 3: Keplerian elements or orbital elements.

These orbital elements belong to two categories:

$$\underbrace{i, \omega, \Omega}_{\text{position in space}} \quad \underbrace{a, e, v}_{\text{position in the orbit}}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 9/36

2.1.5 Link between position and velocity of the satellite, and orbital elements

To express the position $\mathbf{R}$ and velocity $\dot{\mathbf{R}}$ of the satellite in an inertial reference frame, a coordinate transformation involving the six orbital elements must be applied:

$$\{\mathbf{r}, \dot{\mathbf{r}}\} |_{\text{satellite orbit}} \xrightarrow{R_3(-\Omega)R_1(-i)R_3(-\omega)} \{\mathbf{R}, \dot{\mathbf{R}}\} |_{\text{space}}$$

where $R_1(-i)$ is a rotation around the x-axis of an angle $(-i)$ and similarly for $R_3(-\Omega)$ and $R_3(-\omega)$. Or equivalently

$$\mathbf{R} = R_3(-\Omega)R_1(-i)R_3(-\omega)\mathbf{r}$$

with

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix}, R_3(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This follows Kaula's formulation. Note that Balmino (1992) used another sign convention for the matrices R_1, R_2, R_3 , but he obtains the same final result.

Finally, the rotation matrix $Rot = R_3(-\Omega)R_1(-i)R_3(-w)$ is:

$$Rot = \begin{bmatrix} \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i & -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i & \sin i \sin \Omega \\ \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i & -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i & -\sin i \cos \Omega \\ \sin \omega \sin i & \cos \omega \sin i & \cos i \end{bmatrix}$$

By using this transformation, we obtain:

$$\mathbf{R} = a (\cos E - e) \begin{pmatrix} \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i \\ \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i \\ \sin \omega \sin i \end{pmatrix} + a \sqrt{1 - e^2} \sin E \begin{pmatrix} -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i \\ -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i \\ \cos \omega \sin i \end{pmatrix} \quad (7)$$

And its derivative $\dot{\mathbf{R}}$, the velocity will be:

$$\dot{\mathbf{R}} = [Rot] \frac{d\mathbf{r}}{dt} = [Rot] \begin{bmatrix} -a \sin E \cdot \frac{dE}{dt} \\ a \sqrt{1 - e^2} \cos E \cdot \frac{dE}{dt} \\ 0 \end{bmatrix}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 10/36

$$\begin{aligned}
\dot{\mathbf{R}} &= \frac{n a}{1 - e \cos E} \left\{ -\sin E \begin{pmatrix} \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i \\ \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i \\ \sin \omega \sin i \end{pmatrix} + \sqrt{1 - e^2} \cos E \begin{pmatrix} -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i \\ -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i \\ \cos \omega \sin i \end{pmatrix} \right\} \\
&= \frac{n a^2}{r} \left\{ -\sin E \begin{pmatrix} \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i \\ \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i \\ \sin \omega \sin i \end{pmatrix} \right. \\
&\quad \left. + \sqrt{1 - e^2} \cos E \begin{pmatrix} -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i \\ -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i \\ \cos \omega \sin i \end{pmatrix} \right\} \quad (8)
\end{aligned}$$

The position of the satellite in space is determined using all the orbital elements, except for the ratio $\frac{a}{r}$ and the eccentric anomaly E , which must be computed separately based on the satellite's location along its orbit.

2.1.6 Expressions relating r , E , v as well as their cosine and sine in terms of Bessel functions

We now want to know the values of $\cos E$, $\sin E$.

As seen above, the eccentric anomaly E can be expanded as a Fourier series in terms of the mean anomaly M (Equation 4'). According to Balmino (1992), by applying Lagrange's theorem to the Kepler equation, it can be shown that any function $F(E)$ of the eccentric anomaly E , can be expanded as:

$$F(E) = F(M) + \sum_{n=1}^{\infty} \frac{e^n d^{n-1}}{n! dM^{n-1}} [F'(M) \sin^n M]$$

Which converge for $e < e_0$, $e_0 = 0.6627$.

Then, the general expression for E is:

$$E = M + \sum_{n=1}^{\infty} \frac{e^n}{2^{n-1} n!} \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{j} (n-2j)^{n-1} \sin(n-2j)M \quad (9)$$

where the binomial coefficient is:

$$\binom{n}{j} = \frac{n!}{j! (n-j)!} \quad (10)$$

We then substitute this expansion into Equation 9. Then, the $\cos E$ and $\sin E$ can be expanded as series in M :

$$\cos E = -\frac{e}{2} + \sum_{k=1}^{\infty} \frac{1}{k} (J_{k-1}(ke) - J_{k+1}(ke)) \cos kM \quad (11)$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 11/36

$$\sin E = \sum_{k=1}^{\infty} \frac{1}{k} (J_{k-1}(ke) + J_{k+1}(ke)) \sin kM \quad (12)$$

where $J_k(x)$ are the Bessel functions of the first kind of order k defined by:

$$J_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(kt - x \sin t) dt$$

The Bessel functions have the following properties:

$$J_k(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! (n+k)!} \left(\frac{x}{2}\right)^{k+2n} \text{ for } k \geq 0 \quad (13)$$

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^n}{n! 2^n}\right)^2 \text{ for } k = 0 \quad (13)'$$

$$J_{-k}(x) = \sum_{n=0}^{\infty} (-1)^{n+k} \frac{1}{n! (n+k)!} \left(\frac{x}{2}\right)^{k+2n} \text{ for } (-k) < 0 \quad (13)''$$

$$J_{-k}(x) = (-1)^k J_k(x)$$

$$J_{k-1}(x) - J_{k+1}(x) = 2J'_{k-1}(x)$$

We also need to express the ratio $\frac{a}{r}$, which appears in the expression for the velocity vector $\dot{\mathbf{R}}$. Following Duriez (2007) and starting from Kepler's equation (Eq. (4)), we have that $\frac{dM}{dE} = 1 - e \cos E$, and its inverse is $\frac{dE}{dM} = \frac{a}{r}$. Developing $\frac{a}{r}$ in Fourier series of M , it is possible to invert Kepler's equation and obtain E as function of M . Lastly, using that $\left(\frac{a}{r}\right) dM = dE$ and integrating:

$$\frac{a}{r} = 1 + \sum_{k=1}^{\infty} 2J_k(ke) \cos kM \quad (14)$$

By substituting Eqs. (11)-(14) into Eq. (8), we obtain expressions for the satellite's position ($\mathbf{R}$) and velocity ($\dot{\mathbf{R}}$) in the Keplerian case, fully defined in terms of the orbital elements. This formulation is particularly valuable as it lays the groundwork for the next step: introducing a perturbation potential, where the orbit is no longer strictly Keplerian. In this context, it becomes especially useful to derive analytical developments for quantities

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 12/36

such as $\cos v$, $\sin v$, $\left(\frac{r}{a}\right)$, and more generally $\left(\frac{r}{a}\right)^n (\cos mM, \sin mM)$. Those have been extensively addressed not only by Kaula (1966) and Balmino (1992), but also in the online lecture materials provided by Duriez (2007). Below, we present the resulting expressions:

$$\cos v = -e + 2 \left(\frac{1-e^2}{e} \right) \sum_{k=1}^{\infty} J_k(ke) \cos kM \quad (15)$$

$$\sin v = 2\sqrt{1-e^2} \sum_{k=1}^{\infty} \left(\frac{dJ_k(ke)}{de} \right) \sin kM \quad (16)$$

$$\frac{r}{a} = 1 + \frac{e^2}{2} - \sum_{k=1}^{\infty} \frac{e}{k} (J_{k-1}(ke) - J_{k+1}(ke)) \cos kM \quad (17a)$$

$$\frac{a}{r} = 1 + \sum_{k=1}^{\infty} 2J_k(ke) \cos kM \quad (17b)$$

$$\left(\frac{r}{a}\right)^n (\cos mM, \sin mM) = \sum_{k=-\infty}^{\infty} X_k^{n,m}(e) (\cos kM, \sin kM) \quad (18)$$

where $X_k^{n,m}(e)$ are the so-called Hansen coefficients, and are given by (from Duriez (2007)):

$$X_k^{n,m}(e) = (1+q^2)^{-n-1} \sum_{p=0}^{\infty} (-q)^p \sum_{h=0}^p H_{p-h,h}^{n,m} J_{k-m+p-2h}(ke) \quad (19)$$

and

$$H_{p-h,h}^{n,m} = \binom{n+m+1}{p-h} \binom{n-m+1}{h}$$

which is calculated using Eq. (10); or equivalently, from Balmino (1997), which we will use for the calculation of the eccentric function or $G(e)$ -function later in our technical note:

$$X_k^{n,m}(e) = \frac{1}{(1+q^2)^{n+1}} \sum_{p'=0}^{\infty} \sum_{h=0}^{\infty} \binom{n+m+1}{p'} \binom{n-m+1}{h} (-q)^{p'+h} J_{k-m+p'-h}(ke) \quad (19)'$$

where, considering also Eq. (10), and

$$q = \frac{1 - \sqrt{1-e^2}}{e} = \frac{e}{1 + \sqrt{1-e^2}} \quad (20)$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 13/36

The link with the G -functions of Kaula is given by:

$$G_{npk} = X_{n-2p+k}^{-n-1, n-2p}(e) \quad (21)$$

2.2 Satellite orbit perturbed by a gravitational field

2.2.1 Gravitational potential in spherical harmonics

We now consider the case where a gravitational potential perturbs the orbit, affecting both the spacecraft's position and velocity. To begin, we present the gravitational potential U expressed in terms of spherical harmonics, as defined in an Earth-fixed geocentric reference frame:

$$U = \frac{GM_{\oplus}}{r} + V = \frac{GM_{\oplus}}{r} + \frac{GM_{\oplus}}{r} \sum_{n=1}^{\infty} \left(\frac{a_{\oplus}}{r}\right)^n \left\{ \sum_{m=0}^n [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda] P_{nm}(\sin \theta) \right\} \quad (22a)$$

Where the perturbing potential V is given by:

$$\begin{aligned}
 V &= \sum_{n=1}^{\infty} \sum_{m=0}^n V_{nm} \\
 &= \frac{GM_{\oplus}}{r} \sum_{n=1}^{\infty} \left(\frac{a_{\oplus}}{r}\right)^n \left\{ \sum_{m=0}^n [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda] P_{nm}(\sin \theta) \right\} \\
 &= \frac{GM_{\oplus}}{r} \sum_{n=1}^{\infty} \left(\frac{a_{\oplus}}{r}\right)^n \left\{ \sum_{m=0}^n [\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda] \bar{P}_{nm}(\sin \theta) \right\} \quad (22b)
 \end{aligned}$$

with

$$V_{nm} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^n [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda] P_{nm}(\sin \theta) \quad (23a)$$

$$= \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^n [\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda] \bar{P}_{nm}(\sin \theta) \quad (23b)$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 14/36

where $M_{\oplus}$ is the Earth mass, $a_{\oplus}$ is the mean equatorial radius, C_{nm} and S_{nm} are the so-called Stokes coefficients (unnormalized), P_{nm} and $\bar{P}_{nm}$ are respectively, the unnormalized and normalized associated Legendre polynomials, and (θ, λ) the colatitude and longitude, with

$$\bar{P}_{nm} = \sqrt{(2 - \delta_m^0)(2n + 1) \frac{(n - m)!}{(n + m)!}} P_{nm} \quad (24)$$

In this treatment, we assume that $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are constants; however, in practice, they vary due to geodynamic effects.

In Eqs. (22) and (23), the first term corresponds to the perturbation already considered in the Kepler case in Section 2.1.

2.2.2 Force acting on the satellite

The gradient of the potential gives the force acting on the satellite in the non-rotating geocentric reference system:

$$\dot{\mathbf{r}} = \nabla U = \nabla U(r'(t), \theta'(t), \lambda'(t)) \quad (25)$$

Where $r'(t), \theta'(t), \lambda'(t)$ are the distance, latitude and longitude of the spacecraft and are related to the position of the spacecraft $\mathbf{r}(t)$ with the Earth rotating below it. However, the spacecraft position can also be expressed by using the orbital elements as there is a bijective mapping:

$$\{\mathbf{r}(t), \dot{\mathbf{r}}(t)\} = \{x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)\} \rightarrow \{a(t), e(t), i(t), \Omega(t), \omega(t), M(t)\}$$

Note that in the Keplerian motion, the 5 parameters related to the orbit (a, e, i, Ω, ω) remain constant while only M (see Eq. (1)) varies ($\dot{a}, \dot{e}, \dot{i}, \dot{\Omega}, \dot{\omega}, \dot{M} - n = 0$).

As a result of this mapping, the force acting on the satellite —or equivalently, the potential $U(r'(t), \theta'(t), \lambda'(t))$ — can be expressed in terms of the orbital elements $U(a(t), e(t), i(t), \Omega(t), \omega(t), M(t))$. This approach was used by Kaula.

To establish this, Kaula begins with the osculating orbit plane, where the satellite's coordinates are given by Eq. (5). If the potential field U deviates from the Newtonian case, the elliptical orbit will continuously evolve. However, we expect the orbital parameters to change gradually over time.

In inertial space, the satellite coordinates $x(t), y(t), z(t) = x_i(t)$ and its velocity $\dot{x}_i(t)$ are used in the motion equation:

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 15/36

$$\frac{d\dot{x}_i(t)}{dt} = \frac{\partial U}{\partial x_i} \quad (26)$$

By expressing the rates of change $\frac{dx_i}{dt}$ and $\frac{d\dot{x}_i}{dt}$ as functions of the rates of change $\frac{ds_k}{dt}$ of the six Keplerian elements where s_k represents any of $a, e, i, \Omega, \omega, M$, and applying the method of variations of constants, one arrives at the so-called Lagrange equations. These equations relate the variations in the orbital elements to the derivatives of the potential U with respect to them. They yield:

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{na} \frac{\partial V}{\partial M} \\ \frac{de}{dt} &= \frac{1-e^2}{na^2e} \frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e} \frac{\partial V}{\partial \omega} \\ \frac{d\omega}{dt} &= -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i} \frac{\partial V}{\partial i} + \frac{\sqrt{1-e^2}}{na^2e} \frac{\partial V}{\partial e} \\ \frac{di}{dt} &= \frac{\cos i}{na^2\sqrt{1-e^2}\sin i} \frac{\partial V}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial V}{\partial \Omega} \\ \frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial V}{\partial i} \\ \frac{dM}{dt} &= n - \frac{1-e^2}{na^2e} \frac{\partial V}{\partial e} - \frac{2}{na} \frac{\partial V}{\partial a} \end{aligned} \quad (27)$$

Note that these equations are only valid for e and i not too small. They are of the form:

$$\begin{aligned} \left\{ \frac{da}{dt}, \frac{de}{dt}, \frac{di}{dt} \right\} &= f_1 \left(a, e, i, \frac{\partial V}{\partial \Omega}, \frac{\partial V}{\partial \omega}, \frac{\partial V}{\partial M} \right) \\ \left\{ \frac{d\Omega}{dt}, \frac{d\omega}{dt}, \frac{dM}{dt} \right\} &= f_2 \left(a, e, i, \frac{\partial V}{\partial a}, \frac{\partial V}{\partial e}, \frac{\partial V}{\partial i} \right) \end{aligned}$$

The orbital elements (a, e, i) are referred to as the metric orbital elements, while (Ω, ω, M) are known as the **angular orbital elements**. In the first approximation, the metric orbital elements can be considered constant, whereas the angular orbital elements vary slowly with time (Capderou, 2005).

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 16/36

For the cases where e and i are small, it is necessary to switch to a different set of variables. These two situations (e small and i small) can lead to difficulties in defining certain parameters, such as the perigee when e is small and the ascending node when i is small.

Additionally, when analyzing the Lagrange equations, these parameters appear in the denominators as e or $\sin i$. To address this, we need to express the perturbing potential in terms of the new orbital parameters. The choice of these new parameters helps to eliminate the indeterminacies that appear in the equations. For further details, we refer to the book by Capderou (2005).

2.2.3 Expression of the perturbing function in V in terms of osculating orbital elements

For this part, we neglect the Earth's rotation, which can later be accounted for by adding the Greenwich Mean Sidereal Time ($GMST$) to the longitude.

In Eq. (22), we encounter $\cos m\lambda$ and $\sin m\lambda$ and $P_{nm}(\cos \theta)$, which can be expressed using relations derived from the spherical triangle formed by the orbit, the equator, and the satellite's meridian. This is illustrated in Figure 4.

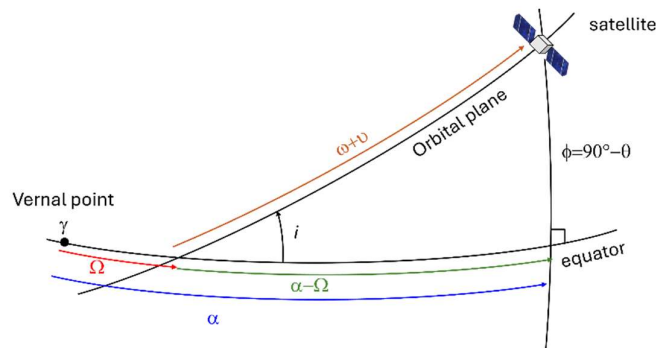


Figure 4: Orbit-equator-meridian triangle.

Where γ is the vernal point, α is the right ascension, ϕ is the latitude of the spacecraft. We thus obtain:

$$\cos(\alpha - \Omega) = \frac{\cos(\omega + \nu)}{\cos \phi} \quad (28)$$

$$\sin(\alpha - \Omega) = \frac{\sin(\omega + \nu) \cos i}{\cos \phi} \quad (29)$$

$$\sin \phi = \sin(\omega + \nu) \sin i \quad (30)$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 17/36

In order to account for the Earth rotation, we express the longitude of the spacecraft as

$$\lambda = \alpha - GMST = (\alpha - \Omega) + (\Omega - GMST) \quad (31)$$

We also use the trigonometric identities:

$$\cos mx = Re \left\{ \sum_{s=0}^m \binom{m}{s} I^s \cos^{m-s} x \sin^s x \right\} \quad (32)$$

$$\sin mx = Re \left\{ \sum_{s=0}^m \binom{m}{s} I^{s-1} \cos^{m-s} x \sin^s x \right\} \quad (33)$$

Where Re denotes the real part, I is $\sqrt{-1}$ and the binomial coefficient $\binom{m}{s}$ is calculated as Eq. (10).

Taking into consideration λ 's value from Eq. (31), we have

$$\cos m\lambda = \cos m(\alpha - \Omega) \cos m(\Omega - GMST) - \sin m(\alpha - \Omega) \sin m(\Omega - GMST) \quad (34)$$

$$\sin m\lambda = \sin m(\alpha - \Omega) \cos m(\Omega - GMST) + \cos m(\alpha - \Omega) \sin m(\Omega - GMST) \quad (35)$$

2.2.3.1 Expression of the perturbing function in V in terms of inclination – $F(i)$ function

Following Kaula (1966), using Eqs. (28)-(35), we obtain

$$\begin{aligned} \cos m\lambda = Re \left\{ \sum_{s=0}^m \binom{m}{s} I^s \frac{\cos^{m-s}(\omega + v) \sin^s(\omega + v) \cos^s i}{\cos^m \phi} \right. \\ \left. \times [\cos m(\Omega - GMST) + I \sin m(\Omega - GMST)] \right\} \quad (36) \end{aligned}$$

$$\begin{aligned} \sin m\lambda = Re \left\{ \sum_{s=0}^m \binom{m}{s} I^s \frac{\cos^{m-s}(\omega + v) \sin^s(\omega + v) \cos^s i}{\cos^m \phi} \right. \\ \left. \times [\sin m(\Omega - GMST) - I \cos m(\Omega - GMST)] \right\} \quad (37) \end{aligned}$$

Substituting Eqs. (30), (36), (37) into Eq. (23) and changing the index in the sum, as outlined by Kaula (1966), we can obtain:

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 18/36

$$V_{nm} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^n \sum_{t=0}^k T_{nmt} \sin^{n-m-2t} i \operatorname{Re}\{[(C_{nm} - IS_{nm}) \cos m(\Omega - GMST) + (S_{nm} + IC_{nm}) \sin m(\Omega - GMST)]\} \\ \times \sum_{s=0}^m \binom{m}{s} I^s \cos^s i \frac{(-I)^{-m-2t+s}}{2^{n-2t}} \sum_{c=0}^{l-m-2t+s} \sum_{d=0}^{m-s} \binom{n-m-2t+s}{c} \binom{m-s}{d} (-1)^c (\cos(n-2t-2c-2d)(\omega+v) + I \sin(n-2t-2c-2d)(\omega+v)) \quad (38)$$

Where the function T_{nmt} is $T_{nmt} = \frac{(2n)!}{(n-m)!2^{2n}n!}$. After some manipulations, V_{nm} becomes:

$$V_{nm} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^n \sum_{p=0}^n F_{nmp}(i) \left\{ \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos((n-2p)(\omega+v) + m(\Omega - GMST)) \right. \\ \left. + \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin((n-2p)(\omega+v) + m(\Omega - GMST)) \right\} \quad (38')$$

Where the Inclination function $F_{nmp}(i)$ is:

$$F_{nmp}(i) = \sum_{t=0}^{\min(p,k)} \frac{(2n-2t)!}{t!(n-t)!(n-m-2t)!2^{2n-2t}} \sin^{n-m-2t} i \\ \times \sum_{s=0}^m \binom{m}{s} \cos^s i \sum_{c=0}^{\max \text{ possible}} \binom{n-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \quad (39)$$

with k being the integer part of $\frac{n-m}{2}$. In practice, the bounds for the variable c are limited to ensure that both binomial coefficients involved are defined and nonzero. Specifically, for the first binomial term, c must satisfy $0 \leq c \leq n-m-2t+s$, while for the second binomial term, the constraint is $0 \leq p-t-c \leq m-s$. Regarding the parameters $[n, m, p]$, the degree $n \in \mathbb{N}$, the order m , which represents the longitudinal variation in the potential, must be $0 \leq m \leq n$, and finally the index p must also lie within $0 \leq p \leq n$.

Example of degree 2 and order 0 (or $n = 2, m = 0$):

$$V_{2m} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^2 \sum_{p=0}^2 F_{2mp}(i) \left\{ \begin{bmatrix} C_{2m} \\ -S_{2m} \end{bmatrix} \begin{matrix} 2-m \text{ even} \\ 2-m \text{ odd} \end{matrix} \cos((2-2p)(\omega+v) + m(\Omega - GMST)) \right. \\ \left. + \begin{bmatrix} S_{2m} \\ C_{2m} \end{bmatrix} \begin{matrix} 2-m \text{ even} \\ 2-m \text{ odd} \end{matrix} \sin((2-2p)(\omega+v) + m(\Omega - GMST)) \right\} \\ F_{2mp}(i) = \sum_{t=0}^{\min(p,k)} \frac{(4-2t)!}{t!(2-t)!(2-m-2t)!2^{4-2t}} \sin^{2-m-2t} i \\ \times \sum_{s=0}^m \binom{m}{s} \cos^s i \sum_{c=0}^{\max \text{ possible}} \binom{2-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k}$$

In that case, $k = 1$:

  ROYAL OBSERVATORY of BELGIUM	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 19/36

$$V_{20} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r} \right)^2 \sum_{p=0}^2 F_{20}(i) \{C_{20} \cos((2-2p)(\omega+v)) + S_{20} \sin((2-2p)(\omega+v))\}$$

(By definition the term S_{n0} is 0, so this last term will disappear.)

$$F_{20}(i) = \sum_{t=0}^{\min(p,1)} \frac{(4-2t)!}{t!(2-t)!(2-2t)! 2^{4-2t}} \sin^{2-2t} i \sum_{c=0}^{\max \text{ possible}} \binom{2-2t}{c} \binom{0}{p-t-c} (-1)^{c-k}$$

$$F_{200}(i) = \frac{-3}{2^3} \sin^2 i \binom{2}{0} = \frac{-3}{8} \sin^2 i$$

$$F_{201}(i) = \frac{3}{4} \sin^2 i + \frac{-1}{2}$$

$$F_{202}(i) = \frac{-3}{8} \sin^2 i$$

The other values of the Inclination Functions or F_{nmp} , can be found in the Table 1 from Kaula (1966):

Table 1: Kaula's $F_{nmp}(i)$ inclination function values, with $n = [2, 4]$.

TABLE 1
Inclination Functions $F_{lmp}(i)$ from Equation (3.62)

l	m	p	$F_{lmp}(i)$
2	0	0	$-3 \sin^2 i / 8$
2	0	1	$3 \sin^2 i / 4 - 1/2$
2	0	2	$-3 \sin^2 i / 8$
2	1	0	$3 \sin i (1 + \cos i) / 4$
2	1	1	$-3 \sin i \cos i / 2$
2	1	2	$-3 \sin i (1 - \cos i) / 4$
2	2	0	$3(1 + \cos i)^2 / 4$
2	2	1	$3 \sin^2 i / 2$
2	2	2	$3(1 - \cos i)^2 / 4$
3	0	0	$-5 \sin^3 i / 16$
3	0	1	$15 \sin^3 i / 16 - 3 \sin i / 4$
3	0	2	$-15 \sin^3 i / 16 + 3 \sin i / 4$
3	0	3	$5 \sin^3 i / 16$
3	1	0	$-15 \sin^2 i (1 + \cos i) / 16$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 20/36

3	1	1	$15 \sin^2 i(1 + 3 \cos i)/16 - 3(1 + \cos i)/4$
3	1	2	$15 \sin^2 i(1 - 3 \cos i)/16 - 3(1 - \cos i)/4$
3	1	3	$-15 \sin^2 i(1 - \cos i)/16$
3	2	0	$15 \sin i(1 + \cos i)^2/8$
3	2	1	$15 \sin i(1 - 2 \cos i - 3 \cos^2 i)/8$
3	2	2	$-15 \sin i(1 + 2 \cos i - 3 \cos^2 i)/8$
3	2	3	$-15 \sin i(1 - \cos i)^2/8$
3	3	0	$15(1 + \cos i)^3/8$
3	3	1	$45 \sin^2 i(1 + \cos i)/8$
3	3	2	$45 \sin^2 i(1 - \cos i)/8$
3	3	3	$15(1 - \cos i)^3/8$
4	0	0	$35 \sin^4 i/128$
4	0	1	$-35 \sin^4 i/32 + 15 \sin^2 i/16$
4	0	2	$(105/64) \sin^4 i - (15/8) \sin^2 i + 3/8$
4	0	3	$-(35/32) \sin^4 i + (15/16) \sin^2 i$
4	0	4	$(35/128) \sin^4 i$
4	1	0	$-(35/32) \sin^3 i(1 + \cos i)$
4	1	1	$(35/16) \sin^3 i(1 + 2 \cos i) - (15/8)(1 + \cos i) \sin i$
4	1	2	$\cos i(15 \sin i/4 - 105 \sin^3 i/16)$
4	1	3	$-(35/16) \sin^3 i(1 - 2 \cos i) + (15/8) \sin i(1 - \cos i)$
4	1	4	$(35/32) \sin^3 i(1 - \cos i)$
4	2	0	$-(105/32) \sin i(1 + \cos i)^2$
4	2	1	$(105/8) \sin^2 i \cos i(1 + \cos i) - (15/8)(1 + \cos i)^2$
4	2	2	$(105/16) \sin^2 i(1 - 3 \cos^2 i) + (15/4) \sin^2 i$
4	2	3	$-(105/8) \sin^2 i \cos i(1 - \cos i) - (15/8)(1 - \cos i)^2$
4	2	4	$-(105/32) \sin^2 i(1 - \cos i)^2$
4	3	0	$(105/16) \sin i(1 + \cos i)^3$
4	3	1	$(105/8) \sin i(1 - 3 \cos^2 i - 2 \cos^3 i)$
4	3	2	$-(315/8) \sin^3 i \cos i$
4	3	3	$-(105/8) \sin i(1 - 3 \cos^2 i + 2 \cos^3 i)$
4	3	4	$-(105/16) \sin i(1 - \cos i)^3$
4	4	0	$(105/16)(1 + \cos i)^4$
4	4	1	$(105/4) \sin^2 i(1 + \cos i)^2$
4	4	2	$(315/8) \sin^4 i$
4	4	3	$(105/4) \sin^2 i(1 - \cos i)^2$
4	4	4	$(105/16)(1 - \cos i)^4$

Note that Kaula nor any of the authors considered on this study did the calculations of $F_{nmp}(i)$, $G_{nmp}(e)$ nor V_{nm} for $n < 2$. We have computed the tables for $F_{1mp}(i)$ and $G_{1m}(e)$ using Mathematica. Those values are shown in Table 2 ($F_{nmp}(i)$), and Table 3 ($G_{nmp}(e)$) up to $n = 3$, to show that for higher orders we obtain the same results as Kaula.

		REFERENCE :	GENESIS-ROB-TN-0006	
		DATE : DD/MM/YEAR	11/07/2025	
		VERSION.ISSUE:	1.1	Page : 21/36
		DRAFT:	5	

Table 2: Inclination functions $F_{nmp}(i)$, for degrees $n = [1, 3]$.

n	m	p	F
1	0	0	$\frac{\sin[i]}{2}$
1	0	1	$-\frac{\sin[i]}{2}$
1	1	0	$\frac{1}{2} (1 + \cos[i])$
1	1	1	$\frac{1}{2} (1 - \cos[i])$
2	0	0	$-\frac{3}{8} \sin[i]^2$
2	0	1	$-\frac{1}{2} + \frac{3 \sin[i]^2}{4}$
2	0	2	$-\frac{3}{8} \sin[i]^2$
2	1	0	$\frac{3}{4} (1 + \cos[i]) \sin[i]$
2	1	1	$-\frac{3}{2} \cos[i] \times \sin[i]$
2	1	2	$\frac{3}{4} (-1 + \cos[i]) \sin[i]$
2	2	0	$\frac{3}{4} (1 + 2 \cos[i] + \cos[i]^2)$
2	2	1	$\frac{3}{4} (2 - 2 \cos[i]^2)$
2	2	2	$\frac{3}{4} (1 - 2 \cos[i] + \cos[i]^2)$
3	0	0	$-\frac{5}{16} \sin[i]^3$
3	0	1	$-\frac{3 \sin[i]}{4} + \frac{15 \sin[i]^3}{16}$
3	0	2	$\frac{3 \sin[i]}{4} - \frac{15 \sin[i]^3}{16}$
3	0	3	$\frac{5 \sin[i]^3}{16}$
3	1	0	$\frac{15}{16} (-1 - \cos[i]) \sin[i]^2$
3	1	1	$\frac{3}{4} (-1 - \cos[i]) + \frac{15}{16} (1 + 3 \cos[i]) \sin[i]^2$
3	1	2	$\frac{3}{4} (-1 + \cos[i]) + \frac{15}{16} (1 - 3 \cos[i]) \sin[i]^2$
3	1	3	$\frac{15}{16} (-1 + \cos[i]) \sin[i]^2$
3	2	0	$\frac{15}{8} (1 + 2 \cos[i] + \cos[i]^2) \sin[i]$
3	2	1	$\frac{15}{8} (1 - 2 \cos[i] - 3 \cos[i]^2) \sin[i]$
3	2	2	$\frac{15}{8} (-1 - 2 \cos[i] + 3 \cos[i]^2) \sin[i]$
3	2	3	$\frac{15}{8} (-1 + 2 \cos[i] - \cos[i]^2) \sin[i]$
3	3	0	$\frac{15}{8} (1 + 3 \cos[i] + 3 \cos[i]^2 + \cos[i]^3)$
3	3	1	$\frac{15}{8} (3 + 3 \cos[i] - 3 \cos[i]^2 - 3 \cos[i]^3)$
3	3	2	$\frac{15}{8} (3 - 3 \cos[i] - 3 \cos[i]^2 + 3 \cos[i]^3)$
3	3	3	$\frac{15}{8} (1 - 3 \cos[i] + 3 \cos[i]^2 - \cos[i]^3)$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 22/36

2.2.3.2 Expression of the perturbing function in V in terms of eccentricity – $G(e)$ function

2.2.3.2.1 Kaula's approach

In Eq. (38), we still need to express, in terms of the orbital elements, $\left(\frac{1}{r}\right)^{n+1}$, as well as the arguments of the cosines and sines entering the expression of v_{nm} . To that aim, we use Eqs. (15)-(18) to obtain:

$$\begin{aligned} \left(\frac{1}{r^{n+1}}\right) \begin{bmatrix} \cos \\ \sin \end{bmatrix} ((n-2p)(\omega+v) + m(\Omega - GMST)) \\ = \left(\frac{1}{a^{n+1}}\right) \sum_t^{\pm n} G_{npt}(e) \begin{bmatrix} \cos \\ \sin \end{bmatrix} ((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST)) \end{aligned}$$

where $G_{npt}(e)$ will be different for the long periods, i.e. those where M is absent ($t = 2p - n$), and the short periods containing M , when $n - 2p + t \neq 0$.

- **For the long periods ($t = 2p - n$):**

$$G_{np(2p-n)}(e) = \frac{1}{(1-e^2)^{n-\frac{1}{2}}} \sum_{d=0}^{p'-1} \binom{n-1}{2d+n-2p'} \binom{2d+n-2p'}{d} \left(\frac{e}{2}\right)^{2d+n-2p'} \quad (40a)$$

in which

$$p' = \begin{cases} p, & \text{for } p \leq \frac{n}{2} \\ n-p, & \text{for } p \geq \frac{n}{2} \end{cases}$$

- **For the short periods ($n - 2p + t \neq 0$):**

$$G_{npt}(e) = (-1)^{|t|} (1+q^2)^n q^{|t|} \sum_{k=0}^{\infty} P_{nptk} Q_{nptk} q^{2k} \quad (40b)$$

where q is given by Eq. (20) and

$$P_{nptk} = \sum_{s=0}^h \frac{(-1)^s}{s!} \binom{2p'-2n}{h-s} \left(\frac{(n-2p'+t')e}{2q} \right)^s$$

with

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 23/36

$$h = \begin{cases} k + t', & \text{for } t' > 0 \\ k, & \text{for } t' < 0 \end{cases}$$

and

$$p' = \begin{cases} p, & t' = t, & \text{for } p \leq \frac{n}{2} \\ n - p, & t' = -t & \text{for } p > \frac{n}{2} \end{cases}$$

In parallel,

$$Q_{nptk} = \sum_{s=0}^h \frac{1}{s!} \binom{-2p'}{h-s} \left(\frac{(n-2p'+t')e}{2q} \right)^s$$

Note that a binomial coefficient with a negative upper argument is defined using the generalized formula $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$, for $k, n \in \mathbb{N}$; thus:

$$\binom{-2p'}{h-s} = (-1)^{h-s} \binom{2p' + h - s - 1}{h-s}$$

with

$$h = \begin{cases} k, & \text{for } t' > 0 \\ k - t', & \text{for } t' < 0 \end{cases}$$

and the same values for p' under the same conditions as shown at the beginning of this page.

Everything can then be expressed as:

$$V_{nm} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right)^n \sum_{p=0}^n F_{nmp}(i) \sum_{t=-\infty}^{\infty} G_{npt}(e) S_{nmpt}(\omega, M, \Omega, GMST) \quad (41)$$

$$\begin{aligned} S_{nmpt}(\omega, M, \Omega, GMST) &= \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST)) \\ &+ \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST)) \quad (42) \end{aligned}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 24/36

2.2.3.2.2 Balmino's approach

$$\begin{aligned}
V_{nm} &= \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^n \sum_{p=0}^n F_{nmp}(i) \left\{ \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos((n-2p)(\omega + v) + m(\Omega - GMST)) \right. \\
&\quad \left. + \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin((n-2p)(\omega + v) + m(\Omega - GMST)) \right\} \\
&= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a}\right)^n \sum_{p=0}^n F_{nmp}(i) \sum_{t=-\infty}^{\infty} G_{npt}(e) \left\{ \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos((n-2p)\omega + (n-2p+t)M \right. \\
&\quad \left. + m(\Omega - GMST)) + \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST)) \right\}
\end{aligned}$$

Using Eqs. (19)' and (21), and using Taylor expansions in terms of the eccentricity (as e is usually quite small – the expressions are valid for $e < 0.6627434$, see Duriez, 2007), we then obtain:

$t > 0$	
$G_{npt}(e) = \left(\frac{-e}{2}\right)^t \sum_{t'=0}^t \binom{2p-2n}{t-t'} \frac{(-1)^{t'}}{t'!} (n-2p+t)^{t'} + \Theta(e^{t+2})$	
$p < n$	$p = n$
$\binom{2p-2n}{t-t'} = (-1)^{t-t'} \binom{2n-2p+t-t'-1}{t-t'}$	$G_{npt}(e) = \left(\frac{-e}{2}\right)^t \frac{(-1)^t}{t!} (t-n)^t + \Theta(e^{t+2})$

$t < 0$	
$G_{npt}(e) = \left(\frac{-e}{2}\right)^{-t} \sum_{t'=0}^{-t} \binom{-2p}{-t-t'} \frac{(n-2p+t)^{t'}}{t'!} + \Theta(e^{-t+2})$	
$p > 0$	$p = 0$
$\binom{-2p}{-t-t'} = (-1)^{-t-t'} \binom{2p-t-t'-1}{-t-t'}$	$G_{npt}(e) = \left(\frac{-e}{2}\right)^{-t} \frac{(t+n)^{-t}}{(-t)!} + \Theta(e^{-t+2})$

1. When $t = 0$:

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 25/36

$$G_{np0}(e) = 1 + \Theta(e^2)$$

The second term needs to be computed. This is done using Eq. (19)', as shown by Balmino (1992) that $X_k^{n,m}(e) = \Theta(e^{|k-m|})$. Then:

$$G_{np}(e) = 1 + [n + (4p - 3n)(n - 4p)] \frac{e^2}{4}$$

Note that for some of these functions, Duriez (2007) also provide expressions, which help us for the next step.

Verification case: degree-2 ($n = 2$), order-0 ($p = 0$), $k = -1$ we should obtain $G_{20-}(e) = \frac{-e}{2} + \frac{e^3}{16}$ (from Kaula's table). Note also that $G_{20-1}(e) = G_{221}(e)$.

From Eq. (21) we have:

$$G_{20-1} = X_1^{-3,2}(e)$$

Recalling Eq. (19)':

$$X_k^{n,m}(e) = \frac{1}{(1+q^2)^{n+1}} \sum_{p'=0}^{\infty} \sum_{h=0}^{\infty} \binom{n+m+1}{p'} \binom{n-m+1}{h} (-q)^{p'+h} J_{k-m+p'-h}(ke) \quad (19)'$$

Following Balmino, from Eq. (19)' and being $\binom{-n'}{h} = (-1)^h \binom{n'+h-1}{h}$, we have:

$$X_{k=1}^{n=-3,m=2}(e) = (1+q^2)^2 \sum_{h=0}^{\infty} \binom{-4}{h} (-q)^h J_{-1-h}(e) = (1+q^2)^2 \sum_{h=0}^{\infty} \binom{3+h}{h} q^h J_{-1-h}(e)$$

Or, following Duriez:

$$X_k^{n,m}(e) = \frac{1}{(1+q^2)^{n+1}} \sum_{p'=0}^{\infty} \sum_{h=0}^{\infty} \binom{n+m+1}{p'-h} \binom{n-m+1}{h} (-q)^{p'} J_{k-m+p'-2h}(ke) \quad (19)''$$

$$X_{k=1}^{n=-3,m=2}(e) = (1+q^2)^2 \sum_{h=0}^{\infty} \binom{3+h}{h} (q)^h J_{-1-h}(e)$$

Using Eq. (13)'' to calculate $J_{-1-h}(e)$, and substituting yields:

$$G_{20-1} = X_{k=1}^{n=-3,m=2}(e) = (1+q^2)^2 \sum_{h=0}^{\infty} \binom{3+h}{h} (q)^h \sum_{t=0}^{\infty} (-1)^{t+h+1} \frac{1}{t!(t+h+1)!} \left(\frac{e}{2}\right)^{2t+h+1}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 26/36

$$\begin{aligned}
&= (1 + q^2)^2 \left\{ \left[\sum_{t=0}^{\infty} (-1)^{t+1} \frac{1}{t! (t+1)!} \left(\frac{e}{2}\right)^{2t+1} \right] + \left[\sum_{t=0}^{\infty} (-1)^t \frac{4q}{t! (t+2)!} \left(\frac{e}{2}\right)^{2t+2} \right] \right. \\
&\quad \left. + \left[\sum_{t=0}^{\infty} (-1)^{t+3} \frac{10 q^2}{t! (t+3)!} \left(\frac{e}{2}\right)^{2t+3} \right] + \dots \right\} \\
&= (1 + q^2)^2 \left\{ \left[-\frac{e}{2} + \frac{1}{2} \left(\frac{e}{2}\right)^3 - \frac{1}{12} \left(\frac{e}{2}\right)^5 + \dots \right] + \left[2q \left(\frac{e}{2}\right)^2 - \frac{2q}{3} \left(\frac{e}{2}\right)^4 + \dots \right] + \left[-\frac{5 q^2}{3} \left(\frac{e}{2}\right)^3 + \frac{5 q^2}{12} \left(\frac{e}{2}\right)^5 + \dots \right] \right. \\
&\quad \left. + \dots \right\} \\
&= (1 + q^2)^2 \left\{ -\frac{e}{2} + 2q \left(\frac{e}{2}\right)^2 + \left(\frac{1}{2} - \frac{5 q^2}{3}\right) \left(\frac{e}{2}\right)^3 - \frac{2q}{3} \left(\frac{e}{2}\right)^4 + \left(-\frac{1}{12} + \frac{5 q^2}{12}\right) \left(\frac{e}{2}\right)^5 + \dots \right\} = \frac{-e}{2} + \frac{e^3}{16} + \dots
\end{aligned}$$

Kaula's table for the eccentricity function $G_{npt}(e)$ does not provide values for $n < 2$. In Table 3, we present the corresponding results for $G_{npt}(e)$ with $n = [1, 3]$. In addition to verifying our results against Kaula's table, we independently cross-checked them with calculations performed by our colleague Antony Trinh, and found full agreement.

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:		1.1 5 Page : 27/36

Table 3: Eccentricity function $G_{npt}(e)$ for degrees $n = [1, 3]$.

n	p	t	G
1	0	-1	0
1	0	0	$1 - \frac{e^2}{2} - \frac{e^4}{64} - \frac{29e^6}{1152} - \frac{2347e^8}{147456} - 0.0113869e^{10} + 0[e]^{11}$
1	0	1	$2e - \frac{3e^3}{2} + \frac{5e^5}{24} - \frac{e^7}{18} - \frac{41e^9}{1920} + 0[e]^{11}$
1	1	-1	$2e - \frac{3e^3}{2} + \frac{5e^5}{24} - \frac{e^7}{18} - \frac{41e^9}{1920} + 0[e]^{11}$
1	1	0	$1 - \frac{e^2}{2} - \frac{e^4}{64} - \frac{29e^6}{1152} - \frac{2347e^8}{147456} - 0.0113869e^{10} + 0[e]^{11}$
1	1	1	0
1	2	-1	$4e - 24e^3 + \frac{241e^5}{6} - \frac{455e^7}{18} + \frac{493e^9}{64} + 0[e]^{11}$
1	2	0	$1 - \frac{17e^2}{2} + \frac{951e^4}{64} - \frac{973e^6}{128} + \frac{25757e^8}{16384} - 0.254191e^{10} + 0[e]^{11}$
1	2	1	$-2e + \frac{9e^3}{2} - \frac{41e^5}{24} + \frac{43e^7}{288} - \frac{101e^9}{1920} + 0[e]^{11}$
2	0	-2	0
2	0	-1	$-\frac{e}{2} + \frac{e^3}{16} - \frac{5e^5}{384} - \frac{143e^7}{18432} - \frac{9697e^9}{1474560} + 0[e]^{11}$
2	0	0	$1 - \frac{5e^2}{2} + \frac{13e^4}{16} - \frac{35e^6}{288} - \frac{5e^8}{576} - \frac{49e^{10}}{3600} + 0[e]^{11}$
2	0	1	$\frac{7e}{2} - \frac{123e^3}{16} + \frac{489e^5}{128} - \frac{1763e^7}{2048} + \frac{13527e^9}{163840} + 0[e]^{11}$
2	0	2	$\frac{17e^2}{2} - \frac{115e^4}{6} + \frac{601e^6}{48} - \frac{1423e^8}{360} + \frac{48619e^{10}}{69120} + 0[e]^{11}$
2	1	-2	$\frac{9e^2}{4} + \frac{7e^4}{4} + \frac{141e^6}{64} + \frac{197e^8}{80} + \frac{62481e^{10}}{23040} + 0[e]^{11}$
2	1	-1	$\frac{3e}{2} + \frac{27e^3}{16} + \frac{261e^5}{128} + \frac{14309e^7}{6144} + \frac{423987e^9}{163840} + 0[e]^{11}$
2	1	0	$\frac{1}{(1-e^2)^{3/2}}$
2	1	1	$\frac{3e}{2} + \frac{27e^3}{16} + \frac{261e^5}{128} + \frac{14309e^7}{6144} + \frac{423987e^9}{163840} + 0[e]^{11}$
2	1	2	$\frac{9e^2}{4} + \frac{7e^4}{4} + \frac{141e^6}{64} + \frac{197e^8}{80} + \frac{62481e^{10}}{23040} + 0[e]^{11}$
2	2	-2	$\frac{17e^2}{2} - \frac{115e^4}{6} + \frac{601e^6}{48} - \frac{1423e^8}{360} + \frac{48619e^{10}}{69120} + 0[e]^{11}$
2	2	-1	$\frac{7e}{2} - \frac{123e^3}{16} + \frac{489e^5}{128} - \frac{1763e^7}{2048} + \frac{13527e^9}{163840} + 0[e]^{11}$
2	2	0	$1 - \frac{5e^2}{2} + \frac{13e^4}{16} - \frac{35e^6}{288} - \frac{5e^8}{576} - \frac{49e^{10}}{3600} + 0[e]^{11}$
2	2	1	$-\frac{e}{2} + \frac{e^3}{16} - \frac{5e^5}{384} - \frac{143e^7}{18432} - \frac{9697e^9}{1474560} + 0[e]^{11}$
2	2	2	0
3	0	-3	0
3	0	-2	$\frac{e^2}{8} + \frac{e^4}{48} + \frac{55e^6}{3072} + \frac{1177e^8}{92160} + 0.00977585e^{10} + 0[e]^{11}$
3	0	-1	$-e + \frac{5e^3}{4} - \frac{7e^5}{48} + \frac{23e^7}{288} + \frac{61e^9}{1440} + 0[e]^{11}$
3	0	0	$1 - 6e^2 + \frac{423e^4}{64} - \frac{125e^6}{64} + \frac{7533e^8}{16384} + \frac{81e^{10}}{1600} + 0[e]^{11}$
3	0	1	$5e - 22e^3 + \frac{607e^5}{24} - \frac{98e^7}{9} + \frac{32753e^9}{11520} + 0[e]^{11}$
3	0	2	$\frac{127e^2}{8} - \frac{3065e^4}{48} + \frac{243805e^6}{3072} - \frac{798865e^8}{18432} + 13.9977e^{10} + 0[e]^{11}$
3	0	3	$\frac{163e^3}{4} - \frac{2577e^5}{16} + \frac{1089e^7}{5} - \frac{181691e^9}{1280} + 0[e]^{11}$
3	1	-3	$\frac{23e^3}{12} + \frac{89e^5}{24} + \frac{5663e^7}{960} + \frac{144431e^9}{17280} + 0[e]^{11}$
3	1	-2	$\frac{11e^2}{8} + \frac{49e^4}{16} + \frac{15665e^6}{3072} + \frac{684503e^8}{92160} + 10.0142e^{10} + 0[e]^{11}$
3	1	-1	$\frac{e}{(1-e^2)^{5/2}}$
3	1	0	$1 + 2e^2 + \frac{239e^4}{64} + \frac{3323e^6}{576} + \frac{1193893e^8}{147456} + 10.683e^{10} + 0[e]^{11}$
3	1	1	$3e + \frac{11e^3}{4} + \frac{245e^5}{48} + \frac{463e^7}{64} + \frac{27929e^9}{2880} + 0[e]^{11}$
3	1	2	$\frac{53e^2}{8} + \frac{39e^4}{16} + \frac{7041e^6}{1024} + \frac{89317e^8}{10240} + 11.3645e^{10} + 0[e]^{11}$
3	1	3	$\frac{77e^3}{6} - \frac{25e^5}{48} + \frac{4751e^7}{480} + \frac{86471e^9}{8640} + 0[e]^{11}$
3	2	-3	$\frac{77e^3}{6} - \frac{25e^5}{48} + \frac{4751e^7}{480} + \frac{86471e^9}{8640} + 0[e]^{11}$
3	2	-2	$\frac{53e^2}{8} + \frac{39e^4}{16} + \frac{7041e^6}{1024} + \frac{89317e^8}{10240} + 11.3645e^{10} + 0[e]^{11}$
3	2	-1	$3e + \frac{11e^3}{4} + \frac{245e^5}{48} + \frac{463e^7}{64} + \frac{27929e^9}{2880} + 0[e]^{11}$
3	2	0	$1 + 2e^2 + \frac{239e^4}{64} + \frac{3323e^6}{576} + \frac{1193893e^8}{147456} + 10.683e^{10} + 0[e]^{11}$
3	2	1	$\frac{e}{(1-e^2)^{5/2}}$
3	2	2	$\frac{11e^2}{8} + \frac{49e^4}{16} + \frac{15665e^6}{3072} + \frac{684503e^8}{92160} + 10.0142e^{10} + 0[e]^{11}$
3	2	3	$\frac{23e^3}{12} + \frac{89e^5}{24} + \frac{5663e^7}{960} + \frac{144431e^9}{17280} + 0[e]^{11}$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 28/36

2.2.4 Degree 1 gravitational potential v_{1m}

Using Eq. (41):

$$V_{nm} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a}\right)^n \sum_{p=0}^n F_{nmp}(i) \sum_{t=-\infty}^{\infty} G_{npt}(e) S_{nmpt}(\omega, M, \Omega, GMST)$$

with (Eq. (42)):

$$S_{nmpt}(\omega, M, \Omega, GMST) = \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST)) \\ + \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin((n-2p)\omega + (n-2p+t)M + m(\Omega - GMST))$$

2.2.4.1 F-functions

$$V_{1m} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right)^1 \sum_{p=0}^1 F_{1m}(i) \left\{ \begin{bmatrix} C_{1m} \\ -S_{1m} \end{bmatrix} \begin{matrix} 1-m \text{ even} \\ 1-m \text{ odd} \end{matrix} \cos((1-2p)(\omega + v) + m(\Omega - GMST)) \right. \\ \left. + \begin{bmatrix} S_{1m} \\ C_{1m} \end{bmatrix} \begin{matrix} 1-m \text{ even} \\ 1-m \text{ odd} \end{matrix} \sin((1-2p)(\omega + v) + m(\Omega - GMST)) \right\} \quad (43)$$

with k being the integer part of $\frac{n-m}{2}$

$$F_{1mp}(i) = \sum_{t=0}^{\min(p,k)} \frac{(2-2t)!}{t!(1-t)!(1-m-2t)! 2^{2-2t}} \sin^{1-m-2t} i \\ \times \sum_{s=0}^m \binom{m}{s} \cos^s i \sum_{c=0}^{\max \text{ possible}} \binom{1-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \quad (44)$$

Order-0 ($m = 0$): as $k = 0$,

$$F_{10p}(i) = \sum_{t=0}^{\min(p,0)} \frac{(2-2t)!}{t!(1-t)!(1-2t)! 2^{2-2t}} \sin^{1-2t} i \sum_{c=0}^{\max \text{ possible}} \binom{1-2t}{c} \binom{0}{p-t-c} (-1)^c$$

we have only $t = 0$, $s = 0$

$$F_{100}(i) = \frac{1}{2} \sin i$$

$$F_{101}(i) = \frac{-1}{2} \sin i$$

$$V_{10} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right) \sum_{p=0}^1 F_{10}(i) \{C_{10} \sin((1-2p)(\omega + v))\} \\ = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right) \{F_{100}(i) \{C_{10} \sin(\omega + v)\} - F_{101}(i) \{C_{10} \sin(\omega + v)\}\} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r}\right) \sin i \{C_{10} \sin(\omega + v)\}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 29/36

Order-1 ($m = 1$): as $k = 0$, k being the integer part of $\frac{n-m}{2}$, we have only $t = 0$

$$F_{11}(i) = \frac{1}{2} \sum_{c=0}^{\max \text{ possible}} \binom{0}{c} \binom{1}{p-c} (-1)^c + \frac{1}{2} \cos i \sum_{c=0}^{\max \text{ possible}} \binom{1}{c} \binom{0}{p-c} (-1)^c$$

$$F_{110}(i) = \frac{1}{2} + \frac{1}{2} \cos i$$

$$F_{111}(i) = \frac{1}{2} + \frac{-1}{2} \cos i$$

$$V_{11} = \frac{GM_{\oplus}}{r} \left(\frac{a_{\oplus}}{r} \right)^1 \left[F_{110}(i) \{ C_{11} \cos((\omega + v) + (\Omega - GMST)) + S_{11} \sin((\omega + v) + (\Omega - GMST)) \} \right. \\ \left. + F_{111}(i) \{ C_{11} \cos(-(\omega + v) + (\Omega - GMST)) + S_{11} \sin(-(\omega + v) + (\Omega - GMST)) \} \right]$$

2.2.4.2 G-functions

$$V_{1m} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) \sum_{p=0}^1 F_{1mp}(i) \sum_{t=-\infty}^{\infty} G_{1p}(e) \left\{ \begin{bmatrix} C_{1m} \\ -S_{1m} \end{bmatrix} \begin{matrix} 1-m \text{ even} \\ 1-m \text{ odd} \end{matrix} \cos((1-2p)\omega + (1-2p+t)M + m(\Omega - GMST)) \right. \\ \left. + \begin{bmatrix} S_{1m} \\ C_{1m} \end{bmatrix} \begin{matrix} 1-m \text{ even} \\ 1-m \text{ odd} \end{matrix} \sin((1-2p)\omega + (1-2p+t)M + m(\Omega - GMST)) \right\}$$

For degree 1 and $p = 0$ or 1 and $t = -1, 0, 1$:

$n=1, p=0, t=-1$,

$$G_{10-1}(e) = 0$$

$n=1, p=0, t=0$,

$$G_{100}(e) = \left(1 - \frac{e^2}{2} \right)$$

$n=1, p=0, t=1$,

$$G_{101}(e) = \left(\frac{e}{2} \right) \sum_{t'=0}^1 \binom{2-t'}{1-t'} \frac{1}{t'!} (2)^{t'} = 2e$$

$n=1, p=1, t=-1$,

$$G_{11-1}(e) = \left(\frac{-e}{2} \right) \sum_{t'=0}^1 (-1)^{1-t'} \binom{2-t'}{1-t'} \frac{(-2)^{t'}}{t'!} = 2e$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 30/36

$n=1, p=1, t=0,$

$$G_{110}(e) = 1 - \frac{e^2}{2}$$

$n=1, p=1, t=1,$

$$G_{111}(e) = 0$$

2.2.4.3 Altogether, gravitational potential of degree 1, V_{1m} :

$$F_{110}(i) = \left(\frac{1}{2} + \frac{1}{2} \cos i\right)$$

$$F_{111}(i) = \left(\frac{1}{2} + \frac{-1}{2} \cos i\right)$$

$$\begin{aligned}
 V_{11} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a}\right) [F_{110}(i)\{G_{10-1}(e)S_{110-1}(\omega, M, \Omega, GMST) + G_{100}(e)S_{1100}(\omega, M, \Omega, GMST) \\
 &\quad + G_{101}(e)S_{1101}(\omega, M, \Omega, GMST)\} \\
 &\quad + F_{111}(i)\{G_{11-1}(e)S_{111-1}(\omega, M, \Omega, GMST) + G_{110}(e)S_{1110}(\omega, M, \Omega, GMST) \\
 &\quad + G_{111}(e)S_{1111}(\omega, M, \Omega, GMST)\}] \\
 &\quad S_{11pt}(\omega, M, \Omega, GMST) \\
 &= C_{11} \cos((1-2p)\omega + (1-2p+t)M + (\Omega - GMST)) \\
 &\quad + S_{11} \sin((1-2p)\omega + (1-2p+t)M + (\Omega - GMST))
 \end{aligned}$$

$$S_{110-1}(\omega, M, \Omega, GMST) = C_{11} \cos(\omega + (\Omega - GMST)) + S_{11} \sin(\omega + (\Omega - GMST))$$

$$S_{1100}(\omega, M, \Omega, GMST) = C_{11} \cos(\omega + M + (\Omega - GMST)) + S_{11} \sin(\omega + M + (\Omega - GMST))$$

$$S_{1101}(\omega, M, \Omega, GMST) = C_{11} \cos(\omega + 2M + (\Omega - GMST)) + S_{11} \sin(\omega + 2M + (\Omega - GMST))$$

$$S_{111-1}(\omega, M, \Omega, GMST) = C_{11} \cos(-\omega - 2M + (\Omega - GMST)) + S_{11} \sin(-\omega - 2M + (\Omega - GMST))$$

$$S_{1110}(\omega, M, \Omega, GMST) = C_{11} \cos(-\omega - M + (\Omega - GMST)) + S_{11} \sin(-\omega - M + (\Omega - GMST))$$

$$S_{1111}(\omega, M, \Omega, GMST) = C_{11} \cos(-\omega + (\Omega - GMST)) + S_{11} \sin(-\omega + (\Omega - GMST))$$

$$\begin{aligned}
 V_{11} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a}\right) \left[\left(\frac{1}{2} + \frac{1}{2} \cos i\right) \left\{ 0 S_{110-1}(\omega, M, \Omega, GMST) + \left(1 - \frac{e^2}{2}\right) S_{1100}(\omega, M, \Omega, GMST) + 2e S_{1101}(\omega, M, \Omega, GMST) \right\} \right. \\
 &\quad + \left(\frac{1}{2} + \frac{-1}{2} \cos i\right) \left\{ 2e S_{111-1}(\omega, M, \Omega, GMST) + \left(1 - \frac{e^2}{2}\right) S_{1110}(\omega, M, \Omega, GMST) \right. \\
 &\quad \left. \left. + 0 S_{1111}(\omega, M, \Omega, GMST) \right\} \right]
 \end{aligned}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 31/36

$$\begin{aligned}
 V_{11} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) & \left[\left(\frac{1}{2} + \frac{1}{2} \cos i \right) \left\{ \left(1 - \frac{e^2}{2} \right) [C_{11} \cos(\omega + M + (\Omega - GMST)) + S_{11} \sin(\omega + M + (\Omega - GMST))] \right. \right. \\
 & + 2e [C_{11} \cos(\omega + 2M + (\Omega - GMST)) + S_{11} \sin(\omega + 2M + (\Omega - GMST))] \Big\} \\
 & + \left(\frac{1}{2} + \frac{-1}{2} \cos i \right) \left\{ 2e [C_{11} \cos(-\omega - 2M + (\Omega - GMST)) + S_{11} \sin(-\omega - 2M + (\Omega - GMST))] \right. \\
 & + \left. \left(1 - \frac{e^2}{2} \right) [C_{11} \cos(-\omega - M + (\Omega - GMST)) + S_{11} \sin(-\omega - M + (\Omega - GMST))] \right\} \Big]
 \end{aligned}$$

with

$$\cos(\omega + M + (\Omega - GMST)) = \cos(\Omega - GMST) \cos(\omega + M) - \sin(\Omega - GMST) \sin(\omega + M)$$

$$\cos(-(\omega + M) + (\Omega - GMST)) = \cos(\Omega - GMST) \cos(\omega + M) + \sin(\Omega - GMST) \sin(\omega + M)$$

$$\sin(\omega + M + (\Omega - GMST)) = \sin(\Omega - GMST) \cos(\omega + M) + \cos(\Omega - GMST) \sin(\omega + M)$$

$$\sin(-\omega - M + (\Omega - GMST)) = \sin(\Omega - GMST) \cos(\omega + M) - \cos(\Omega - GMST) \sin(\omega + M)$$

$$\begin{aligned}
 V_{11} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) & \left[\left(\frac{1}{2} + \frac{1}{2} \cos i \right) \left\{ \left(1 - \frac{e^2}{2} \right) [C_{11} (\cos(\omega + M + (\Omega - GMST)))] + S_{11} (\sin(\omega + M + (\Omega - GMST)))] \right. \right. \\
 & + 2e [C_{11} \cos(\omega + 2M + (\Omega - GMST)) + S_{11} \sin(\omega + 2M + (\Omega - GMST))] \Big\} \\
 & + \left(\frac{1}{2} + \frac{-1}{2} \cos i \right) \left\{ 2e [C_{11} \cos(-\omega - 2M + (\Omega - GMST)) + S_{11} \sin(-\omega - 2M + (\Omega - GMST))] \right. \\
 & + \left. \left(1 - \frac{e^2}{2} \right) [C_{11} (\cos(-\omega - M + (\Omega - GMST)))] + S_{11} (\sin(-\omega - M + (\Omega - GMST)))] \right\} \Big]
 \end{aligned}$$

When the eccentricity is small or equal to 0,

$$\begin{aligned}
 V_{11} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) & \left[\left(\frac{1}{2} + \frac{1}{2} \cos i \right) \{ [C_{11} (\cos(\omega + M + (\Omega - GMST)))] + S_{11} (\sin(\omega + M + (\Omega - GMST)))] \} \right. \\
 & + \left. \left(\frac{1}{2} + \frac{-1}{2} \cos i \right) \{ [C_{11} (\cos(-\omega - M + (\Omega - GMST)))] + S_{11} (\sin(-\omega - M + (\Omega - GMST)))] \} \right] \\
 & = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) [(\cos i) \{ -C_{11} (\sin(\omega + M) \sin((\Omega - GMST))) + S_{11} (\cos(\Omega - GMST) \sin(\omega + M)) \} \\
 & + \{ C_{11} (\cos(\omega + M) \cos((\Omega - GMST))) + S_{11} (\sin(\Omega - GMST) \cos(\omega + M)) \}]
 \end{aligned}$$

$$\begin{aligned}
 V_{11} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) & [(\cos i) \sin(\omega + M) \{ -C_{11} \sin(\Omega - GMST) + S_{11} \cos(\Omega - GMST) \} \\
 & + \cos(\omega + M) \{ C_{11} \cos(\Omega - GMST) + S_{11} \sin(\Omega - GMST) \}]
 \end{aligned}$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:		1.1 5 Page : 32/36

And

$$\begin{aligned}
 V_{10} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) \sum_{p=0}^1 F_{10p}(i) \sum_{t=-\infty}^{\infty} G_{1pt}(e) \{C_{10} \sin((1-2p)\omega + (1-2p+t)M)\} \\
 &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) \left[F_{100}(i) \sum_{t=-\infty}^{\infty} G_{10t}(e) \{C_{10} \sin(\omega + (1+t)M)\} \right. \\
 &\quad \left. + F_{101}(i) \sum_{t=-\infty}^{\infty} G_{11t}(e) \{C_{10} \sin(-\omega + (t-1)M)\} \right]
 \end{aligned}$$

$$\begin{aligned}
 V_{10} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) [F_{100}(i) \{G_{10-1}(e) \{C_{10} \sin(\omega)\} + G_{100}(e) \{C_{10} \sin(\omega + M)\} + G_{101}(e) \{C_{10} \sin(\omega + 2M)\}\} \\
 &\quad + F_{101}(i) \{G_{11-1}(e) \{-C_{10} \sin(\omega + 2M)\} - G_{110}(e) \{C_{10} \sin(\omega + M)\} - G_{111}(e) \{C_{10} \sin(\omega)\}\}]
 \end{aligned}$$

$$F_{100}(i) = \frac{1}{2} \sin i$$

$$F_{101}(i) = \frac{-1}{2} \sin i$$

$$G_{10-1}(e) = 0$$

$$G_{100}(e) = \left(1 - \frac{e^2}{2}\right)$$

$$G_{101}(e) = 2e$$

$$G_{11-1}(e) = 2e$$

$$G_{110}(e) = 1 - \frac{e^2}{2}$$

$$G_{111}(e) = 0$$

$$\begin{aligned}
 V_{10} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) \left[\frac{1}{2} \sin i \{G_{10-1}(e) \{C_{10} \sin(\omega)\} + G_{100}(e) \{C_{10} \sin(\omega + M)\} + G_{101}(e) \{C_{10} \sin(\omega + 2M)\}\} \right. \\
 &\quad \left. + \frac{1}{2} \sin i \{G_{11-1}(e) \{C_{10} \sin(\omega + 2M)\} + G_{110}(e) \{C_{10} \sin(\omega + M)\} + G_{111}(e) \{C_{10} \sin(\omega)\}\} \right]
 \end{aligned}$$

$$\begin{aligned}
 V_{10} &= \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) \left[\frac{1}{2} \sin i \left\{ \left(1 - \frac{e^2}{2}\right) \{C_{10} \sin(\omega + M)\} + 2e \{C_{10} \sin(\omega + 2M)\} \right\} \right. \\
 &\quad \left. + \frac{1}{2} \sin i \left\{ 2e \{C_{10} \sin(\omega + 2M)\} + \left(1 - \frac{e^2}{2}\right) \{C_{10} \sin(\omega + M)\} \right\} \right]
 \end{aligned}$$

$$V_{10} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) C_{10} \sin i \left[\left\{ \left(1 - \frac{e^2}{2}\right) \{ \sin(\omega + M) \} + 2e \{ \sin(\omega + 2M) \} \right\} \right]$$

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 33/36

For a small eccentricity or =0,

$$V_{10} = \frac{GM_{\oplus}}{a} \left(\frac{a_{\oplus}}{a} \right) C_{10} \sin i \{ \sin(\omega + M) \}$$

2.2.4.4 Conclusion

In the expressions above, we first analyzed the effect of the inclination. We observe that V_{10} , which represents the z-component of the geocenter, is proportional to $\sin i$. As a result, a polar orbit is more sensitive to variations in the z-component of the geocenter than any other orbit, while an equatorial orbit would not be affected by the z-component of the geocenter variations.

Additionally, V_{11} has two terms: one that is maximum when cosine=1, and the other when cosine=-1, which corresponds to an equatorial orbit. Therefore, the x- and y-components of the geocenter are more pronounced in an equatorial orbit, while they are more limited (though still present) in a polar orbit.

To give an order of magnitude, we assume that the geocenter's x-, y-, and z-components have the same amplitudes $c_{10} = c_{11} = s_{11} = 1$, and normalize the total degree-1 potential. Figure 5 illustrates the sensitivity to the inclination, considering GENESIS mean motion and over 72 hours (left) or 24 hours (right), in the case of a zero eccentricity.

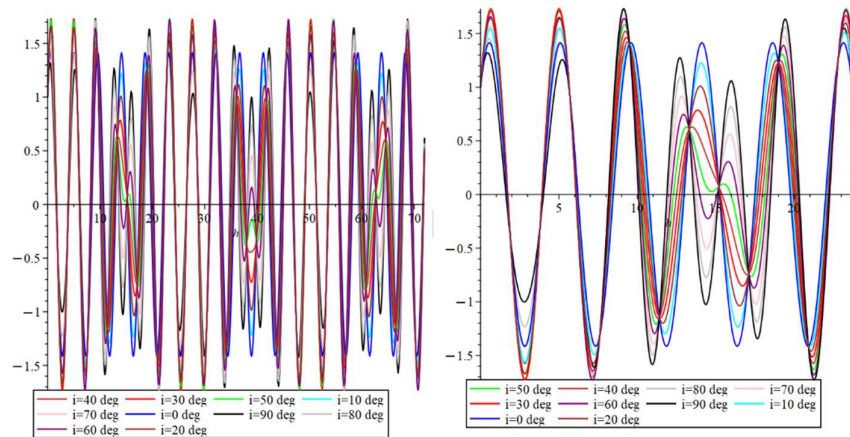


Figure 5. Sensitivity of the orbit inclination to C_{10} , C_{11} , and S_{11} over 3 orbital periods (left) and 1 orbital period (right).

To better understand the relative contributions of the different orders, we have plotted the same functions for degree-1, order-1 (left) and degree-1, order-0 (right).

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 34/36

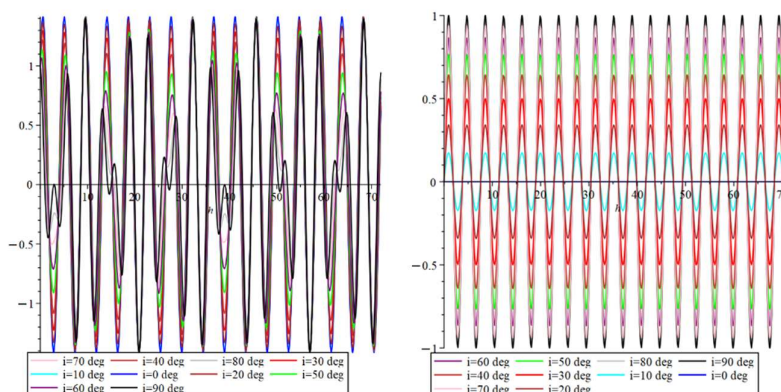


Figure 6. Sensitivity of the orbit inclination to C_{11} and S_{11} (left) and to C_{10} (right), over 3 orbital periods.

From Figure 6, it is clear that the higher sensitivity to degree-1, order-1 (left) is observed for an equatorial or nearly equatorial orbit, while the higher sensitivity to degree-1, order-0 (right) is seen for a polar or nearly polar orbit.

Another way to assess the sensitivity to degree-1, order-1 (left) is by computing the mean integral of the absolute value of each function over one day (see Figure 7). This confirms that the optimal orbit for this case is equatorial, but also favors nearly equatorial orbits or nearly polar orbits.

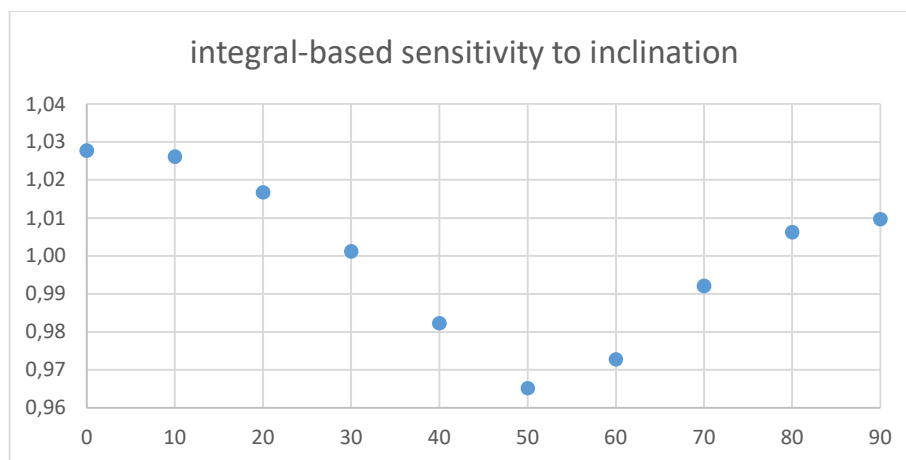


Figure 7: Sensitivity to inclination.

To better incorporate the sensitivity of each order in a global degree-1 sensitivity, we have additionally computed the mean of the sensitivities for each order. These results are compared, in Figure 8, with the mean integral of the absolute value of each function over one day (that was shown above).

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 35/36

Similarly, the Figure 8 illustrates the integral of the absolute value of the functions for different inclinations over one day, separately for degree-1, order-0 (blue) and degree-1, order-1 (orange). For the first case, the computations confirm that the optimal orbit is polar, while for the second case, the optimal orbit is equatorial.

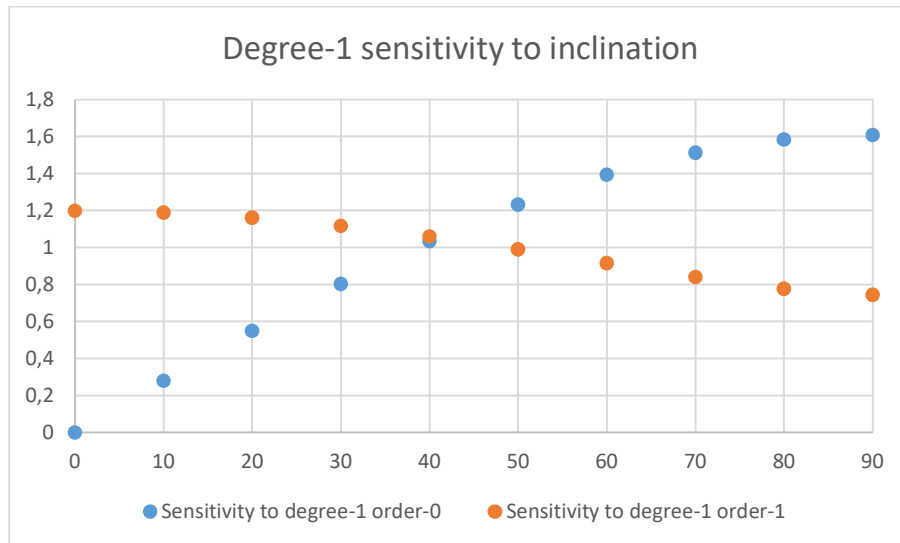
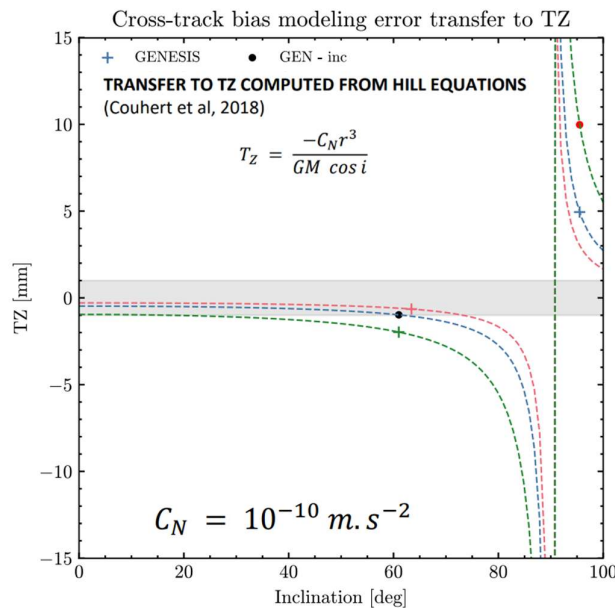


Figure 8: Integral of the absolute value of the functions for different inclinations over one day. Degree-1, order-0 is shown in blue, while orange represents degree-1, order-1.

2.3 Recommendation of the WG on Orbit inclination

With the lack of precise accelerometers on board GENESIS, cross-track perturbing accelerations, related to unknown thermal fluxes from the radiators and inaccuracies in the SRP model (especially the surface properties and self-shadowing), will make minimizing the power of the draconitic oscillations difficult. Because of the high inclination of the orbit, a given 10^{-10} m/s² error will transfer into a draconitic signal of **5 mm amplitude (GENESIS nominal)** in the TZ component of the geocenter motion, against less than **1 mm (GENESIS inclined)** for the same altitude with an inclination of 61°.

 	REFERENCE : DATE : DD/MM/YEAR		GENESIS-ROB-TN-0006 11/07/2025
	VERSION.ISSUE: DRAFT:	1.1 5	Page : 36/36



In addition, if one wants to simultaneously solve for constant cross-track empirical accelerations to reduce the transfer of this mismodeling effect in the TZ component, the resulting geocenter motion estimate will be more scattered in the North-South direction with an orbit at high inclination, because of the correlation previously mentioned.

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