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Limit state analysis of 2D externally statically indeterminate networks using graphic statics

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Abstract
Graphic statics represents a powerful tool in the field of structural design analysis and design, because the reciprocity between form and force diagrams allows for a constant and simultaneous control of both the geometry and the load-bearing behavior of a structure in terms of internal and external forces. When combined to plastic theory, graphic statics can be used for the limit state analysis of strut-and-tie networks used to model the load paths acting inside continuous structures. The method helps finding the set of statically admissible force distributions that respect equilibrium and yield conditions, and gives insight on the maximum external load that can be applied on the structure. The present paper describes a methodology for the limit state analysis of externally statically indeterminate networks using graphic statics. Two case studies then highlight how this methodology can be used to determine the load bearing capacity of a structure.

Keywords: Graphic statics, plastic theory, static indeterminacy, limit analysis, pin-jointed networks, strut-and-tie modeling.

1. Introduction
Using strut-and-tie networks to model the structural behavior of complex structures is a common practice in structural engineering. These networks may be intended as pin-jointed frameworks, or as strut-and-tie models within a continuum of material. For example, researchers as Muttoni et al. [1], and more recently Deschuyteneer et al. [2], have shown that strut-and-tie modeling can easily be combined with graphic statics to provide an elegant and efficient method for the design and analysis of reinforced concrete structures. Indeed, the use of reciprocal form and force diagrams gives direct and simultaneous insight on the static behavior of every structural member.

When dealing with networks presenting a certain degree of static indeterminacy, the distribution of internal forces is not unique, therefore the determination of the optimal force distribution is not straightforward. Within the context of plastic theory, for a material with sufficient ductile behavior, it is possible to perform a limit state analysis on the networks, defining a distribution of internal forces in equilibrium and assuming that no instability can occur in any of the structural members (Heyman [3]). The *static approaches* of limit analysis ensure that any statically admissible load distribution inside the given structure respecting boundary and yield conditions leads to a lower bound estimation of the external load leading to collapse. This type of approaches is consequently regarded as safe, as it can only lead to an underestimation of the effective load bearing capacity of the structure. On the other hand, the *kinematic approaches* aim to determine the collapse conditions of the structure – ultimate loading and collapse mechanism – by analyzing the various configurations of kinematically compatible collapse mechanisms. The kinematic theorem of plasticity then states that the corresponding loading conditions are upper bound values for the actual load bearing capacity of the structure. The combination of these two theorems gives an exact estimation of the collapse conditions of the structure.

In this context, a *static approach* implemented using graphic statics has been developed for evaluating the structural behavior of 2D internally statically indeterminate networks (Rondeaux et al. [4]). The objective of this paper is to extend this methodology to external static indeterminacy of structures caused by superabundant support conditions. The main theoretical concepts related to the limit state analysis of strut-and-tie networks using graphic statics are first reviewed. Two case studies – a simple 2D network with two degrees of external static indeterminacy and a truss on multiple supports – are then developed to illustrate the proposed methodology. The optimization of the corresponding parametric force diagrams gives the magnitude of the load leading to collapse. These results are eventually compared with the ones obtained by the Principle of Virtual Works, which helps to identify the collapse mechanism.

2. Graphical determination of the limit load

By combining graphic statics and limit state analysis, the limit load acting on a given 2D externally statically indeterminate network can be determined using a graphical methodology, as detailed in the six steps of the following procedure:

1. The degree of external static indeterminacy of a structure, subjected to a given loading $\mathbf{q}$, is obtained by counting the number n_e of excess reaction forces in the form diagram $\mathbf{F}$ compared to the isostatic configuration.
2. The n_e+1 associated isostatic equilibrium diagrams of the external forces $\mathbf{F_{ej}}^*$ are drawn at an arbitrary scale.
3. The independent diagrams of the external forces $\mathbf{F_{ej}}^*$ are assembled into one single diagram of external forces $\mathbf{F_e}^*$, by performing a vector addition of the corresponding vectors in $\mathbf{F_{ej}}^*$. As the degree of internal static indeterminacy n_i equals zero, it is straightforward to complete $\mathbf{F_e}^*$ by adding the internal force vectors $\mathbf{f_k}^*$ in order to obtain the complete force diagram $\mathbf{F}^*$.
4. Considering a fixed length for the vector $\mathbf{q}^*$ corresponding to the applied load, each degree of freedom of the force diagram $\mathbf{F}^*$ is detected, which corresponds to an offset of $\mathbf{F_e}^*$ (Mitchell et al. [5]). $\mathbf{F}^*$ can consequently be seen as a parametric force diagram $\mathbf{F_p}^*$, in which each degree of freedom depends on a parameter x_j defined as the position of the node P_j^* along the line of action of one of the excess reaction forces $\mathbf{r_j}^*$. Since the positions of all the other nodes of $\mathbf{F_p}^*$ can be deduced from x_j by geometrical constructions, the variation of x_j describes the overall geometric transformation of $\mathbf{F_p}^*$.
5. As the magnitude of vector $\mathbf{q}^*$ corresponding to the applied load is kept constant, maximizing the limit load is equivalent to maximizing the ratio between $|\mathbf{q}^*|$ and the magnitude $|\mathbf{f_k}^*|$ of the highest internal force $\mathbf{f_k}^*$. An optimization process is then applied on the value of x_j in order to minimize $|\mathbf{f_k}^*|$. Out of this step, the collapse force diagram $\mathbf{F_u}^*$ is obtained.
6. Scaling the entire collapse force diagram $\mathbf{F_u}^*$, so that the highest magnitude of $|\mathbf{f_k}^*|$ is exactly equal to the given admissible normal force $|\mathbf{f_u}^*|$, yields the magnitude $|\mathbf{q_u}^*|$ of the load leading to collapse.

3. Case studies

3.1. Simple 2D network

The above methodology is applied to a two times externally statically indeterminate network, as detailed in Figure 1. Step 1 depicts the form diagram $\mathbf{F}$ of the network, with the two reaction forces $\mathbf{r_1}$ and $\mathbf{r_2}$ chosen as excess in orange. Step 2 shows the 2+1 = 3 associated external form and force diagrams. Step 3 depicts the complete force diagram obtained when assembling the three force diagrams from step 2 (bars in compression are depicted in blue, and the ones in tension in red). Step 4 gives the parameterized force diagram, with x_1 and x_2 the variable positions of point P_1 and P_2 along the line of action of reactions $\mathbf{r_1}^*$ and $\mathbf{r_2}^*$ respectively. Step 5 shows the form and force diagrams obtained after optimization of parameters P_1^* and P_2^*. Finally, Step 6 is the force diagram scaled to obtain the highest magnitude of the applied load $\mathbf{q_u}^*$, which is then the one leading to collapse:

$$|\boldsymbol{q_u}^*| = |\boldsymbol{f_u}^*| \cdot \sqrt{2} \tag{1}$$

with $|\mathbf{f_u}^*|$ the ultimate normal force that can be undertaken by the structural members. The analysis of the collapse form and force diagrams $\mathbf{F_u}$ and $\mathbf{F_u}^*$ reveals that the ultimate force $\mathbf{f_u}^*$ is reached in bars $\mathbf{f_1}$, $\mathbf{f_2}$ and $\mathbf{f_4}$. Considering these bars as being plasticized, the Principle of Virtual Works can also be applied to the corresponding collapse mechanism, which is also depicted in Figure 1:

$$|\boldsymbol{q_u}| \cdot \delta = |\boldsymbol{f_u}| \cdot (\delta\sqrt{2}/2 + \delta\sqrt{2}/4 + \delta\sqrt{2}/4) \tag{2}$$

As both results are equal, this proves that the exact magnitude of the collapse load has been found using the proposed graphical methodology.

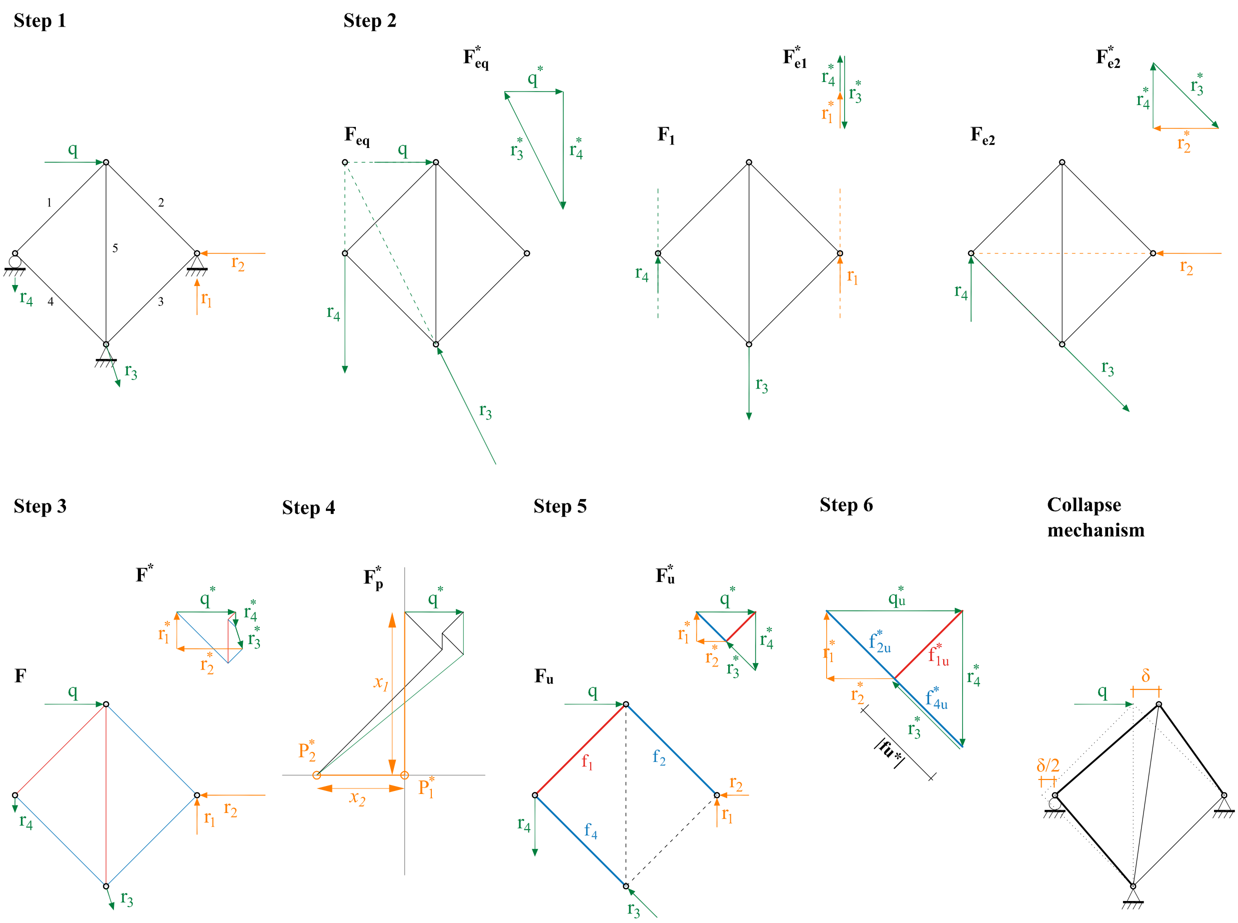

Figure 1: Graphical limit state analysis performed on a two times externally statically indeterminate pin-jointed network – six steps of the procedure and collapse mechanism

3.2. Truss on multiple supports

In the following, the proposed graphical limit state procedure is applied to the structure of Figure 2, for three different support conditions. The form and force diagrams depicted in the figure are the ones corresponding to collapse, on which the ultimate limit load $\mathbf{q_u}^*$ can be measured. The ratios $|\mathbf{q_u}^*|/|\mathbf{f_u}^*|$ obtained for the three support conditions are respectively equal to 0.333, 0.5 and 0.707, with the corresponding plasticized bars depicted in bold in the figure. In practice, this type of result can give insight on the best support configuration to be implemented when designing a truss structure. This corresponds to case C in the present study, as it allows the highest magnitude for the applied load **q**.

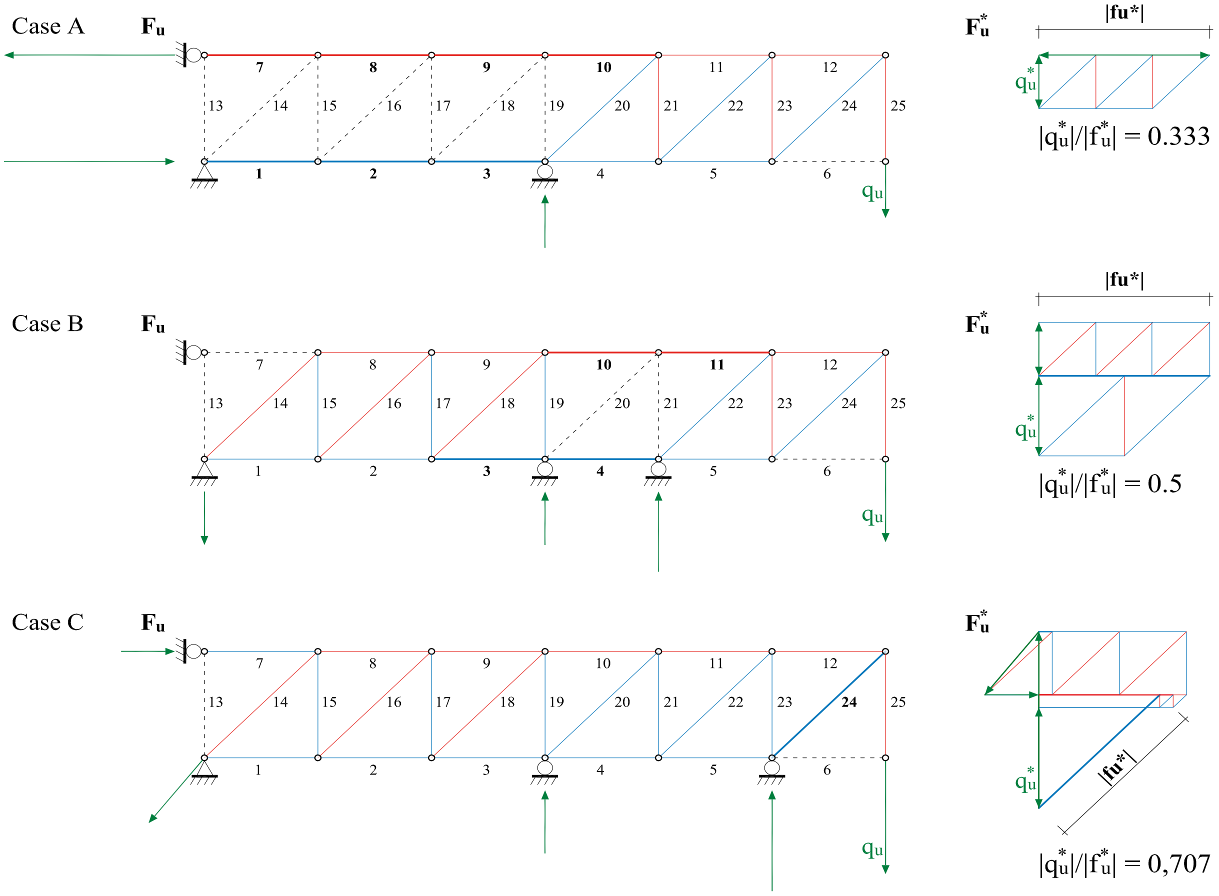

Figure 2: Externally statically indeterminate truss, for three different support conditions

4. Conclusion

Based on the lower bound theorem of limit analysis, this paper presents a graphical methodology for the analysis of strut-and-tie networks in which static indeterminacy is due to an excess of reaction forces at the supports. This methodology can be applied to the whole range of structures that can be modeled as a pin-jointed networks (trusses, arches, beams, walls, etc.) Implemented in a parametrical environment, the methodology allows for the straightforward visualization and the manipulation of the force magnitudes by operating directly on the force diagram. Moreover, it can be used to find the collapse load of the structure.

References

[1] A. Muttoni, J. Schwartz and B. Thürlimann, *Design of concrete structures with stress fields*, Birkhäuser, 1997.

[2] A. Deschuyteneer and D. Zastavni, “Structural robustness at the conceptual stage”, *IABSE workshop: Ignorance, uncertainty and human errors in structural engineering*, Helsinki, Finland, February 15-16, 2017.

[3] J. Heyman, *Basic structural theory*, Cambridge University Press, 2008.

[4] J.-F. Rondeaux, P. D’Acunto, J. Schwartz and D. Zastavni, “Limit state analysis of 2D statically indeterminate networks using graphic statics”, *Interfaces - architecture, engineering, science: Proceedings of the IASS Annual Symposium 2017*, Hamburg, Germany, September 25-28, 2017.

[5] T. Mitchell, W. Baker, A. McRobie and A. Mazurek, “Mechanisms and states of self-stress of planar trusses using graphic statics, part 1”, *International Journal of Space Structures*, vol. 31, no. 2-4, pp. 85-101, 2016.