



ELSEVIER

10 January 2002

PHYSICS LETTERS B

Physics Letters B 524 (2002) 395–399

www.elsevier.com/locate/npe

Soft photon spectrum in orthopositronium and vector quarkonium decays

J. Pestieau, C. Smith

Institut de Physique Théorique, Université catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium

Received 22 November 2001; accepted 27 November 2001

Editor: L. Alvarez-Gaumé

Abstract

QED gauge invariance, when combined with analyticity, leads to constraints on the low energy end of the emitted photon spectra. This is known as Low's theorem. It is shown that the Ore–Powell result, as well as further developments for the orthopositronium differential decay rate, are in contradiction with Low's theorem, i.e., that their predicted soft photon spectra are incorrect.

A solution to this problem is presented. The implications for the orthopositronium lifetime puzzle, the charmonium ρ – π puzzle, the prompt photon spectrum in inclusive quarkonium decays and the extraction of α_S from quarkonium annihilation rates are briefly commented.

© 2002 Published by Elsevier Science B.V. Open access under [CC BY license](https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The single most important concept in contemporary physics is probably gauge invariance. Aside from that, a basic postulate of Quantum Field Theory, and therefore also of QED, is the analyticity of probability amplitudes as functions of their kinematical variables. In the present Letter, we will investigate the consequences of those two principles, analyticity and gauge invariance, for orthopositronium decay into three photons, and quarkonium decay into three photons, two gluons plus a photon or three gluons.

The consequences of gauge invariance are well-known: one gets very tough constraints on the structure of amplitudes in the form of Ward–Takahashi Identities. For instance, any amplitude involving an external photon, $\mathcal{M} = \varepsilon_\mu \mathcal{M}^\mu$, must verify the Ward

Identity $k_\mu \mathcal{M}^\mu = 0$, with k_μ the photon momentum and ε_μ its polarization vector. In addition to gauge invariance, since any probability amplitude is an analytical function, $\mathcal{M}^\mu(k, \dots)$ admit a Laurent expansion in each of its variables. It was F.E. Low who, in the fifties [1], first realized that the Ward identity restricts the form of the first two terms of the Laurent expansion in the external photon energy.

Low's theorem is a model-independent result, valid to all orders: for a complete amplitude, the soft-photon limit only depends on the quantum numbers of the external particles, and not on the details of the intermediate subprocesses. At the level of observables, the low-energy end of the photon spectrum is obtained by combining the amplitude behavior with that of the phase-space. The most characteristic spectra are:

- *Charged particles and photons in external states.* The well-known bremsstrahlung emissions lead to an amplitude in $1/\omega$ for $\omega \rightarrow 0$ (ω is the energy of one of the emitted photons). At the decay rate or cross

E-mail address: smith@fympa.ucl.ac.be (C. Smith).

section level, the IR divergent amplitude generates characteristic IR divergent spectra.

– *Only neutral bosons, including photons in external state.* The amplitude is in ω for $\omega \rightarrow 0$ (ω is again the energy of one of the photon). This statement is much stronger than what is sometimes thought of for a non-bremsstrahlung process. *On general ground, it shows that feeling confident with an IR safe computation is theoretically incorrect.* IR safety at the cross section or decay rate level is definitely not sufficient. When one constructs a model designed to describe some processes among neutral bosons and photons, one must ensure that the amplitude *vanishes* in the soft-photon limit.

The present Letter is organized as follows. First, we show that the lowest order result of Ore and Powell for orthopositronium [2], or vector quarkonium, decay into $\gamma\gamma\gamma$, is in contradiction with Low's theorem. Then, we discuss the sources of the problem, and argue that present theoretical models (and therefore the corrections computed with them) are incomplete. We then propose (and motivate by comparison with standard elementary particle processes) a model that respects Low's theorem. Finally, in the conclusions, we shortly comment about the consequences for orthopositronium lifetime puzzle and heavy quarkonia puzzles.

2. Contradiction between Low's theorem and Ore-Powell differential decay rate

We will concentrate on the orthopositronium decay into three photons. The discussion also applies to vector quarkonium (J/ψ , Υ , ...).

From the requirement of gauge invariance, and because of the quantum numbers of the initial and final particles involved (neutral self-conjugate bosons), Low's theorem predicts that *the decay amplitude must vanish linearly when the energy of one of the photons is going to zero*

$$\mathcal{M}(o\text{-}Ps \rightarrow \gamma\gamma\gamma) \stackrel{\omega \rightarrow 0}{\sim} \mathcal{O}(\omega) \quad (1)$$

with ω the energy of one of the photon. The squared modulus of the amplitude, therefore, behaves as $\mathcal{O}(\omega^2)$ for small photon energy. Aside from that, the three-photon phase-space alone (i.e., with a constant decay

amplitude) gives a differential rate as

$$\left. \frac{d\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma)}{dx} \right|_{\text{phase-space}} \sim x, \quad (2)$$

with $x = 2\omega/M$ the reduced photon energy, M the orthopositronium mass. Combining the amplitude with the phase-space, one finds that the low-energy end of the photon spectrum *must* behave as

$$\frac{d\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma)}{dx} \stackrel{x \rightarrow 0}{\sim} x^3.$$

No matter the model used to compute the orthopositronium decay rate, the differential rate must exhibit this x^3 behavior for small photon energies. As we have already pointed out in [3], the lowest order decay amplitude, as found by Ore and Powell, leads to a differential decay rate in contradiction with the analytical requirement of Low's theorem. Their model is based on the formula [4]:

$$\Gamma(o\text{-}Ps \rightarrow \gamma\gamma\gamma) = \frac{1}{3} |\psi(0)|^2 (4v_{\text{rel}} \sigma(e^-e^+ \rightarrow \gamma\gamma\gamma))_{v_{\text{rel}} \rightarrow 0} \quad (3)$$

with v_{rel} the relative velocity of the e^+e^- in their center-of-mass frame, and $\psi(0)$ the positronium Schrödinger wave function at zero separation (m is the electron mass)

$$|\psi(0)|^2 = \frac{\alpha^3 m^3}{8\pi}. \quad (4)$$

This formula states that in first approximation, the positronium decay rate can be computed from the static limit of the scattering cross section $e^+e^- \rightarrow \gamma\gamma\gamma$. Equivalently, it is found from the squared modulus amplitude for an e^+e^- pair at rest into $\gamma\gamma\gamma$. Summed over photon polarizations, this is easily shown to be [2]

$$\begin{aligned} & \sum_{\text{polarizations}} |\mathcal{M}((e^+e^-)_{v_{\text{rel}}=0} \rightarrow \gamma\gamma\gamma)|^2 \\ &= \frac{(1-x_1)^2}{x_2^2 x_3^2} + \frac{(1-x_2)^2}{x_1^2 x_3^2} + \frac{(1-x_3)^2}{x_1^2 x_2^2}, \end{aligned}$$

which behaves as a constant when one of the x_i is vanishing (as can be seen by using energy-momentum conservation $x_1 + x_2 + x_3 = 2$), while it should vanish as x_i^2 . In turn, the well-known differential rate inherits

an incorrect analytical behavior

$$\frac{d\Gamma(o-Ps \rightarrow \gamma\gamma\gamma)}{dx_1} = \frac{2\alpha^6 m}{9\pi} \Omega(x_1),$$

where the spectrum function is

$$\begin{aligned} \Omega(x_1) &= \int_{1-x_1}^1 dx_2 |\mathcal{M}((e^+e^-)_{v_{\text{rel}}=0} \rightarrow \gamma\gamma\gamma)|_{x_3=2-x_1-x_2}^2 \\ &= \frac{2(2-x_1)}{x_1} + \frac{2(1-x_1)x_1}{(2-x_1)^2} \\ &\quad + 4 \left[\frac{(1-x_1)}{x_1^2} - \frac{(1-x_1)^2}{(2-x_1)^3} \right] \ln(1-x_1) \\ &= \frac{5}{3}x_1 + \mathcal{O}(x_1^2) \quad \text{near } x_1 = 0. \end{aligned} \quad (5)$$

In the Ore–Powell model, the photon energy spectrum vanishes only linearly near zero, instead of the required $\Omega(x_1) = \mathcal{O}(x_1^3)$.

Remark. For completeness, recall that it is this differential rate that gives the total width

$$\begin{aligned} \Gamma(o-Ps \rightarrow \gamma\gamma\gamma) &= \frac{2(\pi^2 - 9)}{9\pi} \alpha^6 m \quad \text{since} \\ \int_0^1 dx_1 \Omega(x_1) &= \pi^2 - 9. \end{aligned}$$

Up to color factors, wave functions, coupling constants, the present analysis can be repeated for the quarkonia decay modes [5]

$$\begin{aligned} \frac{d\Gamma(V \rightarrow \gamma\gamma\gamma)}{dx_1} &= \frac{64}{3} e_Q^6 \alpha^3 \frac{|\phi_0|^2}{M^2} \Omega(x_1), \\ \frac{d\Gamma(V \rightarrow ggg)}{dx_1} &= \frac{160}{81} \alpha_S^3 \frac{|\phi_0|^2}{M^2} \Omega(x_1), \\ \frac{d\Gamma(V \rightarrow gg\gamma)}{dx_1} &= \frac{128}{9} e_Q^2 \alpha_S^2 \frac{|\phi_0|^2}{M^2} \Omega(x_1), \end{aligned}$$

where V is the 1^{--} vector bound state made of the $Q\bar{Q}$ pair, M is the mass of V , ϕ_0 the (unknown) quarkonium configuration space wave function at zero separation and e_Q the heavy quark electric charge in units of the electron one. All these decay spectra do violate the basic requirement of analyticity.

3. Contradiction between Low's theorem and current positronium decay models

The model used by Ore–Powell, based on the factorized formula (3), may seem a bit naive, especially in view of the enormous amount of work done by various groups (see, for example, [3,6–8], and references quoted there). Nevertheless, it is very illustrative of the current approaches in its treatment of intermediate states. Indeed, models derived from Bethe–Salpeter analyses, or from QED non-relativistic effective theory (NRQED, [7]), always connect the process of annihilation of bound charged particles to that of scattering of real, asymptotic charged particles. The difficulty with such approaches is thereby apparent: asymptotic and bound charged particles have drastically different radiation properties: the former exhibit bremsstrahlung-type radiations, while the later do not radiate zero energy photons (for very low-energy photons, a positronium state is just a neutral, self-conjugate boson, hence it does not radiate in that limit).

Bremsstrahlung radiations typically lead to a Laurent expansion for the amplitude as

$$\mathcal{M}^\mu(\omega, \dots) = \mathcal{O}(1/\omega) + \mathcal{O}(1) + \mathcal{O}(\omega).$$

What Low's theorem states is that both the terms of $\mathcal{O}(1/\omega)$ and $\mathcal{O}(1)$ must disappear [1]. While the cancellation of $\mathcal{O}(1/\omega)$ terms is automatic from selection rules, that of $\mathcal{O}(1)$ terms is much more delicate, requiring a non-perturbative treatment of the binding energy. Typically, the $\mathcal{O}(\omega)$ term is of the form [9]

$$\begin{aligned} \mathcal{M}^\mu(\omega, \dots) \\ \stackrel{\omega \rightarrow 0}{\sim} \omega \left(\frac{M^2}{M^2 - 4m^2} + \text{regular terms as } M \rightarrow 2m \right). \end{aligned} \quad (6)$$

Obviously, a perturbative expansion in the binding energy $M - 2m$ is mathematically inconsistent with the soft-photon expansion: if the limit $M \rightarrow 2m$ is taken before $\omega \rightarrow 0$, spurious $\mathcal{O}(1)$ terms arise. Because the basis of current computations is a perturbative expansion in the binding energy $M - 2m$, computed as relativistic and radiative corrections to the Ore–Powell result, one can expect that spurious radiations affect the corrections presented in the literature.

As a comment, notice that the present considerations apply to amplitudes. For orthopositronium, any term less singular than $\mathcal{O}(1/\sqrt{\omega})$ would lead to an IR finite decay rate, because of (2). In other words, unphysical terms can lead to IR finite contributions to the decay rate. Computations presented in the literature sometimes explicitly exhibit such a bad behavior [10].

To illustrate how the Low's theorem is implemented in the presence of charged intermediate states, let us give an example: $K_S^0 \rightarrow e^+e^-\gamma$. The process is modeled by a charged pion loop. The resulting amplitude can be found in many places (see, for example, [3, 11, 12]), and it behaves exactly as predicted by Low's theorem (i.e., as ω^3 near $\omega = 0$), even if in this case $M_K > 2m_\pi$, i.e., the intermediate charged pion pair can be on-shell.

Now, let us imagine that one is willing to compute the decay rate for $K_S^0 \rightarrow e^+e^-\gamma$ by assuming that intermediate on-shell $\pi^+\pi^-$ dominates. The decay process is then factorized as $K_S^0 \rightarrow \pi^+\pi^-$ times $\pi^+\pi^- \rightarrow e^+e^-\gamma$. This is exactly the approximation done to get the Ore–Powell result: $o\text{-}Ps \rightarrow e^+e^-$ times $e^+e^- \rightarrow \gamma\gamma\gamma$. However, the soft-photon spectrum of the factorized approximation is completely wrong, being in contradiction with Low's theorem.

The approximation done was too stringent. To get the correct answer, one must also consider processes like $K_S^0 \rightarrow \pi^+\pi^-\gamma$ times $\pi^+\pi^-(\gamma) \rightarrow e^+e^-(\gamma)$ (where the photon is disconnected). These bremsstrahlung processes interfere destructively with the previous ones, giving a finite *vanishing* complete amplitude in the soft-photon limit. In the framework of dispersion relations [12], this is a simple application of the Cutkosky rule to get the absorptive part of an amplitude. Similarly, we state that the reason why the Ore–Powell result fails to exhibit a correct soft-photon spectrum is because some contributions to the amplitude are missed (like $o\text{-}Ps \rightarrow e^+e^-\gamma$ times $e^+e^-(\gamma) \rightarrow \gamma\gamma(\gamma)$, see [3, 9]).

Higher order corrections to the Ore–Powell result [6, 8] are similarly incomplete. Trying to take binding energy effects into account with models based on any kind of factorization $o\text{-}Ps \rightarrow e^+e^-$ times $e^+e^- \rightarrow \gamma\gamma\gamma$ is hopeless in view of Cutkosky rules. The fact that the on-shell intermediate state method “works” (i.e., is not IR divergent at the decay rate level) for orthopositronium is not a valid argument, since the

amplitudes produced by that method fails to fulfill a basic requirement of QED, namely, Low's theorem.

4. Conclusions and perspectives

From a theoretical perspective, as shown in [3], the introduction of additional contributions (i.e., process like $o\text{-}Ps \rightarrow e^+e^-\gamma$ times $e^+e^-(\gamma) \rightarrow \gamma\gamma(\gamma)$) to the positronium decay amplitude is unavoidable if one is willing to fulfill the basic requirement of Low's theorem. As we have explained, the ω^3 low-energy spectrum is a consequence of the properties of the positronium or quarkonium “as seen from far away”, i.e., as a neutral point-like self-conjugate bosonic particle. Such a particle does not radiate zero-energy photons, and the resulting photon spectrum must exhibit a ω^3 shape near $\omega = 0$. This is equally true for J/ψ , Υ , ..., and our solution is naturally extended to quarkonium theory.

On the practical side, it appears quite obvious that the violation of Low's theorem is very small in positronium decay. In other words, the missing contributions are subleading. As discussed in [3], we can expect them to introduce corrections of the order of the binding energy $E_B = M - 2m$, i.e., α^2 , or beyond. Nevertheless, even if very small, those corrections are relevant to the current theoretical considerations. Indeed, it is at the α^2 level that some discrepancies have been found among experiments, the so-called *orthopositronium lifetime puzzle* [3, 6, 8]. What we claim is that no definite answer to that puzzle could be given at present. Indeed, the additional contributions could turn out to be less suppressed than usually thought, and the current theoretical result for the α^2 correction is not fully reliable.

This state of affair is to be contrasted to the quarkonium case. There, the missing contributions could become sizeable since the binding energy is non-negligible (see (6)). In other words, the relevance of the Ore–Powell result for quarkonia is doubtful for any precision calculation.

For instance, the *photon spectrum in inclusive quarkonium decay into hadrons + photon* [13] will clearly be affected by the missing contributions. Indeed, the modification of the spectrum needed at low energy to fulfill Low's theorem will affect the spectrum also at high energy (since when the photon has

its maximum energy, one of the gluon's energy can go to zero, see the integration ranges in (5)).

Also, the missing contributions should be crucial to solve the ρ - π puzzle [14,15]. The so-called 14% rule is obtained from the ratio of the leptonic mode of the J/ψ and $\psi(2S)$ (which is essentially the ratio of wave functions at zero separation). There is no missed contribution for the leptonic modes. On the other hand, at least 12 additional contributions need to be considered for three-gluon modes, and those will depend on the binding energy, which is quite different for J/ψ and $\psi(2S)$. Those additional contributions, which arise already at the lowest order, are essential to enforce Low's theorem through their destructive interferences with the standard factorized Ore–Powell ones.

Finally, the previous remark also shows that the extraction of α_s [5] from quarkonia branching fractions will be affected, at least partially, by the additional contributions.

In conclusion, we have shown that the implications of gauge invariance and analyticity, in the form of constraints on the low-energy end of photon spectrum are not met by current bound state decay models. Restoring a correct behavior in that low-energy region could lead to potentially interesting advances in both QED and QCD bound state description.

Acknowledgements

We are very pleased to acknowledge useful discussions with Gabriel Lopez Castro, Jean-Marc Gérard, Stéphanie Trine and Jacques Weyers. C.S. acknowledges financial support from FNRS and IISN (Belgium).

References

- [1] F.E. Low, Phys. Rev. 110 (1958) 974.
- [2] A. Ore, J.L. Powell, Phys. Rev. 75 (1949) 1696.
- [3] J. Pestieau, C. Smith, S. Trine, hep-ph/0105034, accepted for publication in Int. J. Mod. Phys. A, and references therein.
- [4] J.A. Wheeler, Ann. N.Y. Acad. Sci. 48 (1946) 219; J. Pirenne, Arch. Sci. Phys. Nat. 29 (1947) 265.
- [5] S.J. Brodsky, D.G. Coyne, T.A. DeGrand, R.R. Horgan, Phys. Lett. B 73 (1978) 203; V.A. Novikov, L.B. Okun, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, V.I. Zakharov, Phys. Rep. C 41 (1978) 1; W. Kwong, P. Mackenzie, R. Rosenfeld, J. Rosner, Phys. Rev. D 37 (1988) 3210; G. Schuler, CERN-TH-7170/94, hep-ph/9403387, and references therein.
- [6] W.E. Caswell, G.P. Lepage, J. Sapirstein, Phys. Rev. Lett. 38 (1977) 488; W.E. Caswell, G.P. Lepage, Phys. Rev. A 20 (1979) 36; G. Adkins, Ann. Phys. 146 (1983) 78; A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys. Rev. Lett. 83 (1999) 1135; G. Adkins, R. Fell, J. Sapirstein, Phys. Rev. Lett. 84 (2000) 5086; B. Kniehl, A. Penin, Phys. Rev. Lett. 85 (2000) 1210; K. Melnikov, A. Yelkhovsky, Phys. Rev. D 62 (2000) 116003; R. Hill, G.P. Lepage, Phys. Rev. D 62 (2000) 111301.
- [7] W.E. Caswell, G.P. Lepage, Phys. Lett. B 167 (1986) 437; P. Labelle, MRST Meeting 1992, CLNS-92-1161.
- [8] I.B. Khriplovich, A.S. Yelkhovsky, Phys. Lett. B 246 (1990) 520; I.B. Khriplovich, A.I. Milstein, J. Exp. Theor. Phys. 79 (1994) 379; V. Antonelli, V. Ivanchenko, E. Kuraev, V. Laliena, Eur. Phys. J. C 5 (1998) 535; V. Antonelli, Int. Work. on Hadronic Atoms and Positronium in the S.M., Dubna, 26–31 May, 1998.
- [9] J. Pestieau, C. Smith, in preparation.
- [10] R.N. Faustov, A.P. Martynenko, V.A. Saleev, Phys. Rev. A 51 (1995) 4520.
- [11] L.M. Sehgal, Phys. Rev. D 7 (1973) 3303; L. Bergström, G. Hulth, Nucl. Phys. B 259 (1985) 137; G. D'Ambrosio, G. Ecker, G. Isidori, H. Neufeld, 2nd DAPHNE Physics Handbook 253–313, hep-ph/9411439.
- [12] K. Nishijima, Fields and Particles: Field Theory and Dispersion Relations, Benjamin, New York, 1969.
- [13] For recent experimental results, comparison with theoretical models and list of references, see: CLEO Collaboration, B. Nemati et al., Phys. Rev. D 55 (1997) 5273; F. Hautmann, Proceedings of International Conference on the Structure and the Interactions of the Photon, Photon '97 Netherlands, 10–15 May, 1997, hep-ph/9708496; M. Kramer, Proceedings of 4th International Symposium on Radiative Corrections, RADCOR '98, Barcelona, 8–12 September, 1998, hep-ph/9901448; S. Wolf, Phys. Rev. D 63 (2001) 074020.
- [14] For recent reviews, see, for example, S.F. Tuan, Commun. Theor. Phys. 33 (2000) 285, and references therein; Y.F. Gu, X.H. Li, Phys. Rev. D 63 (2001) 114019, and references therein.
- [15] J.-M. Gérard, J. Weyers, Phys. Lett. B 462 (1999) 324.