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NONPARAMETRIC GOODNESS-OF-FIT TEST FOR HETEROSCEDASTIC REGRESSION MODELS

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Abstract

For the heteroscedastic nonparametric regression model $Y_{ni} = m(x_{ni})+\sigma(x_{ni})\epsilon_{ni}$, $i = 1, ..., n$, we propose a new test procedure for testing that the regression function m is constant. The test statistic is modeled after the usual lack-of-fit statistic for constant regression in the case of replicated observations, and thus is very easy to compute. The asymptotic theory uses recent developments in the asymptotic theory for analysis of variance when the number of factor levels is large. Comparisons with competing procedures and analysis of a data set are included.

KEY WORDS: Lack-of-fit tests; ANOVA; Nonparametric regression; Nearest neighbor weights; Asymptotic representation; Efficacy.

1 Introduction

Consider the following heteroscedastic nonparametric regression model

$$Y_{ni} = Y(x_{ni}) = m(x_{ni}) + \sigma(x_{ni})\epsilon_{ni}, \;\; i = 1, ..., n, \tag{1.1}$$

where Y is the response, $m(\cdot)$ is an unknown regression function, $x_{n1}, \dots, x_{nn}$ are fixed design points on [0,1], $\sigma^2(\cdot)$ is the variance function, and ϵ_{ni} form a triangular array with row-wise independent random variables with mean 0 and variance 1. The n in the subscript will be omitted from the notation when no confusion is possible.

Of interest is to test the hypothesis of a constant regression function (or no effect):

$$H_0 : m(x) = C \;\; \text{for all } x, \tag{1.2}$$

for some unknown constant $C \in I\!R$. This hypothesis has been considered by quite a few authors. Von Neumann (1941) proposed to test this hypothesis through the ratio of two variance estimators; Raz (1990) used moments matching method to approximate the null permutation distribution of a test statistic for the homoscedastic model; Barry and Hartigan (1990) suggested a likelihood ratio test while assuming the error terms are normally distributed; Buckley (1991) proved that a cusum-based test is locally most powerful in a certain class of Bayesian models; Eubank and Hart (1992) discussed it as a special case of a test based on order selection criteria, a bootstrap version of their test was newly proposed by Chen, Hart and Wang (2001), etc. Chapter 7 of the book of Hart (1997) discussed several tests for (1.2) based on orthogonal series, Dette and Munk (1998a) investigated (1.2) as a special example and made some comparisons with the order selection test of Eubank and Hart. Traditional results often imposed many restrictions on the error structures and were often concerned with equally-spaced design situation only. With the exception of the paper of Chen, Hart and Wang (2001), all the work mentioned above deals with homoscedastic situation only. Dette and Munk (1998b) generalized their results to the heteroscedastic case. The importance of this simple hypothesis lies in the fact that it provides the basis and insight for testing a more general class of hypotheses, i.e., to test if the mean regression function m(x) has some specified parametric form. Such an extension of our test procedure is discussed in Section 2.

The procedure is inspired by a test statistic in the context of analysis of variance where the number of cells tends to infinity. To describe this, suppose there are n cells with cell i having k_{ni} independent observations $V_{i1}, \dots, V_{ik_{ni}}$, $i = 1, \dots, n$, and suppose the observations of two different cells are also independent. Let $N = \sum_{i=1}^{n} k_{ni}$, and define

$$F_n = \frac{MST}{MSE}$$

where

$$MST = \frac{1}{n-1}\sum_{i=1}^{n} k_{ni}(\bar{V}_{i\cdot} - \bar{V}_{\cdot\cdot})^2, \quad MSE = \frac{1}{N-n}\sum_{i=1}^{n}\sum_{j=1}^{k_{ni}}(V_{ij} - \bar{V}_{i\cdot})^2.$$

A number of authors (Boos and Brownie, 1995; Akritas and Arnold, 2000; Akritas and Papadatos, 2001; Bathke 2002) have studied the asymptotic distribution of $\sqrt{n}(F_n - 1)$. Note that since MSE converges to a constant, by Slutsky's theorem the problem reduces to studying the asymptotic distribution of $\sqrt{n}(MST - MSE)$. The basic requirement in these asymptotics is that there be at least two observations per category.

The idea for constructing our test statistic for lack-of-fit test in the present regression setting is to consider each distinct covariate value as a 'category', as proposed in Akritas (2000). Of course, we cannot use the asymptotic theory developed for the ANOVA case because typically there is only one Y-value per covariate value. This can be remedied by considering a window W_i around each x_i consisting of the k_n nearest covariate values. That is, we construct an artificial balanced one-way ANOVA setting with n categories, where the responses in the i-th category are the Y-values that correspond to the covariate values belonging in W_i. This artificial one-way ANOVA has overlapping groups (i.e. neighboring groups have common observations), and thus the asymptotic results for one-way ANOVA with the number of levels tending to infinity still does not apply. However, we can apply the ideas behind Akritas and Papadatos (2001) to derive the asymptotic theory of the test statistic in the present case of overlapping groups. In particular, they showed that under the null hypothesis of equal cell means

$$\sqrt{n}(MST - MSE) = \sqrt{n}\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V} = \sqrt{n}\boldsymbol{V}'\boldsymbol{A}_{\mathbf{D}}\boldsymbol{V} + o_p(1),$$

where $\boldsymbol{V} = (V_{11}, \dots, V_{1k_{n1}}, \dots, V_{n1}, \dots, V_{nk_{nn}})'$, $\boldsymbol{A}$ is the matrix of the quadratic form $MST - MSE$:

$$\boldsymbol{A} = \frac{nk_n - 1}{n(n-1)k_n(k_n-1)} \bigoplus_{i=1}^{n} \boldsymbol{J}_{k_n} - \frac{1}{n(n-1)k_n}\boldsymbol{J}_{nk_n} - \frac{1}{n(k_n-1)}\boldsymbol{I}_{nk_n}, \quad (1.3)$$

and $\boldsymbol{A}_{\mathbf{D}}$ is a block diagonal matrix where the ith block has dimensions $k_{ni} \times k_{ni}$ and equals the corresponding block of the $\boldsymbol{A}$ matrix. Thus, $\sqrt{n}(MST - MSE)$ is, under the null hypothesis, asymptotically equivalent to a scaled and centered sum of independent components and its asymptotic distribution follows. We show that, with appropriate modifications, this approach works in the case of overlapping groups.

As already indicated, we can and will choose the categories in such a way that we have equal sample sizes. Also, in what follows the windows W_i will be understood as sets containing the indices j of the covariate values that belong in the window, that is

$$W_i = \left\{ j : \; I \left[|\widehat{F}_X(x_j) - \widehat{F}_X(x_i)| \leq \frac{k_{ni} - 1}{2n} \right] \right\}, \tag{1.4}$$

if k_{ni} is odd, where $I(A)$ denotes the indicator function of the event A and $\widehat{F}_X$ is the empirical distribution function of the covariate values. Specifically, we take $k_{ni} = k_n$, $\forall i$, and replace the $V_{ij}, j = 1, \ldots, k_n$, by Y_j, $j \in W_i$. The quantities $\bar{V}_{i\cdot}$, and $\bar{V}_{\cdot\cdot}$ in the definition of MST, MSE will be replaced, respectively, by

$$\hat{m}(x_i) = \frac{1}{k_n} \sum_{j=1}^{n} Y_j I(j \in W_i), \quad \bar{\hat{m}} = n^{-1} \sum_{i=1}^{n} \hat{m}(x_i).$$

Also we will keep denoting

$$\boldsymbol{V} = (Y_j, j \in W_1, \ldots, Y_j, j \in W_n)'. \tag{1.5}$$

The asymptotic distribution of the proposed test statistic is derived in Section 2. A comparison with the simpler statistic which uses non-overlapping groups, which essentially discretizes the covariate and applies the one-way ANOVA statistic for testing equality of group means, is also given in Section 2. In Section 3, we investigate the finite sample behavior of the test and compare it with several other tests proposed in the literature, such as the test of Raz (1990), the test of Barry and Hartigan (1990) and the test of Dette and Munk (1998a, 1998b). An analysis of a real dataset from a research project studying the effect of maternal age on the incidence of Down's syndrome is discussed in Section 4. A discussion of possible extensions is given in Section 5, and finally the technical proofs are given in the appendix.

2 Main Results

Let r(x) denote a continuous density on [0,1] which is positive, and consider design points $x_1, x_2, ..., x_n$ satisfying

$$\int_0^{x_i} r(x)dx = \frac{i}{n}, \quad i = 1, ..., n. \tag{2.1}$$

Thus, it is a regular sequence in the sense of Sacks and Ylvisaker(1970). Assume that $\sigma^2(x)$ is Lipschitz continuous, i.e.,

$$|\sigma^2(x) - \sigma^2(y)| \le c|x - y|, \quad \forall\, x, y \in [0, 1], \tag{2.2}$$

for some positive constant c.

We start with the result about the asymptotic equivalence, under H_0 in (1.2), of the quadratic form $(n/k_n)^{1/2}\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V}$ with another quadratic form involving a block diagonal matrix.

Lemma 2.1 *Assume* (2.1), (2.2), $n^{-1}k_n \to 0$, *and that* $E(\epsilon_i^4)$ *are uniformly bounded in* n *and* i. *Then under* $H_0 : m(x) = C$,

$$\left(\frac{n}{k_n}\right)^{1/2} [\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V} - (\boldsymbol{V} - C\mathbf{1}_N)'\boldsymbol{A}_d(\boldsymbol{V} - C\mathbf{1}_N)] \xrightarrow{P} 0,$$

where $\boldsymbol{A}_d$ *is the block diagonal matrix*

$$\boldsymbol{A}_d = diag\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n\}, \text{ with } \boldsymbol{B}_i = \frac{1}{n(k_n - 1)}[\boldsymbol{J}_{k_n} - \boldsymbol{I}_{k_n}],$$

where $\boldsymbol{I}_{k_n}$ *is the* k_n*-dimensional identity matrix, and* $\boldsymbol{J}_{k_n} = \mathbf{1}_{k_n}\mathbf{1}'_{k_n}$ *where* $\mathbf{1}_{k_n}$ *is the* k_n*-dimensional column vector of 1's.*

Note that since $\boldsymbol{A}$ is a contrast matrix, $\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V} = (\boldsymbol{V} - C\mathbf{1}_N)'\boldsymbol{A}(\boldsymbol{V} - C\mathbf{1}_N)$. This equality does not hold true however for $\boldsymbol{A}_d$. This result enables us to show the asymptotic normality of the test statistic.

Theorem 2.2 *Under the assumptions of Lemma 2.1,*
(1) If k_n *is fixed, then*

$$n^{1/2}(MST - MSE) \to N\left(0, \frac{2k_n(2k_n - 1)}{3(k_n - 1)}\tau^2\right),$$

under H_0, where $\tau^2 = \int_0^1 \sigma^4(x)r(x)dx$.
(2) If $k_n \to \infty$, then

$$\left(\frac{n}{k_n}\right)^{1/2}(MST - MSE) \to N\left(0, \frac{4}{3}\tau^2\right),$$

under H_0, where τ^2 is defined as above.

A simple estimator of τ^2 is a modification of the Rice's (1984) estimator, i.e.,

$$\frac{1}{4(n-3)}\sum_{j=2}^{n-2} R_j^2 R_{j+2}^2, \tag{2.3}$$

where $R_j = Y_j - Y_{j-1}$ denote the local residuals, j=2, ...n.

Remark 1. The goodness-of-fit test for no effect described above can actually be generalized to testing the null hypothesis: m(x) belongs to a specified parametric family versus an omnibus alternative. Suppose that $S_\Theta = \{f(\cdot,\theta), \theta \in \Theta\}$ is any parametric family of functions and we wish to test:

$$H_0: \quad m(x) \in S_\Theta,$$

against a general alternative. Let $\hat{\theta}$ be our "best estimator" of the true parameter θ under the null hypothesis, then we may apply the above test to the residuals: $\hat{e}_i = Y_i - m(X_i, \hat{\theta})$, i=1,...,n. This point is quite intuitive and was mentioned, for example, by Hart (1997), Fan and Huang (2001). The only complication is that $\hat{e}_i$ will not be independent in general. This extension will be considered in a forthcoming paper.

Next, we consider the asymptotic behavior of the test statistic under local alternatives.

Theorem 2.3 *Assume the conditions of Theorem 2.2 are satisfied, let $g(x)$ be a Hölder continuous function of order $\delta > 0$ on [0,1],*
(1) If k_n is fixed, then under $H_1 : m(x) = C + (nk_n)^{-1/4}g(x)$,

$$\left(\frac{n}{k_n}\right)^{1/2}(MST - MSE) \to N\left(\gamma^2, \frac{2(2k_n-1)}{3(k_n-1)}\tau^2\right),$$

where τ^2 is defined as in Theorem 2.2 and $\gamma^2 = \int_0^1 g^2(t)r(t)dt - (\int_0^1 g(t)r(t))^2$. Thus, the efficacy of the test statistic is given by

$$\left(\frac{3(k_n-1)}{2(2k_n-1)}\right)^{1/2}\frac{\gamma^2}{\tau}.$$

(2) If $k_n \to \infty$, then under $H_1 : m(x) = C + (nk_n)^{-1/4} g(x)$,

$$\left(\frac{n}{k_n}\right)^{1/2} (MST - MSE) \to N\left(\gamma^2, \frac{4}{3}\tau^2\right),$$

where γ^2 and τ^2 are defined as above. The efficacy of the test statistic is given by

$$\frac{\sqrt{3}\gamma^2}{2\tau}.$$

Remark 2. The efficacy formulas show that as k_n increases so does the efficay. For example, the fixed k_n values of 2, 5 and 10 give efficacies of 0.707, 0.816 and 0.843, respectively. If $k_n \to \infty$, the efficacy is 0.866. Thus, when the convergence to normality is slower (which happens when $k_n \to \infty$) the efficacy is the largest. On the other hand, the rate at which $k_n \to \infty$ does not influence the efficacy, even though it does influence the rate of convergence to normality.

Remark 3. As mentioned in the introduction, the need for considering windows around each covariate value arose from the requirement of the asymptotic theory of ANOVA with large number of factor levels to have more than one observation per group. An alternative way of satisfying this requirement is to discretize the covariate. This results in non-overlapping groups, which allows for direct application of the asymptotic theory for the ANOVA case (Akritas and Papadatos, 2001). Thus, if we consider non-overlapping groups of size k_n, and consider for simplicity the case that $k_n \to \infty$, then under the same sequence of alternatives as in Theorem 2.3,

$$\left(\frac{n}{k_n}\right)^{1/2} (MST - MSE) \to N\left(\gamma^2, 2\tau^2\right),$$

so its efficacy is $\gamma^2/(\sqrt{2}\tau)$, implying it is about 82% as efficient as the statistic based on overlapping windows.

3 Simulations

In this section, we compare the achieved level and power of our test with some other tests in the literature for various sample sizes n, window sizes k_n and error distributions, both homoscedastic and heteroscedastic. In particular, for the homoscedastic case we adopted the simulation designs considered in Raz (1990), Barry and Hartigan (1990) and Dette and Munk (1998a) so we can directly compare the performance of our test statistic with theirs. By using their simulation designs we make feasible a comparison of our test statistic with other tests discussed in the aforementioned papers, such as von Neumann's (1941) ratio test (in the paper of Barry and Hartigan) and the order selection test of Eubank and Hart (1992) (in the paper of Dette and Munk, 1998a).

For the heteroscedastic case, we compare our test statistic with that of Dette and Munk (1998b). Since these authors do not report simulations, their test statistic was incorporated in our simulations.

We mention that, as is clear from their definition given in (1.4), in order for the windows W_i to be of equal size the first window must be $W_{(k_n+1)/2}$ and the last $W_{n-(k_n+1)/2}$. Thus our test procedure tests for the equality of a total of $n - k_n + 1$ group means.

The simulations demonstrate that our test statistic has good power properties, and in several situations, outperforms the competitors. Our simulations are based on 5,000 runs, random numbers are generated using Matlab.

3.1 Homoscedastic case

3.1.1 Level of the Test

The sample sizes considered are n=50, 60, 80, 100. We used the same error distributions as those used in Raz (1990), namely the standard normal, the $sinh^{-1}$-normal and the lognormal, the design points are equally-spaced on the interval (0,1]: $x_i = i/n$, i=1,..., n. The $sinh^{-1}$-normal distribution is symmetric with moderately large kurtosis. The lognormal distribution is highly skewed with very large kurtosis. As in the paper of Raz (1990), we generated $sinh^{-1}$-normal distribution using the method of Johnson, Ramberg and Wang (1982). For comparison purpose, we standardized the errors distributions so that they all have mean 0 and variance 1. The performance of the test depends on the window size k_n, a smoothing parameter, therefore we presented our results for a range of window sizes for each simulated scenario.

The simulation results for nominal level 0.05 are presented in Table 1. In general, it seems that window sizes of 9 and 11 observations are too large for the sample sizes we considered, since they tend to be conservative, especially for the smaller sample sizes. Window sizes of 5 and 7 observations seem to perform well throughout, though not quite as accurate as Raz's (1990) randomization test. Similar results were observed for the nominal level 0.01, but are not reported.

Put Table 1 about here

3.1.2 Power of the Test

We performed three power simulation studies involving the alternatives:

$$\left.\begin{array}{lll} \text{alternative 1:} & m(x) & = 2.001x, \\ \text{alternative 2:} & m(x) & = 9.513(x-\overline{x})^2, \\ \text{alternative 3:} & m(x) & = 1.155sin[4(j-1)*\pi/n], \end{array}\right\} \tag{3.1}$$

$$\left.\begin{array}{lllll} \text{alternative 4:} & m(x) & = x, & \text{alternative 5:} & m(x) = 2e^{5x}/(1+e^{5x})-1, \\ \text{alternative 6:} & m(x) & = 64x^3(1-x)^3, & & \end{array}\right\} \tag{3.2}$$

$$\left.\begin{array}{lll} \text{alternative 7:} & m(x) & = (1+sin(3\pi x))/2; \\ \text{alternative 8:} & m(x) & = \begin{cases} 15x/6, & \text{if } 0 \le x \le 0.2, \\ (5-10x)/6, & \text{if } 0.2 \le x \le 0.4, \\ (-9+25x)/6, & \text{if } 0.4 \le x \le 0.6, \\ (-18+20x)/6, & \text{if } 0.6 \le x \le 0.8, \\ (-2+5x)/6, & \text{if } 0.8 \le x \le 1.0. \end{cases} \end{array}\right\} \tag{3.3}$$

$$\text{alternative 9:} \quad m(x) = 1+\theta cos(10\pi x), \tag{3.4}$$

The first compares the power of our test statistic with that of Raz's (1990) randomization test. The comparison is achieved by using the simulation design that was used in his paper. In particular, the design points are equally spaced on the interval (0,1], used the same three error structures as in the last section, and generated data from alternatives (3.1), which are the same three alternatives considered by Raz (1990).

Put Table 2 about here

Raz only provided results for n=30 in his paper. In Table 2 we present results for both n=30 and 60. As it can be seen there, the power is higher when the error is nonnormal. Also, the power is smallest when the alternative is linear and largest when it is sinusoidal. Compared with Raz's test, the power of our test is a little bit lower for the first two alternatives when $n = 30$ but it behaves better for the sinusoidal alternative hypothesis. It should be pointed out that Raz's test depends heavily both on the span size (smoothing parameter) and the estimating procedures (local regression, order 2 kernel or order 4 kernel), no single choice of span size or estimating procedure gave nearly uniformly better results for the three alternatives and this makes his approach less attractive for implementation in practice. Moreover, our test statistic is much simpler to compute. The power of our test improves dramatically when n=60.

The second simulation study, compares our test with the B-test developed in Barry and Hartigan (1990) and several other tests discussed in their paper. Again the comparison is achieved by using the simulation design that was used in their paper. In particular, the design points are generated by $x_i = \frac{2i-1}{2(n+1)}$, $i = 1, ..., n+1$, the errors are $N(0, \sigma^2)$ with σ values: 0.1, 0.5, 1.0 and 2.0, and consider the five alternatives given in (3.2) and (3.3).

Put Table 3 about here

Put Table 4 about here

Tables 3 and 4 present the rejection rates for testing at the 0.05 significance level. It is clear from these tables that the power decreases rapidly as the noise level increases, the same phenomenon was observed in the paper of Barry and Hartigan (table2, p1335). Compared with their results, our test has significantly larger power for alternatives 6, 7 and 8. For the other two types of alternatives, our test has somewhat lower power. The B-test is based on maximum likelihood thus the assumption of normal error distribution is required. In addition, the design points have to be equally spaced. Our test is more flexible in both error structures and mechanisms of design points and it can be much more easily calculated.

In our third simulation study we generated random data from the same setting discussed in the paper of Dette and Munk (1998a). Thus, the design points are $x_i = (i - 1/2)/n$, i=1,...,n, with n=100, the alternatives considered are all of the type (3.4) with $\theta = 0.00, 0.25, 0.50, 0.75, 1.00$, and the error terms are i.i.d. standard normal. Table 5 presents the rejection rates for testing at nominal levels $\alpha = 0.05$ and $\alpha = 0.10$.

Put Table 5 about here

Compared with the results in Table 2 of Dette and Munk (1998a, pp. 795), the power of our test is remarkably higher. As pointed out in that paper, the order selection test of Eubank and Hart (1992) is extremely sensitive with respect to the choice and ordering of the orthogonal basis functions, and therefore it loses some of its appeal in many practical applications. Our test is computationally simpler than that of Eubank and Hart(1992).

3.2 Heteroscedastic case

In this section we compare our test procedure with that of Dette and Munk (1998b) in the heteroscedastic fixed design case. (The Dette and Munk procedure is also applicable for more general hypotheses but for $d = 1$ and $g_1 = 1$ it pertains to the test of the constant regression hypothesis.) Their procedure rejects when

$$\frac{\sqrt{n}\widehat{M}_n^2}{\hat{\tau}} > z_{1-\alpha}, \tag{3.5}$$

where $\widehat{M}_n^2 = n^{-1}\sum_{i=1}^n (Y_i - \overline{Y}_n)^2 - [2(n-1)]^{-1}\sum_{i=2}^n (Y_i - Y_{i-1})^2$, $\hat{\tau}$ is the square root of (2.3), and $z_{1-\alpha}$ is the $1 - \alpha$ quantile of the standard normal distribution. Our simulation assumes several different functional forms for $\sigma^2(x)$ and takes ϵ to be i.i.d. N(0,1). The sample sizes are taken to be 60, 80, 100. The design points are equally spaced on [0,1], and the nominal level is 0.05.

From Table 6, we observe that our test maintains the type I error fairly accurately while the Dette and Munk test is somewhat liberal. Parenthetically, we remark that simulations in Munk (2002) indicate that the use of jacknife correction and of bootstrap improve the performance of the Dette and Munk test. Table 7 displays the result of estimated power for three different functional forms of $m(x)$ and the same three different forms of variance function as in Table 6. In these two tables, the test of Dette and Munk is abbreviated as "DM" test. The power of our test is significantly higher than that of the Dette and Munk test in all cases.

Put Table 6 about here

Put Table 7 about here

4 A Case Study

In the 1960's, a large scale study of the effect of maternal age on the incidence of Down's syndrome, a genetic disorder caused by an extra chromosome 21 or a part of chromosome 21 being translocated to another chromosome, was conducted at the British Columbia Health Surveillance Registry (Geyer, 1991). There are 30 different age groups, the proportion of down's syndrome among the babies born to mothers of that age group was calculated. Of interest is to test if the incidence of down's syndrome is affected by maternal age. Another hypothesis of interest is that the incidence of down's syndrome is only sharply rises after age 30. Using age as covariate and proportion of the disease as response variable, we performed classical F test (denoted by CF, corresponding to fit a linear regression model and test if the slope is zero), our nonparametric test (denoted by NP) and the test of Dette and Munk (denoted by DM). For the overall data, the p-value of the CF test is 0; the p-values of NP test are all 0 for $k_n = 3, 5, 7$ no matter we assume homoscedasticity or not; the p-values of DM test are both 0 no matter we assume homoscedasticity or not. All three approaches suggest that the incidence of down's syndrome is influenced by maternal age. For the data corresponding to the first 14 age groups (up to age 30.5), the p-value of the CF test is 0.379; the p-value of NP test is 0.7925 for $k_n = 3$, 0.7205 for $k_n = 5$, 0.7808 for $k_n = 7$ if we assume homoscedasticity and is 0.9083 for $k_n = 3$, 0.7683 for $k_n = 5$, 0.8283 for $k_n = 7$ if we don't assume homoscedasticity; the p-value for DM test is 0.4735 if we assume homoscedasticity and 0.4587 if we don't assume homoscedasticity. All three approaches agree that the influence of maternal age only becomes obvious after age 30.

Put Figure 1 about here

5 Discussion

We have presented a statistic for testing the hypothesis of a constant regression function. It is closely related to the classical lack-of-fit statistic used in the case of replicated observations, and to the one-way ANOVA test statistic. In fact, the only difference lies in the fact that the replications or groups are artificially generated from observations with neighboring covariate values. Thus, our statistic is very easily computed with any software package.

The statistic belongs to the category of test statistics based on difference or ratio of two estimators of the integrated variance function $\int \sigma^2(t)r(t)dt$. One is asymptotically unbiased under the null hypothesis only, and the other is so under both the null and alternative hypotheses. Indeed, MSE is asymptotically unbiased under general nonparametric alternatives while MST is asymptotically unbiased under the null model only and asymptotically positively biased under the alternative. An early example of a test statistic of this class is the ratio test of von Neumann (1941). A more recent example is the statistic of Dette and Munk (1998a,b) given in (3.5).

A natural application of the results presented here is to test whether two curves are equal when two data sets are observed at the same design points. Assume we observe triple variables: (x_i, Y_i, Z_i), i=1,...,n. If

$$\begin{aligned} Y_i &= m_1(x_i) + \sigma_1(x_i)\epsilon_i, \\ Z_i &= m_2(x_i) + \sigma_2(x_i)\epsilon_i, \end{aligned}$$

then to test the hypothesis $m_1(x) = m_2(x), \forall\ x$, we may apply our test with $C = 0$ to $Y_i - Z_i$, i=1,...,n. This type of problem arises frequently in practice; see Hart (1997), §9.6.

The extension to more complicated hypotheses and to more covariates is being investigated. The present methodology, and its extension to higher dimensions, will be applied for testing the hypothesis of no covariate effect in the nonparametric ANCOVA model of Akritas, Arnold and Du (2000).

Appendix: Proofs

In all proofs that follow, we will take k_n be odd for simplicity.

Proof of Lemma 2.1. It is easy to see that the block diagonal elements of $\boldsymbol{A}_d$ equal those of $\boldsymbol{A}$. Hence, it suffices to prove that the off diagonal blocks of $\boldsymbol{A}$ are negligible. For $i_1 \neq i_2$, every element of the block (i_1, i_2) equals $c = -\frac{1}{n(n-1)k_n}$. Hence,

$$\begin{aligned}
&\left(\frac{n}{k_n}\right)^{1/2} (\boldsymbol{V} - C\mathbf{1}_N)'(\boldsymbol{A} - \boldsymbol{A}_d)(\boldsymbol{V} - C\mathbf{1}_N) \\
&= \left(\frac{n}{k_n}\right)^{1/2} c \sum_{i_1 \neq i_2} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} (Y_{j_1} - C)(Y_{j_2} - C) I(j_1 \in W_{i_1}) I(j_2 \in W_{i_2}),
\end{aligned}$$

where $W_i = \{j : w_{ij} \neq 0\}$. It is easy to show that $E\left[\left(\frac{n}{k_n}\right)^{1/2} (\boldsymbol{V} - C\mathbf{1}_N)'(\boldsymbol{A} - \boldsymbol{A}_d)(\boldsymbol{V} - C\mathbf{1}_N)\right] = o(1)$, while

$$\begin{aligned}
&\frac{n}{k_n} E[(\boldsymbol{V} - C\mathbf{1}_N)'(\boldsymbol{A} - \boldsymbol{A}_d)(\boldsymbol{V} - C\mathbf{1}_N)]^2 \\
&= \frac{n}{k_n} c^2 \sum_{i_1 \neq i_2} \sum_{i_3 \neq i_4} \sum_{j_1, \dots, j_4 = 1}^{n} E[(Y_{j_1} - C)(Y_{j_2} - C)(Y_{j_3} - C)(Y_{j_4} - C) \\
&\qquad \times I(j_k \in W_{i_k}, k = 1, \dots, 4)]. \qquad \text{(A.1)}
\end{aligned}$$

The expected value in this sum is different from zero, only if $Y_{j_1}, \dots, Y_{j_4}$ consists of two pairs of equal observations, or $j_1 = j_2 = j_3 = j_4$. Since there are $O(n^2 k_n^4)$ terms for the former case to happen and $O(nk_n^4)$ for the latter case to happen, the order of (A.1) is

$$O\left(\frac{n}{k_n} \frac{1}{n^4 k_n^2} n^2 k_n^4\right) = O(k_n n^{-1}) = o(1),$$

and this finishes the proof.

Proof of Theorem 2.2. We can assume without loss of generality that $C = 0$.

$$\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V} = \frac{1}{n(k_n-1)}\sum_{i=1}^{n}\sum_{j_1\neq j_2}^{n} Y_{j_1}Y_{j_2}I(j_1, j_2 \in W_i),$$

it's obvious that $E(\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V}) = 0$.

$$\begin{aligned}
E[\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V}]^2 &= \frac{1}{n^2(k_n-1)^2}\sum_{i_1,i_2}^{n}\sum_{j_1\neq\ell_1}^{n}\sum_{j_2\neq\ell_2}^{n} E[Y_{j_1}Y_{\ell_1}Y_{j_2}Y_{\ell_2}]I(j_s \in W_{i_s}, \ell_s \in W_{i_s}, s = 1, 2) \\
&= \frac{2}{n^2(k_n-1)^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j\neq\ell}^{n}\sigma^2(x_j)\sigma^2(x_\ell)I(j, \ell \in W_{i_1}\cap W_{i_2}) \\
&= \frac{2}{n^2(k_n-1)^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j\neq\ell}^{n}\sigma^2(x_j)\left(\sigma^2(x_j) + O\left(\frac{k_n}{n}\right)\right)I(j, \ell \in W_{i_1}\cap W_{i_2}) \\
&= \frac{2}{n^2(k_n-1)^2}\sum_{j=1}^{n}\sigma^4(x_j)\sum_{l\neq j}^{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} I(j, l \in W_{i_1}\cap W_{i_2}) + O\left(\frac{k_n^2}{n^2}\right) \\
&= \frac{2}{n^2(k_n-1)^2}\sum_{j=1}^{n}\sigma^4(x_j)2\left[1 + 2^2 + 3^2 + \cdots + (k_n-1)^2\right] + O\left(\frac{k_n^2}{n^2}\right) \\
&= \frac{2}{n^2(k_n-1)^2}\frac{k_n(k_n-1)(2k_n-1)}{3}\sum_{j=1}^{n}\sigma^4(x_j) + O\left(\frac{k_n^2}{n^2}\right),
\end{aligned}$$

where the third equality is a result of the assumptions that $\sigma^2(x)$ is Lipschitz continuous and the design density is r bounded away from zero, while the second last equality follows from the fact that if $1 \leq |j_1 - j_2| = s \leq k_n - 1$, then they are in $(k_n - s)^2$ pairs of windows (including two identical windows) whose intersection includes both j_1 and j_2. Thus,

$$\begin{aligned}
E\left(\sqrt{\frac{n}{k_n}}\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V}\right)^2 &= \frac{2(2k_n-1)}{3(k_n-1)}\frac{1}{n}\sum_{j=1}^{n}\sigma^4(x_j) + O\left(\frac{k_n}{n}\right) \\
&= \frac{2(2k_n-1)}{3(k_n-1)}\sum_{j=1}^{n}\sigma^4(x_j)\int_{x_{j-1}}^{x_j} r(t)dt + o(1) \\
&\to \begin{cases} \frac{4}{3}\tau^2, & \text{if } k_n \to \infty \\ \frac{2(2k_n-1)}{3(k_n-1)}\tau^2, & \text{if } k_n \text{ is fixed.} \end{cases}
\end{aligned}$$

It remains to verify the asymptotic normality of $(n/k_n)^{1/2}(\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V})$. We will make use of

Markov's blocking technique (see proof of Theorem 27.5 in Billingsley, 1986). Write

$$\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V} = n^{-1}\sum_{i=1}^{n} A_i = n^{-1}S_n,$$

where

$$A_i = \frac{1}{k_n - 1}\sum_{j_1 \neq j_2}^{n} Y_{j_1}Y_{j_2}I(j_1, j_2 \in W_i).$$

We will establish the weak convergence of $(nk_n)^{-1/2}S_n$. First note that

$$E(S_n^4) \;=\; O\left(\frac{1}{k_n^4}\right)\sum_{i_1,i_2,i_3,i_4}\sum_{j_1\neq l_1}\ldots\sum_{j_4\neq l_4} E(Y_{j_1}Y_{l_1}\ldots Y_{j_4}Y_{l_4})I(j_k, l_k \in W_{i_k}, k = 1,2,3,4).$$

The nonzero terms in the sum must be one of the following forms: $E(Y_{j_1}^4 Y_{j_2}^4)$, which is of order $O(nk_n^5)$; $E(Y_{j_1}^4 Y_{j_2}^2 Y_{j_3}^2)$ or $E(Y_{j_1}^3 Y_{j_2}^3 Y_{j_3}^2)$, which are both of order $O(nk_n^6)$; and $E(Y_{j_1}^2 Y_{j_2}^2 Y_{j_3}^2 Y_{j_4}^2)$, which is of order $O(n^2k_n^6)$. Thus by the boundedness assumptions of the moments of error terms, $E(S_n^4) \leq K_1(n^2k_n^2)$, for some positive constant K_1. Next, define

$$U_{ni} = A_{(i-1)(b_n+l_n)+1} + \ldots + A_{(i-1)(b_n+l_n)+b_n}$$
$$V_{ni} = A_{(i-1)(b_n+l_n)+b_n+1} + \ldots + A_{i(b_n+l_n)},$$

$i = 1, \ldots, r_n$, where

$$b_n \sim n^{2/3}k_n^{1/3}, \quad l_n \sim k_n, \quad r_n \sim b_n^{-1}n = n^{1/3}k_n^{-1/3},$$

and assume for simplicity that n is a multiple of $b_n + l_n$. Then,

$$S_n = \sum_{i=1}^{r_n} U_{ni} + \sum_{i=1}^{r_n} V_{ni}.$$

The idea is to show that $\sum_{i=1}^{r_n} V_{ni} = o_P((nk_n)^{-1/2})$, such that by Slutsky's result, the asymptotic normality of S_n will follow from that of $\sum_{i=1}^{r_n} U_{ni}$. First consider

$$P\left((nk_n)^{-1/2}\left|\sum_{i=1}^{r_n} V_{ni}\right| \geq \varepsilon\right) \leq \sum_{i=1}^{r_n} P(|V_{ni}| \geq \varepsilon(nk_n)^{1/2}r_n^{-1})$$
$$\leq K\varepsilon^{-4}(nk_n)^{-2}r_n^5(l_nk_n)^2 = O(n^{-2}r_n^5k_n^2) = O(k_n^{1/3}n^{-1/3}) = o(1),$$

where the second inequality follows by Markov's inequality using the fact that $E(V_{ni}^4) \leq K(l_nk_n)^2$ (for some $K > 0$ and for all i), which follows in a similar way as for $E(S_n^4)$. Hence,

$$\sum_{i=1}^{r_n} V_{ni} = o_P((nk_n)^{-1/2}).$$

Since $U_{n1}, \dots, U_{nr_n}$ are independent, the asymptotic normality of $\sum_{i=1}^{r_n} U_{ni}$ can be established by verifying Lyapounov's condition:

$$\sum_{i=1}^{r_n} \frac{E(U_{ni}^4)}{\left[\sum_{i=1}^{r_n} E(U_{nj}^2)\right]^2} \to 0, \text{ as } n \to \infty. \tag{A.2}$$

Since $E(S_n^2) = n^2 \mathrm{Var}(\boldsymbol{V}'\boldsymbol{A}_d\boldsymbol{V}) \sim nk_n$ and since $E(S_n^4) \sim (nk_n)^2$, it follows in a similar way that $E(U_{ni}^2) \sim b_n k_n$ and that $E(U_{ni}^4) \sim (b_n k_n)^2$. Hence, the order of (A.2) is

$$O\left(r_n \frac{b_n^2 k_n^2}{r_n^2 b_n^2 k_n^2} \right) = O(r_n^{-1}) = o(1).$$

Hence, the asymptotic normality of $\sum_{i=1}^{r_n} U_{ni}$ follows, which finishes the proof.

Before stating the proof of Theorem 2.3, we prove a lemma which will be needed in the proof.

Lemma A.1 *For any Lipschitz continuous function $g(x)$ on [0,1], we have*

$$k_n^{-1} \sum_{j=1}^{n} g(x_j) I(j \in W_i) - g(x_i) = O\left(\frac{k_n}{n}\right),$$

uniformly in $i = 1, \dots, n$. In this lemma, $g(x_i)$ can be replaced with any $g(x_m)$ with $x_m \in W_i$.

Proof. Using the Lipschitz condition and the mean value theorem, the left side is less than or equal to

$$\begin{aligned}
& \frac{1}{k_n} \sum_{j=1}^{n} |g(x_j) - g(x_i)| I\left[|\hat{F}_X(x_j) - \hat{F}_X(x_i)| \le \frac{k_n - 1}{2n} \right] \\
\le \; & \frac{M}{k_n \inf_x r(x)} \sum_{j=1}^{n} \frac{|j-i|}{n} I\left[|\hat{F}_X(x_j) - \hat{F}_X(x_i)| \le \frac{k_n - 1}{2n} \right] \\
\le \; & \frac{M_1}{nk_n} (2 + 4 + \dots + 2(k_n - 1)) = O\left(\frac{k_n}{n}\right),
\end{aligned}$$

where M and M_1 are some positive constants. This completes the proof.

Proof of Theorem 2.3. The proof is given for the case $k_n \to \infty$, the case k_n is fixed can be proved the same way. Let $\boldsymbol{V}$ be defined by (1.5) and set

$$\boldsymbol{g} = ((g(x_j), j \in W_1, \dots, g(x_j), j \in W_n)'.$$

Thus, under $H_1 : m(x) = C + (nk_n)^{-1/4} g(x)$, we have $E(\boldsymbol{V}) = C\mathbf{1}_N + (nk_n)^{-1/4}\boldsymbol{g}$. Further denote $\boldsymbol{Z} = \boldsymbol{V} - C\mathbf{1}_N - (nk_n)^{-1/4}\boldsymbol{g}$ so that

$$\begin{aligned} MST - MSE &= \boldsymbol{V}'\boldsymbol{A}\boldsymbol{V} \\ &= \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} + 2(nk_n)^{-1/4}\boldsymbol{g}'\boldsymbol{A}\boldsymbol{Z} + (nk_n)^{-1/2}\boldsymbol{g}'\boldsymbol{A}\boldsymbol{g}. \end{aligned} \tag{A.3}$$

We will show that

$$(nk_n)^{-1/4}\boldsymbol{g}'\boldsymbol{A}\boldsymbol{Z} = o_P(n^{-1/2}k_n^{1/2}) \tag{A.4}$$

$$(nk_n)^{-1/2}\boldsymbol{g}'\boldsymbol{A}\boldsymbol{g} = n^{-1/2}k_n^{1/2}\gamma^2 + o_P(n^{-1/2}k_n^{1/2}) \tag{A.5}$$

from which it will follow that (A.3) equals

$$\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} + n^{-1/2}k_n^{1/2}\gamma^2 + o_P(n^{-1/2}k_n^{1/2}).$$

Since $E(\boldsymbol{Z}) = \mathbf{0}_N$, $\boldsymbol{Z}$ satisfies the null hypothesis, hence it follows from Theorem 2.2 and Slutsky's theorem that the above converges in distribution to the stated normal distribution, concluding the proof of the theorem.

We start with verifying (A.4). Using (1.3), write

$$\begin{aligned} &\boldsymbol{g}'\boldsymbol{A}\boldsymbol{Z} \\ &= \frac{nk_n - 1}{n(n-1)k_n(k_n-1)} \sum_{i=1}^n \left[\sum_{j=1}^n g(x_j) I(j \in W_i)\right] \left[\sum_{k=1}^n Z_k I(k \in W_i)\right] \\ &\quad - \frac{1}{n(n-1)k_n} \left[k_n \sum_{i=1}^n g(x_i)\right] \left[k_n \sum_{i=1}^n Z_i\right] - \frac{k_n}{n(k_n-1)} \sum_{i=1}^n g(x_i) Z_i. \end{aligned} \tag{A.6}$$

Using Lemma A.1, the sum in the first term can be written as

$$\begin{aligned} &k_n \sum_{i=1}^n \left[g(x_i) + O(n^{-1}k_n)\right] \left[\sum_{k=1}^n Z_k I(k \in W_i)\right] \\ &= k_n \sum_{k=1}^n \left[\sum_{i=1}^n g(x_i) I(i \in W_k)\right] Z_k + k_n^2 O(n^{-1}k_n) \sum_{k=1}^n Z_k \\ &= k_n^2 \sum_{k=1}^n g(x_k) Z_k + O_P(n^{-1/2}k_n^3), \end{aligned}$$

where the last equality is an application of central limit theorem for independent but not identically distributed random variables. Hence, (A.6) equals

$$\frac{(nk_n-1)k_n}{n(n-1)(k_n-1)}\sum_{i=1}^{n} g(x_i)Z_i - \frac{k_n}{n(n-1)}\left[\sum_{i=1}^{n} g(x_i)\right]\left[\sum_{i=1}^{n} Z_i\right]$$
$$-\frac{k_n}{n(k_n-1)}\sum_{i=1}^{n} g(x_i)Z_i + O_P(n^{-3/2}k_n^2)$$
$$= \frac{nk_n}{n-1}\left\{\left[n^{-1}\sum_{i=1}^{n} g(x_i)Z_i\right] - \left[n^{-1}\sum_{i=1}^{n} g(x_i)\right]\left[n^{-1}\sum_{i=1}^{n} Z_i\right]\right\} + O_P(n^{-3/2}k_n^2).$$

Denote $D = [n^{-1}\sum_{i=1}^{n} g(x_i)Z_i] - [n^{-1}\sum_{i=1}^{n} g(x_i)][n^{-1}\sum_{i=1}^{n} Z_i]$, then it's easy to check that $E(D) = 0$, $Var(D) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2(x_i)(g(x_i) - n^{-1}\sum_{j=1}^{n} g(x_j))^2 = O(n^{-1})$, thus $D = O_p(n^{-1/2})$ and

$$\boldsymbol{g}'\boldsymbol{A}\boldsymbol{Z} = O_p(k_n n^{-1/2}) + O_P(n^{-3/2}k_n^2).$$

So $(A.4)$ is satisfied provided $n^{-1}k_n \to 0$. Similarly, we have

$$\boldsymbol{g}'\boldsymbol{A}\boldsymbol{g} \;=\; \frac{nk_n}{n-1}\left\{\left[n^{-1}\sum_{i=1}^{n} g^2(x_i)\right] - \left[n^{-1}\sum_{i=1}^{n} g(x_i)\right]^2\right\} + O(n^{-3/2}k_n^2).$$

Using the definition (2.1) of the design points it is straightforward to show that

$$\begin{aligned}
\left|\int_0^1 u(t)r(t)dt - \frac{1}{n}\sum_{i=1}^{n} u(x_i)\right| &= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} u(t)r(t)dt - \sum_{i=1}^{n} u(x_i)\int_{x_{i-1}}^{x_i} r(t)dt\right| \\
&= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} (u(t)-u(x_i))r(t)dt\right| \\
&\le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |u(t)-u(x_i)|r(t)dt \\
&\le c\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |t-x_i|^{\delta} r(t)dt \\
&\le \frac{c}{(n\min_p r(p))^{\delta}}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} r(t)dt \\
&= \frac{c}{(n\min_p r(p))^{\delta}}\int_0^1 r(t)dt = O(n^{-\delta}),
\end{aligned}$$

where we assume $x_0 = 0$, and $u(t)$ is Hölder continuous of order $\delta > 0$. Thus

$$\left[n^{-1}\sum_{i=1}^{n} g^2(x_i)\right] - \left[n^{-1}\sum_{i=1}^{n} g(x_i)\right]^2 = \int_0^1 g^2(t)r(t)dt - \left(\int_0^1 g(t)r(t)dt\right)^2 + O(n^{-\delta}),$$

and so

$$\boldsymbol{g}'\boldsymbol{A}\boldsymbol{g} = k_n\gamma^2 + O(k_n n^{-\delta}) + O(n^{-3/2}k_n^2),$$

which establishes $(A.5)$. This concludes the proof.

References

[1] Akritas, M. G. (2000), *"Nonparametric Regression and Lack-of-fit Tests"*, unpublished lecture notes.

[2] Akritas, M. G., Arnold, S (2000), "Asymptotics for Analysis of Variance when the Number of Levels is Large", *Journal of the American Statistical Association,* 95, 212-226

[3] Akritas, M. G., Arnold, S. and Du, Y. (2000), "Nonparametric Models and Methods for Nonlinear Analysis of Covariance", *Biometrica,* 87, 507-526

[4] Akritas, M. G. and Papadatos, N. (2001), "Heteroscedastic One-way ANOVA and Lack-of-fit Test", *submitted.*

[5] Barry, Daniel. and Hartigan, J. A. (1990), "An Omnibus Test for Departures from Constant Mean", *The Annals of Statistics,* 18, 1340–1357.

[6] Bathke, Arne. (2002), "ANOVA for a Large Number of Treatments. *Mathematical Methods of Statistics",* Volume 11, No 1, 118-132.

[7] Billingsley, P. (1986), *Probability and Measures.* New York: John Wiley.

[8] Boos, D. D. and Brownie, C. (1995), "ANOVA and Rank Tests when the Number of Treatments is Large", *Statistics & Probability Letters,* 23, 183–191.

[9] Brown, R. L. Durbin, J. and Evans, J. M. (1975), "Techniques for Testing the Constancy of Regression Relationships over Time (with discussion)", *Journal of the Royal Statistical Society,* Ser. B, 37, 149–63.

[10] Buckley, M. J. (1991), "Detecting a Smooth Signal: Optimality of Cusum Based Procedures", *Biometrika,* 78, 253–262.

[11] Chen, C. F., Hart, J. F. and Wang, S. (2001), "Bootstrapping the Order Selection Test", preprint,
available at *www.stat.tamu.edu/ftp/pub/rjcarroll/2001.papers.directory/p52vn2.pdf*

[12] Dette, H. and Munk, A. (1998a), "Validation of Linear Regression Modles", *The Annals of Statistics,* 26, 778–800.

[13] Dette, H. and Munk, A. (1998b), "A Simple Goodness-of-fit Test for Linear Models under a Random Design Assumption", *Annals of the Institute of Statistical Mathematics,* 50, 253–276.

[14] Eubank, R. L. Hart, J. D. (1992), "Testing Goodness-of-fit in Regression via Order Selection Criteria", *The Annals of Statistics,* 20, 1412–1425.

[15] Fan, J. Q. and Huang, L. S. (2001), "Goodness-of-Fit Tests for Parametric Regression Models", *Journal of the American Statistical Association,* 96, 640–652.

[16] Geyer, C. J. (1991), " Constrained Maximum Likelihood Exemplified by Isotonic Convex Logistic Regression", *Journal of the American Statistical Association,* 86, 717–724.

[17] Hart, J. (1997). *Nonparametric Smoothing and Lack-of-fit Test*, New York: Springer-Verlag.

[18] Johnson, M. E., Ramberg, J. S. and Wang, C. (1982), "The Johnson Translation System in Monte Carlo Studies", *Communications in Statistics,* Part B, 11, 521–525.

[19] Munk, A. (1990), "Testing the Goodness of Fit of Parametric Regression Models with Random Toeplitz Forms", *Scandinavian Journal of Statistics,* 29, 501–535.

[20] Raz, J. (1990), "Testing for No Effect when Estimating a Smooth Function by Nonparametric Regression: a Randomization Approach", *Journal of the American Statistical Association,* 85, 132–138.

[21] Rice, J. (1984), "Bandwidth Choice for Nonparametric Regression", *The Annals of Statistics,* 12, 1215–1230.

[22] Sacks, J. and Ylvisaker, D. (1970), "Designs for Regression Problems with Correlated Errors III", *Annals of the Institute of Statistical Mathematics,* 41, 2057–2074.

[23] Sen, A. and Srivastava, M. S. (1975), "On Tests for Detecting Changing in Mean", *The Annals of Statistics,* 3, 98–108.

[24] Stute, W. (1982), "The Oscillation Behavior of Empirical Process", *The Annals of Statistics,* 10, 86–107.

[25] von Neumann, J. (1941), "Distribution of the Ratio of Mean Squared Successive Difference to Variance", *Annals of Mathematical Statistics,* 12, 367–395.

Table 1: Estimated Level for Nominal Level 0.05, Homoscedastic Case

	ε	N(0,1)	Lognormal	Sinh^{-1}-normal
n	kn	level	level	level
50	5	0.0600	0.0462	0.0482
	7	0.0500	0.0348	0.0332
	9	0.0404	0.0272	0.0320
	11	0.0328	0.0198	0.0234
60	5	0.0570	0.0474	0.0514
	7	0.0476	0.0374	0.0406
	9	0.0400	0.0326	0.0348
	11	0.0368	0.0250	0.0270
80	5	0.0674	0.0508	0.0498
	7	0.0498	0.0456	0.0426
	9	0.0426	0.0346	0.0360
	11	0.0394	0.0336	0.0326
100	5	0.0658	0.0530	0.0610
	7	0.0536	0.0418	0.0452
	9	0.0536	0.0370	0.0458
	11	0.0474	0.0342	0.0380

Table 2: Estimated Power under Alternatives (3.1) at Nominal Level 0.05.

		error	N(0,1)	lognormal	$\sinh^{-1}$-normal
	n	kn	power	power	power
Alternative 1	60	5	0.8576	0.8622	0.8660
		7	0.8764	0.8710	0.8636
		9	0.8612	0.8784	0.8676
		11	0.8606	0.8648	0.8624
	30	3	0.5780	0.7204	0.6630
		5	0.5748	0.7372	0.6800
		7	0.5314	0.7112	0.6448
		9	0.4728	0.6584	0.5806
Alternative 2	60	5	0.9468	0.9108	0.9294
		7	0.9396	0.9084	0.9266
		9	0.9136	0.8952	0.8998
		11	0.8858	0.8736	0.8854
	30	3	0.6776	0.8048	0.7482
		5	0.6110	0.7496	0.6950
		7	0.4722	0.6826	0.5792
		9	0.3004	0.5524	0.4122
Alternative 3	60	5	0.9960	0.9696	0.9780
		7	0.9976	0.9714	0.9848
		9	0.9982	0.9714	0.9838
		11	0.9962	0.9582	0.9764
	30	3	0.8826	0.8926	0.8908
		5	0.8644	0.8694	0.8628
		7	0.6200	0.7598	0.7106
		9	0.0616	0.1906	0.0896

Table 3: Estimated Power under Alternatives (3.2) at Nominal Level 0.05.

		σ	0.10	0.50	1.0	2.0
	n	kn	power	power	power	power
alternative 4	100	7	100	97.8400	40.4200	12.7800
		9	100	98.4000	43.4000	12.5200
		11	100	98.5400	43.2200	11.9600
		13	100	98.3400	44.3800	12.1800
	20	3	100	44.4000	16.6600	9.3600
		5	100	38.4000	11.9800	6.0200
		7	100	29.6400	7.5800	3.0400
alternative 5	100	7	100	96.6000	37.7800	11.6200
		9	100	96.8000	39.6400	10.6200
		11	100	97.2000	37.9600	11.2200
		13	100	96.9000	38.1200	10.0600
	20	3	100	41.5800	15.2000	9
		5	100	30.6200	9.6000	5.3200
		7	100	21.6400	5.6000	2.7200
alternative 6	100	7	100	99.9800	65.5400	17.5000
		9	100	99.9800	68.6400	18.7600
		11	100	99.9400	69.6000	18.2400
		13	100	100	70.2600	18.5000
	20	3	100	63.9000	21.8400	10.5400
		5	100	51.4200	13.2800	6.1800
		7	100	17.7600	4.2800	2.6000

Table 4: Estimated Power under Alternatives (3.3) at Nominal Level 0.05.

		σ	0.10	0.50	1.0	2.0
alternative 7	n	kn	power	power	power	power
	100	7	100	99.9400	62.4200	18.5200
		9	100	100	66.0200	18.8800
		11	100	99.9600	68.8400	19.6000
		13	100	99.9800	70.6000	19.1000
	20	3	100	58.7200	22.1200	10.4600
		5	100	49.9400	13	6.0400
		7	100	13.7600	4.1200	2.0400
alternative 8	100	7	100	100	80.9800	23.8600
		9	100	100	80.4400	23.3000
		11	100	100	81.2200	22.4200
		13	100	100	79.5600	22.0200
	20	3	100	41.9200	17.0600	10.1000
		5	21.2200	14.9000	8.5600	5.1800
		7	0.0600	5.4600	4.2000	2.5200

Table 5: Estimated Power under Alternatives (3.4) for $n = 100$.

	θ	0.00	0.25	0.50	0.75	1.00
α	kn	level	power	power	power	power
0.05	5	0.0634	0.1372	0.4560	0.8760	0.9940
	7	0.0548	0.1062	0.3908	0.8116	0.9914
	9	0.0450	0.0856	0.2660	0.6850	0.9480
0.10	5	0.1046	0.1944	0.5580	0.9182	0.9966
	7	0.0852	0.1650	0.4910	0.8872	0.9940
	9	0.0794	0.1336	0.3740	0.7870	0.9732

Table 6: Estimated Level for Nominal Level 0.05, Heteroscedastic Case

	σ(x)	$e^x/(e-1)$	$2e^{2x}/(e^2-1)$	$3x^2$
n		level	level	level
60	kn =7	0.0642	0.0648	0.0666
	kn =9	0.0540	0.0532	0.0574
	kn =11	0.0420	0.0432	0.0440
	DM test	0.0960	0.1282	0.1422
80	kn =7	0.0634	0.0692	0.0764
	kn =9	0.0524	0.0534	0.0598
	kn =11	0.0468	0.0516	0.0468
	DM test	0.0912	0.1218	0.1384
100	kn =9	0.0560	0.0636	0.0660
	kn =11	0.0544	0.0556	0.0548
	kn =13	0.0472	0.0440	0.0474
	DM test	0.0816	0.1124	0.1250

Table 7: Estimated Power for Nominal Level 0.05, Heteroscedastic case

m(x)	σ(x) n, kn		$e^x/(e-1)$	$2e^{2x}/(e^2-1)$	$3x^2$
2x	60	kn =7	0.7946	0.6540	0.4652
		kn =9	0.7984	0.6488	0.4724
		kn =11	0.8036	0.6636	0.4780
		DM test	0.6150	0.4964	0.3766
	100	kn =9	0.9538	0.8528	0.6490
		kn =11	0.9638	0.8742	0.6660
		kn =13	0.9666	0.8772	0.6802
		DM test	0.7384	0.5780	0.4188
$64x^3(1-x)^3$	60	kn =7	0.4050	0.3072	0.2182
		kn =9	0.4112	0.3024	0.1942
		kn =11	0.3808	0.2730	0.1786
		DM test	0.2856	0.2656	0.2354
	100	kn =9	0.6034	0.4342	0.2800
		kn =11	0.6192	0.4588	0.2914
		kn =13	0.6384	0.4676	0.2770
		DM test	0.3494	0.2846	0.2228
$\frac{1+\sin(3\pi x)}{2}$	60	kn =7	0.4016	0.3018	0.2024
		kn =9	0.4238	0.2930	0.2022
		kn =11	0.4114	0.3034	0.1854
		DM test	0.2528	0.2388	0.2060
	100	kn =9	0.5966	0.4222	0.2780
		kn =11	0.6108	0.4548	0.2832
		kn =13	0.6470	0.4702	0.2726
		DM test	0.3018	0.2498	0.2134

Figure 1: Scatter Plot of Proportion of Disease Incidence vs Maternal Age