Université catholique de Louvain

Non-formal Rankin-Cohen deformation

Valentin DENDONCKER

Supervisor:
Prof. Pierre Bieliavsky

Dissertation submitted in partial fulfillment of the requirements
for the degree of Docteur en Sciences

Examination committee:

Prof. Pierre-Emmanuel Caprace (Université catholique de Louvain)
Prof. Michael Pevzner (Université de Reims Champagne-Ardenne)
Prof. Xiang Tang (Washington University in St. Louis)
Prof. Michel Willem (Université catholique de Louvain), chair

May 2018

Remerciements

Au-delà des nombreuses contrées mathématiques que cette thèse m'a permis d'arpenter, c'est certainement l'aventure humaine que je retiendrai le plus de ces années passées en tant que jeune chercheur. Car malgré les diverses épreuves qu'il m'aura fallu surmonter pour mener à bien ce projet de longue haleine, c'est grâce aux nombreuses discussions et aux belles rencontres que mon doctorat restera toujours à mes yeux un agréable souvenir. Les premières lignes de cet ouvrage vous sont donc consacrées, vous tous sans qui ce manuscrit n'aurait probablement jamais vu le jour.

Mes premières pensées se tournent bien évidemment vers toi Pierre. Je me souviens de la première fois où je t'ai vu comme si c'était hier. Je me rappelle très bien avoir été instantanément captivé par ton enthousiasme communicatif et la passion que tu dégageais en parlant de mathématiques. Plusieurs années après, rien n'a changé : la flamme mathématique qui t'anime est toujours aussi vive. Quel plaisir de te côtoyer, toi pour qui les équations, les raisonnements mathématiques et les résultats qui en découlent s'enchaînent naturellement et mélodieusement à la façon des fugues de Bach si chères à ton cœur. Quelle aisance déconcertante ! Je te remercie de m'avoir ouvert les portes de ton univers et de m'avoir guidé dans les méandres de la géométrie non commutative. Merci aussi de m'avoir démontré à de multiples reprises à quel point les star-produits et les intégrales oscillantes pouvaient sentir bon le curry vert (ou rouge selon les goûts !), le lait de coco et le basilic thaï. Nos repas mathématico-gastronomiques resteront pour moi de très agréables souvenirs. Je tiens également à te remercier pour le jury que tu as composé en vue de lire et de commenter cette thèse. À cet égard, j'en profite pour remercier les professeurs Pierre-Emmanuel Caprace, Michael Pevzner, Xiang Tang et Michel Willem pour avoir fait de la défense privée un moment à la fois instructif et constructif. Enfin, merci Pierre pour la bonne dynamique que tu as su insuffler au sein de la charmante équipe de jeunes chercheurs du groupe de géométrie.

C'est donc à présent tout naturellement à vous, chers collègues géomètres, que je souhaite m'adresser. J'ai vraiment eu beaucoup de chance de passer ces

années de recherche à vos côtés. Entre discussions passionnantes, parties de plaisir, virées à l'étranger (en train bien sûr, peur de l'avion oblige), séminaires ou encore verres après le travail, je n'ai vraiment pas vu le temps passer. Merci à vous Alban, Florian, Jérémy, Liebrecht et Stéphane pour tous ces moments partagés.

Et bien évidemment, merci à toi Natacha. Plus qu'une collègue, tu as été ma voisine de bureau depuis le début mais surtout une oreille attentive durant tout le doctorat. Merci pour toutes nos conversations, dans les bons comme dans les moins bons moments. Tu as sans nul doute été un soutien indispensable tout au long de ces années et je suis très heureux d'avoir partagé cette aventure avec toi.

Mais heureusement, les terres cyclotronniennes ne sont pas uniquement peuplées de géomètres. Il y aussi les analystes, les algébristes, les catégoriciens, les topologues, et j'en passe. Merci à vous tous pour votre bonne humeur quotidienne. Je tiens tout particulièrement à te remercier toi Valérian. J'ai beaucoup apprécié nos fréquentes discussions autour d'une petite bière après les longues journées de calculs (oui bon d'accord, autour "de petites bières", mais toujours avec modération !). Je suis ravi d'avoir rencontré un collègue il y a quelques années qui fait aujourd'hui partie de mes amis.

Bien évidemment, je tiens également à remercier les charmantes secrétaires du cyclotron, pour tout leur travail au quotidien. Merci d'aussi bien vous occuper de nous les mathématiciens, que ce soit en veillant à ce que nous ayons toujours suffisamment de café, mais aussi en nous aidant à remplir les formulaires, notes de frais et autres fiches de surveillance santé. Il faut bien l'admettre, l'administration n'est pas notre tasse de thé (ni même de café d'ailleurs). Merci à vous, Cathy, Carine et Martine.

Il est à présent grand temps de me tourner vers tous ceux qui m'ont permis au quotidien de m'évader du monde parfois embrumé des équations.

Merci à vous mes amis non mathématiciens. J'ai une chance incroyable d'être aussi bien entouré. Vous n'imaginez pas le plaisir que je ressens à chaque fois que je vous vois. Ce qui m'a le plus frappé depuis que j'ai entamé mon doctorat, c'est l'intérêt que vous avez porté à mon travail. Vous avez régulièrement pris des nouvelles de mes recherches et avez même eu le courage de me demander quel était mon sujet de thèse. Merci de m'avoir offert aussi souvent l'occasion de me sentir écouté. Sans vous et sans votre soutien, ces années de recherche n'auraient pas eu la même saveur.

Merci également à ma famille, aux Belges comme aux Français qui m'ont si chaleureusement adopté. J'ai toujours trouvé auprès de vous soutien, amour et réconfort.

Mais surtout, merci à vous maman et papa. Maman, tu es tout simplement exceptionnelle. Depuis toujours tu m'es indispensable. Je ne te remercierai jamais assez pour tout ton amour, ton soutien indéfectible et ta bienveillance infinie. Merci pour tout ce que tu m'apportes au quotidien, j'ai une chance incroyable. Papa, j'aurais tant aimé que tu puisses lire ces quelques lignes. Merci pour tout ce que tu m'as transmis. Merci de m'avoir rendu amoureux de la musique et de la cuisine. Merci de m'avoir fait comprendre à quel point les amis sont une denrée rare et précieuse. Merci de m'avoir légué une part de toi.

Enfin, ces dernières lignes sont pour toi Pauline. Tu es tout simplement la plus belle personne que je connaisse. Merci pour ton écoute, tes conseils, ta tendresse et ta bienveillance sans limites, à mon égard ou à l'égard des autres. Merci pour ta joie de vivre et pour tous les bons moments que nous partageons. En plus d'être mon épouse, tu es ma meilleure amie. J'ai une chance incroyable de passer chaque journée à tes côtés. Merci aussi pour tout le temps que tu as consacré à relire ce texte, alors que chaque équation (et tu es bien placée pour savoir qu'il y en a un certain nombre dans les pages qui suivent) doivent pour toi comme pour toute personne "normale" (bah oui, il faut bien l'admettre, un mathématicien n'est pas ce que l'on qualifierait spontanément d'individu "normal") ressembler uniquement à une succession de hiéroglyphes dénués de sens. Pour ma part, jamais je n'aurais été capable de lire avec autant d'attention un texte aussi incompréhensible ! Merci également de m'avoir aussi souvent écouté parler de mes star-produits avec autant d'intérêt. Tu as une patience exceptionnelle. Du fond du cœur, je te remercie pour tout ton amour.

Contents

Introduction **1**

1 Rankin-Cohen brackets and formal deformation quantization **11**
1.1 Modular forms and Rankin-Cohen brackets 11
1.1.1 Modular forms 11
1.1.2 Rankin-Cohen brackets 14
1.2 Link with formal deformation 16
1.2.1 Cohen, Manin and Zagier 17
1.2.2 Connes and Moscovici 17
1.2.3 Geometrical interpretation of the Rankin-Cohen deformation 24

2 Bieliavsky-Gayral star-products and non-formal deformation quantization **43**
2.1 The formal version 43
2.1.1 Symplectic symmetric space structure 43
2.1.2 The formal star-product 46
2.2 Non-formal aspects 56
2.2.1 Weyl quantization 56
2.2.2 Non-formal BG deformation quantization 58
2.2.3 An adapted Schwartz space for the tracial BG star-product 72

3 Non-formal Rankin-Cohen star-product **91**
3.1 The retract method 91
3.2 Intertwining of RC and BG star-products via retract 97
3.2.1 Characterization of the morphisms defining the retract equations 97
3.2.2 Resolution of the retract equations 103
3.2.3 Inversion of the retract solutions 119
3.2.4 Intertwiners of RC and BG 124

3.3 Non-formal aspects of the intertwiners 128
3.4 The non-formal RC star-product 142
3.4.1 Formula and integral representation 142
3.4.2 Topological algebras 154
3.4.3 Gel'fand-Shilov and Paley-Wiener spaces 160

4 An alternative integral representation for the Rankin-Cohen star-product 165
4.1 Construction of the integral representation 165
4.2 Link between both integral representations for the RC star-product 172

Bibliography 179

Introduction

In classical mechanics, the space of observables of a dynamical system is encoded mathematically by means of a space of smooth functions on a symplectic manifold (M, ω). Within this setting, each point of M represents a state of the system, that is, a possible value of position and momentum variables. Endowed with the Poisson bracket $\{\cdot, \cdot\}$ associated with the symplectic form ω, the space $(C^\infty(M), \{\cdot, \cdot\})$ of smooth functions on M (with values in $\mathbb{C}$) then naturally carries a structure of *associative and commutative Poisson algebra*, whose product law is the classical pointwise product of functions. The dynamics of the system is implemented by a Hamiltonian $H \in C^\infty(M)$. It means that the time evolution of an observable $t \in \mathbb{R} \mapsto f_t \in C^\infty(M)$ is described by the following equation:

$$\frac{d}{dt} f_t = \{H, f_t\}.$$

In the Heisenberg picture of quantum mechanics (appeared in the 1920s), the mathematical framework is different. Indeed, at that level, the states of a dynamical system are symbolized by elements of a Hilbert space $\mathcal{H}$. The observables are then described by means of linear operators acting on the space $\mathcal{H}$. Each eigenvalue of such an operator corresponds to a possible value of the underlying observable. The space of observables is thus naturally endowed with a structure of *noncommutative algebra*, for the composition law of operators. The dynamics is written in terms of a quantum Hamiltonian H^q, which is in this context a linear operator. The time evolution of an observable $t \in \mathbb{R} \mapsto A_t$ is then given by the equation below:

$$\frac{d}{dt} A_t = \frac{i}{\hbar} [H^q, A_t],$$

where $\hbar$ is the Planck constant and with $[H^q, A_t] = H^q \circ A_t - A_t \circ H^q$.

As we can see, the fact that classical and quantum mechanics underly respectively a *commutative* and a *noncommutative* algebra is due (within the above

Heisenberg picture of quantum mechanics) to a radical change in the nature of objects: the smooth functions on the one hand and the linear operators on the other hand. But in the late seventies, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer propose in [5] another interpretation. Instead of viewing the quantum level as a change in the nature of the observables, they suggest that quantum mechanics may be formulated by means of an algebra of formal power series in some *deformation parameter* ν, with coefficients in the space of smooth functions on the underlying symplectic manifold. In general, ν is proportional to $\hbar$, the latter then being considered as a parameter (and not a constant) in the construction. This is the birth of the so-called *formal deformation quantization* theory. Within this context, no reference to a Hilbert space is made, both classical and quantum mechanics are described in terms of classical observables. The quantum framework then appears as a deformation of the classical one, that is, it follows from a twisting of the classical pointwise product into a noncommutative and associative law on the space of classical observables. More precisely, such a deformation quantization of a symplectic manifold (M, ω) is defined mathematically as a bilinear map

$$\star_\nu : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]] : (f, g) \mapsto f \star_\nu g := \sum_{k=0}^{+\infty} \nu^k C_k(f, g),$$

where ν is a formal deformation parameter and where each element $C_k(f, g)$ belongs to the space $C^\infty(M)$ of smooth functions on M. Extended $\mathbb{C}[[\nu]]$-linearly to $C^\infty(M)[[\nu]]$, the product $\star_\nu$ is such that the space $(C^\infty(M)[[\nu]], \star_\nu)$ becomes an associative algebra. Moreover, $\star_\nu$ constitutes a *deformation* of the classical pointwise product *in the direction of the Poisson bracket*, in the sense that for $f, g \in C^\infty(M)$, $\lim_{\nu \to 0} f \star_\nu g = fg$ and $C_1(f, g) - C_1(g, f) = \{f, g\}$ (up to a constant), where $\{\cdot, \cdot\}$ is the Poisson bracket associated with the symplectic form ω. The product $\star_\nu$ is then called a (formal) *star-product* on (M, ω). It is worth mentioning that in 1977, Kontsevich has shown through its *formality theorem* that any Poisson manifold (so, in particular, any symplectic manifold) is endowed with such a (not necessarily unique) formal star-product.

The most popular example of such a star-product is undoubtedly the *Moyal star-product* $\star_\nu^0$ on the flat manifold $\mathbb{R}^{2n} = \{(q, p) \mid q, p \in \mathbb{R}^n\}$ (with n a non-zero positive integer), endowed with its standard symplectic form $dq \wedge dp$. This

star-product is defined as follows:

$$\begin{aligned}\star_\nu^0 : C^\infty\left(\mathbb{R}^{2n}\right) \times C^\infty\left(\mathbb{R}^{2n}\right) &\to C^\infty\left(\mathbb{R}^{2n}\right)[[\nu]] \\ (f,g) &\mapsto f \star_\nu^0 g := fg + \sum_{k=1}^{+\infty} \nu^k C_k^0(f,g),\end{aligned}$$

where

$$(\omega_{ij})_{i,j} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

and

$$C_k^0(f,g) := \frac{1}{k!} \sum_{I_k, J_k} \omega_{i_k j_k} \cdots \omega_{i_k j_k} \partial_{i_1 \cdots i_k} f \partial_{j_1 \cdots j_k} g,$$

with $I_k = (i_1, \cdots, i_k)$, $J_k = (j_1, \cdots, j_k)$, $1 \leq i_1, \cdots, i_k, j_1, \cdots, j_k \leq 2n$. Extended $\mathbb{C}[[\nu]]$-linearly, $\star_\nu^0$ then defines an associative product on $C^\infty\left(\mathbb{R}^{2n}\right)[[\nu]]$. Moreover, it constitutes a deformation of the classical pointwise product in the direction of the Poisson bracket associated with the symplectic form $dq \wedge dp$. Note that the symplectic manifold $\left(\mathbb{R}^{2n}, dq \wedge dp\right)$ is the phase space of the dynamical system associated with a free particle moving in $\mathbb{R}^n$. Within the Heisenberg picture of quantum mechanics, the quantization of this system is thus modeled by a Hilbert space $\mathcal{H}$ of linear operators. In the case of a free particle in $\mathbb{R}^n$, this Hilbert space $\mathcal{H}$ is equal to $L^2\left(\mathbb{R}^n\right)$. Thanks to a work due to Weyl, the Moyal star-product can be related to the Heisenberg picture of this quantized situation. Indeed, denoting as $\mathcal{B}\left(\mathcal{H}\right)$ the space of bounded operators on $\mathcal{H}$, it turns out that there exists a correspondence

$$W_\hbar : C^\infty(M) \to \mathcal{B}\left(\mathcal{H}\right),$$

called the *Weyl quantization map*, between the spaces of classical and quantum observables. This map is defined as follows:

$$W_\hbar(f)\varphi(q_0) := \frac{1}{\pi\hbar} \int_{\mathbb{R}^{2n}} e^{\frac{ip}{\hbar}(q_0 - q)} f\left(\frac{q_0 + q}{2}, p\right) \varphi(q) dq dp,$$

where $\varphi \in L^2(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^{2n})$ is a classical observable such that the integral makes sense. Here is now the spectacular property regarding the above Weyl quantization map: for two functions f, g belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ (which is a subspace of $C^\infty\left(\mathbb{R}^{2n}\right)$), we have the identity

$$W_\hbar(f) \circ W_\hbar(g) = W_\hbar(f \star_\hbar^W g),$$

where $f \star_\hbar^W g$ is again a Schwartz function on $\mathbb{R}^{2n}$. This function is given for any $(q_0, p_0) \in \mathbb{R}^{2n}$ by

$$\left(f \star_\hbar^W g\right)(q_0,p_0) := \\ \frac{1}{(\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar} S^W\left((q_0,p_0),(q,p),(q',p')\right)} f(q,p) g(q',p') dq dp dq' dp',$$

with

$$S^W(x,y,z) := dq \wedge dp\,(x,y) + dq \wedge dp\,(y,z) + dq \wedge dp\,(z,x).$$

The law $\star_\hbar^W$ then defines an associative and noncommutative product on the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, inherited from the noncommutativity of the composition law at the level of operators acting on the Hilbert space $\mathcal{H}$. But that is not all: it turns out that for given $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, the map $\left[\hbar \in \mathbb{R} \mapsto f \star_\hbar^W g \in \mathcal{S}(\mathbb{R}^{2n})\right]$ is smooth. Moreover, its Taylor series at 0 coincides with $f \star_\nu^0 g$ (the Moyal star-product), for $\nu := \frac{\hbar}{2i}$. The interest of the Weyl quantization is thus twofold. On one hand, it allows to interpret the Moyal star-product as the reading at the level of (a subspace of) classical observables of the noncommutative composition law for the quantum observables. On the other hand, it shows that on a well-chosen subspace of the classical observables (namely here the space $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ of Schwartz functions on $\mathbb{R}^{2n}$), there exists a *convergent* expression for the formal series defining the Moyal star-product, by means of an integral representation. Actually, this convergent version of the Moyal star-product constitutes an example of a so-called *non-formal star-product*. Unlike the formal case where no particular convergence of the series associated with the star-product is required, the non-formal one is such that the star-multiplication of two functions is again a *genuine function* (at least on a suitable function space), rather than a *formal power series* in the deformation parameter. As suggested by the Moyal-Weyl situation, a non-formal star-product can be interpreted as a convergent form of an underlying formal star-product.

The present work outlines the explicit construction of a convergent version of Zagier's Rankin-Cohen formal deformation. For this, we focus on a two-dimensional non-flat symplectic manifold, namely the affine group of the real line "$ax + b$". This two-dimensional Lie group will be denoted as $\mathbb{S}$ throughout this text, whereas its symplectic form will be denoted as $\omega^{\mathbb{S}}$. Notice that the Lie group $\mathbb{S}$ naturally constitutes a *symmetry group* (or group of Hamiltonian symplectomorphisms) of the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. From a physical point of view, it is then important that the symmetries encoded by $\mathbb{S}$ are preserved after a quantization of the underlying dynamical system. At the level of a (formal) star-product $\star_\nu$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, it is then required that for all $x \in \mathbb{S}$ and

for all $\varphi, \psi \in C^\infty(\mathbb{S})$, the following identity holds:

$$L_x^\star \varphi \star_\nu L_x^\star \psi = L_x^\star \left(\varphi \star_\nu \psi\right),$$

where $L_x : \mathbb{S} \to \mathbb{S} : x_0 \mapsto x x_0$ denotes the left action of $\mathbb{S}$. Such a star-product is called *left invariant star-product* on the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

In this work, we will consider two different formal star-products on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ that appear in the literature and fulfill the above condition of left invariance. The first one, whose construction is (partially) outlined in the first chapter of this work, is due to Zagier, Connes and Moscovici, but based on the researches of Rankin and Cohen concerning the *modular forms* (cf. Definition 1.1.1). In a few lines, here is the fascinating story of this star-product. In 1977, the works of Rankin and Cohen provide a sequence of bilinear operations indexed by positive integers (called since then *Rankin-Cohen brackets*), allowing to associate a modular form to two given modular forms (which is quite complicated). The surprising fact (noticed among others by Zagier and Eholzer) about those Rankin-Cohen brackets $[\cdot,\cdot]_n$ is that (for instance) the formula $\sum_{n=0}^{+\infty}[\cdot,\cdot]_n$ defines a new associative product on the space of modular forms. This therefore suggests a link between the works of Rankin and Cohen and the formal deformation quantization theory. A few years later, Connes and Moscovici then develop an extended notion of Rankin-Cohen brackets (denoted as RC_n, for n a positive integer) to the space of Hecke operators. From this, they deduce a formal deformation of a certain type of algebras, based on the brackets RC_n (cf. [32]). In turn, as noticed by Bieliavsky, Tang and Yao in [25], the result of Connes and Moscovici leads to a left invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, via a reduced form RC_n^{red} of the elements RC_n.

Proposition A ([25])**.** The deformation $\star_\nu^{RC} := \sum_{n=0}^{+\infty} \nu^n RC_n^{red}$ defines a left invariant star-product on the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

A very interesting fact regarding the star-product $\star_\nu^{RC}$ (that we will call *Rankin-Cohen star-product* on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ in this text) is that it admits a more geometrical interpretation formulated by means of the Moyal star-product.

Proposition B ([25], Proposition 3.4)**.** The star-product $\star_\nu^{RC}$ corresponds to the restriction to the upper half-plane of $\mathbb{R}^2$ of the two-dimensional Moyal star-product.

In chapter two, we present the second left invariant star-product associated with the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. This star-product was originally discovered by Bieliavsky in [11], and constitutes now a very particular example of a far

more general theory developed by Bieliavsky and Gayral in [20]. Here is a rough idea of the construction. First of all, note that the Lie group $\mathbb{S}$ corresponds to the Iwasawa component of $SL_2(\mathbb{R})$. Setting $\mathrm{Lie}(SL_2(\mathbb{R})) =: \mathfrak{sl}_2(\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{H, E, F\}$ with brackets given by

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H,$$

it means that $\mathrm{Lie}(\mathbb{S}) =: \mathfrak{s} = \mathrm{span}_{\mathbb{R}}\{H, E\}$ and thus $\mathbb{S} = \exp\left(\mathrm{span}_{\mathbb{R}}\{H, E\}\right)$. This leads to the following global coordinate system on $\mathbb{S}$:

$$\mathfrak{s} \simeq \mathbb{R}^2 \to \mathbb{S} : (a, \ell) \mapsto \exp(aH)\exp(\ell E).$$

Since $\mathbb{S}$ is two-dimensional, a natural way to get a star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is to consider the (two-dimensional) Moyal star-product, written in terms of the above coordinate system on $\mathbb{S}$. However, this "Moyal star-product" $\star_\nu^{\mathbb{S}}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is *not* left invariant. The method used by Bieliavsky then consists in modifying this star-product by means of a *formal equivalence* (that is, an operator of the form $T_\nu = \mathrm{Id} + \sum_{k=1}^{+\infty} \nu^k T_k$, where each T_k is a differential operator), in such a way that the formula

$$f \star_\nu^{BG} g := T_\nu \left(T_\nu^{-1} f \star_\nu^{\mathbb{S}} T_\nu^{-1} g\right)$$

yields a left invariant formal star-product on the space $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. The remarkable fact concerning the star-product $\star_\nu^{BG}$ is that it admits a convergent form, as in the case of Moyal-Weyl. Indeed, in their article [24], Bieliavsky and Massar show that if we restrict $\star_\nu^{BG}$ to the space $\mathcal{E}_\nu^{BG} := T_\nu\left(\mathcal{S}(\mathbb{S})\right)$ (where $\mathcal{S}(\mathbb{S})$ is the standard Schwartz space on $\mathbb{S} \simeq \mathbb{R}^2$ with respect to the previous global coordinate system), the star-product $\star_\nu^{BG}$ admits a representation in terms of an oscillatory integral.

Theorem C ([24], Theorem 3.1)**.** For $f, g \in \mathcal{E}_\nu^{BG}$, the formula

$$f \star_\nu^{BG} g := T_\nu \left(T_\nu^{-1} f \star_\nu^{\mathbb{S}} T_\nu^{-1} g\right)$$

defines an associative algebra structure on $\mathcal{E}_\nu^{BG}$. At the level of smooth compactly supported functions, the star-product admits the following integral

representation (within the global coordinate system $\mathbb{S} = \{(a, \ell)\} \simeq \mathbb{R}^2$):

$$(\varphi \star_\nu^{BG} \psi)(a_0, \ell_0) = \frac{1}{(2\pi i\nu)^2} \int_{\mathbb{R}^4} \cosh(2(a_1 - a_2)) e^{\frac{1}{2\nu}(\sinh(2(a_0-a_1))\ell_2 + \sinh(2(a_2-a_0))\ell_1 + \sinh(2(a_1-a_2))\ell_0)} \times \varphi(a_1, \ell_1)\psi(a_2, \ell_2) da_1 d\ell_1 da_2 d\ell_2,$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{S}) := C_c^{+\infty}(\mathbb{S})$. Moreover, this star-product is invariant under the left action L of $\mathbb{S}$.

On $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, there exists thus two formal star-products, namely $\star_\nu^{RC}$ (the *Rankin-Cohen star-product*) and $\star_\nu^{BG}$ (the *Bieliavsky-Gayral star-product*), whose second one admits a convergent form by means of an integral representation (on a suitable function space). The natural question arising from that is then to wonder whether the star-product $\star_\nu^{RC}$ also admits a convergent expression or not, at least on a suitable subspace of $C^\infty(\mathbb{S})$. This constitutes precisely the subject of chapters three and four of the present work, which both bring, by means of two different strategies, a positive answer to the question.

In chapter three, the idea consists in making use of the non-formal aspect of the star-product $\star_\nu^{BG}$, to give rise to a non-formal Rankin-Cohen star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. For this, we need first to compute in an explicit manner all the (formal) $\mathbb{S}$-equivariant operators that intertwine both Bieliavsky-Gayral and Rankin-Cohen star-products, that is, the set of invertible operators T that commute with the left action L of $\mathbb{S}$ and satisfying in addition the relation

$$\varphi \star_\nu^{RC} \psi = T\left(T^{-1}\varphi \star_\nu^{BG} T^{-1}\psi\right),$$

for all $\varphi, \psi \in C^\infty(\mathbb{S})[[\nu]]$. Such an operator T is called an $\mathbb{S}$-*equivalence* between $\star_\nu^{BG}$ and $\star_\nu^{RC}$. For this purpose, we outline in section 3.1 the so-called *retract method*. This method allows to characterize the (kernels of the) operators intertwining two star-products by means of a system of partial differential equations, through the study of different symmetries of the star-products (cf. Definition 1.2.30). In section 3.2, we then apply this method to the particular case of $\star_\nu^{RC}$ and $\star_\nu^{BG}$, which leads to the expressions of the desired operators. After that, we study in section 3.3 their analytical properties in order to turn them into non-formal operators, that is, operators given in terms of a convergent integral and well-defined on a suitable function space. The reason for this is that the intertwining of the non-formal Bieliavsky-Gayral star-product via one

of the integral operators previously obtained yields a *non-formal* version of $\star_\nu^{RC}$. This is the subject of section 3.4, where the following result is stated.

Theorem D. On a particular *partial Paley-Wiener type function space* $\mathcal{E}_\nu^{RC} \subset C^\infty(\mathbb{S})$, the star-product $\star_\nu^{RC}$ admits the following integral representation (within the global coordinate system $\mathbb{S} = \{(a, \ell)\} \simeq \mathbb{R}^2$):

$$(2\pi i\nu)^2(\varphi \star_\nu^{RC} \psi)(a_0, \ell_0) = \int_{\mathbb{R}^4} \frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1 - a_2)} e^{\frac{1}{\nu}[\tanh(a_1-a_2)\ell_0 + \tanh(a_2)\ell_1 - \tanh(a_1)\ell_2]} \times \varphi\left(a_1 + a_0 + \log\frac{\cosh(a_1 - a_2)}{\cosh(a_2)}, \ell_1\right) \times \psi\left(a_2 + a_0 + \log\frac{\cosh(a_1 - a_2)}{\cosh(a_1)}, \ell_2\right) da_1 d\ell_1 da_2 d\ell_2.$$

Endowed with this product, the space $\left(\mathcal{E}_\nu^{RC}, \star_\nu^{RC}\right)$ then becomes an associative algebra. Moreover, this star-product is invariant under the left action of $\mathbb{S}$.

The above theorem constitutes one of the most important results of the present work. Indeed, it yields a non-formal Rankin-Cohen star-product, well-defined on a suitable subspace of $C^\infty(\mathbb{S})$. The precise link between the integral representation of Theorem D and the formal star-product $\star_\nu^{RC}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ coming from [25] (and due to Connes-Moscovici's Rankin-Cohen formula) is given by the following result.

Theorem E. For given functions $\varphi, \psi \in \mathcal{E}_\nu^{RC}$, the function mapping $\hbar \in \mathbb{R}$ to the integral representation $\varphi \star_{\frac{\hbar}{i}}^{RC} \psi$ is smooth. Its Taylor series at 0 defines an associative formal star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ which coincides with $\star_\nu^{RC}$, for $\nu := \frac{\hbar}{i}$.

We conclude the third chapter with some topological considerations. We show, among other things, that a particular intertwiner computed via the retract method in the second section establishes an isometry (with respect to the $L^2(\mathbb{S})$-inner product) between $\mathcal{E}_\nu^{RC}$ and the function space adapted to (a particular version of) the Bieliavsky-Gayral star-product. This gives rise to an associative Hilbert algebra $\left(\mathcal{H}_\nu^{RC}, \star_\nu^{RC}\right)$, associated with the Rankin-Cohen deformation quantization.

In the fourth chapter of this work, the strategy adopted to deal with the non-formality of the Rankin-Cohen star-product differs from the chapter three. Indeed, in chapter four, we start from Proposition B, which connects this star-product with a restriction of the two-dimensional Moyal star-product. The idea is then to use the Weyl formula (which is the non-formal counterpart of the

Moyal star-product) to get an integral representation for Rankin-Cohen. This is what is done in section 4.1 by building an identification between the functions on $\mathbb{S}$ and the even functions on $\mathbb{R}^2$. The reading of the Weyl product through this identification then leads to a non-formal Rankin-Cohen star-product $\tilde{\star}_\nu^{RC}$, which extends as an associative product to the space $L^2(\mathbb{S})$ of square-integrable functions on $\mathbb{S}$. The link between the non-formal star-products constructed in sections 3.4 and 4.1 is established in section 4.2, in which we state the following result.

Theorem F. The restriction of the star-product $\tilde{\star}_\nu^{RC}$ to the space $\mathcal{E}_\nu^{RC}$ coincides with the star-product $\star_\nu^{RC}$. That is, for all $\varphi, \psi \in \mathcal{E}_\nu^{RC}$, the following identity holds:

$$\varphi \tilde{\star}_\nu^{RC} \psi = \varphi \star_\nu^{RC} \psi.$$

The space $\left(\mathcal{E}_\nu^{RC}, \tilde{\star}_\nu^{RC}\right)$ is then a subalgebra of $\left(L^2(\mathbb{S}), \tilde{\star}_\nu^{RC}\right)$.

This theorem means that there exists an extension $\tilde{\star}_\nu^{RC}$ of the star-product $\star_\nu^{RC}$ obtained in the third chapter. However, as a consequence of the retract method, it turns out that there is no non-formal equivalence between this extended product and the Bieliavsky-Gayral star-product. By this, we mean that *there is no* $\mathbb{S}$*-equivariant integral operator* T such that, for all φ, ψ belonging to some dense subspace of $L^2(\mathbb{S})$, the following identity holds at the non-formal level:

$$\varphi \tilde{\star}_\nu^{RC} \psi = T\left(T^{-1}\varphi \star_\nu^{BG} T^{-1}\psi\right).$$

As shown in chapter three, such an equivalence can only be realized at the level of a certain *partial Paley-Wiener type function space* $\mathcal{E}_\nu^{RC}$, associated with the non-formal star-product $\star_\nu^{RC}$.

To sum up, the present work adds a new example to the list of non-formal star-products developed in the context of symplectic Lie groups. Moreover, the work done in chapter three establishes a precise link between this new non-formal star-product and the construction of Pierre Bieliavsky and Victor Gayral realized in [20]. This therefore opens the way for new investigations. For instance, it could be interesting to compare our non-formal Rankin-Cohen star-product with the integral formulas for the Rankin-Cohen quantization based on covariant symbolic calculus given in [46], [50] and [51]. Also, in their book [20], authors have developed a way to use their star-product in order to deform Fréchet algebras which the Lie group $\mathbb{S}$ acts on in a strongly continuous and isometrical manner. We can then naturally ask whether the same kind of result holds for the Rankin-Cohen star-product (although we think that this is not the case). Moreover, in a very recent paper ([15]), Bieliavsky, Bonneau, D'Andrea

and Gayral have built multiplicative unitaries (leading to quantum Kählerian Lie groups) from (an alternative version of) the non-formal star-product $\star_\nu^{BG}$. It could thus be interesting to know if such a construction is possible for the non-formal version of the Rankin-Cohen star-product.

Chapter 1

Rankin-Cohen brackets and formal deformation quantization

This first chapter outlines the surprising way in which Rankin-Cohen brackets fit within the framework of deformation quantization.

1.1 Modular forms and Rankin-Cohen brackets

The notion of Rankin-Cohen brackets has its origins in a work of Rankin in the 1950's concerning modular forms. Such objects appear in many fields of mathematics and especially in number theory through their link with elliptic curves. We begin this section by introducing the general definition of a modular form ([27]).

1.1.1 Modular forms

We denote by $\mathbb{H}$ the upper half-plane of $\mathbb{C}$, i.e. $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. It is well-known that the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations (or Möbius transformations) in the following way: for $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$,

$$A.z := \frac{az+b}{cz+d}. \tag{1.1}$$

This action is well-defined since c and d can't be simultaneously equal to 0. To see that $A.z \in \mathbb{H}$, it suffices to remark that $\mathrm{Im}\,(A.z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$. The transitivity

is obtained by a straightforward computation. Remark that $A \in SL_2(\mathbb{R})$ and $-A$ act on $z \in \mathbb{H}$ in the same manner. It is then sufficient to consider the action of the group $PSL_2(\mathbb{R}) := SL_2(\mathbb{R})/\{\pm I\}$.

In order to define the notion of modular form, we now pass to a discrete subgroup of $SL_2(\mathbb{R})$, namely the so-called *full modular group* $\Gamma := SL_2(\mathbb{Z})$. This group takes its name from the fact that the quotient $\Gamma\backslash\mathbb{H}$ can be interpreted as the moduli space of elliptic curves over $\mathbb{C}$ (or Riemann surfaces). This means that points of $\Gamma\backslash\mathbb{H}$ classify isomorphism classes of such elliptic curves. To have an intuition of this phenomenon, it is first necessary to know that an elliptic curve over $\mathbb{C}$ is of the form $\mathbb{C}/\Lambda$, for $\Lambda \subset \mathbb{C}$ a lattice. It turns out that two lattices Λ_1, Λ_2 lead to the same curve if there are related by a homothety, i.e. if there exists $\lambda \in \mathbb{C}_0$ such that $\Lambda_2 = \lambda\Lambda_1$. Thus, to each $z \in \mathbb{H}$ one can associate a lattice $\Lambda_z := \mathbb{Z}z + \mathbb{Z}1$ and the quotient $\mathbb{C}/\Lambda_z$, consisting of an elliptic curve. Actually, every elliptic curve over $\mathbb{C}$ can be obtained in this way but not uniquely. Indeed, as mentioned above, such a curve is of the form $\mathbb{C}/\Lambda$, with Λ defined up to homotheties. If we choose an oriented basis $\{\omega_1, \omega_2\}$ of Λ with $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$, i.e. such that $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then the lattice $\omega_2^{-1}\Lambda = \Lambda_z$, with $z := \frac{\omega_1}{\omega_2}$ and we get the identification $\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda_z$. But one can show that a different choice of oriented basis for Λ corresponds to an element of the form $A.z$, with $A \in \Gamma$ ([27]).

A *modular function* is a complex-valued function f defined on $\mathbb{H}$ which is invariant under the action of Γ given by (1.1), i.e. satisfying $f(A.z) = f(z)$ for $A \in \Gamma$ and $z \in \mathbb{H}$. Such a function admits several interpretations thanks to the discussion above. Indeed, it can also be seen as a complex-valued function defined on $\Gamma\backslash\mathbb{H}$, as a complex-valued function defined on the set of isomorphism classes of elliptic curves over $\mathbb{C}$, and also as a function $\tilde{f}$ defined on lattices in $\mathbb{C}$ such that $\tilde{f}(\lambda\Lambda) = \tilde{f}(\Lambda)$ with Λ a lattice and $\lambda \in \mathbb{C}_0$. To a function $\tilde{f}$ on lattices, one associates the corresponding invariant function f on $\mathbb{H}$ by setting $f(z) := \tilde{f}(\Lambda_z)$. For the reverse direction, given a function f, one defines the corresponding $\tilde{f}$ by $\tilde{f}(\Lambda) := f\left(\frac{\omega_1}{\omega_2}\right)$ with $\{\omega_1, \omega_2\}$ an oriented basis of Λ. In the literature, a modular function f on $\mathbb{H}$ is also required to be meromorphic and of exponential growth at infinity, i.e. such that $f(x+iy) = O(e^{cy})$ as $y \to +\infty$ and $f(x+iy) = O\left(e^{\frac{c}{y}}\right)$ as $y \to 0$ for some $c > 0$. However, the invariance condition imposed on modular functions is too restrictive to lead to interesting arithmetic properties. That is precisely the reason why one needs to introduce the *modular forms*. As in the case of rational numbers and integers, those forms are such that every modular function is given by a quotient of two modular forms. For this purpose, the invariance condition imposed on modular functions

is relaxed for the modular forms. Indeed, viewed as a function on lattices, a modular form $\tilde{f}$ has to satisfy the identity $\tilde{f}(\lambda\Lambda) = \lambda^{-k}\tilde{f}(\Lambda)$, with $\lambda \in \mathbb{C}_0$ and for some integer k called the *weight* of $\tilde{f}$. If we want the corresponding invariant function f on $\mathbb{H}$, it suffices to set $f(z) := \tilde{f}(\Lambda_z)$ for $z \in \mathbb{H}$. At the level of f, the so-called *modularity condition* is then given by

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$. Again, in the literature, the term *modular form* requires an additional growth condition and the holomorphy of the function. More precisely, we have the following definition.

Definition 1.1.1 ([57])**.** A modular form of weight $k \in \mathbb{N}$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that $|f(z)|$ is bounded by a polynomial in $\max\left\{1, \mathrm{Im}(z)^{-1}\right\}$ and satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$.

If we introduce the standard *slash operator* defined for a modular form f of weight k, $\gamma \in \Gamma$ and $z \in \mathbb{H}$ by

$$f|_k\,\gamma(z) := \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

then the modularity condition for f becomes

$$f|_k\,\gamma = f.$$

In what follows, we will often drop the subscript k for the slash operator.

It turns out that for odd k, only the trivial form $f \equiv 0$ fulfills the conditions of the previous definition and apart from this particular case, there exists no simple non-trivial example of modular form.

Example 1.1.2. One important example is given by the so-called *discriminant function* Δ defined for $z \in \mathbb{H}$ by

$$\Delta(z) := e^{2\pi i z}\prod_{n=1}^{+\infty}\left(1-e^{2\pi i z}\right)^{24}. \tag{1.2}$$

This is a form of weight 12 and it played an important role in the work of Ramanujan about modular forms.

Example 1.1.3. Two other important examples are the so-called *Eisenstein series* of weight 4 and 6, respectively given by

$$E_4(z) = 1 + 240 \sum_{n=1}^{+\infty} n^3 \frac{q^n}{1-q^n},\ E_6(z) = 1 - 504 \sum_{n=1}^{+\infty} n^5 \frac{q^n}{1-q^n},$$

where $z \in \mathbb{H}$ and for $q := e^{2\pi i z}$.

The space of modular forms of weight k is denoted by $\mathcal{M}_k(\Gamma)$ and the algebra of all modular forms is then given by $\mathcal{M}(\Gamma) = \bigoplus_{k\geq 0} \mathcal{M}_k(\Gamma)$.

It is worth mentioning that the above definition makes sense if we replace Γ by any congruent subgroup of $SL_2(\mathbb{Z})$. A particular family of such groups is the set of congruent subgroups $\Gamma(n)$ of level n, defined for each integer $n \geq 1$ by

$$\Gamma(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\},$$

where the congruence is understood entry by entry. In particular, $\Gamma(1) = SL_2(\mathbb{Z})$. Then, for each $n \geq 1$, we adapt Definition 1.1.1 by setting $\mathcal{M}(\Gamma(n)) := \bigoplus \mathcal{M}_k(\Gamma(n))$, the *graded algebra of modular forms of level n*. Remark that $\mathcal{M}(\Gamma(1)) = \mathcal{M}(\Gamma)$, as the linear fractional action of matrices Id and $-\mathrm{Id}$ on $\mathbb{H}$ are the same. For a more general congruent subgroup $\tilde{\Gamma}$, an element in $\mathcal{M}\left(\tilde{\Gamma}\right)$ is called a $\tilde{\Gamma}$-*modular form*.

We now turn to the description of Rankin-Cohen brackets.

1.1.2 Rankin-Cohen brackets

Given two modular forms $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$, a natural idea to combine them in order to obtain a new modular form consists in taking their pointwise product. Of course, it is straightforward to show that fg is then a modular form of weight $k + \ell$. Since a modular form is required to be holomorphic, another idea to produce a new modular form from f and g is to consider bilinear combinations of f, g and their derivatives. But there is a problem: *the derivative of a modular form is not modular anymore*. Indeed, if we set

$$f' := \frac{1}{2\pi i} \frac{df}{dz},$$

we can observe that

$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2}f'(z) + \frac{k}{2\pi i}c(cz+d)^{k+1}f(z),$$

and because of the second term, f' can't be a modular form of weight $k+2$ as suggested by the first term. There are several ideas to deal with it. A first one consists in modifying the classical differentiation operator so that it preserves modularity. For instance, the *Serre derivative* (which historically goes back to Ramanujan) defined for any modular form $f \in \mathcal{M}_k(\Gamma)$ by

$$\vartheta_k f := f' - \frac{k}{12}E_2 f, \tag{1.3}$$

where $E_2(z) := \frac{1}{2\pi i}\frac{d}{dz}(\log \Delta(z))$ denotes the Eisenstein series of weight 2, is such that $\vartheta_k f \in \mathcal{M}_{k+2}(\Gamma)$. The operator $\vartheta_k : \mathcal{M}_k(\Gamma) \to \mathcal{M}_{k+2}(\Gamma)$ is then a derivation of degree 2.

A second idea to tackle the problem of non-modularity of the classical derivative is to consider some particular combinations of modular forms. Actually, the question of knowing what kind of bilinear operations give rise to a modular form from two given modular forms goes back to Rankin in the 1950's ([47], [57]). During those years, Rankin delivers a construction allowing to obtain a new modular form from a single given modular form and its derivatives. In 1977, based on Rankin's work, Cohen provides an answer to the question concerning two given modular forms by defining a sequence of *brackets* (i.e. bilinear operations) indexed by positive integers ([25]). This is the birth of *Rankin-Cohen brackets*. More precisely, we have the following proposition.

Proposition 1.1.4 ([27])**.** For $f \in \mathcal{M}_k(\Gamma)$, $g \in \mathcal{M}_\ell(\Gamma)$ and for every integer $n \geq 0$, the function $[f,g]_n$ defined by

$$[f,g]_n := \sum_{r=0}^{n}(-1)^r\binom{k+n-1}{n-r}\binom{\ell+n-1}{r}f^{(r)}g^{(n-r)}$$

(with $f^{(r)} := \left(\frac{1}{2\pi i}\frac{d}{dz}\right)^r f$) belongs to $\mathcal{M}_{k+\ell+2n}(\Gamma)$.

The proof of this result is given in details in [57]. We can easily notice different properties of those brackets: $[f,g]_0$ is equal to the classical pointwise product fg and $[f,g]_1 = kfg' - \ell f'g$ is a modular form of weight $k+\ell+2$. Moreover, as pointed out in [27], the bracket $[.,.]_1$ is clearly anti-symmetric and a straightforward computation shows that it satisfies the Jacobi identity, which makes the space $\mathcal{M}(\Gamma)$ into a (graded) Lie algebra. But there is more: for a

given element $f \in \mathcal{M}_k(\Gamma)$, then for $g \in \mathcal{M}_\ell(\Gamma)$ and $h \in \mathcal{M}_m(\Gamma)$,

$$\begin{aligned}[f,gh]_1 &= kf(gh)' - (\ell+m)f'gh \\ &= kfg'h + kfgh' - \ell f'gh - mf'gh \\ &= [f,g]_1 h + g[f,h]_1,\end{aligned}$$

which means that the map $g \mapsto [f,g]_1$ is a derivation with respect to the classical pointwise multiplication. Therefore, the space $\mathcal{M}(\Gamma)$ becomes a Poisson algebra. For the n-th Rankin-Cohen brackets, we have

$$[f,g]_n = (-1)^n [g,f]_n$$

and for $n > 0$,

$$[f,1]_n = [1,f]_n = 0$$

(see [45, 58] for details).

1.2 Link with formal deformation

The purpose of this section is to show how Rankin-Cohen brackets lead to a particular (universal) deformation formula. We recall first what is meant by a *formal deformation*.

Definition 1.2.1 ([36], Definition 1.13)**.** Let $\mathcal{A}$ be a unital, associative and commutative algebra over a field $\mathbb{K}$ with multiplication $\mu_0 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and unit $1_{\mathcal{A}}$. Then a *formal deformation* of $\mathcal{A}$ is an associative and unital $\mathbb{K}[[\nu]]$-algebra structure on the formal power series module $\mathcal{A}[[\nu]]$ with multiplication of the form

$$\mu_\nu = \mu_0 + \sum_{k=1}^{+\infty} \nu^k \mu_k,$$

where each $\mu_k : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a $\mathbb{K}$-bilinear map extended to be $\mathbb{K}[[\nu]]$-bilinear.

After a deformation, the associativity is then preserved but not necessarily the commutativity. When $\mathcal{A}$ is endowed with an extra Poisson structure $\{.,.\}$, the formal deformation is then required to be *in the direction of the Poisson bracket*, which means that μ_1 has to satisfy the equation

$$\mu_1(a,b) - \mu_1(b,a) = c\,\{a,b\}$$

for a certain constant c depending on the chosen normalization and with $a, b \in \mathcal{A}$.

1.2.1 Cohen, Manin and Zagier

Thanks to the discussion of the previous section, for the particular Poisson algebra $\mathcal{A} := \mathcal{M}(\Gamma)$ with the pointwise multiplication μ_0, we know that the zeroth Rankin-Cohen bracket coincides with μ_0 and that the first Rankin-Cohen bracket defines a Poisson structure. It seems then natural to ask wether those two brackets appear as the first two terms of some (associative) formal deformation of modular forms in the direction of the Poisson bracket. This question has been treated through the use of different techniques by several authors, including P. Cohen, Y. Manin, D. Zagier, A. and J. Unterberger and M. Pevzner (see for instance [27, 45, 50]). The (surprising) result given by Cohen, Manin and Zagier is based on the fact that there exists an equivalence between the space of formal power series with coefficients in modular forms and a certain space of invariant pseudo-differential operators ([29]). Their theorem is the following.

Theorem 1.2.2 ([27])**.** Let ν and κ be two formal parameters. For $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$, the multiplication $\star_{\nu,\kappa}$ defined by

$$f \star_{\nu,\kappa} g := \sum_{n=0}^{+\infty} t_n(k,\ell;\nu,\kappa)[f,g]_n$$

where the coefficients $t_n(k,\ell;\nu,\kappa) \in \mathbb{Q}(k,\ell)[\nu,\kappa]$ are given by

$$t_n(k,\ell;\nu,\kappa) := \nu^n \sum_{0\leq j\leq n/2} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\frac{\kappa}{\nu}-1}{j}\binom{-\frac{\kappa}{\nu}}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-\ell-\frac{1}{2}}{j}\binom{n+k+\ell-\frac{3}{2}}{j}},$$

is a formal deformation of the space of modular forms.

This means that there exists a two-parameter family of formal deformations based on Rankin-Cohen brackets. Notice that the choice $\kappa = 0$ leads to

$$\mu_{0,\nu}(f,g) := f \star_{\nu,0} g = \sum_{n=0}^{+\infty} \nu^n [f,g]_n,$$

which implies in particular that the sum of Rankin-Cohen brackets defines a formal deformation of the algebra $(\mathcal{M}(\Gamma), [.,.]_0)$.

1.2.2 Connes and Moscovici

In their paper [30] in which they deal with the problem of computing the index of transversally elliptic operators on foliations, Connes and Moscovici define a family of Hopf algebras $\mathcal{H}_k$ for each integer k, together with an

action on the algebra of the transverse frame bundle of any codimension n foliation. Let us briefly describe the particular Hopf algebra $\mathcal{H}_1$: as an algebra, it corresponds exactly to the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n; n \geq 1\}$ and brackets

$$[Y, X] = X,\ [Y, \delta_n] = n\delta_n,\ [X, \delta_n] = \delta_{n+1},\ [\delta_n, \delta_m] = 0;\ n, m \geq 1.$$

The product $\cdot : \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{H}_1$ is that of the universal enveloping algebra. The coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\begin{aligned}
\Delta Y &= Y \otimes 1 + 1 \otimes Y, \\
\Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\
\Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1,
\end{aligned}$$

and the multiplicativity property

$$\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2),$$

where $h_1, h_2 \in \mathcal{H}_1$. The counit $\epsilon : \mathcal{H}_1 \to \mathbb{C}$ consists in taking the value of the constant term (i.e. $\epsilon(h) =$ constant term of $h \in \mathcal{H}_1$) and the antipode $S : \mathcal{H}_1 \to \mathcal{H}_1$ is determined by

$$S(X) = -X + \delta_1 Y,\ S(Y) = -Y,\ S(\delta_1) = -\delta_1,$$

and the anti-isomorphism property

$$S(h_1 h_2) = S(h_1)S(h_2),$$

where $h_1, h_2 \in \mathcal{H}_1$. The space $(\mathcal{H}_1, \cdot, \Delta, S, 1, \epsilon)$ then defines a Hopf algebra.

In [31], the same authors associate to any congruence subgroup $\tilde{\Gamma}$ of $SL_2(\mathbb{Z})$ an algebra $\mathcal{A}\left(\tilde{\Gamma}\right)$, which consists in an extension of both the ring of classical Hecke operators and the algebra $\mathcal{M}\left(\tilde{\Gamma}\right)$ of $\tilde{\Gamma}$-modular forms. This algebra acts on $\mathcal{M}\left(\tilde{\Gamma}\right)$ extending the action of classical Hecke operators. Here is a brief description of their construction. Define the inductive limit $\mathcal{M} := \varinjlim_{n \to +\infty} \mathcal{M}(\Gamma(n))$. This corresponds to the algebra of modular forms of all levels. We set $G^+(\mathbb{Q}) := GL^+(2, \mathbb{Q})$, where the superscript $+$ means that we consider the identity component.

Definition 1.2.3 ([31])**.** Let $\tilde{\Gamma}$ be a congruence subgroup. A *Hecke operator form of level* $\tilde{\Gamma}$ is a map

$$F : \tilde{\Gamma}\backslash G^+(\mathbb{Q}) \to \mathcal{M} : \tilde{\Gamma}\alpha \mapsto F_\alpha \in \mathcal{M}$$

with finite support and satisfying the covariance condition

$$F_{\alpha\gamma} = F_\alpha|\,\gamma,\ \forall \alpha \in G^+(\mathbb{Q}),\ \gamma \in \Gamma.$$

The space $\mathcal{A}\left(\tilde{\Gamma}\right)$ of Hecke operator forms of level $\tilde{\Gamma}$ then defines an algebra ([31], Proposition 2) that Connes and Moscovici call the *modular Hecke algebra of level* $\tilde{\Gamma}$, which contains as subalgebra the algebra of $\tilde{\Gamma}$-modular forms $\mathcal{M}\left(\tilde{\Gamma}\right)$.

One important point in [31] is that Connes and Moscovici observe the existence of a non-trivial action of the Hopf algebra $\mathcal{H}_1$ constructed in [30] on the algebra $\mathcal{A}\left(\tilde{\Gamma}\right)$. It means that $\mathcal{A}\left(\tilde{\Gamma}\right)$ is a left $\mathcal{H}_1$-module algebra, in the sense of the following definition.

Definition 1.2.4 ([36], Definition 1.1)**.** Let $\mathcal{B} := \mathcal{B}\left(\mu_\mathcal{B}, \Delta_\mathcal{B}, 1_\mathcal{B}, \epsilon_\mathcal{B}\right)$ be a bialgebra over a commutative field $\mathbb{K}$, with multiplication $\mu_\mathcal{B} : \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$, comultiplication $\Delta_\mathcal{B} : \mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$, unit $1_\mathcal{B}$ ant counit $\epsilon_\mathcal{B} : \mathcal{B} \to \mathbb{K}$. A left $\mathcal{B}$-module $\mathcal{A}$ is a left $\mathcal{B}$-module algebra if $\mathcal{A} = \mathcal{A}(\mu_A, 1_A)$ is a unital $\mathbb{K}$-algebra such that for all $b \in \mathcal{B}$,

1) $b \cdot 1_\mathcal{A} = (\epsilon(b)) \cdot 1_\mathcal{A}$;

2) $b \cdot (aa') = \sum \left(b_{(1)} \cdot a\right)\left(b_{(2)} \cdot a'\right)$ where $\Delta_\mathcal{B}(b) = \sum b_{(1)} \otimes b_{(2)}$.

We now present the way that Connes and Moscovici define the action of $\mathcal{H}_1$ on $\mathcal{A}\left(\tilde{\Gamma}\right)$. Firstly, they define the action of the generators X and Y of $\mathcal{H}_1$ on the space of modular forms: for f a modular form of weight k, they set

$$Y(f) := \frac{k}{2} f \tag{1.4}$$

and

$$X(f) := \vartheta_k(f), \tag{1.5}$$

where ϑ denotes the Serre derivative given by (1.3). Explicitly, we then have

$$X(f) := \frac{1}{2\pi i}\frac{df}{dz} - \frac{1}{6\pi i}\frac{d}{dz}(\log\Delta).f = \frac{1}{2\pi i}\frac{df}{dz} - \frac{1}{12\pi i}\frac{d}{dz}(\log\Delta).Yf,$$

where Δ denotes the discriminant function (1.2). Remark that with these definitions, using the fact that $X(f)$ is of weight $k+2$,

$$[Y,X]f = Y(X(f)) - X(Y(f)) = \frac{k+2}{2}X(f) - \frac{k}{2}X(f) = X(f)$$

and then $[Y,X] = X$. Now, for each element $\alpha \in G^+(\mathbb{Q})$, define the function μ_α on $\mathbb{H}$ by

$$\mu_\alpha(z) := \frac{1}{12\pi i}\frac{d}{dz}\log\left(\frac{\Delta|\,\alpha}{\Delta}\right).$$

The action of the generators of $\mathcal{H}_1$ on $\mathcal{A}\left(\tilde{\Gamma}\right)$ is given for $F \in \mathcal{A}\left(\tilde{\Gamma}\right)$ and $\alpha \in G^+(\mathbb{Q})$ by the following operators:

$$\begin{aligned} Y(F)_\alpha &:= Y\left(F_\alpha\right), \\ X(F)_\alpha &:= X\left(F_\alpha\right), \\ \delta_m(F)_\alpha &:= \mu_{m,\alpha}\left(F_\alpha\right), \end{aligned} \tag{1.6}$$

where $\mu_{m,\alpha} := X^{m-1}(\mu_\alpha)$, $\forall m \in \mathbb{N}$.

Theorem 1.2.5 ([31], Theorem 12)**.** Let $\tilde{\Gamma}$ be any congruence subgroup.

1) The formulas (1.6) define a Hopf action of $\mathcal{H}_1$ on the algebra $\mathcal{A}\left(\tilde{\Gamma}\right)$.

2) The derivation $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2$ is inner and implemented by $\omega_4 := -\frac{E_4}{72} \in \mathcal{A}\left(\tilde{\Gamma}\right)$ (where E_4 denotes the Eisenstein series of weight 4 defined in Example 1.1.3), i.e.

$$\delta_2'(a) = [a,\omega_4],\ \forall a \in \mathcal{A}\left(\tilde{\Gamma}\right).$$

Remark 1.2.6 ([57])**.** The restricted Hopf action of $\mathcal{H}_1$ to the space of modular forms[1] (given by formulas (1.6)) coincides with the Hopf action of the universal enveloping algebra generated by Y and X with bracket $[Y,X] = X$, determined by formulas (1.4) and (1.5). This comes from the fact that $\mu_\alpha \equiv 0$ for all $\alpha \in SL_2(\mathbb{Z})$.

As noticed by Connes and Moscovici, Theorem 1.2.5 suggests a generalization of the notion of Rankin-Cohen brackets to the modular Hecke algebra. This observation is based on a work of Zagier. Indeed, in [58], the latter shows that using the element $\Phi := \frac{\omega_4}{2} \in \mathcal{M}_4(\Gamma)$, the Rankin-Cohen brackets can be

[1] Recall that the space of modular forms constitutes a subalgebra of the modular Hecke algebra.

reformulated as follows ([58], Proposition 1): for $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$,

$$[f,g]_n = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f_r g_s, \tag{1.7}$$

where $f_r \in \mathcal{M}_{k+2r}(\Gamma)$, $g_s \in \mathcal{M}_{\ell+2s}(\Gamma)$ $(r,s \geq 0)$ are defined recursively by

$$f_{r+1} = Xf_r + r(r+k-1)\Phi f_{r-1},\ g_{s+1} = Xg_s + s(s+\ell-1)\Phi g_{s-1}$$

with initial conditions $f_0 = f$ and $g_0 = g$ and where $X : \mathcal{M}_k(\Gamma) \to \mathcal{M}_{k+2}(\Gamma)$ is given by (1.5). Thanks to Remark 1.2.6, the formula (1.7) then allows to express the Rankin-Cohen brackets by means of the restricted Hopf action of $\mathcal{H}_1$ to the space of modular forms and the element $\Phi \in \mathcal{M}_4(\Gamma)$. Note that this modular form Φ of weight 4 used by Zagier is very reminiscent with the particular inner derivation δ_2' considered by Connes and Moscovici. These last two then suggest in [32] the following extension of Zagier's formula for Rankin-Cohen brackets: for $a, b \in \mathcal{A}\left(\tilde{\Gamma}\right)$, they set

$$RC_n(a,b) := \sum_{k=0}^{n} \frac{A_k}{k!}(2Y+k)_{n-k}(a) \frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k(b), \tag{1.8}$$

with $(Z)_k := Z(Z+1)\dots(Z+k-1)$ and where A_k and B_k are obtained by induction using $A_{-1} := 0$, $A_0 := 1$,

$$A_{k+1} := S(X)A_k - k\omega_4^0 \left(Y - \frac{k-1}{2}\right) A_{k-1}, \tag{1.9}$$

where ω_4^0 is the right multiplication by ω_4, $B_0 := 1$, $B_1 := X$ and

$$B_{k+1} := XB_k - k\omega_4 \left(Y - \frac{k-1}{2}\right) B_{k-1}. \tag{1.10}$$

In the particular case of modular forms, the formula (1.8) coincides with (1.7).

Actually, Connes and Moscovici go much further: they remark that the formula (1.8) admits a generalization for every algebra $\mathcal{A}$ endowed with a *projective structure* in the following sense.

Definition 1.2.7 ([32])**.** Let $\mathcal{H}_1$ be the Hopf algebra described above. We say that an algebra $\mathcal{A}$ is endowed with a $\mathcal{H}_1$-*projective structure* if $\mathcal{H}_1$ acts on $\mathcal{A}$ and if the derivation $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2$ is inner and implemented by an element $\Omega \in \mathcal{A}$, i.e.

$$\delta_2'(a) = [\Omega, a] = \Omega a - a\Omega,\ \forall a \in \mathcal{A},$$

with

$$\delta_k(\Omega) = 0, \ \forall k \in \mathbb{N}.$$

Given such an algebra $\mathcal{A}$, the generalized Rankin-Cohen brackets $RC_n(a,b)$ for $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$ are then defined by the formula (1.8), but where ω_4 is replaced by Ω in the expressions (1.9) and (1.10). Explicitly, it gives

$$A_{k+1} := S(X)A_k - k\Omega^0 \left(Y - \frac{k-1}{2}\right) A_{k-1}, \tag{1.11}$$

$A_{-1} := 0$, $A_0 := 1$,

$$B_{k+1} := XB_k - k\Omega \left(Y - \frac{k-1}{2}\right) B_{k-1}, \tag{1.12}$$

$B_0 := 1$ and $B_1 := X$. Using these new brackets, they generalize the deformed products of Theorem 1.2.2.

Theorem 1.2.8 ([32], Theorem 10)**.** Let $\mathcal{A}$ be an algebra endowed with a $\mathcal{H}_1$-projective structure. Then, for any formal parameters ν and κ, the product formula

$$a \star_\nu^\kappa b := \sum_{n=0}^{+\infty} \nu^n RC_n \left(\mathbf{t}_n^\kappa \left(Y \otimes 1, 1 \otimes Y\right)(a \otimes b)\right) \tag{1.13}$$

gives a formal deformation of $\mathcal{A}$, where RC_n are defined by (1.8) (for A_k and B_k given respectively by (1.11) and (1.12)),

$$\mathbf{t}_n^\kappa(x,y) := \left(-\frac{1}{4}\right)^n \sum_{j=0}^{+\infty} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-x-\frac{1}{2}}{j}\binom{-y-\frac{1}{2}}{j}\binom{n+x+y-\frac{3}{2}}{j}}$$

are the coefficients defined in [29] and

$$\binom{x}{j} := \frac{\prod_{m=0}^{j-1}(x-m)}{j!}.$$

When $\kappa = \frac{1}{2}$ or $\kappa = \frac{3}{2}$, then for all $n \in \mathbb{N}$ and for all x, y,

$$\mathbf{t}_n^\kappa(x,y) = 1,$$

which leads to the following particular formula.

Corollary 1.2.9. The particular choice $\kappa = \frac{1}{2}$ (or $\kappa = \frac{3}{2}$) reduces the formula (1.13) to

$$a \star_\nu b = \sum_{n=0}^{+\infty} \nu^n RC_n(a,b) \tag{1.14}$$

(where we have dropped the superscript 1/2 in $\star_\nu^{1/2}$).

Before stating one important last result of Connes and Moscovici concerning Rankin-Cohen brackets, we recall the notion of *universal deformation formula* in the sense of [36].

Definition 1.2.10 ([36], Definition 1.16)**.** Let $\mathcal{B} := \mathcal{B}(\mu_\mathcal{B}, \Delta_\mathcal{B}, 1_\mathcal{B}, \epsilon_\mathcal{B})$ be a bialgebra over a commutative field $\mathbb{K}$, with multiplication $\mu_\mathcal{B} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$, comultiplication $\Delta_\mathcal{B} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$, unit $1_\mathcal{B}$ ant counit $\epsilon_\mathcal{B} : \mathcal{B} \to \mathbb{K}$. A *universal deformation formula* (UDF) based on a bialgebra $\mathcal{B}$ is an element $F \in (\mathcal{B}\otimes\mathcal{B})[[\nu]]$ of the form

$$F = 1_\mathcal{B} \otimes 1_\mathcal{B} + \sum_{n=1}^{+\infty} \nu^n F_n,$$

with ν a formal parameter and where each $F_n \in \mathcal{B} \otimes \mathcal{B}$ satisfies

$$(\epsilon_\mathcal{B} \otimes \mathrm{Id})\, F = 1 \otimes 1_\mathcal{B},\ (\mathrm{Id} \otimes \epsilon_\mathcal{B})\, F = 1_\mathcal{B} \otimes 1$$

and

$$((\Delta_\mathcal{B} \otimes \mathrm{Id})\,(F))\,(F \otimes 1_\mathcal{B}) = ((\mathrm{Id} \otimes \Delta_\mathcal{B})\,(F))\,(1_\mathcal{B} \otimes F)\,.$$

As mentioned by Giaquinto and Zhang, such an element F can then be used to *twist* the multiplication of any left $\mathcal{B}$-module algebra $\mathcal{A}$.

Proposition 1.2.11 ([36])**.** Let $\mathcal{B}$ be a bialgebra over a commutative field $\mathbb{K}$ and let ν be a formal parameter. If $\mathcal{A} := \mathcal{A}(\mu_\mathcal{A}, 1_\mathcal{A})$ is a left $\mathcal{B}$-module algebra (where $\mu_\mathcal{A} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is extended to be $\mathbb{K}[[\nu]]$-linear) and F a UDF based on $\mathcal{B}$, then the formula

$$\mu_\nu(a, b) := (\mu_A \circ F)(a \otimes b),\ \forall a, b \in A,$$

defines a formal deformation of $\mathcal{A}$.

The conditions imposed on the element F ensure that the twisted product is again associative.

Connes and Moscovici prove in [32] that the extended Rankin-Cohen brackets provide a UDF based on $\mathcal{H}_1$, allowing to twist the product of any algebra $\mathcal{A}$ endowed with a $\mathcal{H}_1$-projective structure. More precisely, one has the following result.

Theorem 1.2.12 ([32])**.** The element

$$RC := \sum_{n=0}^{+\infty} \nu^n RC_n, \tag{1.15}$$

where

$$RC_n := \sum_{k=0}^{n} \frac{A_k}{k!}(2Y+k)_{n-k} \otimes \frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k,$$

$$A_{k+1} := S(X)A_k - k\Omega^0 \left(Y - \frac{k-1}{2}\right) A_{k-1},$$

$$B_{k+1} := XB_k - k\Omega \left(Y - \frac{k-1}{2}\right) B_{k-1},$$

defines a UDF based on $\mathcal{H}_1$ for any projective $\mathcal{H}_1$-action, with Ω the element underlying the $\mathcal{H}_1$-projective structure. Therefore, if $\mathcal{A}$ is an algebra endowed with a $\mathcal{H}_1$-projective structure implemented by an element $\Omega \in \mathcal{A}$, then the formula (1.14) extended $\mathbb{K}[[\nu]]$-linearly defines an associative product on $\mathcal{A}[[\nu]]$, i.e. a formal deformation of $\mathcal{A}$.

1.2.3 Geometrical interpretation of the Rankin-Cohen deformation

The purpose of this section is to explain how the Rankin-Cohen deformation (for a particular choice of $\mathcal{H}_1$) fits into the geometrical context of *formal deformation quantization* on symplectic manifolds. Such a deformation, also called a *star-product*, is a particular *formal deformation* (in the sense of Definition 1.2.1), said to be *in the direction of the Poisson bracket* (coming from the symplectic form associated with the considered symplectic manifold).

We start by recalling some geometrical tools (we refer for instance to [6] for more details).

Given a symplectic manifold (M, ω), the non-degeneracy of ω implies that for each $f \in C^\infty(M)$ (i.e. the space of smooth functions on M valuated in $\mathbb{C}$), there exists a unique vector field $X_f \in \Gamma^\infty(TM)$, called *Hamiltonian vector field associated with* f, such that

$$\iota(X_f)\,\omega = -df,$$

where $\iota(X_f)\,\omega$ denotes the contraction of X_f with the form ω. So we can define a map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : (f,g) \mapsto \{f,g\} := \omega(X_f, X_g),$$

which defines a Poisson structure on the manifold M. This Poisson structure (or Poisson bracket) is said to be *associated with the symplectic form* ω. Moreover,

the application mapping $f \in C^\infty(M)$ to $X_f \in \Gamma^\infty(TM)$ is a Lie algebra morphism, i.e.

$$X_{\{f,g\}} = [X_f, X_g],$$

with $f, g \in C^\infty(M)$ (see for instance [33] for details). The Poisson bracket can be written in terms of a Hamiltonian vector field using the following identity:

$$\{f, g\} = dg\,(X_f) =: X_f(g).$$

Now, we consider a Lie group G acting on a manifold M via a map $\tau : G \times M \to M : (g, x) \mapsto \tau_g(x)$. If we denote by $\mathfrak{g}$ the Lie algebra of G (i.e. $\mathfrak{g} := \mathrm{Lie}(G)$), we get for each $X \in \mathfrak{g}$ an application

$$X^\star : M \to TM : x \mapsto X^\star_x := \left.\frac{d}{dt}\right|_0 \tau_{\exp(-tX)}(x),$$

which defines a vector field on M called *fondamental vector field associated with* X. The map

$$\mathfrak{g} \to \Gamma^\infty(TM) : X \mapsto X^\star$$

is then a Lie algebra homomorphism. If in addition the action of G on M is transitive, we get the identification

$$T_x(M) = \{X^\star_x \mid X \in \mathfrak{g}\}.$$

Definition 1.2.13. Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M via a map τ. This action is called *symplectic* if for each $g \in G$, τ_g is a symplectomorphism.

Definition 1.2.14. Let (M, ω) be a symplectic manifold together with a symplectic action of a Lie group G with associated Lie algebra $\mathfrak{g}$. This action is called *Hamiltonian* if there exists a Lie algebra homomorphism

$$\lambda : (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\}),$$

with $\{\cdot, \cdot\}$ the Poisson bracket associated with ω, satisfying the relation

$$\iota\,(X^\star)\,\omega = -d\,(\lambda(X)),$$

for each $X \in \mathfrak{g}$. This Lie algebra homomorphism is then called *moment map* and in what follows the element $\lambda(X)$ will be denoted by λ_X.

Remark 1.2.15. Within the context of the previous definition, the fondamental vector field $X^\star$ is then Hamiltonian and associated with the function λ_X, i.e.

$X^\star = X_{\lambda_X}$. Hence, for all $f \in C^\infty(M)$,

$$X^\star(f) = \{\lambda_X, f\}\,.$$

Star-product on a symplectic manifold

In the context of symplectic manifolds, a *star-product* is defined as follows.

Definition 1.2.16 ([5])**.** Let (M, ω) be a symplectic manifold. A *differential star-product* on M (or a *star-product* on M in short) is a bilinear map

$$\star_\nu : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]] : (f, g) \mapsto f \star_\nu g := \sum_{k=0}^{+\infty} \nu^k C_k(f, g),$$

where ν is a formal parameter called *deformation parameter* and with

$$C^\infty(M)[[\nu]] := \left\{ \sum_{k=0}^{+\infty} \nu^k f_k \mid f_k \in C^\infty(M)\ \forall k \in \mathbb{N} \right\},$$

such that

(i) for each $k \in \mathbb{N}\backslash\{0\}$, the map

$$C_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

defines a bidifferential operator on M;

(ii) the law $\star_\nu$, extended $\mathbb{C}[[\nu]]$-linearly to the space of formal power series $C^\infty(M)[[\nu]]$, gives an associative product on $C^\infty(M)[[\nu]]$;

(iii) the so-called *classical limit conditions* are satisfied, i.e.

$$C_0(f, g) = fg \quad \text{and} \quad C_1(f, g) - C_1(g, f) = c\{f, g\},$$

where $c \in \mathbb{C}$ is a constant depending on the chosen normalization (in general, $c = 1$ or $c = 2$) and where $\{\cdot, \cdot\}$ means the Poisson structure associated with ω;

(iv) $1 \star_\nu f = f \star_\nu 1 = f$.

Remark 1.2.17. From now on, the notation $C^\infty(M)$ will be used to denote the space of smooth functions on M valuated in $\mathbb{C}$.

It turns out that on a symplectic manifold it may exist more than one star-product, which justifies the usefulness of the following notion of *equivalence* between two star-products.

Definition 1.2.18. Let $\star_\nu$ and $\star'_\nu$ be two star-products on a symplectic manifold (M, ω). There are said to be *equivalent* if there exists a linear formal operator T on $C^\infty(M)[[\nu]]$ of the form

$$T = \sum_{k=0}^{+\infty} \nu^k T_k, \tag{1.16}$$

such that

(i) $T_0 = \mathrm{Id}$;

(ii) T_k is a $\mathbb{C}[[\nu]]$-linear differential operator for all $k \in \mathbb{N}$;

(iii) T *intertwines* $\star_\nu$ and $\star'_\nu$, that is, for $f, g \in C^\infty(M)[[\nu]]$,

$$T\left(f \star_\nu g\right) = Tf \star'_\nu Tg.$$

Remark 1.2.19. Within the context of the previous definition, the fact that $T_0 = \mathrm{Id}$ implies that T is formally invertible, which implies that the intertwining property (iii) is equivalent to

$$f \star_\nu g = T^{-1}\left(Tf \star'_\nu Tg\right).$$

This condition will be denoted shortly by $\star_\nu = T^{-1}\left(\star'_\nu\right)$ or equivalently by $\star'_\nu = T\left(\star_\nu\right)$.

Conversely, given a star-product $\star_\nu$ on a symplectic manifold (M, ω) and T an operator of the form (1.16) satisfying conditions (i) and (ii) of Definition 1.2.18, it can be shown that $T\left(\star_\nu\right)$ is a star-product on (M, ω) if and only if T_k vanishes on constants for all $k \geq 1$ (see for instance [7]). Remark that this condition is equivalent to require that T leaves the constants unchanged.

Let us now present two important constructions of such star-products. The first one is the *Moyal star-product* on $\mathbb{R}^{2n}$ and is one of the simplest example of deformation quantization. It is constructed as follows: let ω be the standard constant symplectic form on the flat space $\mathbb{R}^{2n}$, i.e. such that its matrix representation $\left(\omega_{ij}\right)_{i,j}$ in the canonical basis is given by

$$\left(\omega_{ij}\right)_{i,j} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

Let also $\star_\nu^0$ be the map defined as

$$\star_\nu^0 : C^\infty\left(\mathbb{R}^{2n}\right) \times C^\infty\left(\mathbb{R}^{2n}\right) \to C^\infty\left(\mathbb{R}^{2n}\right)[[\nu]]$$
$$(f,g) \mapsto f \star_\nu^0 g := fg + \sum_{k=1}^{+\infty} \nu^k C_k^0(f,g), \tag{1.17}$$

where

$$C_k^0(f,g) := \frac{1}{k!} \sum_{I_k,J_k} \omega_{i_k,j_k} \cdots \omega_{i_k j_k} \partial_{i_1 \cdots i_k} f \partial_{j_1 \cdots j_k} g,$$

with $I_k = (i_1, \cdots, i_k)$, $J_k = (j_1, \cdots, j_k)$, $1 \leq i_1, \cdots, i_k, j_1, \cdots, j_k \leq 2n$ and where $\partial_{i_1 \cdots i_k}$ (respectively $\partial_{j_1 \cdots j_k}$) means the composition of partial differential operators in the variables $i_1, \ldots, i_k$ (respectively $j_1 \cdots j_k$). Remark that each C_k^0 $(k \geq 1)$ defines a bidifferential operator such that $C_k^0(f,1) = C_k^0(1,f) = 0$. Moreover, by observing that the canonical Poisson bracket on $\mathbb{R}^{2n}$ associated with ω can be written as follows:

$$\{f,g\} = \sum_{I_1,J_1} \omega_{i_1 j_1} \partial_{i_1} f \partial_{j_1} g = \sum_{i,j=1}^{2n} \omega_{ij} \partial_i f \partial_j g = C_1^0(f,g),$$

we get the identity

$$C_1^0(f,g) - C_1^0(g,f) = 2\{f,g\}.$$

It turns out that extended $\mathbb{C}[[\nu]]$-linearly to $C^\infty\left(\mathbb{R}^{2n}\right)[[\nu]]$, the law $\star_\nu^0$ is associative and then defines a star-product on $\mathbb{R}^{2n}$ (see for instance [26] for more details).

Definition 1.2.20. The star-product $\star_\nu^0$ is called the *Moyal star-product* on $\mathbb{R}^{2n}$.

Remark 1.2.21. An alternative formulation of $\star_\nu^0$ is given for $f, g \in C^\infty\left(\mathbb{R}^{2n}\right)$ by

$$(f \star_\nu^0 g)(z) = \exp\left(\nu \omega_{ij} \frac{\partial}{\partial_{x_i}} \frac{\partial}{\partial_{y_j}}\right) (f(x)g(y))\Big|_{x=y=z}.$$

For instance, on the space $\left(\mathbb{R}^2 = \{q,p\}, \omega = dq \wedge dp\right)$, the explicit expression of the Moyal star-product is given by

$$f \star_\nu^0 g = fg + \sum_{k=1}^{+\infty} \frac{\nu^k}{k!} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \partial_q^{k-j} \partial_p^j f \partial_q^j \partial_p^{k-j} g, \tag{1.18}$$

where $f, g \in C^\infty\left(\mathbb{R}^2\right)[[\nu]]$.

Another important family of examples of star-products on symplectic manifolds comes from the work of Fedosov. Roughly speaking, a Fedosov star-product

consists in gluing together local expressions of the Moyal star-product. We briefly describe its construction here and refer to [34, 38] for more details.

Let (M,ω) be a $2n$-dimensional symplectic manifold. For each $x \in M$, we associate to the symplectic vector space T_xM (the fiber associated with x in the tangent bundle) a unital associative algebra W_x over $\mathbb{C}$, called the *Weyl algebra* associated with x, whose elements are formal power series (with parameter ν) of the form

$$a(x,y,\nu) = \sum_{2k+l\geq 0} \nu^k a_{k,i_1,\cdots,i_l}(x) y_{i_1}\cdots y_{i_l},$$

where the coefficients $a_{k,i_1,\cdots,i_l}$ are symmetric covariant l-tensor fields on M and $y := (y_1,\cdots,y_{2n}) \in T_xM$ is a tangent vector. The product law $\circ$ of two such elements is defined using the expression of Moyal star-product given in Remark 1.2.21:

$$\begin{aligned}(a\circ b)(x,y,\nu) &:= \left(\exp\left(\frac{\nu}{2}\Omega_{ij}\frac{\partial}{\partial y_i}\frac{\partial}{\partial z_j}\right) a(x,y,\nu)b(x,z,\nu)\right)\Big|_{y=z} \\ &= \sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{\nu}{2}\right)^k \Omega_{i_1j_1}\cdots\Omega_{i_kj_k}\frac{\partial^k a(x,y,\nu)}{\partial y_{i_1}\cdots\partial y_{i_k}}\frac{\partial^k b(x,y,\nu)}{\partial y_{i_1}\cdots\partial y_{i_k}},\end{aligned}$$

with $\Omega_{ij} := (\omega^{-1})_{ij}$. The *Weyl bundle* W over (M,ω) is then defined to be the bundle for which the fiber at point $x \in M$ is W_x. The algebra of smooth sections of W with fiberwise multiplication $\circ$ is denoted by $\Gamma^\infty(W)$. It is worth mentioning that the center of this algebra coincides with $C^\infty(M)[[\nu]]$. One of the key ingredients in the construction given by Fedosov is the notion of *Fedosov connection*, which requires to look at the algebra of smooth sections $\Gamma^\infty(W\otimes\Lambda) = \bigoplus_{r=0}^{2n}\Gamma^\infty(W\otimes\Lambda^r)$, where Λ^r is the set of smooth r-forms on M. Such a section has locally the form

$$s(x,y,\nu) = \sum_{2k+l\geq 0} \nu^k a_{k,i_1,\cdots,i_l,j_1,\cdots,j_r}(x) y_{i_1}\cdots y_{i_l} dx_{j_1}\wedge\cdots\wedge dx_{j_r},$$

where the coefficients are covariant tensors symmetric in $i_1,\cdots,i_l$ and antisymmetric in $j_1,\cdots,j_r$. Two elements in $\Gamma^\infty(W\otimes\Lambda)$ can be multiplied together by extending the law $\circ$ with the exterior multiplication, i.e. by setting

$$(a\otimes\lambda)\circ(b\otimes\lambda') := (a\circ b)\otimes(\lambda\wedge\lambda').$$

We then define the commutator on $\Gamma^\infty(W\otimes\Lambda)$ by

$$[s,s'] := s\circ s' - (-1)^{r_s r_{s'}} s'\circ s,$$

if $s \in \Gamma^\infty(W \otimes \Lambda^{r_s})$ and $s' \in \Gamma^\infty(W \otimes \Lambda^{r_{s'}})$, and also two operations $\delta, \delta^\star : \Gamma^\infty(W \otimes \Lambda) \to \Gamma^\infty(W \otimes \Lambda)$ by

$$\delta(s) := \sum_k dx_k \wedge \frac{\partial s}{\partial y_k},\ \delta^\star(s) := \sum_k y_k \iota\left(\frac{\partial}{\partial x_k}\right) s,$$

where $\iota\left(\frac{\partial}{\partial x_k}\right)$ means the contraction of the section with the vector filed $\frac{\partial}{\partial x_k}$. An *Abelian connection* (of Fedosov) on the Weyl algebra bundle is a linear map $D : \Gamma^\infty(W \otimes \Lambda) \to \Gamma^\infty(W \otimes \Lambda)$ satisfying the Leibniz rule and such that

$$D^2 s = 0,$$

for all sections $s \in \Gamma^\infty(W \otimes \Lambda)$. An important result obtained by Fedosov in [34] tells us that it is always possible to construct such an Abelian connection on W from a chosen *symplectic connection* on M. We recall that this latter notion is defined as follows.

Definition 1.2.22. A connection ∇ on (M, ω) is called *symplectic* if

a) ∇ is torsion free, i.e.

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0;$$

b) the symplectic form ω is parallel (which is denoted by $\nabla\omega = 0$), i.e.

$$\nabla_X \omega(Y, Z) := X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0;$$

where X, Y, Z are vector fields on M.

Symplectic connections always exist but are not unique, although under some extra conditions on the manifold M there is sometimes a canonical choice of such a connection. Given a symplectic connection ∇ on M and a basis $\{X_i\}$ of its tangent bundle, we can consider the *Christoffel symbols* Γ_{ij}^k associated with ∇ determined by

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k.$$

We introduce the element

$$\bar{\Gamma} := \frac{1}{2} \sum_{ijkr} \omega_{ki} \Gamma_{rj}^k y_i y_j dx_r,$$

as well as the connection δ on W given by

$$\partial a := da - \frac{1}{\nu}[\bar{\Gamma}, a].$$

This connection extends as a covariant exterior derivative (also denoted by ∂) on the entire space of sections $\Gamma^\infty(W \otimes \Lambda)$ by using the Leibniz rule:

$$\partial(a \otimes \lambda) = \partial(a) \wedge \lambda + a \otimes d\lambda.$$

In [34], Fedosov shows that there exists a certain $r \in \Gamma^\infty\left(W \otimes \Lambda^1\right)$ (uniquely determined by the equation (3.4) in [34]), such that the element

$$D := \partial - \delta - \frac{1}{\nu}[r, \cdot]$$

defines an Abelian connection on W. This allows us to present the main object of Fedosov's construction: the subalgebra $W_D \subset \Gamma^\infty(W)$ of flat sections of W, i.e. the set

$$W_D := \{a \in \Gamma^\infty(W) \mid Da = 0\}.$$

This space is central for the construction of the Fedosov star-product because of the following theorem.

Theorem 1.2.23 ([34], Theorem 3.3)**.** Given a flat connection D, for any $f \in C^\infty(M)[[\nu]]$, there is a unique $a \in W_D$ such that $\sigma(a) := a(x, 0, \nu) = f(x)$.

This theorem tells us that the symbol map $\sigma : \Gamma^\infty(W) \to C^\infty(M)[[\nu]]$ is a linear isomorphism when restricted to W_D. Hence, it can be used to transport the algebra structure of W_D to $C^\infty(M)[[\nu]]$. The *Fedosov star-product* $\star_\nu^F$ (depending on the symplectic connection ∇) is then defined on $C^\infty(M)[[\nu]]$ as follows:

$$f \star_\nu^F g := \sigma\left(\sigma^{-1}(f) \circ \sigma^{-1}(g)\right). \tag{1.19}$$

A geometrical reformulation for Rankin-Cohen deformation

We consider the Hopf algebra $\mathcal{H}_1$ of Connes and Moscovici in the particular case where all the δ_n are trivial, i.e. $\delta_n = 0$ for all $n \geq 1$. In this case, the Hopf algebra $\mathcal{H}_1$ reduces to the universal enveloping algebra of the Lie algebra with basis $\{X, Y\}$ and bracket $[Y, X] = X$. Actually, this Lie algebra is that associated with the so-called "$ax + b$" Lie group, or *affine group of the real line*. In what follows the "$ax + b$" Lie group will be denoted by $\mathbb{S}$ and its Lie algebra Lie($\mathbb{S}$) by $\mathfrak{s}$. Within this notations, we have $\mathcal{H}_1 = \mathcal{U}(\mathfrak{s})$. For any algebra

$\mathcal{A}$ endowed with a $\mathcal{U}(\mathfrak{s})$-projective action[2], we can apply Theorem 1.2.12 in order to get a formal deformation of $\mathcal{A}$. Using the triviality of all the δ_n, the elements RC_n defining the universal deformation formula $RC \in \mathcal{U}(\mathfrak{s}) \otimes \mathcal{U}(\mathfrak{s})[[\nu]]$ (cf. formula (1.15)) given by

$$RC = \sum_{n=0}^{+\infty} \nu^n RC_n,$$

are in this particular context reduced to the following expressions:

$$RC_n =: RC_n^{red} = \sum_{k=0}^{n} \left[\frac{(-1)^k}{k!} X^k \left(2Y + k\right)_{n-k} \otimes \frac{1}{(n-k)!} X^{n-k} \left(2Y + n - k\right)_k \right]. \tag{1.20}$$

Indeed, since $\delta_n = 0$ for all $n \geq 1$ and $\Omega = 0$, we have $S(X) := -X + \delta_1 Y = -X$,

$$A_{k+1} := S(X)A_k - k\Omega^0 \left(Y - \frac{k-1}{2}\right) A_{k-1} = S(X)A_k = -XA_k,$$

with $A_{-1} := 0$, $A_0 := 1$ and

$$B_{k+1} := XB_k - k\Omega \left(Y - \frac{k-1}{2}\right) B_{k-1} = XB_k,$$

with $B_0 := 1$ and $B_1 := X$. The Connes-Moscovici-Rankin-Cohen formal deformation $\star_\nu^{RC}$ of $\mathcal{A}$ is then given by

$$f \star_\nu^{RC} g := \sum_{n=0}^{+\infty} \nu^n RC_n^{red}(f, g), \tag{1.21}$$

for $f, g \in \mathcal{A}[[\nu]]$.

Surprisingly, this deformation $\star_\nu^{RC}$ is strongly linked to the Moyal (and the Fedosov) star-product described above, as shown by Bieliavsky, Tang and Yao in [25]. We describe here briefly their development but refer to their full paper and [57] for more details. In the notations of [25], let (V, ω) be the symplectic vector space $\left(\mathbb{R}^2 = \{(q,p)\}, \omega := dq \wedge dp\right)$ and $\mathfrak{h}_V := \mathrm{span}_{\mathbb{R}}\{Q, P\} \oplus \mathbb{R}Z$ the corresponding Heisenberg algebra, i.e. with brackets given by

$$[Q, P] = \omega(Q, P)Z = Z, \ [Y, Z] = 0 \ \forall \ Y \in \mathrm{span}_{\mathbb{R}}\{Q, P\}.$$

[2]By this, we mean a $\mathcal{U}(\mathfrak{s})$-action together with a $\mathcal{U}(\mathfrak{s})$-projective structure (cf. Definition 1.2.7).

Setting $\mathfrak{sl}_2(\mathbb{R}) := \mathrm{span}_{\mathbb{R}}\{H, E, F\}$ with brackets

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H,$$

we form the semi-direct product

$$\mathfrak{g}_{RC} := \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{h}_V, \tag{1.22}$$

for which the table is given by

$$\begin{aligned}
&[H, E] = 2E,\ [H, F] = -2F,\ [H, Q] = Q,\ [H, P] = -P,\\
&[E, F] = H,\ [E, Q] = 0,\ [E, P] = Q,\ [F, Q] = P,\ [F, P] = 0,\\
&[Q, P] = Z,\ [Y, Z] = 0\ \forall\ Y \in \mathfrak{g}_{RC}.
\end{aligned} \tag{1.23}$$

The infinitesimal linear action $\mathfrak{g}_{RC} \to \Gamma^\infty(TV) : Y \mapsto Y^\star$ then underlies a Hamiltonian action in the sense that there exists an associated moment map $\lambda : \mathfrak{g}_{RC} \to C^\infty(V) : Y \mapsto \lambda_Y$. The explicit expressions of the fundamental vector fields are

$$H^\star = -q\partial_q + p\partial_p,\ E^\star = -p\partial_q,\ F^\star = -q\partial_p,$$

$$Q^\star = -\partial_q,\ P^\star = -\partial_p,\ Z^\star = 0,$$

and the associated moment maps are given by

$$\lambda_H = qp,\ \lambda_E = \frac{1}{2}p^2,\ \lambda_F = -\frac{1}{2}q^2,\ \lambda_Q = p,\ \lambda_P = -q,\ \lambda_Z = 1.$$

Remark that $[X, Y]^\star = [X^\star, Y^\star]$ and $\{\lambda_X, \lambda_Y\} = \lambda_{[X,Y]}$ for all $X, Y \in \mathfrak{g}_{RC}$ (where $\{\cdot, \cdot\}$ is the Poisson bracket associated with ω). It turns out that the "ax+b" Lie group $\mathbb{S}$ corresponds to the Iwasawa component of $SL_2(\mathbb{R})$, i.e. $\mathfrak{s} = \mathrm{span}_{\mathbb{R}}\{H, E\}$ and thus $\mathbb{S} := \exp(\mathrm{span}_{\mathbb{R}}\{H, E\})$. Within the coordinate system

$$\mathfrak{s} \simeq \mathbb{R}^2 \to \mathbb{S} : (a, \ell) \mapsto \exp(aH)\exp(\ell E), \tag{1.24}$$

the group law of $\mathbb{S}$ reads

$$(a, \ell)(a', \ell') = (a + a', e^{-2a'}\ell + \ell').$$

This group acts locally on V by matrix multiplications. Indeed, for

$$(a, \ell) := \begin{pmatrix} e^a & \ell e^a \\ 0 & e^{-a} \end{pmatrix} \in \mathbb{S}$$

(where $\mathbb{S}$ is viewed as a subgroup of $SL_2(\mathbb{R})$) and

$$(q,p) = \begin{pmatrix} q \\ p \end{pmatrix} \in V,$$

we have

$$(a,\ell).(q,p) = \begin{pmatrix} e^a & \ell e^a \\ 0 & e^{-a} \end{pmatrix} . \begin{pmatrix} q \\ p \end{pmatrix} = \left((q+p\ell)e^a, pe^{-a}\right).$$

We denote by $\mathcal{O}$ the $\mathbb{S}$-orbit of the element $(0,1)$; so

$$\mathcal{O} = \mathbb{S}.(0,1) = \{(\ell e^a, e^{-a}) \mid a,\ell \in \mathbb{R}\}, \tag{1.25}$$

which corresponds exactly to the upper half-plane of V (the action is simply transitive). Setting

$$\underline{\phi} : \mathbb{S} \to \mathcal{O} : (a,\ell) \mapsto \begin{pmatrix} e^a & \ell e^a \\ 0 & e^{-a} \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\ell e^a, e^{-a}), \tag{1.26}$$

we obtain a symplectic form $\omega^{\mathbb{S}} := \underline{\phi}^\star\, \omega|_{\mathcal{O}} = da \wedge d\ell$ and

$$\underline{\phi}((a,\ell)(a',\ell')) = (a,\ell).\underline{\phi}(a',\ell').$$

The simple transitivity of the action of $\mathbb{S}$ on $\mathcal{O}$ leads to a $\mathbb{S}$-equivariant identification between $(\mathcal{O}, \omega|_{\mathcal{O}})$ and $(\mathbb{S}, \omega^{\mathbb{S}})$.

If we denote by $\lambda^{RC} : \mathfrak{g}_{RC} \to C^\infty(\mathbb{S})$ the transported moment map, i.e. $\lambda^{RC}_X := \phi^\star\,(\lambda_X|_{\mathcal{O}})$ for $X \in \mathfrak{g}_{RC}$, we get the following expressions:

$$\lambda^{RC}_H = \ell,\ \lambda^{RC}_E = \frac{1}{2}e^{-2a},\ \lambda^{RC}_F = -\frac{1}{2}\ell^2 e^{2a},\ \lambda^{RC}_Q = e^{-a},\ \lambda^{RC}_P = -\ell e^a,\ \lambda^{RC}_Z = 1. \tag{1.27}$$

The associated fundamental vector fields in coordinates (a,ℓ) read

$$\begin{aligned} &H^\star = -\partial_a,\ E^\star = -e^{-2a}\partial_\ell,\ F^\star = \ell e^{2a}\left(\partial_a - \ell\partial_\ell\right),\ Q^\star = -e^{-a}\partial_\ell, \\ &P^\star = e^a\left(\partial_a - \ell\partial_\ell\right),\ Z^\star = 0. \end{aligned} \tag{1.28}$$

Consider now the algebra $C^\infty(\mathbb{S})$. Of course, $\mathfrak{s}$ acts on this algebra by derivations, under the identification between $\mathfrak{s}$ and $\Gamma^\infty(T\mathbb{S})^{\mathbb{S}}$, the space of left invariant vector fields. A straightforward computation leads to the following expression for the two generators $\tilde{H}$ and $\tilde{E}$ of $\Gamma^\infty(T\mathbb{S})^{\mathbb{S}}$:

$$\tilde{H}_{(a,\ell)} = \partial_a - 2\ell\partial_\ell,\ \tilde{E}_{(a,\ell)} = \partial_\ell. \tag{1.29}$$

The elements of $\mathcal{U}(\mathfrak{s})$ then act on $C^\infty(\mathbb{S})$ as left invariant differential operators. Moreover, this action is projective in the sense of Definition 1.2.7, the element $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 = 0$ being simply implemented by $\Omega := 0 \in C^\infty(\mathbb{S})$. Since $[H, E] = 2E$, we set $X := \tilde{E}$ and $Y := \frac{1}{2}\tilde{H}$ in order to get the relation $[Y, X] = X$. This allows to consider Connes-Moscovici's Rankin-Cohen formal deformation (1.21) with $\mathcal{A} := C^\infty(\mathbb{S})$.

Proposition 1.2.24. The deformation $\star_\nu^{RC} := \sum_{n=0}^{+\infty} \nu^n RC_n^{red}$ defines a star-product on the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

Proof. For $\varphi, \psi \in C^\infty(\mathbb{S})$, we compute easily that

$$RC_0^{red}(\varphi, \psi) = \varphi\psi.$$

Moreover,

$$\begin{aligned} RC_1^{red}(\varphi, \psi) - RC_1^{red}(\psi, \varphi) &= 4\left(Y(\varphi)X(\psi) - Y(\psi)X(\varphi)\right) \\ &= 2\left(\tilde{H}(\varphi)\tilde{E}(\psi) - \tilde{E}(\varphi)\tilde{H}(\psi)\right) \\ &= 2\left(\partial_a\varphi\partial_\ell\psi - \partial_\ell\varphi\partial_a\psi\right) \\ &= 2\left\{\varphi, \psi\right\}, \end{aligned}$$

with $\{\cdot, \cdot\}$ associated with $\omega^{\mathbb{S}}$. In addition to the properties of $\star_\nu^{RC}$ showed by Connes and Moscovici in their Theorem 1.2.12, it implies that $\star_\nu^{RC}$ defines a star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ in the sense of Definition 1.2.16. ■

Definition 1.2.25. The star-product $\star_\nu^{RC}$ on the symplectic manifold $(\mathbb{S}, \omega^{\mathbb{S}} = da \wedge d\ell)$ is called the *Rankin-Cohen star-product* (or *RC star-product* in short).

We now introduce some additional notions about star-products.

Definition 1.2.26. Let $\star_\nu$ be a star-product on a symplectic manifold (M, ω). A *derivation* D of $\star_\nu$ is a linear operator on $C^\infty(M)[[\nu]]$ of the form

$$D = \sum_{k=0}^{+\infty} \nu^k D_k,$$

where for every k, D_k is a $\mathbb{C}[[\nu]]$-linear differential operator, with D satisfying the following relation for every $f, g \in C^\infty(M)[[\nu]]$:

$$D\left(f \star_\nu g\right) = Df \star_\nu g + f \star_\nu Dg.$$

The space of derivations of $\star_\nu$ will be denoted by $\mathrm{Der}\,(\star_\nu)$. It is endowed with the bracket $[\cdot,\cdot]$ given for $D, D' \in \mathrm{Der}\,(\star_\nu)$ by

$$[D, D'] = D \circ D' - D' \circ D.$$

Definition 1.2.27. For $\star_\nu$ a star-product on a symplectic manifold (M,ω), we define the $\star_\nu$-commutator $[\cdot,\cdot]_{\star_\nu}$ by

$$[\varphi,\psi]_{\star_\nu} := \varphi \star_\nu \psi - \psi \star_\nu \varphi,$$

for $\varphi, \psi \in C^\infty(M)[[\nu]]$.

Lemma 1.2.28. For $\star_\nu$ a star-product on a symplectic manifold (M,ω) and $\varphi \in C^\infty(M)[[\nu]]$ the map

$$\begin{aligned}[\varphi,\cdot]_{\star_\nu} : C^\infty(M)[[\nu]] &\to C^\infty(M)[[\nu]] \\ \psi &\mapsto [\varphi,\psi]_{\star_\nu}\end{aligned}$$

is a derivation of the star-product $\star_\nu$.

Proof. We have

$$\begin{aligned}[\varphi,\psi]_{\star_\nu} \star \phi + \psi \star_\nu [\varphi,\phi]_{\star_\nu} &= (\varphi \star_\nu \psi - \psi \star_\nu \varphi) \star_\nu \phi + \psi \star_\nu (\varphi \star_\nu \phi - \phi \star_\nu \varphi) \\ &= \varphi \star_\nu \psi \star_\nu \phi - \psi \star_\nu \phi \star_\nu \varphi \\ &= [\varphi, \psi \star_\nu \phi]_{\star_\nu}.\end{aligned}$$

■

Remark 1.2.29. In the literature, a derivation of a star-product $\star_\nu$ on a symplectic manifold (M,ω) implemented by an element $\varphi \in C^\infty(M)[[\nu]]$, i.e. of the form $[\varphi,\cdot]_{\star_\nu}$, is called an *inner derivation*.

Definition 1.2.30 ([38], Definition 64)**.** A star-product $\star_\nu$ on a symplectic manifold (M,ω) is said to be *invariant* under an action τ of a Lie group G on M, if for all $g \in G$, the diffeomorphism τ_g is a *symmetry* of $\star_\nu$, i.e. it satisfies

$$(\tau_g)^\star \varphi \star_\nu (\tau_g)^\star \psi = \tau_g(\varphi \star_\nu \psi),$$

for all $\varphi, \psi \in C^\infty(M)[[\nu]]$. In this case, we will say that $\star_\nu$ is G-invariant and that G is the *symmetry group* of $\star_\nu$.
Similarly, the star-product is said to be *invariant* under an action of $\mathfrak{g} = \mathrm{Lie}(G)$ on M (that is the data of a homomorphism $\mathfrak{g} \to \Gamma^\infty(TM) : X \mapsto X^\star$) if each

$X^\star$ is a derivation of $\star_\nu$. In that case, we will say that $\star_\nu$ is $\mathfrak{g}$-invariant and that $\mathfrak{g}$ is the *symmetry algebra* of $\star_\nu$.

In [25], the authors show the following result concerning the Rankin-Cohen deformation on $\mathbb{S}$.

Lemma 1.2.31 ([25], Lemma 3.3)**.** For all $X \in \mathfrak{g}_{RC}$, $\varphi \in C^\infty(\mathbb{S})[[\nu]]$ and for all $n \neq 1$,

$$RC_n^{red}\left(\lambda_X^{RC}, \varphi\right) - RC_n^{red}\left(\varphi, \lambda_X^{RC}\right) = 0.$$

This lemma implies that the star-product $\star_\nu^{RC}$ is $\mathfrak{g}_{RC}$-invariant. Indeed, for $X \in \mathfrak{g}_{RC}$ and $\varphi \in C^\infty(\mathbb{S})[[\nu]]$,

$$\begin{aligned}\frac{1}{2\nu}\left[\lambda_X^{RC}, \varphi\right]_{\star_\nu^{RC}} &= \frac{1}{2\nu}\sum_{n=0}^{+\infty}\nu^n\left(RC_n^{red}\left(\lambda_X^{RC}, \varphi\right) - RC_n^{red}\left(\varphi, \lambda_X^{RC}\right)\right)\\ &= \frac{1}{2}\left(RC_1^{red}\left(\lambda_X^{RC}, \varphi\right) - RC_1^{red}\left(\varphi, \lambda_X^{RC}\right)\right)\\ &= \{\lambda_X, \varphi\},\end{aligned}$$

where the last line comes from the proof of Proposition 1.2.24. Thanks to the property of the moment map, we get

$$\frac{1}{2\nu}\left[\lambda_X^{RC}, \varphi\right]_{\star_\nu^{RC}} = X^\star(\varphi),$$

with $X^\star$ the fundamental vector field associated with X (remark that both Poisson bracket and derivation are naturally extended $\mathbb{C}[[\nu]]$-linearly to $C^\infty(\mathbb{S})[[\nu]]$). But this means exactly that $X^\star$ is both a derivation of the classical pointwise product and a derivation of $\star_\nu^{RC}$. The star-product $\star_\nu^{RC}$ is then $\mathfrak{g}_{RC}$-invariant in the sense of Definition 1.2.30. Under the identification between $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ and $(\mathcal{O}, \omega|_{\mathcal{O}}) \subset (V, \omega)$ given by $\underline{\phi}$, the (transported) star-product $\star_\nu^{RC}$ then defines a (local) $\mathfrak{g}_{RC}$-invariant star-product on (a restriction of) the symplectic flat space (V, ω). But in [37], Simone Gutt showed that this invariance characterize the Moyal star-product $\star_\nu^0$, which implies that $\star_\nu^{RC}$ corresponds via $\underline{\phi}$ to the restriction on $\mathcal{O}$ of $\star_\nu^0$.

Proposition 1.2.32 ([25], Proposition 3.4)**.** The reduced Connes-Moscovici-Rankin-Cohen deformation realized on $\mathcal{O} \subset V$ coincides with the restriction to $\mathcal{O}$ of the Moyal star-product on (V, ω).

More explicitly, if we restrict the Moyal star-product on (the open set) $(\mathcal{O}, \omega|_{\mathcal{O}})$ by setting $\star_\nu^{\mathcal{O}} := \star_\nu^0\big|_{C^\infty(\mathcal{O})\times C^\infty(\mathcal{O})}$, the star-product $\star_\nu^{RC}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is then given by $\star_\nu^{\mathrm{RC}} = \underline{\phi}^\star\left(\star_\nu^{\mathcal{O}}\right)$, which establishes a direct (and unexpected) link between a particular case of Connes-Moscovici's Rankin-Cohen deformation

and the Moyal star-product. Moreover, this construction connects the work of Connes and Moscovici to the work of Giaquinto and Zhang, who presented in [36] the following deformation formula.

Theorem 1.2.33 ([36], Theorem 2.20)**.** Let X, Y be two generators of $\mathfrak{s} = \mathrm{Lie}(\mathbb{S})$, satisfying $[Y, X] = X$. Then, the following expression defines a universal deformation formula of the Hopf algebra associated with $\mathfrak{s}$:

$$F := \sum_{k=0}^{+\infty} \frac{t^k}{k!} F_k = 1 \otimes 1 + tX \wedge Y + \frac{t^2}{2!} \left(X^2 \otimes Y_2 - 2XY_1 \otimes XY_1 + Y_2 \otimes X^2 \right) + \dots,$$

where $F_k := \sum_{r=0}^{k} (-1)^r \binom{k}{r} X^{k-r} Y_r \otimes X^r Y_{k-r}$, $X \wedge Y := X \otimes Y - Y \otimes X$ and $Y_r := Y(Y+1)\dots(Y+r-1)$.

In [25], authors remarked that the above formula admits a realization in terms of the Moyal formula.

Proposition 1.2.34 ([25], Proposition 3.1)**.** The element F of the previous theorem can be realized by the standard Moyal product.

Proof. It suffices to consider the space $\mathbb{R} \times \mathbb{R}_0+$ (in coordinates (x, y)) on which X and Y act as $\frac{1}{y}\frac{\partial}{\partial x}$ and $-y\frac{\partial}{\partial y}$, respectively (see [25] for details). Then, using the identity

$$Y_r = (-y)^r \frac{\partial^r}{\partial y^r},$$

a straightforward computation shows that F in this representation is equal to the Moyal product. ■

The previous proposition together with Proposition 1.2.32 then imply that there exists an equivalence between the deformation of Giaquinto and Zhang and that of Connes and Moscovici.

As explained in [25], we can also give the explicit link between $\star_\nu^{RC}$ and Fedosov's construction. Indeed, the Lie group $\mathbb{S}$ acts on $\mathbb{R}$ by

$$\exp(aH + \ell E).x := e^{2a}x + \ell e^a,$$

where $x \in \mathbb{R}$. If we lift this action to the cotangent bundle $T^\star\mathbb{R}$ which we identify to $\mathbb{R}^2$, we get

$$\exp(aH + \ell E).(x_1, x_2) = (e^{2a}x_1 + \ell e^a, e^{-2a}x_2),$$

where $(x_1, x_2) \in \mathbb{R}^2$. It turns out that the $\mathbb{S}$-orbit $\tilde{\mathcal{O}}$ of $\tilde{o} := (0,1) \in \mathbb{R}^2$ under this action identifies as a $\mathbb{S}$-homogeneous space to $\mathcal{O} \subset V$ via the map

$$\tilde{\phi} : \mathcal{O} \to \tilde{\mathcal{O}} : g.(0,1) \mapsto g.\tilde{o}.$$

In terms of (q,p)-coordinates, this gives

$$\tilde{\phi}(q,p) = \left(\frac{q}{2p}, p^2\right).$$

The left invariant vector fields $\tilde{H}$ and $\tilde{E}$ on $\mathbb{S}$ may be transported to $\tilde{\mathcal{O}}$ via $\tilde{\phi} \circ \underline{\phi}$, which gives the expressions

$$\tilde{H} = -2x_2\partial_{x_2},\ \tilde{E} = \frac{1}{x_2}\partial_{x_1}.$$

We then define

$$Y := \frac{1}{2}\tilde{H},\ X := \tilde{E},$$

which allows to consider the reduced Connes-Moscovici-Rankin-Cohen deformation realized on $\tilde{\mathcal{O}}$.
Now, we transport the restriction to $\mathcal{O}$ of the standard symmetric flat connection on V, which we denote by $\nabla^{\mathcal{O}}$ (satisfying $\nabla^{\mathcal{O}}_{\partial_q}\partial_q = \nabla^{\mathcal{O}}_{\partial_p}\partial_p = \nabla^{\mathcal{O}}_{\partial_q}\partial_p = 0$), to $\tilde{\mathcal{O}}$ via $\tilde{\phi}$, i.e. we consider $\nabla^{\tilde{\mathcal{O}}} := \tilde{\phi}\left(\nabla^{\mathcal{O}}\right)$. This connection is determined by the relations

$$\nabla^{\tilde{\mathcal{O}}}_{\partial_{x_1}}\partial_{x_1} = 0,\ \nabla^{\tilde{\mathcal{O}}}_{\partial_{x_1}}\partial_{x_2} = \frac{1}{2x_2}\partial_{x_1},\ \nabla^{\tilde{\mathcal{O}}}_{\partial_{x_2}}\partial_{x_2} = -\frac{1}{2x_2}\partial_{x_2}.$$

It turns out that this connection is symplectic and under the identification of $\tilde{\mathcal{O}}$ with $\mathbb{R} \times \mathbb{R}_0^+$, we can then apply Fedosov's method to construct a star-product on $\left(\mathbb{R} \times \mathbb{R}_0^+, \tilde{\omega} := dx \wedge dy\right)$. Bieliavsky, Tang and Yao then showed that this Fedosov star-product coincides with the reduced Connes-Moscovici-Rankin-Cohen deformation realized on $\tilde{\mathcal{O}}$.

Proposition 1.2.35 ([25])**.** The reduced Connes-Moscovici-Rankin-Cohen deformation realized on $\tilde{\mathcal{O}}$ is identical to Fedosov's construction of the star-product on $\left(\tilde{\mathcal{O}}, \tilde{\omega}\right)$ using the symplectic connection $\nabla^{\tilde{\mathcal{O}}}$.

Before introducing extra properties of Connes-Moscovici's Rankin-Cohen deformation, we recall some concepts about star-products.

Definition 1.2.36 ([41], Definition 2.3.10)**.** Let (M, ω) be a symplectic manifold together with a Hamiltonian action of a Lie group G with associated Lie algebra

$\mathfrak{g}$, implemented by the moment map

$$\lambda : (\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\}),$$

and let $\star_\nu$ be a star-product on (M,ω). A *quantum moment map* is a linear homomorphism

$$\Lambda : (\mathfrak{g}, [\cdot,\cdot]) \to \left(C^\infty(M)[[\nu]], \frac{1}{2\nu}[\cdot,\cdot]_{\star_\nu}\right)$$
$$X \mapsto \Lambda(X) =: \Lambda_X,$$

such that

$$\Lambda_X = \lambda_X + O(\nu),$$

with $O(\nu) := \left\{\sum_{k=1}^{+\infty} \nu^k f \mid f \in C^\infty(M)\right\}$. When there exists such a map, the star-product $\star_\nu$ is said to be *covariant* w.r.t. the map Λ, or Λ-*covariant* in short.

Definition 1.2.37. Let (M,ω) be a symplectic manifold together with a Hamiltonian action of a Lie group G with associated Lie algebra $\mathfrak{g}$. A star-product $\star_\nu$ on (M,ω) is said to be G-*strongly invariant* if it is both G-invariant and Λ-covariant for a certain quantum moment map Λ.
Similarly, the star-product $\star_\nu$ is said to be $\mathfrak{g}$-*strongly invariant* if it is $\mathfrak{g}$-invariant and Λ-covariant for a certain quantum moment map Λ.

Definition 1.2.38. Let (M,ω) be a symplectic manifold, $\star_\nu$ a star-product on (M,ω) and $\mathfrak{g}$ a Lie algebra acting on M (i.e. with the data of a homomorphism $\mathfrak{g} \to \Gamma^\infty(TM) : X \mapsto X^\star$). The star-product $\star_\nu$ is said to be $\mathfrak{g}$-*Hamiltonian* if there exists a Lie algebra homomorphism

$$\lambda : (\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\})$$
$$X \mapsto \lambda(X) =: \lambda_X$$

(where $\{\cdot,\cdot\}$ is the Poisson bracket associated with ω), such that for every $X \in \mathfrak{g}$,

$$\frac{1}{2\nu}[\lambda_X, \cdot]_{\star_\nu} = X^\star.$$

Definition 1.2.39. Let (G,ω^G) be a symplectic Lie group. A *twist* on (G,ω^G) is a left invariant star-product (i.e. invariant w.r.t. the left action of G on itself) and Hamiltonian w.r.t. the right invariant vector fields.

Proposition 1.2.40. The star-product $\star_\nu^{RC}$ is $\mathfrak{g}_{RC}$-strongly invariant.

Proof. For this proof we use the interpretation of $\star_\nu^{\mathrm{RC}}$ in terms of the standard Moyal star-product, i.e. $\star_\nu^{\mathrm{RC}} = \underline{\phi}^\star\left(\star_\nu^{\mathcal{O}}\right)$. A straightforward computation yields the following expression for the star-commutator of the Moyal star-product:

$$\frac{1}{2\nu}[f,g]_{\star_\nu^0} = \frac{1}{\nu}\sum_{\substack{k\geq 1\\ k \text{ odd}}}\frac{\nu^k}{k!}\sum_{j=0}^{\frac{k-1}{2}}(-1)^j\begin{pmatrix} k \\ j\end{pmatrix}\left(\partial_q^{k-j}\partial_p^j f\partial_q^j\partial_p^{k-j}g - \partial_q^j\partial_p^{k-j}f\partial_q^{k-j}\partial_p^j g\right), \tag{1.30}$$

where $f,g\in C^\infty\left(\mathbb{R}^2\right)[[\nu]]$ with $\mathbb{R}^2$ in coordinates $\{(q,p)\}$.

For every $X\in\mathfrak{g}_{RC}$, we get

$$\begin{aligned}\frac{1}{2\nu}\left[\lambda_X^{RC},\varphi\right]_{\star_\nu^{RC}} &= \underline{\phi}^\star\left(\frac{1}{2\nu}\left[\lambda_X,\left(\underline{\phi}^{-1}\right)^\star\varphi\right]_{\star_\nu^{\mathcal{O}}}\right)\\ &= \underline{\phi}^\star\left\{\lambda_X,\left(\underline{\phi}^{-1}\right)^\star\varphi\right\}\\ &= \left\{\lambda_X^{RC},\varphi\right\}\\ &= X^\star(\varphi).\end{aligned}$$

Thus, every fundamental vector field is a derivation of the star-product $\star_\nu^{RC}$, which implies its $\mathfrak{g}_{RC}$-invariance. Using formula (1.30), we have for $X_1,X_2\in\mathfrak{g}_{RC}$:

$$\frac{1}{2\nu}\left[\lambda_{X_1}^{RC},\lambda_{X_2}^{RC}\right]_{\star_\nu^{RC}} = \left\{\lambda_{X_1}^{RC},\lambda_{X_2}^{RC}\right\} = \lambda_{[X_1,X_2]}^{RC},$$

which leads to the λ^{RC}-covariance of $\star_\nu^{RC}$. Remark that λ^{RC} is clearly a quantum moment map, which completes the proof. ∎

Proposition 1.2.41. The star-product $\star_\nu^{\mathrm{RC}}$ is a twist on $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$.

Proof. We denote the right action of $\mathbb{S}$ (on itself) by

$$R_{x_0}:\mathbb{S}\to\mathbb{S}:x\mapsto R_{x_0}(x)=xx_0, \tag{1.31}$$

(for $x_0\in\mathbb{S}$) and we define the right invariant vector field $\underline{X}$ associated with $-X\in\mathfrak{s}=Lie(\mathbb{S})$ by

$$\underline{X}_x := R_{x*e}(-X) = \left.\frac{d}{dt}\right|_0\exp(-tX).x,$$

for $x\in\mathbb{S}$. The expression of $\underline{H}$ and $\underline{E}$ are given by

$$\underline{H}_{(a,\ell)} = -\partial_a,\qquad \underline{E}_{(a,\ell)} = -e^{-2a}\partial_\ell,$$

and we obtain a Lie algebra homomorphism

$$\mathfrak{s} \to \Gamma^\infty(T\mathbb{S}) : X \mapsto \underline{X}.$$

Remark that the right invariant vector fields are the same as the fundamental vector fields associated with $\omega^{\mathbb{S}}$, thanks to the fact that

$$\underline{X}_x = \left.\frac{d}{dt}\right|_0 L_{\exp(-tX)}(x),$$

where

$$L_{x_0} : \mathbb{S} \to \mathbb{S} : x \mapsto L_{x_0}(x) = x_0 x,$$

(for $x_0 \in \mathbb{S}$) denotes the left action of $\mathbb{S}$ (on itself). Thanks to the previous proposition, we get directly

$$\frac{1}{2\nu}\left[\lambda_X^{RC}, \varphi\right]_{\star_\nu^{RC}} = \underline{X}(\varphi),$$

for all $X \in \mathfrak{s}$. This means that the star-product $\star_\nu^{RC}$ is $\mathfrak{s}$-Hamiltonian. Remark that the (left) $\mathfrak{s}$-invariance of the star-product $\star_\nu^{RC}$ corresponds to the fact that right invariant vector fields are derivations of $\star_\nu^{RC}$. By simple connectedness of $\mathbb{S}$, we get directly the (left) $\mathbb{S}$-invariance of $\star_\nu^{RC}$, which completes the proof. ■

Remark 1.2.42. The form $\omega^{\mathbb{S}} = da \wedge d\ell$ being left invariant, the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is actually a *symplectic Lie group*.

As we have seen in this first chapter, the construction of Connes and Moscovici based on Rankin-Cohen brackets (as well as the construction of Giaquinto and Zhang) leads to a twist on the symplectic Lie group $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. Under the $\mathbb{S}$-equivariant identification (1.26), this deformation corresponds to a restriction of the 2-dimensional Moyal star-product to the upper half-plane of $\mathbb{R}^2$. Moreover, the local symmetries of this star-product (i.e. its Lie algebra of invariance) are implemented by the Lie algebra $\mathfrak{g}_{RC}$. It turns out that there exists another construction of twist on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, admitting another type of (local) symmetries. This other star-product belongs to a far more general family of star-products constructed by Pierre Bieliavsky and Victor Gayral in [20]. In this book, authors have developed a deformation quantization method for actions of Kählerian Lie groups. The next chapter is devoted to the presentation of their work in the particular case of the affine group of the real line.

Chapter 2

Bieliavsky-Gayral star-products and non-formal deformation quantization

In this chapter, we present another type of star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, obtained by Pierre Bieliavsky and Victor Gayral, constituting a particular case of their far more general construction developed in [20].

2.1 The formal version

2.1.1 Symplectic symmetric space structure

Definition 2.1.1 ([10], Definition 1.1)**.** A *symplectic symmetric space* is a triple (M, s, ω) where

(i) M is a connected smooth manifold,

(ii) s is a smooth map

$$s : M \times M \to M : (x, y) \mapsto s_x(y) := s(x, y),$$

such that for every $x \in M$, the partial map $s_x : M \to M$ is an involutive diffeomorphism admitting x as isolated fixed point (the diffeomorphism s_x is called the *symmetry at point* x) and for all points $x, y \in M$, the relation $s_x \circ s_y \circ s_x = s_{s_x(y)}$ holds.

(iii) ω is a closed and non-degenerate 2-form on M, invariant under the symmetries, i.e. satisfying

$$s_x^\star \omega = \omega, \quad \forall x \in M.$$

A morphism between two symplectic symmetric spaces is defined as a symplectomorphism that intertwines the symmetries. For a given symplectic symmetric space (M, s, ω), we denote as $\mathrm{Aut}(M, s, \omega)$ its automorphism group.

As in chapter 1, let us consider the symplectic Lie group $\mathbb{S}$ (in (a, ℓ)-coordinates (1.24)) endowed with the symplectic structure $\omega^{\mathbb{S}} = da \wedge d\ell$ and the group law given by

$$L_{(a,\ell)}(a', \ell') := (a, \ell).(a', \ell') = \left(a + a', e^{-2a'}\ell + \ell'\right) \tag{2.1}$$

(where L denotes the left action as in the previous chapter). We introduce the map (cf. [13])

$$s_{(a,\ell)}(a', \ell') = (2a - a', 2\cosh(2a - 2a')\ell - \ell').$$

Proposition 2.1.2 ([20], Proposition 3.9)**.** The triple $(\mathbb{S}, s, \omega^{\mathbb{S}})$ defines a symplectic symmetric space.

Proposition 2.1.3 ([20], Proposition 6.1)**.** The smallest subgroup $G(M)$ of $\mathrm{Aut}(M, s, \omega)$ that is stable under σ and that acts transitively on M is a Lie subgroup of $\mathrm{Aut}(M, s, \omega)$. It coincides with the group generated by products of an even number of symmetries:

$$G(M) = \mathrm{gr}\left\{s_x \circ s_y \mid x, y \in M\right\}.$$

The group $G(M)$ is called the *transvection group of* M.

Definition 2.1.4 ([20], Definition 6.3)**.** A *symplectic involutive Lie algebra* (shortly a *siLa*) is a triple $(\mathfrak{g}, \sigma, \varpi)$ where $(\mathfrak{g}, \sigma)$ is an *involutive Lie algebra* (shortly an *iLa*) i.e. $\mathfrak{g}$ is a finite dimensional real Lie algebra, σ is an involutive automorphism of $\mathfrak{g}$, and where $\varpi \in \Lambda^2\mathfrak{g}^\star$ is a Chevalley 2-cocycle on $\mathfrak{g}$ (valued in the trivial representation on $\mathbb{R}$) such that, denoting by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the ± 1-eigenspace decomposition of $\mathfrak{g}$ associated with the involution $\sigma =: id_{\mathfrak{k}} \oplus (-id_{\mathfrak{p}})$, the cocycle ϖ contains $\mathfrak{k}$ in its radical and restricts to $\mathfrak{p} \times \mathfrak{p}$ as a

non-degenerate 2-form. A morphism between two such siLa's is a Lie algebra homomorphism which intertwines both involutions and 2-cocycles.

Definition 2.1.5 ([20], Definition 6.4)**.** A *transvection symplectic triple* is a siLa $(\mathfrak{g}, \sigma, \varpi)$ where

(i) the action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful, and

(ii) $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

Proposition 2.1.6 ([20], Proposition 6.6)**.** There is a bijection between the isomorphism classes of simply connected symplectic symmetric spaces and the isomorphism classes of transvection symplectic triples.

We now come back to the group $\mathbb{S}$. It turns out[1] that the Lie algebra $\mathfrak{g}_{BG}$ of the transvection group[2] of $\mathbb{S}$ is a 1-dimensional split extension of two copies of $\mathbb{R}$:

$$\mathfrak{g}_{BG} = \mathbb{R}H \ltimes (\mathbb{R}E \oplus \mathbb{R}F)\,,$$

with table given by

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = 0,$$

which means that $\mathfrak{g}_{BG} = \mathfrak{so}(1,1) \ltimes \mathbb{R}^2$, the Poincaré algebra. The involution σ of $\mathfrak{g}_{BG}$ is given by

$$\sigma\left(aH + (xE \oplus yH)\right) = -aH + (yE \oplus xH)\,.$$

We have the ± 1-eigenspaces decomposition $\mathfrak{g}_{BG} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} := \{xE \oplus xF \mid x \in \mathbb{R}\}\,, \quad \mathfrak{p} := \mathbb{R}H + \{xE \oplus -xF \mid x \in \mathbb{R}\}\,.$$

Lastly, we define $\varpi \in \Lambda^2\mathfrak{g}^\star$ by

$$\varpi\left(H, E \oplus -F\right) = 1$$

and by zero everywhere else on $\mathfrak{g}_{BG} \times \mathfrak{g}_{BG}$. So ϖ is $\mathfrak{k}$-invariant and its restriction to $\mathfrak{p} \times \mathfrak{p}$ is non-degenerate. Moreover, $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and the action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful. Thus the siLa $(\mathfrak{g}_{BG}, \sigma, \varpi)$ is a transvection symplectic triple. We set $G_{BG} = \exp\{\mathfrak{g}_{BG}\} = SO(1,1) \ltimes \mathbb{R}^2$ and $K = \exp\{\mathfrak{k}\}$. We identify $\mathfrak{g}_{BG}$ with G_{BG} via the global chart:

$$aH + t_1E \oplus t_2F \mapsto \exp\{aH\}\exp\{t_1E \oplus t_2F\},$$

[1] We refer to [20], section 7.1 for more details.
[2] See Proposition 2.1.2.

where $a, t_1, t_2 \in \mathbb{R}$. The group law of G_{BG} in these coordinate chart reads

$$(a, t_1, t_2).(a', t_1', t_2') = \left(a + a', e^{-2a'}t_1 + t_1', e^{2a'}t_2 + t_2'\right),$$

and the inversion map is given by

$$(a, t_1, t_2)^{-1} = \left(-a, -e^{2a}t_1, -e^{-2a}t_2\right).$$

The involution σ is given by

$$\sigma(a, t_1, t_2) = (-a, t_2, t_1).$$

Now, we consider the following global coordinate system on G_{BG}/K:

$$G_{BG}/K \to \mathbb{R}^2 : (a, t_1, t_2)K \mapsto (a, t_1 - t_2).$$

From the formula $s_{gK}(g'K) = g\sigma\left(g^{-1}g'\right)K$, we obtain the following isomorphism of symplectic symmetric space.

Proposition 2.1.7 ([20], Proposition 7.2)**.** Under the identification $\mathbb{S} \simeq \mathbb{R}^2 \simeq G_{BG}/K$, the symplectic symmetric space underlying the siLa $(\mathfrak{g}_{BG}, \sigma, \varpi)$ is isomorphic the the symplectic symmetric space $(\mathbb{S}, s, \omega^{\mathbb{S}})$.

Under the identification $\mathbb{S} \simeq G_{BG}/K$, the action of G_{BG} on $\mathbb{S}$ reads

$$(a, t_1, t_2).(a_0, \ell_0) = \left(a + a_0, e^{-2a_0}t_1 - e^{2a_0}t_2 + \ell_0\right).$$

The fundamental vector fields associated with this action are given by

$$H^\star_{(a,\ell)} = -\partial_a,\ E^\star_{(a,\ell)} = -e^{-2a}\partial_\ell,\ F^\star_{(a,\ell)} = -e^{2a}\partial_\ell, \tag{2.2}$$

and the moment map $\lambda^{BG} : (\mathfrak{g}_{BG}, [\cdot,\cdot]) \to (C^\infty(\mathbb{S}), \{\cdot,\cdot\})$ (where the Poisson bracket is the one associated with $\omega^{\mathbb{S}}$) is then defined by the following expressions:

$$\lambda^{BG}_H = \ell,\ \lambda^{BG}_E = \frac{1}{2}e^{-2a},\ \lambda^{BG}_F = -\frac{1}{2}e^{2a}. \tag{2.3}$$

2.1.2 The formal star-product

We present here in a heuristic way the method originally used by Pierre Bieliavsky in [11] to give rise to a twist on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. This method is based on star-representation theoretical techniques and makes use of the covariance of the formal Moyal product (as we will see below). The main idea consists in constructing an equivalence (in the sense of Definition 1.2.18) such that the

transport of the Moyal star-product under this equivalence yields a (left) $\mathbb{S}$-invariant star-product. Recall that this left invariance is locally implemented by the fact that right invariant vector fields are derivations of the star-product.

Consider the standard Moyal star-product on $\mathbb{R}^2$ given by the expression (1.18), but expressed in terms of the coordinates (a, ℓ). Explicitly, if we denote this star-product by $\star_\nu^{\mathbb{S}}$, this gives

$$\varphi \star_\nu^{\mathbb{S}} \psi = \varphi\psi + \sum_{k=1}^{+\infty} \frac{\nu^k}{k!} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \partial_a^{k-j} \partial_\ell^j \varphi \partial_a^j \partial_\ell^{k-j} \psi, \tag{2.4}$$

for $\varphi, \psi \in C^\infty(\mathbb{S})$. The $\star_\nu^{\mathbb{S}}$-commutator is then given by

$$\begin{aligned} &\frac{1}{2\nu}[\varphi, \psi]_{\star_\nu^{\mathbb{S}}} = \\ &\frac{1}{\nu} \sum_{\substack{k \geq 1 \\ k \text{ odd}}} \frac{\nu^k}{k!} \sum_{j=0}^{\frac{k-1}{2}} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \left(\partial_a^{k-j} \partial_\ell^j \varphi \partial_a^j \partial_\ell^{k-j} \psi - \partial_a^j \partial_\ell^{k-j} \varphi \partial_a^{k-j} \partial_\ell^j \psi \right). \end{aligned} \tag{2.5}$$

Remark that for each $X, Y \in \mathfrak{g}_{BG}$, the linearity of $\lambda^{BG} : \mathfrak{g}_{BG} \to C^\infty(\mathbb{S})$ together with the fact that $\lambda_H^{BG}, \lambda_E^{BG}$ and λ_F^{BG} all depend on a single coordinate imply that all the terms in $k \geq 3$ of the star-commutator $\frac{1}{2\nu}[\lambda_X^{BG}, \lambda_Y^{BG}]_{\star_\nu^{\mathbb{S}}}$ are equal to 0. So it is simply given by

$$\begin{aligned} \frac{1}{2\nu}[\lambda_X^{BG}, \lambda_Y^{BG}]_{\star_\nu^{\mathbb{S}}} &= \partial_a \lambda_X^{BG} \partial_\ell \lambda_Y^{BG} - \partial_\ell \lambda_X^{BG} \partial_a \lambda_Y^{BG} \\ &= \left\{ \lambda_X^{BG}, \lambda_Y^{BG} \right\}. \end{aligned}$$

Thanks to the property of the moment map, we get the following relation:

$$\frac{1}{2\nu}[\lambda_X^{BG}, \lambda_Y^{BG}]_{\star_\nu^{\mathbb{S}}} = \lambda_{[X,Y]}^{BG}, \tag{2.6}$$

showing the λ^{BG}-covariance of the Moyal star-product (in (a, ℓ)-coordinates). Let us now define a map ρ in the following way:

$$\begin{aligned} \rho : \mathfrak{g}_{BG} &\to \mathrm{Der}\left(\star_\nu^{\mathbb{S}}\right) \\ X &\mapsto \rho_X := \frac{1}{2\nu} \left[\lambda_X^{BG}, \cdot\right]_{\star_\nu^{\mathbb{S}}}. \end{aligned}$$

Lemma 2.1.8. The map ρ defined above is a *representation* of $\mathfrak{g}_{BG}$ on the space $\mathrm{Der}\left(\star_\nu^{\mathbb{S}}\right)$.

Proof. The linearity comes from that of the moment map λ^{BG}. Moreover, using (2.6), we notice that for $X, Y \in \mathfrak{g}_{BG}$,

$$\begin{aligned}\rho_{[X,Y]} &= \frac{1}{2\nu}\left[\lambda^{BG}_{[X,Y]}, \cdot\right]_{\star^{\mathbb{S}}_\nu} \\ &= \frac{1}{2\nu}\left[\frac{1}{2\nu}\left[\lambda^{BG}_X, \lambda^{BG}_Y\right]_{\star^{\mathbb{S}}_\nu}, \cdot\right]_{\star^{\mathbb{S}}_\nu} \\ &= \left[\frac{1}{2\nu}\left[\lambda^{BG}_X, \cdot\right]_{\star^{\mathbb{S}}_\nu}, \frac{1}{2\nu}\left[\lambda^{BG}_Y, \cdot\right]_{\star^{\mathbb{S}}_\nu}\right] \\ &= [\rho_X, \rho_Y],\end{aligned}$$

which completes the proof. ■

From the formula (2.5), a straightforward computation leads to the following expressions:

$$\rho_H = -\partial_a,\ \rho_E = -\frac{e^{-2a}}{2\nu}\sinh(2\nu\partial_\ell),\ \rho_F = -\frac{e^{2a}}{2\nu}\sinh(2\nu\partial_\ell), \tag{2.7}$$

where

$$\sinh(2\nu\partial_\ell) := \sum_{k=0}^{+\infty} \frac{(2\nu)^{2k+1}}{(2k+1)!}\partial_\ell^{2k+1}.$$

If we want to interpret the $\star^{\mathbb{S}}_\nu$-representation ρ as a multiplicative representation, it seems appropriate to introduce the (formal) partial Fourier transform $\mathcal{F}$ in the ℓ-variable defined (for reasonable function $\varphi \in C^\infty(\mathbb{S})$) by

$$\mathcal{F}\varphi(a,\xi) = \int_{\mathbb{R}} e^{-i\xi\ell}\varphi(a,\ell)d\ell, \tag{2.8}$$

naturally extended $\mathbb{C}[[\nu]]$-linearly to $C^\infty(\mathbb{S})[[\nu]]$. We also denote as $\tilde{\mathbb{S}} = \{(a,\xi) \mid a,\xi \in \mathbb{R}\}$ the space of coordinates on which the partial Fourier transform of a function (on $\mathbb{S}$) is defined. Note that the (formal) inverse partial Fourier transform $\mathcal{F}^{-1}$ is then given by

$$\mathcal{F}^{-1}\psi(a,\ell) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell\xi}\psi(a,\xi)d\xi.$$

The interest of this transform is based on the identities

$$\mathcal{F}\circ\partial_\ell = \mathcal{M}_{i\xi}\circ\mathcal{F} \quad \text{and} \quad \mathcal{F}\circ\mathcal{M}_\ell = i\partial_\xi\circ\mathcal{F} \tag{2.9}$$

(satisfied for reasonable functions), where $\mathcal{M}_f$ denotes the multiplication operator by a function f.

From now on, we choose $\nu \in i\mathbb{R}_0$ (i.e. a purely imaginary non-zero number) and we introduce the smooth change of variables

$$\Psi_\nu : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}} : (a, \xi) \mapsto \left(a, \frac{\sinh(2i\nu\xi)}{2i\nu} \right), \tag{2.10}$$

with smooth inverse given by

$$\Psi_\nu^{-1} : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}} : (a, \xi) \mapsto \left(a, \frac{\operatorname{arcsinh}(2i\nu\xi)}{2i\nu} \right). \tag{2.11}$$

The operator T_ν defined as

$$T_\nu := \mathcal{F}^{-1} \circ \left(\Psi_\nu^{-1} \right)^\star \circ \mathcal{F} \tag{2.12}$$

then admits as explicit expression

$$T_\nu \varphi(a_0, \ell_0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\frac{\sinh(2i\nu\xi)}{2\nu} \ell_0 - i\xi\ell} \cosh(2i\nu\xi) \varphi(a_0, \ell) d\ell d\xi, \tag{2.13}$$

while its inverse $T_\nu^{-1} = \mathcal{F}^{-1} \circ \Psi_\nu^\star \circ \mathcal{F}$ is given by

$$T_\nu^{-1} \psi(a_0, \ell_0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{\sinh(2i\nu\xi)}{2\nu} \ell + i\xi\ell_0} \psi(a_0, \ell) d\ell d\xi. \tag{2.14}$$

Both integral expressions (2.13) and (2.14) are well-defined on the space of Schwartz functions on $\mathbb{S} = \{(a, \ell)\} \simeq \mathbb{R}^2$, which we denote by $\mathcal{S}(\mathbb{S})$ (that is, the space of Schwartz functions in variables (a, ℓ), see [11]). Moreover, they both admit a formal asymptotic expansion (denoted also by T_ν and T_ν^{-1} by a slight abuse of notations)

$$T_\nu : C^\infty(\mathbb{S})[[\nu]] \to C^\infty(\mathbb{S})[[\nu]] : \varphi \mapsto \varphi + \sum_{k=1}^{+\infty} \nu^k U_k \varphi \tag{2.15}$$

and

$$T_\nu^{-1} : C^\infty(\mathbb{S})[[\nu]] \to C^\infty(\mathbb{S})[[\nu]] : \varphi \mapsto \varphi + \sum_{k=1}^{+\infty} \nu^k \tilde{U}_k \varphi, \tag{2.16}$$

where for all $k \geq 1$, U_k and $\tilde{U}_k$ are in the space of differential operators on $\mathbb{S}$ (see [13, 16]). Using the decomposition (2.12) together with the expressions (2.7), we get[3]

$$T_\nu \circ \rho_H \circ T_\nu^{-1} = -\partial_a = H^\star.$$

[3]Recall that the expressions of the fundamental vector fields are given by (2.2).

Moreover,

$$\begin{aligned} T_\nu \circ \rho_E \circ T_\nu^{-1} &= \mathcal{F}^{-1} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{M}_{-\frac{e^{-2a}}{2\nu}\sinh(2i\nu\xi)} \circ \Psi_\nu^\star \circ \mathcal{F} \\ &= \mathcal{F}^{-1} \circ \mathcal{M}_{-e^{-2a}i\xi} \circ \mathcal{F} \\ &= -e^{-2a}\partial_\ell \\ &= E^\star, \end{aligned}$$

and a similar computation yields

$$T_\nu \circ \rho_F \circ T_\nu^{-1} = F^\star.$$

Remark that for $X \in \mathfrak{g}_{BG}$ and $\varphi \in C^\infty(\mathbb{S})[[\nu]]$,

$$\left(T_\nu \circ \rho_X \circ T_\nu^{-1}\right)\varphi = \frac{1}{2\nu}\left(T_\nu\left(\lambda_X \star_\nu^{\mathbb{S}} T_\nu^{-1}\varphi\right) - T_\nu\left(T_\nu^{-1}\varphi \star_\nu^{\mathbb{S}} \lambda_X\right)\right),$$

and thus

$$\begin{aligned} &\left(T_\nu \circ \rho_X \circ T_\nu^{-1}\right)\varphi = \\ &\frac{1}{2\nu}\left(T_\nu\left(T_\nu^{-1}\left(T_\nu\lambda_X\right) \star_\nu^{\mathbb{S}} T_\nu^{-1}\varphi\right) - T_\nu\left(T_\nu^{-1}\varphi \star_\nu^{\mathbb{S}} T_\nu^{-1}\left(T_\nu\lambda_X\right)\right)\right). \end{aligned}$$

Since T_ν leaves the constants unchanged (cf. [11]), $T_\nu\left(\star_\nu^{\mathbb{S}}\right)$ defines a star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, equivalent (in the sense of Definition 1.2.18) to $\star_\nu^{\mathbb{S}}$ by construction. Using the notation of Remark 1.2.19, we get

$$\begin{aligned} \left(T_\nu \circ \rho_X \circ T_\nu^{-1}\right)\varphi &= \frac{1}{2\nu}\left(T_\nu\lambda_X T_\nu\left(\star_\nu^{\mathbb{S}}\right)\varphi - \varphi T_\nu\left(\star_\nu^{\mathbb{S}}\right)T_\nu\lambda_X\right) \\ &= \frac{1}{2\nu}\left[T_\nu\lambda_X, \varphi\right]_{T_\nu\left(\star_\nu^{\mathbb{S}}\right)}. \end{aligned}$$

All the above development together with the linearity of ρ and T_ν lead to the following result.

Proposition 2.1.9. For all $X \in \mathfrak{g}_{BG}$,

$$\frac{1}{2\nu}\left[T_\nu\lambda_X, \cdot\right]_{T_\nu\left(\star_\nu^{\mathbb{S}}\right)} = X^\star.$$

Note that this proposition means exactly that the star-product $T_\nu\left(\star_\nu^{\mathbb{S}}\right)$ is $\mathfrak{g}_{BG}$-invariant.

Definition 2.1.10. The star-product $\star_\nu^{\mathrm{BG}} := T_\nu\left(\star_\nu^{\mathbb{S}}\right)$ on the symplectic manifold $(\mathbb{S}, \omega^{\mathbb{S}} = da \wedge d\ell)$ is called the *Bieliavsky-Gayral star-product* (or *BG star-product* in short).

Proposition 2.1.11. The star-product $\star_\nu^{BG}$ is strongly $\mathfrak{g}_{BG}$-invariant.

Proof. Remark first that Proposition 2.1.9 implies that every fundamental vector field is a derivation of $\star_\nu^{BG}$, which leads to its $\mathfrak{g}_{BG}$-invariance. So it remains to prove the $\mathfrak{g}_{BG}$-covariance. Let us define the map

$$\begin{aligned}\Lambda^{BG} : \mathfrak{g}_{BG} &\to C^\infty(\mathbb{S})[[\nu]] \\ X &\mapsto \Lambda_X^{BG} := \frac{1}{2\nu}\left[T_\nu\lambda_X^{BG}, \cdot\right]_{\star_\nu^{BG}}.\end{aligned}$$

Thanks to the asymptotic expansion (2.15) of T_ν, we get directly that Λ_X^{BG} is of the form

$$\Lambda_X^{BG} = \lambda_X^{BG} + \mathcal{O}(\nu).$$

Now, by construction of $\star_\nu^{BG}$, we have for $X, Y \in \mathfrak{g}_{BG}$,

$$\begin{aligned}\frac{1}{2\nu}\left[\Lambda_X^{BG}, \Lambda_Y^{BG}\right]_{\star_\nu^{BG}} &= \frac{1}{2\nu}\left[T_\nu\lambda_X^{BG}, T_\nu\lambda_Y^{BG}\right]_{T_\nu(\star_\nu^{\mathbb{S}})} \\ &= T_\nu\left(\frac{1}{2\nu}\left[\lambda_X^{BG}, \lambda_Y^{BG}\right]_{\star_\nu^{\mathbb{S}}}\right).\end{aligned}$$

The λ^{BG}-covariance of the Moyal star-product on $\mathbb{S}$ (see (2.6)) then yields

$$\begin{aligned}T_\nu\left(\frac{1}{2\nu}\left[\lambda_X^{BG}, \lambda_Y^{BG}\right]_{\star_\nu^{\mathbb{S}}}\right) &= T_\nu\lambda_{[X,Y]}^{BG} \\ &= \Lambda_{[X,Y]}^{BG}.\end{aligned}$$

The map Λ^{BG} is then a quantum moment map, which completes the proof. ■

Remark 2.1.12. Thanks to the connectedness of the group G_{BG}, the $\mathfrak{g}_{BG}$-invariance of $\star_\nu^{BG}$ is equivalent to its G_{BG}-invariance.

Proposition 2.1.13. The star-product $\star_\nu^{\mathrm{BG}}$ is a twist on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

Proof. We start by showing that for every $X \in \mathfrak{s}$,

$$T_\nu\lambda_X^{BG} = \lambda_X^{BG}.$$

The linearity of λ^{BG} and T_ν implies that it is sufficient to show the above identity only for $X = H$ and $X = E$. To this purpose, recall first that the partial Fourier transform given by (2.8) admits an extension to the space of tempered distributions $\mathcal{S}'(\mathbb{S})$ on $\mathbb{S}$ (we refer to [48] for more details). Since $\lambda_E^{BG}(a, \ell) = \frac{1}{2}e^{-2a}$, this function does not depend on the ℓ-variable. Hence,

thanks to the expression (2.13) of T_ν, we get directly

$$T_\nu \lambda_E^{BG} = \lambda_E^{BG} T_\nu 1 = \lambda_E^{BG},$$

using $T_\nu 1 = 1$ (which holds in $\mathcal{S}'(\mathbb{S})$). For $\lambda_H^{BG}(a,\ell) = \ell$, we have (again in $\mathcal{S}'(\mathbb{S})$)

$$\mathcal{F}\lambda_H^{BG}(a,\xi) = \int_{\mathbb{R}} e^{-i\xi\ell}\ell d\ell = 2\pi i \partial_\xi \delta_0(\xi),$$

where δ_0 is the Dirac delta function at 0. Due the the fact that the diffeomorphism of $\mathbb{R}$

$$\Psi_{\nu,2}(\xi) := \frac{\sinh(2i\nu\xi)}{2i\nu}$$

has a unique root in $\xi = 0$ and satisfies

$$\partial_\xi \Psi_{\nu,2}(0) = 1,$$

a straightforward computation leads to

$$T_\nu \lambda_H^{BG}(a,\ell) = \mathcal{F}^{-1}\left(\left(\Psi_\nu^{-1}\right)^\star \mathcal{F}\lambda_H^{BG}\right)(a,\ell) = \ell = \lambda_H^{BG}(a,\ell).$$

Using Proposition 2.1.9, we get for all $X \in \mathfrak{s}$ the following identity:

$$\frac{1}{2\nu}\left[\lambda_X^{BG}, \cdot\right]_{\star_\nu^{BG}} = X^\star.$$

Recall that as mentioned in the proof of Proposition 1.2.41, the right invariant vector fields on $\mathbb{S}$ coincide with the fundamental vector fields, i.e.

$$\underline{H} = H^\star \quad \text{and} \quad \underline{E} = E^\star.$$

Since the left $\mathfrak{s}$-invariance of $\star_\nu^{BG}$ corresponds to the fact that the right invariant vector fields are derivations of $\star_\nu^{BG}$, the simple connectedness of $\mathbb{S} = \exp(\mathfrak{s})$ then leads to the left $\mathbb{S}$-invariance of the star-product. That concludes the proof. ■

The previous proposition along with Proposition 1.2.41 show that $\star_\nu^{RC}$ and $\star_\nu^{BG}$ are both twists on the symplectic Lie group $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. In particular, they are both left $\mathbb{S}$-invariant (or $\mathbb{S}$-invariant in short) star-products (that is, invariant under the left action on $\mathbb{S}$) and we can then naturally wonder when such star-products are equivalent. The following result answers this question. It has been shown by Nest and Tsygan in [43], Deligne and then Bertelson, Cahen and Gutt in [9]. The proof makes use of cohomological arguments and relies on the fact that on a symplectic manifold, every star-product is equivalent to a Fedosov star-product.

Proposition 2.1.14 ([9])**.** Let $\star_\nu$ and $\star'_\nu$ be two star-products on a symplectic manifold (M,ω). If the second de Rham cohomology space $H^2_{dR}(M)$ is trivial, then $\star_\nu$ and $\star'_\nu$ are equivalent in the sense of Definition 1.2.18.

But there is more: by adapting this result in the case of left invariant star-product on a symplectic manifold (M,ω), Bertelson, Bieliavsky and Gutt proved in [8] that under some additional cohomological properties, two such invariant star-product are actually *G-equivalent*, in the following sense.

Definition 2.1.15 ([8], Definition 2.2)**.** Let (M,ω) be a symplectic manifold. Two G-invariant star-products $\star_\nu$ and $\star'_\nu$ on M are said to be *G-equivalent* if there exists a sequence $\{T_k\}_{k\geq 1}$ of G-invariant differential operators on $C^\infty(M)$ (extended naturally to $C^\infty(M)[[\nu]]$) vanishing on constants such that, defining

$$T := \mathrm{Id} + \sum_{k=1}^{+\infty} \nu^k T_k,$$

the following identity holds for all $f, g \in C^\infty(M)[[\nu]]$:

$$T\left(f \star_\nu g\right) = Tf \star'_\nu Tg.$$

Such an operator T is called an *G-equivalence*.

We can now enunciate the theorem concerning the G-equivalence of G-invariant star-products.

Theorem 2.1.16 ([8], Theorem 4.1)**.** Let (M,ω) be a symplectic manifold. The set of G-equivalence classes of G-invariant (formal) star-products on M is canonically parametrized by the set of sequences of elements belonging to the second de Rham cohomology space of the G-invariant de Rham complex on M.

This theorem is very useful in the case of the symplectic Lie group $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$ for which we consider the left action of $\mathbb{S}$ on itself. Indeed, we have the following result.

Lemma 2.1.17. Let us denote by $H^2_{dR}(\mathbb{S})^{\mathbb{S}}$ the (left) $\mathbb{S}$-invariant second de Rham cohomology space of $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$. We have

$$H^2_{dR}(\mathbb{S})^{\mathbb{S}} \simeq \{0\}\,.$$

Proof. Thanks to Theorem 15.1 of [28], it remains to study the second cohomology of the Lie algebra $\mathfrak{s}$ of $\mathbb{S}$. We will show that every Chevalley-Eilenberg 2-cocycle on $\mathfrak{s}$ is a coboundary. Recall that ω_2 is a 2-cocycle on $\mathfrak{s}$ if ω_2 is an

antisymmetric and bilinear map on $\mathfrak{s}$ (with values in $\mathbb{R}$), such that for every $X, Y, Z \in \mathfrak{s}$, the following identity holds:

$$\omega_2\left([X,Y],Z\right) + \omega_2\left([Z,X],Y\right) + \omega_2\left([Y,Z],X\right) = 0.$$

Let ω_2 be such a 2-cocycle and let us define

$$\begin{aligned}\omega_1 : \mathfrak{s} = \mathrm{span}_{\mathbb{R}}\{H,E\} &\to \mathbb{R}\\ aH + \ell E &\mapsto \omega_2\left(H, \ell E\right).\end{aligned}$$

The linearity of ω_1 comes directly from that of $\omega_2(H,\cdot)$. Moreover, denoting by $\delta\omega_1$ the Chevalley-Eilenberg coboundary of ω_1, we get for $X = a_X H + \ell_X E$ and $Y = a_Y H + \ell_Y E$:

$$\begin{aligned}\delta\omega_1(X,Y) &= \frac{1}{2}\omega_1\left([X,Y]\right)\\ &= \frac{1}{2}\omega_1\left(2(a_X\ell_Y - a_Y\ell_X)E\right)\\ &= \omega_2\left(H, (a_X\ell_Y - a_Y\ell_X)E\right),\end{aligned}$$

where we have used the relation $[H,E] = 2E$. But by bilinearity and antisymmetry of ω_2,

$$\omega_2(a_X H + \ell_X E, a_Y H + \ell_Y E) = (a_X\ell_Y - a_Y\ell_X)\omega_2(H,E),$$

which implies that

$$\delta\omega_1(X,Y) = \omega_2(X,Y).$$

The cocycle ω_2 is thus a coboundary, which completes the proof. ■

Thanks to Theorem 2.1.16, Lemma 2.1.17 implies that we can pass from any left invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ to another by using an equivalence which commutes with the left action of $\mathbb{S}$. In particular, this is the case for the two star-products $\star_\nu^{RC}$ and $\star_\nu^{BG}$. The first goal of this thesis will actually consist in finding the expression of such an equivalence between these two star-products. This work is interesting in itself, because it will allow to link explicitly $\star_\nu^{RC}$ and $\star_\nu^{BG}$, which are the only two known twists on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, both characterized by their symmetries, as witnessed by the following results.

Proposition 2.1.18 ([18], Lemma 5.3)**.** The formal star-product $\star_\nu^{RC}$ is (up to a redefinition of the deformation parameter) the only locally $\mathbb{S} \ltimes H_V$-invariant (with H_V the Heisenberg group associated with V) formal star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

Proof. This is a direct consequence of the fact that the Moyal star-product is completely determined by its strongly local invariance under the group generated by H, E, Q, P and Z and acting linearly on $\left(\mathbb{R}^2, dq \wedge dp\right)$. ■

Proposition 2.1.19 ([12], Proposition 2.2)**.** Every formal G_{BG}-invariant star-product $\sharp_\nu$ on $\mathbb{S}$ is of the form $T\left(\star_\nu^{BG}\right)$ with

$$T = \sum_{k=0}^{+\infty} \nu^k \tilde{T}_k,$$

where $\tilde{T}_k \in \mathbb{C}\left[\tilde{E}\right]$. That is, by an equivalence of the form

$$T = \mathcal{F}^{-1} \circ \mathcal{M}_{\mathcal{T}} \circ \mathcal{F},$$

where $\mathcal{T} = \sum_{k=0}^{+\infty} \mathcal{T}_k$ is some formal function on $\tilde{\mathbb{S}}$ depending only on the variable ξ (i.e. $\mathcal{T} \in C^\infty_{(\xi)}(\mathbb{R})[[\nu]]$) such that $\mathcal{T}_0 = 1$.

Proof. Let us first consider a star-product $\sharp^0_\nu$ in the same class of G_{BG}-invariance of $\star_\nu^{BG}$. We know by Theorem 2.1.16 that both are intertwined by an operator which is left invariant (i.e. an element of $\mathcal{U}(\mathfrak{s})[[\nu]]$), commuting with the F-action. Recall that

$$F^\star_{(a,\ell)} = \frac{1}{2} e^{2a} \partial_\ell.$$

If we take a look at the left invariant vector fields:

$$\tilde{H}_{(a,\ell)} = \partial_a - 2\ell\partial_\ell, \ \tilde{E}_{(a,\ell)} = \partial_\ell,$$

we see that each term of T must be polynomial in $\tilde{E}$ and must have the identity as term of order 0 in ν. This means that intertwining T with $\mathcal{F}$ yields a formal polynomial with constant leading term equal to 1. But one knows that the G_{BG}-equivalence classes of G_{BG}-invariant star-products on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ are canonically parametrized by the formal series with coefficients in the second G_{BG}-equivariant[4] de Rham cohomology space $H^2_{dR}(\mathbb{S})^{G_{BG}}$ (cf. Theorem 2.1.16). The only generator of this space is the symplectic form $\omega^{\mathbb{S}} = da \wedge d\ell$ (see [41], Lemma 2.3.13 for the proof of this fact) and thus a change of the G_{BG}-equivalence class remains to a change of the deformation parameter (see for instance [14]). ■

Actually, the real aim to find the explicit expression of a left invariant intertwiner (i.e. an equivalence that commutes with the left action) between $\star_\nu^{RC}$ and $\star_\nu^{BG}$ is motivated by a remarkable and very interesting property

[4] Or G_{BG}-invariant.

of the star-product $\star_\nu^{BG}$ which has not yet been mentioned. It turns out that there exists a reformulation of this star-product in terms of a so-called *oscillatory integral*. Within this setting and in a suitable functional framework, the star multiplication of two functions then gives rise to a *genuine function* (depending on the deformation parameter), rather than a *formal power series* in the deformation parameter. This is part of another field of study in geometrical deformation quantization called the *non-formal deformation quantization*. The next section is devoted to the non-formal aspect of the Bieliavsky-Gayral construction in the particular case of the symplectic Lie group $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$.

2.2 Non-formal aspects

Recall briefly that in classical mechanics, the space of observables of a dynamical system is described mathematically by a space of smooth functions on a symplectic manifold (M, ω). The points of the phase space M correspond to states of the system. The space of observables then forms naturally a *commutative* Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$. The dynamics of such a classical system is implemented by a Hamiltonian $H \in C^\infty(M)$. More precisely, the time evolution of an observable $t \in \mathbb{R} \mapsto f_t \in C^\infty(M)$ is given by the following equation:

$$\frac{d}{dt} f_t = \{H, f_t\}\,. \tag{2.17}$$

At the quantum level, the observables are described (in the Heisenberg representation) in terms of linear operators on a Hilbert space $\mathcal{H}$. The possible values of an observable correspond to the eigenvalues of the corresponding operator. The space of observables then forms a *noncommutative* algebra, endowed with the commutator of operators. In this quantum setting, the dynamics is described with a Hamiltonian H^q which is a linear operator, i.e. the time evolution of an observable $t \in \mathbb{R} \mapsto A_t$ is given by the following equation:

$$\frac{d}{dt} A_t = \frac{i}{\hbar}\left[H^q, A_t\right]. \tag{2.18}$$

2.2.1 Weyl quantization

Dynamics (2.17) and (2.18) naturally suggest that quantum mechanics may be derived from classical mechanics by means of a *quantization map*, that is a correspondence

$$W_\hbar : C^\infty(M) \to \mathcal{B}(\mathcal{H})$$

(where $\mathcal{B}(\mathcal{H})$ denotes the space of bounded operators on $\mathcal{H}$) mapping a classical observable to a quantum one, and satisfying the relation

$$W_\hbar(\{f,g\}) = \frac{i}{\hbar}\left[W_\hbar(f), W_\hbar(g)\right].$$

Consider for instance the quantization of a free particle in $\mathbb{R}^n$. In that case, on the classical side, the phase space corresponds to the symplectic manifold $\left(\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n, \omega = dq \wedge dp\right)$ (with $q = (q_1 \dots q_n)$ and $p = (p_1 \dots p_n)$) and on the quantum side, the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ is the space of square integrable functions on the configuration space. The Weyl quantization map is then given by

$$W_\hbar(f)\varphi(q_0) := \frac{1}{\pi\hbar}\int_{\mathbb{R}^{2n}} e^{\frac{ip}{\hbar}(q_0-q)} f\left(\frac{q_0+q}{2}, p\right)\varphi(q)dqdp,$$

with $\varphi \in L^2(\mathbb{R}^n)$ and f a classical observable such that this integral makes sense. It turns out that for $f, g \in \mathcal{S}\left(\mathbb{R}^{2n}\right) \subset C^\infty\left(\mathbb{R}^{2n}\right)$ (where $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ is the Schwartz space on $\mathbb{R}^{2n}$, that is, the space of *rapidly decreasing functions*), we have

$$W_\hbar(f) \circ W_\hbar(g) = W_\hbar(f \star_\hbar^W g),$$

where

$$\left(f \star_\hbar^W g\right)(q_0,p_0) := \\ \frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar}S^W\left((q_0,p_0),(q,p),(q',p')\right)} f(q,p)g(q',p')dqdpdq'dp' \tag{2.19}$$

and with

$$S^W(x,y,z) := \omega(x,y) + \omega(y,z) + \omega(z,x), \tag{2.20}$$

for $x, y, z \in \mathbb{R}^{2n}$ seen as vectors. Although not very obvious, the link between this integral expression (the so-called *Weyl product*) and the Moyal product (see Definition 1.2.20) is very close. The reason is the following ([4, 42]): for given $f, g \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$ and $(q_0, p_0) \in \mathbb{R}^{2n}$, the map $\hbar \mapsto \left(f \star_\hbar^W g\right)(q_0,p_0)$ is smooth and its Taylor expansion around $\hbar = 0$ coincides with the expression of the standard Moyal product $\star_\nu^0$ defined in chapter 1 by (1.17), with $\nu := \frac{\hbar}{2i}$. In addition, we emphasize that using oscillatory integral techniques, it can be shown that for $f, g \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$, the star-product $f \star_\hbar^W g$ is again a Schwartz function. In other words, the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ endowed with the star-product $\star_\hbar^W$ forms an associative algebra (the associativity coming from that of the composition law of operators).

The first interest of this construction is that it allows to interpret a star-product as a noncommutative law at the level of classical observables, coming from the noncommutative operatorial composition law of the quantum observables. Instead of viewing the quantum framework as a change in the nature of the observables (that is, by using operators rather than functions), the quantization is understood as a deformation of the initial algebra structure of the classical observables. Both classical and quantum theories are then described using *algebras of functions*. The previous construction also shows that unlike the case of *formal* deformation quantization[5] (as described in chapter 1) where no particular convergence of the series is required, there exists certain situations for which, under some restrictions on the space of classical observables, the star-product becomes *convergent*, that is, the star-multiplication of two functions is again a function (given in terms of an oscillatory integral), rather than a formal power series. Hereafter such a convergent star-product (together with its associated space of functions) will be called a *non-formal deformation quantization* or a *non-formal star-product* in short.

2.2.2 Non-formal BG deformation quantization

We present here an overview of the construction of Pierre Bieliavsky and Marc Massar (see [23, 24] for details) of a non-formal (left invariant) star-product on the symplectic Lie group $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. This construction constitutes a particular case of the work done more recently by Pierre Bieliavsky and Victor Gayral in [20].

Recall that the (formal) BG star-product is defined by $\star_\nu^{\mathrm{BG}} := T_\nu\left(\star_\nu^{\mathbb{S}}\right)$ (see Definition 2.1.10). In order to get an integral expression for this star-product, we first consider the convergent version of the Moyal star-product $\star_\nu^{\mathbb{S}}$ defined by (2.4), which amounts to particularize to expression (2.19) in the case $n = 1$, with $\nu := \frac{\hbar}{2i}$ and in the global $\{(a, \ell)\}$ coordinate system. For $\varphi, \psi \in \mathcal{S}(\mathbb{S})$, this integral expression is given by[6]

$$\left(\varphi \star_\nu^{\mathbb{S}} \psi\right)(a_0, \ell_0) = \frac{1}{(2\pi i\nu)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{-\frac{1}{\nu} S^{\mathbb{S}}((a_0,\ell_0),(a_1,\ell_1),(a_2,\ell_2))} \varphi(a_1, \ell_1)\psi(a_2, \ell_2) da_1 da_2 d\ell_1 d\ell_2, \tag{2.21}$$

[5] We add the term *formal* to emphasize the difference with the non-formal framework.

[6] By abuse of notations we denote both asymptotic expansion and integral expression in the same way.

with

$$\begin{aligned} S^{\mathbb{S}}((a_0,\ell_0),(a_1,\ell_1),(a_2,\ell_2)) &= \omega^{\mathbb{S}}((a_0,\ell_0),(a_1,\ell_1)) + \omega^{\mathbb{S}}((a_1,\ell_1),(a_2,\ell_2)) \\ &\quad + \omega^{\mathbb{S}}((a_2,\ell_2),(a_0,\ell_0)) \\ &= a_0\ell_1 - a_1\ell_0 + a_1\ell_2 - a_2\ell_1 + a_2\ell_0 - a_0\ell_2. \end{aligned}$$

Now, using the expression (2.13), we set

$$\mathcal{E}_\nu^{BG} := T_\nu\left(\mathcal{S}(\mathbb{S})\right).$$

Lemma 2.2.1 ([13], Lemma 3.1)**.** The following inclusions hold for all $\nu \in i\mathbb{R}_0$:

(i) $\Psi_\nu^\star\left(\mathcal{S}(\tilde{\mathbb{S}})\right) \subset \mathcal{S}(\tilde{\mathbb{S}})$;

(ii) $\left(\Psi_\nu^{-1}\right)^\star\left(\mathcal{S}(\tilde{\mathbb{S}})\right) \subset \mathcal{S}'(\tilde{\mathbb{S}})$;

where Ψ_ν and Ψ_ν^{-1} are defined respectively by (2.10) and (2.11).

Lemma 2.2.2 ([24], Lemma 3.2)**.** We have

(i) $\mathcal{S}(\mathbb{S}) \subset \mathcal{E}_\nu^{BG} \subset \mathcal{S}'(\mathbb{S})$;

(ii) the map $T_\nu^{-1} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S})$ defined by (2.14) extends to $\mathcal{E}_\nu^{BG}$ as a linear isomorphism $T_\nu^{-1} : \mathcal{E}_\nu^{BG} \to \mathcal{S}(\mathbb{S})$;

(iii) $T_\nu^{-1} \circ T_\nu = \mathrm{Id}_{\mathcal{S}(\mathbb{S})}$ and $T_\nu \circ T_\nu^{-1}\big|_{\mathcal{S}(\mathbb{S})} = \mathrm{Id}_{\mathcal{S}(\mathbb{S})}$.

In particular, both integral operators T_ν and T_ν^{-1} are well-defined (i.e. integrals (2.13) and (2.14) exist) and are inverse of each other on the Schwartz space $\mathcal{S}(\mathbb{S})$. By intertwining the star-product $\star_\nu^{\mathbb{S}}$ (given by (2.21)) with these operators (when it makes sense), we get[7]

$$\begin{aligned} &(\varphi \star_\nu^{BG} \psi)(a_0,\ell_0) = T_\nu\left(T_\nu^{-1}\varphi \star_\nu^{\mathbb{S}} T_\nu^{-1}\psi\right)(a_0,\ell_0) = \\ &\frac{1}{(2\pi)^5(i\nu)^2}\int \cosh(2i\nu\xi) e^{\frac{\sinh(2i\nu\xi)}{2\nu}\ell_0 - i\xi\ell - \frac{\sinh(2i\nu\xi_1)}{2\nu}\ell_3 + i\xi_1\ell_1 - \frac{\sinh(2i\nu\xi_2)}{2\nu}\ell_4 + i\xi_2\ell_2} \\ &\qquad \times e^{-\frac{1}{\nu}(a_0\ell_1 - a_1\ell + a_1\ell_2 - a_2\ell_1 + a_2\ell - a_0\ell_2)}\varphi(a_1,\ell_3)\psi(a_2,\ell_4) \\ &\qquad\qquad \times da_1 da_2 d\ell d\ell_1 d\ell_2 d\ell_3 d\ell_4 d\xi d\xi_1 d\xi_2. \end{aligned}$$

[7] In order to lighten the notations, we will often drop the domain (of integration) of our integrals. When such an integral appears in what follows, the underlying domain will always be understood as $\mathbb{R} \times \ldots \times \mathbb{R}$ n times, with n the number of variables of integration.

Computing the distributional integrals in ℓ, ℓ_1 and ℓ_2-variables, we get respectively

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\ell[\xi-\frac{a_1-a_2}{i\nu}]}d\ell = \delta_0\left(\xi - \frac{a_1-a_2}{i\nu}\right),$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell_1[\xi_1-\frac{a_0-a_2}{i\nu}]}d\ell_1 = \delta_0\left(\xi_1 - \frac{a_0-a_2}{i\nu}\right),$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell_2[\xi_2-\frac{a_1-a_0}{i\nu}]}d\ell_2 = \delta_0\left(\xi_2 - \frac{a_1-a_0}{i\nu}\right).$$

Thanks to the identities above (and after changing the name of integration variables), the star-product formula becomes

$$\begin{aligned}(\varphi \star_\nu^{BG} \psi)(a_0,\ell_0) = \\ \frac{1}{(2\pi i\nu)^2}\int \cosh(2(a_1-a_2))e^{\frac{1}{2\nu}(\sinh(2(a_0-a_1))\ell_2+\sinh(2(a_2-a_0))\ell_1+\sinh(2(a_1-a_2))\ell_0)} \\ \times \varphi(a_1,\ell_1)\psi(a_2,\ell_2)da_1da_2d\ell_1d\ell_2.\end{aligned} \tag{2.22}$$

Let us denote by $d_{\mathbb{S}}$ the Lebesgue measure on $\mathbb{S} \simeq \mathbb{R}^2$. Within coordinates (1.24) (that is, with $x = (a,\ell)$), this gives

$$d_{\mathbb{S}}(x) = dad\ell.$$

Remark 2.2.3. The Lebesgue measure $d_{\mathbb{S}}(x)$ coincides with the left invariant Haar measure on $\mathbb{S}$, denoted by $d^L_{\mathbb{S}}(x)$. Indeed, for $x_0 \in \mathbb{S}$, a straightforward computation shows that for all $\varphi \in \mathcal{D}(\mathbb{S}) := C_c^{+\infty}(\mathbb{S})$,

$$\int_{\mathbb{S}} L^\star_{x_0}\varphi(x)d_{\mathbb{S}}(x) = \int_{\mathbb{S}} \varphi(x)d_{\mathbb{S}}(x).$$

By construction, the star-product $\star_\nu^{BG}$ is left invariant (for the left action on $\mathbb{S}$) and we have the following theorem.

Theorem 2.2.4 ([24], Theorem 3.1)**.** For $f,g \in \mathcal{E}_\nu^{BG}$, the formula

$$f \star_\nu^{BG} g := T_\nu\left(T_\nu^{-1}f \star_\nu^{\mathbb{S}} T_\nu^{-1}g\right)$$

defines an associative algebra structure on $\mathcal{E}_\nu^{BG}$. Moreover, at the level of smooth compactly supported functions, the star-product admits the following

oscillatory integral expression:

$$(\varphi \star_\nu^{BG} \psi)(a_0, \ell_0) = \\ \frac{1}{(2\pi i\nu)^2} \int \cosh(2(a_1 - a_2)) e^{\frac{1}{2\nu}(\sinh(2(a_0-a_1))\ell_2 + \sinh(2(a_2-a_0))\ell_1 + \sinh(2(a_1-a_2))\ell_0)} \\ \times \varphi(a_1, \ell_1)\psi(a_2, \ell_2) d_{\mathbb{S}}x_1 d_{\mathbb{S}}x_2, \tag{2.23}$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{S}) := C_c^{+\infty}(\mathbb{S})$ and with, within coordinates (1.24), $x_j = (a_j, \ell_j)$ for $j \in \{1, 2\}$. This star-product is (in particular[8]) invariant under the left action (2.1) of $\mathbb{S}$, that is for all $x \in \mathbb{S}$,

$$L_x^\star \left(\varphi \star_\nu^{BG} \psi\right) = L_x^\star \varphi \star_\nu^{BG} L_x^\star \psi.$$

The link between this integral expression and the BG star-product at the formal level given by intertwining the formal Moyal star-product on $\mathbb{S}$ is given in the following theorem.

Theorem 2.2.5 ([16], Theorem 4.9)**.** For every smooth compactly supported functions $\varphi, \psi \in \mathcal{D}(\mathbb{S})$ and for every $x \in \mathbb{S}$, the function

$$\mathbb{R} \to \mathbb{C} : \theta \mapsto (\varphi \star_{\frac{\theta}{i}}^{BG} \psi)(x)$$

is smooth. Its Taylor series at 0 defines an associative formal star-product $\star_\nu^{BG}$ on $C^\infty(\mathbb{S})[[\nu]]$ with $\nu := \frac{\theta}{i}$, whose expression is of the form[9]

$$\varphi \star_\nu^{BG} \psi = \varphi\psi + \nu \{\varphi, \psi\} + O\left(\nu^2\right),$$

with $\{\cdot, \cdot\}$ the Poisson bracket associated with the symplectic form $\omega^{\mathbb{S}}$.

Sketch of proof. We focus on the second part of the theorem (i.e. the result about the Taylor series) which illustrates some oscillatory integral techniques. Firstly, note that thanks to the left invariance of the BG star-product, it suffices to show the result at the neutral element $x = e$ (corresponding to $(0, 0)$ within coordinates (1.24)). We set $\theta := i\nu \in \mathbb{R}_0$ and changing the variables following

[8]The maximal group of invariance of this star-product is the transvection group $G_{BG} \supset \mathbb{S}$.
[9]Recall that $O\left(\nu^2\right) := \left\{\sum_{k=2}^{+\infty} \nu^k f \mid f \in C^\infty(\mathbb{S})\right\}$.

$\ell_j \mapsto 2\theta\ell_j$ (for $j \in \{1,2\}$), the integral formula (2.23) at $(a,\ell) = (0,0)$ becomes

$$(\varphi \star^{BG}_{\frac{\theta}{i}} \psi)(0,0) = \\ \frac{1}{\pi^2}\int \cosh(2(a_1-a_2))e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)}\varphi(a_1,2\theta\ell_1)\psi(a_2,2\theta\ell_2)d_{\mathbb{S}}x_1 d_{\mathbb{S}}x_2, \tag{2.24}$$

with, within coordinates (1.24), $x_j = (a_j,\ell_j)$ and where $\varphi,\psi \in \mathcal{D}(\mathbb{S})$. Using the distributional identities

$$\int_{\mathbb{R}} e^{i\sinh(2a_2)\ell_1} d\ell_1 = \pi\delta_0(a_2) \tag{2.25}$$

and

$$\int_{\mathbb{R}} e^{-i\sinh(2a_1)\ell_2} d\ell_2 = \pi\delta_0(a_1), \tag{2.26}$$

the evaluation of (2.24) at $\theta = 0$ reduces to $(\varphi.\psi)(0,0)$. Taking now the derivative of (2.24) with respect to θ under the integral sign[10], we get

$$\frac{d}{d\theta}(\varphi \star^{BG}_{\frac{\theta}{i}} \psi)(0,0) = \\ \frac{1}{\pi^2}\int \cosh(2(a_1-a_2))e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)} \\ \times \left[2\ell_1\partial_2\varphi(a_1,2\theta\ell_1)\psi(a_2,2\theta\ell_2) + 2\ell_2\varphi(a_1,2\theta\ell_1)\partial_2\psi(a_2,2\theta\ell_2)\right] d_{\mathbb{S}}x_1 d_{\mathbb{S}}x_2. \tag{2.27}$$

Remark now the following identities:

$$\ell_1 = \frac{1}{2i\cosh(2a_2)}\partial_{a_2} e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)}, \tag{2.28}$$

and

$$\ell_2 = -\frac{1}{2i\cosh(2a_1)}\partial_{a_1} e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)}. \tag{2.29}$$

[10] It can be done properly for θ in a neighborhood of 0, by means of the Lebesgue dominated convergence.

Splitting the integral (2.27) and using identities (2.28) and (2.29) together with integrations by parts, we get

$$\begin{aligned}
&\frac{d}{d\theta}(\varphi \star^{BG}_{\frac{\theta}{i}} \psi)(0,0) = \\
&-\frac{1}{\pi^2}\int e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)}\partial_{a_2}\left(\frac{\cosh(2(a_1-a_2))}{i\cosh(2a_2)}\psi(a_2, 2\theta\ell_2)\right) \\
&\qquad\qquad \times \partial_2\varphi(a_1, 2\theta\ell_1) d_{\mathbb{S}}x_1 d_{\mathbb{S}}x_2 \\
&+\frac{1}{\pi^2}\int e^{i(\sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2)}\partial_{a_1}\left(\frac{\cosh(2(a_1-a_2))}{i\cosh(2a_1)}\varphi(a_1, 2\theta\ell_1)\right) \\
&\qquad\qquad \times \partial_2\psi(a_2, 2\theta\ell_2) d_{\mathbb{S}}x_1 d_{\mathbb{S}}x_2.
\end{aligned}$$

Using both distributional identities (2.25) and (2.26) together with an evaluation at $\theta = 0$, it yields

$$\begin{aligned}
\frac{d}{d\theta}(\varphi \star^{BG}_{\frac{\theta}{i}} \psi)(0,0)\Big|_{\theta=0} &= \frac{1}{i}\left(\partial_1\varphi(0,0)\partial_2\psi(0,0) - \partial_2\varphi(0,0)\partial_1\psi(0,0)\right) \\
&= \frac{1}{i}\left\{\varphi, \psi\right\}(0,0).
\end{aligned}$$

Using $\frac{\theta}{i} = \nu$, an iteration of this kind of reasoning leads to the following asymptotic expansion:

$$\varphi \star^{BG}_{\nu} \psi \sim \varphi\psi + \nu\left\{\varphi, \psi\right\} + O\left(\nu^2\right).$$

■

Following [11], we denote by $\langle \cdot, \cdot \rangle_{\mathcal{S}(\mathbb{S})}$ the restriction to $\mathcal{S}(\mathbb{S})$ of the usual inner product on $L^2(\mathbb{S})$, i.e.

$$\langle \varphi, \psi \rangle_{\mathcal{S}(\mathbb{S})} := \int_{\mathbb{S}} \varphi(x)\bar{\psi}(x) d_{\mathbb{S}}(x). \tag{2.30}$$

Thanks to the linear bijection $T_\nu^{-1} : \mathcal{E}_\nu^{BG} \to \mathcal{S}(\mathbb{S})$, we can then transport this structure to get an inner product on $\mathcal{E}_\nu^{BG}$, defined as

$$\langle f, g \rangle_{\mathcal{E}_\nu^{BG}} := \langle T_\nu^{-1} f, T_\nu^{-1} g \rangle_{\mathcal{S}(\mathbb{S})}. \tag{2.31}$$

This turns $\mathcal{E}_\nu^{BG}$ into a pre-Hilbert space and we denote by $\mathcal{H}_\nu^{BG}$ its Hilbert completion.

Proposition 2.2.6 ([11], Proposition 7.2)**.** We have the following results:

(i) the inclusions $\mathcal{D}(\mathbb{S}) \subset \mathcal{S}(\mathbb{S}) \subset \mathcal{E}_\nu^{BG} \subset \mathcal{H}_\nu^{BG}$ are dense;

(ii) the algebra structure $\star_\nu^{BG}$ on $\mathcal{E}_\nu^{BG}$ extends to $\mathcal{H}_\nu^{BG}$. Endowed with the extended product again denoted by $\star_\nu^{BG}$, the space $\mathcal{H}_\nu^{BG}$ becomes an associative topological algebra;

(iii) the transvection group G_{BG} acts on $\left(\mathcal{H}_\nu^{BG}, \star_\nu^{BG}\right)$ by algebra automorphisms.

Remark 2.2.7. The formula (2.31) shows that the space $\mathcal{H}_\nu^{BG}$ differs from the space $L^2(\mathbb{S})$ due to the non-unitarity of T_ν^{-1} (with respect to the usual inner product of $L^2(\mathbb{S})$).

Let us now slightly generalize the above construction. Recall that Proposition 2.1.19 tells us that at the formal level, every G_{BG}-invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is given by means of an G_{BG}-equivalence of the form

$$T_{\mathcal{T}} := \mathcal{F}^{-1} \circ \mathcal{M}_{\mathcal{T}} \circ \mathcal{F},$$

where $\mathcal{T} = \sum_{k=0}^{+\infty} \mathcal{T}_k$ is some formal function on $\tilde{\mathbb{S}}$ such that $\mathcal{T} \in C^\infty_{(\xi)}(\mathbb{R})[[\nu]]$. Every G_{BG}-invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is then formally obtained by the formula

$$T_{\mathcal{T}}\left(\star_\nu^{BG}\right) = \left(T_{\mathcal{T}} \circ T_\nu\right)\left(\star_\nu^{\mathbb{S}}\right) = \left(\mathcal{F}^{-1} \circ \mathcal{M}_{\mathcal{T}} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F}\right)\left(\star_\nu^{\mathbb{S}}\right),$$

using decomposition (2.12) of T_ν. When $\mathcal{T}$ (which can depend on ν) belongs to the space $\mathcal{O}_M(\mathbb{R})$ of Schwartz multipliers of $\mathcal{S}(\mathbb{R})$, a slight generalization of Lemmas 2.2.1 and 2.2.2 (item (i)) implies that the following linear map defined on the Schwartz space $\mathcal{S}(\mathbb{S})$ takes its values in the space of tempered distributions $\mathcal{S}'(\mathbb{S})$:

$$\mathcal{F}^{-1} \circ \mathcal{M}_{\mathcal{T}} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}'(\mathbb{S}).$$

Following [13] (Lemma 3.2) and [20] (chapter 4), we introduce the space

$$\mathcal{O}_{C,\gamma}(\mathbb{R}^m) := \left\{ f \in C^\infty(\mathbb{R}^m) \mid \exists r > 0 : \forall \alpha \in \mathbb{N}^m, \exists C_\alpha > 0 : |\partial^\alpha f(x)| \leq C_\alpha \left(1 + |x|\right)^{r - \gamma|\alpha|} \right\}.$$

That is, the subspace of smooth functions (with values in $\mathbb{C}$), the derivatives of which are uniformly polynomially bounded.

Definition 2.2.8 ([20], Definition 4.2)**.** We denote by $\mathbf{\Theta}$, the subspace of $C^\infty(\mathbb{R})$ constituted by the elements ϑ such that $\exp \circ \pm\, \vartheta$ belong to the space $\mathcal{O}_{C,1}(\mathbb{R})$, and normalized such that $\vartheta(0) = 0$.

It is worth mentioning that $\mathcal{O}_{C,1}(\mathbb{R})$ is clearly a subset of the Schwartz multipliers[11] $\mathcal{O}_M(\mathbb{R})$.

Remark 2.2.9. With a slight abuse of notation, we will say that a function $\vartheta \in C^\infty\left(\tilde{\mathbb{S}}\right) \simeq C^\infty\left(\mathbb{R}^2\right)$ depending only on the last variable of $\tilde{\mathbb{S}}$ (i.e. the variable ξ) belongs to $\mathbf{\Theta}$ if the functions $[\mathbb{R} \to \mathbb{C} : \xi \mapsto \vartheta(a,\xi)]$ (that is, ϑ viewed as a function of only one variable) belong to $\mathbf{\Theta}$ for all $a \in \mathbb{R}$.

When $\mathcal{T}$ is of the form $\exp(\vartheta)$, with $\vartheta \in \mathbf{\Theta}$, we have the following result.

Lemma 2.2.10 ([20], Lemma 4.3)**.** Let $\vartheta \in \mathbf{\Theta}$. Then the inverse of the map of

$$T_{\nu,\vartheta} := \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(\vartheta)} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F}, \tag{2.32}$$

given by

$$T_{\nu,\vartheta}^{-1} = \mathcal{F}^{-1} \circ \Psi_\nu^\star \circ \mathcal{M}_{\exp(-\vartheta)} \circ \mathcal{F}, \tag{2.33}$$

defines a continuous linear injection from $\mathcal{S}(\mathbb{S})$ to itself[12].

By generalizing what has been done above, we set

$$\mathcal{E}_{\nu,\vartheta}^{BG} := T_{\nu,\vartheta}\left(\mathcal{S}(\mathbb{S})\right),$$

the range subspace of $T_{\nu,\vartheta}$ in the tempered distribution space $\mathcal{S}'(\mathbb{S})$, for which we have the following inclusions:

$$\mathcal{S}(\mathbb{S}) \subset \mathcal{E}_{\nu,\vartheta}^{BG} \subset C^\infty(\mathbb{S}).$$

We also define an inner product on $\mathcal{E}_{\nu,\vartheta}^{BG}$ by

$$\langle f, g\rangle_{\mathcal{E}_{\nu,\vartheta}^{BG}} := \langle T_{\nu,\vartheta}^{-1} f, T_{\nu,\vartheta}^{-1} g\rangle_{\mathcal{S}(\mathbb{S})}, \tag{2.34}$$

which is well-defined thanks to Lemma 2.2.10.

We will now determine a particular function $\vartheta \in \mathbf{\Theta}$ such that the associated operator $T_{\nu,\vartheta}^{-1}$ becomes unitary on $L^2(\mathbb{S})$, turning in that case the completion of $\mathcal{E}_{\nu,\vartheta}^{BG}$ (with respect to the inner product (2.34)) into the space $L^2(\mathbb{S})$ itself. For this purpose, we first recall that the (partial) Fourier transform and its inverse satisfy respectively

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{S}(\tilde{\mathbb{S}})} = 2\pi \langle f, g\rangle_{\mathcal{S}(\mathbb{S})} \tag{2.35}$$

[11] The functions belonging to the space of Schwartz multipliers are also called *tempered functions*, or *slowly increasing functions*. See Definition 2.2.27 for more details.

[12] Note that the operator $T_{\nu,\vartheta}$ in the particular case $\vartheta \equiv 0 \in \mathbf{\Theta}$ corresponds to the operator T_ν defined by (2.12).

for $f, g \in \mathcal{S}(\mathbb{S})$, and

$$\langle \mathcal{F}^{-1} f, \mathcal{F}^{-1} g \rangle_{\mathcal{S}(\mathbb{S})} = \frac{1}{2\pi} \langle f, g \rangle_{\mathcal{S}(\tilde{\mathbb{S}})}, \tag{2.36}$$

for $f, g \in \mathcal{S}(\tilde{\mathbb{S}})$. Combining this with the identity (2.34) and the decomposition (2.33), we deduce that for $f, g \in \mathcal{S}(\mathbb{S})$:

$$\begin{aligned} \langle f, g \rangle_{\mathcal{E}^{BG}_{\nu,\vartheta}} &= \frac{1}{2\pi} \langle \left(\Psi_\nu^\star \circ \mathcal{M}_{\exp(-\vartheta)} \circ \mathcal{F} \right) f, \left(\Psi_\nu^\star \circ \mathcal{M}_{\exp(-\vartheta)} \circ \mathcal{F} \right) g \rangle_{\mathcal{S}(\tilde{\mathbb{S}})} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| e^{-\vartheta(\Psi_\nu(a,\xi))} \right|^2 \mathcal{F}f \left(\Psi_\nu(a,\xi) \right) \overline{\mathcal{F}g} \left(\Psi_\nu(a,\xi) \right) da d\xi. \end{aligned}$$

Changing the variables following $\Psi_\nu(a,\xi) \mapsto (a,\xi)$ yields

$$\langle f, g \rangle_{\mathcal{E}^{BG}_{\nu,\vartheta}} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| e^{-\vartheta(a,\xi)} \right|^2 \left| \mathrm{Jac}_{\Psi_\nu^{-1}}(a,\xi) \right| \mathcal{F}f(a,\xi) \overline{\mathcal{F}g}(a,\xi) da d\xi, \tag{2.37}$$

and a straightforward computation leads to

$$\left| \mathrm{Jac}_{\Psi_\nu^{-1}}(a,\xi) \right| = \frac{1}{\sqrt{1 - 4\nu^2\xi^2}},$$

which depends only on the variable ξ. Let us then define $\vartheta_0 \in C^\infty\left(\tilde{\mathbb{S}}, \mathbb{R}\right)$ (i.e. a *real-valued function*) by

$$\vartheta_0(a,\xi) := \frac{1}{2} \log\left(\left| \mathrm{Jac}_{\Psi_\nu^{-1}}(a,\xi) \right| \right) = -\frac{1}{4} \log\left(1 - 4\nu^2\xi^2\right). \tag{2.38}$$

Since

$$\exp\left(\pm\vartheta_0(\xi)\right) = \left(1 - 4\nu^2\xi^2\right)^{\mp\frac{1}{4}}$$

and $\vartheta_0(0) = 0$, the function ϑ_0 (viewed as a function of its last variable only) clearly belongs to $\mathbf{\Theta}$. In the particular case $\vartheta = \vartheta_0$, the equation (2.37) simplifies to

$$\begin{aligned} \langle f, g \rangle_{\mathcal{E}^{BG}_{\nu,\vartheta}} &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathcal{F}f(a,\xi) \overline{\mathcal{F}g}(a,\xi) da d\xi \\ &= \int_{\mathbb{R}^2} f(a,\ell) \bar{g}(a,\ell) da d\ell \\ &= \langle f, g \rangle_{\mathcal{S}(\mathbb{S})}, \end{aligned}$$

where we have used the relation (2.35). The Hilbert completion $\mathcal{H}^{BG}_{\nu,\vartheta_0}$ of the pre-Hilbert space $\mathcal{E}^{BG}_{\nu,\vartheta_0}$ then corresponds to the space $L^2(\mathbb{S})$ on which T^{-1}_{ν,ϑ_0}

extends as a unitary operator. Moreover, for

$$\star^{BG}_{\nu,\vartheta_0} := T_{\nu,\vartheta_0}\left(\star^{\mathbb{S}}_{\nu}\right), \tag{2.39}$$

$\left(\mathcal{E}^{BG}_{\nu,\vartheta_0}, \star^{BG}_{\nu,\vartheta_0}\right)$ becomes an associative algebra[13], endowed with the Fréchet algebra structure transported from $\mathcal{S}(\mathbb{S}) \simeq \mathcal{S}\left(\mathbb{R}^2\right)$ under the operator T_{ν,ϑ_0}. A similar computation to that made for $\star^{BG}_{\nu}$ leads to the following integral representation at the level of compactly supported functions:

$$\begin{aligned}(\varphi \star^{BG}_{\nu,\vartheta_0} \psi)(a_0,\ell_0) = \\ \frac{1}{(2\pi i\nu)^2}\int \sqrt{\cosh(2(a_0-a_1))\cosh(2(a_2-a_0))\cosh(2(a_1-a_2))} \\ \times e^{\frac{1}{2\nu}(\sinh(2(a_0-a_1))\ell_2+\sinh(2(a_2-a_0))\ell_1+\sinh(2(a_1-a_2))\ell_0)} \\ \times \varphi(a_1,\ell_1)\psi(a_2,\ell_2)da_1da_2d\ell_1d\ell_2\end{aligned}$$

(where $\varphi,\psi \in \mathcal{D}(\mathbb{S})$). By changing the variables following $a_j - a_0 \mapsto a_j$ first and then $\ell_j \mapsto \ell_j - \ell_0 e^{-2a_j}$ for $j \in \{1,2\}$, we get

$$\begin{aligned}(\varphi \star^{BG}_{\nu,\vartheta_0} \psi)(a_0,\ell_0) = \\ \frac{1}{(2\pi i\nu)^2}\int \sqrt{\cosh(2a_1)\cosh(2a_2)\cosh(2(a_1-a_2))}e^{\frac{1}{2\nu}(\sinh(2a_2)\ell_1-\sinh(2a_1)\ell_2)} \\ \times R^{\star}_{(a_1,\ell_1)}\varphi(a_0,\ell_0)R^{\star}_{(a_2,\ell_2)}\psi(a_0,\ell_0)da_1da_2d\ell_1d\ell_2,\end{aligned}$$

where R denotes the right action (1.31) of $\mathbb{S}$ on itself[14]. The following results summarize the above construction.

Lemma 2.2.11 ([20], Lemma 4.3)**.** The operator T^{-1}_{ν,ϑ_0} extends as a unitary operator on $L^2(\mathbb{S}) = \mathcal{H}^{BG}_{\nu,\vartheta_0}$.

In particular, the above lemma means that the inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}^{BG}_{\nu,\vartheta_0}}$ defined by (2.34) coincides with $\langle\cdot,\cdot\rangle_{\mathcal{S}(\mathbb{S})}$, that is, the restriction to the Schwartz space of the usual inner product on $L^2(\mathbb{S})$.

Theorem 2.2.12 ([20], Theorem 4.5)**.** Let $\nu \neq 0$ and $\vartheta_0 \in \mathbf{\Theta}$ defined by (2.38). Then,

(i) for all compactly supported functions $\varphi,\psi \in \mathcal{D}(\mathbb{S}) \subset \mathcal{E}^{BG}_{\nu,\vartheta_0}$, we have the integral representation

$$\varphi \star^{BG}_{\nu,\vartheta_0} \psi = \int_{\mathbb{S}\times\mathbb{S}} K^{BG}_{\nu,\vartheta_0}(x_1,x_2)R^{\star}_{x_1}\varphi R^{\star}_{x_2}\psi d_{\mathbb{S}}(x_1)d_{\mathbb{S}}(x_2), \tag{2.40}$$

[13]Remark that the space $\left(\mathcal{E}^{BG}_{\nu,\vartheta_0}, \star^{BG}_{\nu,\vartheta_0}\right)$ for $\vartheta_0 \equiv 0$ corresponds to $\left(\mathcal{E}^{BG}_{\nu}, \star^{BG}_{\nu}\right)$.

[14]Within coordinates (1.24), we have $R_{(a,\ell)}(a_0,\ell_0) = \left(a_0 + a, \ell + \ell_0 e^{-2a}\right)$.

where the *two-point kernel* is given by

$$K^{BG}_{\nu,\vartheta_0}(x_1,x_2) := (2\pi i\nu)^{-2}\, A^{BG}_{\nu,\vartheta_0}(x_1,x_2)\exp\left(\frac{1}{2\nu}S^{BG}_{\nu,\vartheta_0}(x_1,x_2)\right),$$

with, in the coordinates $x_i = (a_i, \ell_i)$, the *amplitude* and the *phase* defined respectively by

$$A^{BG}_{\nu,\vartheta_0}(x_1,x_2) := \sqrt{\cosh(2a_1)\cosh(2a_2)\cosh(2(a_1-a_2))} \tag{2.41}$$

and

$$S^{BG}_{\nu,\vartheta_0}(x_1,x_2) := \sinh(2a_2)\ell_1 - \sinh(2a_1)\ell_2; \tag{2.42}$$

(ii) the product $\star^{BG}_{\nu,\vartheta_0}$ is invariant under the transvection group $G_{BG} \supset \mathbb{S}$. That is, for all $g \in G_{BG}$ and for all $\varphi, \psi \in \mathcal{D}(\mathbb{S})$, we have

$$\tau^\star_g\varphi \star^{BG}_{\nu,\vartheta_0} \tau^\star_g\psi = \tau^\star_g\left(\varphi \star^{BG}_{\nu,\vartheta_0} \psi\right).$$

Remark 2.2.13. It is possible to generalize the previous theorem by considering $\vartheta_0 - \vartheta$ (instead of ϑ_0) for any element $\vartheta \in \mathbf{\Theta}$. It can then be shown that the associated operator $T^{-1}_{\nu,\vartheta_0-\vartheta}$ extends in that case as a unitary operator on $L^2(\mathbb{S})$ if and only if ϑ is purely imaginary (see [20], Lemma 4.3). The integral representation of the star-product remains valid up to a multiplicative factor (depending on ϑ) in the amplitude. We refer to [20], chapter 4 for more details.

Regarding the Hilbert completion $\mathcal{H}^{BG}_{\nu,\vartheta_0} = L^2(\mathbb{S})$, we have the following result.

Proposition 2.2.14 ([11], Proposition 7.2)**.** The algebra structure $\star^{BG}_{\nu,\vartheta_0}$ on $\mathcal{E}^{BG}_{\nu,\vartheta_0}$ extends to $\mathcal{H}^{BG}_{\nu,\vartheta_0}$. Endowed with the extended product again denoted by $\star^{BG}_{\nu,\vartheta_0}$, the space $\mathcal{H}^{BG}_{\nu,\vartheta_0}$ becomes an associative topological algebra.

Before mentioning an additional property from [11] concerning the space $\mathcal{H}^{BG}_{\nu,\vartheta_0}$, we recall the definition of a *-*algebra*.

Definition 2.2.15. A *-*algebra* A is an associative algebra (over the field $\mathbb{C}$) endowed with an involution $^\star : A \to A$, such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$,

$$(a+b)^\star = a^\star + b^\star,\ (ab)^\star = b^\star a^\star \text{ and } (\lambda a)^\star = \bar{\lambda}a^\star.$$

Proposition 2.2.16 ([11], Proposition 7.3)**.** The triple $\left(\mathcal{H}^{BG}_{\nu,\vartheta_0}, \star^{BG}_{\nu,\vartheta_0}, \bar{\cdot}\right)$ is a *-algebra, where $\bar{\cdot}$ denotes the complex conjugation.

Traciality

Let us now present an additional property of the Weyl star-product (2.19) (we consider here the case $n = 1$). Consider two Schwartz functions $f, g \in \mathcal{S}\left(\mathbb{R}^2\right)$. Thanks to the following distributional identities for $q, q', p, p' \in \mathbb{R}$:

$$\int_{\mathbb{R}} e^{-\frac{2i}{\hbar}(p-p')q_0} dq_0 = \pi\hbar\delta_0\left(p - p'\right),$$

$$\int_{\mathbb{R}} e^{-\frac{2i}{\hbar}(q'-q)p_0} dp_0 = \pi\hbar\delta_0\left(q' - q\right),$$

we get[15]

$$\begin{aligned}\int_{\mathbb{R}^2} \left(f \star_\hbar^W g\right)(q_0, p_0) dq_0 dp_0 \\ &= \int e^{-\frac{2i}{\hbar}(qp'-pq')} f(q,p) g(q',p') \delta_0\left(q' - q\right) \delta_0\left(p - p'\right) dq dp dq' dp' \\ &= \int_{\mathbb{R}^2} f(q,p) g(q,p) dq dp. \end{aligned} \tag{2.43}$$

This means that the integral of the star-product $\star_\hbar^W$ is equal to the integral of the classical pointwise product.

Definition 2.2.17 ([17])**.** Let (M, ω) be a symplectic manifold with an associated (formal) star-product $\star_\nu$ on $C^\infty(M)[[\nu]]$. A *trace form* Tr (or a *trace* in short) for $\star_\nu$ is a $\mathbb{C}[[\nu]]$-linear functional

$$\mathrm{Tr} : C^\infty(M)[[\nu]] \to \mathbb{C}[[\nu]]$$

such that

$$\mathrm{Tr}\left(\varphi \star_\nu \psi\right) = \mathrm{Tr}\left(\psi \star_\nu \varphi\right).$$

Remark 2.2.18. Notice that the above computation concerning the Weyl star-product shows that the integral operator Tr^W defined as

$$\mathrm{Tr}^W : \mathcal{S}\left(\mathbb{R}^2\right) \to \mathbb{C} : f \mapsto \mathrm{Tr}^W(f) := \int_{\mathbb{R}^2} f(q,p) dq dp$$

is a trace at the non-formal level for $\star_\hbar^W$ (that is, a linear functional which, when applied to the associated formal star-product, is a trace in the sense of Definition

[15] Remark that the integral makes sense thanks to the fact that the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, which is stable under multiplication, is a subset of $L^1\left(\mathbb{R}^2\right)$ endowed with the standard Lebesgue measure.

2.2.17), thanks to the fact that using (2.43), we have for $f, g \in \mathcal{S}\left(\mathbb{R}^2\right)$:

$$\mathrm{Tr}^W\left(f \star_\hbar^W g\right) = \int_{\mathbb{R}^2} f(q,p)g(q,p)dqdp = \mathrm{Tr}^W\left(g \star_\hbar^W f\right).$$

Definition 2.2.19. Let (M,ω) be a symplectic manifold with an associated (formal or non-formal) star-product $\star_\nu$. We will say that $\star_\nu$ is a *tracial star-product*, or equivalently a *strongly closed star-product*, if the following identity holds (whenever it makes sense):

$$\int_M (f \star_\nu g)(x)d\mu_\omega(x) = \int_M (f.g)(x)d\mu_\omega(x),$$

where μ_ω is the Liouville measure associated with the symplectic form ω.

Thanks to the identity

$$\int_{\mathbb{R}^2}\left(f \star_\hbar^W g\right)(q,p)dqdp = \int_{\mathbb{R}^2}(fg)(q,p)dqdp, \tag{2.44}$$

the product $\star_\hbar^W$ is thus a tracial star-product (or a strongly closed star-product).

Let us now construct a trace for $\star_{\nu,\vartheta_0}^{BG}$. Recall first that for $\varphi, \psi \in \mathcal{S}(\mathbb{S})$,

$$\varphi \star_{\nu,\vartheta_0}^{BG} \psi = T_{\nu,\vartheta_0}\left(T_{\nu,\vartheta_0}^{-1}\varphi \star_\nu^{\mathbb{S}} T_{\nu,\vartheta_0}^{-1}\psi\right).$$

Using the identification $\mathbb{S} \simeq \mathbb{R}^2$ by virtue of the coordinates (1.24), it seems then natural to define the linear functional[16]

$$\mathrm{Tr}_{\vartheta_0}^{BG} : \mathcal{S}(\mathbb{S}) \to \mathbb{C} : \varphi \mapsto \mathrm{Tr}^W\left(T_{\nu,\vartheta_0}^{-1}\varphi\right), \tag{2.45}$$

i.e. $\mathrm{Tr}_{\vartheta_0}^{BG} := \mathrm{Tr}^W \circ T_{\nu,\vartheta_0}^{-1}$. It is straightforward to see that this functional defines a trace for $\star_{\nu,\vartheta_0}^{BG}$. Indeed, since $\star_\nu^{\mathbb{S}}$ denotes (at the non-formal level) the Weyl star-product on $\mathbb{S} \simeq \mathbb{R}^2$ in coordinates $\{(a,\ell)\}$, we have for $\varphi, \psi \in \mathcal{S}(\mathbb{S})$:

$$\begin{aligned}
\mathrm{Tr}_{\vartheta_0}^{BG}\left(\varphi \star_{\nu,\vartheta_0}^{BG} \psi\right) &= \mathrm{Tr}^W\left(T_{\nu,\vartheta_0}^{-1}\varphi \star_\nu^{\mathbb{S}} T_{\nu,\vartheta_0}^{-1}\psi\right) \\
&= \mathrm{Tr}^W\left(T_{\nu,\vartheta_0}^{-1}\psi \star_\nu^{\mathbb{S}} T_{\nu,\vartheta_0}^{-1}\varphi\right) \\
&= \mathrm{Tr}_{\vartheta_0}^{BG}\left(\psi \star_{\nu,\vartheta_0}^{BG} \varphi\right).
\end{aligned}$$

We will now show that $\star_{\nu,\vartheta_0}^{BG}$ is a tracial star-product. To this purpose, let us first mention the following result.

[16] Recall that Lemma 2.2.10 tells us that $T_{\nu,\vartheta_0}^{-1} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S})$. Note also that by a slight abuse of notation, the functional Tr^W stands here for $\int_{\mathbb{S}}$.

Lemma 2.2.20. For $\varphi \in \mathcal{S}(\mathbb{S})$ and $\vartheta \in \mathbf{\Theta}$ satisfying $\bar{\vartheta}(-x) = \vartheta(x)$ for all $x \in \mathbb{R}$, we have

$$\overline{T^{-1}_{\nu,\vartheta}\varphi} = T^{-1}_{\nu,\vartheta}\bar{\varphi}.$$

Proof. By virtue of the decomposition (2.33), a straightforward computation leads to the following explicit expression:

$$T^{-1}_{\nu,\vartheta}\varphi(a,\ell) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left(\ell\xi - \frac{\sinh(2i\nu\xi)}{2i\nu}\ell'\right)} e^{-\vartheta\left(\frac{\sinh(2i\nu\xi)}{2i\nu}\right)}\varphi(a,\ell')d\ell' d\xi.$$

We then get[17]

$$\overline{T^{-1}_{\nu,\vartheta}\varphi}(a,\ell) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\left(\ell\xi - \frac{\sinh(2i\nu\xi)}{2i\nu}\ell'\right)} e^{-\bar{\vartheta}\left(\frac{\sinh(2i\nu\xi)}{2i\nu}\right)}\bar{\varphi}(a,\ell')d\ell' d\xi.$$

Changing the variables following $-\xi \mapsto \xi$, we get:

$$\begin{aligned}\overline{T^{-1}_{\nu,\vartheta}\varphi}(a,\ell) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left(\ell\xi - \frac{\sinh(2i\nu\xi)}{2i\nu}\ell'\right)} e^{-\bar{\vartheta}\left(-\frac{\sinh(2i\nu\xi)}{2i\nu}\right)}\bar{\varphi}(a,\ell')d\ell' d\xi \\ &= T^{-1}_{\nu,\vartheta}\bar{\varphi}(a,\ell),\end{aligned}$$

thanks to the fact that by assumptions,

$$\bar{\vartheta}\left(-\frac{\sinh(2i\nu\xi)}{2i\nu}\right) = \vartheta\left(\frac{\sinh(2i\nu\xi)}{2i\nu}\right).$$

■

Proposition 2.2.21. The star-product $\star^{BG}_{\nu,\vartheta_0}$ is strongly closed.

Proof. For $\varphi, \psi \in \mathcal{S}(\mathbb{S})$, we have

$$\begin{aligned}\mathrm{Tr}^{BG}_{\vartheta_0}\left(\varphi \star^{BG}_{\nu,\vartheta_0} \psi\right) &= \mathrm{Tr}^{W}\left(T^{-1}_{\nu,\vartheta_0}\varphi \star^{\mathbb{S}}_{\nu} T^{-1}_{\nu,\vartheta_0}\psi\right) \\ &= \int_{\mathbb{S}}\left(T^{-1}_{\nu,\vartheta_0}\varphi\right)(x)\left(T^{-1}_{\nu,\vartheta_0}\psi\right)(x)d_{\mathbb{S}}(x) \\ &= \langle T^{-1}_{\nu,\vartheta_0}\varphi, \overline{T^{-1}_{\nu,\vartheta_0}\psi}\rangle_{\mathcal{S}(\mathbb{S})},\end{aligned}$$

by traciality of $\star^{\mathbb{S}}_{\nu}$ (which corresponds to $\star^{W}_{\frac{\hbar}{2i}}$ under the usual identification $\mathbb{S} \simeq \mathbb{R}^2$). Using the expression (2.38) for ϑ_0 (viewed as a function of its last variable only), it is easy to observe that, being an even and real-valued function,

$$\overline{\vartheta_0}(-\xi) = \vartheta_0(\xi),$$

[17] Recall that $i\nu \in \mathbb{R}_0$.

for all $\xi \in \mathbb{R}$. Lemma 2.2.20 together with Lemma 2.2.11 then imply that

$$\begin{aligned}\langle T^{-1}_{\nu,\vartheta_0}\varphi, \overline{T^{-1}_{\nu,\vartheta_0}\psi}\rangle_{\mathcal{S}(\mathbb{S})} &= \langle T^{-1}_{\nu,\vartheta_0}\varphi, T^{-1}_{\nu,\vartheta_0}\bar{\psi}\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \langle \varphi, \bar{\psi}\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \int_{\mathbb{S}} \varphi(x)\psi(x) d_{\mathbb{S}}(x).\end{aligned}$$

Observe now that thanks to the identity

$$\left(\mathrm{Tr}^W \circ \mathcal{F}^{-1}\right)\varphi = \int_{\mathbb{R}} \varphi(a,0)da$$

holding for all $\varphi \in \mathcal{S}(\mathbb{S})$ (and with $\mathbb{S} \simeq \mathbb{R}^2$), the trace operator

$$\mathrm{Tr}^{BG}_{\vartheta_0} = \mathrm{Tr}^W \circ T^{-1}_{\nu,\vartheta_0} = \int_{\mathbb{S}} \circ\, \mathcal{F}^{-1} \circ \Psi^{\star}_{\nu} \circ \mathcal{M}_{\exp(-\vartheta_0)} \circ \mathcal{F}$$

reduces to

$$\mathrm{Tr}^{BG}_{\vartheta_0} = \int_{\mathbb{S}}, \tag{2.46}$$

which completes the proof. ■

Remark 2.2.22. Observe that a similar computation shows that $\star^{BG}_{\nu}$ (which coincides with $\star^{BG}_{\nu,\vartheta}$ for $\vartheta \equiv 0$) *is not* a tracial star-product.

2.2.3 An adapted Schwartz space for the tracial BG star-product

As already mentioned, the standard Schwartz space on $\mathbb{R}^2$ is closed under the Weyl star-product formula (2.19) (with $n = 1$). The purpose of this subsection consists in defining an adapted Schwartz space for the tracial BG star-product, that is, a space of *rapidly decreasing functions*, closed under the tracial BG star-product formula (2.40). For this, we describe (a part of) the work of Bieliavsky and Gayral developed in [20]. Although similar (or even sometimes equivalent) to other notions of Schwartz space on Lie groups, the originality of their construction relies on the fact that their definition is formulated in terms of the phase function S^{BG}_{ν,ϑ_0} (2.42) only. This phase function appears naturally in a star-product formula of oscillatory integral type (for instance (2.19) or (2.40)), and its expression is related to the geometry of the underlying symplectic Lie group (as initially observed and developed by Weinstein in [54] and thereafter studied by Bieliavsky in [11]).

Before presenting several notions (sometimes technical) required to define precisely a Schwartz type function space adapted to the Bieliavsky-Gayral tracial star-product, let us first briefly expose the strategy used to go beyond the *formal* level of an oscillatory integral type expression. Roughly speaking, such an expression is of the form

$$\int Ae^{iS}F, \tag{2.47}$$

where A is a possibly unbounded smooth function, S is a real-valued smooth function and where F has to be suitably chosen to give a non-formal meaning to the integral. The difficulty consists in finding the largest possible space of functions F such that the expression (2.47) makes sense. Of course, the unboundedness of A implies the necessity for F to be *rapidly decreasing*. But that is not all : it turns out that in many cases, it is also useful that the expression (2.47) (viewed as a function) enjoys some regularity conditions. The easiest idea seems then to restrict, and sometimes drastically, the set of possible functions F. Hopefully, that is not the only way of proceeding. A genius method consists in introducing a differential operator D such that $D(AF)$ belongs to the space L^1 of integrable functions and with an associated formal adjoint $D^\dagger$ (with respect to the L^2-inner product) satisfying

$$D^\dagger\left(e^{iS}\right) = e^{iS}.$$

If it exists, such an operator then allows to rewrite (2.47) as

$$\int Ae^{iS}F = \int e^{iS}D(AF),$$

by making use of integration by parts arguments. In general, this technique then allows to keep a sufficiently large space of possible functions F.

Before describing the suitable space of functions for the oscillatory integral expression of the tracial Bieliavsky-Gayral star-product, we start by recalling some generalities (following [20]) concerning the universal enveloping algebra of a Lie algebra. Let $\mathfrak{g}$ be the Lie algebra associated with a Lie group G given with an ordered basis $\{X_1, \cdots, X_m\}$, to which we associate the Poincaré-Birkhoff-Witt basis of $\mathcal{U}(\mathfrak{g})$ defined as $\left\{X^\beta \mid \beta \in \mathbb{N}^m\right\}$, with $X := X_1^{\beta_1} \cdots X_m^{\beta_m}$. This induces the filtration

$$\mathcal{U}(\mathfrak{g}) = \bigcup_{k\in\mathbb{N}} \mathcal{U}_k(\mathfrak{g}),$$

in the sense that for $k \leq l$, $\mathcal{U}_k(\mathfrak{g}) \subset \mathcal{U}_l(\mathfrak{g})$, where for $k \in \mathbb{N}$,

$$\mathcal{U}_k(\mathfrak{g}) := \left\{ \sum_{|\beta| \leq k} C_\beta X^\beta \mid C_\beta \in \mathbb{R} \right\},$$

with $|\beta| := \beta_1 + \cdots + \beta_m$. We endow the finite dimensional vector space $\mathcal{U}_k(\mathfrak{g})$ with the ℓ_1-norm $|\cdot|_k$ defined within the basis $\left\{X^\beta \mid \beta \in \mathbb{N}^m\right\}$ as follows: for

$$X = \sum_{|\beta| \leq k} C_\beta X^\beta \in \mathcal{U}_k(\mathfrak{g}),$$

we set

$$|X|_k := \sum_{|\beta| \leq k} |C_\beta|.$$

The family of norms $\{|\cdot|_k\}_{k \in \mathbb{N}}$ is then compatible with the filtration of $\mathcal{U}(\mathfrak{g})$ in the sense that if $X \in \mathcal{U}_k(\mathfrak{g})$, then $|X|_k = |X|_l$ whenever $l \geq k$. Now, for a subspace $V \subset \mathfrak{g}$, we denote by $\mathcal{U}(V)$ the unital subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by V, i.e.

$$\mathcal{U}(V) := \mathrm{span}_{\mathbb{R}} \{X_1 \cdots X_n \mid X_j \in V, n \in \mathbb{N}\}.$$

In what follows, for the sake of completeness, we introduce a sequence of definitions and intermediary results that will allow us to enunciate two very important propositions (cf. Propositions 2.2.42 and 2.2.43) concerning the tracial Bieliavsky-Gayral star-product.

Definition 2.2.23 ([20], Definition 2.1)**.** Let G be a connected real Lie group. An element $\mu \in C^\infty\left(G, \mathbb{R}_0^+\right)$ is called a *weight* if it satisfies the following properties:

(i) for every $X \in \mathcal{U}(\mathfrak{g})$, there exists $C_L, C_R > 0$ such that

$$\left|\tilde{X}\mu\right| \leq C_L \mu \text{ and } |\underline{X}\mu| \leq C_R \mu,$$

where $\underline{X}$ is the right invariant differential operator associated with X;

(ii) there exists $C, L, R > 0$ such that for all $g, h \in G$, we have

$$\mu(gh) \leq C \mu(g)^L \mu(h)^R.$$

A pair (L, R) as in the item (ii) is called a *sub-multiplicative degree* of the weight. A weight with sub-multiplicative degree $(1, 1)$ is called a *sub-multiplicative weight*.

Example 2.2.24. On the group $\mathbb{S}$, consider the so-called *modular weight on* $\mathbb{S}$ defined as

$$\mathfrak{d}_{\mathbb{S}} : \mathbb{S} \to \mathbb{R}_0^+ : (a, \ell) \mapsto \sqrt{3 + e^{4a}\left(1 + 4\ell^2\right) + 4\ell^2 + e^{-4a}}.$$

In [20], it is shown in Lemma 2.4 that this function is a sub-multiplicative weight on $\mathbb{S}$.

Given a weight μ on a Lie group G, we can define the following space of smooth functions whose all derivatives are *bounded* by μ (up to a constant).

Definition 2.2.25 ([20], Section 2.1)**.** Let G be a real connected Lie group given with a weight μ. We set[18]

$$\mathcal{B}^{\mu}(G) := \left\{F \in C^{\infty}(G) \mid \forall X \in \mathcal{U}\left(\mathfrak{g}\right), \exists C > 0 : \left|\tilde{X}F\right| \leq C\mu\right\}.$$

When the weight $\mu \equiv 1$, we simply denote this space by $\mathcal{B}(G)$. We endow the space $\mathcal{B}^{\mu}(G)$ with the natural topology associated with the following semi-norms:

$$\|F\|_{k,\mu} := \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sup_{g \in G} \left\{ \frac{\left|\tilde{X}F\right|}{\mu(g)\left|X\right|_k} \right\},$$

with $k \in \mathbb{N}$.

For the particular weight $\mu \equiv 1$, we have the following important result that we quote from [20].

Proposition 2.2.26 ([20], Proposition 2.47)**.** The multiplication $\star^{BG}_{\nu,\vartheta_0}$ is a continuous, bilinear and associative product on the space $\mathcal{B}(\mathbb{S})$.

In order to define the Schwartz space adapted to $\star^{BG}_{\nu,\vartheta_0}$, we first recall the notion of *tempered function*.

Definition 2.2.27. A *tempered function* $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function whose every derivative is bounded by a polynomial function. That is, a smooth function $f = (f_1, \cdots, f_m)$ is tempered if for every multi-index $\alpha \in \mathbb{N}^n$ and $k \in \{1, \cdots, m\}$, there exists a polynomial $p_{\alpha,k}$ such that for all $x \in \mathbb{R}^n$,

$$\left|\partial^{\alpha} f_k(x)\right| \leq p_{\alpha,k}(x).$$

[18]The notation $C^{\infty}(G)$ stands for the complex-valued smooth functions on G. Moreover, recall that $\left(\tilde{X}F\right)(x)$ means the derivation at a point x of a function F with respect to a left invariant vector field $\tilde{X}$ associated with an element $X \in \mathfrak{g}$; explicitly, it is given by $\left(\tilde{X}F\right)(x) := \frac{d}{dt}\Big|_0 F\left(x.\exp(tX)\right)$.

Definition 2.2.28 ([20], Definition 2.17)**.** A real and connected Lie group G is called *tempered* if there exists a global coordinate system $\underline{\Psi} : \mathbb{R}^m \to G$ such that the multiplication and inverse operations seen through $\underline{\Psi}$ are tempered functions in the sense of the previous definition. That is, G is tempered if for all $x, y \in \mathbb{R}^m$,

$$m : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m : (x, y) \mapsto m(x, y) := \underline{\Psi}^{-1}\left(\underline{\Psi}(x).\underline{\Psi}(y)\right)$$

and

$$\underline{i} : \mathbb{R}^m \to \mathbb{R}^m : x \mapsto \underline{i}(x) := \underline{\Psi}\left(\left(\underline{\Psi}(x)^{-1}\right)\right.$$

(respectively the multiplication and inverse of G seen through $\underline{\Psi}$) are tempered functions in the sense of Definition 2.2.27.
A smooth real-valued or complex-valued function f on a tempered Lie group G is called a *tempered function* if $f \circ \underline{\Psi}$ is tempered on $\mathbb{R}^m$ in the sense of Definition 2.2.27, under the identification $\mathbb{C} \simeq \mathbb{R}^2$ (that is, the function f is tempered on G if f is bounded by a polynomial in $\underline{\Psi}^{-1}$-coordinates).

This definition is motivated by the following fact.

Lemma 2.2.29 ([20], Lemma 2.16)**.** Let G be a connected and real Lie group with Lie algebra $\mathfrak{g}$ and let $\underline{\Psi} : \mathbb{R}^m \to G$ be a global diffeomorphism. Then the multiplication and inverse operations seen through $\underline{\Psi}$ are tempered functions if and only if for every element $A \in \mathcal{U}(\mathfrak{g})$, their derivatives along the left invariant differential operator $\tilde{A}$ associated with A are bounded by a function which is polynomial within the chart $\underline{\Psi}$.

Remark 2.2.30. As mentioned in [20] (see Remark 2.18), it can be shown that a smooth function f on a tempered Lie group G is tempered if and only if for any $X \in \mathcal{U}(\mathfrak{g})$, its derivative along $\tilde{X}$ (the left invariant differential operator associated with X) is bounded by a polynomial function within the global chart $\underline{\Psi} : \mathbb{R}^m \to G$.

We now introduce the notion of *tempered pair*.

Definition 2.2.31 ([20], Definition 2.22)**.** Let G be a connected real Lie group with Lie algebra $\mathfrak{g}$ and S be a real-valued smooth function on G. The pair (G, S) is called *tempered pair* if the following two properties are satisfied:

(i) the map

$$\underline{\Phi} : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \left(\tilde{X}S\right)(x)\right] \tag{2.48}$$

is a global diffeomorphism, where $\mathfrak{g}^\star$ is the dual space of $\mathfrak{g}$;

(ii) the inverse map $\underline{\Phi}^{-1} : \mathfrak{g}^\star \simeq \mathbb{R}^m \to G$ endows G with the structure of a tempered Lie group.

Remark 2.2.32. It can be shown (see [20], Remark 2.23) that within the context of the previous definition, the function S is itself automatically tempered (in the sense of Definition 2.2.28).

Let us illustrate this notion by an example concerning the Lie group $\mathbb{S}$.

Lemma 2.2.33. On $\mathbb{S}$, we consider the function $\mathbf{S}$ defined (in terms of coordinates (1.24)) by

$$\mathbf{S} : \mathbb{S} \to \mathbb{R} : (a, \ell) \mapsto \sinh(2a)\ell.$$

The pair $(\mathbb{S}, \mathbf{S})$ is tempered.

Proof. We begin by computing the expression of the map $\underline{\Phi}$ defined by (2.48) in this case. For the basis $\{H, E\}$ of $\mathfrak{s} = \mathrm{Lie}(\mathbb{S})$, we denote by $\{H^\star, E^\star\}$ the corresponding basis of the dual space $\mathfrak{s}^\star$. The expression of $\underline{\Phi} : \mathbb{S} \to \mathfrak{s}^\star$ is thus given for each $(a, \ell) \in \mathbb{S}$ by

$$\begin{aligned}\underline{\Phi}(a, \ell) &= \left(\tilde{H}\mathbf{S}\right)(a, \ell)H^\star + \left(\tilde{E}\mathbf{S}\right)(a, \ell)E^\star \\ &= 2e^{-2a}\ell H^\star + \sinh(2a)E^\star,\end{aligned}$$

where we have used the explicit expressions (1.29) of $\tilde{H}$ and $\tilde{E}$. Under the canonical identification

$$\mathfrak{s}^\star \xrightarrow{\sim} \mathbb{R}^2 : xH^\star + yE^\star \mapsto (x, y),$$

the expression of $\underline{\Phi}$ becomes

$$\underline{\Phi} : \mathbb{S} \to \mathbb{R}^2 : (a, \ell) \mapsto \left(2e^{-2a}\ell, \sinh(2a)\right),$$

which is clearly a diffeomorphism with inverse given by

$$\underline{\Phi}^{-1} : \mathbb{R}^2 \to \mathbb{S} : (x, y) \mapsto \left(\frac{1}{2}\mathrm{arcsinh}(y), \frac{x}{2}e^{\mathrm{arcsinh}(y)}\right).$$

Note that the function

$$e^{\mathrm{arcsinh}(y)} = y + \sqrt{1 + y^2}$$

is clearly a tempered function on $\mathbb{R}$. Let us now show that the inverse map $\underline{\Phi}^{-1}$ endows $\mathbb{S}$ with a structure of tempered Lie group. The multiplication seen

through $\underline{\Phi}^{-1}$ is given for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ by

$$\begin{aligned} m\left((x_1, y_1), (x_2, y_2)\right) &= \underline{\Phi}\left(\underline{\Phi}^{-1}(x_1, y_1).\underline{\Phi}^{-1}(x_2, y_2)\right) \\ &= \left(x_1 e^{-2\mathrm{arcsinh}(y_2)} + x_2 e^{-\mathrm{arcsinh}(y_1)}, \sinh\left(\mathrm{arcsinh}(y_1) + \mathrm{arcsinh}(y_2)\right)\right). \end{aligned}$$

The first component is obviously a tempered function. Moreover, since

$$\sinh\left(\mathrm{arcsinh}(y_1) + \mathrm{arcsinh}(y_2)\right) = y_1\sqrt{1+y_2^2} + y_2\sqrt{1+y_1^2}$$

is clearly a tempered function too, we deduce directly the temperedness of the multiplication m. A similar proof leads to the fact that the inverse map of $\mathbb{S}$ seen through $\underline{\Phi}^{-1}$ is tempered as well. ■

As an illustration of Remark 2.2.32, note that in the context of the previous lemma, the function $\mathbf{S} : \mathbb{S} \to \mathbb{R}$ is tempered. Indeed, for $\underline{\Phi}$ the diffeomorphism described in the above proof, we have

$$\mathbf{S} \circ \underline{\Phi}^{-1} : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \mathbf{S}\left(\underline{\Phi}^{-1}(x, y)\right) = \frac{xy}{2} e^{\mathrm{arcsinh}(y)}.$$

But

$$\frac{xy}{2} e^{\mathrm{arcsinh}(y)} = \frac{xy}{2}\left(y + \sqrt{1+y^2}\right),$$

which can obviously be bounded by a polynomial in x and y, as well as all of its derivatives. This means exactly that the map $\mathbf{S}$ is tempered.

Given a tempered pair (G, S), we consider a vector space decomposition

$$\mathfrak{g} = \bigoplus_{n=0}^{N} V_n, \tag{2.49}$$

and for every $n \in \{0, \cdots, N\}$, an ordered basis $\left\{e_j^n\right\}_{j=1,\cdots,\dim(V_n)}$ of V_n. We thus get a global coordinate system on G given by

$$x_n^j := \left(\widetilde{e_j^n} S\right)(x), \tag{2.50}$$

for $n \in \{0, \cdots, N\}$ and $j \in \{1, \cdots, \dim(V_n)\}$. We choose a scalar product on each V_n and let $|\cdot|_n$ be the associated Euclidean norm. For every $n \in \{0, \cdots, N\}$, we also define the space

$$V^{(n)} := \bigoplus_{k=0}^{n} V_k.$$

Definition 2.2.34 ([20], Definition 2.24)**.** A tempered pair (G, S) is called *admissible* if there exists a decomposition (2.49) with associated coordinate

system (2.50), such that for every $n \in \{0, \cdots, N\}$, there exists an element $X_n \in \mathcal{U}(V_n) \subset \mathcal{U}(\mathfrak{g})$ whose associated multiplier α_n, defined by

$$\tilde{X}_n \mathbf{E} =: \alpha_n \mathbf{E},$$

where $\mathbf{E} := e^{iS}$, satisfies the following properties:

(i) there exists $C_n > 0$ and $\rho_n > 0$ such that

$$|\alpha_n| \geq C_n \left(1 + |x_n|_n^{\rho_n}\right),$$

where $x_n := \left(x_j^n\right)_{j=1,\cdots,\dim(V_n)}$;

(ii) for all $n \in \{0, \cdots, N\}$, there exists a tempered function $\mu_n \in C^\infty\left(G, \mathbb{R}_0^+\right)$ such that

(ii.1) for every $A \in \mathcal{U}\left(V^{(n)}\right) \subset \mathcal{U}(\mathfrak{g})$, there exists $C_A > 0$ such that

$$\left|\tilde{A}\alpha_n\right| \leq C_A \left|\alpha_n\right| \mu_n;$$

(ii.2) μ_n is independent of the variables $\left\{x_r^j\right\}_{j=1,\cdots,\dim(V_r)}$ for all $r \leq n$. That is, for all $r \leq n$ and for all $j \in \{1, \cdots, \dim(V_r)\}$, it satisfies

$$\frac{\partial \mu_n}{\partial x_r^j} = 0.$$

We now pass to the notion of *tameness*.

Definition 2.2.35 ([20], Definition 2.34)**.** A tempered Lie group G with associated diffeomorphism $\underline{\Phi} : G \to \mathfrak{g}^\star$ given by (2.48) is called *tame* if there exists an Euclidean norm $|\cdot|$ on $\mathfrak{g}^\star$, a tempered weight $\mu_{\underline{\Phi}}$ and two positive constants C, ρ such that

$$C\left(1 + |\underline{\Phi}|^2\right)^{\frac{\rho}{2}} \leq \mu_{\underline{\Phi}}.$$

Before introducing the *Schwartz space associated with a tempered pair* (G, S), we give (as an illustration of the previous "technical" definitions) an example of an admissible and tame tempered pair.

Lemma 2.2.36. The tempered pair $(\mathbb{S}, \mathbf{S})$ is admissible and tame.

Proof. We start with the admissibility. Recall that for the group $\mathbb{S}$, the associated global diffeomorphism $\underline{\Phi} : \mathbb{S} \to \mathbb{R}^2$ admits the following expression (within coordinates (1.24)):

$$\underline{\Phi}(a, \ell) = \left(2e^{-2a}\ell, \sinh(2a)\right).$$

Consider the decomposition $\mathfrak{s} = V_0 \oplus V_1$ with $V_0 := \mathbb{R}H$ and $V_1 := \mathbb{R}E$. In the notations of Definition 2.2.34, we thus have the following expressions for the global coordinate system on $\mathbb{S}$:

$$x_0 = 2e^{-2a}\ell,\ x_1 = \sinh(2a).$$

Using the expressions (1.29) of $\tilde{H}$ and $\tilde{E}$, we get for $k \in \mathbb{N}$:

$$\tilde{H}^k x_0 = (-1)^k 2^{2k+1} e^{-2a}\ell$$

and

$$\tilde{H}^k x_1 = 2^k \begin{cases} \cosh(2a) & k \text{ odd} \\ \sinh(2a) & k \text{ even} \end{cases},\ \tilde{E}^k x_1 = 0.$$

For every $X \in \mathcal{U}(V_0)$, we thus deduce that there exists $C_X^0 > 0$ such that

$$\left|\tilde{X}x_0\right| \leq C_X^0 \left(1 + |x_0|\right), \tag{2.51}$$

while for every $X \in \mathcal{U}\left(V^{(1)}\right)$ (with in this case $V^{(1)} = \mathfrak{s}$), we observe that there exists $C_X^1 > 0$ such that

$$\left|\tilde{X}x_1\right| \leq C_X^1 \left(1 + |x_1|\right). \tag{2.52}$$

Define now $X_0 := 1 - H^2 \in \mathcal{U}(V_0)$ and $X_1 := 1 - E^2 \in \mathcal{U}(V_1)$. For $\mathbf{E} := e^{i\sinh(2a)\ell}$, a straightforward computation shows that the associated multipliers α_n (for $n \in \{0,1\}$), defined by

$$\alpha_n := \mathbf{E}^{-1}\tilde{X}_n\mathbf{E},$$

admit the following expressions:

$$\alpha_0 = 1 + |x_0|^2 + 4ix_0,\ \alpha_1 = 1 + |x_1|^2.$$

Remark then that

$$\frac{1}{|\alpha_0|^2} = \frac{1}{\left(1 + |x_0|^2\right)^2 + |x_0|^2} \leq \frac{1}{\left(1 + |x_0|^2\right)^2},$$

which implies that $|\alpha_0| \geq 1 + |x_0|^2$. Thus, the multipliers α_n satisfy the condition (i) of Definition 2.2.34, for $\rho_n = 2$ and $C_n = 1$. Now, using inequalities (2.51) and (2.52), we remark easily that for every $X \in \mathcal{U}(V_0)$, there exists $\tilde{C}_X^0 > 0$ such that

$$\left|\tilde{X}\alpha_0\right| \leq \tilde{C}_X^0 \left|\alpha_0\right|,$$

while for every $X \in \mathcal{U}\left(V^{(1)}\right)$, we can find $\tilde{C}_X^1 > 0$ such that

$$\left|\tilde{X}\alpha_1\right| \leq \tilde{C}_X^1 \left|\alpha_1\right|.$$

The condition (ii) of Definition 2.2.34 is then fulfilled for the tempered weights $\mu_0 \equiv 1$ and $\mu_1 \equiv 1$ and the pair $(\mathbb{S}, \mathbf{S})$ is then admissible.
We now pass to the tameness. Remark that

$$1 + \left|\underline{\Phi}(a,\ell)\right|^2 = 1 + 4e^{-4a}\ell^2 + \sinh^2(2a).$$

Thanks to the expression of $\mathfrak{d}_{\mathbb{S}}$ given in Example 2.2.24, we thus observe that there exists $C > 0$ such that the rough estimate

$$C\left(1 + \left|\underline{\Phi}(a,\ell)\right|^2\right)^{\frac{1}{2}} \leq \mathfrak{d}_{\mathbb{S}}^4(a,\ell)$$

is satisfied for all $(a,\ell) \in \mathbb{S}$. Since $\mathfrak{d}_{\mathbb{S}}^4$ is a (positive) power of a weight, this function is a weight too (see Remark 2.2 in [20]). Moreover, Lemma 2.21 in [20] tells us that on a tempered Lie group (in our case, the group $\mathbb{S}$), the modular weight $\mathfrak{d}_{\mathbb{S}}$ is a tempered function. The function $\mathfrak{d}_{\mathbb{S}}^4$ is thus a tempered function too and this shows the tameness of the pair $(\mathbb{S}, \mathbf{S})$ for the tempered weight $\mu^{\mathbb{S}} := \mathfrak{d}_{\mathbb{S}}^4$. ■

Definition 2.2.37 ([20], Definition 2.40)**.** Let (G,S) be a tempered pair and $\mathbf{E} := e^{iS}$. For all $X \in \mathcal{U}(\mathfrak{g})$, define

$$\alpha_X := \mathbf{E}^{-1}\tilde{X}\mathbf{E} \in C^\infty(G).$$

Then, the space $\mathcal{S}^S(G)$ of smooth functions on G defined as

$$\mathcal{S}^S(G) := \left\{ f \in C^\infty(G) \mid \forall X, Y \in \mathcal{U}(\mathfrak{g}), \forall n \in \mathbb{N}, \sup_{g \in G} \left|\alpha_X^n(g)\left(\tilde{Y}f\right)(g)\right| < +\infty \right\}$$

is called the *Schwartz space associated with the tempered pair* (G,S).

The following result shows that the Schwartz space associated with an admissible tempered pair (G,S) is isomorphic to the standard Schwartz space of the Euclidean space $\mathfrak{g}^\star \simeq \mathbb{R}^m$.

Lemma 2.2.38 ([20], Lemma 2.41)**.** Let (G,S) be an admissible tempered pair and $\underline{\Phi} : G \to \mathfrak{g}^\star$ be the diffeomorphism given by (2.48). Fixing an Euclidean structure on $\mathfrak{g}^\star \simeq \mathbb{R}^m$, we denote by $\mathcal{S}(\mathfrak{g}^\star)$ the standard Schwartz space on $\mathfrak{g}^\star$.

Then, $\mathcal{S}^S(G)$ coincides with

$$\mathcal{S}^{\underline{\Phi}}(G) := \left\{ f \in C^\infty(G) \mid f \circ \underline{\Phi}^{-1} \in \mathcal{S}\left(\mathfrak{g}^\star\right) \right\}.$$

In particular, endowed with the transported topology, $\mathcal{S}^S(G)$ is a nuclear Fréchet space. If moreover the pair (G, S) is tame, then the transported topology on $\mathcal{S}^S(G)$ is equivalent to the one associated with the semi-norms:

$$\|f\|_{k,n} := \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sup_{g \in G} \left\{ \frac{\mu_{\underline{\Phi}}(g)^n \left| \left(\tilde{X} f\right)(g) \right|}{|X|_k} \right\},$$

with $k, n \in \mathbb{N}$ and $\mu_{\underline{\Phi}}$ the tempered weight associated with the tameness.

Recall now that if G is a real connected Lie group, then the product $G \times G$ is naturally endowed with a structure of real connected Lie group too. So, for such a Lie group G, let us consider a tempered, admissible and tame pair $(G \times G, \tilde{S})$ (where $\tilde{S}$ is now a real-valued function defined on $G \times G$). Definition 2.2.37 provides a way to associate a Schwartz space to such a pair. But using the continuity of the partial evaluation maps, we can also define a "one-variable" Schwartz space as follows.

Definition 2.2.39 ([20], Definition 2.48)**.** Let $(G \times G, \tilde{S})$ be a tempered, admissible and tame pair. Then, the *Schwartz space on G associated with $\tilde{S}$* is defined as

$$\mathcal{S}^{\tilde{S}}(G) := \left\{ [G \to \mathbb{C} : g \mapsto f(g, e)] \mid f \in \mathcal{S}^{\tilde{S}}(G \times G) \right\}.$$

We endow the latter with the topology induced by the semi-norms:

$$\|\varphi\|_{k,n} := \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sup_{g \in G} \left\{ \frac{\mu_{\underline{\Phi},1}(g)^n \left| \left(\tilde{X} \varphi\right)(g) \right|}{|X|_k} \right\},$$

with $k, n \in \mathbb{N}$ and where $\mu_{\underline{\Phi},1}(g) := \mu_{\underline{\Phi}}(g, e)$, for $\mu_{\underline{\Phi}}$ the tempered weight on $G \times G$ associated with the tameness of the pair $(G \times G, \tilde{S})$ and e the neutral element of G.

We now come back to the "two-point" phase function S^{BG}_{ν,ϑ_0} given by (2.42), for which we have the following important result, constituting a particular case of Theorem 3.29 of the book [20] (to which we refer for the proof).

Proposition 2.2.40 ([20], Theorem 3.29)**.** Let S^{BG}_{ν,ϑ_0} be the function given by (2.42). Then the pair $\left(\mathbb{S} \times \mathbb{S}, S^{BG}_{\nu,\vartheta_0}\right)$ is tempered, admissible and tame.

Concerning the amplitude function A^{BG}_{ν,ϑ_0}, we have the following lemma (which is a particular case of Lemma 4.8 of the reference [20]).

Lemma 2.2.41 ([20], Lemma 4.8)**.** The amplitude function A^{BG}_{ν,ϑ_0}, given by (2.41), belongs to the space $\mathcal{B}^{\mu_0}(\mathbb{S}\times\mathbb{S})$, for the tempered weight μ_0 defined by

$$\mu_0 : \mathbb{S}\times\mathbb{S} \to \mathbb{R}_0^+ : (x_1, x_2) \mapsto \cosh(a_1)\cosh(a_2),$$

with, within coordinates (1.24), $x_j = (a_j, \ell_j)$ for $j \in \{1, 2\}$.

Let $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ be the one-variable Schwartz space on $\mathbb{S}$ associated with the tempered, admissible and tame pair $\left(\mathbb{S}\times\mathbb{S}, S^{BG}_{\nu,\vartheta_0}\right)$, as in Definition 2.2.39. Proposition 2.2.40 along with Lemma 2.2.41 imply the following most important result of this subsection.

Proposition 2.2.42 ([20], Proposition 4.10)**.** Endowed with the multiplication $\star^{BG}_{\nu,\vartheta_0}$, the space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ becomes an associative Fréchet algebra which acts continuously on $\mathcal{B}^{\mu}(\mathbb{S})$ (for μ a tempered weight), via

$$L_{\star^{BG}_{\nu,\vartheta_0}}(F) : \varphi \mapsto F \star^{BG}_{\nu,\vartheta_0} \varphi,$$

where $\varphi \in \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ and $F \in \mathcal{B}^{\mu}(\mathbb{S})$. In particular, $\left(\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}), \star^{BG}_{\nu,\vartheta_0}\right)$ is an ideal of $\left(\mathcal{B}(\mathbb{S}), \star^{BG}_{\nu,\vartheta_0}\right)$.

The space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ then constitutes a space of smooth functions adapted to the tracial star-product $\star^{BG}_{\nu,\vartheta_0}$, in the sense that this space is closed under the $\star^{BG}_{\nu,\vartheta_0}$-multiplication. Remark now that the constant functions obviously belong to the space $\mathcal{B}(\mathbb{S})$ (which contains the space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$). Thanks to the previous result, for $c \in \mathbb{C}$ a constant (identified with the function $[\mathbb{S} \to \mathbb{C} : x \mapsto c]$) and $\varphi \in \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$, both expressions $\varphi \star^{BG}_{\nu,\vartheta_0} c$ and $c \star^{BG}_{\nu,\vartheta_0} \varphi$ make sense at the non-formal level. The following result shows in particular that, as at the formal level, the constant function 1 is the neutral element for $\star^{BG}_{\nu,\vartheta_0}$.

Proposition 2.2.43 ([20], Proposition 4.11)**.** Identifying an element $c \in \mathbb{C}$ with the function $[\mathbb{S} \to \mathbb{C} : x \mapsto c]$ in $\mathcal{B}(\mathbb{S})$, we have

$$F \star^{BG}_{\nu,\vartheta_0} c = Fc \quad \text{and} \quad c \star^{BG}_{\nu,\vartheta_0} F = cF,$$

for $F \in \mathcal{B}(\mathbb{S})$. In particular, the element $[\mathbb{S} \to \mathbb{C} : x \mapsto 1]$ is the unit of $\left(\mathcal{B}(\mathbb{S}), \star^{BG}_{\nu,\vartheta_0}\right)$.

Characterization of the adapted Schwartz space on $\mathbb{S}$

Let us describe more explicitly the one-point Schwartz function space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ associated with the two-point pair $\left(\mathbb{S}\times\mathbb{S}, S^{BG}_{\nu,\vartheta_0}\right)$. This space is defined as

$$\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}) := \left\{ [\mathbb{S}\to\mathbb{C} : x\mapsto f(x,e)] \mid f\in \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}\times\mathbb{S})\right\}, \tag{2.53}$$

and endowed with the topology induced by the semi-norms:

$$\|\varphi\|_{k,n} := \sup_{X\in\mathcal{U}_k(\mathfrak{s})}\sup_{x\in\mathbb{S}}\left\{\frac{\mu_1^{\mathbb{S}\times\mathbb{S}}(x)^n\left|\left(\tilde{X}\varphi\right)(x)\right|}{|X|_k}\right\},$$

where $\mu_1^{\mathbb{S}\times\mathbb{S}}(x) := \mu^{\mathbb{S}\times\mathbb{S}}(x,e)$, for $\mu^{\mathbb{S}\times\mathbb{S}}$ the tempered weight on $\mathbb{S}\times\mathbb{S}$ associated with the tameness of the pair $(\mathbb{S}\times\mathbb{S}, S^{BG}_{\nu,\vartheta_0})$. In [20], Corollary 3.28, it is mentioned that this tempered weight is given by

$$\mu^{\mathbb{S}\times\mathbb{S}} = \mathfrak{d}^4_{\mathbb{S}\times\mathbb{S}}, \tag{2.54}$$

with, for $x,y\in\mathbb{S}$,

$$\mathfrak{d}_{\mathbb{S}\times\mathbb{S}}(x,y) := \sqrt{\mathfrak{d}^2_{\mathbb{S}}(x)+\mathfrak{d}^2_{\mathbb{S}}(y)-1}, \tag{2.55}$$

where $\mathfrak{d}_{\mathbb{S}}$ is the modular weight on $\mathbb{S}$ given in Example 2.2.24. So, for $x\in\mathbb{S}$,

$$\mu_1^{\mathbb{S}\times\mathbb{S}}(x) = \mathfrak{d}^4_{\mathbb{S}}(x)+\mathfrak{d}^4_{\mathbb{S}}(e)+1+2\mathfrak{d}^2_{\mathbb{S}}(x)\mathfrak{d}^2_{\mathbb{S}}(e)-2\mathfrak{d}^2_{\mathbb{S}}(x)-2\mathfrak{d}^2_{\mathbb{S}}(e). \tag{2.56}$$

Actually, on $\mathbb{S}$, we can also consider two other Schwartz type function spaces. Indeed, using Definition 2.2.37 and Lemma 2.2.33, we have a Schwartz space $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ associated with the tempered pair $(\mathbb{S},\mathbf{S})$. This space is defined as

$$\mathcal{S}^{\mathbf{S}}(\mathbb{S}) := \left\{\varphi\in C^\infty(\mathbb{S}) \mid \forall X,Y\in\mathcal{U}(\mathfrak{s}), \forall n\in\mathbb{N}, \sup_{x\in\mathbb{S}}\left|\alpha_X^n(x)\left(\tilde{Y}\varphi\right)(x)\right| < +\infty\right\}, \tag{2.57}$$

with $\alpha_X := e^{-i\mathbf{S}}\tilde{X}e^{i\mathbf{S}}$. On $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$, Lemmas 2.2.36 and 2.2.38 imply that the topology is the one associated with the semi-norms[19]:

$$\|\varphi\|_{k,n} := \sup_{X\in\mathcal{U}_k(\mathfrak{s})}\sup_{x\in\mathbb{S}}\left\{\frac{\mu^{\mathbb{S}}(x)^n\left|\left(\tilde{X}\varphi\right)(x)\right|}{|X|_k}\right\}.$$

In reference [21], we can find another Schwartz space on $\mathbb{S}$ defined as follows. First, consider the multipliers $\tilde{\alpha}_Y\in C^\infty(\mathbb{S}\times\mathbb{S})$ defined for $Y=(Y_1,Y_2)\in\mathfrak{s}\oplus\mathfrak{s}$

[19] Recall that $\mu^{\mathbb{S}} := \mathfrak{d}^4_{\mathbb{S}}$.

and $x_1, x_2 \in \mathbb{S}$ by

$$\tilde{Y} e^{iS^{BG}_{\nu,\vartheta_0}(x_1,x_2)} =: -i\tilde{\alpha}_Y(x_1,x_2) e^{iS^{BG}_{\nu,\vartheta_0}(x_1,x_2)}.$$

For instance, we compute easily that

$$\tilde{\alpha}_{(H,0)}(x_1,x_2) = 2\left(\cosh(2a_1)\ell_2 + \sinh(2a_2)\ell_1\right)$$

and

$$\tilde{\alpha}_{(E,0)}(x_1,x_2) = -\sinh(2a_2),$$

with $x_j = (a_j, \ell_j)$ (for $j \in \{1,2\}$), within coordinates (1.24). Now, for $X \in \mathfrak{s}$ and $x \in \mathbb{S}$, we set

$$\tilde{\alpha}_X(x) := \tilde{\alpha}_{(X,0)}(e,x).$$

Within coordinates (1.24), we get:

$$\tilde{\alpha}_H(a,\ell) = 2\ell,\ \tilde{\alpha}_E(a,\ell) = -\sinh(2a). \tag{2.58}$$

Following [21], we then define another Schwartz space on $\mathbb{S}$.

Definition 2.2.44. We denote by $\mathcal{S}^{BG}(\mathbb{S})$ the space of smooth functions defined as

$$\mathcal{S}^{BG}(\mathbb{S}) := \left\{ \varphi \in C^\infty(\mathbb{S}) \mid \forall j \in \mathbb{N}^2, \forall X \in \mathcal{U}(\mathfrak{s}), \sup_{x\in\mathbb{S}} \left|\tilde{\alpha}(x)^j \left(\tilde{X}\varphi\right)(x)\right| < +\infty \right\},$$

where, for $j = (j_1, j_2)$, $\tilde{\alpha}^j := \tilde{\alpha}_H^{j_1}\tilde{\alpha}_E^{j_2}$.

We endow the latter with the topology associated with the semi-norms:

$$\|\varphi\|_{k,j} := \sup_{X\in\mathcal{U}_k(\mathfrak{s})} \sup_{x\in\mathbb{S}} \left\{ \frac{\left|\tilde{\alpha}(x)^j \left(\tilde{X}\varphi\right)(x)\right|}{|X|_k} \right\}, \tag{2.59}$$

for $k, j \in \mathbb{N}$.

On the group $\mathbb{S}$, we thus have defined three Schwartz type function spaces, namely $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$, $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ and $\mathcal{S}^{BG}(\mathbb{S})$. It turns out that they are all topologically isomorphic to each other.

Proposition 2.2.45. The three spaces $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$, $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ and $\mathcal{S}^{BG}(\mathbb{S})$ (given respectively by (2.53), (2.57) and Definition 2.2.44) are isomorphic as topological spaces. Moreover, they are all isomorphic to the usual Schwartz space $\mathcal{S}\left(\mathbb{R}^2\right)$.

Proof. We will first show that $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \simeq \mathcal{S}^{BG}(\mathbb{S})$. For this, remark that the space $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ defined by (2.57) corresponds to the following set of smooth functions:

$$\left\{\varphi \in C^{\infty}(\mathbb{S}) \mid \forall Y \in \mathcal{U}(\mathfrak{s}), \forall n \in \mathbb{N}, \sup_{x\in\mathbb{S}} \left|\left(\mu^{\mathbb{S}}(x)\right)^{n} \left(\tilde{Y}\varphi\right)(x)\right| < +\infty\right\}. \quad (2.60)$$

Indeed, a straightforward computation shows that each $\alpha_X = \mathbf{E}^{-1}\tilde{X}\mathbf{E}$, with $X \in \mathcal{U}(\mathfrak{s})$ and $\mathbf{E} := e^{i\mathbf{S}}$, is of the form[20]

$$\alpha_X(a,\ell) = \sum_{K,M,N} c_{k,m,n}\ell^k \left(e^{-2a}\right)^m \sinh^n(2a),$$

where K, M, N are some finite sets of integers, $k \in K$, $m \in M$, $n \in N$ and where $c_{k,m,n} \in \mathbb{C}$ for all k, m, n. Since

$$\ell \leq 3 + 4\ell^2 \leq \mathfrak{d}^2_{\mathbb{S}}(a,\ell) \leq \mathfrak{d}^4_{\mathbb{S}}(a,\ell) = \mu^{\mathbb{S}}(a,\ell)$$

and

$$e^{-2a} \leq 2\sinh(2a) \leq 2\cosh(2a) \leq 2\cosh(4a) \leq \mathfrak{d}^2_{\mathbb{S}}(a,\ell) \leq \mu^{\mathbb{S}}(a,\ell),$$

any power of $|\alpha_X|$ can be bounded by a certain polynomial in $\mu^{\mathbb{S}}$.
By Lemma 2.2.33, we also know that the group $\mathbb{S}$ is tempered, for the tempered structure associated with the map

$$\underline{\Phi} : \mathbb{S} \to \mathbb{R}^2 : (a,\ell) \mapsto \left(2e^{-2a}\ell, \sinh(2a)\right).$$

But a straightforward computation leads to the identities

$$2e^{-2a}\ell = -i\alpha_H(a,\ell),\ \sinh(2a) = -i\alpha_E(a,\ell).$$

Moreover, Lemma 2.21 in [20] tells us that on a tempered Lie group, the associated modular weight is a tempered function. Of course, as any power of a tempered function is again a tempered function, this lemma implies in particular that the function $\mu^{\mathbb{S}}$ is tempered on the tempered pair $(\mathbb{S}, \mathbf{S})$. In other words, this function is bounded by a certain polynomial in $|\underline{\Phi}_1(a,\ell)| = |\alpha_H(a,\ell)|$ and $|\underline{\Phi}_2(a,\ell)| = |\alpha_E(a,\ell)|$. This shows that the sets of functions (2.57) and (2.60) coincide.
Now, remark that for $j = (j_1, j_2) \in \mathbb{N}^2$,

$$\left|\tilde{\alpha}(a,\ell)^j\right| = \left|\tilde{\alpha}_H(a,\ell)^{j_1}\tilde{\alpha}_E(a,\ell)^{j_2}\right| = \left|(2\ell)^{j_1}(\sinh(2a))^{j_2}\right|.$$

[20] Recall that $\mathbf{S}(a,\ell) = \sinh(2a)\ell$, see Lemma 2.2.33.

Since

$$|2\ell| \leq 3 + 4\ell^2 \leq \mathfrak{d}^2_{\mathbb{S}}(a,\ell) \leq \mathfrak{d}^4_{\mathbb{S}}(a,\ell) = \mu^{\mathbb{S}}(a,\ell)$$

and

$$|\sinh(2a)| \leq \cosh(2a) \leq \cosh(4a) \leq \mathfrak{d}^2_{\mathbb{S}}(a,\ell) \leq \mu^{\mathbb{S}}(a,\ell),$$

we have

$$\left|\tilde{\alpha}^j\right| \leq \left(\mu^{\mathbb{S}}(a,\ell)\right)^{j_1+j_2}.$$

It implies that $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \subset \mathcal{S}^{BG}(\mathbb{S})$. For the reverse inclusion, remark that

$$e^{\pm 4a} \leq 2\cosh(4a) \leq 4\cosh^2(2a) = 4\left(1+\sinh^2(2a)\right) = 4\left(1+\tilde{\alpha}_E(a,\ell)^2\right).$$

Since $2\ell = \tilde{\alpha}_H(a,\ell)$, it implies that

$$\mu^{\mathbb{S}}(a,\ell) = \left(3 + e^{4a}\left(1+\ell^2\right) + 4\ell^2 + e^{-4a}\right)^2 \leq p\left(\left|\tilde{\alpha}_H(a,\ell)\right|, \left|\tilde{\alpha}_E(a,\ell)\right|\right),$$

for a certain polynomial p. This shows that any power of $\mu^{\mathbb{S}}$ can be bounded by a polynomial in $|\tilde{\alpha}_H|$ and $|\tilde{\alpha}_E|$ and thus $\mathcal{S}^{BG}(\mathbb{S}) \subset \mathcal{S}^{\mathbf{S}}(\mathbb{S})$. Remark moreover that the above reasoning proves that the topologies induced by the semi-norms of $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ and $\mathcal{S}^{BG}(\mathbb{S})$ are the same, which leads to the topological isomorphism $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \simeq \mathcal{S}^{BG}(\mathbb{S})$.
Now, we show that $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \simeq \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$. First, notice that as in the case of $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$, the space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}\times\mathbb{S})$ can be defined as the set of smooth functions $f \in C^\infty(\mathbb{S}\times\mathbb{S})$ such that $\forall Y \in \mathcal{U}(\mathfrak{s}\oplus\mathfrak{s})$ and $\forall n \in \mathbb{N}$,

$$\sup_{(x,y)\in\mathbb{S}\times\mathbb{S}} \left|\left(\mu^{\mathbb{S}\times\mathbb{S}}(x,y)\right)^n \left(\tilde{Y} f\right)(x,y)\right| < +\infty. \tag{2.61}$$

Using the fact that $\mathfrak{d}_{\mathbb{S}}(e) = \sqrt{5}$, the expression of $\mu_1^{\mathbb{S}\times\mathbb{S}}$ given by (2.56) implies the existence of constants $c_1, c_2 > 0$ such that for all $x \in \mathbb{S}$,

$$\mu_1^{\mathbb{S}\times\mathbb{S}}(x) = \mathfrak{d}^4_{\mathbb{S}}(x) + c_1\mathfrak{d}^2_{\mathbb{S}}(x) + c_2.$$

This in turn implies that there exists a constant $c > 0$ such that for all $x \in \mathbb{S}$,

$$\mu^{\mathbb{S}}(x) \leq \mu_1^{\mathbb{S}\times\mathbb{S}} \leq c\mu^{\mathbb{S}}(x). \tag{2.62}$$

For $f \in \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}\times\mathbb{S})$, remark now that a particularization of (2.61) implies that the associated "one-variable" function $\underline{f} := f(\cdot, e)$ satisfies[21]

$$\sup_{x\in\mathbb{S}} \left|\left(\mu_1^{\mathbb{S}\times\mathbb{S}}(x)\right)^n \left(\tilde{X}\underline{f}\right)(x)\right| < +\infty,$$

[21] Recall that $\mu^{\mathbb{S}\times\mathbb{S}}(x,e) =: \mu_1^{\mathbb{S}\times\mathbb{S}}(x)$.

for all $X \in \mathcal{U}(\mathfrak{s})$ and for all $n \in \mathbb{N}$. The first inequality in (2.62) then shows that

$$\sup_{x\in\mathbb{S}} \left| \left(\mu^{\mathbb{S}}(x)\right)^n \left(\tilde{X}\underline{f}\right)(x) \right| < +\infty,$$

which means that $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}) \subset \mathcal{S}^{\mathbf{S}}(\mathbb{S})$.
For the reverse inclusion, notice that for $\varphi \in \mathcal{S}^{\mathbf{S}}(\mathbb{S})$ non identically equal to 0, we can assume without loss of generality that $\varphi(e) \neq 0$. The function $\varphi \otimes \varphi$ defined as

$$\varphi \otimes \varphi : \mathbb{S} \times \mathbb{S} : (x, y) \mapsto \varphi \otimes \varphi(x, y) := \varphi(x)\varphi(y),$$

then belongs to $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S} \times \mathbb{S})$. It is straightforward by observing that, thanks to identities (2.54) and (2.55), $\mu^{\mathbb{S}\times\mathbb{S}}(x, y)$ is equal to a certain polynomial in $\mathfrak{d}_{\mathbb{S}}(x)$ and $\mathfrak{d}_{\mathbb{S}}(y)$. The function $\mu^{\mathbb{S}\times\mathbb{S}}(x, y)$ can thus be bounded by a polynomial in $\mu^{\mathbb{S}}(x)$ and $\mu^{\mathbb{S}}(y)$. Since $\frac{1}{\varphi(e)}\varphi \otimes \varphi(x, e) = \varphi(x)$, we thus have shown that $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \subset \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$. Moreover, inequalities (2.62) imply that the topologies induced by the semi-norms of $\mathcal{S}^{\mathbf{S}}(\mathbb{S})$ and $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ are the same, which leads to the second topological isomorphism $\mathcal{S}^{\mathbf{S}}(\mathbb{S}) \simeq \mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$.
The last claim of the proposition comes from Lemmas 2.2.36 and 2.2.38. ■

Remark 2.2.46. The above proposition allows to have a better understanding of the behavior of the functions belonging to the Schwartz space $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S})$ (given by (2.53)), adapted to the strongly closed star-product $\star^{BG}_{\nu,\vartheta_0}$. Indeed, it turns out that thanks to the isomorphism $\mathcal{S}^{S^{BG}_{\nu,\vartheta_0}}(\mathbb{S}) \simeq \mathcal{S}^{BG}(\mathbb{S})$, this space corresponds to the usual Schwartz space in coordinates (x, y) with $x = \sinh(2a)$ and $y = \ell$ (see [21]). That is, setting

$$\underline{\widetilde{\Phi}} : \mathbb{S} \to \mathbb{R}^2 : (a, \ell) \mapsto (x, y) := (\sinh(2a), \ell),$$

the function $\varphi \circ \underline{\widetilde{\Phi}}^{-1}$ belongs to the usual Schwartz space on $\mathbb{R}^2$ whenever $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. This means that any derivative of $\varphi(a, \ell)$ (with respect to the left invariant differential operators) is bounded by a polynomial in $(x, y) = (\sinh(2a), \ell)$.

We now show that the space $\mathcal{S}^{BG}(\mathbb{S})$ injects continuously into the standard Schwartz space on $\mathbb{S}$ (within coordinates (1.24)).

Lemma 2.2.47. The inclusion $\mathcal{S}^{BG}(\mathbb{S}) \hookrightarrow \mathcal{S}(\mathbb{S})$ is continuous.

Proof. Let $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. Using Definition 2.2.44, we know that this space corresponds to the usual Schwartz space in coordinates $(x, y) := (\sinh(2a), \ell)$. Within the notations of Definition 2.2.44 and using expressions (2.58), remark

that since $\tilde{H}_{(a,\ell)} = \partial_a - 2\ell\partial_\ell$ and $\tilde{E}_{(a,\ell)} = \partial_\ell$, we have

$$\partial_a = \tilde{H}_{(a,\ell)} + \alpha_H(a,\ell)\tilde{E}_{(a,\ell)}.$$

Moreover, for all $a \in \mathbb{R}$, $|a| \leq |\sinh(a)| = |\alpha_E(a,\ell)|$. So, for given $k, p, m, n \in \mathbb{N}$ and for all $(a,\ell) \in \mathbb{S}$, we have

$$\left|a^k \ell^p \partial_a^m \partial_\ell^n \varphi(a,\ell)\right| \leq \left|\alpha_E^k(a,\ell)\alpha_H^p(a,\ell)\left(\tilde{H}_{(a,\ell)} + \alpha_H(a,\ell)\tilde{E}_{(a,\ell)}\right)^m \tilde{E}_{(a,\ell)}^n \varphi\right|,$$

where the latter is bounded thanks to the fact that $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. That completes the proof, by definition of the semi-norms inducing the Fréchet topologies of $\mathcal{S}(\mathbb{S})$ and $\mathcal{S}^{BG}(\mathbb{S})$. ■

Remark 2.2.48. Combining Propositions 2.2.42, 2.2.43 and 2.2.45, we observe that $\mathcal{S}^{BG}(\mathbb{S})$ is then an associative and closed Fréchet algebra for the product $\star^{BG}_{\nu,\vartheta_0}$. Moreover, for $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$, the following identities hold:

$$\varphi \star^{BG}_{\nu,\vartheta_0} 1 = 1 \star^{BG}_{\nu,\vartheta_0} \varphi = \varphi.$$

From now on, in order to lighten the notations, the star-product $\star^{BG}_{\nu,\vartheta_0}$ will be simply denoted by $\star^{BG}_{\nu}$.

As announced previously, the purpose of this work is twofold. On one hand, we want to describe the set of $\mathbb{S}$-invariant intertwiners between BG and RC star-products. On the other hand, we want to discuss the existence (or not) of an oscillatory integral type formula for the RC deformation, together with an algebra of (smooth) *functions*[22], closed for the RC deformation, turning $\star^{RC}_{\nu}$ into a non-formal star-product. As we will see, both goals are closely related to each other. In the next chapter, we present a method, named *retract method* (or *retract* in short), that will allow us to obtain the desired set of $\mathbb{S}$-invariant intertwiners by solving some systems of partial differential equations (PDE's in short).

[22]We insist on the fact that we use the terminology *function* to contrast with *formal power series*.

Chapter 3

Non-formal Rankin-Cohen star-product

In this chapter, we first introduce a method allowing to compute in an explicit way the set of all $\mathbb{S}$-invariant intertwiners between the star-products $\star_\nu^{BG}$ and $\star_\nu^{RC}$. After that, we focus on the analytical aspects of such an intertwiner in order to derive a non-formal expression for the Rankin-Cohen deformation.

3.1 The retract method

In this section, we generalize a method used by the authors in [19] to derive the expression of all invariant deformation quantizations on the hyperbolic plane. Roughly speaking, their method consists in computing first the set of invariant deformation quantizations on a curvature contraction of the hyperbolic plane, that is, the same manifold but endowed with a connection whose some components of the Riemann curvature tensor are equal to zero. The (intuitive) advantage is that the every geometric theory on such a contraction appears as simpler than in the initial framework. Then, in order to recover the deformation quantizations on the initial space (without curvature contraction), the authors of [19] compute the set of operators that reverse the contraction process[1]. The important point is that the method used for computing those operators relies on the fact that both hyperbolic plane and its curvature contraction share *common symmetries*, in the sense that there exists a Lie group acting simultaneously on both the initial manifold and the contracted one.

[1] That is the reason why we will call such a procedure a *retract method*.

In what follows, we describe a way to get the expressions of (invariant) intertwiners between two invariant star-products, via the resolution of particular systems of (distributional) PDE's.

Let $\left(M_1,\omega^1\right),\left(M_2,\omega^2\right)$ be two symplectic manifolds and G_1,G_2 two Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, acting respectively (possibly locally only[2]) on M_1,M_2, through some (local) maps τ^1,τ^2:

$$\tau^1 : G_1 \times M_1 \to M_1 : (g_1,x_1) \mapsto \tau^1_{g_1}(x_1),$$

$$\tau^2 : G_2 \times M_2 \to M_2 : (g_2,x_2) \mapsto \tau^2_{g_2}(x_2).$$

The groups G_1 and G_2 are called *symmetry groups* (or *local symmetry groups* in the case of a local action) of M_1 and M_2. Assume that M_1 and M_2 share common symmetries, that is, that there exists a Lie group $\mathbb{B}$ injecting simultaneously in G_1 and G_2 (i.e. such that there exists two maps $\mathbb{B}\hookrightarrow G_1$, $\mathbb{B}\hookrightarrow G_2$) such that the (transported) actions[3]

$$\tau^j : \mathbb{B}\times M_j \to M_j : (x,p_j) \mapsto \tau^j_x(p_j)$$

for $j\in\{1,2\}$ are (at least locally) simply transitive. We then get the $\mathbb{B}$-equivariant identifications

$$M_1 \simeq \mathbb{B} \simeq M_2.$$

Consider now two formal star-products $\left(\mathbb{A}_1,\star^1_\nu\right),\left(\mathbb{A}_2,\star^2_\nu\right)$ (with deformation parameter ν) on M_1,M_2 respectively (where, for $j\in\{1,2\}$, $\mathbb{A}_j\subset C^\infty(M_j)[[\nu]]$ is an algebra). Remark that under the previous identifications, for $j\in\{1,2\}$, $\mathbb{A}_j\subset C^\infty\left(\mathbb{B}\right)[[\nu]]$ (under the left regular representation). We assume moreover that $\mathcal{D}(\mathbb{B})\subset\mathbb{A}_j$, for $j\in\{1,2\}$.
Denoting by $\mathfrak{b}:=\mathrm{Lie}(\mathbb{B})$, we get for $j\in\{1,2\}$ the following injections:

$$\mathfrak{g}_j \hookrightarrow \mathrm{Der}\left(\star^j_\nu\right) : Y_j \mapsto Y_j^\star.$$

In particular,

$$\mathfrak{b} \hookrightarrow \mathrm{Der}\left(\star^j_\nu\right) : X \mapsto X^\star.$$

[2]By local action, we mean an action of the Lie algebra $\mathfrak{g}_j$ by derivations on M_j, for $j\in\{1,2\}$.

[3]We mean actions through the injections $\mathbb{B}\hookrightarrow G_1$ and $\mathbb{B}\hookrightarrow G_2$, that we again denote by τ^j for $j\in\{1,2\}$, with a slight abuse of notation.

Assume now that $\star_\nu^1$ and $\star_\nu^2$ are in the same $\mathbb{B}$-equivalence class. Then there exists an equivalence between $\star_\nu^1$ and $\star_\nu^2$ (for the left regular representation):

$$T : \mathbb{A}_1 \xrightarrow{\sim} \mathbb{A}_2,$$

such that for all $x \in \mathbb{B}$,

$$L_x^\star T = T L_x^\star. \tag{3.1}$$

The above relation is referred to as the $\mathbb{B}$-*equivariance* of T.

Remark 3.1.1. In equation (3.1), we stress that the left actions in the LHS and the RHS are not exactly the same (because there are seen through two different identifications), but are denoted identically with a slight abuse of notation. In the same way, we will sometimes speak of such an operator T as an $\mathbb{B}$-equivalence, in the sense of Definition 2.1.15, in which the *invariance* of the sequence of the differential operators $\{T_k\}_k$ defining the operator T will be understood as the *equivariance* of the operators T_k's (in the sense of the equation (3.1)).

On $\mathcal{D}(\mathbb{B}) \subset \mathbb{A}_j$ (for $j \in \{1, 2\}$), we can express T as a (formal) convolution operator. Indeed, the Schwartz kernel theorem allows to express T in terms of a formal distributional kernel. More precisely, for $\varphi \in \mathcal{D}(\mathbb{B})$ and $x \in \mathbb{B}$, we have[4]

$$T\varphi(x) = \int_{\mathbb{B}} u(x, y)\varphi(y) d_{\mathbb{B}}^L(y), \tag{3.2}$$

for u a distribution in the whole generality. On one hand, note that for $x_0 \in \mathbb{B}$, we then have:

$$L_{x_0}^\star \left(T\varphi\right)(x) = \int_{\mathbb{B}} u(x_0 x, y)\varphi(y) d_{\mathbb{B}}^L(y).$$

On the other hand, by left invariance of $d_{\mathbb{B}}^L(y)$, we also have

$$T\left(L_{x_0}^\star \varphi\right)(x) = \int_{\mathbb{B}} u(x, y)\varphi(x_0 y) d_{\mathbb{B}}^L(y) = \int_{\mathbb{B}} u(x, x_0^{-1} y)\varphi(y) d_{\mathbb{B}}^L(y).$$

But the $\mathbb{B}$-equivariance of T (3.1) implies in turn the following (distributional) equality:

$$u(x, x_0^{-1} y) = u(x_0 x, y).$$

Remark that for the particular choice $x = e$, the previous identity becomes

$$u(e, x_0^{-1} y) = u(x_0, y). \tag{3.3}$$

[4]Recall that $d_{\mathbb{B}}^L$ denotes the left invariant Haar measure on the group $\mathbb{B}$.

Remark 3.1.2. For f a function on $\mathbb{B}$, we denote by $\check{f}$ the function defined for all $x \in \mathbb{B}$ by $\check{f}(x) := f\left(x^{-1}\right)$.

For all $x \in \mathbb{B}$, we set $\check{u_T}(x) := u(e, x)$, i.e.

$$u_T(x) := u\left(e, x^{-1}\right). \tag{3.4}$$

Combining identities (3.2), (3.3) and (3.4), we get the following convolution form for the operator T:

$$T\varphi(x) = \int_{\mathbb{B}} u_T\left(y^{-1}x\right)\varphi(y) d^L_{\mathbb{B}}(y).$$

Remark 3.1.3. Hereafter, we will sometimes denote such an operator T by $u_T \times \cdot$, to insist on the fact that it is a convolution operator with associated convolution kernel u_T.

This operator T allows us to define a Lie algebra morphism as follows:

$$\begin{aligned} D : \mathfrak{g}_2 &\to \mathrm{Der}\left(\star^1_\nu\right) \\ Y &\mapsto D_Y := T^{-1} Y^\star T. \end{aligned}$$

Remark that for $X \in \mathfrak{b}$, $D_X = X^\star$, the infinitesimal translation of the $\mathbb{B}$-equivariance of T meaning exactly that for all $X \in \mathfrak{b}$,

$$TX^\star = X^\star T.$$

The map D then extends to $\mathfrak{g}_2$ the injection $\mathfrak{b} \hookrightarrow \mathrm{Der}\left(\star^1_\nu\right)$, into a Lie algebra morphism. Assume now that all derivations of $\star^1_\nu$ are inner (in the sense of Remark 1.2.29). In particular, taking $Z \in \mathfrak{g}_1$, we can write $Z^\star$ in the following way:

$$Z^\star = \frac{1}{2\nu}\left[\Phi_Z, \cdot\right]_{\star^1_\nu},$$

for some map $\Phi_Z \in C^\infty(M_1)[[\nu]]$ defined up to additive formal constants. If Φ_X is known explicitly for all X in the basis of $\mathfrak{b}$, it is then possible to characterize all possible Lie algebra morphisms D extending $\mathfrak{b} \hookrightarrow \mathrm{Der}\left(\star^1_\nu\right)$, in terms of a system of ordinary differential equations (ODE's). Indeed, let V_2 be a vector space such that $\mathfrak{g}_2 = \mathfrak{b} \oplus V_2$ (as a vectorial decomposition) and let $\{Y_l\}_l$ be a basis of V_2 and $\{X_k\}_k$ a basis of $\mathfrak{b}$. Denoting by $[\cdot,\cdot]_2$ the Lie bracket on $\mathfrak{g}_2$, we have

$$[X_k, Y_l]_2 = \sum_a C^a_{k,l} X_a + \sum_b \underline{C}^b_{k,l} Y_b,$$

for some constants $C_{k,l}^a, \underline{C}_{k,l}^b$. Moreover, for $Y \in \mathfrak{g}_2$, $D_Y \in \mathrm{Der}\left(\star_\nu^1\right)$ and we have

$$D_Y = \frac{1}{2\nu}\left[\Phi_Y, \cdot\right]_{\star_\nu^1}.$$

Since D is a Lie algebra morphism, we have

$$D_{[X_k, Y_l]_2} = \left[D_{X_k}, D_{Y_l}\right], \tag{3.5}$$

where the bracket in the RHS is the commutator of derivations. Firstly, the linearity of D yields

$$\begin{aligned} D_{[X_k,Y_l]_2} &= \sum_a C_{k,l}^a D_{X_a} + \sum_b \underline{C}_{k,l}^b D_{Y_b} \\ &= \frac{1}{2\nu}\left[\sum_a C_{k,l}^a \Phi_{X_a} + \sum_b \underline{C}_{k,l}^b \Phi_{Y_b}, \cdot\right]_{\star_\nu^1}. \end{aligned}$$

Secondly, using the Jacobi identity, we get

$$\begin{aligned} \left[D_{X_k}, D_{Y_l}\right] &= D_{X_k} \circ D_{Y_l} - D_{Y_l} \circ D_{X_k} \\ &= \frac{1}{2\nu}\left[\Phi_{X_k}, \frac{1}{2\nu}\left[\Phi_{Y_l}, \cdot\right]_{\star_\nu^1}\right]_{\star_\nu^1} - \frac{1}{2\nu}\left[\Phi_{Y_l}, \frac{1}{2\nu}\left[\Phi_{X_k}, \cdot\right]_{\star_\nu^1}\right]_{\star_\nu^1} \\ &= \frac{1}{2\nu}\left[\frac{1}{2\nu}\left[\Phi_{X_k}, \Phi_{Y_l}\right]_{\star_\nu^1}, \cdot\right]_{\star_\nu^1} \\ &= \frac{1}{2\nu}\left[X_k^\star\left(\Phi_{Y_l}\right), \cdot\right]_{\star_\nu^1}. \end{aligned}$$

For all X_k in the basis of $\mathfrak{b}$ and Y_l in the basis of V_2, (3.5) then leads to the following ODE's:

$$X_k^\star\left(\Phi_{Y_l}\right) = \sum_a C_{k,l}^a \Phi_{X_a} + \sum_b \underline{C}_{k,l}^b \Phi_{Y_b}, \tag{3.6}$$

up to additive formal constants. Once the solutions Φ_{Y_l} obtained for all Y_l in the basis of V_2, we can deduce the expression of D_Y for all $Y \in \mathfrak{g}_2$ by use of linear extensions.

It remains then to compute the expression of the kernel u_T associated with the operator T. It is actually possible if the Haar measure $d_{\mathbb{B}}^L(y)$ is preserved under $Y^\star$ for all $Y \in \mathfrak{g}_2$. Indeed, remark that using the same arguments as before, the inverse T^{-1} of T is also a convolution operator with associated kernel denoted

by $u_{T^{-1}}$. Moreover, for $\varphi \in \mathcal{D}(\mathbb{B})$, $x \in \mathbb{B}$ and $Y \in \mathfrak{g}_2$,

$$D_Y\left(T^{-1}\varphi\right)(x) = T^{-1}\left(Y^\star\varphi\right)(x)$$
$$D_Y|_x \int_{\mathbb{B}} u_{T^{-1}}\left(x^{-1}y\right)\varphi(y)d^L_{\mathbb{B}}(y) = \int_{\mathbb{B}} u_{T^{-1}}\left(x^{-1}y\right) Y^\star_y\varphi d^L_{\mathbb{B}}(y),$$

where $|_x$ means that the operator is applied by considering the function following it as a function depending only on the x-variable. If $Y^\star$ preserves $d^L_{\mathbb{B}}(y)$, then

$$\int_{\mathbb{B}} u_{T^{-1}}\left(x^{-1}y\right) Y^\star_y\varphi d^L_{\mathbb{B}}(y) = -\int_{\mathbb{B}} Y^\star|_y\, u_{T^{-1}}\left(x^{-1}y\right)\varphi(y)d^L_{\mathbb{B}}(y).$$

This leads to the following distributional equation on $u_{T^{-1}}$, for any $Y \in \mathfrak{g}_2$:

$$D_Y|_x\, u_{T^{-1}}\left(x^{-1}y\right) = -\,Y^\star|_y\, u_{T^{-1}}\left(x^{-1}y\right). \tag{3.7}$$

The solutions of those *retract equations* leading to an associated invertible operator then give explicit (formal) expressions of intertwiners between $\star^1_\nu$ and $\star^2_\nu$. All the above development is summarized by the following proposition.

Proposition 3.1.4 (Retract)**.** Let $(M_1, \star^1_\nu), (M_2, \star^2_\nu)$ be two symplectic manifolds given with an associated star-product (for which ν is the deformation parameter). Let G_1 and G_2 be two (possibly local only) symmetry groups of M_1 and M_2, with respective Lie algebras $\mathfrak{g}_1 := \mathrm{Lie}\,(G_1)$, $\mathfrak{g}_2 := \mathrm{Lie}\,(G_2)$, and $\mathbb{B}$ a Lie group (with associated Lie algebra $\mathfrak{b}$) injecting simultaneously into G_1 and G_2, such that its action on both manifolds M_1 and M_2 is simply transitive. Assume in addition that

(i) $\star^1_\nu$ and $\star^2_\nu$ are in the same class of $\mathbb{B}$-equivalence;

(ii) all derivations of $\star^1_\nu$ are inner and for all X in the basis of $\mathfrak{b}$, $X^\star = \frac{1}{2\nu}\left[\Phi_X, \cdot\right]_{\star^1_\nu}$, for some known maps $\Phi_X \in C^\infty(M_1)[[\nu]]$ defined up to additive formal constants ;

(iii) the action of $\mathfrak{g}_2$ preserves the left invariant Haar measure on $\mathbb{B}$.

Then, for each $\mathbb{B}$-equivariant intertwiner T such that $T\left(\star^1_\nu\right) = \star^2_\nu$, both T and T^{-1} are convolution operators. Moreover, for all $x, y \in \mathbb{B} \simeq M_1 \simeq M_2$ and for all $Y \in \mathfrak{g}_2$, the kernel $u_{T^{-1}}$ of T^{-1} is solution of a distributional equation of type

$$D_Y|_x\, u_{T^{-1}}\left(x^{-1}y\right) = -\,Y^\star|_y\, u_{T^{-1}}\left(x^{-1}y\right),$$

where $D : \mathfrak{g}_2 \to \mathrm{Der}\left(\star^1_\nu\right)$ is a certain Lie algebra morphism extending the injection $\mathfrak{b} \hookrightarrow \mathrm{Der}\left(\star^1_\nu\right)$, associated with the solution of a system of ODE's of type (3.6).

3.2 Intertwining of RC and BG star-products via retract

In this section, we apply the retract method (that is, Proposition 3.1.4) to the particular case of RC and BG star-products. For this, we choose $M_1 = M_2 := \mathbb{S}$ endowed with the symplectic structure $\omega^{\mathbb{S}}$, $\mathbb{A}_1 = \mathbb{A}_2 := C^\infty(\mathbb{S})[[\nu]]$, $\mathbb{B} := \mathbb{S}$, $\mathfrak{g}_1 := \mathfrak{g}_{BG}$, $\mathfrak{g}_2 := \mathfrak{g}_{RC}$, $\star_\nu^1 := \star_\nu^{BG}$ and $\star_\nu^2 := \star_\nu^{RC}$. It is worth mentioning that in what follows, in many situations, the computations will be understood in a heuristic way. To be more precise, a lot of integrals will be handled without any justification regarding their existence and well-definedness in a precise analytical context. The main reason is that the only goal of the present section consists in providing an explicit expression for the intertwiners of $\star_\nu^{BG}$ and $\star_\nu^{RC}$, for which we will provide an in-depth analysis (regarding their properties and the meaning of the integrals used) in section 3.3.

We start by describing the set of all derivations D extending the injection $\mathfrak{s} \hookrightarrow \mathrm{Der}\left(\star_\nu^{BG}\right)$.

3.2.1 Characterization of the morphisms defining the retract equations

Let us first consider the algebra $\mathfrak{g}_{RC}$ (1.22), that is the algebra of invariance of the RC star-product. This algebra decomposes as $\mathfrak{s} \oplus V_{RC}$, with, as vector spaces, $\mathfrak{s} = \mathrm{span}_{\mathbb{R}}\{H, E\}$ and $V_{RC} := \mathrm{span}_{\mathbb{R}}\{F, Q, P, Z\}$ (the table for those elements is given in (1.23)). If we denote by $\{X_k\}_k$ (for $k \in \{1, 2\}$) the basis of $\mathfrak{s}$ and $\{Y_l\}_l$ (for $l \in \{1, \ldots, 4\}$) the one of V_{RC}[5], then for all k and for all l, we can write

$$[X_k, Y_l] = \sum_a C_{kl}^a X_a + \sum_b \underline{C}_{kl}^b Y_b,$$

with $a \in \{1, 2\}$, $b \in \{1, \ldots, 4\}$ and where $C_{k,l}^a, \underline{C}_{k,l}^b$ are some real constants. We thus have to find all the Lie algebra morphisms D's such that

$$D_{[X_k, Y_l]} = [D_{X_k}, D_{Y_l}].$$

We will need the following result from [56].

Theorem 3.2.1 ([56], Theorem 8.2)**.** Let M be a symplectic manifold endowed with a star-product $\star_\nu$. If $H^1_{dR}(M) = \{0\}$, then all derivations of $\star_\nu$ are inner.

[5]We mean $X_1 := H$, $X_2 := E$, $Y_1 := F$, $Y_2 := Q$, $Y_3 := P$, $Y_4 := Z$.

In the case of the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, since $H^1_{dR}(\mathbb{S}) \simeq H^1_{dR}\left(\mathbb{R}^2\right)$, the Poincaré lemma implies that

$$H^1_{dR}(\mathbb{S}) = \{0\}.$$

From the previous theorem, we can thus deduce that for any $Y \in \mathfrak{g}_{RC}$, the associated derivation D_Y is inner, i.e. there exists some $\Phi_Y \in C^\infty(\mathbb{S})[[\nu]]$ such that

$$D_Y = \frac{1}{2\nu}\left[\Phi_Y, \cdot\right]_{\star_\nu^{\mathrm{BG}}}.$$

Remark 3.2.2. As mentioned in the previous section, for an element $X \in \mathfrak{s}$, the associated derivation reduces to $D_X = X^\star$. Moreover, the star-product $\star_\nu^{BG}$ being a twist on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ (see Proposition 2.1.13), we get

$$D_X = X^\star = \frac{1}{2\nu}\left[\lambda^{BG}_X, \cdot\right]_{\star_\nu^{BG}},$$

for all $X \in \mathfrak{s}$ and with λ^{BG}_X defined by (2.3). Thus, the maps Φ mentioned in the item (ii) of Proposition 3.1.4 are in the case of $\mathfrak{s}$ given by λ^{BG}. We thus set

$$\Phi_H := \lambda^{BG}_H,\ \Phi_E := \lambda^{BG}_E,$$

up to additive formal constants.

Remark 3.2.3. Thanks to Proposition 2.1.18, there is no need to consider the derivation D_F (which is actually quite complicated). Indeed, the solution of the retract equations (3.7) will yield an intertwiner T such that the star-product $T\left(\star_\nu^{BG}\right)$ will be locally $\mathbb{S} \ltimes H_V$-invariant. Proposition 2.1.18 will then automatically implies the $SL_2(\mathbb{R}) \ltimes H_V$-invariance of this star-product. In other words, we can restrict ourselves to the subalgebra $\tilde{\mathfrak{g}}_{RC} \subset \mathfrak{g}_{RC}$ of invariance for $\star_\nu^{RC}$, with $\tilde{\mathfrak{g}}_{RC} := \mathrm{span}_{\mathbb{R}}\{H, E, Q, P, Z\}$.

In our particular case, recalling that (within coordinates (1.24)) $H^\star_{(a,\ell)} = -\partial_a$ and $E^\star_{(a,\ell)} = -e^{-2a}\partial_\ell$ (see (2.2)), equations (3.6) lead to the following systems for elements Q and P (we will consider the element Z later):

$$1)\begin{cases} -\partial_a\Phi_Q &= \Phi_Q + c(H,Q) \\ -e^{-2a}\partial_\ell\Phi_Q &= c(E,Q) \end{cases}, \quad 2)\begin{cases} -\partial_a\Phi_P &= -\Phi_P + c(H,P) \\ -e^{-2a}\partial_\ell\Phi_P &= \Phi_Q + c(E,P) \end{cases},$$

where, for all $k \in \{1,2\}$ and for all $l \in \{1,\ldots,4\}$, $c(X_k, Y_l) \in \mathbb{C}[[\nu]]$. In order to get solutions for systems 1 and 2, a straightforward computation shows that we have to choose $c(E,Q) = 0$ and $c(E,P) = c(H,Q)$. Within the global chart

(1.24), the general solution of the system 1 is then of the form

$$\Phi_Q(a,\ell) = K_1 e^{-a} - c\,(H,Q)\,,$$

for some constant $K_1 \in \mathbb{C}[[\nu]]$, while the general solution of the system 2 is given by

$$\Phi_P(a,\ell) = (K_2 - K_1\ell)\, e^{a} + c\,(H,P)\,,$$

for some constant $K_2 \in \mathbb{C}[[\nu]]$.

Remark 3.2.4. Note that in the case of the retract equations (3.7), we are more interested in the expressions of the derivations $\frac{1}{2\nu}\left[\Phi_Y,\cdot\right]_{\star_\nu^{BG}}$ (with $Y \in \mathfrak{g}_{RC}$) than in the functions Φ_Y themselves. Since for any $C \in \mathbb{C}[[\nu]]$, we have

$$\frac{1}{2\nu}\left[\Phi_Y + C,\cdot\right]_{\star_\nu^{BG}} = \frac{1}{2\nu}\left[\Phi_Y,\cdot\right]_{\star_\nu^{BG}}\,,$$

we make the particular choices $c\,(H,Q) := 0$ and $c\,(H,P) := 0$ for the maps Φ_Q and Φ_P. That reduces their respective expressions to

$$\Phi_Q(a,\ell) = K_1 e^{-a},\ \Phi_P(a,\ell) = (K_2 - K_1\ell)\, e^{a}. \tag{3.8}$$

Accordingly, we define the following derivations of $\star_\nu^{BG}$:

$$D_Q := \frac{1}{2\nu}\left[\Phi_Q,\cdot\right]_{\star_\nu^{BG}}\,,\ D_P := \frac{1}{2\nu}\left[\Phi_P,\cdot\right]_{\star_\nu^{BG}}\,.$$

Remark 3.2.5. Since we look for *morphisms*[6] $D : \tilde{\mathfrak{g}}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right)$, we require in particular the relation

$$D_{[Q,P]} = [D_Q, D_P]$$

to be fulfilled. Using the fact that D is inner (with respect to a map Φ), the Jacobi identity implies that up to an additive formal constant, the following identity holds:

$$\Phi_{[Q,P]} = \frac{1}{2\nu}\left[\Phi_Q,\Phi_P\right]_{\star_\nu^{BG}}\,.$$

By relations (1.23), we know that $[Q,P] = Z$ and thus $\Phi_{[Q,P]} = \Phi_Z$. But a straightforward computation shows that $\left[\Phi_Q,\Phi_P\right]_{\star_\nu^{BG}} \in \mathbb{C}[[\nu]]$. A similar argument to that used in Remark 3.2.4 then leads us to set $\Phi_Z := 0$, which implies $D_Z = 0$.

We thus have shown the following proposition.

[6] Recall that $\tilde{\mathfrak{g}}_{RC} := \mathrm{span}_{\mathbb{R}}\{H,E,Q,P,Z\}$.

Proposition 3.2.6. The family of morphisms $D : \tilde{\mathfrak{g}}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right)$, extending

$$\mathfrak{s} \to \mathrm{Der}\left(\star_\nu^{BG}\right) : X \mapsto X^\star,$$

is parametrized by $\mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$. More precisely, such a derivation is always of the form D^{K_1,K_2} with $(K_1, K_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$, defined within the global chart (1.24) as the linear extension of the set of following maps[7]

$$D_H^{K_1,K_2}\Big|_{(a,\ell)} = H^\star_{(a,\ell)},\ D_E^{K_1,K_2}\Big|_{(a,\ell)} = E^\star_{(a,\ell)},\ D_Z^{K_1,K_2}\Big|_{(a,\ell)} = 0,$$
$$D_Q^{K_1,K_2}\Big|_{(a,\ell)} = \frac{1}{2\nu}\left[K_1 e^{-a}, \cdot\right]_{\star_\nu^{BG}},\ D_P^{K_1,K_2}\Big|_{(a,\ell)} = \frac{1}{2\nu}\left[\left(K_2 - K_1\ell\right) e^{a}, \cdot\right]_{\star_\nu^{BG}}.$$

Using the fact that $\star_\nu^{\mathrm{BG}} := T_\nu\left(\star_\nu^{\mathbb{S}}\right)$ for $T_\nu := \mathcal{F}^{-1} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F}$ (see (2.12)), we have

$$D_Q = T_\nu \circ \frac{1}{2\nu}\left[T_\nu^{-1}\left(\Phi_Q\right), \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ T_\nu^{-1}$$

and

$$D_P = T_\nu \circ \frac{1}{2\nu}\left[T_\nu^{-1}\left(\Phi_P\right), \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ T_\nu^{-1},$$

for the $\star_\nu^{\mathbb{S}}$-commutator given in (2.5).

Lemma 3.2.7. The operator T_ν^{-1} leaves Φ_Q and Φ_P unchanged.

Proof. First of all, notice that T_ν^{-1} leaves all the functions depending only on the a-variable unchanged. Moreover, for $\varphi(a,\ell) := \ell$, we have $T_\nu^{-1}\varphi = \varphi$. Indeed, at the distributional level, we compute that

$$T_\nu^{-1}\varphi(a,\ell) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell\xi}\mathcal{F}\varphi\left(a, \frac{\sinh(2i\nu\xi)}{2i\nu}\right)d\xi.$$

But

$$\begin{aligned}\mathcal{F}\varphi\left(a, \frac{\sinh(2i\nu\xi)}{2i\nu}\right) &= \int_{\mathbb{R}} e^{-i\frac{\sinh(2i\nu\xi)}{2i\nu}z} z dz \\ &= \frac{2\pi i}{\cosh(2i\nu\xi)}\partial_\xi\left[\delta_0\left(\frac{\sinh(2i\nu\xi)}{2i\nu}\right)\right] \\ &= \frac{2\pi i}{\cosh(2i\nu\xi)}\partial_\xi\delta_0(\xi).\end{aligned}$$

[7] By this, we mean that $D_Y^{K_1,K_2} := \sum_k c_k^Y D_{X_k}^{K_1,K_2} + \sum_l c_l^Y D_{Y_l}^{K_1,K_2}$, for $Y = \sum_k c_k^Y X_k + \sum_l c_l^Y Y_l$. Remark that we have added superscripts in the notations to insist on the dependence parameters K_1 and K_2.

Then

$$\begin{aligned} T_\nu^{-1}\varphi(a,\ell) &= i\int_{\mathbb{R}} e^{i\ell\xi}\frac{1}{\cosh(2i\nu\xi)}\partial_\xi\delta_0(\xi)d\xi \\ &= -i\left(\frac{i\ell e^{i\ell\xi}\cosh(2i\nu\xi) - e^{i\ell\xi}2i\nu\sinh(2i\nu\xi)}{\cosh^2(2i\nu\xi)}\right)\Big|_{\xi=0} \\ &= \varphi(a,\ell). \end{aligned}$$

The lemma is then a direct consequence of the expressions of Φ_Q and Φ_P given in (3.8). ■

We then obtain the following expressions:

$$D_Q = \mathcal{F}^{-1}\circ\left(\Psi_\nu^{-1}\right)^\star\circ\mathcal{F}\circ\frac{1}{2\nu}\left[\Phi_Q,\cdot\right]_{\star_\nu^{\mathbb{S}}}\circ\mathcal{F}^{-1}\circ\Psi_\nu^\star\circ\mathcal{F},$$

$$D_P = \mathcal{F}^{-1}\circ\left(\Psi_\nu^{-1}\right)^\star\circ\mathcal{F}\circ\frac{1}{2\nu}\left[\Phi_P,\cdot\right]_{\star_\nu^{\mathbb{S}}}\circ\mathcal{F}^{-1}\circ\Psi_\nu^\star\circ\mathcal{F}.$$

Let us now compute those expressions explicitly. First, we have (in (a,ℓ)-coordinates)

$$\frac{1}{2\nu}\left[\Phi_Q,\cdot\right]_{\star_\nu^{\mathbb{S}}} = -\frac{K_1}{\nu}e^{-a}\sinh\left(\nu\partial_\ell\right)$$

and

$$\frac{1}{2\nu}\left[\Phi_P,\cdot\right]_{\star_\nu^{\mathbb{S}}} = \left(K_2 - K_1\ell\right)\frac{e^a}{\nu}\sinh\left(\nu\partial_\ell\right) + K_1e^a\partial_a\cosh\left(\nu\partial_\ell\right),$$

where for all $c\in\mathbb{C}$,

$$\sinh\left(c\nu\partial_\ell\right) := \sum_{n\in\mathbb{N}}\frac{(c\nu)^{2n+1}}{(2n+1)!}\partial_\ell^{2n+1} \quad\text{and}\quad \cosh\left(c\nu\partial_\ell\right) := \sum_{n\in\mathbb{N}}\frac{(c\nu)^{2n}}{(2n)!}\partial_\ell^{2n}.$$

It implies that (for the corresponding (a,ξ)-coordinates)

$$\mathcal{F}\circ\frac{1}{2\nu}\left[\Phi_Q,\cdot\right]_{\star_\nu^{\mathbb{S}}}\circ\mathcal{F}^{-1} = -\frac{K_1}{\nu}e^{-a}\sinh\left(i\nu\xi\right)\mathrm{Id}$$

and

$$\begin{aligned}\mathcal{F}\circ\frac{1}{2\nu}\left[\Phi_P,\cdot\right]_{\star_\nu^{\mathbb{S}}}\circ\mathcal{F}^{-1} = \frac{e^a}{\nu}\sinh(i\nu\xi)\left(K_2\mathrm{Id} - iK_1\partial_\xi\right) \\ + K_1e^a\cosh(i\nu\xi)\left(\mathrm{Id}+\partial_a\right).\end{aligned}$$

From the latter, we deduce:

$$\left(\Psi_\nu^{-1}\right)^\star\circ\mathcal{F}\circ\frac{1}{2\nu}\left[\Phi_Q,\cdot\right]_{\star_\nu^{\mathbb{S}}}\circ\mathcal{F}^{-1}\circ\Psi_\nu^\star = -\frac{K_1}{\nu}e^{-a}\sinh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right)\mathrm{Id} \quad (3.9)$$

and

$$\left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F} \circ \frac{1}{2\nu}\left[\Phi_P, \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ \mathcal{F}^{-1} \circ \Psi_\nu^\star = \frac{K_2}{\nu} e^a \sinh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right) \mathrm{Id} + K_1 e^a \cosh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right)(\mathrm{Id} + \partial_a) + \frac{K_1}{i\nu} e^a \sinh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right)\sqrt{1-4\nu^2\xi^2}\,\partial_\xi. \tag{3.10}$$

Setting

$$\tilde{D}_Q := \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F} \circ \frac{1}{2\nu}\left[\Phi_Q, \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ \mathcal{F}^{-1} \circ \Psi_\nu^\star$$

and

$$\tilde{D}_P := \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F} \circ \frac{1}{2\nu}\left[\Phi_P, \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ \mathcal{F}^{-1} \circ \Psi_\nu^\star,$$

we get the explicit expressions of the derivations up to a partial Fourier transform:

$$D_Q = \mathcal{F}^{-1} \circ \tilde{D}_Q \circ \mathcal{F},\ D_P = \mathcal{F}^{-1} \circ \tilde{D}_P \circ \mathcal{F}.$$

Computation of a particular morphism

Here, we point out a way to compute a particular morphism $D : \tilde{\mathfrak{g}}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right)$, extending $\mathfrak{s} \hookrightarrow \mathrm{Der}\left(\star_\nu^{BG}\right)$, without using the retract method. The idea is as follows: if we are able to find a quantum moment map $\Lambda^{RC} : \mathfrak{g}_{RC} \to C^\infty(\mathbb{S})[[\nu]]$ (see Definition 1.2.36) so that $\star_\nu^{BG}$ is Λ^{RC}-covariant, then the map

$$\underline{D} : \mathfrak{g}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right) : Y \mapsto \frac{1}{2\nu}\left[\Lambda_Y^{RC}, \cdot\right]_{\star_\nu^{BG}}$$

defines automatically a Lie algebra morphism, extending $\mathfrak{s} \hookrightarrow \mathrm{Der}\left(\star_\nu^{BG}\right)$.
It turns out that obtaining such a map Λ^{BG} is not so hard. Indeed, after a short computation, we remark first that $\star_\nu^{\mathbb{S}} = T_\nu^{-1}\left(\star_\nu^{BG}\right)$ is "almost" λ^{RC}-covariant, in the sense that if we set[8]

$$\tilde{\lambda}^{RC} : \mathfrak{g}_{RC} \to C^\infty(\mathbb{S})[[\nu]] : Y \mapsto \tilde{\lambda}_Y^{RC} := \lambda_Y^{RC} + \frac{\nu^2}{2} e^{2a}[Y]_F,$$

where $[Y]_F$ denotes the component of Y in the direction of the basis element F, we get the $\tilde{\lambda}^{RC}$-covariance of $\star_\nu^{\mathbb{S}}$. In particular, we thus have

$$\tilde{\lambda}^{RC}\big|_{\tilde{\mathfrak{g}}_{RC}} \equiv \lambda^{RC}\big|_{\tilde{\mathfrak{g}}_{RC}}. \tag{3.11}$$

[8] Recall that λ^{RC} is defined by (1.27).

Accordingly, we get the Λ^{RC}-covariance of $\star_\nu^{BG}$ by setting

$$\Lambda^{RC} : \mathfrak{g}_{RC} \to C^\infty(\mathbb{S})[[\nu]] : Y \mapsto \Lambda_Y^{RC} := T_\nu\left(\tilde{\lambda}_Y^{RC}\right),$$

by use of the fact that for $Y_1, Y_2 \in \mathfrak{g}_{RC}$,

$$\frac{1}{2\nu}\left[\Lambda_{Y_1}^{RC}, \Lambda_{Y_2}^{RC}\right]_{\star_\nu^{BG}} = T_\nu\left(\frac{1}{2\nu}\left[\tilde{\lambda}_{Y_1}^{RC}, \tilde{\lambda}_{Y_2}^{RC}\right]_{\star_\nu^{\mathbb{S}}}\right) = T_\nu\left(\tilde{\lambda}_{[Y_1,Y_2]}^{RC}\right) = \Lambda_{[Y_1,Y_2]}^{RC}.$$

Now, we set

$$\underline{D} : \mathfrak{g}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right) : Y \mapsto \frac{1}{2\nu}\left[\Lambda_Y^{RC}, \cdot\right]_{\star_\nu^{BG}},$$

i.e.

$$\underline{D}_Y = \mathcal{F}^{-1} \circ \left(\Psi_\nu^{-1}\right)^\star \circ \mathcal{F} \circ \frac{1}{2\nu}\left[\tilde{\lambda}_Y^{RC}, \cdot\right]_{\star_\nu^{\mathbb{S}}} \circ \mathcal{F}^{-1} \circ \Psi_\nu^\star \circ \mathcal{F}.$$

Of course, using (1.27) and (3.11), $\underline{D}_H = H^\star$, $\underline{D}_E = E^\star$ and $\underline{D}_Z = 0$. Moreover, setting

$$\tilde{\underline{D}}_Q := \mathcal{F} \circ \underline{D}_Q \circ \mathcal{F}^{-1} \quad \text{and} \quad \tilde{\underline{D}}_P := \mathcal{F} \circ \underline{D}_P \circ \mathcal{F}^{-1},$$

similar computations to that performed for $D_Q^{K_1,K_2}$ and $D_P^{K_1,K_2}$ lead to the following expressions:

$$\tilde{\underline{D}}_Q\big|_{(a,\xi)} = -\frac{e^{-a}}{\nu}\sinh\left(\frac{\mathrm{arcsinh}(2i\nu\xi)}{2}\right)\mathrm{Id}$$

and

$$\tilde{\underline{D}}_P\big|_{(a,\xi)} = \\ e^a\cosh\left(\frac{\mathrm{arcsinh}(2i\nu\xi)}{2}\right)(\mathrm{Id} + \partial_a) + \frac{e^a}{i\nu}\sinh\left(\frac{\mathrm{arcsinh}(2i\nu\xi)}{2}\right)\sqrt{1 - 4\nu^2\xi^2}\,\partial_\xi.$$

Remark 3.2.8. We have $\underline{D} = D^{1,0}$, where $D^{1,0} := D^{K_1,K_2}\big|_{K_1=1,K_2=0}$, for D^{K_1,K_2} given in Proposition 3.2.6.

3.2.2 Resolution of the retract equations

Set-up of the equations

Recall that given a derivation D^{K_1,K_2} (with $(K_1, K_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$) and an element $Y \in \tilde{\mathfrak{g}}_{RC}$, the (inverse of the) corresponding intertwiner[9] $T_{K_1,K_2} =$

[9] See Remark 3.1.3 for the notation.

$u_{K_1,K_2} \times \cdot$ satisfies the following identities:

$$D_Y^{K_1,K_2}\left(T^{-1}_{K_1,K_2}\varphi\right)(x_0) = T^{-1}_{K_1,K_2} Y^\star\varphi(x_0)$$
$$D_Y^{K_1,K_2}\Big|_{x_0}\int_{\mathbb{S}} u_{K_1,K_2}\left(x^{-1}x_0\right)\varphi(g)d^L_{\mathbb{S}}(x) = \int_{\mathbb{S}} u_{K_1,K_2}\left(x^{-1}x_0\right) Y^\star\varphi(x)d^L_{\mathbb{S}}(x)$$
$$\int_{\mathbb{S}} D_Y^{K_1,K_2}\Big|_{x_0} u_{K_1,K_2}\left(x^{-1}x_0\right)\varphi(x)d^L_{\mathbb{S}}(x) = -\int_{\mathbb{S}} Y^\star|_x\, u_{K_1,K_2}\left(x^{-1}x_0\right)\varphi(x)d^L_{\mathbb{S}}(x),$$

where the last line comes from the following observation: here, the left invariant Haar measure $d^L_{\mathbb{S}}(x)$ coincides with the Liouville measure on $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$. Indeed, since $Y^\star$ is a (local) Hamiltonian action, if we denote by $\mathcal{L}$ the Lie derivative, the Cartan formula along with the fact that $\omega^{\mathbb{S}}$ is closed lead to:

$$\mathcal{L}_{Y^\star}\omega^{\mathbb{S}} = \iota\left(Y^\star\right)d\omega^{\mathbb{S}} + d\left(\iota\left(Y^\star\right)\omega^{\mathbb{S}}\right) = d\left(-d\lambda^{RC}_Y\right) = 0.$$

The form $\omega^{\mathbb{S}}$ is then locally invariant under the action of $\mathfrak{g}_{RC}$.
For any $Y \in \tilde{\mathfrak{g}}_{RC}$, we then get the following associated retract equation (in the sense of distributions):

$$D_Y^{K_1,K_2}\Big|_{x_0} u_{K_1,K_2}\left(x^{-1}x_0\right) = -\,Y^\star|_x\, u_{K_1,K_2}\left(x^{-1}x_0\right), \tag{3.12}$$

where $x_0, x \in \mathbb{S}$. The linearity of D^{K_1,K_2} allows us to consider only the retract equations associated with the basis elements of $\tilde{\mathfrak{g}}_{RC}$, or even only with the elements Q and P, by virtue of the triviality of the equation (3.12) for $Y = H, E, Z$.
In order to lighten the notations, we define the two following one-variable smooth functions:

$$\widetilde{\sinh}(\xi) := \sinh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right) = \operatorname{sign}(i\nu\xi_0)\sqrt{\frac{-1+\sqrt{1-4\nu^2\xi_0^2}}{2}}, \tag{3.13}$$

where *sign* denotes the *sign function* on $\mathbb{R}$, and

$$\widetilde{\cosh}(\xi) := \cosh\left(\frac{\operatorname{arcsinh}(2i\nu\xi)}{2}\right) = \sqrt{\frac{1+\sqrt{1-4\nu^2\xi_0^2}}{2}}. \tag{3.14}$$

Within these notations and setting (as before) $\tilde{D}^{K_1,K_2} := \mathcal{F}\circ\tilde{D}^{K_1,K_2}\circ\mathcal{F}^{-1}$, we get from (3.9) and (3.10):

$$\tilde{D}_Q^{K_1,K_2}\Big|_{(a,\xi)} = -\frac{K_1}{\nu}e^{-a}\widetilde{\sinh}(\xi)\mathrm{Id}, \tag{3.15}$$

and

$$\tilde{D}_P^{K_1,K_2}\Big|_{(a,\xi)} =$$
$$K_1 e^a \widetilde{\cosh}(\xi)\,(\mathrm{Id}+\partial_a) + \frac{K_1}{i\nu} e^a \widetilde{\sinh}(\xi)\sqrt{1-4\nu^2\xi^2}\,\partial_\xi + \frac{K_2}{\nu} e^a \widetilde{\sinh}(\xi)\mathrm{Id}. \quad (3.16)$$

Recall also that[10]

$$Q^\star|_{(a,\ell)} = -e^{-a}\partial_\ell \quad \text{and} \quad P^\star|_{(a,\ell)} = e^a\left(\partial_a - \ell\partial_\ell\right).$$

The following important result summarizes all the above development.

Theorem 3.2.9. (i) Any intertwiner T such that $T\left(\star_\nu^{BG}\right)$ is a $\mathfrak{g}_{RC}$-invariant star-product admits as an inverse a convolution operator $T^{-1} = u \times \cdot$, with u a solution of a system of equations of the form:

$$\begin{cases} D_Q^{K_1,K_2}\Big|_{x_0} u\left(x^{-1}x_0\right) = -\,Q^\star|_x\, u\left(x^{-1}x_0\right) \\ D_P^{K_1,K_2}\Big|_{x_0} u\left(x^{-1}x_0\right) = -\,P^\star|_x\, u\left(x^{-1}x_0\right) \end{cases}, \quad (3.17)$$

for some $K_1, K_2 \in \mathbb{C}[[\nu]]$, where

$$D_Q^{K_1,K_2} = \mathcal{F}^{-1}\circ\tilde{D}_Q^{K_1,K_2}\circ\mathcal{F} \quad \text{and} \quad D_P^{K_1,K_2} = \mathcal{F}^{-1}\circ\tilde{D}_P^{K_1,K_2}\circ\mathcal{F},$$

for $\tilde{D}_Q^{K_1,K_2}$, $\tilde{D}_P^{K_1,K_2}$ given respectively by (3.15) and (3.16) (within the global chart on $\tilde{\mathbb{S}}$), and $Q^\star, P^\star$ given in (1.28) (within the global chart on $\mathbb{S}$).

(ii) Let $U = u \times \cdot$ be an operator such that u satisfies the system (3.17). If U is invertible, then $U^{-1}\left(\star_\nu^{BG}\right)$ coincides (up to a redefinition of the deformation parameter) with $\star_\nu^{RC}$.

Proof. The item (i) is a direct application of the retract method (Proposition 3.1.4). For item (ii), we notice that if $U := u \times \cdot$ is invertible, then

$$D_Y U = U Y^\star$$

and

$$U^{-1} D_Y = Y^\star U^{-1},$$

[10] See (1.28).

for $Y \in \{H, E, Q, P, Z\}$. By linearity, we then deduce that those equations hold for any $Y \in \tilde{\mathfrak{g}}_{RC}$. Hence, since

$$Y^\star = U^{-1} D_Y U,$$

we observe that for $Y \in \tilde{\mathfrak{g}}_{RC}$, the derivation $Y^\star$ of the classical pointwise product is also a derivation of $U^{-1}\left(\star_\nu^{BG}\right)$. Proposition 2.1.18 then leads to the desired result. ■

Explicit resolution of the equations

We start with the following observation.

Lemma 3.2.10. For given (K_1, K_2), let u_{K_1,K_2} be a solution of (3.17) such that the associated $\mathbb{S}$-equivariant convolution operator $U_{K_1,K_2} := u_{K_1,K_2} \times \cdot$ is invertible. Then, u_{K_1,K_2} is the unique solution[11] of (3.17), up to a multiplicative formal constant.

Proof. Let K_1, K_2 be two formal constants such that the associated system of equations (3.17) admits (at least) two solutions u and u' leading to two convolution operators $U := u \times \cdot$ and $U' := u' \times \cdot$. Then, the form of the retract equations implies that both $U^{-1}\left(\star_\nu^{BG}\right)$ and $U'^{-1}\left(\star_\nu^{BG}\right)$ define $\mathfrak{g}_{RC}$-invariant star-products. Hence, Proposition 2.1.18 implies that they both coincide with $\star_\nu^{RC}$, up to a redefinition of the deformation parameter. Without loss of generality, we then assume that $U^{-1}\left(\star_\nu^{BG}\right) = U'^{-1}\left(\star_\nu^{BG}\right)$ and we set

$$\Phi := U'^{-1} U \in \mathrm{Aut}\left(\star_\nu^{RC}\right),$$

where $\mathrm{Aut}\left(\star_\nu^{RC}\right)$ denotes the set of *automorphisms* of $\star_\nu^{RC}$ (that is, the set of intertwiners of $\star_\nu^{RC}$ with itself). Trivially, for all $X \in \mathfrak{s}$, the following identity holds:

$$\Phi X^\star = X^\star \Phi.$$

Formally, we can thus write Φ as a convolution operator (on $\mathbb{S}$); we set $\Phi =: w \times \cdot$. Moreover, by assumptions, the retract equations are satisfied for U and U'. Thus, if we simply denote D^{K_1,K_2} by D, we have for any $Y \in \mathfrak{g}_{RC}$:

$$U^{-1} D_Y U = Y^\star$$

and

$$U'^{-1} D_Y U' = Y^\star.$$

[11] Up to a redefinition of the deformation parameter.

That implies

$$\Phi Y^\star = U'^{-1} U Y^\star = U'^{-1} D_Y U = Y^\star U'^{-1} U = Y^\star \Phi.$$

In other words, Φ commutes with elements of $\mathfrak{g}_{RC}$.
For $Y \in \mathfrak{g}_{RC}$, $x \in \mathbb{S}$ and $\varphi \in \mathcal{D}(\mathbb{S})$, we have

$$Y_x^\star \Phi(\varphi) = \int_{\mathbb{S}} w\left(y^{-1}x\right) \varphi(y) d_{\mathbb{S}}^L(y) = \int_{\mathbb{S}} Y^\star|_x \, w\left(y^{-1}x\right) \varphi(y) d_{\mathbb{S}}^L(y)$$

and

$$\Phi\left(Y^\star \varphi\right)(x) = \int_{\mathbb{S}} w\left(y^{-1}x\right) Y_y^\star(\varphi) d_{\mathbb{S}}^L(y) = -\int_{\mathbb{S}} Y^\star|_y \, w\left(y^{-1}x\right) \varphi(y) d_{\mathbb{S}}^L(y).$$

The fact that Φ commutes with $\mathfrak{g}_{RC}$ leads to the following equation:

$$Y^\star|_x \, w\left(y^{-1}x\right) + Y^\star|_y \, w\left(y^{-1}x\right) = 0, \tag{3.18}$$

for all $Y \in \mathfrak{g}_{RC}$. Although useless for $Y \in \mathfrak{s}$ (because Φ is a convolution on $\mathbb{S}$), this relation is interesting for $Y \in \mathfrak{g}_{RC} \backslash \mathfrak{s}$. Indeed, recall first that locally, we have an action

$$\tau : G_{RC} \times \mathbb{S} \to \mathbb{S},$$

where G_{RC} is the Lie group such that $\mathrm{Lie}\left(\mathfrak{g}_{RC}\right) = G_{RC}$. Then, remark that

$$\begin{aligned} Y^\star|_x \, w\left(y^{-1}x\right) &= Y_x^\star \left(L_{y^{-1}}^\star w\right) \\ &= \left.\frac{d}{dt}\right|_0 L_{y^{-1}}^\star w\left(\tau_{\exp(-tY)}(x)\right) \\ &= \left.\frac{d}{dt}\right|_0 w\left(y^{-1}\tau_{\exp(-tY)}(x)\right). \end{aligned}$$

Moreover, since[12]

$$w\left(y^{-1}x\right) = \check{w}\left(x^{-1}y\right) = L_{x^{-1}}^\star \check{w}(y),$$

[12]Recall that, as mentioned in Remark 3.1.2, $\check{w}(x) := w\left(x^{-1}\right)$.

we also have

$$\begin{aligned} Y^\star|_y\, w\left(y^{-1}x\right) &= Y_y^\star\left(L_{x^{-1}}^\star \check{w}\right) \\ &= \left.\frac{d}{dt}\right|_0 L_{x^{-1}}^\star \check{w}\left(\tau_{\exp(-tY)}(y)\right) \\ &= \left.\frac{d}{dt}\right|_0 w\left(\left(\tau_{\exp(-tY)}(y)\right)^{-1} x\right). \end{aligned}$$

Hence, for $x = e$, observe that equation (3.18) becomes

$$\left.\frac{d}{dt}\right|_0 w\left(y^{-1}\tau_{\exp(-tY)}(e)\right) + \left.\frac{d}{dt}\right|_0 w\left(\left(\tau_{\exp(-tY)}(y)\right)^{-1}\right) = 0. \tag{3.19}$$

For any $Y \in \mathrm{Stab}_{\mathfrak{g}_{RC}}(e)$ (that is, the Lie algebra of the stabilizer subgroup[13] of the neutral element e), the equation (3.19) becomes

$$\left.\frac{d}{dt}\right|_0 w\left(\left(\tau_{\exp(-tY)}(y)\right)^{-1}\right) = 0,$$

i.e.

$$Y_y^\star \check{w} = 0,$$

for all $y \in \mathbb{S}$. So, for all $\varphi \in \mathcal{D}(\mathbb{S})$,

$$\int_{\mathbb{S}} \check{w}(y) \left.\frac{d}{dt}\right|_0 \varphi\left(\tau_{\exp(-tY)}(y)\right) d_{\mathbb{S}}^L(y) = 0,$$

which means that $\mathrm{supp}\,(\check{w}) \subset \{e\}$. Remark that the form of the equation (3.18) implies that we also have $\mathrm{supp}(w) \subset \{e\}$.
Since $\Phi = w \times \cdot$ for a certain distributional kernel w, we can write that for all $\varphi \in \mathcal{D}(\mathbb{S})$,

$$\Phi(\varphi)(x) = \int_{\mathbb{S}} w\left(y^{-1}x\right)\varphi(y) d_{\mathbb{S}}^L(y) = \int_{\mathbb{S}} w\left(y^{-1}\right)\varphi(xy) d_{\mathbb{S}}^L(y) = \langle \check{w}, L_x^\star \varphi\rangle,$$

where we have used the standard notation $\langle\cdot,\cdot\rangle$ for the pairing between a distribution and a function. Now, we choose a function $\varphi_0 \in \mathcal{D}(\mathbb{S})$ and consider the map $[x \in \mathbb{S} \mapsto \langle \check{w}, L_x^\star \varphi_0\rangle]$. Let Σ be the support of this map. For $x \in \Sigma$, $\langle \check{w}, L_x^\star \varphi_0\rangle \neq 0$, which implies that $\{e\} \in \mathrm{supp}\,(L_x^\star \varphi_0)$. So $\{x\} \in \mathrm{supp}\,(\varphi_0)$ and then $\Sigma \subset \mathrm{supp}\,(\varphi_0)$. In other words, the map $[x \in \mathbb{S} \mapsto \langle \check{w}, L_x^\star \varphi_0\rangle]$ restricts the support of φ_0. According to Peetre theorem, we deduce that $\Phi \in \mathrm{DO}(\mathbb{S})^{\mathbb{S}}[[\nu]]$, i.e. the set of formal left invariant differential operators on $\mathbb{S}$. It is well-known

[13] Recall that the stabilizer subgroup of a given element is a Lie subgroup a G_{RC}. In our case, the dimension of this group is equal to three; see for instance [53], Theorem 3.62.

that this set identifies to $\mathcal{U}(\mathfrak{s})[[\nu]]$, and so there exists a unique element $\tilde{X}_\Phi \in \mathcal{U}(\mathfrak{s})[[\nu]]$ such that $\Phi = \tilde{X}_\Phi$ (the equality is understood as an identification). Since Φ commutes with the whole algebra $\mathfrak{g}_{RC}$, the same is true for $\tilde{X}_\Phi$. Recall now that the (uncompleted) generators of fundamental vector fields are given by

$$H^\star = -\partial_a,\ E^\star = -e^{-2a}\partial_\ell,\ F^\star = \ell e^{2a}\left(\partial_a - \ell\partial_\ell\right),\ Q^\star = -e^{-a}\partial_\ell,$$
$$P^\star = e^a\left(\partial_a - \ell\partial_\ell\right),$$

while the left invariant vector fields are given by

$$\tilde{H} = \partial_a - 2\ell\partial_\ell,\ \tilde{E} = \partial_\ell.$$

Notice that $Q^\star = e^a E^\star$ and $F^\star = \ell e^a P^\star$. Since $\tilde{X}_\Phi$ commutes with any element of $\mathfrak{g}_{RC}$, we have in particular

$$\left[\tilde{X}_\Phi, Q^\star\right] = 0.$$

Since

$$\left[\tilde{X}_\Phi, Q^\star\right] = \left[\tilde{X}_\Phi, e^a E^\star\right] = \tilde{X}_\Phi\left(e^a E^\star\right) - e^a E^\star \tilde{X}_\Phi = \tilde{X}_\Phi\left(e^a E^\star\right) - e^a \tilde{X}_\Phi E^\star,$$

we deduce that

$$\tilde{X}_\Phi\left(e^a E^\star\right) = e^a \tilde{X}_\Phi E^\star.$$

A short computation then shows that $\tilde{X}_\Phi$ has to be polynomial in $\tilde{E}$ only. Moreover, since $\left[\tilde{X}_\Phi, F^\star\right] = 0$, a similar reasoning shows that $\tilde{X}_\Phi\left(\ell e^a P^\star\right) = \ell e^a \tilde{X}_\Phi P^\star$. Again, a short computation implies that $\tilde{X}_\Phi$ can not depend on $\tilde{E}$, which means that $\tilde{X}_\Phi \in \mathbb{C}[[\nu]]$. In others words, $\Phi = U'^{-1}U \in \mathbb{C}[[\nu]]$ and so U and U' are equal up to a multiplicative formal constant. ■

This lemma means that for purely geometrical reasons, to each particular $(K_1, K_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$, the associated system (3.17) leads, at most, to one relevant solution (up to a multiplicative constant), that is, a solution associated with an $\mathbb{S}$-equivalence between $\star_\nu^{BG}$ and $\star_\nu^{RC}$. We now pass to the explicit resolution.

By means of the global coordinates (1.24), we set $x = (a, \ell)$ and $x_0 = (a_0, \ell_0)$. Then, a partial Fourier transform in the ℓ_0-variable applied to the LHS and RHS of equations (3.17) (we denote by ξ_0 the conjugated variable associated

with ℓ_0 under the partial Fourier transform), leads to

$$\tilde{D}_Y^{K_1,K_2}\Big|_{(a_0,\xi_0)} W_{K_1,K_2}(a,\ell,a_0,\xi_0) + Y^\star|_{(a,\ell)}\, W_{K_1,K_2}(a,\ell,a_0,\xi_0) = 0, \quad (3.20)$$

which holds for $Y = Q$, $Y = P$ and for every $(a_0, \xi_0) \in \tilde{\mathbb{S}}$, where

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0) := \int_{\mathbb{R}} e^{-i\xi_0\ell_0} u_{K_1,K_2}\left((a,\ell)^{-1}(a_0,\ell_0)\right) d\ell_0, \quad (3.21)$$

and with u_{K_1,K_2} a solution of (3.17) for some choice of K_1 and K_2. For $Y = Q$, the equation (3.20) becomes[14] (using (1.28) and (3.15))

$$-\frac{K_1}{\nu} e^{-a_0}\widetilde{\sinh}(\xi_0) W_{K_1,K_2}(a,\ell,a_0,\xi_0) - e^{-a}\partial_\ell W_{K_1,K_2}(a,\ell,a_0,\xi_0) = 0, \quad (3.22)$$

and for $Y = P$ (using (1.28) and (3.16)), (3.20) becomes

$$\begin{aligned} &K_1 e^{a_0}\widetilde{\cosh}(\xi_0) W_{K_1,K_2}(a,\ell,a_0,\xi_0) + K_1 e^{a_0}\widetilde{\cosh}(\xi_0)\partial_{a_0} W_{K_1,K_2}(a,\ell,a_0,\xi_0) \\ &+ \frac{K_1}{i\nu} e^{a_0}\widetilde{\sinh}(\xi_0)\sqrt{1-4\nu^2\xi_0^2}\,\partial_{\xi_0} W_{K_1,K_2}(a,\ell,a_0,\xi_0) \\ &+ \frac{K_2}{\nu} e^{a_0}\widetilde{\sinh}(\xi_0) W_{K_1,K_2}(a,\ell,a_0,\xi_0) \\ &+ e^a\partial_a W_{K_1,K_2}(a,\ell,a_0,\xi_0) - \ell e^a\partial_\ell W_{K_1,K_2}(a,\ell,a_0,\xi_0) = 0. \end{aligned} \quad (3.23)$$

Remark 3.2.11. Formally, using (3.21), we have

$$\begin{aligned} W_{K_1,K_2}(a,\ell,a_0,\xi_0) &= \int_{\mathbb{R}} e^{-i\xi_0\ell_0} u_{K_1,K_2}\left((a,\ell)^{-1}(a_0,\ell_0)\right) d\ell_0 \\ &= \int_{\mathbb{R}} e^{-i\xi_0\ell_0} u_{K_1,K_2}\left(a_0 - a, -e^{-2(a_0-a)}\ell + \ell_0\right) d\ell_0 \\ &= e^{-i\xi_0\ell e^{2(a-a_0)}} \int_{\mathbb{R}} e^{-i\xi_0\ell_0} u_{K_1,K_2}(a_0-a,\ell_0) d\ell_0. \end{aligned}$$

Setting $\alpha := a - a_0$, we get the following form for W_{K_1,K_2}:

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0) = e^{-i\xi_0\ell e^{2\alpha}}\widetilde{W}_{K_1,K_2}(\alpha,\xi_0), \quad (3.24)$$

where

$$\widetilde{W}_{K_1,K_2}(\alpha,\xi_0) := \int_{\mathbb{R}} e^{-i\xi_0\ell_0} u_{K_1,K_2}(-\alpha,\ell_0) d\ell_0.$$

[14] Recall that the functions $\widetilde{\sinh}$ and $\widetilde{\cosh}$ are defined respectively by (3.13) and (3.14).

If we substitute W_{K_1,K_2} with the RHS of (3.24) within the equation (3.22), a multiplication of the LHS and the RHS (of (3.22)) by $-\nu e^{a_0}$ leads to

$$\left(K_1\widetilde{\sinh}(\xi_0) - i\nu\xi_0 e^{\alpha}\right) e^{-i\xi_0\ell e^{2\alpha}}\widetilde{W}_{K_1,K_2}(\alpha,\xi_0) = 0$$

(with $\alpha := a - a_0$ as in Remark 3.2.11), or simply

$$\left(K_1\widetilde{\sinh}(\xi_0) - i\nu\xi_0 e^{\alpha}\right)\widetilde{W}_{K_1,K_2}(\alpha,\xi_0) = 0. \tag{3.25}$$

For $\xi_0 = 0$, this latter equation is trivially satisfied. However, for $\xi_0 \neq 0$, this equation implies that $\widetilde{W}_{K_1,K_2}$ has to be a distribution whose support is a subset of $\mathcal{W} := \left\{(\alpha,\xi_0) \in \tilde{\mathbb{S}} \mid K_1\widetilde{\sinh}(\xi_0) - i\nu\xi_0 e^{\alpha} = 0\right\}$. Thanks to the expression (3.13), we can see that $\widetilde{\sinh}(\xi_0)$ is a non-zero real number (for $\xi_0 \neq 0$), admitting the same sign as $i\nu\xi_0 e^{\alpha}$. For a fixed non-zero ξ_0, the identity characterizing the set $\mathcal{W}$ thus admits a solution in α if and only if K_1 is a *strictly positive real number* (possibly depending on ν)[15]. Let us now recall the following result.

Lemma 3.2.12. All weak solutions $w \in \mathcal{D}'(\mathbb{R})$ of the equation

$$rw(r) = 0 \tag{3.26}$$

are of the form $w(r) = c\delta_0(r)$, for c a constant and δ_0 the Dirac delta distribution with support in $r = 0$.

Proof. Suppose that w satisfies (3.26) and choose a function $f_0 \in \mathcal{D}(\mathbb{R})$ such that $f_0(0) = 1$. For every $\varphi \in \mathcal{D}(\mathbb{R})$, the function $\varphi - \varphi(0)f_0$ is equal to 0 in $r = 0$. By applying a Taylor development to the latter around $r = 0$, we observe that there exists a function $\psi \in C^{\infty}(\mathbb{R})$ such that

$$\varphi - \varphi(0)f_0 = r\psi.$$

Remark that this equation implies that $\psi \in \mathcal{D}(\mathbb{R})$. So we have

$$\langle w, r\psi\rangle = 0 = \langle w,\varphi\rangle - \langle w,\varphi(0)f_0\rangle.$$

Setting $c := \langle w, f_0\rangle$, we obtain

$$\langle w,\varphi\rangle = c\varphi(0),$$

which implies $w = c\delta$. ∎

[15] In other words, this is the required condition on K_1 to have a non-trivial solution $\widetilde{W}_{K_1,K_2}$ associated with a non-zero ξ_0.

For fixed ξ_0, the equation (3.25) only depends on the variable $\alpha \in \mathbb{R}$. Remark that for strictly positive real number K_1 and $\xi_0 \neq 0$, the function

$$f_{K_1,\xi_0}(\alpha) := K_1\widetilde{\sinh}(\xi_0) - i\nu\xi_0 e^{\alpha}$$

is injective and possesses as unique root

$$\alpha = \log\left(K_1 \frac{\widetilde{\sinh}(\xi_0)}{i\nu\xi_0}\right).$$

Setting $r := f_{K_1,\xi_0}(\alpha)$ and then

$$w_{K_1,K_2,\xi_0}(r) := \widetilde{W}_{K_1,K_2}\left(f^{-1}_{K_1,\xi_0}(r), \xi_0\right),$$

the equation (3.25) becomes

$$r w_{K_1,K_2,\xi_0}(r) = 0.$$

Lemma 3.2.12 then implies that all weak solutions w_{K_1,K_2,ξ_0} are of the form

$$w_{K_1,K_2,\xi_0}(r) = c_{K_1,K_2,\xi_0}\delta_0(r),$$

where c_{K_1,K_2,ξ_0} is independent of r (and thus independent of α). Since

$$\widetilde{W}_{K_1,K_2}(\alpha,\xi_0) = w_{K_1,K_2,\xi_0}\left(f_{K_1,\xi_0}(\alpha)\right),$$

all weak solutions of (3.25) are of the form

$$\begin{aligned}\widetilde{W}_{K_1,K_2}(\alpha,\xi_0) &= \chi_{K_1,K_2}(\xi_0)\delta_0\left(K_1\widetilde{\sinh}(\xi_0) - i\nu\xi_0 e^{\alpha}\right)\\ &= \chi_{K_1,K_2}(\xi_0)\delta_0\left(\alpha - \log\left(K_1\frac{\widetilde{\sinh}(\xi_0)}{i\nu\xi_0}\right)\right),\end{aligned}$$

for a certain distribution χ_{K_1,K_2}. For $\xi_0 \neq 0$, we define the following (strictly positive) function:

$$\eta_\nu(\xi_0) := \frac{\widetilde{\sinh}(\xi_0)}{i\nu\xi_0}. \tag{3.27}$$

Thereby (again for $\xi_0 \neq 0$), the solution of (3.22) is given by

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0) = \chi_{K_1,K_2}(\xi_0)e^{-i\xi_0\ell e^{2(a-a_0)}}\delta_0\left(a - a_0 - \log\eta_\nu(\xi_0)\right). \tag{3.28}$$

Notice that taking into account the form of W_{K_1,K_2} obtained in Remark 3.2.11, we have $\partial_a W_{K_1,K_2} = -\partial_{a_0} W_{K_1,K_2}$. Moreover, a straightforward reformulation

of (3.22) yields

$$\partial_\ell W_{K_1,K_2}(a,\ell,a_0,\xi_0) = -\frac{K_1}{\nu} e^{a-a_0}\widetilde{\sinh}(\xi_0) W_{K_1,K_2}(a,\ell,a_0,\xi_0). \tag{3.29}$$

Now, if we multiply the LHS and RHS of the equation (3.23) by e^{-a} and if we replace the element $\partial_\ell W_{K_1,K_2}$ by the RHS of (3.29), the equation (3.23) becomes

$$\begin{aligned}
&\left(K_1 e^{a_0-a}\widetilde{\cosh}(\xi_0) + \frac{K_1}{\nu}\ell e^{a-a_0}\widetilde{\sinh}(\xi_0)\right) W_{K_1,K_2}(a,\ell,a_0,\xi_0)\\
&+ \left(K_1 e^{a_0-a}\widetilde{\cosh}(\xi_0) - 1\right)\partial_{a_0} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\\
&+ \frac{K_1}{i\nu} e^{a_0-a}\widetilde{\sinh}(\xi_0)\sqrt{1-4\nu^2\xi_0^2}\,\partial_{\xi_0} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\\
&+ \frac{K_2}{\nu} e^{a_0-a}\widetilde{\sinh}(\xi_0) W_{K_1,K_2}(a,\ell,a_0,\xi_0) = 0.
\end{aligned} \tag{3.30}$$

Hence, in particular, it remains to look for a distribution χ such that for each $\varphi \in \mathcal{D}(\mathbb{S})$,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(K_1 e^{a_0-a}\widetilde{\cosh}(\xi_0) + \frac{K_1}{\nu}\ell e^{a-a_0}\widetilde{\sinh}(\xi_0)\right) W_{K_1,K_2}(a,\ell,a_0,\xi_0)\varphi(a,\ell)dad\ell\\
&+ K_1 e^{a_0}\widetilde{\cosh}(\xi_0)\partial_{a_0}\int_{\mathbb{R}^2} e^{-a} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\varphi(a,\ell)dad\ell\\
&- \partial_{a_0}\int_{\mathbb{R}^2} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\varphi(a,\ell)dad\ell\\
&+ \frac{K_1}{i\nu}\widetilde{\sinh}(\xi_0)\sqrt{1-4\nu^2\xi_0^2}\,\partial_{\xi_0}\int_{\mathbb{R}^2} e^{a_0-a} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\varphi(a,\ell)dad\ell\\
&+ \frac{K_2}{\nu} e^{a_0}\widetilde{\sinh}(\xi_0)\int_{\mathbb{R}^2} e^{-a} W_{K_1,K_2}(a,\ell,a_0,\xi_0)\varphi(a,\ell)dad\ell = 0.
\end{aligned} \tag{3.31}$$

Assume first that $\xi_0 \neq 0$. To lighten the notations, we set

$$\underline{\varphi}(a_0,\xi_0,\ell) := e^{-iK_1^2\ell\xi_0\eta_\nu^2(\xi_0)}\varphi(a_0 + \log\left(K_1\eta_\nu(\xi_0)\right),\ell). \tag{3.32}$$

Using (3.28) along with an integration w.r.t. the a-variable, the equation (3.31) becomes

$$\chi_{K_1,K_2}(\xi_0)\int_{\mathbb{R}}\left(\frac{\widetilde{\cosh}(\xi_0)}{\eta_\nu(\xi_0)}+\frac{K_1^2}{\nu}\ell\eta_\nu(\xi_0)\widetilde{\sinh}(\xi_0)+\frac{K_2}{\nu K_1}\frac{\widetilde{\sinh}(\xi_0)}{\eta_\nu(\xi_0)}\right)\underline{\varphi}(a_0,\xi_0,\ell)d\ell$$
$$+\chi_{K_1,K_2}(\xi_0)\left(\frac{\widetilde{\cosh}(\xi_0)}{\eta_\nu(\xi_0)}e^{a_0}\partial_{a_0}\int_{\mathbb{R}}e^{-a_0}\underline{\varphi}(a_0,\xi_0,\ell)d\ell-\partial_{a_0}\int_{\mathbb{R}}\underline{\varphi}(a_0,\xi_0,\ell)d\ell\right)$$
$$+\frac{\widetilde{\sinh}(\xi_0)}{i\nu}\sqrt{1-4\nu^2\xi_0^2}\partial_{\xi_0}\int_{\mathbb{R}}\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\underline{\varphi}(a_0,\xi_0,\ell)d\ell=0. \tag{3.33}$$

Remark 3.2.13. For $\varphi\in\mathcal{D}(\mathbb{S})$, we have the following identities[16]:

$$e^{a_0}\partial_{a_0}\int_{\mathbb{R}}e^{-a_0}\underline{\varphi}(a_0,\xi_0,\ell)d\ell=$$
$$-\int_{\mathbb{R}}\underline{\varphi}(a_0,\xi_0,\ell)d\ell+\int_{\mathbb{R}}e^{-iK_1^2\ell\xi_0\eta_\nu^2(\xi_0)}\partial_1\varphi(a_0+\log\left(K_1\eta_\nu(\xi_0)\right),\ell)d\ell$$

(where ∂_1 means the partial derivative w.r.t. the first variable),

$$\partial_{a_0}\int_{\mathbb{R}}\underline{\varphi}(a_0,\xi_0,\ell)d\ell=\int_{\mathbb{R}}e^{-iK_1^2\ell\xi_0\eta_\nu^2(\xi_0)}\partial_1\varphi(a_0+\log\left(K_1\eta_\nu(\xi_0)\right),\ell)d\ell,$$

and

$$\partial_{\xi_0}\int_{\mathbb{R}}\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\underline{\varphi}(a_0,\xi_0,\ell)d\ell=$$
$$\int_{\mathbb{R}}\left[\frac{d}{d\xi_0}\left(\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\right)-K_1^2i\ell\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\frac{d}{d\xi_0}\left(\xi_0\eta_\nu^2(\xi_0)\right)\right]\underline{\varphi}(a_0,\xi_0,\ell)d\ell$$
$$+\frac{\chi_{K_1,K_2}(\xi_0)\frac{d}{d\xi_0}\eta_\nu(\xi_0)}{\eta_\nu^2(\xi_0)}\int_{\mathbb{R}}e^{-iK_1^2\ell\xi_0\eta_\nu^2(\xi_0)}\partial_1\varphi(a_0+\log\left(K_1\eta_\nu(\xi_0)\right),\ell)d\ell.$$

Using the identities of the previous remark within the equation (3.33), we see that the coefficient in front of the term

$$\int_{\mathbb{R}}e^{-iK_1^2\ell\xi_0\eta_\nu^2(\xi_0)}\partial_1\varphi(a_0+\log\left(K_1\eta_\nu(\xi_0)\right),\ell)d\ell,$$

is given by

$$\chi_{K_1,K_2}(\xi_0)\left(\frac{\widetilde{\cosh}(\xi_0)}{\eta_\nu(\xi_0)}+\frac{\widetilde{\sinh}(\xi_0)}{i\nu}\sqrt{1-4\nu^2\xi_0^2}\frac{\frac{d}{d\xi_0}\eta_\nu(\xi_0)}{\eta_\nu^2(\xi_0)}-1\right).$$

We now show that this coefficient is always equal to 0.

[16] We use the notation $\underline{\varphi}$ defined in (3.32).

Lemma 3.2.14. For all $\xi_0 \neq 0$,

$$\frac{\widetilde{\cosh}(\xi_0)}{\eta_\nu(\xi_0)} + \frac{\widetilde{\sinh}(\xi_0)}{i\nu}\sqrt{1-4\nu^2\xi_0^2}\frac{\frac{d}{d\xi_0}\eta_\nu(\xi_0)}{\eta_\nu^2(\xi_0)} = 1.$$

Proof. First, notice that by (3.13) and (3.27),

$$\eta_\nu(\xi_0) = \frac{1}{i\nu\xi_0}\mathrm{sign}(i\nu\xi_0)\sqrt{\frac{-1+\sqrt{1-4\nu^2\xi_0^2}}{2}} = \sqrt{\frac{1-\sqrt{1-4\nu^2\xi_0^2}}{2\nu^2\xi_0^2}}.$$

A multiplication by $\sqrt{\frac{1+\sqrt{1-4\nu^2\xi_0^2}}{1+\sqrt{1-4\nu^2\xi_0^2}}}$ then leads to

$$\eta_\nu(\xi_0) = \sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}} = \frac{1}{\widetilde{\cosh}(\xi_0)}, \tag{3.34}$$

from which we deduce that

$$\frac{d}{d\xi_0}\eta_\nu(\xi_0) = -\frac{i\nu\widetilde{\sinh}(\xi_0)}{\widetilde{\cosh}^{\,2}(\xi_0)\sqrt{1-4\nu^2\xi_0^2}}.$$

A straightforward computation then yields

$$\frac{\widetilde{\cosh}(\xi_0)}{\eta_\nu(\xi_0)} + \frac{\widetilde{\sinh}(\xi_0)}{i\nu}\sqrt{1-4\nu^2\xi_0^2}\frac{\frac{d}{d\xi_0}\eta_\nu(\xi_0)}{\eta_\nu^2(\xi_0)} = \widetilde{\cosh}^{\,2}(\xi_0) - \widetilde{\sinh}^{\,2}(\xi_0) = 1.$$

■

In the same way, it can be shown that the coefficient in front of the term $\int_{\mathbb{R}} \underline{\varphi}(a_0,\xi_0,\ell)\ell d\ell$, which is given by

$$\frac{K_1^2}{\nu}\chi_{K_1,K_2}(\xi_0)\left(\eta_\nu(\xi_0)\widetilde{\sinh}(\xi_0) - \frac{1}{\eta_\nu(\xi_0)}\frac{d}{d\xi_0}(\xi_0\eta_\nu^2(\xi_0))\widetilde{\sinh}(\xi_0)\sqrt{1-4\nu^2\xi_0^2}\right),$$

is equal to 0 for all $\xi_0 \neq 0$. After reduction, the unique remaining term in the LHS of (3.33) is a multiple of $\int_{\mathbb{R}} \underline{\varphi}(a_0,\xi_0,\ell)d\ell$, in front of which the coefficient is given by

$$\left(\frac{\widetilde{\sinh}(\xi_0)}{i\nu}\sqrt{1-4\nu^2\xi_0^2}\frac{d}{d\xi_0}\left(\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\right) + \frac{K_2}{\nu K_1}\widetilde{\sinh}(\xi_0)\frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)}\right).$$

Hence, by setting

$$\tilde{\chi}_{K_1,K_2}(\xi_0) := \frac{\chi_{K_1,K_2}(\xi_0)}{\eta_\nu(\xi_0)},$$

it remains to solve the following ODE:

$$\sqrt{1-4\nu^2\xi_0^2}\frac{d}{d\xi_0}\tilde{\chi}_{K_1,K_2}(\xi_0)+\frac{iK_2}{K_1}\tilde{\chi}_{K_1,K_2}(\xi_0)=0.$$

The general solution of the latter is given by

$$\tilde{\chi}_{K_1,K_2}(\xi_0)=Ke^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)},$$

for K an arbitrary constant. Thereby, for non-zero fixed ξ_0, we get the following expression for χ_{K_1,K_2}:

$$\chi_{K_1,K_2}(\xi_0)=K\eta_\nu(\xi_0)e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)}.$$

Assume now that $\xi_0=0$. In this case, as already observed, the equation (3.22) is trivially satisfied. Now, using[17]

$$W_{K_1,K_2}(a,\ell,a_0,0)=e^{-i\xi_0\ell e^{2\alpha}}\widetilde{W}_{K_1,K_2}(\alpha,\xi_0)\Big|_{\xi_0=0}=\widetilde{W}_{K_1,K_2}(\alpha,0),$$

along with the relation $\partial_{a_0}W_{K_1,K_2}=-\partial_a W_{K_1,K_2}$, the equation (3.30) becomes

$$K_1e^{-\alpha}\widetilde{W}_{K_1,K_2}(\alpha,0)-\left(K_1e^{-\alpha}-1\right)\partial_\alpha\widetilde{W}_{K_1,K_2}(\alpha,0)=0. \tag{3.35}$$

Remark that for K' an arbitrary constant, the element

$$\widetilde{W}_{K_1,K_2}(\alpha,0):=K'\delta_0(\alpha-\log(K_1)) \tag{3.36}$$

fulfills the condition (3.35). Indeed, the Leibniz rule extended to distributions implies

$$\partial_\alpha\left(e^{-\alpha}\widetilde{W}_{K_1,K_2}(\alpha,0)\right)=-e^{-\alpha}\widetilde{W}_{K_1,K_2}(\alpha,0)+e^{-\alpha}\partial_\alpha\widetilde{W}_{K_1,K_2}(\alpha,0).$$

The LHS of (3.35) is thus equal to

$$-K_1\partial_\alpha\left(e^{-\alpha}\widetilde{W}_{K_1,K_2}(\alpha,0)\right)+\partial_\alpha\widetilde{W}_{K_1,K_2}(\alpha,0),$$

with

$$e^{-\alpha}\tilde{W}_{K_1,K_2}(\alpha,0)=\frac{K}{K_1}\delta_0(\alpha-\log(K_1)),$$

by virtue of the expression of $\tilde{W}_{K_1,K_2}(\alpha,0)$ given by (3.36).

[17]Recall that $\alpha=a-a_0$.

The following remark will allow us to connect the solution W_{K_1,K_2} (or $\widetilde{W}_{K_1,K_2}$) obtained for $\xi_0 \neq 0$ with that for $\xi_0 = 0$.

Remark 3.2.15. Using (3.14) and (3.34), we have

$$\lim_{\xi_0 \to 0} \eta_\nu(\xi_0) = \lim_{\xi_0 \to 0} \frac{1}{\widetilde{\cosh}(\xi_0)} = 1.$$

It implies that the function η_ν defined on $\mathbb{R}_0$ by (3.27) admits a continuous (and even smooth) extension to $\mathbb{R}$. This extension (again denoted by η_ν with a slight abuse of notation) is given by

$$\eta_\nu : \mathbb{R} \to \mathbb{R}_0^+ : \xi_0 \mapsto \sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}}. \tag{3.37}$$

So, if for any $\xi_0 \in \mathbb{R}$, we set

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0) := \\ K\eta_\nu(\xi_0) e^{-i\left[\xi_0 \ell e^{2(a-a_0)} + \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)\right]} \delta_0\left(a - a_0 - \log(K_1\eta_\nu(\xi_0))\right),$$

with K an arbitrary constant, we see that

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0)|_{\xi_0=0} = \widetilde{W}_{K_1,K_2}(a,\ell,a_0,\xi_0)\Big|_{\xi_0=0} = K\delta_0(a-a_0-\log(K_1)),$$

which coincides with the expression of $\widetilde{W}_{K_1,K_2}(a-a_0,0)$ given by (3.36), for $K' = K$.
Explicitly, we thus have the following (ν-depending) expression for W_{K_1,K_2}:

$$W_{K_1,K_2}(a,\ell,a_0,\xi_0) = K\sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}} e^{-i\left[\frac{2\xi_0}{1+\sqrt{1-4\nu^2\xi_0^2}}\ell + \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)\right]} \\ \times \delta_0\left[a - a_0 - \log(K_1) - \frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}\right)\right].$$

Now, using (3.21), observe that it suffices to apply an inverse partial Fourier transform to $W_{K_1,K_2}(0,0,a_0,\xi_0)$ w.r.t. the ξ_0-variable to get the expression of the corresponding convolution kernel. From now on, in order to stress the ν-dependence of the latter, it will be denoted as $u_{\nu,\vec{K}}$, for $\vec{K} := (K_1,K_2)$. We then have:

$$u_{\nu,\vec{K}}(a_0,\ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell_0\xi_0} W_{K_1,K_2}(0,0,a_0,\xi_0)d\xi_0.$$

Up to a multiplicative formal constant K, we obtain the following more explicit expression:

$$u_{\nu,\vec{K}}(a_0,\ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi_0\ell_0 - \frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)}\sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}} \times \delta_0\left(a_0 + \log(K_1) + \frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}\right)\right)d\xi_0. \tag{3.38}$$

The associated convolution operator $U_{\nu,\vec{K}} := u_{\nu,\vec{K}} \times \cdot$ is thus computed as follows:

$$\begin{aligned} U_{\nu,\vec{K}}\varphi(a_0,\ell_0) &= \int_{\mathbb{R}^2} u_{\nu,\vec{K}}\left((a,\ell)^{-1}(a_0,\ell_0)\right)\varphi(a,\ell)dad\ell \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^3} e^{i\left[\xi_0\left(\ell_0 - \ell e^{2(a-a_0)}\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi_0)\right]}\eta_\nu(\xi_0) \\ &\qquad \times \delta_0\left[a - a_0 - \log(K_1\eta_\nu(\xi_0))\right]\varphi(a,\ell)d\xi_0 dad\ell. \end{aligned}$$

Using (3.37) and performing the integral w.r.t. the a-variable, we get the following result.

Proposition 3.2.16. Let K_1 be a strictly positive real number (possibly depending on ν) and $K_2 \in \mathbb{C}[[\nu]]$. Then, setting $\vec{K} := (K_1, K_2)$, each solution of (3.17) is a formal multiple of the operator $U_{\nu,\vec{K}}$ given formally by

$$\begin{aligned} &U_{\nu,\vec{K}}\varphi(a_0,\ell_0) = \\ &\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[\xi\left(\ell_0 - \frac{2K_1^2}{1+\sqrt{1-4\nu^2\xi^2}}\ell\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi)\right]}\sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}} \\ &\qquad \times \varphi\left(a_0 + \log(K_1) + \frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}\right), \ell\right)d\ell d\xi. \end{aligned}$$

Moreover, when K_1 is not a strictly positive real number, (3.17) admits only the trivial solution.

We insist on the fact that at this moment, $U_{\nu,\vec{K}}$ has no reason to be invertible. However, if for some particular choice of constants K_1 and K_2, the inverse operator $U^{-1}_{\nu,\vec{K}}$ exists, then Theorem 3.2.9 will imply that $U^{-1}_{\nu,\vec{K}}\left(\star^{BG}_\nu\right) = \star^{RC}_\nu$.

Remark 3.2.17. To be an intertwiner in the sense of Definition 1.2.18, a necessary condition on the operator $U_{\nu,\vec{K}}$ is that[18]

$$\lim_{\nu\to 0} U_{\nu,\vec{K}} = \mathrm{Id}.$$

In particular, for all $\varphi \in \mathcal{D}(\mathbb{S})$ and for all $x \in \mathbb{S}$, it is thus required that

$$\lim_{\nu\to 0} U_{\nu,\vec{K}}\varphi(x) = \lim_{\nu\to 0} \int_{\mathbb{S}} u_{\nu,\vec{K}}\left(y^{-1}x\right)\varphi(y) d^L_{\mathbb{S}}(y) = \varphi(x).$$

The Lebesgue dominated convergence then implies that

$$\lim_{\nu\to 0} u_{\nu,\vec{K}} = \delta_e, \tag{3.39}$$

with e the neutral element of $\mathbb{S}$. Since for all $\xi_0 \in \mathbb{R}$,

$$\lim_{\nu\to 0} \frac{\operatorname{arcsinh}(2i\nu\xi_0)}{2i\nu} = \xi_0$$

and

$$\lim_{\nu\to 0} \frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}} = 1,$$

a quick look at the expression (3.38) shows that K_1 and K_2 have to be chosen in order to fulfill the following limit conditions:

$$\lim_{\nu\to 0} K_1 = 1, \ \lim_{\nu\to 0} K_2 = 0.$$

Indeed, in that case, for all $(a_0, \ell_0) \in \mathbb{R}^2$,

$$\lim_{\nu\to 0} u_{\nu,\vec{K}}(a_0,\ell_0) = \frac{\delta_0(a_0)}{2\pi}\int_{\mathbb{R}} e^{i\xi_0\ell_0} d\xi_0 = \delta_{(0,0)}(a_0,\ell_0),$$

which, by virtue of the global chart (1.24), yields the desired identity (3.39).

3.2.3 Inversion of the retract solutions

Our goal consists now in finding a way to invert the solution given in Proposition 3.2.16, whenever it is possible. Before stating the following result, recall that in chapter 2, we have mentioned the existence of a strongly closed star-product $\underline{\star}^{BG}_{\nu} := \star^{BG}_{\nu,\vartheta_0}$ on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$. Lemma 2.1.17 along with Theorem 2.1.16 imply that this strongly closed star-product is $\mathbb{S}$-equivalent to $\star^{BG}_{\nu}$ (also defined in chapter 2, see Definition 2.1.10). In what follows, we will denote this $\mathbb{S}$-equivalence by

[18] See item (i) of Definition 1.2.18.

$\underline{T}_\nu$, satisfying $\underline{T}_\nu\left(\star_\nu^{BG}\right) = \underline{\star}_\nu^{BG}$. Moreover, the trace form $\mathrm{Tr}_{\vartheta_0}^{BG}$ (see (2.45)) will be simply denoted as $\underline{\mathrm{Tr}}^{BG}$ in order to lighten the notation.

Proposition 3.2.18. Every $\mathbb{S}$-equivalence T such that $T\left(\star_\nu^{BG}\right) = \star_\nu^{RC}$ is of the form

$$T\psi(x_0) = K_0\left(\left[\underline{T}_\nu u_{\nu,\vec{K}}\right]^\vee \times \underline{T}_\nu\psi\right)(x_0),$$

for some formal constants $K_0 \in \mathbb{C}[[\nu]]$, $\vec{K} = (K_1, K_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$, with K_1 a strictly positive real number (possibly depending on ν).

Proof. Let U be an $\mathbb{S}$-equivalence such that $U\left(\underline{\star}_\nu^{BG}\right) = \star_\nu^{RC}$. Formally, U is thus a convolution operator. So, there exists a certain distribution v such that for all $\varphi \in \mathcal{D}(\mathbb{S})$, we have (when it makes sense)[19]

$$\begin{aligned} U\varphi(x_0) &= \int_{\mathbb{S}} v\left(x^{-1}x_0\right)\varphi(x)d_{\mathbb{S}}^L(x) \\ &= \int_{\mathbb{S}}\left(L^\star_{x_0^{-1}}\check{v}\underline{\star}_\nu^{BG}\varphi\right)(x)d_{\mathbb{S}}^L(x), \end{aligned}$$

by virtue of the traciality of $\underline{\star}_\nu^{BG}$ (see Proposition 2.2.21). Thus, using (2.46), we get

$$U\varphi(x_0) = \underline{\mathrm{Tr}}^{BG}\left(L^\star_{x_0^{-1}}\check{v}\underline{\star}_\nu^{BG}\varphi\right).$$

Now, using the fact that $\underline{T}_\nu\left(\star_\nu^{BG}\right) = \underline{\star}_\nu^{BG}$ along with the $\mathbb{S}$-equivariance[20] of $\underline{T}_\nu^{-1}$, this leads to

$$\begin{aligned} U\varphi(x_0) &= \underline{\mathrm{Tr}}^{BG}\left(\underline{T}_\nu\left(\underline{T}_\nu^{-1}\left(L^\star_{x_0^{-1}}\check{v}\right)\star_\nu^{BG}\underline{T}_\nu^{-1}\varphi\right)\right) \\ &=: \mathrm{Tr}^{BG}\left(L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right)\star_\nu^{BG}\underline{T}_\nu^{-1}\varphi\right), \end{aligned}$$

where $\mathrm{Tr}^{BG} := \underline{\mathrm{Tr}}^{BG} \circ \underline{T}_\nu$, which defines obviously a trace for $\star_\nu^{BG}$ (in the sense of Definition 2.2.17). Remark now that any $\mathbb{S}$-equivalence $T : \star_\nu^{BG} \to \star_\nu^{RC}$ admits the decomposition $T := U \circ \underline{T}_\nu$, with U an $\mathbb{S}$-equivalence such that $U\left(\underline{\star}_\nu^{BG}\right) = \star_\nu^{RC}$. So, for $\psi \in \mathcal{D}(\mathbb{S})$,

$$T\psi(x_0) = \mathrm{Tr}^{BG}\left(L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right)\star_\nu^{BG}\psi\right). \tag{3.40}$$

As mentioned at the beginning of this chapter, such an operator T can be found by solving a system of "inverse" retract equations of type

$$TD_Y = Y^\star T, \tag{3.41}$$

[19] It is worth mentioning that the star-product $\underline{\star}_\nu^{BG}$ extends to the space of distributions, constituting a *multiplier space* for $\underline{\star}_\nu^{BG}$; we refer to [21], section 4.2 for more details.

[20] See (3.1).

for $Y \in \mathfrak{g}_{RC}$ and with $D : \mathfrak{g}_{RC} \to \mathrm{Der}\left(\star_\nu^{BG}\right)$ a Lie algebra morphism extending the injection $\mathfrak{s} \hookrightarrow \mathrm{Der}\left(\star_\nu^{BG}\right)$. Using (3.40), equation (3.41) becomes

$$\mathrm{Tr}^{BG}\left(L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right) \star_\nu^{BG} D_Y|_x\, \psi\right) = \mathrm{Tr}^{BG}\left(Y^\star|_{x_0}\left(L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right)\right) \star_\nu^{BG} \psi\right). \tag{3.42}$$

Now, we notice the following fact: for $Y \in \mathfrak{g}_{RC}$, D_Y is a derivation of $\star_\nu^{BG}$. Theorem 3.2.1 then implies that D_Y is an inner derivation and thus equal (up to a formal constant) to a $\star_\nu^{BG}$-commutator. The cyclicality of the trace Tr^{BG} thus implies that

$$\mathrm{Tr}^{BG} \circ D_Y = 0.$$

From this, we deduce that for two functions ψ_1, ψ_2, we have

$$0 = \mathrm{Tr}^{BG}\left(D_Y(\psi_1 \star_\nu^{BG} \psi_2)\right) = \mathrm{Tr}^{BG}\left(D_Y\psi_1 \star_\nu^{BG} \psi_2\right) + \mathrm{Tr}^{BG}\left(\psi_1 \star_\nu^{BG} D_Y\psi_2\right),$$

i.e.

$$\mathrm{Tr}^{BG}\left(\psi_1 \star_\nu^{BG} D_Y\psi_2\right) = -\mathrm{Tr}^{BG}\left(D_Y\psi_1 \star_\nu^{BG} \psi_2\right).$$

Combining the above identity with (3.42) yields the following one, satisfied for all $\psi \in \mathcal{D}(\mathbb{S})$:

$$-\mathrm{Tr}^{BG}\left(D_Y|_x\, L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right) \star_\nu^{BG} \psi\right) = \mathrm{Tr}^{BG}\left(Y^\star|_{x_0}\left(L^\star_{x_0^{-1}}\left(\underline{T}_\nu^{-1}\check{v}\right)\right) \star_\nu^{BG} \psi\right).$$

In turn, using the decomposition[21] $\mathrm{Tr}^{BG} := \underline{\mathrm{Tr}}^{BG} \circ \underline{T}_\nu$ and the traciality of $\underline{\star}_\nu^{BG}$, this equation implies that the following distributional equation has to be satisfied:

$$D_Y|_x\left(\underline{T}_\nu^{-1}\check{v}\right)(x_0^{-1}x) = -\,Y^\star|_{x_0}\left(\underline{T}_\nu^{-1}\check{v}\right)(x_0^{-1}x),$$

with $x_0, x \in \mathbb{S}$. Remark now that this equation is exactly a retract equation for $\underline{T}_\nu^{-1}\check{v}$, and thus Proposition 3.2.16 implies that there exists $K_0 \in \mathbb{C}[[\nu]]$ and $\vec{K} \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$, such that

$$\underline{T}_\nu^{-1}\check{v} = K_0 u_{\nu,\vec{K}}. \tag{3.43}$$

Moreover, setting $\vec{K} = (K_1, K_2)$, we know that K_1 has to be a strictly positive real number (possibly depending on ν). Indeed, if this is not the case, Proposition 3.2.16 implies that $u_{\nu,\vec{K}} \equiv 0$, which in turn implies that $v \equiv 0$, by virtue of the identity (3.43). But this is impossible due to the fact that by assumptions, $U = v \times \cdot$ is an $\mathbb{S}$-equivalence between $\underline{\star}_\nu^{BG}$ and $\star_\nu^{RC}$. Lastly, notice that (3.43)

[21] Recall that $\underline{\mathrm{Tr}}^{BG} = \int_{\mathbb{S}}$ and $\underline{T}_\nu\left(\star_\nu^{BG}\right) = \underline{\star}_\nu^{BG}$.

leads to the announced form for the operator T, since for all $\psi \in \mathcal{D}(\mathbb{S})$,

$$\begin{aligned} T\psi(x_0) &= \mathrm{Tr}^{BG}\left(L^{\star}_{x_0^{-1}}\left(\underline{T}_{\nu}^{-1}\check{v}\right) \star_{\nu}^{BG} \psi\right) \\ &= \int_{\mathbb{S}} v\left(x^{-1}x_0\right) \underline{T}_{\nu}\psi(x) d^L_{\mathbb{S}}(x) \\ &= K_0 \int_{\mathbb{S}} \left[\underline{T}_{\nu} u_{\nu,\vec{K}}\right]^{\vee}\left(x^{-1}x_0\right) \underline{T}_{\nu}\psi(x) d^L_{\mathbb{S}}(x). \end{aligned}$$

■

In some sense, the previous proposition means that there exists a "correspondence" between the space of kernels $u_{\nu,\vec{K}}$ defining the "direct" intertwiners $U_{\nu,\vec{K}}$ and that of kernels defining the inverted intertwiners $U^{-1}_{\nu,\vec{K}}$. In order to use this result, we start by computing the expression of the intertwiner $\underline{T}_{\nu}$, satisfying $\underline{T}_{\nu}\left(\star_{\nu}^{BG}\right) = \underline{\star}_{\nu}^{BG}$. First, recall that by (2.39), we have

$$\underline{\star}_{\nu}^{BG} = T_{\nu,\vartheta_0}\left(\star_{\nu}^{\mathbb{S}}\right).$$

Moreover, by Definition 2.1.10, we also have

$$\star_{\nu}^{BG} = T_{\nu}\left(\star_{\nu}^{\mathbb{S}}\right).$$

So, we get

$$\left(T_{\nu,\vartheta_0} \circ T_{\nu}^{-1}\right)\left(\star_{\nu}^{BG}\right) = \underline{\star}_{\nu}^{BG},$$

which implies

$$\underline{T}_{\nu} = T_{\nu,\vartheta_0} \circ T_{\nu}^{-1}.$$

Using the decompositions (2.12) and (2.32), we get

$$\begin{aligned} \underline{T}_{\nu} &= \left(\mathcal{F}^{-1} \circ \mathcal{M}_{\exp(\vartheta_0)} \circ \left(\Psi^{\star}_{\nu}\right)^{-1} \circ \mathcal{F}\right) \circ \left(\mathcal{F}^{-1} \circ \Psi^{\star}_{\nu} \circ \mathcal{F}\right) \\ &= \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(\vartheta_0)} \circ \mathcal{F}. \end{aligned} \tag{3.44}$$

Thanks to the expression (2.38) of ϑ_0, we get for all $(a,\xi) \in \mathbb{R}^2$:

$$\exp(\vartheta_0)(a,\xi) = \frac{1}{\sqrt[4]{1-4\nu^2\xi^2}},$$

which leads to the following explicit expression for $\underline{T}_{\nu}$:

$$\underline{T}_{\nu}\psi(a,\ell) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\xi(\ell-\ell')} \frac{1}{\sqrt[4]{1-4\nu^2\xi^2}} \psi(a,\ell') d\xi d\ell'. \tag{3.45}$$

We now pass to the inversion of the intertwiners $U_{\nu,\vec{K}}$. By Proposition 3.2.18, we know that if it exists, the inverse of such an operator $U_{\nu,\vec{K}}$ is given by

$$U^{-1}_{\nu,\vec{K}} = K_0 \left(\left[\underline{T}_\nu u_{\nu,\vec{K}'} \right]^\vee \times \cdot \right) \circ \underline{T}_\nu, \tag{3.46}$$

for some formal constants $K_0 \in \mathbb{C}[[\nu]]$, $\vec{K}' = (K'_1, K'_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$, with K'_1 a strictly positive real number. Using together the expressions (3.38) and (3.45), we compute that, formally and up to a multiplicative formal constant:

$$\underline{T}_\nu u_{\nu,\vec{K}}(a,\ell) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi\ell - \frac{K'_2}{2\nu K'_1}\operatorname{arcsinh}(2i\nu\xi)} \frac{\eta_\nu(\xi)}{\sqrt[4]{1-4\nu^2\xi^2}} \delta_0\left(a + \log(K'_1\eta_\nu(\xi))\right) d\xi,$$

with η_ν given by (3.37). Then, using (3.46), we get[22]:

$$\begin{aligned} &U^{-1}_{\nu,\vec{K}}\psi(a_0,\ell_0) = \\ &\frac{K_0}{(2\pi)^2} \int e^{i\xi\left(\ell - e^{-2(a-a_0)}\ell_0\right) + i\xi'(\ell-\ell') - \frac{K'_2}{2\nu K'_1}\operatorname{arcsinh}(2i\nu\xi)} \frac{\eta_\nu(\xi)}{\sqrt[4]{1-4\nu^2\xi^2}\sqrt[4]{1-4\nu^2\xi'^2}} \\ &\qquad \times \delta_0\left(a - a_0 + \log(K'_1\eta_\nu(\xi))\right) \psi(a,\ell') d\xi' d\ell' d\xi da d\ell. \end{aligned}$$

Performing the integral w.r.t. the ℓ-variable yields

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\xi+\xi')\ell} d\ell = \delta_0(\xi'+\xi).$$

Therefore, since ϑ_0 is an even function, an integration w.r.t. the ξ'-variable followed by an integration w.r.t. the a-variable lead to the following proposition.

Proposition 3.2.19. Let K_1 be a strictly positive real number (possibly depending on ν) and $K_2 \in \mathbb{C}[[\nu]]$. Then, when it exists, the inverse of $U_{\nu,\vec{K}}$ is given up to a formal constant by the following formal operator:

$$\begin{aligned} &\tilde{U}_{\nu,\vec{K}'}\psi(a_0,\ell_0) = \\ &\int_{\mathbb{R}^2} e^{i\left[\xi\left(\ell - \frac{2K'^2_1}{1+\sqrt{1-4\nu^2\xi^2}}\ell_0\right) - \frac{K'_2}{2i\nu K'_1}\operatorname{arcsinh}(2i\nu\xi)\right]} \sqrt{\frac{2}{(1-4\nu^2\xi^2)\left(1+\sqrt{1-4\nu^2\xi^2}\right)}} \\ &\qquad \times \psi\left(a_0 - \log(K'_1) - \frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}\right), \ell\right) d\ell d\xi, \end{aligned}$$

[22] We omit the domain of the integrals to lighten the notation. Recall that in this case, the underlying domain will always be understood as $\mathbb{R}\times\ldots\times\mathbb{R}$ n times, with n the number of variables in the integrals.

for some $\vec{K}' := (K_1', K_2')$, with K_1' a strictly positive real number (possibly depending on ν) and $K_2' \in \mathbb{C}[[\nu]]$.

3.2.4 Intertwiners of RC and BG

Propositions 3.2.16 and 3.2.19 give the explicit form of both intertwiner and inverse intertwiner between the star-product $\star_\nu^{BG}$ and a $\mathfrak{g}_{RC}$-invariant star-product. Moreover, we know that the set of all intertwiners is indexed by two formal parameters $(K_1, K_2) \in \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]]$.

Let K_1, K_2 be two formal parameters such that the solution of equations (3.17) leads to an invertible operator[23]; let call it $\tau_{\nu,\vec{K}}$, with $\vec{K} := (K_1, K_2)$. By Proposition 3.2.16, we know that there exists a formal constant K such that $\tau_{\nu,\vec{K}} = KU_{\nu,\vec{K}}$ (with $U_{\nu,\vec{K}}$ as in Proposition 3.2.16). Then, setting $T_{\nu,\vec{K}} := \tau^{-1}_{\nu,\vec{K}}$, we know that $T_{\nu,\vec{K}}\left(\star_\nu^{BG}\right)$ is a $\mathfrak{g}_{RC}$-invariant star-product, which, by virtue of Proposition 2.1.18, has to coincide with $\star_\nu^{RC}$ (up to a redefinition of the deformation parameter). Our goal is now to determine the map

$$\mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]] \to \mathbb{C}[[\nu]] \times \mathbb{C}[[\nu]] : (K_1, K_2) \mapsto (K_1', K_2'),$$

such that

$$T_{\nu,\vec{K}} = \tilde{K}\tilde{U}_{\nu,\vec{K}'},$$

for a certain formal constant $\tilde{K}$, with $\vec{K}' := (K_1', K_2')$ and with $U_{\nu,\vec{K}'}$ as in Proposition 3.2.19. For this, we will simply impose that $\tau_{\nu,\vec{K}} \circ T_{\nu,\vec{K}} = \mathrm{Id}$ and $T_{\nu,\vec{K}} \circ \tau_{\nu,\vec{K}} = \mathrm{Id}$.

Firstly, thanks to the expressions given in Propositions 3.2.16 and 3.2.19 and setting $\vec{K} := (K_1, K_2)$, $\vec{K}' = (K_1', K_2')$, we have heuristically[24]

$$\begin{aligned}
&U_{\nu,\vec{K}}\left(\tilde{U}_{\nu,\vec{K}'}\psi\right)(a_0, \ell_0) = \\
&\frac{1}{2\pi}\int e^{i\left[\xi\left(\ell_0 - K_1^2\eta_\nu^2(\xi)\ell\right) - \frac{K_2}{2i\nu K_1}\mathrm{arcsinh}(2i\nu\xi) + \xi'\left(\ell' - K_1'^2\eta_\nu^2(\xi')\ell\right) - \frac{K_2'}{2i\nu K_1'}\mathrm{arcsinh}(2i\nu\xi')\right]} \\
&\qquad \times \frac{\eta_\nu(\xi)\eta_\nu(\xi')}{\sqrt{1-4\nu^2\xi'^2}}\psi\left(a_0 + \log(K_1\eta_\nu(\xi)) - \log(K_1'\eta_\nu(\xi')), \ell'\right) d\ell d\ell' d\xi d\xi',
\end{aligned}$$

with, as a reminder, $\eta_\nu(\xi_0) := \sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi_0^2}}}$. Performing the integral w.r.t. the ℓ-variable yields

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\ell\left(K_1\xi\eta_\nu^2(\xi) + K_1'^2\xi'\eta_\nu(\xi')\right)} d\ell = \delta_0\left(K_1\xi\eta_\nu^2(\xi) + K_1'^2\xi'\eta_\nu(\xi')\right).$$

[23] Recall that this condition is satisfied if K_1 is a strictly positive real number possibly depending on the deformation parameter ν.

[24] Again, we omit the domain of the integrals.

A short reflection allows to see that the only way to simplify the expression of $U_{\nu,\vec{K}} \circ \tilde{U}_{\nu,\vec{K}'}$ consists in choosing $K_1' := K_1$ (recall that by Proposition 3.2.16, K_1 has to be a strictly positive real number). In that case,

$$\delta_0\left(K_1\xi\eta_\nu^2(\xi) + K_1'^2\xi'\eta_\nu(\xi')\right) = \frac{1}{K_1^2}\delta_0\left(\xi\eta_\nu^2(\xi) + \xi'\eta_\nu^2(\xi')\right).$$

Since

$$\frac{d}{d\xi}\left(\xi\eta_\nu^2(\xi)\right) = \frac{\eta_\nu^2(\xi)}{\sqrt{1-4\nu^2\xi^2}}, \tag{3.47}$$

which is an even function, we get

$$\frac{1}{K_1^2}\delta_0\left(\xi\eta_\nu^2(\xi) + \xi'\eta_\nu^2(\xi')\right) = \frac{\sqrt{1-4\nu^2\xi'^2}}{K_1^2\eta_\nu^2(\xi')}\delta_0(\xi+\xi').$$

Hence, performing the integral w.r.t. the ℓ-variable followed by an integration w.r.t. the ξ-variable, we are left with:

$$U_{\nu,\vec{K}}\left(\tilde{U}_{\nu,\vec{K}'}\psi\right)(a_0,\ell_0) = \frac{1}{K_1^2}\int_{\mathbb{R}^2} e^{i\left[\xi'(\ell'-\ell_0)+\frac{K_2-K_2'}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi')\right]}\psi(a_0,\ell')d\ell' d\xi'.$$

From this, we see that we have to choose $K_2' := K_2$. This leads to the choice $\vec{K}' = \vec{K}$. Then, performing the integral w.r.t. the ξ'-variable, we finally obtain

$$\begin{aligned} U_{\nu,\vec{K}}\left(\tilde{U}_{\nu,\vec{K}}\psi\right)(a_0,\ell_0) &= \frac{2\pi}{K_1^2}\int_{\mathbb{R}}\psi(a_0,\ell')\delta_0(\ell'-\ell_0)d\ell' \\ &= \frac{2\pi}{K_1^2}\psi(a_0,\ell_0). \end{aligned}$$

Setting

$$T_{\nu,\vec{K}} := \tilde{U}_{\nu,\vec{K}}, \tag{3.48}$$

and

$$\tau_{\nu,\vec{K}} := \frac{K_1^2}{2\pi}U_{\nu,\vec{K}}, \tag{3.49}$$

the above computation shows heuristically that $\tau_{\nu,\vec{K}}$ is a formal left inverse for $T_{\nu,\vec{K}}$.

Now, we consider the operator $T_{\nu,\vec{K}} \circ \tau_{\nu,\vec{K}}$ with $T_{\nu,\vec{K}}$ and $\tau_{\nu,\vec{K}}$ given respectively

by (3.48) and (3.49). We have

$$T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\right)(a_0,\ell_0) = \frac{K_1^2}{(2\pi)^2}\int e^{i\left[\xi\left(\ell-K_1^2\eta_\nu^2(\xi)\ell_0\right)+\xi'\left(\ell-K_1^2\eta_\nu^2(\xi')\ell'\right)-\frac{K_2}{2i\nu K_1}\left(\operatorname{arcsinh}(2i\nu\xi)+\operatorname{arcsinh}(2i\nu\xi')\right)\right]} \times \frac{\eta_\nu(\xi)\eta_\nu(\xi')}{\sqrt{1-4\nu^2\xi^2}}\varphi\left(a_0-\log(K_1\eta_\nu(\xi))+\log(K_1\eta_\nu(\xi')),\ell'\right)d\ell d\ell' d\xi d\xi'.$$

Performing the integral w.r.t. the ℓ-variable yields

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell(\xi+\xi')}d\ell = \delta_0(\xi+\xi').$$

Performing now the integral w.r.t. the ξ-variable, we are left with:

$$T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\right)(a_0,\ell_0) = \frac{K_1^2}{2\pi}\int_{\mathbb{R}^2} e^{iK_1^2\xi'\eta_\nu^2(\xi')(\ell_0-\ell')}\frac{\eta_\nu^2(\xi')}{\sqrt{1-4\nu^2\xi'^2}}\varphi(a_0,\ell')d\ell' d\xi'.$$

We then change the variables following $\xi'\eta_\nu^2(\xi')\mapsto z$. Using (3.47), we have

$$\frac{d}{d\xi'}\left(\xi'\eta_\nu^2(\xi')\right) = \frac{\eta_\nu^2(\xi')}{\sqrt{1-4\nu^2\xi'^2}}.$$

Now, notice that the range of the strictly increasing function $\left[\xi'\in\mathbb{R}\mapsto\xi'\eta_\nu^2(\xi')\right]$ is included in $\left]-\frac{1}{|i\nu|},\frac{1}{|i\nu|}\right[$, since (using (3.37))

$$\lim_{\xi'\to\pm\infty}\xi'\eta_\nu^2(\xi') = \lim_{\xi'\to\pm\infty}\frac{2\xi'}{1+\sqrt{1-4\nu^2\xi'^2}} = \pm\frac{1}{|i\nu|}. \tag{3.50}$$

This leads to

$$\begin{aligned} T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\right)(a_0,\ell_0) &= \frac{K_1^2}{2\pi}\int_{\mathbb{R}\times\left]\frac{-1}{i\nu},\frac{1}{i\nu}\right[} e^{iK_1^2z(\ell_0-\ell')}\varphi(a_0,\ell')d\ell' dz \\ &= \frac{K_1^2}{2\pi}\int_{-\frac{1}{|i\nu|}}^{\frac{1}{|i\nu|}} e^{iK_1^2z\ell_0}\mathcal{F}\varphi(a_0,K_1^2z)dz. \end{aligned}$$

We can see that if we want to simplify this integral, the only way consists in requiring some conditions on the function φ. It is important to understand that although our approach is formal, the choice to write all operators using integrals is not so innocent. Indeed, since our goal is to obtain a non-formal expression for the Rankin-Cohen star-product, this strategy will allow us to get at the same time the expression of the star-product in terms of an integral formula as well as (if it exists) an adapted function space. The above computation shows

that at the non-formal level, we will have to ask that

$$\operatorname{supp}(\mathcal{F}\varphi) \subset \mathbb{R} \times \left[\frac{-K_1^2}{i\nu}, \frac{K_1^2}{i\nu}\right]. \tag{3.51}$$

In that case, we get (formally)

$$\begin{aligned} T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\right)(a_0,\ell_0) &= \frac{K_1^2}{2\pi}\int_{-\frac{1}{|i\nu|}}^{\frac{1}{|i\nu|}} e^{iK_1^2 z\ell_0}\mathcal{F}\varphi(a_0,K_1^2 z)dz \\ &= \frac{K_1^2}{2\pi}\int_{\mathbb{R}} e^{iK_1^2 z\ell_0}\mathcal{F}\varphi(a_0,K_1^2 z)dz \\ &= \frac{K_1^2}{2\pi}\int_{\mathbb{R}^2} e^{iK_1^2 z(\ell_0-\ell')}\varphi(a_0,\ell')d\ell' dz. \end{aligned}$$

Since the computation of the integral w.r.t. the z-variable yields

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{iK_1^2 z(\ell_0-\ell')}dz = \frac{1}{K_1^2}\delta_0(\ell_0-\ell'),$$

we finally obtain

$$T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\right)(a_0,\ell_0) = \varphi(a_0,\ell_0).$$

This means exactly that under a restriction on φ, $\tau_{\nu,\vec{K}}$ defined by (3.49) is also a formal right inverse for $T_{\nu,\vec{K}}$.

Remark 3.2.20. Note that if as in Remark 3.2.17, we choose $\vec{K}$ such that the limit condition

$$\lim_{\nu\to 0}\vec{K} = \lim_{\nu\to 0}(K_1,K_2) = (1,0)$$

is satisfied, that implies that both intertwiners $T_{\nu,\vec{K}}$ and $\tau_{\nu,\vec{K}}$, given respectively by (3.48) and (3.49), admit the correct limits:

$$\lim_{\nu\to 0} T_{\nu,\vec{K}} = \mathrm{Id},\ \lim_{\nu\to 0}\tau_{\nu,\vec{K}} = \mathrm{Id}.$$

The following theorem summarizes all the previous development.

Theorem 3.2.21. Any $\mathbb{S}$-equivalence between $\star_\nu^{BG}$ and a $\mathfrak{g}_{RC}$-invariant star-product on $\left(\mathbb{S},\omega^{\mathbb{S}}\right)$ is a linear operator $T_{\nu,\vec{K}}$ (defined such that the star-product

$T_{\nu,\vec{K}}\left(\star_\nu^{BG}\right)$ is $\mathfrak{g}_{RC}$-invariant), given formally by

$$\begin{aligned} &T_{\nu,\vec{K}}\psi(a_0,\ell_0) = \\ &\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[\xi\left(\ell-\frac{2K_1^2}{1+\sqrt{1-4\nu^2\xi^2}}\ell_0\right)-\frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi)\right]}\sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}} \\ &\qquad\times\sqrt{\frac{1}{1-4\nu^2\xi^2}}\psi\left(a_0-\log(K_1)-\frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}\right),\ell\right)d\ell d\xi, \end{aligned} \tag{3.52}$$

where $\vec{K}:=(K_1,K_2)$, with K_1 a strictly positive real number possibly depending on ν and $K_2\in\mathbb{C}[[\nu]]$, satisfies the limit condition

$$\lim_{\nu\to 0}\vec{K}=(1,0). \tag{3.53}$$

The formal inverse of $T_{\nu,\vec{K}}$, denoted by $\tau_{\nu,\vec{K}}$, is the linear operator given formally by

$$\begin{aligned} &\tau_{\nu,\vec{K}}\varphi(a_0,\ell_0) = \\ &\frac{K_1^2}{2\pi}\int_{\mathbb{R}^2} e^{i\left[\xi\left(\ell_0-\frac{2K_1^2}{1+\sqrt{1-4\nu^2\xi^2}}\ell\right)-\frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu\xi)\right]}\sqrt{\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}} \\ &\qquad\times\varphi\left(a_0+\log(K_1)+\frac{1}{2}\log\left(\frac{2}{1+\sqrt{1-4\nu^2\xi^2}}\right),\ell\right)d\ell d\xi. \end{aligned} \tag{3.54}$$

For $\varphi,\psi\in C^\infty(\mathbb{S})[[\nu]]$, the formal product $\sharp_{\nu,\vec{K}}$ defined as

$$\varphi\sharp_{\nu,\vec{K}}\psi := T_{\nu,\vec{K}}\left(\tau_{\nu,\vec{K}}\varphi\star_\nu^{BG}\tau_{\nu,\vec{K}}\psi\right)$$

is then $\mathfrak{g}_{RC}$-invariant. By Proposition 2.1.18, it implies that up to a redefinition of the deformation parameter ν, $\sharp_{\nu,\vec{K}}$ coincides with $\star_\nu^{RC}$. We will now give a non-formal meaning to this formula through a more in-depth analytical study of the operators $T_{\nu,\vec{K}}$ and $\tau_{\nu,\vec{K}}$.

3.3 Non-formal aspects of the intertwiners

The purpose of this section is to show the existence (on some well-chosen spaces of smooth functions) of the integrals defining the operators computed in the previous section via retract, by means of oscillatory integral techniques.

Recall that in chapter 2, we have introduced the space $\mathcal{S}^{BG}(\mathbb{S})$ of smooth functions on $\mathbb{S}$ (cf. Definition 2.2.44), adapted to the strongly closed version of the Bieliavsky-Gayral deformation quantization, namely the star-product $\underline{\star}_\nu^{BG}$. In order to make use of this space for the analytical study outlined within the present section, we will focus on the intertwiners between $\underline{\star}_\nu^{BG}$ and $\star_\nu^{RC}$, rather than those between $\star_\nu^{BG}$ and $\star_\nu^{RC}$. For this, let us recall that

$$\underline{T}_\nu\left(\star_\nu^{BG}\right) = \underline{\star}_\nu^{BG},$$

where $\underline{T}_\nu$ is the operator given by (3.45). It follows that the intertwiners $\underline{T}_{\nu,\vec{K}}$ of $\underline{\star}_\nu^{BG}$ and $\star_\nu^{RC}$ are given by

$$\underline{T}_{\nu,\vec{K}} := T_{\nu,\vec{K}} \circ \underline{T}_\nu^{-1}. \tag{3.55}$$

Thanks to the decomposition (3.44) of $\underline{T}_\nu$, its inverse $\underline{T}_\nu^{-1}$ decomposes as

$$\underline{T}_\nu^{-1} = \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(-\vartheta_0)} \circ \mathcal{F}.$$

Combining (3.52) with the above operator, and setting[25]

$$\tilde{\eta}_\nu : \mathbb{R} \to \mathbb{R}_0^+ : r \mapsto \eta_\nu(r)\exp(\vartheta_0(r)) = \sqrt{\frac{2}{1-4\nu^2r^2+\sqrt{1-4\nu^2r^2}}}, \tag{3.56}$$

we get the following expression for $\underline{T}_{\nu,\vec{K}}$:

$$\begin{aligned}&\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0) = \\ &\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[r\left(z-K_1^2\eta_\nu^2(r)\ell_0\right)-\frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r)\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right)drdz,\end{aligned} \tag{3.57}$$

for η_ν defined by (3.37) (and K_1 a ν-depending strictly positive real number).

Remark 3.3.1. It is worth mentioning that the above operator could have been directly obtained by use of the retract method, but the computations would have been more complicated than in the case of the intertwiner between $\star_\nu^{BG}$ and $\star_\nu^{RC}$.

The following lemma shows that, under a slight restriction on the choice of K_2, the integral in the RHS of (3.57) exists whenever φ belongs to the standard Schwartz space on $\mathbb{S}$ (that is, the Schwartz space in (a,ℓ)-coordinates (1.24)).

[25] Recall that we denote in the same way the function ϑ_0 defined on $\mathbb{S} \simeq \mathbb{R}^2$ and the one defined on $\mathbb{R}$.

Lemma 3.3.2. For $\varphi \in \mathcal{S}(\mathbb{S})$ and K_2 a real number (possibly ν-depending), the integrand in (3.57) is bounded by a function in $L^1\left(\mathbb{R}^2\right)$, turning the RHS of (3.57) into an existing integral.

Proof. Let us first consider the following integral:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\left(\mathrm{Id}-\partial_z^2\right)\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right)drdz, \tag{3.58}$$

in which, in order to lighten the notation, we have set

$$\mathbf{E} := e^{i\left[r\left(z-K_1^2\eta_\nu^2(r)\ell_0\right)-\frac{K_2}{2i\nu K_1}\mathrm{arcsinh}(2i\nu r)\right]}. \tag{3.59}$$

Since any (classical) derivative of a Schwartz function is again a Schwartz function, $\left(\mathrm{Id}-\partial_z^2\right)\varphi =: \tilde{\varphi}$ is Schwartz on $\mathbb{S}\simeq\mathbb{R}^2$. It means that for all $k,p,m,n\in\mathbb{N}$,

$$\sup_{(a,\ell)}\left|a^k\ell^p\partial_a^m\partial_\ell^n\tilde{\varphi}(a,\ell)\right| < +\infty.$$

In particular,

$$\sup_{(a,\ell)}\left|(1+\ell^2)\tilde{\varphi}(a,\ell)\right| < +\infty.$$

Setting $c_0 := \sup_{(a,\ell)}\left|(1+\ell^2)\tilde{\varphi}(a,\ell)\right|$, we get, for any $(a,\ell)\in\mathbb{R}^2$:

$$|\tilde{\varphi}(a,\ell)| \leq \frac{c_0}{1+\ell^2},$$

where the function $\frac{1}{1+\ell^2}$ belongs to $L^1(\mathbb{R})$. Having said that, we can then observe that

$$\left|\left(\mathrm{Id}-\partial_z^2\right)\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right)\right| \leq \frac{c_0}{1+z^2},$$

which holds for any a_0 and any r. Now, remark that $|\tilde{\eta}_\nu(r)|\leq 1$ for all $r\in\mathbb{R}$. Moreover, for K_2 a real number, the element $\mathbf{E}$ is a pure phase[26] and then $|\mathbf{E}|=1$. It follows that there exists a positive real constant c such that for all $r,z\in\mathbb{R}$:

$$\left|\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\left(\mathrm{Id}-\partial_z^2\right)\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right)\right| \leq \frac{c}{(1+r^2)(1+z^2)},$$

[26]Recall that $i\nu$ and K_1 are real numbers.

where the latter belongs to $L^1\left(\mathbb{R}^2\right)$. To conclude the proof, it suffices to observe that thanks to the identity

$$\frac{1}{1+r^2}\left(\mathrm{Id}-\partial_z^2\right)\mathbf{E}=\mathbf{E},$$

along with the self-adjointness[27] of the differential operator $\mathrm{Id}-\partial_z^2$ (w.r.t. the $L^2(\mathbb{R})$-inner product), the expression (3.58) coincides with the RHS in (3.57) (with $\vec{K}:=(K_1,K_2)$). ■

We can now show that the Schwartz property is retained by the operators $\underline{T}_{\nu,\vec{K}}$.

Proposition 3.3.3. Let $\varphi\in\mathcal{S}(\mathbb{S})$ and K_2 be a ν-depending real number. Then, $\underline{T}_{\nu,\vec{K}}\varphi\in\mathcal{S}(\mathbb{S})$, with $\vec{K}:=(K_1,K_2)$ and $\underline{T}_{\nu,\vec{K}}$ given by (3.57).

Proof. Let K_1 be a strictly positive (possibly) ν-depending real number, let K_2 be a (possibly) ν-depending complex number and let φ be a Schwartz function on $\mathbb{S}\simeq\mathbb{R}^2$. Using the proof of Lemma 3.3.2, we know that the function $\underline{T}_{\nu,\vec{K}}\varphi$ given by (3.57) (with $\vec{K}:=(K_1,K_2)$) admits the equivalent expression:

$$\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0)=\frac{1}{2\pi}\int\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\left(\mathrm{Id}-\partial_z^2\right)\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right)drdz,$$

with $\mathbf{E}$ defined by (3.59) and where we have dropped the domain of the integral, which is here equal to $\mathbb{R}^2$.
We start by showing that $\underline{T}_{\nu,\vec{K}}\varphi\in C^\infty(\mathbb{S})$. First of all, in order to lighten the notation, we set $D:=\mathrm{Id}-\partial_z^2$ and

$$f_\varphi(a_0,\ell_0,r,z):=\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}D\varphi\left(a_0-\log(K_1\eta_\nu(r)),z\right).$$

Recall that in the proof of Lemma 3.3.2, we have shown that there exists a constant $c\in\mathbb{R}^+$, independent of a_0 and ℓ_0, such that, for all r,z:

$$|f_\varphi(a_0,\ell_0,r,z)|\leq\frac{c}{(1+r^2)(1+z^2)}.$$

The Lebesgue dominated convergence[28] then implies that $\underline{T}_{\nu,\vec{K}}\varphi$ is continuous in (a_0,ℓ_0) for all $a_0,\ell_0\in\mathbb{R}$, and thus $\underline{T}_{\nu,\vec{K}}\varphi\in C^0(\mathbb{S})$.

[27] It can be shown by use of an integration by parts argument.
[28] See for instance [55], Theorem 2.2.17.

Consider now the following limit:

$$\lim_{h\to 0}\frac{1}{h}\left[\underline{T}_{\nu,\vec{K}}\varphi(a_0+h,\ell_0)-\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0)\right]= \\ \frac{1}{2\pi}\lim_{h\to 0}\int\frac{1}{h}\left[f_\varphi(a_0+h,\ell_0,r,z)-f_\varphi(a_0,\ell_0,r,z)\right]drdz. \tag{3.60}$$

By smoothness of f_φ, the mean-value theorem implies that there exists $a_h \in]a_0, a_0+h[$ such that

$$\frac{1}{h}\left[f_\varphi(a_0+h,\ell_0,r,z)-f_\varphi(a_0,\ell_0,r,z)\right]=\partial_1 f_\varphi(a_h,\ell_0,r,z),$$

where ∂_1 denotes the partial derivative w.r.t. the first variable. But

$$|\partial_1 f_\varphi(a_h,\ell_0,r,z)|=\left|\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\partial_1\left(D\varphi\right)\left(a_h-\log(K_1\eta_\nu(r)),z\right)\right|,$$

where $\partial_1\left(D\varphi\right)$ is of course again a Schwartz function. Thus, there exists $c_{1,0}\in\mathbb{R}^+$, independent of a_0, a_h and ℓ_0, such that, for all $r,z\in\mathbb{R}$:

$$\left|\mathbf{E}\frac{\tilde{\eta}_\nu(r)}{1+r^2}\partial_1\left(D\varphi\right)\left(a_h-\log(K_1\eta_\nu(r)),z\right)\right|\leq\frac{c_{1,0}}{(1+r^2)(1+z^2)}.$$

The Lebesgue dominated convergence then implies that

$$\lim_{h\to 0}\int\frac{1}{h}\left[f_\varphi(a_0+h,\ell_0,r,z)-f_\varphi(a_0,\ell_0,r,z)\right]drdz=\int\partial_1 f_\varphi(a_0,\ell_0,r,z)drdz,$$

where, as mentioned above, $|\partial_1 f_\varphi|$ is bounded by a $L^1\left(\mathbb{R}^2\right)$-function (in the r,z-variables), independent of a_0 and ℓ_0. It follows that the limit (3.60) exists and the dominated convergence theorem implies moreover that $\partial_a\underline{T}_{\nu,\vec{K}}\varphi\in C^0(\mathbb{S})$. Now, we consider the limit

$$\lim_{h\to 0}\frac{1}{h}\left[\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0+h)-\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0)\right]= \\ \frac{1}{2\pi}\lim_{h\to 0}\int\frac{1}{h}\left[f_\varphi(a_0,\ell_0+h,r,z)-f_\varphi(a_0,\ell_0,r,z)\right]drdz. \tag{3.61}$$

By the mean-value theorem, we have

$$\frac{1}{h}\left[f_\varphi(a_0,\ell_0+h,r,z)-f_\varphi(a_0,\ell_0,r,z)\right]=\partial_2 f_\varphi(a_0,\ell_h,r,z),$$

for a certain $\ell_h\in]\ell_0,\ell_0+h[$, and where ∂_2 denotes the partial derivative w.r.t. the second variable. Using the fact that for all $r\in\mathbb{R}$, $\left|r\eta_\nu^2(r)\right|\leq 1$, we get, for

all $r, z \in \mathbb{R}$:

$$|\partial_2 f_\varphi(a_0, \ell_h, r, z)| = \left|K_1^2 i r \eta_\nu^2(r) f_\varphi(a_0, \ell_h, r, z)\right| \leq \frac{c_{0,1}}{(1+r^2)(1+z^2)},$$

for a certain positive constant $c_{0,1}$ independent of a_0, ℓ_0 and ℓ_h. The dominated convergence theorem then implies:

$$\lim_{h\to 0} \int \frac{1}{h} \left[f_\varphi(a_0, \ell_0 + h, r, z) - f_\varphi(a_0, \ell_0, r, z)\right] dr dz = \int \partial_2 f_\varphi(a_0, \ell_0, r, z) dr dz.$$

Since (as mentioned above) $|\partial_2 f_\varphi|$ is bounded by a $L^1\left(\mathbb{R}^2\right)$-function (in the r, z-variables), independent of a_0 and ℓ_0, a new use of the Lebesgue theorem implies that the limit (3.61) exists, which in turn implies that $\partial_\ell \underline{T}_{\nu,\vec{K}}\varphi \in C^0(\mathbb{S})$. We thus have shown that $\underline{T}_{\nu,\vec{K}}\varphi \in C^1(\mathbb{S})$.
Remark now that for $m, n \in \mathbb{N}$,

$$\partial_1^m \partial_2^n f_\varphi(a_0, \ell_0, r, z) = \left(-iK_1^2 r \eta_\nu^2(r)\right)^n \mathbf{E} \frac{\tilde{\eta}_\nu(r)}{1+r^2} \partial_1^m (D\varphi)\left(a_0 - \log(K_1 \eta_\nu(r)), z\right),$$

where, obviously, $\partial_1^m(D\varphi)$ is again a Schwartz function. Then, there exists $c_{m,n} \in \mathbb{R}^+$, independent of a_0 and ℓ_0, such that

$$|\partial_1^m \partial_2^n f_\varphi(a_0, \ell_0, r, z)| \leq \frac{c_{m,n}}{(1+r^2)(1+z^2)},$$

for all $r, z \in \mathbb{R}$. An induction on the degree of the partial derivatives then shows that $\underline{T}_{\nu,\vec{K}}\varphi \in C^\infty(\mathbb{S})$.
It remains to show that for all $k, p, m, n \in \mathbb{N}$,

$$\sup_{(a,\ell)} \left|a^k \ell^p \partial_a^m \partial_\ell^n (\underline{T}_{\nu,\vec{K}}\varphi)(a, \ell)\right| < +\infty.$$

For this, remark that thanks to Lemma 3.3.2 and the Fubini theorem, if we start by performing the integral w.r.t. the z-variable within the expression of $\underline{T}_{\nu,\vec{K}}\varphi$ given by (3.57), it leads to[29]:

$$\underline{T}_{\nu,\vec{K}}\varphi(a, \ell) = \int e^{i\left[-K_1^2 r \eta_\nu^2(r)\ell - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]} \tilde{\eta}_\nu(r) \hat{\varphi}\left(a - \log(K_1 \eta_\nu(r)), r\right) dr, \qquad (3.62)$$

with $\hat{\varphi} := \mathcal{F}^{-1}\varphi$ which is again a Schwartz function on $\mathbb{S}$, by property of the partial Fourier transform. An obvious but important fact about η_ν, $\log \eta_\nu$, $\tilde{\eta}_\nu$ and arcsinh is that those functions are tempered in the sense of Definition

[29] Again, we drop the domain of the integral which is here equal to $\mathbb{R}$.

2.2.27 (this is clear thanks to the expressions (3.37) and (3.56)), and thus belong to the space $\mathcal{O}_M(\mathbb{R})$ of Schwartz multipliers on $\mathbb{R}$. In particular, it implies that, viewed as a function in the r-variable only, $\tilde{\eta}_\nu \hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)$ is a Schwartz function on $\mathbb{R}$ (for a fixed a). Having said that, we set $\mu(r) := r\eta_\nu^2(r)$, $\underline{\mathbf{E}} := e^{-i\mu(r)\ell}$, $\tilde{\mathbf{E}} := e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu r)}$ and $A := i\left(\frac{d}{dr}\mu(r)\right)^{-1}\partial_r$. Remark that both $\underline{\mathbf{E}}$ and $\tilde{\mathbf{E}}$ are pure phases. A straightforward computation leads to the identity $A^2\underline{\mathbf{E}} = \ell^2\underline{\mathbf{E}}$, which in turn implies that

$$\frac{1}{1+\ell^2}\left[\mathrm{Id} + A^2\right](\underline{\mathbf{E}}) = \underline{\mathbf{E}}.$$

Any power of the differential operator $\frac{1}{1+\ell^2}\left[\mathrm{Id} + A^2\right]$ thus leaves the element $\underline{\mathbf{E}}$ unchanged. Remark now that the functions $\left(\frac{d}{dr}\mu\right)^{-1}$ and $\frac{d^2}{dr^2}\mu$ both belong to the space $\mathcal{O}_M(\mathbb{R})$. Moreover, for a fixed a, the function $\tilde{\mathbf{E}}\tilde{\eta}_\nu\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)$ is a Schwartz function on $\mathbb{R}$ (w.r.t. the r-variable). Setting

$$A^\dagger := -i\left(\left(\frac{d}{dr}\mu(r)\right)^{-2}\left(\frac{d^2}{dr^2}\mu(r)\right)\mathrm{Id} - \left(\frac{d}{dr}\mu(r)\right)^{-1}\partial_r\right)$$

(that is, the adjoint operator of A w.r.t. the $L^2(\mathbb{R})$-inner product), an integration by parts argument then yields the following identity for all $q \in \mathbb{N}$:

$$\underline{T}_{\nu,\vec{K}}(a,\ell) = \frac{1}{(1+\ell^2)^q}\int \underline{\mathbf{E}}\left[\mathrm{Id} + \left(A^\dagger\right)^2\right]^q\left(\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)\right)dr. \tag{3.63}$$

Note that the integrand in the above RHS is again a Schwartz function on $\mathbb{R}$, w.r.t. the r-variable. Since $\hat{\varphi}$ is a Schwartz function on $\mathbb{S} \simeq \mathbb{R}^2$, this integrand can be written by means of a linear combination of elements of the form $\partial_1^{n_1}\partial_2^{n_2}\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)$ (with $n_1, n_2 \in \mathbb{N}$), with coefficients in the space $\mathcal{O}_M(\mathbb{R})$ (w.r.t. the r-variable). Using the fact that $\partial_1^{n_1}\partial_2^{n_2}\hat{\varphi} \in \mathcal{S}(\mathbb{S})$ (for any $n_1, n_2 \in \mathbb{N}$), the function $\partial_1^{n_1}\partial_2^{n_2}\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)$ can be bounded by a $L^1(\mathbb{R})$-function (w.r.t. the r-variable), independent of a. Hence, the RHS in (3.63) can be bounded by a $L^1(\mathbb{R})$-function (w.r.t. the r-variable), independent of a and ℓ. The Lebesgue dominated convergence along with the mean-value theorem then imply that for all $m, j \in \mathbb{N}$:

$$\partial_a^m\partial_\ell^j\int \underline{\mathbf{E}}\left[\mathrm{Id} + \left(A^\dagger\right)^2\right]^q\left(\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right)\right)dr = \int(-i\mu(r))^j\underline{\mathbf{E}}\left[\mathrm{Id} + \left(A^\dagger\right)^2\right]^q\left(\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\partial_1^m(\hat{\varphi})\left(a - \log(K_1\eta_\nu(r)), r\right)\right)dr.$$

The Leibniz rule then leads to

$$\begin{aligned}
&\partial_a^m \partial_\ell^n \left(\underline{T}_{\nu,\vec{K}} \varphi \right)(a,\ell) = \\
&\sum_{j=0}^{n} \binom{n}{j} \left(\frac{d}{d\ell} \right)^{n-j} \left(\frac{1}{(1+\ell^2)^q} \right) \\
&\quad \times \int (-i\mu(r))^j \underline{\mathbf{E}} \left[\mathrm{Id} + \left(A^\dagger \right)^2 \right]^q \left(\tilde{\mathbf{E}} \tilde{\eta}_\nu(r) \partial_1^m(\hat{\varphi}) \left(a - \log(K_1 \eta_\nu(r)), r \right) \right) dr.
\end{aligned}$$

For all $j \in \mathbb{N}$, remark that $\left| (-i\mu(r))^j \underline{\mathbf{E}} \right| \leq 1$. Moreover, for any choice of $p, n \in \mathbb{N}$, note that if we choose $q := p$, then for all $j \in \mathbb{N}$ with $j \leq n$, there exists a certain $c_j \in \mathbb{R}^+$ such that for all $\ell \in \mathbb{R}$, $\left| \ell^p \left(\frac{d}{d\ell} \right)^{n-j} \left(\frac{1}{(1+\ell^2)^q} \right) \right| \leq c_j$. So, we obtain the following estimate:

$$\begin{aligned}
&\left| a^k \ell^p \partial_a^m \partial_\ell^n (\underline{T}_{\nu,\vec{K}} \varphi)(a,\ell) \right| \leq \\
&c_1 \int \left| a^k \left[\mathrm{Id} + \left(A^\dagger \right)^2 \right]^p \left(\tilde{\mathbf{E}} \tilde{\eta}_\nu(r) \partial_1^m(\hat{\varphi}) \left(a - \log(K_1 \eta_\nu(r)), r \right) \right) \right| dr, \qquad (3.64)
\end{aligned}$$

where $c_1 := \sum_{j=0}^{n} c_j \binom{n}{j}$, with $c_j \in \mathbb{R}^+$. Remark now that for all $a \in \mathbb{R}$,

$$|a|^k \leq \left(|a - \log(K_1 \eta_\nu(r))| + |\log(K_1 \eta_\nu(r))| \right)^k,$$

where, as a reminder, the function $\log(K_1 \eta_\nu(r))$ belongs to $\mathcal{O}_M(\mathbb{R})$. In particular, we can find a certain $c' \in \mathbb{R}^+$ such that $|\log(K_1 \eta_\nu(r))| \leq c'(1+r^2)$ for all $r \in \mathbb{R}$. Thus, by Newton's binomial formula, we get

$$|a|^k \leq c_2 \sum_{t=0}^{k} \binom{k}{t} |a - \log(K_1 \eta_\nu(r))|^{k-t} (1+r^2)^t,$$

for a certain $c_2 \in \mathbb{R}^+$. Therefore,

$$\begin{aligned}
&\left| a^k \ell^p \partial_a^m \partial_\ell^n \left(\underline{T}_{\nu,\vec{K}} \varphi \right)(a,\ell) \right| \leq \\
&c_1 c_2 \sum_{t=0}^{k} \binom{k}{t} \int \Big| (a - \log(K_1 \eta_\nu(r)))^{k-t} (1+r^2)^t \\
&\qquad \times \left[\mathrm{Id} + \left(A^\dagger \right)^2 \right]^p \left(\tilde{\mathbf{E}} \tilde{\eta}_\nu(r) \partial_1^m(\hat{\varphi}) \left(a - \log(K_1 \eta_\nu(r)), r \right) \right) \Big| dr. \qquad (3.65)
\end{aligned}$$

To conclude the proof, it suffices to remark that $\varphi \in \mathcal{S}(\mathbb{S})$ implies that for all $m \in \mathbb{N}$, $\partial_1^m(\hat{\varphi}) \in \mathcal{S}(\mathbb{S})$. Since the function $\tilde{\eta}_\nu$, the element $\tilde{\mathbf{E}}$ as well as the coefficients of the differential operator $\mathrm{Id} + \left(A^\dagger \right)^2$ are tempered functions

on $\mathbb{R}$ w.r.t. the r-variable, we can find for all $t \in \mathbb{N}$ a function $\psi_t \in L^1(\mathbb{R})$ (independent of a and ℓ) such that the integrand in (3.65) is bounded by ψ_t, for all $r \in \mathbb{R}$. In other words, for all $a, \ell \in \mathbb{R}$,

$$\left| a^k \ell^p \partial_a^m \partial_\ell^n (\underline{T}_{\nu,\vec{K}} \varphi)(a,\ell) \right| \leq c_1 c_2 \sum_{t=0}^{k} \binom{k}{t} \int \psi_t(r) dr < +\infty,$$

where the majorant is independent of (a, ℓ). We thus have shown that

$$\sup_{(a,\ell)\in\mathbb{S}} \left| a^k \ell^p \partial_a^m \partial_\ell^n (\underline{T}_{\nu,\vec{K}} \varphi)(a,\ell) \right| < +\infty.$$

■

Thanks to Remark 2.2.46, we know that the space $S^{BG}(\mathbb{S})$ (see Definition 2.2.44) corresponds to the usual Schwartz space in coordinates $(x, y) = (\sinh(2a), \ell)$. In other words, the functions in $S^{BG}(\mathbb{S})$ are Schwartz functions (in the usual sense) with an extra property of exponential decreasing in their first variable. Since the operator $\underline{T}_{\nu,\vec{K}}$ does not really affect the first variable of the functions on $\mathbb{S} \simeq \mathbb{R}^2$ (by means of coordinates (1.24)), this observation leads to the following corollary.

Corollary 3.3.4. Let $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$ and K_2 be a possibly ν-depending real number. Then, $\underline{T}_{\nu,\vec{K}} \varphi \in \mathcal{S}^{BG}(\mathbb{S})$, with $\vec{K} := (K_1, K_2)$ and $\underline{T}_{\nu,\vec{K}}$ given by (3.57).

Proof. Let φ be a function belonging to $\mathcal{S}^{BG}(\mathbb{S})$. Lemma 2.2.47 and Proposition 3.3.3 imply that $\underline{T}_{\nu,\vec{K}} \varphi$ belongs to $\mathcal{S}(\mathbb{S})$. Then, it remains to show that for all $k, p, m, n \in \mathbb{N}$,

$$\sup_{(a,\ell)\in\mathbb{S}} \left| \sinh^k(2a) \ell^p \tilde{H}^m \tilde{E}^n (\underline{T}_{\nu,\vec{K}} \varphi)(a,\ell) \right| < +\infty.$$

Note that regarding the expressions (1.29) of the left invariant vector fields $\tilde{H}$ and $\tilde{E}$, it is then sufficient to show that for all $k, p, m, n \in \mathbb{N}$,

$$\sup_{(a,\ell)\in\mathbb{S}} \left| \sinh^k(2a) \ell^p \partial_a^m \partial_\ell^n (\underline{T}_{\nu,\vec{K}} \varphi)(a,\ell) \right| < +\infty.$$

For this, we start by replacing a by $\sinh(2a)$ in the inequality (3.64) and we consider the following estimate:

$$\begin{aligned}|\sinh(2a)| &= |\sinh(2(a-\log(K_1\eta_\nu(r)))+2\log(K_1\eta_\nu(r)))| \\ &\leq |\sinh(2(a-\log(K_1\eta_\nu(r))))\cosh(2\log(K_1\eta_\nu(r)))| \\ &\quad + |\cosh(2(a-\log(K_1\eta_\nu(r))))\sinh(2\log(K_1\eta_\nu(r)))|. \end{aligned} \tag{3.66}$$

Note that the functions $|\sinh|$ and cosh have the same asymptotic behavior. Moreover, thanks to the identities (2.9), the (inverse) partial Fourier transform[30] $\mathcal{F}^{-1}$ maps $\mathcal{S}^{BG}\left(\tilde{\mathbb{S}}\right)$ to $\mathcal{S}^{BG}(\mathbb{S})$. Hence, in the inequality (3.64), we notice that for all $m \in \mathbb{N}$, $\partial_1^m\left(\hat{\varphi}\right) \in \mathcal{S}^{BG}(\mathbb{S})$ whenever $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. Remark also that the functions $\sinh(2\log(K_1\eta_\nu(r)))$ and $\cosh(2\log(K_1\eta_\nu(r)))$ are linear combinations of powers of functions $\eta_\nu(r)$ and $\eta_\nu^{-1}(r)$, that are tempered as functions on $\mathbb{R}$. A similar proof to that made for Proposition 3.3.3 then leads to the desired result. ■

Remark 3.3.5. From now on, K_1 and K_2 will always denote two possibly ν-depending real numbers satisfying the limit condition (3.53), with K_1 strictly positive.

In the previous section, we have seen that in order to be fulfilled, the identity $\underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\varphi\right) = \varphi$ requires at least a certain condition on the support of the partial Fourier transform of φ (see (3.51)). It turns out that this condition is naturally satisfied for the functions belonging to the space $\underline{T}_{\nu,\vec{K}}\left(\mathcal{S}(\mathbb{S})\right)$. Indeed, Proposition 3.3.3 along with the property of the partial Fourier transform imply that for $\varphi \in \mathcal{S}(\mathbb{S})$, $\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right) \in \mathcal{S}(\mathbb{S})$, with $\vec{K} := (K_1, K_2)$. Remark that for such a function $\varphi \in \mathcal{S}(\mathbb{S})$, we have:

$$\begin{aligned}&\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)(a,\xi) = \\ &\int_{\mathbb{R}} e^{-i\xi\ell}\left(\int_{\mathbb{R}} e^{-iK_1^2 r\eta_\nu^2(r)\ell}\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a-\log(K_1\eta_\nu(r)),r\right)dr\right)d\ell, \end{aligned} \tag{3.67}$$

where $\hat{\varphi} := \mathcal{F}^{-1}\varphi$ and $\tilde{\mathbf{E}} := e^{-\frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)}$. Using the same notations as in the proof of Proposition 3.3.3, we observe that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|e^{-i\xi\ell}e^{-iK_1^2 r\eta_\nu^2(r)\ell}\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a-\log(K_1\eta_\nu(r)),r\right)\right|drd\ell \\ &\qquad = \int_{\mathbb{R}^2}\left|\frac{1}{(1+\ell^2)}\left[\mathrm{Id}+\left(A^\dagger\right)^2\right]\left(\tilde{\mathbf{E}}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a-\log(K_1\eta_\nu(r)),r\right)\right)\right|drd\ell, \end{aligned}$$

[30] We stress the fact that this is a Fourier transform in the second variable only.

where the latter is an existing integral[31]. So we can use the Fubini theorem within the expression (3.67) and compute first the integral w.r.t. the ℓ-variable. This leads to the distributional identity:

$$\int_{\mathbb{R}} e^{-i\left(\xi+K_1^2\mu(r)\right)\ell} d\ell = 2\pi\delta_0\left(\xi + K_1^2\mu(r)\right).$$

Notice that the limit (3.50) implies that the latter distribution is identically zero for $\xi \notin \left]-\frac{K_1^2}{i\nu}, \frac{K_1^2}{i\nu}\right[$. It means that

$$\operatorname{supp}\left(\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)\right) \subset \mathbb{R} \times \left[-\frac{K_1^2}{|i\nu|}, \frac{K_1^2}{|i\nu|}\right],$$

by means of the global chart (1.24) (that is, for $\tilde{\mathbb{S}} \simeq \mathbb{S} \simeq \mathbb{R}^2$). We thus have shown the following lemma.

Lemma 3.3.6. For any[32] $\psi \in \underline{T}_{\nu,\vec{K}}\left(\mathcal{S}(\mathbb{S})\right)$ and under the identifications $\tilde{\mathbb{S}} \simeq \mathbb{S} \simeq \mathbb{R}^2$ w.r.t. (1.24), we have:

$$\operatorname{supp}\left(\mathcal{F}\psi\right) \subset \mathbb{R} \times \left[-\frac{K_1^2}{|i\nu|}, \frac{K_1^2}{|i\nu|}\right].$$

Recall that the operator $\underline{T}_{\nu,\vec{K}}$ satisfies $\underline{T}_{\nu,\vec{K}}\left(\star_\nu^{BG}\right) = \star_\nu^{RC}$. Using (3.55), its formal inverse, that we will denote as $\underline{\tau}_{\nu,\vec{K}}$, is then given by

$$\underline{\tau}_{\nu,\vec{K}} := \underline{T}_\nu \circ \tau_{\nu,\vec{K}}.$$

More explicitly, a computation leads to the following expression for $\underline{\tau}_{\nu,\vec{K}}$:

$$\begin{aligned}&\underline{\tau}_{\nu,\vec{K}}\psi(a_0,\ell_0) = \\ &\frac{K_1^2}{2\pi}\int_{\mathbb{R}^2} e^{i\left[r\left(\ell_0 - K_1^2\eta_\nu^2(r)z\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r)\psi\left(a_0 + \log(K_1\eta_\nu(r)), z\right) dr dz,\end{aligned} \tag{3.68}$$

with $\tilde{\eta}_\nu$ defined by (3.56).

Remark 3.3.7. Let $\gamma_{\nu,\vec{K}}$ be the smooth map defined by

$$\begin{aligned}\gamma_{\nu,\vec{K}} : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (a,\xi) &\mapsto (a + \log(K_1\eta_\nu(\xi)), K_1^2 r\eta_\nu^2(\xi)),\end{aligned}$$

[31] See the proof of Proposition 3.3.3 and the expression (3.63) with $q := 1$.
[32] Recall that $K_1 \in \mathbb{R}_0^+$ and $K_2 \in \mathbb{R}$, both possibly ν-depending.

and let $\mathcal{M}_{\nu,\vec{K}}$ be the operator defined for any smooth function f on $\tilde{\mathbb{S}} \simeq \mathbb{R}^2$ by

$$\mathcal{M}_{\nu,\vec{K}}(f) := \left[(a,\xi) \mapsto K_1^2 \tilde{\eta}_\nu(\xi) e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi)} f(a,\xi)\right].$$

At least formally, a simple computation shows that $\underline{\tau}_{\nu,\vec{K}}$ given by (3.68) admits the following decompostion:

$$\underline{\tau}_{\nu,\vec{K}} = \mathcal{F}^{-1} \circ \mathcal{M}_{\nu,\vec{K}} \circ \gamma^\star_{\nu,\vec{K}} \circ \mathcal{F}. \tag{3.69}$$

The decomposition (3.69) of $\underline{\tau}_{\nu,\vec{K}}$ given in the above remark yields the following result at the non-formal level.

Proposition 3.3.8. The operator $\underline{\tau}_{\nu,\vec{K}}$ is a non-formal left inverse for $\underline{T}_{\nu,\vec{K}}$. More precisely, we have:

$$\left(\underline{\tau}_{\nu,\vec{K}} \circ \underline{T}_{\nu,\vec{K}}\right)\Big|_{\mathcal{S}(\mathbb{S})} = \mathrm{Id}_{\mathcal{S}(\mathbb{S})}.$$

Proof. Let φ be a function belonging to $\mathcal{S}(\mathbb{S})$. In the notations of Remark 3.3.7, we start by computing the expression of $\gamma^\star_{\nu,\vec{K}}\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)(a,\xi)$, where $(a,\xi) \in \tilde{\mathbb{S}} \simeq \mathbb{R}^2$. Using the expression (3.67) and performing[33] the integral w.r.t. the ℓ-variable followed by the integral w.r.t. the r-variable, we get

$$\gamma^\star_{\nu,\vec{K}}\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)(a,\xi) = \frac{2\pi}{K_1^2\tilde{\eta}_\nu(\xi)} e^{\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu\xi)} \hat{\varphi}(a,-\xi).$$

It implies that

$$\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)\right)(a,\xi) = 2\pi\hat{\varphi}(a,-\xi),$$

which is obviously a Schwartz function. Finally, notice that an inverse partial Fourier transform applied to the latter expression yields:

$$\mathcal{F}^{-1}\left(\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)\right)\right)(a,\ell) = \varphi(a,\ell),$$

which holds for all $(a,\ell) \in \mathbb{S} \simeq \mathbb{R}^2$. That completes the proof, thanks to the decomposition (3.69) of $\underline{\tau}_{\nu,\vec{K}}$. ∎

In what follows, we will show that the operator $\underline{\tau}_{\nu,\vec{K}}$ is also a non-formal right inverse for $\underline{T}_{\nu,\vec{K}}$, at least on a particular functional space that we have to specify. That is the reason why we start by introducing the following definition.

[33] We use here the Fubini theorem.

Definition 3.3.9. For $\sigma \in \mathbb{R}^+$, we denote as $\mathcal{S}_\sigma$ the space of all functions $f \in \mathcal{S}^{BG}(\mathbb{S})$ (cf. Definition 2.2.44) such that $\mathrm{supp}(f) \subset \mathbb{R} \times [-\sigma, \sigma]$, with $\mathbb{S}$ viewed as $\mathbb{R}^2$ by means of the chart (1.24). Moreover, we will denote as $\underline{\mathcal{S}}_\sigma$ the space of all functions $f \in \mathcal{S}^{BG}(\mathbb{S})$ such that $\mathrm{supp}(f) \subset \mathbb{R} \times [-\epsilon, \epsilon]$, for a certain $0 \leq \epsilon < \sigma$. Lastly, we will denote respectively as $\mathcal{S}^\sigma$ and $\underline{\mathcal{S}}^\sigma$ the spaces $\mathcal{F}^{-1}(\mathcal{S}_\sigma)$ and $\mathcal{F}^{-1}(\underline{\mathcal{S}}_\sigma)$, that is, the ranges of $\mathcal{S}_\sigma$ and $\underline{\mathcal{S}}_\sigma$ under the inverse partial Fourier transform[34].

Remark 3.3.10. As a consequence of the *Hardy uncertainty principle* (see for instance [40]), the space $\underline{\mathcal{S}}^\sigma$ (for $\sigma \in \mathbb{R}^+$) does not contain the space $\mathcal{D}(\mathbb{S})$ of smooth compactly supported functions

Lemma 3.3.11. For $\nu \neq 0$, the following inclusion holds:

$$\underline{T}_{\nu,\vec{K}}\left(\mathcal{D}(\mathbb{S})\right) \subset \underline{\mathcal{S}}^{K_1^2/|i\nu|}.$$

Proof. For a function $\psi = \underline{T}_{\nu,\vec{K}}\varphi$ with $\varphi \in \mathcal{D}(\mathbb{S})$, note first that Corollary 3.3.4 implies that ψ belongs to the space $\mathcal{S}^{BG}(\mathbb{S})$. Now, notice that for $a, \xi \in \mathbb{R}$, we have (by virtue of the expression (3.57) of $\underline{T}_{\nu,\vec{K}}$):

$$\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)(a,\xi) = \int_{\mathbb{R}}\int_B e^{-i\xi\ell} e^{i\left[-K_1^2 r\eta_\nu^2(r)\ell - \frac{K_2}{2i\nu K_1}\mathrm{arcsinh}(2i\nu r)\right]} \tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right) dr d\ell,$$

for a certain compact set $B \subset \mathbb{R}$ such that $\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right) = 0$ for $r \notin B$. Indeed, the function $\hat{\varphi}$ is compactly supported in its first variable (because the partial Fourier transform affects only the second variable), and for a fixed $a \in \mathbb{R}$, we have:

$$\lim_{r \to \pm\infty} a - \log(K_1\eta_\nu(r)) = +\infty.$$

Performing the integral w.r.t. the ℓ-variable in the above expression yields

$$\mathcal{F}\left(\underline{T}_{\nu,\vec{K}}\varphi\right)(a,\xi) = 2\pi \int_B e^{-\frac{K_2}{2\nu K_1}\mathrm{arcsinh}(2i\nu r)} \tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(K_1\eta_\nu(r)), r\right) \delta_0\left(\xi + K_1^2 r\eta_\nu^2(r)\right) dr.$$

Since the range of the map $\left[B \to \mathbb{R} : r \mapsto K_1^2 r\eta_\nu^2(r)\right]$ is included in an interval $[-\epsilon, \epsilon]$ for a certain $0 \leq \epsilon < \frac{K_1^2}{|i\nu|}$, the distribution $\delta_0\left(\xi + K_1^2 r\eta_\nu^2(r)\right)$ is identically zero for ξ outside the interval $[-\epsilon, \epsilon]$. We thus have shown that $\psi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$. ■

Now, we show the announced result concerning the operator $\underline{\tau}_{\nu,\vec{K}}$.

[34] This notation is inspired by that of Gel'fand and Shilov used in [35].

Proposition 3.3.12. For a given deformation parameter $\nu \neq 0$, the following holds[35]:

$$\left(\underline{T}_{\nu,\vec{K}} \circ \underline{\tau}_{\nu,\vec{K}}\right)\Big|_{\underline{\mathcal{S}}^{K_1^2/|i\nu|}} = \mathrm{Id}|_{\underline{\mathcal{S}}^{K_1^2/|i\nu|}}.$$

Proof. First of all, remark that for a Schwartz function φ on $\mathbb{S}$, a change of variables following $r \mapsto -r$ within the expression (3.57), followed by an integration w.r.t. the z-variable, lead to the following expression for $\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0)$ (with $(a_0,\ell_0) \in \mathbb{R}^2$):

$$\underline{T}_{\nu,\vec{K}}\varphi(a_0,\ell_0) = \\ \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\left[K_1^2 r\eta_\nu^2(r)\ell_0 + \frac{K_2}{2i\nu K_1}\mathrm{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r)\mathcal{F}\varphi\left(a_0 - \log(K_1\eta_\nu(r)), r\right)dr.$$

Now, we choose a function $\psi \in \mathcal{S}^\sigma$, for $0 \leq \sigma < \frac{K_1^2}{|i\nu|}$ (i.e. $\psi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$). We thus have the identity $\psi = \mathcal{F}^{-1}\psi_0$, for a certain $\psi_0 \in \mathcal{S}_\sigma$. Notice that in particular[36], $\psi_0, \psi \in \mathcal{S}(\mathbb{S})$. Thanks to the decomposition (3.69) of $\underline{\tau}_{\nu,\vec{K}}$, we have

$$\underline{\tau}_{\nu,\vec{K}}\psi = \mathcal{F}^{-1}\left(\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right)\right),$$

with

$$\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right)(a,\xi) = \\ K_1^2\tilde{\eta}_\nu(\xi)e^{-\frac{K_2}{2\nu K_1}\mathrm{arcsinh}(2i\nu\xi)}\psi_0(a + \log(K_1\eta_\nu(\xi)), K_1^2\xi\eta_\nu^2(\xi)).$$

Let us show that the function $\underline{\tau}_{\nu,\vec{K}}\psi$ belongs to the standard Schwartz space $\mathcal{S}(\mathbb{S})$. First, by using the limit (3.50) along with the fact that $\psi_0 \in \mathcal{S}_\sigma$, we can see that the function $\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right)(a,\xi)$ vanishes identically for ξ outside an interval $[-\tilde{\sigma},\tilde{\sigma}]$, for a certain $0 \leq \tilde{\sigma} < +\infty$. Moreover[37],

$$\partial_a\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right) = \mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\partial_1\psi_0\right),$$

and for all $a, \xi \in \mathbb{R}$,

$$|a| \leq |a + \log(K_1\eta_\nu(\xi))| + |\log(K_1\eta_\nu(\xi))|.$$

Given that the (classical) partial derivatives are operators that restrict the support and that (in particular) $\psi_0 \in \mathcal{S}(\mathbb{S})$, we can deduce that for all $k,p,m,n \in$

[35] See Definition 3.3.9 for the notation.
[36] See Lemma 2.2.47.
[37] Recall that ∂_1 means the partial derivative w.r.t. the first variable.

$\mathbb{N}$,

$$\sup_{(a,\xi)\in\tilde{\mathbb{S}}}\left|a^k\xi^p\partial_a^m\partial_\xi^n\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right)(a,\xi)\right|<+\infty.$$

It implies that $\mathcal{M}_{\nu,\vec{K}}\left(\gamma^\star_{\nu,\vec{K}}\psi_0\right)\in\mathcal{S}(\tilde{\mathbb{S}})$, which in turn implies that $\underline{\tau}_{\nu,\vec{K}}\psi\in\mathcal{S}(\mathbb{S})$. Notice now that

$$\mathcal{F}\left(\underline{\tau}_{\nu,\vec{K}}\psi\right)(a_0-\log(K_1\eta_\nu(r)),r)= \\ K_1^2\tilde{\eta}_\nu(r)e^{-\frac{K_2}{2\nu K_1}\operatorname{arcsinh}(2i\nu r)}\psi_0\left(a_0,K_1^2r\eta_\nu^2(r)\right).$$

If we plug this expression into that of $\underline{T}_{\nu,\vec{K}}$ for $\varphi:=\underline{\tau}_{\nu,\vec{K}}\psi$, we get:

$$\underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\psi\right)(a_0,\ell_0)=\frac{K_1^2}{2\pi}\int_{\mathbb{R}}e^{iK_1^2r\eta_\nu^2(r)\ell_0}\tilde{\eta}_\nu^2(r)\psi_0\left(a_0,K_1^2r\eta_\nu^2(r)\right)dr.$$

This integral exists thanks to the fact that (as mentioned above), the function $\psi_0\left(a_0,K_1^2r\eta_\nu^2(r)\right)$ vanishes identically for r outside a certain finite interval $[-\tilde{\sigma},\tilde{\sigma}]$. Changing the variables following $K_1^2r\eta_\nu^2(r)\mapsto r'$ and using (3.47) along with (3.56), we get:

$$\begin{aligned}\underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\psi\right)(a_0,\ell_0)&=\frac{1}{2\pi}\int_{-\frac{K_1^2}{|i\nu|}}^{\frac{K_1^2}{|i\nu|}}e^{ir'\ell_0}\psi_0(a_0,r')dr'\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ir'\ell_0}\psi_0(a_0,r')dr'\\&=\mathcal{F}^{-1}\psi_0(a_0,\ell_0)\\&=\psi(a_0,\ell_0),\end{aligned}$$

where, for the second equality, we have used the fact that $\psi_0\in\mathcal{S}_\sigma$, with $\sigma<\frac{K_1^2}{|i\nu|}$. ■

In the next section, we will transport the algebra structure $\left(\mathcal{S}^{BG}(\mathbb{S}),\underline{\star}_\nu^{BG}\right)$ under the operator $\underline{T}_{\nu,\vec{K}}$. Thanks to the identity $\underline{T}_{\nu,\vec{K}}\left(\underline{\star}_\nu^{BG}\right)=\star_\nu^{RC}$, that will generate a non-formal Rankin-Cohen star-product formula, together with an adapted space of functions for this non-formal deformation quantization.

3.4 The non-formal RC star-product

3.4.1 Formula and integral representation

The purpose of this section is to give rise to a non-formal expression (that is, an expression in terms of an integral) for the Rankin-Cohen deformation

quantization formula $\star_\nu^{RC}$ given in Proposition 1.2.24. To this end, recall that we can transport the space of Schwartz functions $\mathcal{S}(\mathbb{S})$ under the operator[38] $\underline{T}_{\nu,\vec{K}}$ in a non-formal way (see for instance Lemma 3.3.2). From chapter 2, we know that the space of smooth functions $\mathcal{S}^{BG}(\mathbb{S})$ is a closed algebra for the product $\underline{\star}_\nu^{BG}$. It seems then natural to transport this space by means of the operator $\underline{T}_{\nu,\vec{K}}$, in order to produce a closed functional space for the star-product $\underline{T}_{\nu,\vec{K}}\left(\underline{\star}_\nu^{BG}\right) = \star_\nu^{RC}$. Thanks to Corollary 3.3.4, note that $\underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right) \subset \mathcal{S}^{BG}(\mathbb{S})$. We start by showing that $\underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$ contains the space $\underline{\mathcal{S}}^{K_1^2/|i\nu|}$.

Lemma 3.4.1. For $\nu \neq 0$, the following inclusion holds:

$$\underline{\mathcal{S}}^{K_1^2/|i\nu|} \subset \underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right).$$

Proof. Let us consider a positive real number σ satisfying $\sigma < \frac{K_1^2}{|i\nu|}$, along with a function $\psi \in \mathcal{S}^\sigma$ (in the notations of Definition 3.3.9). By Proposition 3.3.12, we know that $\psi = \underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\psi\right)$. The fact that $\underline{\tau}_{\nu,\vec{K}}\psi$ belongs to $\mathcal{S}^{BG}(\mathbb{S})$ is based on similar arguments to that used for the proof of Proposition 3.3.12. Indeed, since we already know that $\underline{\tau}_{\nu,\vec{K}}\psi \in \mathcal{S}(\mathbb{S})$, it remains to show that this function and any of its (classical) partial derivatives decrease faster than a hyperbolic sine in their first variable. By replacing a by $\sinh(2a)$ in the proof of Proposition 3.3.12, the desired result then follows from the inequality (3.66) and from the fact that the inverse partial Fourier transform maps obviously $\mathcal{S}^{BG}(\tilde{\mathbb{S}})$ to $\mathcal{S}^{BG}(\mathbb{S})$. ■

Now, we set

$$\mathcal{E}_{\nu,\vec{K}} := \underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right), \tag{3.70}$$

that is, the range subspace of $\underline{T}_{\nu,\vec{K}}$ in the Schwartz space $\mathcal{S}(\mathbb{S})$. Corollary 3.3.4 and Lemmas 3.3.11 and 3.4.1 imply the following inclusions:

$$\underline{T}_{\nu,\vec{K}}\left(\mathcal{D}(\mathbb{S})\right) \subset \underline{\mathcal{S}}^{K_1^2/|i\nu|} \subset \mathcal{E}_{\nu,\vec{K}} \subset \mathcal{S}^{BG}(\mathbb{S}). \tag{3.71}$$

Lemma 3.4.2. The function space $\mathcal{S}^{BG}(\mathbb{S})$ is stable under the left action (2.1) of $\mathbb{S}$. That is, for any $x_0 \in \mathbb{S}$, we have the inclusion

$$L_{x_0}^\star\left(\mathcal{S}^{BG}(\mathbb{S})\right) \subset \mathcal{S}^{BG}(\mathbb{S}).$$

[38] Recall that from now on, $\vec{K} := (K_1, K_2)$ with K_1 and K_2 two possibly ν-depending real numbers satisfying the limit condition (3.53), with K_1 strictly positive.

Proof. Recall that for $x_0 = (a_0, \ell_0)$, $x = (a, \ell)$ (within coordinates (1.24)) and φ a function on $\mathbb{S}$, we have (using the left action (2.1)):

$$L^{\star}_{x_0}\varphi(g) = \varphi(x_0 g) = \varphi\left(a_0 + a, e^{-2a}\ell_0 + \ell\right).$$

If $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$ (see Definition 2.2.44), then the function $L^{\star}_{x_0}\varphi$ belongs also to $\mathcal{S}^{BG}(\mathbb{S})$. Indeed, it comes from the rough estimates[39]

$$|\sinh(2a)| \leq |\sinh(2(a_0+a))\cosh(2a_0)| + |\cosh(2(a_0+a))\sinh(2a_0)|,$$

$$|\ell| \leq \left|e^{-2a}\ell_0 + \ell\right| + e^{-2a}|\ell_0| \quad \text{and} \quad e^{-2a} \leq 4\left(1 + \sinh^2(2a)\right),$$

holding for all $a_0, a, \ell_0, \ell \in \mathbb{R}$, along with the fact that for any $X \in \mathcal{U}(\mathfrak{s})$, the associated vector field $\tilde{X}$ is left invariant. ■

Being a convolution operator, $\underline{T}_{\nu,\vec{K}}$ is thus left equivariant, as shown in the next proposition.

Proposition 3.4.3. For all $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$ and for all $x_0 \in \mathbb{S}$, the following identity holds:

$$\underline{T}_{\nu,\vec{K}}\left(L^{\star}_{x_0}\varphi\right) = L^{\star}_{x_0}\left(\underline{T}_{\nu,\vec{K}}\varphi\right).$$

Proof. Let φ be a function belonging to the space $\mathcal{S}^{BG}(\mathbb{S})$. Thanks to Lemma 3.4.2, we know that for all $x_0 \in \mathbb{S}$, the function $L^{\star}_{x_0}\varphi$ belongs to $\mathcal{S}^{BG}(\mathbb{S})$ too. For given $(a, \ell) \in \mathbb{R}^2$, setting $x_0 := (a_0, \ell_0)$ (by means of coordinates (1.24)) and using the left action (2.1), we can thus consider the expression[40]

$$\begin{aligned}
&\underline{T}_{\nu,\vec{K}}\left(L^{\star}_{(a_0,\ell_0)}\varphi\right)(a,\ell) = \\
&\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[r\left(z - K_1^2\eta_\nu^2(r)\ell\right) - \frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r) \\
&\qquad \times \varphi\left(a_0 + a - \log(K_1\eta_\nu(r)), e^{-2(a-\log(K_1\eta_\nu(r)))}\ell_0 + z\right) dr dz,
\end{aligned}$$

which is an existing integral. Notice that

$$e^{-2(a-\log(K_1\eta_\nu(r)))} = K_1^2\eta_\nu^2(r)e^{-2a}.$$

[39] Notice that the functions $\cosh(x)$ and $|\sinh(x)|$ have the same asymptotic behavior.
[40] See (3.57).

Then, performing a change of variables following $z \mapsto z - K_1^2\eta_\nu^2(r)e^{-2a}\ell_0$, we get

$$\begin{aligned}&\underline{T}_{\nu,\vec{K}}\left(L^{\star}_{(a_0,\ell_0)}\varphi\right)(a,\ell) = \\ &\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left[r\left(z-K_1^2\eta_\nu^2(r)(e^{-2a}\ell_0+\ell)\right)-\frac{K_2}{2i\nu K_1}\operatorname{arcsinh}(2i\nu r)\right]}\tilde{\eta}_\nu(r) \\ &\qquad\qquad \times\varphi\left(a_0+a-\log(K_1\eta_\nu(r)),z\right)drdz,\end{aligned}$$

which corresponds exactly to $\underline{T}_{\nu,\vec{K}}\varphi\left(L_{(a_0,\ell_0)}(a,\ell)\right)$. ■

We can now enunciate one of the most important result of the present work.

Theorem 3.4.4. Let K_1 and K_2 be two possibly ν-depending real numbers satisfying the limit condition $\lim_{\nu\to 0}(K_1,K_2)=(1,0)$, with K_1 strictly positive and $\nu\in i\mathbb{R}_0$ a non-zero deformation parameter.

(i) For[41] $\varphi,\psi\in\mathcal{E}_{\nu,\vec{K}}$ (with $\vec{K}:=(K_1,K_2)$), the formula

$$\varphi\sharp_{\nu,\vec{K}}\psi := \underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\varphi\underline{\star}_\nu^{BG}\underline{\tau}_{\nu,\vec{K}}\psi\right)$$

defines an associative product on $\mathcal{E}_{\nu,\vec{K}}$. The algebra $\left(\mathcal{E}_{\nu,\vec{K}},\sharp_{\nu,\vec{K}}\right)$ is then endowed with the Fréchet algebra structure transported under $\underline{T}_{\nu,\vec{K}}$ from $\mathcal{S}^{BG}(\mathbb{S})$[42].

(ii) The star-product $\sharp_{\nu,\vec{K}}$ is $\mathfrak{g}_{RC}$-invariant and, up to a redefinition of the deformation parameter, it coincides with $\star_\nu^{RC}$.

(iii) At the level of functions belonging to[43] $\underline{\mathcal{S}}^{K_1^2/|i\nu|}$, the star-product admits the following integral representation[44]:

$$\begin{aligned}&\varphi\sharp_{\nu,\vec{K}}\psi = \\ &\frac{K_1^4}{(2\pi i\nu)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)}e^{\frac{K_1^2}{\nu}[\tanh(a_2)\ell_1-\tanh(a_1)\ell_2]} \\ &\qquad\times R^{\star}_{\left(a_1+\log\frac{\cosh(a_1-a_2)}{\cosh(a_2)},\ell_1\right)}\varphi R^{\star}_{\left(a_2+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)},\ell_2\right)}\psi da_1d\ell_1da_2d\ell_2.\end{aligned}$$

[41] Recall that $\mathcal{E}_{\nu,\vec{K}}=\underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$; see (3.70).
[42] See Definition 2.2.44 and semi-norms (2.59).
[43] See Definition 3.3.9 for the notation.
[44] Recall that R denotes the right action of $\mathbb{S}$.

(iv) Moreover, this star-product is invariant under the left action[45] of $\mathbb{S}$. That is, for all $x \in \mathbb{S}$, we have:

$$L_x^\star \left(\varphi \sharp_{\nu,\vec{K}} \psi\right) = L_x^\star \varphi \sharp_{\nu,\vec{K}} L_x^\star \psi.$$

Proof. The item (i) comes from the associativity of $\underline{\star}_\nu^{BG}$ (see chapter 2, section 2.2). The item (ii) comes from the construction of $\sharp_{\nu,\vec{K}}$ itself (see the retract method used in chapter 3, section 2) and Proposition 2.1.18. For the item (iii), we start by choosing σ such that $\sigma < \frac{K_1^2}{|i\nu|}$ and two functions $\varphi, \psi \in \mathcal{S}^\sigma$. Observe that using (2.40) (in which we perform the integrals w.r.t. the ℓ_1, ℓ_2-variables), (2.41) and (2.42), we get[46]:

$$\begin{aligned}
&(2\pi i\nu)^2 \left(\underline{\mathcal{T}}_{\nu,\vec{K}} \varphi \underline{\star}_\nu^{BG} \underline{\mathcal{T}}_{\nu,\vec{K}} \psi\right)(a_0, \ell_0) = \\
&\int \sqrt{\cosh(2a_1 - 2a_0)\cosh(2a_2 - 2a_0)\cosh(2a_1 - 2a_2)} \\
&\times e^{-\frac{1}{2\nu}[\sinh(2a_1-2a_0)\ell_2 - \sinh(2a_2-2a_0)\ell_1 - \sinh(2a_1-2a_2)\ell_0]} \\
&\times \mathcal{F}(\underline{\mathcal{T}}_{\nu,\vec{K}}\varphi)\left(a_1 + a_0, -\frac{\sinh(2a_2)}{2i\nu}\right) \mathcal{F}(\underline{\mathcal{T}}_{\nu,\vec{K}}\psi)\left(a_2 + a_0, \frac{\sinh(2a_1)}{2i\nu}\right) da_1 da_2,
\end{aligned}$$

where $a_0, \ell_0 \in \mathbb{R}$. Remark that for all $a \in \mathbb{R}$, a straightforward computation leads to

$$\eta_\nu\left(\frac{\sinh(2a)}{2i\nu}\right) = \frac{1}{\cosh(a)} \tag{3.72}$$

and

$$\frac{\sinh(2a)}{2i\nu}\eta_\nu^2\left(\frac{\sinh(2a)}{2i\nu}\right) = \frac{1}{i\nu}\tanh(a), \tag{3.73}$$

for η_ν defined by (3.37). We also have

$$\tilde{\eta}_\nu\left(\frac{\sinh(2a)}{2i\nu}\right) = \frac{1}{\cosh(a)\sqrt{\cosh(2a)}}, \tag{3.74}$$

with $\tilde{\eta}_\nu$ defined by (3.56). Thanks to the decomposition (3.69), we thus have

$$\begin{aligned}
&\mathcal{F}(\underline{\mathcal{T}}_{\nu,\vec{K}}\varphi)\left(a_1 + a_0, -\frac{\sinh(2a_2)}{2i\nu}\right) = \\
&\frac{K_1^2}{\cosh(a_2)\sqrt{\cosh(2a_2)}} e^{\frac{K_2}{\nu K_1}a_2} \mathcal{F}\varphi\left(a_1 + a_0 + \log\frac{K_1}{\cosh(a_2)}, -\frac{K_1^2}{i\nu}\tanh(a_2)\right),
\end{aligned}$$

[45] See (2.1).

[46] As usual for long computations, we drop the domain of the integrals and use the same convention as in the rest of the text.

and a very similar identity for ψ. Plugging this into the previous integral expression, we get:

$$\begin{aligned}
&(2\pi i\nu)^2 \left(\underline{\tau}_{\nu,\vec{K}}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}}\psi\right)(a_0,\ell_0) = \\
&K_1^4 \int \frac{\sqrt{\cosh(2a_1-2a_2)}}{\cosh(a_1)\cosh(a_2)} e^{\frac{1}{2\nu}\left[\sinh(2a_1-2a_2)\ell_0 - \frac{K_2}{K_1^2}(2a_1-2a_2)\right]} \\
&\qquad \times \mathcal{F}\varphi\left(a_1+a_0+\log\frac{K_1}{\cosh(a_2)}, -\frac{K_1^2}{i\nu}\tanh(a_2)\right) \\
&\qquad \times \mathcal{F}\psi\left(a_2+a_0+\log\frac{K_1}{\cosh(a_1)}, \frac{K_1^2}{i\nu}\tanh(a_1)\right) da_1 da_2, \qquad (3.75)
\end{aligned}$$

which is an existing integral thanks to the fact that[47] $\varphi,\psi \in \mathcal{S}^\sigma$ (which implies that the integrals w.r.t. the a_1, a_2-variables are performed over a compact set). Now, using the expression (3.57) for the operator $\underline{T}_{\nu,\vec{K}}$, we obtain (for $a,\ell \in \mathbb{R}$):

$$\begin{aligned}
&(2\pi i\nu)^2 \underline{T}_{\nu,\vec{K}} \left(\underline{\tau}_{\nu,\vec{K}}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}}\psi\right)(a,\ell) = \\
&\frac{K_1^4}{2\pi} \int e^{i\ell_0\left[r+\frac{1}{2i\nu}\sinh(2a_1-2a_2)\right]} e^{-K_1^2 r\eta_\nu^2(r)\ell - \frac{K_2}{2\nu K_1}[\operatorname{arcsinh}(2i\nu r)+(2a_1-2a_2)]}\tilde{\eta}_\nu(r) \\
&\qquad \times \frac{\sqrt{\cosh(2a_1-2a_2)}}{\cosh(a_1)\cosh(a_2)} \mathcal{F}\varphi\left(a_1+a-\log(\eta_\nu(r)\cosh(a_2)), -\frac{K_1^2}{i\nu}\tanh(a_2)\right) \\
&\qquad\qquad \times \mathcal{F}\psi\left(a_2+a-\log(\eta_\nu(r)\cosh(a_1)), \frac{K_1^2}{i\nu}\tanh(a_1)\right) da_1 da_2 d\ell_0 dr,
\end{aligned} \qquad (3.76)$$

which is again an existing integral (it can be shown by means of arguments based on oscillatory integrals techniques similar to that already used in this work). Performing first the integral in the ℓ_0-variable, we get the following distributional identity:

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\ell_0\left[r+\frac{1}{2i\nu}\sinh(2a_1-2a_2)\right]} d\ell_0 = \delta_0\left(r+\frac{1}{2i\nu}\sinh(2a_1-2a_2)\right).$$

Plugging this into the above expression (3.76) and performing the integral w.r.t. the r-variable, we finally get (using again (3.72), (3.73) and (3.74)) the following

[47] Recall that the range of the hyperbolic tangent is contained in $]-1,1[$.

simplified expression:

$$\frac{(2\pi i\nu)^2}{K_1^4}\underline{T}_{\nu,\vec{K}}\left(\underline{\tau}_{\nu,\vec{K}}\varphi\underline{\star}_\nu^{BG}\underline{\tau}_{\nu,\vec{K}}\psi\right)(a,\ell) = \int \frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)} e^{\frac{K_1^2}{\nu}[\tanh(a_1-a_2)\ell+\tanh(a_2)\ell_1-\tanh(a_1)\ell_2]} \times \varphi\left(a_1+a+\log\frac{\cosh(a_1-a_2)}{\cosh(a_2)},\ell_1\right) \times \psi\left(a_2+a+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)},\ell_2\right) da_1 d\ell_1 da_2 d\ell_2. \tag{3.77}$$

Performing a change of variables following

$$\ell_1 \mapsto \ell_1 + \ell\exp\left[-2\left(a_1+\log\frac{\cosh(a_1-a_2)}{\cosh(a_2)}\right)\right]$$

and

$$\ell_2 \mapsto \ell_2 + \ell\exp\left[-2\left(a_2+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)}\right)\right],$$

we get the announced expression in the item (iii), thanks to the expression (1.31) of the right action R on $\mathbb{S}$, which in coordinates (1.24) reads

$$R_{(a,\ell)}(a',\ell') = \left(a'+a, \ell+\ell' e^{-2a}\right).$$

Lastly, notice that for $\varphi =: \underline{T}_{\nu,\vec{K}}\varphi_0$ and $\psi =: \underline{T}_{\nu,\vec{K}}\psi_0$ belonging to $\underline{T}_{\nu,\vec{K}}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$, we have

$$\varphi\sharp_{\nu,\vec{K}}\psi := \underline{T}_{\nu,\vec{K}}\left(\varphi_0\underline{\star}_\nu^{BG}\psi_0\right).$$

The item (iv) is then a direct consequence of the left invariance[48] of $\underline{\star}_\nu^{BG}$ (see Theorem 2.2.12, item (ii)) and that of $\underline{T}_{\nu,\vec{K}}$ (see Proposition 3.4.3). ■

Remark 3.4.5. (i) As mentioned in the previous proof, the expression in the item (iii) of Theorem 3.4.4 constitues an existing integral, whenever $\varphi,\psi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$. Indeed, recall first that this integral is equivalent to the RHS of (3.77). Since (in particular) φ and ψ are Schwartz functions, they are both bounded by an integrable function in their second variable (respectively ℓ_1 and ℓ_2). Moreover, there exists a positive constant c such that for all $a_1, a_2 \in \mathbb{R}$,

$$\frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)} \leq \frac{c}{(1+a_1)^2(1+a_2)^2},$$

[48] Or left equivariance.

where the latter is an integrable function.

(ii) Notice also that the RHS of the formula in the item (iii) of the previous theorem does not depend on the parameter K_2. Moreover, the expression of the star-product $\sharp_{\nu,\vec{K}}$ is the same for any choice of the parameter K_1, up to a redefinition of the deformation parameter ν (it suffices to set $\nu_{K_1} := \frac{\nu}{K_1^2}$ as being the new deformation parameter). In what follows, without any loss of generality, we will then focus on the particular star-product $\sharp_{\nu,\vec{K}_0}$, with $\vec{K}_0 := (1,0)$, corresponding to the limit case (3.53).

Proposition 3.4.6. (i) For every $\varphi \in \mathcal{D}(\mathbb{S})$, the function of $\theta \in \mathbb{R}$ given by $\underline{T}_{\frac{\theta}{i},\vec{K}_0}\varphi$ belongs to $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}))$.

(ii) For every ψ_1, ψ_2 belonging to the space $\underline{T}_{\nu,\vec{K}_0}(\mathcal{D}(\mathbb{S}))$, the function of $\theta \in \mathbb{R}$ given by $\psi_1 \sharp_{\frac{\theta}{i},\vec{K}_0} \psi_2$ belongs to $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}))$.

Proof. We use similar arguments to that of the proof of Proposition 3.3.3. Indeed, starting with the item (i), notice that for every $(a,\ell) \in \mathbb{S} \simeq \mathbb{R}^2$, we have:

$$\underline{T}_{\frac{\theta}{i},\vec{K}_0}\varphi(a,\ell) = \int_{\mathbb{R}} e^{-ir\eta^2_{\frac{\theta}{i}}(r)\ell} \tilde{\eta}_{\frac{\theta}{i}}(r)\hat{\varphi}\left(a - \log(\eta_{\frac{\theta}{i}}(r)), r\right) dr.$$

The integrand is smooth in the variables (a,ℓ,θ), as well as all of its partial derivatives. Moreover, we have shown previously that $\tilde{\eta}_{\frac{\theta}{i}}(r)\hat{\varphi}\left(a - \log(\eta_{\frac{\theta}{i}}(r)), r\right)$ is a Schwartz function w.r.t. the r-variable (for any a and for any θ) and that $r\eta^2_{\frac{\theta}{i}}(r)$ is tempered (w.r.t. the r-variable). We deduce that any partial derivative (w.r.t. a, ℓ and θ) of the integrand is bounded by an integrable function. Thus, the function $\left[(x,\theta) \mapsto \underline{T}_{\frac{\theta}{i},\vec{K}}\varphi(x)\right]$ (with $x \in \mathbb{S} \simeq \mathbb{R}^2$) belongs to $C^\infty(\mathbb{S}\times\mathbb{R})$ (by virtue of the Lebesgue dominated convergence). The desired result then follows from the reference [49], Theorem 40.1.

For the item (ii), remark first that Lemma 3.3.11 implies that $\underline{T}_{\nu,\vec{K}_0}(\mathcal{D}(\mathbb{S})) \subset \mathcal{S}^\sigma$, for a certain $0 \leq \sigma < \frac{1}{|\theta|}$. The product $\psi_1 \sharp_{\frac{\theta}{i},\vec{K}_0} \psi_2$ is thus given by the expression in the item (iii) of Theorem 3.4.4. After a change of variables following $\ell_j \mapsto \theta\ell_j$ for $j \in \{1,2\}$, we get

$$\begin{aligned}
&\left(\psi_1 \sharp_{\frac{\theta}{i},\vec{K}_0} \psi_2\right)(a,\ell) = \\
&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)} e^{i(\tanh(a_2)\ell_1 - \tanh(a_1)\ell_2)} \\
&\times R^\star_{\left(a_1 + \log\frac{\cosh(a_1-a_2)}{\cosh(a_2)}, \theta\ell_1\right)}\psi_1(a,\ell) R^\star_{\left(a_2+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)}, \theta\ell_2\right)}\psi_2(a,\ell) da_1 d\ell_1 da_2 d\ell_2.
\end{aligned} \tag{3.78}$$

Again here, the integrand is smooth in the variables (a,ℓ,θ), as well as all of its partial derivatives. Moreover, each of these functions are bounded by an

integrable function. Indeed, setting

$$\mathbf{E} := e^{i(\tanh(a_2)\ell_1 - \tanh(a_1)\ell_2)},$$

remark that for $A_1 := \frac{i}{\tanh'(a_1)}\partial_{a_1}$ (which makes sense since the first derivative[49] of the hyperbolic tangent is nowhere zero) and $A_2 := -\frac{i}{\tanh'(a_2)}\partial_{a_2}$, the operator $\mathbf{A} := \left(\mathrm{Id} + A_1^2\right)\left(\mathrm{Id} + A_2^2\right)$ satisfies

$$\mathbf{AE} = \left(1 + \ell_1^2\right)\left(1 + \ell_2^2\right)\mathbf{E}.$$

Recalling that the integrand in (3.78) is compactly supported w.r.t. the a_1, a_2-variables (see (3.75) in the proof of Theorem 3.4.4), and setting

$$F(x, x_1, x_2, \theta) := \frac{R^\star_{\left(a_1 + \log\frac{\cosh(a_1 - a_2)}{\cosh(a_2)}, \theta\ell_1\right)}\psi_1(a,\ell)R^\star_{\left(a_2 + \log\frac{\cosh(a_1 - a_2)}{\cosh(a_1)}, \theta\ell_2\right)}\psi_2(a,\ell)}{\cosh(a_1)\cosh(a_2)\cosh(a_1 - a_2)}$$

with $x := (a, \ell)$ and $x_j := (a_j, \ell_j)$ for $j \in \{1, 2\}$, an integration by parts argument leads to the following alternative expression for $\sharp_{\frac{\theta}{i}, \vec{K}_0}$:

$$\left(\psi_1 \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi_2\right)(a,\ell) =$$
$$\frac{(-1)^n}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\frac{1}{\left(1+\ell_1^2\right)\left(1+\ell_2^2\right)}\right)^n \mathbf{E}\mathbf{A}^n F(x, x_1, x_2, \theta)dx_1 dx_2,$$

valid for any $n \in \mathbb{N}$. It follows that the function $\left[(x,\theta) \mapsto \left(\psi_1 \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi_2\right)(x)\right]$ defines an element of $C^\infty(\mathbb{S}\times\mathbb{R})$. Theorem 40.1 in [49] then implies that the function $\left[\theta \mapsto \psi_1 \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi_2\right]$ belongs to $C^\infty\left(\mathbb{R}, C^\infty(\mathbb{S})\right)$. ■

The first item of the previous proposition then allows to consider the Taylor series at 0 of the function $\underline{T}_{\frac{\theta}{i}, \vec{K}_0}\varphi$ (with $\varphi \in \mathcal{D}(\mathbb{S})$). This asymptotic expansion then shows that $\underline{T}_{\frac{\theta}{i}, \vec{K}_0}$ is a formal equivalence, in the sense of Definition 1.2.18 (with $\nu := \frac{\theta}{i}$).

We can now establish the precise link (suggested at several times previously in this work) between the integral representation of $\sharp_{\nu, \vec{K}_0}$ and the formal version of the Rankin-Cohen star-product[50].

Theorem 3.4.7. For every functions $\varphi, \psi \in \underline{T}_{\nu, \vec{K}_0}\left(\mathcal{D}(\mathbb{S})\right)$ and for every $x \in \mathbb{S}$, the function $\left[\mathbb{R} \to \mathbb{C} : \theta \mapsto \left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x)\right]$ is smooth. Its Taylor series at 0

[49] We denote this derivative by $\tanh'$ to slightly lighten the notation.
[50] See Definition 1.2.25.

defines an associative formal star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ which coincides with $\star_\nu^{RC}$, for $\nu := \frac{\theta}{i}$.

Proof. The smoothness of the map $\left[\mathbb{R} \to \mathbb{C} : \theta \mapsto \left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x)\right]$ comes from Proposition 3.4.6, item (ii). Now, recall that

$$\left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x) = \underline{T}_{\frac{\theta}{i}, \vec{K}_0}\left(\underline{\tau}_{\frac{\theta}{i}, \vec{K}_0} \varphi \underline{\star}_{\frac{\theta}{i}}^{BG} \underline{\tau}_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x).$$

By virtue of the associativity of (the formal version of) $\underline{\star}_{\frac{\theta}{i}}^{BG}$ (see for instance [16], Theorem 4.9), the Taylor series at 0 of $\left[\mathbb{R} \to \mathbb{C} : \theta \mapsto \left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x)\right]$ then yields an associative formal star-product, which, by construction of $\underline{T}_{\frac{\theta}{i}, \vec{K}_0}$ and $\underline{\tau}_{\frac{\theta}{i}, \vec{K}_0}$, is (locally) $\mathfrak{g}_{RC}$-invariant. We then deduce from Proposition 2.1.18 that the formal version of $\sharp_{\nu, \vec{K}_0}$ coincides with $\star_\nu^{RC}$. ■

Remark 3.4.8. Of course, we can also compute explicitly the Taylor series at 0 of the function $\left[\mathbb{R} \to \mathbb{C} : \theta \mapsto \left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(x)\right]$ by means of a stationary phase method. Let us first illustrate it for the 0-order term. Using (3.78) and setting $x_1 := (a_1, \ell_1)$, $x_2 := (a_2, \ell_2)$, we observe that[51]

$$\left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(e)\Big|_{\theta=0} =$$
$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1 - a_2)} e^{i(\tanh(a_2)\ell_1 - \tanh(a_1)\ell_2)}$$
$$\times R^\star_{\left(a_1 + \log \frac{\cosh(a_1 - a_2)}{\cosh(a_2)}, 0\right)} \varphi(0,0) R^\star_{\left(a_2 + \log \frac{\cosh(a_1 - a_2)}{\cosh(a_1)}, 0\right)} \psi(0,0) da_1 da_2.$$

Since for all $a \in \mathbb{R}$,

$$\delta_0(\tanh(a)) = \delta_0(a),$$

performing the integrals w.r.t. the variables ℓ_1 and ℓ_2 yields

$$\left(\varphi \sharp_{\frac{\theta}{i}, \vec{K}_0} \psi\right)(e)\Big|_{\theta=0} = (\varphi\psi)(e).$$

The left invariance[52] of $\sharp_{\frac{\theta}{i}, \vec{K}_0}$ then implies that the above formula again holds after replacing e by x, for any $x \in \mathbb{S}$. For the 1-order term of the Taylor series,

[51] Recall that e denotes the neutral element of $\mathbb{S}$.
[52] See Theorem 3.4.4, item (iv).

notice that

$$\frac{\partial}{\partial\theta}\left(\varphi\sharp_{\frac{\theta}{i},\vec{K}_0}\psi\right)(e)\Big|_{\theta=0} =$$
$$\frac{1}{(2\pi)^2}\int\frac{\partial}{\partial\theta}\left[\frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)}e^{i(\tanh(a_2)\ell_1-\tanh(a_1)\ell_2)}\right.$$
$$\left.\times R^\star_{\left(a_1+\log\frac{\cosh(a_1-a_2)}{\cosh(a_2)},\theta\ell_1\right)}\varphi(0,0)R^\star_{\left(a_2+\log\frac{\cosh(a_1-a_2)}{\cosh(a_1)},\theta\ell_2\right)}\psi(0,0)\right]\Big|_{\theta=0}da_1da_2.$$

Using the identities

$$\frac{i}{\tanh'(a_1)}\partial_{a_1}e^{i(\tanh(a_2)\ell_1-\tanh(a_1)\ell_2)} = \ell_2 e^{i(\tanh(a_2)\ell_1-\tanh(a_1)\ell_2)}$$

and

$$-\frac{i}{\tanh'(a_2)}\partial_{a_2}e^{i(\tanh(a_2)\ell_1-\tanh(a_1)\ell_2)} = \ell_1 e^{i(\tanh(a_2)\ell_1-\tanh(a_1)\ell_2)},$$

we deduce (after a straightforward computation) that

$$\frac{\partial}{\partial\theta}\left(\varphi\sharp_{\frac{\theta}{i},\vec{K}_0}\psi\right)(e)\Big|_{\theta=0} = \{\varphi,\psi\}(e),$$

where $\{\cdot,\cdot\}$ is the Poisson bracket associated with the symplectic form $\omega^{\mathbb{S}} = da\wedge d\ell$ (within coordinates (1.24)). The left invariance of $\sharp_{\frac{\theta}{i},\vec{K}_0}$ implies that the above formula still holds after replacing e by x, for any $x\in\mathbb{S}$. The computations for the other terms of the Taylor series are similar.

The star-product $\sharp_{\nu,\vec{K}_0}$ is then the non-formal equivalent (that is, the integral representation) of the formal star-product $\star_\nu^{RC}$ introduced in chapter 1. This is what motivates the following definition.

Definition 3.4.9. The star-product $\sharp_{\nu,\vec{K}_0}$ (cf. the item (iii) of Theorem 3.4.4 with $\vec{K} := \vec{K}_0$) will be hereafter called the *non-formal Rankin-Cohen star-product* (or *non-formal RC star-product* in short). By a slight abuse of notation, it will also be denoted by $\star_\nu^{RC}$. Moreover, the associated space $\mathcal{E}_{\nu,\vec{K}_0}$ will be denoted as $\mathcal{E}_\nu^{RC}$.

Unless the constant 1 does not belong to $\mathcal{E}_\nu^{RC}$ (with 1 identified with the function $[\mathbb{S}\to\mathbb{C}: x\mapsto 1]$), we will now show that for any $\varphi\in\mathcal{E}_\nu^{RC}$, both expressions $\varphi\star_\nu^{RC}1$ and $1\star_\nu^{RC}\varphi$ make sense (where $\star_\nu^{RC}$ stands for the non-formal RC star-product), and satisfy the following identities:

$$\varphi\star_\nu^{RC}1 = 1\star_\nu^{RC}\varphi = \varphi.$$

This means that the constant function 1 plays the role of a neutral element for the (non-formal version of the) star-product $\star_\nu^{RC}$.

Proposition 3.4.10. Let φ be a function belonging to the space $\mathcal{E}_\nu^{RC}$. Identifying an element $c \in \mathbb{C}$ with the constant function $[\mathbb{S} \to \mathbb{C} : x \mapsto c]$, both elements $\varphi \star_\nu^{RC} c$ and $c \star_\nu^{RC} \varphi$ belong to $\mathcal{E}_\nu^{RC}$. Moreover, we have:

$$\varphi \star_\nu^{RC} c = c \star_\nu^{RC} \varphi = c\varphi \in \mathcal{E}_\nu^{RC}.$$

Proof. We start with the element $\varphi \star_\nu^{RC} c$, where $\varphi \in \mathcal{E}_\nu^{RC}$ and $c \in \mathbb{C}$. First, recall that the operator $\underline{\tau}_{\nu,\vec{K}_0}$ decomposes as

$$\underline{\tau}_{\nu,\vec{K}_0} = \mathcal{F}^{-1} \circ \mathcal{M}_{\nu,\vec{K}_0} \circ \gamma^\star_{\nu,\vec{K}_0} \circ \mathcal{F},$$

by virtue of the decomposition (3.69) (see also Remark 3.3.7 for the notations). Thanks to the fact that $\mathcal{F}$ (along with its inverse) extends to the space $\mathcal{S}'(\mathbb{S})$ of tempered distributions on $\mathbb{S} \simeq \mathbb{R}^2$ (by means of coordinates (1.24)), we get the following distributional identity (where c is viewed as a constant function on $\mathbb{S}$):

$$\mathcal{F}(c)(a,\xi) = c \int_{\mathbb{R}} e^{-i\xi\ell} d\ell = 2\pi c \delta_0(\xi),$$

for all $(a,\xi) \in \tilde{\mathbb{S}} \simeq \mathbb{R}^2$. Now, notice that the function $\left[\mathbb{R} \to \mathbb{R} : r \mapsto r\eta_\nu^2(r)\right]$ admits a unique root at $r = 0$. Moreover, since $\frac{d}{dr}(r\eta_\nu^2(r))\big|_{r=0} = 1$, $\tilde{\eta}_\nu(0) = 1$ and $\vec{K}_0 = (1,0)$, we get:

$$\mathcal{M}_{\nu,\vec{K}_0}\left(\gamma^\star_{\nu,\vec{K}_0}\mathcal{F}(c)\right)(a,\xi) = 2\pi c\delta_0(\xi).$$

Applying an inverse partial Fourier transform $\mathcal{F}^{-1}$ then leads to:

$$\underline{\tau}_{\nu,\vec{K}_0} c = c,$$

for all $c \in \mathbb{C}$. In a similar way, we can show that $\underline{T}_{\nu,\vec{K}_0} c = c$, for any $c \in \mathbb{C}$. Having said that, recall now the identity

$$\varphi \star_\nu^{RC} c := \underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0} c\right), \tag{3.79}$$

where $\star_\nu^{RC} = \sharp_{\nu,\vec{K}_0}$. Since $\varphi \in \mathcal{E}_\nu^{RC} := \underline{T}_{\nu,\vec{K}_0}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$, there exists a function $\varphi_0 \in \mathcal{S}^{BG}(\mathbb{S})$ such that $\varphi = \underline{T}_{\nu,\vec{K}_0}\varphi_0$. Using Proposition 3.3.8 and Lemma 2.2.47, we thus have $\underline{\tau}_{\nu,\vec{K}_0}\varphi = \varphi_0 \in \mathcal{S}^{BG}(\mathbb{S})$. Since any constant $c = \underline{\tau}_{\nu,\vec{K}_0} c$ belongs to the space $\mathcal{B}(\mathbb{S})$ (see Definition 2.2.25), Propositions 2.2.42, 2.2.43

and 2.2.45 imply that

$$\underline{\tau}_{\nu,\vec{K}_0}\varphi\underline{\star}_\nu^{BG}\underline{\tau}_{\nu,\vec{K}_0}c = c\underline{\tau}_{\nu,\vec{K}_0}\varphi = c\varphi_0 \in \mathcal{S}^{BG}(\mathbb{S}).$$

In turn, the linearity of $\underline{T}_{\nu,\vec{K}_0}$ implies that

$$\underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\varphi\underline{\star}_\nu^{BG}\underline{\tau}_{\nu,\vec{K}_0}c\right) = c\varphi \in \mathcal{E}_\nu^{RC},$$

which, through (3.79), yields the identity $\varphi \star_\nu^{RC} c = c\varphi$. The proof for the element $c \star_\nu^{RC} \varphi$ is entirely similar. ■

3.4.2 Topological algebras

In this subsection, we introduce a pre-Hilbert structure on the space $\mathcal{E}_\nu^{RC}$. For this, we come back to the star-product $\underline{\star}_\nu^{BG}$, i.e. the strongly closed version of the Bieliavsky-Gayral deformation quantization. Recall[53] that the pre-Hilbert structure associated with the space $\mathcal{S}^{BG}(\mathbb{S})$ is $\langle\cdot,\cdot\rangle_{\mathcal{S}(\mathbb{S})}$, that is, the restriction to the Schwartz space $\mathcal{S}(\mathbb{S})$ of the usual inner product on $L^2(\mathbb{S})$ (see (2.30)). In the sake of simplicity, we start by focusing on $\underline{\mathcal{S}}^{1/|i\nu|}$, which is a subset of $\mathcal{E}_\nu^{RC}$ (see (3.71)). Using the intertwiner $\underline{T}_{\nu,\vec{K}_0}$ previously computed, a natural way to define an inner product on this subspace then consists in setting

$$\langle\varphi,\psi\rangle_{\underline{\mathcal{S}}^{1/|i\nu|}} := \langle\underline{\tau}_{\nu,\vec{K}_0}\varphi,\underline{\tau}_{\nu,\vec{K}_0}\psi\rangle_{\mathcal{S}(\mathbb{S})}, \tag{3.80}$$

for $\varphi,\psi \in \underline{\mathcal{S}}^{1/|i\nu|}$. Note that the above RHS makes sense. Indeed, since φ,ψ belong (in particular) to $\mathcal{E}_\nu^{RC}$, there exists $\varphi_0,\psi_0 \in \mathcal{S}^{BG}(\mathbb{S})$ such that $\varphi = \underline{T}_{\nu,\vec{K}_0}\varphi_0$ and $\psi = \underline{T}_{\nu,\vec{K}_0}\psi_0$. Hence, Proposition 3.3.8 implies that $\underline{\tau}_{\nu,\vec{K}_0}\varphi = \varphi_0$ and $\underline{\tau}_{\nu,\vec{K}_0}\psi = \psi_0$, with $\varphi_0,\psi_0 \in \mathcal{S}(\mathbb{S})$ (see Lemma 2.2.47). Moreover, the linearity of $\underline{\tau}_{\nu,\vec{K}_0}$ along with Proposition 3.3.12 imply that $\langle\cdot,\cdot\rangle_{\underline{\mathcal{S}}^{1/|i\nu|}}$ is indeed an inner product.

Proposition 3.4.11. For $\varphi,\psi \in \underline{\mathcal{S}}^{1/|i\nu|}$, the following identity holds:

$$\langle\underline{\tau}_{\nu,\vec{K}_0}\varphi,\underline{\tau}_{\nu,\vec{K}_0}\psi\rangle_{\mathcal{S}(\mathbb{S})} = \langle\varphi,\psi\rangle_{\mathcal{S}(\mathbb{S})}.$$

Proof. Let $\varphi,\psi \in \underline{\mathcal{S}}^{1/|i\nu|}$. Within the notations of Remark 3.3.7, notice first that

$$\langle\mathcal{M}_{\nu,\vec{K}_0}\left(\gamma^\star_{\nu,\vec{K}_0}\mathcal{F}\varphi\right),\mathcal{M}_{\nu,\vec{K}_0}\left(\gamma^\star_{\nu,\vec{K}_0}\mathcal{F}\psi\right)\rangle_{\mathcal{S}(\mathbb{S})} = 2\pi\langle\varphi,\psi\rangle_{\mathcal{S}(\mathbb{S})}.$$

[53] See chapter 2.

Indeed, a straightforward computation shows that the LHS is equal to

$$\int_{\mathbb{R}^2} \tilde{\eta}_\nu^2(r)\mathcal{F}\varphi(a+\log(\eta_\nu(r)), r\eta_\nu^2(r))\overline{\mathcal{F}\psi}(a+\log(\eta_\nu(r)), r\eta_\nu^2(r))dadr = \int_{\mathbb{R}^2} \tilde{\eta}_\nu^2(r)\mathcal{F}\varphi(a, r\eta_\nu^2(r))\overline{\mathcal{F}\psi}(a, r\eta_\nu^2(r))dadr.$$

Note that the above integrals exist; indeed, since $\varphi, \psi \in \underline{\mathcal{S}}^{1/|i\nu|}$, both $\mathcal{F}\varphi$ and $\mathcal{F}\psi$ are in particular Schwartz functions on $\mathbb{S}$ and the integral in the r-variable is only computed over a certain compact set of $\mathbb{R}$.
A change of variables following $r\eta_\nu^2(r) \mapsto r'$ then leads to

$$\int_{\mathbb{R}^2} \tilde{\eta}_\nu^2(r)\mathcal{F}\varphi(a, r\eta_\nu^2(r))\overline{\mathcal{F}\psi}(a, r\eta_\nu^2(r))dadr = \int_{-\frac{1}{|i\nu|}}^{\frac{1}{|i\nu|}} \int_{-\infty}^{+\infty} \mathcal{F}\varphi(a, r')\overline{\mathcal{F}\psi}(a, r')dadr'.$$

Thanks to the formula (2.35) and the hypothesis made on φ, ψ, we see that the above RHS coincides with

$$\int_{\mathbb{R}^2} \mathcal{F}\varphi(a, r')\overline{\mathcal{F}\psi}(a, r')dadr' = \langle \mathcal{F}\varphi, \mathcal{F}\psi\rangle_{\mathcal{S}(\mathbb{S})} = 2\pi\langle\varphi, \psi\rangle_{\mathcal{S}(\mathbb{S})}.$$

The proposition then follows from the decomposition (3.69) and the formula (2.36). ■

This means that the inner product $\langle\cdot,\cdot\rangle_{\underline{\mathcal{S}}^{1/|i\nu|}}$ on $\underline{\mathcal{S}}^{1/|i\nu|}$ given by (3.80) coincides with the restriction of $\langle\cdot,\cdot\rangle_{\mathcal{S}(\mathbb{S})}$ to $\underline{\mathcal{S}}^{1/|i\nu|}$.

Definition 3.4.12. Endowed with the (restriction of the) inner product $\langle\cdot,\cdot\rangle_{\mathcal{S}(\mathbb{S})}$, the space $\underline{\mathcal{S}}^{1/|i\nu|}$ becomes a pre-Hilbert space whose Hilbert completion is denoted as $\mathcal{H}_\nu^{RC}$.

Remark 3.4.13. As already mentioned in Remark 3.3.10, the space $\underline{\mathcal{S}}^{1/|i\nu|}$ does not contain the space $\mathcal{D}(\mathbb{S})$ of smooth compactly supported functions. Moreover, since $\underline{\mathcal{S}}_{1/|i\nu|} = \mathcal{F}\left(\underline{\mathcal{S}}^{1/|i\nu|}\right)$ also does not contain the space $\mathcal{D}(\mathbb{S})$, the Hilbert completion $\mathcal{H}_\nu^{RC}$ does not coincide with the space $L^2(\mathbb{S})$.

Proposition 3.4.14. The linear operator $\underline{T}_{\nu,\vec{K}_0} : \mathcal{D}(\mathbb{S}) \to \mathcal{H}_\nu^{RC}$ extends as a continuous operator from $L^2(\mathbb{S})$ to $\mathcal{H}_\nu^{RC}$.

Proof. Let $\varphi, \psi \in \mathcal{D}(\mathbb{S})$. Using (3.71), we know that both $\underline{T}_{\nu,\vec{K}_0}\varphi$ and $\underline{T}_{\nu,\vec{K}_0}\psi$ belong to $\underline{\mathcal{S}}^{1/|i\nu|}$. Hence, Propositions 3.3.8 and 3.4.11 imply that

$$\begin{aligned}\langle \underline{T}_{\nu,\vec{K}_0}\varphi, \underline{T}_{\nu,\vec{K}_0}\psi\rangle_{\underline{\mathcal{S}}^{1/|i\nu|}} &= \langle \underline{T}_{\nu,\vec{K}_0}\varphi, \underline{T}_{\nu,\vec{K}_0}\psi\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \langle \underline{\tau}_{\nu,\vec{K}_0}\left(\underline{T}_{\nu,\vec{K}_0}\varphi\right), \underline{\tau}_{\nu,\vec{K}_0}\left(\underline{T}_{\nu,\vec{K}_0}\psi\right)\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \langle \varphi, \psi\rangle_{\mathcal{S}(\mathbb{S})}.\end{aligned}$$

The operator $\underline{T}_{\nu,\vec{K}_0}$ is then continuous from $\mathcal{D}(\mathbb{S})$ to $\mathcal{H}^{RC}_\nu$, w.r.t. the $L^2(\mathbb{S})$-topology. Moreover, it satisfies $\left\|\underline{T}_{\nu,\vec{K}_0}\right\| = 1$ (where $\|\cdot\|$ denotes here the operator norm). Since $\mathcal{D}(\mathbb{S})$ is a dense subset of $L^2(\mathbb{S})$, Proposition 3.2.3 in [55] then implies that the continuous operator $\underline{T}_{\nu,\vec{K}_0}$ admits a unique continuous extension to $L^2(\mathbb{S})$ that we denote again by $\underline{T}_{\nu,\vec{K}_0}$, with an operator norm equal to 1. ■

Corollary 3.4.15. The inclusions[54]

$$\underline{T}_{\nu,\vec{K}_0}\left(\mathcal{D}(\mathbb{S})\right) \subset \underline{\mathcal{S}}^{1/|i\nu|} \subset \mathcal{E}^{RC}_\nu \subset \mathcal{H}^{RC}_\nu$$

are dense w.r.t. the $L^2(\mathbb{S})$-topology.

Proof. Given that $\mathcal{D}(\mathbb{S})$ is dense in $\mathcal{S}(\mathbb{S})$ and $\mathcal{D}(\mathbb{S}) \subset \mathcal{S}^{BG}(\mathbb{S}) \subset \mathcal{S}(\mathbb{S})$, the space $\mathcal{D}(\mathbb{S})$ is dense in $\mathcal{S}^{BG}(\mathbb{S})$ (w.r.t. the $L^2(\mathbb{S})$-topology). Hence, the continuity of $\underline{T}_{\nu,\vec{K}_0}$ obtained in Proposition 3.4.14 implies that $\underline{T}_{\nu,\vec{K}_0}\left(\mathcal{D}(\mathbb{S})\right)$ is dense in $\mathcal{E}^{RC}_\nu := \underline{T}_{\nu,\vec{K}_0}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$. This shows that the two first inclusions of the corollary are dense.
Now, for $\varphi \in \mathcal{E}^{RC}_\nu$, we know that there exists $\varphi_0 \in \mathcal{S}^{BG}(\mathbb{S})$ such that $\varphi = \underline{T}_{\nu,\vec{K}_0}\varphi_0$. Then, there exists a sequence $\{\varphi_n\}_n$, $\varphi_n \in \mathcal{D}(\mathbb{S})$, satisfying $\lim_n \varphi_n = \varphi_0$. Hence, the continuity of $\underline{T}_{\nu,\vec{K}_0}$ yields:

$$\varphi = \underline{T}_{\nu,\vec{K}_0}\varphi_0 = \underline{T}_{\nu,\vec{K}_0}\left(\lim_n \varphi_n\right) = \lim_n \underline{T}_{\nu,\vec{K}_0}\varphi_n,$$

where, for each n, $\underline{T}_{\nu,\vec{K}_0}\varphi_n \in \underline{\mathcal{S}}^{1/|i\nu|}$. That implies $\mathcal{E}^{RC}_\nu \subset \mathcal{H}^{RC}_\nu$. Since by definition $\underline{\mathcal{S}}^{1/|i\nu|}$ is dense in $\mathcal{H}^{RC}_\nu$, so is $\mathcal{E}^{RC}_\nu$. ■

Remark 3.4.16. The (extended) operator $\underline{T}_{\nu,\vec{K}_0} : L^2(\mathbb{S}) \to \mathcal{H}^{RC}_\nu$ is invertible. Indeed, by Proposition 3.3.12, its restriction $\underline{T}_{\nu,\vec{K}_0} : \mathcal{D}(\mathbb{S}) \to \underline{\mathcal{S}}^{1/|i\nu|}$ (where the latter is a dense subset of $\mathcal{H}^{RC}_\nu$) admits $\underline{\tau}_{\nu,\vec{K}_0}$ as inverse operator. Since its operator norm is equal to 1 (see the proof of Proposition 3.4.14), $\underline{T}_{\nu,\vec{K}_0}$ defines a linear isometry. Its (extended) inverse will be again denoted by $\underline{\tau}_{\nu,\vec{K}_0}$. Note however that $\underline{T}_{\nu,\vec{K}_0} : L^2(\mathbb{S}) \to \mathcal{H}^{RC}_\nu$ is a nonunitary operator.

[54] Recall that the two first inclusions of the corollary hold by (3.71).

Now, recall that for the BG deformation, $\left(L^2(\mathbb{S}), \underline{\star}_\nu^{BG}\right)$ constitutes an associative Hilbert algebra, as it has already been mentioned in Proposition 2.2.14. Using the operator $\underline{T}_{\nu,\vec{K}_0}$ connecting the BG and RC deformations, we can then extend the star-product $\star_\nu^{RC}$ to the Hilbert completion $\mathcal{H}_\nu^{RC}$.

Proposition 3.4.17. The algebra structure $\star_\nu^{RC}$ on $\mathcal{E}_\nu^{RC}$ extends to $\mathcal{H}_\nu^{RC}$. Endowed with the extended product again denoted by $\star_\nu^{RC}$, the space $\mathcal{H}_\nu^{RC}$ becomes an associative topological algebra.

Proof. Recall once again that for $\varphi, \psi \in \mathcal{E}_\nu^{RC}$,

$$\varphi \star_\nu^{RC} \psi = \underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi\right).$$

Under the (extended) operator $\underline{T}_{\nu,\vec{K}_0} : L^2(\mathbb{S}) \to \mathcal{H}_\nu^{RC}$ coming from Proposition 3.4.14 and the (extended) product $\underline{\star}_\nu^{BG}$ coming from Proposition 2.2.14, the RHS of the above identity implies that the star-product $\star_\nu^{RC}$ extends to the completion $\mathcal{H}_\nu^{RC}$ as an associative product. ■

As in the case of $\left(L^2(\mathbb{S}), \underline{\star}_\nu^{BG}\right)$, we can also show that the complex conjugation is a *-involution for $\left(\mathcal{H}_\nu^{RC}, \star_\nu^{RC}\right)$. Before stating this result, we start by showing the following lemma.

Lemma 3.4.18. For every $\varphi \in L^2(\mathbb{S})$, we have $\overline{\underline{T}_{\nu,\vec{K}_0}\varphi} = \underline{T}_{\nu,\vec{K}_0}\bar{\varphi}$.

Proof. By continuity of the complex conjugation (w.r.t. the $L^2(\mathbb{S})$-topology), it is sufficient to show the identity for $\varphi \in \mathcal{D}(\mathbb{S})$. For such a function and thanks to the formula (3.57) in the particular case $\vec{K} := \vec{K}_0 = (1, 0)$, we have:

$$\overline{\underline{T}_{\nu,\vec{K}_0}\varphi}(a_0, \ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ir\left(z - \eta_\nu^2(r)\ell_0\right)}\tilde{\eta}_\nu(r)\bar{\varphi}\left(a_0 - \log(K_1\eta_\nu(r)), z\right) dr dz.$$

Using the fact that the real-valued functions η_ν and $\tilde{\eta}_\nu$ are both even functions, a change of variables following $-r \mapsto r$ leads to

$$\overline{\underline{T}_{\nu,\vec{K}_0}\varphi}(a_0, \ell_0) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ir\left(z - \eta_\nu^2(r)\ell_0\right)}\tilde{\eta}_\nu(r)\bar{\varphi}\left(a_0 - \log(K_1\eta_\nu(r)), z\right) dr dz,$$

which is exactly equal to $\underline{T}_{\nu,\vec{K}_0}\bar{\varphi}(a_0, \ell_0)$. ■

We then obtain the following result.

Proposition 3.4.19. The triple $\left(\mathcal{H}_\nu^{RC}, \star_\nu^{RC}, \bar{\cdot}\right)$ is a *-algebra.

Proof. It suffices to show that the complex conjugation is an involution for $\star_\nu^{RC}$ (the rest follows from Definition 3.4.12 and Proposition 3.4.17). At the level of

functions belonging to $\mathcal{E}_\nu^{RC}$, the previous lemma implies that

$$\overline{\varphi \star_\nu^{RC} \psi} = \overline{\underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi\right)} = \underline{T}_{\nu,\vec{K}_0}\left(\overline{\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi}\right).$$

Since $\varphi \in \mathcal{E}_\nu^{RC}$, there exists $\varphi_0 \in \mathcal{S}^{BG}(\mathbb{S})$ such that $\varphi = \underline{T}_{\nu,\vec{K}_0}\varphi_0$. By virtue of Proposition 3.3.8 and Lemma 3.4.18, we thus have:

$$\overline{\underline{\tau}_{\nu,\vec{K}_0}\varphi} = \overline{\varphi_0} = \underline{\tau}_{\nu,\vec{K}_0}\left(\underline{T}_{\nu,\vec{K}_0}\overline{\varphi_0}\right) = \underline{\tau}_{\nu,\vec{K}_0}\left(\overline{\underline{T}_{\nu,\vec{K}_0}\varphi_0}\right) = \underline{\tau}_{\nu,\vec{K}_0}\overline{\varphi}. \tag{3.81}$$

Moreover, thanks to Proposition 2.2.16, we know that the complex conjugation is an involution for $\underline{\star}_\nu^{BG}$. We then get the following identity:

$$\underline{T}_{\nu,\vec{K}_0}\left(\overline{\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi}\right) = \underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\overline{\psi} \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\overline{\varphi}\right) = \overline{\psi} \star_\nu^{RC} \overline{\varphi},$$

and we thus have shown that

$$\overline{\varphi \star_\nu^{RC} \psi} = \overline{\psi} \star_\nu^{RC} \overline{\varphi}.$$

The proposition follows from the extension of this identity to (the functions belonging to) the space $\mathcal{H}_\nu^{RC}$. ■

Trace form for RC

The goal of this short subsection is to show that the star-product $\star_\nu^{RC}$ is strongly closed. We start with the following fact.

Lemma 3.4.20. On the space $\mathcal{S}^{BG}(\mathbb{S})$, the following identity holds:

$$\int_{\mathbb{S}} \circ\, \underline{T}_{\nu,\vec{K}_0} = \int_{\mathbb{S}} .$$

Proof. Let $\varphi \in \mathcal{S}^{BG}(\mathbb{S})$. By Proposition 3.3.3, we know that $\underline{T}_{\nu,\vec{K}_0}\varphi$ is a Schwartz function and thus belongs to the space $L^1(\mathbb{S}) \simeq L^1(\mathbb{R}^2)$. Explicitly, using the expression (3.62) in the particular case $\vec{K} = \vec{K}_0$, we get

$$\int_{\mathbb{R}^2} \underline{T}_{\nu,\vec{K}_0}\varphi(a,\ell)\,da\,d\ell = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{ir\eta_\nu^2(r)\ell}\tilde{\eta}_\nu(r)\hat{\varphi}\left(a - \log(\eta_\nu(r)), r\right)dr\right)da\,d\ell,$$

where $\hat{\varphi} := \mathcal{F}^{-1}\varphi$. Using oscillatory integral techniques similar to that in the proof of Proposition 3.3.3, we can show that the Fubini theorem applies to the RHS of the above identity (see for instance the expression (3.63)). After a change of variables following $a - \log(K_1\eta_\nu(r)) \mapsto a$, followed by the integral

w.r.t. the ℓ-variable, the RHS (of the above identity) becomes

$$2\pi \int_{\mathbb{R}} \hat{\varphi}(a,0)da = \int_{\mathbb{R}^2} \varphi(a,\ell)dad\ell,$$

where we have used the (distributional identity) $\delta_0(r\eta_\nu^2(r)) = \delta_0(r)$. ■

We can now show the traciality of $\star_\nu^{RC}$.

Proposition 3.4.21. The star-product $\star_\nu^{RC}$ is strongly closed. That is, for $\varphi, \psi \in \mathcal{E}_\nu^{RC}$, the following identity holds:

$$\int_{\mathbb{S}} \left(\varphi \star_\nu^{RC} \psi\right)(x) d_{\mathbb{S}}(x) = \int_{\mathbb{S}} (\varphi.\psi)(x) d_{\mathbb{S}}(x).$$

The map $\mathrm{Tr}^{RC} := \int_{\mathbb{S}}$ thus defines a trace form for $\star_\nu^{RC}$.

Proof. For $\varphi, \psi \in \mathcal{E}_\nu^{RC}$, the element $\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi$ belongs to $\mathcal{S}^{BG}(\mathbb{S}) \subset \mathcal{S}(\mathbb{S})$. Using Lemma 3.4.20 (and the link between $\star_\nu^{RC}$ and $\underline{\star}_\nu^{BG}$ through the operator $\underline{T}_{\nu,\vec{K}_0}$), we get[55]:

$$\begin{aligned}\int_{\mathbb{S}} \left(\varphi \star_\nu^{RC} \psi\right)(x) d_{\mathbb{S}}(x) &= \int_{\mathbb{S}} \left(\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi\right)(x) d_{\mathbb{S}}(x) \\ &= \int_{\mathbb{S}} \left(\underline{\tau}_{\nu,\vec{K}_0}\varphi.\underline{\tau}_{\nu,\vec{K}_0}\psi\right)(x) d_{\mathbb{S}}(x),\end{aligned}$$

where the last identity comes from Proposition 2.2.21. Combining now Remark 3.4.16 and the identity (3.81), we obtain:

$$\begin{aligned}\int_{\mathbb{S}} \left(\underline{\tau}_{\nu,\vec{K}_0}\varphi.\underline{\tau}_{\nu,\vec{K}_0}\psi\right)(x) d_{\mathbb{S}}(x) &= \langle \underline{\tau}_{\nu,\vec{K}_0}\varphi, \overline{\underline{\tau}_{\nu,\vec{K}_0}\psi}\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \langle \varphi, \overline{\psi}\rangle_{\mathcal{S}(\mathbb{S})} \\ &= \int_{\mathbb{S}} (\varphi.\psi)(x) d_{\mathbb{S}}(x).\end{aligned}$$

■

Remark 3.4.22. Notice that the trace form Tr^{RC} comes from the observation that under the identity

$$\varphi \star_\nu^{RC} \psi = \underline{T}_{\nu,\vec{K}_0}\left(\underline{\tau}_{\nu,\vec{K}_0}\varphi \underline{\star}_\nu^{BG} \underline{\tau}_{\nu,\vec{K}_0}\psi\right),$$

[55] Recall that $\underline{\star}_\nu^{BG} := \star_{\nu,\vartheta_0}^{BG}$ (see chapter 2).

the map $\mathrm{Tr}^{BG}_{\vartheta_0} \circ \underline{T}_{\nu,\vec{K}_0}$ naturally defines a trace form for $\star^{RC}_{\nu}$ (see Proposition 2.2.21 for the notation). Since $\mathrm{Tr}^{BG}_{\vartheta_0} = \int_{\mathbb{S}}$, Lemma 3.4.20 thus leads to:

$$\mathrm{Tr}^{BG}_{\vartheta_0} \circ \underline{T}_{\nu,\vec{K}_0} = \int_{\mathbb{S}} =: \mathrm{Tr}^{RC}.$$

3.4.3 Gel'fand-Shilov and Paley-Wiener spaces

We end this chapter by connecting the several function spaces involved in the construction of the non-formal RC star-product with the literature (especially [2], [3] and [35]).

Gel'fand-Shilov spaces of type $\mathcal{S}$

We refer to [35] for details.

Definition 3.4.23 ([35], Section 4.2)**.** For $\sigma \geq 0$, the space $\mathcal{S}_{0,\sigma}$ consists of all functions $f \in \mathcal{D}(\mathbb{R}) \subset C^{\infty}(\mathbb{R})$ such that $\mathrm{supp}(f) \subset [-\sigma, \sigma]$.
The space $\mathcal{S}^{0,\sigma}$ consists of all entire functions ϕ on $\mathbb{C}$ such that there exists $C > 0$ with

$$|(x+iy)^k \phi(x+iy)| \leq C \exp(\sigma|y|)$$

(where $k = 0, 1, 2, \ldots$).

Lemma 3.4.24 ([35], Section 4.6)**.** Denoting by F the Fourier transform on $\mathbb{R}$, we have

$$F\left(\left(\mathcal{S}^{0,\sigma}\right)_{\mathbb{R}}\right) = \mathcal{S}_{0,\sigma},$$

where $\left(\mathcal{S}^{0,\sigma}\right)_{\mathbb{R}}$ means the restriction of functions in $\mathcal{S}^{0,\sigma}$ to the real axis.

Remark 3.4.25 ([35], Section 4.2)**.** A function in $\mathcal{S}^{0,\sigma}$ is completely determined by its restriction to the real axis, which leads to the identification $\mathcal{S}^{0,\sigma} \simeq \left(\mathcal{S}^{0,\sigma}\right)_{\mathbb{R}}$. The previous lemma then becomes

$$F\left(\mathcal{S}^{0,\sigma}\right) = \mathcal{S}_{0,\sigma}.$$

The functions in $\mathcal{S}^{0,\sigma}$ (identified with $\left(\mathcal{S}^{0,\sigma}\right)_{\mathbb{R}}$) are said to be *bandlimited to* $[-\sigma, \sigma]$. For $\sigma > 0$, a function f is called *strictly bandlimited to* $[-\sigma, \sigma]$ if it is bandlimited to $[-\epsilon, \epsilon]$ for some $0 \leq \epsilon < \sigma$. We denote the space of functions that are strictly bandlimited to $[-\sigma, \sigma]$ as $\underline{\mathcal{S}}^{0,\sigma}$.

Remark 3.4.26 ([35], Section 4.8)**.** For $\sigma > 0$, the space $\mathcal{S}^{0,\sigma}$ does not contain any compactly supported functions.

Remark 3.4.27. For $\alpha > 1$ (the cases where $\alpha \leq 1$ are uninteresting because the involved spaces are trivial), we define the space $\mathcal{S}_\alpha^{0,\sigma}$ to be the set of all entire functions ϕ on $\mathbb{C}$ such that there exists $C > 0$, $a > 0$ with

$$|\phi(x+iy)| \leq C \exp\left(-a|x|^{\frac{1}{\alpha}} + \sigma|y|\right).$$

We have

$$\bigcup_{\alpha>1} \mathcal{S}_\alpha^{0,\sigma} \subset \mathcal{S}^{0,\sigma}.$$

Again, a function in $\mathcal{S}_\alpha^{0,\sigma}$ is completely determined by its restriction to the real axis and we have the identification $\mathcal{S}_\alpha^{0,\sigma} \simeq \left(\mathcal{S}_\alpha^{0,\sigma}\right)_{\mathbb{R}}$.

Now, for $\varphi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$ and $a \in \mathbb{R}$ (under the global chart (1.24)), we set

$$\varphi_a : \mathbb{R} \to \mathbb{R} : \ell \mapsto \varphi_a(\ell) := \varphi(a, \ell).$$

We thus have $\mathcal{F}\varphi(a,\xi) = F\varphi_a(\xi)$. Note that in particular, for all $a \in \mathbb{R}$, $\varphi_a \in \underline{\mathcal{S}}^{0,\frac{K_1^2}{|i\nu|}}$. The space $\underline{\mathcal{S}}^{K_1^2/|i\nu|}$ involved in the construction of the non-formal Rankin-Cohen star-product can thus be viewed as a *partial space of type* $\mathcal{S}$. We mention that this observation allows us to connect the present work with [22], in which authors describe function spaces of type $\mathcal{S}$ related to convergent star-products on $\mathbb{S}$.

Paley-Wiener spaces

We refer to [2] and [3] for details.

For a function $f \in L^2(\mathbb{R})$, we define the *support of* f, $\mathrm{supp}(f)$, to be the smallest closed set outside which the function f vanishes almost everywhere. We set also

$$R_f := \sup_{x \in \mathrm{supp}(f)} |x|,$$

to be the *radius of the support of* f. Let

$$L_c^2(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) \mid \mathrm{supp}(f) \text{ is compact} \right\};$$

$$L_R^2(\mathbb{R}) := \left\{ f \in L_c^2(\mathbb{R}) \mid R_f = R \right\},$$

with $R \geq 0$.

Definition 3.4.28 ([2], Definition 2.2)**.** The L^2-Paley-Wiener space $PW^2(\mathbb{R})$ is the space of all functions $f \in C^\infty(\mathbb{R})$ such that, setting $\Delta := \frac{d^2}{dx^2}$ the Laplacian on $\mathbb{R}$,

(i) $\Delta^n f \in L^2(\mathbb{R}),\ \forall n \in \mathbb{N}$;

(ii) $R_f^{\Delta} := \lim_{n\to+\infty} \|\Delta^n f\|_{L^2(\mathbb{R})}^{\frac{1}{2n}} < +\infty$.

For $\sigma \geq 0$, we set also

$$PW_\sigma^2(\mathbb{R}) := \left\{ f \in PW^2(\mathbb{R}) \mid R_f^{\Delta} = \sigma \right\}.$$

In the literature, the space $PW_\sigma^2(\mathbb{R})$ is sometimes called L^2-*Bernstein space* and is denoted by $B_\sigma^2(\mathbb{R})$. Denoting again by F the Fourier transform on $\mathbb{R}$, we have the following Paley-Wiener theorem in the real case.

Theorem 3.4.29 ([2], Theorem 2.3)**.** The inverse Fourier transform F^{-1} is a bijection of $L_c^2(\mathbb{R})$ onto $PW^2(\mathbb{R})$, mapping $L_\sigma^2(\mathbb{R})$ onto $PW_\sigma^2(\mathbb{R})$.

We thus obtain the following characterization: $R_{Ff} = \sigma$ if and only if $f \in PW_\sigma^2(\mathbb{R})$.

Remark 3.4.30. The classical theorem of Paley-Wiener-Schwartz in the complex case implies that $PW_\sigma^2(\mathbb{R})$ consists of all functions in $L^2(\mathbb{R})$ that can be extended to holomorphic functions of exponential type σ on $\mathbb{C}$.

In order to express a refinement of the previous theorem, we introduce the following definition.

Definition 3.4.31 ([2], Definition 2.6)**.** The Paley-Wiener space $PW(\mathbb{R})$ is the space of all functions $f \in C^\infty(\mathbb{R})$ such that

(i) $(1+|x|)^m\, \Delta^n f \in L^2(\mathbb{R}),\ \forall m, n \in \mathbb{N}$;

(ii) $R_f^{\Delta} := \lim_{n\to+\infty} \|\Delta^n f\|_{L^2(\mathbb{R})}^{\frac{1}{2n}} < +\infty$.

We set

$$PW_\sigma(\mathbb{R}) := \left\{ f \in PW(\mathbb{R}) \mid R_f^{\Delta} = \sigma \right\}.$$

Remark that the extra requirement on the functions in $PW(\mathbb{R})$ ensure that $Ff \in C^\infty(\mathbb{R})$.

Theorem 3.4.32 ([2], Theorem 2.7)**.** The inverse Fourier transform F^{-1} is a bijection of $\mathcal{D}(\mathbb{R})$ onto $PW(\mathbb{R})$, mapping the space

$$\mathcal{D}_\sigma(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) \mid \mathrm{supp}(f) \subset [-\sigma, \sigma] \}$$

onto $PW_\sigma(\mathbb{R})$.

Since by definition $S_{0,\sigma} = \mathcal{D}_\sigma(\mathbb{R})$, Lemma 3.4.24 and Theorem 3.4.32 imply that

$$S^{0,\sigma} \simeq PW_\sigma(\mathbb{R}).$$

Moreover, in [44], we can find the following result.

Proposition 3.4.33. For $\sigma \geq 0$, $\underline{\mathcal{S}}^{0,\sigma}$ is a $L^2(\mathbb{R})$-dense subspace of $PW^2_\sigma(\mathbb{R})$.

Notice that if $\varphi \in \underline{\mathcal{S}}^{K_1^2/|i\nu|}$, then for all $a \in \mathbb{R}$, the function φ_a belongs to the space $PW^2_{\frac{K_1^2}{|i\nu|}}(\mathbb{R})$. Therefore, the space $\underline{\mathcal{S}}^{K_1^2/|i\nu|}$ can also be interpreted as a *partial Paley-Wiener type function space*.

In the last chapter, we outline an alternative way of building an integral representation for the Rankin-Cohen deformation quantization. The construction is based on the result given by Proposition 1.2.32, which links the Rankin-Cohen formal star-product with a restriction of the Moyal product on $\mathbb{R}^2$.

Chapter 4

An alternative integral representation for the Rankin-Cohen star-product

In chapter 3, the integral representation for Rankin-Cohen relies on the fact that there exists a link (by means of an intertwiner) between the Rankin-Cohen deformation on the symplectic manifold $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ and that of Bieliavsky and Gayral outlined in [20]. The particular interest of the BG star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ is the non-formal aspect of its construction, which plays a crucial role in the integral formula obtained for $\star_\nu^{RC}$ and given in Theorem 3.4.4. But if we have a look at the geometrical interpretation of the formal Rankin-Cohen star-product given by Proposition 1.2.32, we can notice that $\star_\nu^{RC}$ can be directly written in terms of the Moyal star-product on the symplectic (flat) manifold $\left(\mathbb{R}^2 = \{q, p\}, \omega = dq \wedge dp\right)$. This suggests another way (in some sense more natural) of building an integral representation for the RC star-product by means of the Weyl product, which constitutes the non-formal version of the Moyal product (see chapter 2, section 2.2). This chapter is devoted to this construction.

4.1 Construction of the integral representation

Recall that thanks to Proposition 1.2.32, the formal Rankin-Cohen star-product admits the following expression:

$$\varphi \star_\nu^{RC} \psi = \underline{\phi}^\star \left((\underline{\phi}^{-1})^\star \varphi \star_\nu^{\mathcal{O}} (\underline{\phi}^{-1})^\star \psi \right) = \underline{\phi}^\star \left(\left[(\underline{\phi}^{-1})^\star \varphi \star_\nu^0 (\underline{\phi}^{-1})^\star \psi \right] \big|_{\mathcal{O}} \right),$$

where the set $\mathcal{O}$ is the upper half-plane of $\mathbb{R}^2$ (see (1.25)), the map $\underline{\phi} : \mathbb{S} \to \mathcal{O}$ is defined by (1.26), and where $\star_\nu^0$ denotes the Moyal star-product on the symplectic manifold $\left(\mathbb{R}^2 = \{q,p\}, \omega = dq \wedge dp\right)$, as in Definition 1.2.20. In view of this formula, it is thus tempting to pass from the (formal) Moyal product to its convergent version (the Weyl product), in order to produce an integral formula for $\star_\nu^{RC}$. But unfortunately, the expression[1] $\left[(\underline{\phi}^{-1})^\star \varphi \star_{2i\nu}^W (\underline{\phi}^{-1})^\star \psi\right]\big|_{\mathcal{O}}$ does not make sense, because neither $(\underline{\phi}^{-1})^\star \varphi$ nor $(\underline{\phi}^{-1})^\star \psi$ is defined on the whole space $\mathbb{R}^2$. In order to avoid this problem, we introduce the following map:

$$\begin{aligned} \Sigma : C^\infty(\mathcal{O}) &\to C^\infty(\mathbb{R}^2) \\ \varphi &\mapsto \widetilde{\varphi} + s^\star \widetilde{\varphi}, \end{aligned} \tag{4.1}$$

where $\widetilde{\varphi}$ denotes the extension by 0 on the whole space $\mathbb{R}^2$ of the function φ, and $s := -\operatorname{id}|_{\mathbb{R}^2}$ is the central symmetry of $\mathbb{R}^2$.

Remark 4.1.1. Notice that for any function $\varphi \in C^\infty(\mathcal{O})$, $\Sigma(\varphi)$ is an even function, in the sense that

$$s^\star \Sigma(\varphi) = \Sigma(\varphi).$$

For $\varphi, \psi \in C^\infty(\mathcal{O})$, we have:

$$\begin{aligned} \Sigma(\varphi) \star_\nu^0 \Sigma(\psi) &= \widetilde{\varphi} \star_\nu^0 \widetilde{\psi} + s^\star \widetilde{\varphi} \star_\nu^0 \widetilde{\psi} + \widetilde{\varphi} \star_\nu^0 s^\star \widetilde{\psi} + s^\star \widetilde{\varphi} \star_\nu^0 s^\star \widetilde{\psi} \\ &= \widetilde{\varphi} \star_\nu^0 \widetilde{\psi} + s^\star \widetilde{\varphi} \star_\nu^0 s^\star \widetilde{\psi}. \end{aligned}$$

Indeed, $s^\star \widetilde{\varphi}$ and $\widetilde{\psi}$ (respectively $\widetilde{\varphi}$ and $s^\star \widetilde{\psi}$) have disjoint supports and then $s^\star \widetilde{\varphi} \star_\nu^0 \widetilde{\psi} \equiv 0$ (respectively $\widetilde{\varphi} \star_\nu^0 s^\star \widetilde{\psi} \equiv 0$). It implies that

$$\left[\Sigma(\varphi) \star_\nu^0 \Sigma(\psi)\right]\big|_{\mathcal{O}} = \widetilde{\varphi} \star_\nu^0 \widetilde{\psi} = \varphi \star_\nu^{\mathcal{O}} \psi,$$

and then the formal Rankin-Cohen star-product can be re-expressed as

$$\varphi \star_\nu^{RC} \psi = \underline{\phi}^\star \left(\left[\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right) \star_\nu^0 \Sigma\left((\underline{\phi}^{-1})^\star \psi\right)\right]\big|_{\mathcal{O}} \right).$$

But, at least at the level of $\mathcal{D}(\mathcal{O})$ (the space of smooth compactly supported functions on $\mathcal{O}$), notice that the RHS is exactly the asymptotic expansion of

$$\underline{\phi}^\star \left(\left[\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right) \star_{2i\nu}^W \Sigma\left((\underline{\phi}^{-1})^\star \psi\right)\right]\big|_{\mathcal{O}} \right).$$

[1] Recall that the Moyal product $\star_\nu^0$ coincides with the asymptotic expansion of $\star_{2i\nu}^W$ around $i\nu = 0$ (see chapter 2, subsection 2.2.1).

We then get a non-formal formula for the Rankin-Cohen star-product, which we will denote as $\tilde{\star}_\nu^{RC}$:

$$\varphi \tilde{\star}_\nu^{RC} \psi := \underline{\phi}^\star \left(\left[\Sigma \left((\underline{\phi}^{-1})^\star \varphi \right) \star_{2i\nu}^W \Sigma \left((\underline{\phi}^{-1})^\star \psi \right) \right] \Big|_{\mathcal{O}} \right). \tag{4.2}$$

Remark 4.1.2. Since $s = -\operatorname{id}|_{\mathbb{R}^2} \in SL_2(\mathbb{R})$, the (well-known) $SL_2(\mathbb{R})$-invariance of $\star_{2i\nu}^W$ implies that $\Sigma \left((\underline{\phi}^{-1})^\star \varphi \right) \star_\nu^W \Sigma \left((\underline{\phi}^{-1})^\star \psi \right)$ is an even function.

Lemma 4.1.3. The map

$$\begin{aligned} \tilde{\Sigma} : L^2(\mathbb{S}) &\to L^2(\mathbb{R}^2) \\ \varphi &\mapsto \frac{\sqrt{2}}{2} \Sigma \left((\underline{\phi}^{-1})^\star \varphi \right) \end{aligned}$$

defines an isometric injection, with $\operatorname{Im}(\tilde{\Sigma}) = L^2(\mathbb{R}^2)_{\text{even}}$ (the set of square integrable and even functions on $\mathbb{R}^2$).

Proof. Denote by[2] ω the symplectic form $dq \wedge dp$ on $\mathbb{R}^2$ and $\omega^{\mathbb{S}} := \underline{\phi}^\star \, \omega|_{\mathcal{O}}$. We also denote by μ_ω and $\mu_{\omega^{\mathbb{S}}}$ the Liouville measures associated respectively to ω and $\omega^{\mathbb{S}}$. First, notice that thanks to the definition of the map $\underline{\phi}$ (1.26):

$$\underline{\phi} : \mathbb{S} \to \mathcal{O} : (a, \ell) \mapsto (\ell e^a, e^{-a}),$$

we have

$$\left| \operatorname{Jac}_{\underline{\phi}}(a, \ell) \right| := \det \begin{pmatrix} \ell e^a & e^a \\ -e^{-a} & 0 \end{pmatrix} = 1.$$

[2] We use the same notations as in chapter 2, section 1.2.

Then, for $\varphi \in L^2(\mathbb{S})$, we get:

$$\begin{aligned}
\left\|\tilde{\Sigma}(\varphi)\right\|^2_{L^2(\mathbb{R}^2)} &= \frac{1}{2}\int_{\mathbb{R}^2} \left|\Sigma\left(\left(\underline{\phi}^{-1}\right)^\star \varphi\right)\right|^2 d\mu_\omega \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \Bigg(\left|\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi\right|^2 + \underbrace{\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi\overline{s^\star\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi}}_{\equiv 0} \\
&\qquad + \underbrace{\overline{\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi}s^\star\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi}_{\equiv 0} + \left|s^\star\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi\right|^2 \Bigg) d\mu_\omega \\
&= \int_{\mathbb{R}^2} \left|\widetilde{\left(\underline{\phi}^{-1}\right)^\star}\varphi\right|^2 d\mu_\omega \\
&= \int_{\mathcal{O}} \left|\left(\underline{\phi}^{-1}\right)^\star \varphi\right|^2 d\mu_\omega|_{\mathcal{O}} \\
&= \int_{\mathbb{S}} |\varphi|^2 \, d\mu_{\omega^{\mathbb{S}}} \\
&= \|\varphi\|^2_{L^2(\mathbb{S})} .
\end{aligned}$$

■

Remark 4.1.4. The following identities hold[3]:

(i) for $f \in L^2(\mathcal{O})$,
$$\Sigma(f)|_{\mathcal{O}} = f;$$

(ii) for $g \in L^2(\mathbb{R}^2)_{\text{even}}$,
$$\Sigma\left(g|_{\mathcal{O}}\right) = g.$$

In particular, it implies that for any $\varphi \in L^2(\mathbb{S})$, we have:

$$\underline{\phi}^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star\varphi\right)\big|_{\mathcal{O}}\right) = \varphi.$$

Before stating the following result, we recall that as mentioned in chapter 2, subsection 2.2.1, the Schwartz space $\left(\mathcal{S}\left(\mathbb{R}^2\right), \star^W_{2i\nu}\right)$ is an associative algebra. Moreover, its algebra structure extends to $L^2\left(\mathbb{R}^2\right)$ (we refer for instance to [39] for more details), for which we denote again the product law by $\star^W_{2i\nu}$. Having said that, notice that all the above reasoning leads to the following fact.

Proposition 4.1.5. The product law $\tilde{\star}^{RC}_\nu$ extends to $L^2(\mathbb{S})$. Endowed with the extended product again denoted by $\tilde{\star}^{RC}_\nu$, the space $\left(L^2(\mathbb{S}), \tilde{\star}^{RC}_\nu, \bar{\cdot}\right)$ becomes a *-algebra associated with the Rankin-Cohen deformation.

[3]We use the fact that in an L^2-space, two functions equal almost everywhere are identified with each other.

Proof. Using Lemma 4.1.3 and Remark 4.1.4, it suffices to transport the algebra structure on $\left(L^2\left(\mathbb{R}^2\right), \star_{2i\nu}^W\right)$ via the map Σ (given by (4.1)), in order to obtain the algebra structure on $\left(L^2(\mathbb{S}), \tilde{\star}_\nu^{RC}\right)$. Moreover, notice that for a smooth function f defined on $\mathcal{O}$, we have:

$$\overline{\Sigma(f)} = \Sigma\left(\bar{f}\right).$$

Lastly, thanks to the fact that the complex conjugation is an involution for the Weyl product[4], we get the mentioned *-algebra structure on $\left(L^2(\mathbb{S}), \tilde{\star}_\nu^{RC}\right)$. ■

Recall now that $SL_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication:

$$\begin{aligned}\rho : SL_2(\mathbb{R}) \times \mathbb{R}^2 &\to \mathbb{R}^2\\ (A, x) &\mapsto \rho_A(x) := A \cdot x.\end{aligned}$$

For $f \in L^2\left(\mathbb{R}^2\right)_{\text{even}}$, $A \in SL_2(\mathbb{R})$ and $x \in \mathbb{R}^2$, notice that

$$\rho_A^\star f(-x) = f\left(A \cdot (-x)\right) = f\left(-A \cdot x\right) = f\left(A \cdot x\right) = \rho_A^\star f(x),$$

which means that $\rho_A^\star f$ is even too. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^2} \left|\rho_A^\star f\right|^2 (q,p) dq dp &= \int_{\mathbb{R}^2} |f|^2 \left(A \cdot \begin{pmatrix} q \\ p \end{pmatrix}\right) dq dp\\ &= \int_{\mathbb{R}^2} |f|^2 (q', p') dq' dp',\end{aligned}$$

thanks to the fact that $\det(A) = 1$. In other words, we get an action of $SL_2(\mathbb{R})$ on $L^2\left(\mathbb{R}^2\right)_{\text{even}}$ induced by $\rho^\star$:

$$\begin{aligned}\rho^\star : SL_2(\mathbb{R}) \times L^2\left(\mathbb{R}^2\right)_{\text{even}} &\to L^2\left(\mathbb{R}^2\right)_{\text{even}}\\ (A, f) &\mapsto \rho_A^\star f.\end{aligned}$$

Using (up to a constant) the correspondence between $L^2(\mathbb{S})$ and $L^2(\mathbb{R}^2)_{\text{even}}$ given by Lemma 4.1.3, we can transport the above action in an action of $SL_2(\mathbb{R})$ on $L^2(\mathbb{S})$. Hence, if we denote again this action by $\rho^\star$, we get:

$$\begin{aligned}\rho^\star : SL_2(\mathbb{R}) \times L^2(\mathbb{S}) &\to L^2(\mathbb{S})\\ (A, \varphi) &\mapsto \rho_A^\star \varphi := \left[\mathbb{S} \to \mathbb{C} : x \mapsto \Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\left(A \cdot \underline{\phi}(x)\right)\right]. \quad (4.3)\end{aligned}$$

[4]This is a well-known fact, see for instance [52].

Remark 4.1.6. Notice that the restriction of $\rho^\star$ to $\mathbb{S}$ coincides with the left regular action. Indeed, for $x \in \mathbb{S}$ (with $\mathbb{S}$ viewed as a subgroup of $SL_2(\mathbb{R})$), we have

$$\rho_x^\star \varphi(x_0) = \Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\left(x \cdot \underline{\phi}(x_0)\right).$$

But using the expression (1.26), we observe that

$$x \cdot \underline{\phi}(x_0) = x \cdot \left(x_0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = (xx_0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\phi}(xx_0).$$

Since $\underline{\phi}(xx_0) \in \mathcal{O}$, the first item of Remark 4.1.4 then implies that

$$\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\left(x \cdot \underline{\phi}(x_0)\right) = \Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\left(\underline{\phi}(xx_0)\right) = \varphi(xx_0) = L_x^\star \varphi(x_0).$$

Proposition 4.1.7. The star-product $\tilde{\star}_\nu^{RC}$ is $SL_2(\mathbb{R})$-invariant.

Proof. For $A \in SL_2(\mathbb{R})$, $x_0 \in \mathbb{S}$ and $\varphi, \psi \in L^2(\mathbb{S})$, we have:

$$\left(\rho_A^\star \varphi \tilde{\star}_\nu^{RC} \rho_A^\star \psi\right)(x_0) = \left(\left[\Sigma\left((\underline{\phi}^{-1})^\star \left(\rho_A^\star \varphi\right)\right) \star_{2i\nu}^W \Sigma\left((\underline{\phi}^{-1})^\star \left(\rho_A^\star \psi\right)\right)\right]\Big|_{\mathcal{O}}\right)\left(\underline{\phi}(x_0)\right).$$

Note that for all $y \in \mathcal{O}$,

$$(\underline{\phi}^{-1})^\star \left(\rho_A^\star \varphi\right)(y) = \Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)(A \cdot y) = \rho_A^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\right)(y),$$

which is an even L^2-function on $\mathbb{R}^2$. Hence,

$$\Sigma\left(\rho_A^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\right)\right) = \rho_A^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\right),$$

and the same holds for ψ. The $SL_2(\mathbb{R})$-invariance of $\star_{2i\nu}^W$ then implies that

$$\rho_A^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)\right) \star_{2i\nu}^W \rho_A^\star\left(\Sigma\left((\underline{\phi}^{-1})^\star \psi\right)\right) = \rho_A^\star\left[\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right) \star_{2i\nu}^W \Sigma\left((\underline{\phi}^{-1})^\star \psi\right)\right].$$

Hence,

$$\left(\rho_A^\star \varphi \tilde{\star}_\nu^{RC} \rho_A^\star \psi\right)(x_0) = \left(\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right) \star_{2i\nu}^W \Sigma\left((\underline{\phi}^{-1})^\star \psi\right)\right)\left(A \cdot \underline{\phi}(x_0)\right).$$

Since

$$\rho_A^\star\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(x_0) = \Sigma\left((\underline{\phi}^{-1})^\star \left(\varphi \tilde{\star}_\nu^{RC} \psi\right)\right)\left(A \cdot \underline{\phi}(x_0)\right),$$

it remains to show that

$$\Sigma\left((\underline{\phi}^{-1})^\star \left(\varphi \tilde{\star}_\nu^{RC} \psi\right)\right) = \Sigma\left((\underline{\phi}^{-1})^\star \varphi\right) \star_{2i\nu}^W \Sigma\left((\underline{\phi}^{-1})^\star \psi\right).$$

For this, note that by the definition (4.2) of $\tilde{\star}_\nu^{RC}$ and by using the item (ii) of Remark 4.1.4, we have:

$$\begin{aligned}\Sigma\left((\underline{\phi}^{-1})^\star\left(\varphi\tilde{\star}_\nu^{RC}\psi\right)\right) &= \Sigma\left(\left[\Sigma\left((\underline{\phi}^{-1})^\star\varphi\right)\star_{2i\nu}^W\Sigma\left((\underline{\phi}^{-1})^\star\psi\right)\right]\big|_{\mathcal{O}}\right)\\ &= \Sigma\left((\underline{\phi}^{-1})^\star\varphi\right)\star_{2i\nu}^W\Sigma\left((\underline{\phi}^{-1})^\star\psi\right),\end{aligned}$$

which completes the proof. ■

Remark 4.1.8. (i) The above proposition means that the space $L^2\left(\mathbb{R}^2\right)_{\text{even}}$ allows to realize an $SL_2(\mathbb{R})$-invariant extension of $\star_{2i\nu}^W\big|_{\mathcal{O}}$ (the latter being only $\mathfrak{sl}_2(\mathbb{R})$-invariant).

(ii) It could be interesting to compare the star-product $\tilde{\star}_\nu^{RC}$ with the formulas given in [1].

In the following result, we show that $\tilde{\star}_\nu^{RC}$ is a tracial star-product.

Proposition 4.1.9. The star-product $\tilde{\star}_\nu^{RC}$ is strongly closed. That is, at the level of compactly supported functions on $\mathbb{S}$, the following identity holds:

$$\int_{\mathbb{S}}\left(\varphi\tilde{\star}_\nu^{RC}\psi\right)(x)d_{\mathbb{S}}(x) = \int_{\mathbb{S}}\left(\varphi.\psi\right)(x)d_{\mathbb{S}}(x).$$

The map $\widetilde{\mathrm{Tr}}^{RC} := \int_{\mathbb{S}}$ thus defines a trace form for $\tilde{\star}_\nu^{RC}$.

Proof. For $\varphi,\psi\in\mathcal{D}(\mathbb{S})$, we have:

$$\int_{\mathbb{S}}\left(\varphi\tilde{\star}_\nu^{RC}\psi\right)(x)d_{\mathbb{S}}(x) = \int_{\mathbb{S}}\left(\Sigma\left((\underline{\phi}^{-1})^\star\varphi\right)\star_{2i\nu}^W\Sigma\left((\underline{\phi}^{-1})^\star\psi\right)\right)(\underline{\phi}(x))d_{\mathbb{S}}(x).$$

We recall also that for $x=(a,\ell)$ (through the global chart (1.24)), the definition of the map $\underline{\phi}$ (1.26) implies:

$$\left|\mathrm{Jac}_{\underline{\phi}}(a,\ell)\right| = \det\begin{pmatrix}\ell e^a & e^a\\ -e^{-a} & 0\end{pmatrix} = 1.$$

Hence, using the evenness of the function $\Sigma\left((\underline{\phi}^{-1})^{\star}\varphi\right) \star_{2i\nu}^{W} \Sigma\left((\underline{\phi}^{-1})^{\star}\psi\right)$ (see Remark 4.1.2) and the traciality of $\star_{2i\nu}^{W}$ (2.44), we obtain:

$$\begin{aligned}
\int_{\mathbb{S}} \left(\varphi \tilde{\star}_{\nu}^{RC} \psi\right)(x) d_{\mathbb{S}}(x) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(\Sigma\left((\underline{\phi}^{-1})^{\star}\varphi\right) \star_{2i\nu}^{W} \Sigma\left((\underline{\phi}^{-1})^{\star}\psi\right)\right)(q,p) dqdp \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left(\Sigma\left((\underline{\phi}^{-1})^{\star}\varphi\right) .\Sigma\left((\underline{\phi}^{-1})^{\star}\psi\right)\right)(q,p) dqdp \\
&= \int_{\mathbb{R}^2} \left(\widetilde{(\underline{\phi}^{-1})^{\star}}\varphi . \widetilde{(\underline{\phi}^{-1})^{\star}}\psi\right)(q,p) dqdp \\
&= \int_{\mathcal{O}} \left((\underline{\phi}^{-1})^{\star}\varphi . (\underline{\phi}^{-1})^{\star}\psi\right)(q,p) dqdp \\
&= \int_{\mathbb{S}} (\varphi.\psi)(x) d_{\mathbb{S}}(x),
\end{aligned}$$

which completes the proof. ■

The next subsection is devoted to the connection between the non-formal product $\star_{\nu}^{RC}$ constructed in chapter 3 and the product $\tilde{\star}_{\nu}^{RC}$ exhibited above.

4.2 Link between both integral representations for the RC star-product

Let us consider two functions φ, ψ belonging to the space $\underline{\mathcal{S}}^{1/|i\nu|} \subset \mathcal{E}_{\nu}^{RC}$. In particular, since $\mathcal{E}_{\nu}^{RC} := \underline{T}_{\nu,\vec{K}_0}\left(\mathcal{S}^{BG}(\mathbb{S})\right)$, both functions φ and ψ belong to the Schwartz space $\mathcal{S}(\mathbb{S})$ (see Proposition 3.3.3). By virtue of the expression (4.2) and the formula (2.19) for the Weyl product, their $\tilde{\star}_{\nu}^{RC}$-multiplication is given explicitly by

$$\begin{aligned}
\left(\varphi \tilde{\star}_{\nu}^{RC} \psi\right)(a,\ell) = \frac{1}{(2\pi i\nu)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{-\frac{1}{\nu}S^{W}\left((\ell e^{a},e^{-a}),(q,p),(q',p')\right)} \Sigma\left((\underline{\phi}^{-1})^{\star}\varphi\right)(q,p) \\
\times \Sigma\left((\underline{\phi}^{-1})^{\star}\psi\right)(q',p') dqdpdq'dp',
\end{aligned} \tag{4.4}$$

where $(a,\ell) \in \mathbb{R}^2 \simeq \mathbb{S}$ (by means of the coordinates (1.24)) and with

$$S^{W}\left((\ell e^{a}, e^{-a}),(q,p),(q',p')\right) = \ell e^{a}p - e^{-a}q + qp' - pq' + e^{-a}q' - \ell e^{a}p',$$

by using (2.20) with the usual symplectic form $\omega = dq \wedge dp$ on $\mathbb{R}^2$. Note that the integral in the above RHS exists. Indeed, first of all, recall that using (4.1), we have

$$\Sigma\left((\underline{\phi}^{-1})^{\star}\varphi\right) = \widetilde{(\underline{\phi}^{-1})^{\star}}\varphi + s^{\star}\widetilde{(\underline{\phi}^{-1})^{\star}}\varphi,$$

with $s := -\mathrm{id}|_{\mathbb{R}^2}$ (and the same for the function ψ). For $(q,p) \in \mathcal{O}$ (the upper-half plane of $\mathbb{R}^2$), notice that[5]

$$(\underline{\phi}^{-1})^\star \varphi(q,p) = \varphi\left(-\log(p), qp\right).$$

Thanks to an iterative use of the relations

$$\partial_q \left(\varphi \circ \underline{\phi}^{-1}\right)(q,p) = e^{-(-\log(p))} \left(\partial_2 \varphi \circ \underline{\phi}^{-1}\right)(q,p)$$

and

$$\partial_p \left(\varphi \circ \underline{\phi}^{-1}\right)(q,p) = -e^{-\log(p)} \left(\partial_1 \varphi \circ \underline{\phi}^{-1}\right)(q,p) + qpe^{-\log(p)} \left(\partial_2 \varphi \circ \underline{\phi}^{-1}\right)(q,p),$$

we then observe that for all $k,m,n \in \mathbb{N}$ and for all $q \in \mathbb{R}$,

$$\lim_{p\to 0} \frac{1}{p^k} \partial_q^m \partial_p^n (\underline{\phi}^{-1})^\star \varphi(q,p) = \lim_{p\to 0} e^{-k\log(p)} \partial_q^m \partial_p^n (\underline{\phi}^{-1})^\star \varphi(q,p) = 0.$$

That leads to the fact that for all $j,k,m,n \in \mathbb{N}$,

$$\sup_{(q,p)\in\mathcal{O}} \left| q^j p^k \partial_q^m \partial_p^n (\underline{\phi}^{-1})^\star \varphi(q,p) \right| < +\infty,$$

which in turn implies that the function $\Sigma\left((\underline{\phi}^{-1})^\star \varphi\right)$ belongs to $\mathcal{S}(\mathbb{R}^2)$, the usual Schwartz space on the plane (of course, the same holds for the function $\Sigma\left((\underline{\phi}^{-1})^\star \psi\right)$).
If we now come back to the identity (4.4), notice that the RHS splits into four integrals (due to the extension by zero coming from the definition (4.1) of the map Σ). Namely, if we set

$$\begin{aligned}
D_1 &:= \{(q,p,q',p') \in \,]-\infty,+\infty[\times]0,+\infty[\times]-\infty,+\infty[\times]0,+\infty[\}\,,\\
D_2 &:= \{(q,p,q',p') \in \,]-\infty,+\infty[\times]-\infty,0[\times]-\infty,+\infty[\times]0,+\infty[\}\,,\\
D_3 &:= \{(q,p,q',p') \in \,]-\infty,+\infty[\times]-\infty,0[\times]-\infty,+\infty[\times]-\infty,0[\}\,,\\
D_4 &:= \{(q,p,q',p') \in \,]-\infty,+\infty[\times]0,+\infty[\times]-\infty,+\infty[\times]-\infty,0[\}\,,
\end{aligned}$$

we get (for $\varphi,\psi \in \underline{\mathcal{S}}^{1/|i\nu|}$)[6]:

$$\begin{aligned}
\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(a,\ell) = \frac{1}{(2\pi i\nu)^2} \sum_{j=1}^4 \int_{D_j} & e^{-\frac{1}{\nu}\left(\ell e^a p - e^{-a} q + qp' - pq' + e^{-a} q' - \ell e^a p'\right)} \\
& \times (\underline{\phi}^{-1})^\star \varphi(\mathrm{sign}(p)q, |p|)(\underline{\phi}^{-1})^\star \psi(\mathrm{sign}(p')q', |p'|) dq dp dq' dp',
\end{aligned}$$

[5] Recall that the map $\underline{\phi}$ is defined by (1.26).
[6] Note that we make use of the Fubini theorem.

where sign denotes the sign function on $\mathbb{R}$. Changing the variables following $q \mapsto e^a q$, $p \mapsto e^{-a} p$, $q' \mapsto e^a q'$ and $p' \mapsto e^{-a} p'$ (note that the Jacobian determinant of this change of variables is equal to 1), we obtain

$$\begin{aligned}\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(a,\ell) =& \frac{1}{(2\pi i\nu)^2} \sum_{j=1}^{4} \int_{D_j} e^{-\frac{1}{\nu}\left(\ell p - q + qp' - pq' + q' - \ell p'\right)} \\ & \times \varphi\left(a - \log(|p|), qp\right) \psi\left(a - \log(|p'|), q'p'\right) dqdpdq'dp'.\end{aligned}$$

A second change of variables following $q \mapsto \frac{q}{p}$ and $q' \mapsto \frac{q'}{p'}$ (for which the Jacobian determinant is given by pp') then leads to:

$$\begin{aligned}\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(a,\ell) =& \frac{1}{(2\pi i\nu)^2} \sum_{j=1}^{4} \int_{D_j} \frac{1}{pp'} e^{-\frac{1}{\nu}\left(\ell p - \frac{q}{p} + \frac{q}{p} p' - p\frac{q'}{p'} + \frac{q'}{p'} - \ell p'\right)} \\ & \times \varphi\left(a - \log(|p|), q\right) \psi\left(a - \log(|p'|), q'\right) dqdpdq'dp'.\end{aligned}$$

Now, we denote the four quadrants of the plane $\mathbb{R}^2 = \{(p,p')\}$ as follows:

$$\begin{aligned} & d_1 := \{(p,p') \in \,]0,+\infty[\times]0,+\infty[\}\,,\ d_2 := \{(p,p') \in \,]-\infty,0[\times]0,+\infty[\}\,, \\ & d_3 := \{(p,p') \in \,]-\infty,0[\times]-\infty,0[\}\,,\ d_4 := \{(p,p') \in \,]0,+\infty[\times]-\infty,0[\}\,. \end{aligned}$$

Performing the integrals w.r.t. the q,q'-variables, we get:

$$\begin{aligned}\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(a,\ell) =& \frac{1}{(2\pi i\nu)^2} \sum_{j=1}^{4} \int_{d_j} \frac{1}{pp'} e^{-\frac{1}{\nu}(p-p')\ell} \mathcal{F}\varphi\left(a - \log(|p|), \frac{1}{i\nu}.\frac{p'-1}{p}\right) \\ & \times \mathcal{F}\psi\left(a - \log(|p'|), \frac{1}{i\nu}.\frac{1-p}{p'}\right) dpdp',\end{aligned}$$

where, as usual, $\mathcal{F}$ means the partial Fourier transform w.r.t. the second variable. Recall that thanks to the fact that $\varphi, \psi \in \underline{\mathcal{S}}^{1/|i\nu|}$, the support of both functions $\mathcal{F}\varphi$ and $\mathcal{F}\psi$ is contained within the strip $\mathbb{R} \times [-\sigma,\sigma]$, for a certain $0 \leq \sigma < \frac{1}{|i\nu|}$. Note that the only elements $(p,p') \in \mathbb{R}^2$ which fulfill both conditions $\left|\frac{p'-1}{p}\right| < 1$ and $\left|\frac{p-1}{p'}\right| < 1$ constitute a subset of the first quadrant d_1. Hence, the integrals taken over the other quadrants vanish (the integrands being identically zero). Let us denote by $\widetilde{d_1}$ the subset of $d_1 =]0,+\infty[\times]0,+\infty[$ defined as follows:

$$\widetilde{d_1} := \left\{(p,p') \in \ d_1 \text{ s.t. } \left|\frac{p'-1}{p}\right| < 1 \text{ and } \left|\frac{p-1}{p'}\right| < 1\right\}.$$

We are therefore left with

$$\left(\varphi \tilde{\star}_\nu^{RC} \psi\right)(a,\ell) = \frac{1}{(2\pi i\nu)^2} \int_{\widetilde{d}_1} \frac{1}{pp'} e^{-\frac{1}{\nu}(p-p')\ell} \mathcal{F}\varphi\left(a - \log(p), \frac{1}{i\nu}.\frac{p'-1}{p}\right) \times \mathcal{F}\psi\left(a - \log(p'), \frac{1}{i\nu}.\frac{1-p}{p'}\right) dpdp'.$$

We will now rewrite this expression with the help of the following lemma.

Lemma 4.2.1. The set $\widetilde{d}_1$ is diffeomorphic to $\mathbb{R}^2$ through the map

$$\widetilde{\phi_1} : \mathbb{R}^2 \to \widetilde{d}_1 : (a_1, a_2) \mapsto \left(\frac{e^{-a_1}\cosh(a_2)}{\cosh(a_1 - a_2)}, \frac{e^{-a_2}\cosh(a_1)}{\cosh(a_1 - a_2)}\right).$$

Proof. Let us start with the injectivity. For this, assume that $(a_1, a_2), (a_1', a_2') \in \mathbb{R}^2$ are such that $\widetilde{\phi_1}(a_1, a_2) = \widetilde{\phi_1}(a_1', a_2') =: (p, p')$. Notice that

$$p - p' = -\tanh(a_1 - a_2) = -\tanh(a_1' - a_2'),$$

which implies that $a_1 - a_2 = a_1' - a_2'$. From the identity $\widetilde{\phi_1}(a_1, a_2) = \widetilde{\phi_1}(a_1', a_2')$, we then deduce that

$$e^{-a_1}\cosh(a_1) = e^{-a_1'}\cosh(a_1') \quad \text{and} \quad e^{-a_2}\cosh(a_2) = e^{-a_2'}\cosh(a_2'),$$

which in turn implies that $a_1 = a_1'$ and $a_2 = a_2'$. The map $\widetilde{\phi_1}$ is then injective. The rest is then a direct consequence of the following observations: for any $a_1, a_2 \in \mathbb{R}$,

$$a_1 = \operatorname{arctanh}\left(\frac{1-p}{p'}\right) \quad \text{and} \quad a_2 = \operatorname{arctanh}\left(\frac{1-p'}{p}\right), \tag{4.5}$$

with $(p, p') := \widetilde{\phi_1}(a_1, a_2)$. Moreover, for all $(a_1, a_2) \in \mathbb{R}^2$,

$$\left|\operatorname{Jac}_{\widetilde{\phi_1}}(a_1, a_2)\right| = \frac{e^{-a_1-a_2}}{\cosh^3(a_1 - a_2)} \neq 0. \tag{4.6}$$

■

Setting $(p, p') := \widetilde{\phi_1}(a_1, a_2)$, the use of (4.5), (4.6), together with the identity

$$\frac{1}{pp'} = \frac{\cosh^2(a_1 - a_2)}{e^{-a_1-a_2}\cosh(a_1)\cosh(a_2)},$$

yields the above rewriting for $\varphi\tilde{\star}_\nu^{RC}\psi$:

$$\begin{aligned}\left(\varphi\tilde{\star}_\nu^{RC}\psi\right)(a,\ell) = \\ \frac{1}{(2\pi i\nu)^2}\int_{\mathbb{R}^2}\frac{1}{\cosh(a_1)\cosh(a_2)\cosh(a_1-a_2)}e^{\frac{1}{\nu}\tanh(a_1-a_2)\ell} \\ \times\mathcal{F}\varphi\left(a+a_1+\log\left(\frac{\cosh(a_1-a_2)}{\cosh(a_2)}\right),-\frac{1}{i\nu}\tanh(a_2)\right) \\ \times\mathcal{F}\psi\left(a+a_2+\log\left(\frac{\cosh(a_1-a_2)}{\cosh(a_1)}\right),\frac{1}{i\nu}\tanh(a_1)\right)da_1da_2.\end{aligned}$$

By writing more explicitly the partial Fourier transforms (by means of integrals) in the above RHS, we can then notice that the integral representation of $\left(\varphi\tilde{\star}_\nu^{RC}\psi\right)(a,\ell)$ coincides exactly with the expression of $\left(\varphi\star_\nu^{RC}\psi\right)(a,\ell)$ given in the item (iii) of Theorem 3.4.4 (with $\vec{K}:=\vec{K}_0=(1,0)$). This leads to the following result.

Theorem 4.2.2. The restriction of the star-product $\tilde{\star}_\nu^{RC}$ to the space $\mathcal{E}_\nu^{RC}$ coincides with the star-product $\star_\nu^{RC}$. That is, for all $\varphi,\psi\in\mathcal{E}_\nu^{RC}$, the following holds[7]:

$$\varphi\tilde{\star}_\nu^{RC}\psi=\varphi\star_\nu^{RC}\psi.$$

The space $\left(\mathcal{E}_\nu^{RC},\star_\nu^{RC}\right)=\left(\mathcal{E}_\nu^{RC},\tilde{\star}_\nu^{RC}\right)$ is then a subalgebra of $\left(L^2(\mathbb{S}),\tilde{\star}_\nu^{RC}\right)$.

Proof. Thanks to Corollary 3.4.15, we know that the space $\underline{\mathcal{S}}^{1/|i\nu|}$ is dense in $\mathcal{E}_\nu^{RC}$. Hence, for $\varphi=\lim_n\varphi_n,\psi=\lim_m\psi_m\in\mathcal{E}_\nu^{RC}$, with $\varphi_n,\psi_m\in\underline{\mathcal{S}}^{1/|i\nu|}$ for all n,m, the continuity of the star-products $\star_\nu^{RC}$ and $\tilde{\star}_\nu^{RC}$ (see respectively Proposition 3.4.17 and Proposition 4.1.5) along with the previous computation lead to:

$$\begin{aligned}\varphi\star_\nu^{RC}\psi&=\left(\lim_n\varphi_n\right)\star_\nu^{RC}\left(\lim_m\psi_m\right)\\&=\lim_{n,m}\left(\varphi_n\star_\nu^{RC}\psi_m\right)\\&=\lim_{n,m}\left(\varphi_n\tilde{\star}_\nu^{RC}\psi_m\right)\\&=\left(\lim_n\varphi_n\right)\tilde{\star}_\nu^{RC}\left(\lim_m\psi_m\right)\\&=\varphi\tilde{\star}_\nu^{RC}\psi.\end{aligned}$$

For $\varphi,\psi\in\mathcal{E}_\nu^{RC}$, the product $\varphi\tilde{\star}_\nu^{RC}\psi$ then belongs to $\mathcal{E}_\nu^{RC}$. This completes the proof. ■

[7]Recall that $\star_\nu^{RC}$ and $\tilde{\star}_\nu^{RC}$ are given respectively by the item (iii) of Theorem 3.4.4 and the formula (4.2).

This means that $\tilde{\star}_\nu^{RC}$ defines an extension of the non-formal star-product $\star_\nu^{RC}$. Nonetheless, we note an important point. Recall that in chapter 3, we have computed all the intertwiners between the (strongly closed version of the) non-formal BG star-product and a $\mathfrak{g}_{RC}$-invariant star-product on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$ (namely, the RC star-product). We can see, therefore, that it is possible to intertwine non-formally $\underline{\star}_\nu^{BG}$ and $\tilde{\star}_\nu^{RC}$ only at the level of the subalgebra $\mathcal{E}_\nu^{RC}$ of $L^2(\mathbb{S})$ (the latter being the algebra associated with $\tilde{\star}_\nu^{RC}$). Hence, in addition to providing a link between two non-formal star-products on $\left(\mathbb{S}, \omega^{\mathbb{S}}\right)$, the work done in chapter 3 have yielded an unexpected subalgebra of $L^2(\mathbb{S})$, stable for the $\tilde{\star}_\nu^{RC}$-product.

Denote now as $\left(\mathcal{H}_\nu^{RC}\right)^\perp$ the orthogonal complement of $\mathcal{H}_\nu^{RC}$ in the space $L^2(\mathbb{S})$. Note that this space contains all the Schwartz functions (i.e. functions in the space $\mathcal{S}(\mathbb{S})$) admitting a partial Fourier transform with a support contained outside the strip $\mathbb{R} \times \left[-\frac{1}{|i\nu|}, \frac{1}{|i\nu|}\right]$ (w.r.t. the identification $\tilde{\mathbb{S}} \simeq \mathbb{S} \simeq \mathbb{R}^2$ by means of the coordinates (1.24)). Indeed, for such a function ψ and for $\varphi \in \underline{\mathcal{S}}^{1/|i\nu|} \subset \mathcal{H}_\nu^{RC}$, the formula (2.35) implies

$$\langle \psi, \varphi \rangle_{L^2(\mathbb{S})} = \frac{1}{2\pi} \langle \mathcal{F}\psi, \mathcal{F}\varphi \rangle_{L^2(\mathbb{S})} = 0.$$

We end this section with the following proposition.

Proposition 4.2.3. The space $\left(\mathcal{H}_\nu^{RC}\right)^\perp$ is a right $\mathcal{H}_\nu^{RC}$-module w.r.t. $\tilde{\star}_\nu^{RC}$, i.e. we have the following map:

$$\begin{aligned} \left(\mathcal{H}_\nu^{RC}\right)^\perp \times \mathcal{H}_\nu^{RC} &\to \left(\mathcal{H}_\nu^{RC}\right)^\perp \\ (\psi, \varphi) &\mapsto \psi \tilde{\star}_\nu^{RC} \varphi. \end{aligned}$$

Proof. We consider the trace form $\widetilde{\mathrm{Tr}}^{RC}$ given in Proposition 4.1.9 (i.e. $\widetilde{\mathrm{Tr}}^{RC}(f) = \int_{\mathbb{S}} f$). Extending this trace to the space $L^1(\mathbb{S})$, Proposition 4.1.9 implies that for $f, g \in L^2(\mathbb{S})$, $\widetilde{\mathrm{Tr}}^{RC}(f \tilde{\star}_\nu^{RC} \bar{g}) = \langle f, g \rangle_{L^2(\mathbb{S})}$. Hence, for $\psi \in \left(\mathcal{H}_\nu^{RC}\right)^\perp$ and $\varphi_1, \varphi_2 \in \mathcal{H}_\nu^{RC}$, we have

$$\begin{aligned} \langle \psi \tilde{\star}_\nu^{RC} \varphi_1, \overline{\varphi_2} \rangle_{L^2(\mathbb{S})} &= \widetilde{\mathrm{Tr}}^{RC}\left((\psi \tilde{\star}_\nu^{RC} \varphi_1) \tilde{\star}_\nu^{RC} \varphi_2 \right) \\ &= \widetilde{\mathrm{Tr}}^{RC}\left(\psi \tilde{\star}_\nu^{RC} (\varphi_1 \tilde{\star}_\nu^{RC} \varphi_2) \right) \\ &= \langle \psi, \overline{\varphi_1 \tilde{\star}_\nu^{RC} \varphi_2} \rangle_{L^2(\mathbb{S})}. \end{aligned}$$

Using Theorem 4.2.2, we know that $\varphi_1 \tilde{\star}_\nu^{RC} \varphi_2 = \varphi_1 \star_\nu^{RC} \varphi_2$. So this element belongs to $\mathcal{H}_\nu^{RC}$. Since this space is stable under the complex involution[8], we get $\langle \psi, \overline{\varphi_1 \tilde{\star}_\nu^{RC} \varphi_2} \rangle_{L^2(\mathbb{S})} = 0$, which implies that the element $\psi \tilde{\star}_\nu^{RC} \varphi_1$ belongs to the space $\left(\mathcal{H}_\nu^{RC}\right)^\perp$. ■

[8] See Proposition 3.4.19.

Bibliography

[1] A. Alekseev and A. Lachowska. Invariant $\star$-products on coadjoint orbits and the Shapovalov pairing. *Commentarii Mathematici Helvetici*, 80:795–810, 2005.

[2] N. B. Andersen. Real Paley-Wiener theorems for the inverse Fourier transform on a Riemannian symmetric space. *Pacific Journal of Mathematics*, 213(1):1–14, 2004.

[3] N. B. Andersen and M. de Jeu. Real Paley-Wiener theorems and local spectral radius formulas. *Transactions of the American Mathematical Society*, 362(7):3613–3640, 2010.

[4] D. Arnal and J. C. Cortet. Représentations $\star$ des groupes exponentiels. *Journal of Functional Analysis*, 92:103–135, 1990.

[5] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Quantum mechanics as a deformation of classical mechanics. *Letters in Mathematical Physics*, 1(6):521–530, 1977.

[6] N. Berline and M. Vergne. Hamiltonian manifolds and moment map. Lecture notes, July 2011.

[7] M. Bertelson. Paramétrisation des classes d'équivalence de star-produit. Master's thesis, Université libre de Bruxelles, 1995.

[8] M. Bertelson, P. Bieliavsky, and S. Gutt. Parametrizing equivalence classes of invariant star products. *Letters in Mathematical Physics*, (46):339–345, 1998.

[9] M. Bertelson, M. Cahen, and S. Gutt. Equivalence of star products. *Classical and Quantum Gravity*, 14(1A):A93, 1997.

[10] P. Bieliavsky. *Espaces symétriques symplectiques*. PhD thesis, Université libre de Bruxelles, 1995.

[11] P. Bieliavsky. Strict quantization of solvable symmetric spaces. *Journal of Symplectic Geometry*, 1:269–320, 2002.

[12] P. Bieliavsky. Deformation quantization for actions of the affine group. *arXiv:0709.1110v1 [math.QA]*, 2007.

[13] P. Bieliavsky. Non-formal deformation quantizations of solvable Ricci-type symplectic symmetric spaces. *Journal of Physics: Conference Series*, 103(1), 2008.

[14] P. Bieliavsky and Ph. Bonneau. On the geometry of the characteristic class of a star product on a symplectic manifold. *Reviews in Mathematical Physics*, 15(2):199–215, 2003.

[15] P. Bieliavsky, Ph. Bonneau, F. D'Andrea, and V. Gayral. Quantum Kählerian Lie groups from multiplicative unitaries. *arXiv:1705.08326v2 [math.OA]*, 2017.

[16] P. Bieliavsky, Ph. Bonneau, and Y. Maeda. Universal deformation formulae, symplectic Lie groups and symmetric spaces. *Pacific Journal of Mathematics*, 230(2):41–57, 2007.

[17] P. Bieliavsky, M. Bordemann, S. Gutt, and S. Waldmann. Traces for star products on the dual of a Lie algebra. *arXiv:0202126v2 [math.QA]*, 2002.

[18] P. Bieliavsky, A. de Goursac, Y. Maeda, and F. Spinnler. Non-formal star-exponential on contracted one-shetted hyperboloids. *arXiv:1501.07491v1 [math.QA]*, 2015.

[19] P. Bieliavsky, S. Detournay, and Ph. Spindel. The deformation quantizations of the hyperbolic plane. *Communications in Mathematical Physics*, 289(2):529–559, 2009.

[20] P. Bieliavsky and V. Gayral. *Deformation Quantization for Actions of Kählerian Lie groups*, volume 236. Memoirs of the American Mathematical Society, 2015.

[21] P. Bieliavsky, V. Gayral, A. de Goursac, and F. Spinnler. *Developments and Retrospectives in Lie Theory: Geometric and Analytic Methods*, chapter "Harmonic Analysis on Homogeneous Complex Bounded Domains and Noncommutative Geometry", pages 41–76. Springer International Publishing, 2014.

[22] P. Bieliavsky and Y. Maeda. Convergent star product algebras on "ax + b". *arXiv:0209295v1 [math.QA]*, 2002.

[23] P. Bieliavsky and M. Massar. Strict deformation quantization for actions of a class of symplectic Lie groups. *arXiv:math/0011144v3 [math.QA]*, 2001.

[24] P. Bieliavsky and M. Massar. Oscillatory integral formulae for left-invariant star products on a class of Lie groups. *Letters in Mathematical Physics*, 58(2):115–128, 2011.

[25] P. Bieliavsky, X. Tang, and Y. Yao. Rankin-Cohen brackets and formal quantization. *arXiv:0506506v4 [math.QA]*, 2008.

[26] M. Bordemann. Quantification par déformation. *African Journal of Mathematical Physics*, 2(2):21–61, 2005.

[27] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier. *The 1-2-3 of Modular Forms*, chapter "Elliptic Modular Forms and Their Applications", pages 1–103. Springer, 2008.

[28] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Transactions of the American Mathematical Society*, 63(1):85–124, 1948.

[29] P. Cohen, Y. Manin, and D. Zagier. *Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman*, chapter "Automorphic Pseudodifferential Operators", pages 17–47. Birkhäuser Boston, Boston, MA, 1997.

[30] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Communications in Mathematical Physics*, 198(1):199–246, 1998.

[31] A. Connes and H. Moscovici. Modular Hecke algebras and their Hopf symmetry. *Moscow Mathematical Journal*, 4(1):67–109, 2004.

[32] A. Connes and H. Moscovici. Rankin-Cohen brackets and the Hopf algebra of transverse geometry. *Moscow Mathematical Journal*, 4(1):111–130, 2004.

[33] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10. Berkeley Mathematics Lecture Notes, 1999.

[34] B.V. Fedosov. A simple geometrical construction of deformation quantization. *Journal of Differential Geometry*, 40(2):213–238, 1994.

[35] I.M. Gel'fand and G.E. Shilov. *Generalized functions*, volume 2 : Spaces of fundamental and generalized functions. Academic Press Inc., 1968.

[36] A. Giaquinto and J. Zhang. Bialgebra actions, twists, and universal deformation formulas. *Journal of Pure and Applied Algebra*, 128:133–151, 1998.

[37] S. Gutt. *Déformations formelles de l'algèbre des fonctions différentiables sur une variété symplectique*. PhD thesis, Université libre de Bruxelles, 1983.

[38] S. Gutt. Deformation quantization: an introduction. Lecture notes, 2005.

[39] F. Hansen. Quantum mechanics in phase space. *Reports on Mathematical Physics*, 19(3):361–381, 1984.

[40] G. H. Hardy. A theorem concerning Fourier transforms. *Journal of the London Mathematical Society*, s1-8(3):227–231, 1933.

[41] S. Korvers. *Quantifications par déformations formelles et non formelles de la boule unité de* $\mathbb{C}^n$. PhD thesis, Université catholique de Louvain, 2014.

[42] J. M. Maillard. On the twisted convolution product and the Weyl transformation of tempered distributions. *Journal of Geometry and Physics*, 3:231–261, 1986.

[43] R. Nest and B. Tsygan. Algebraic index theorem for families. *Advances in Mathematics*, 113(2):151–205, 1995.

[44] S. Norvidas. Approximation of entire functions of exponential type by exponential sums. *Lithuanian Mathematical Journal*, 51(4):522–532, 2011.

[45] M. Pevzner. Rankin-Cohen brackets and associativity. *Letters in Mathematical Physics*, 85(2):195–202, 2008.

[46] M. Pevzner. Rankin-Cohen brackets and representations of conformal Lie groups. *Annales mathématiques Blaise Pascal*, 19(2):455–484, 2012.

[47] R.A. Rankin. The construction of automorphic forms from the derivatives of a given form. *The Michigan Mathematical Journal*, 4(2):181–186, 1957.

[48] L. Schwartz. *Théorie des distributions.* Hermann, 1997.

[49] F. Trèves. *Topological vector spaces, distributions and kernels.* Academic Press New York, 1967.

[50] A. Unterberger and J. Unterberger. Algebras of symbols and modular forms. *Journal d'Analyse Mathématique*, 68(1):121–143, 1996.

[51] G. van Dijk and M. Pevzner. Ring structures for holomorphic discrete series and Rankin-Cohen brackets. *Journal of Lie Theory*, 17:283–305, 2007.

[52] S. Waldmann. *Quantum Mathematical Physics: A Bridge between Mathematics and Physics*, chapter "Recent Developments in Deformation Quantization", pages 421–439. Springer International Publishing, 2016.

[53] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Springer-Verlag New York, 1983.

[54] A. Weinstein. Traces and triangles in symmetric symplectic spaces. *Contemporary Mathematics*, 179:261–270, 1994.

[55] M. Willem. *Functional Analysis.* Springer New York, 2013.

[56] P. Xu. Fedosov $\star$-products and quantum momentum maps. *Communications in Mathematical Physics*, 197:167–197, 1998.

[57] Y. Yao. *Autour des déformations de Rankin-Cohen.* PhD thesis, École Polytechnique, 2007.

[58] D. Zagier. Modular forms and differential operators. *Proceedings Mathematical Sciences*, 104(1):57–75, 1994.