BOUSSINESQ'S EQUATION FOR WATER WAVES: ASYMPTOTICS IN SECTOR V

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ABSTRACT. We consider the Boussinesq equation on the line for a broad class of Schwartz initial data for which (i) no solitons are present, (ii) the spectral functions have generic behavior near ± 1, and (iii) the solution exists globally. In a recent work, we identified ten main sectors describing the asymptotic behavior of the solution, and for each of these sectors we gave an exact expression for the leading asymptotic term. In this paper, we give a proof for the formula corresponding to the sector $\frac{x}{t} \in (0, \frac{1}{\sqrt{3}})$.

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1. INTRODUCTION

The Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}, \tag{1.1}$$

where $u(x,t)$ is a real-valued function and subscripts denote partial derivatives, is an approximate model for water waves introduced by J. Boussinesq in the 1870's [5]. This nonlinear equation models small-amplitude dispersive waves in shallow water that can propagate in both the right and left directions [23]. The Boussinesq equation is ill-posed, and is therefore often referred to as the "bad" Boussinesq equation. Partly due to this ill-posedness, the mathematical literature on (1.1) is limited: we refer to [1, 3, 20] for results on soliton solutions, to [32] for a Lax pair, and to [24, 31] for nonexistence results of global solutions for certain initial-boundary value problems. Until recently [9], the problem of obtaining long-time asymptotics for the solution of (1.1) was still open, see e.g. Deift's list of open problems in [13].

In [9], we developed an inverse scattering approach to the initial value problem for (1.1). Among other things, we proved in [9] that for a large class of initial data, the solution is unique and exists globally. The asymptotic behavior of such solutions is described by ten main asymptotic sectors, and for each of these sectors the leading asymptotic term was given in [9, Theorem 2.14]. For reasons of length, the proofs of some of these asymptotic formulas were omitted in [9]. The purpose of this paper is to give the proof for the formula corresponding to the sector $\frac{x}{t} \in (0, \frac{1}{\sqrt{3}})$ (this sector was denoted "sector V" in [9]). The proof is based on a Deift–Zhou [15] steepest descent analysis of an associated Riemann–Hilbert (RH) problem. A brief overview of the main steps in the derivation can be found in Section 3; here we merely mention that one novelty of the proof is that only certain of the jump matrices are initially factorized, with the relevant factors paving the way for

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the subsequent transformations of the Riemann-Hilbert problem, while the other jumps are only adjusted according to these transformations and are then handled via estimates.

Results on long-time asymptotics for other integrable equations can be found in e.g. [2, 4, 6–8, 12, 14–19, 21, 25, 27–30].

2. Main result

We consider real-valued initial data in the Schwartz class,

$$u_0(x) := u(x,0) \in \mathcal{S}(\mathbb{R}), \qquad u_1(x) := u_t(x,0) \in \mathcal{S}(\mathbb{R}). \tag{2.1}$$

We further assume that $\int_{\mathbb{R}} u_1(x)dx = 0$. As noted in [9], this assumption is natural because it ensures that the total mass $\int_{\mathbb{R}} u(x,t)dx$ is conserved in time. The approach of [9] provides a representation for the solution to the initial value problem in terms of the solution n of a row-vector RH problem. This RH problem depends on two scalar reflection coefficients, $r_1(k)$ and $r_2(k)$, which are defined as follows (see [9] for details).

2.1. **Definition of r_1 and r_2.** Let $\omega := e^{\frac{2\pi i}{3}}$ and define $\{l_j(k), z_j(k)\}_{j=1}^3$ by

$$l_j(k) = i\frac{\omega^j k + (\omega^j k)^{-1}}{2\sqrt{3}}, \qquad z_j(k) = i\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{4\sqrt{3}}, \qquad k \in \mathbb{C} \setminus \{0\}. \tag{2.2}$$

Let $P(k)$ and $\mathsf{U}(x,k)$ be given by

$$P(k) = \begin{pmatrix} 1 & 1 & 1 \\ l_1(k) & l_2(k) & l_3(k) \\ l_1(k)^2 & l_2(k)^2 & l_3(k)^2 \end{pmatrix}, \quad \mathsf{U}(x,k) = P(k)^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{u_{0x}}{4} - \frac{iv_0}{4\sqrt{3}} & -\frac{u_0}{2} & 0 \end{pmatrix} P(k),$$

where $v_0(x) = \int_{-\infty}^x u_1(x')dx'$. Define $X(x,k), X^A(x,k)$ as the unique solutions to the Volterra integral equations

$$X(x,k) = I - \int_x^\infty e^{(x-x')\mathcal{L}(k)}(\mathsf{U}X)(x',k)e^{-(x-x')\mathcal{L}(k)}dx',$$
$$X^A(x,k) = I + \int_x^\infty e^{-(x-x')\mathcal{L}(k)}(\mathsf{U}^T X^A)(x',k)e^{(x-x')\mathcal{L}(k)}dx',$$

where $(\cdot)^T$ is the transpose operation and $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$. Define $s(k)$ and $s^A(k)$ by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\mathcal{L}(k)}(\mathsf{U}X)(x,k)e^{x\mathcal{L}(k)}dx, \quad s^A(k) = I + \int_{\mathbb{R}} e^{x\mathcal{L}(k)}(\mathsf{U}^T X^A)(x,k)e^{-x\mathcal{L}(k)}dx.$$

The two spectral functions $\{r_j(k)\}_1^2$ are defined by

$$\begin{cases} r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, & k \in \hat{\Gamma}_1 \setminus \hat{\mathcal{Q}}, \\ r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, & k \in \hat{\Gamma}_4 \setminus \hat{\mathcal{Q}}, \end{cases} \tag{2.3}$$

where the contours $\hat{\Gamma}_j = \Gamma_j \cup \partial\mathbb{D}$ and the set $\hat{\mathcal{Q}}$ are defined in Section 2.4.

2.2. **Assumptions.** As in [9, Theorem 2.14], our results will be valid under the following assumptions:

- (i) There are no solitons: we suppose that $s(k)_{11}$ is nonzero for $k \in (\bar{D}_2 \cup \partial\mathbb{D}) \setminus \hat{\mathcal{Q}}$, where D_2 is the open set shown in Figure 1 and $\partial\mathbb{D}$ is the unit circle.
- (ii) The spectral functions s and s^A have generic behavior near $k = 1$ and $k = -1$: we suppose for $k_\star = 1$ and $k_\star = -1$ that

$$\lim_{k\to k_\star}(k-k_\star)s(k)_{11} \neq 0, \quad \lim_{k\to k_\star}(k-k_\star)s(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s(k)_{33} \neq 0,$$
$$\lim_{k\to k_\star}(k-k_\star)s^A(k)_{11} \neq 0, \quad \lim_{k\to k_\star}(k-k_\star)s^A(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{33} \neq 0.$$

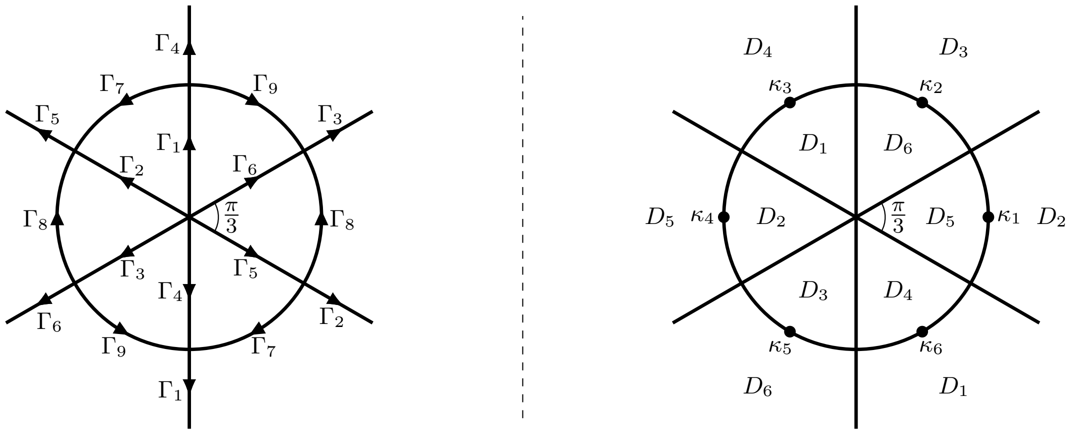

Figure 1. *The contour* $\Gamma = \cup_{j=1}^{9} \Gamma_j$ *in the complex* k*-plane (left) and the open sets* D_n*,* $n = 1, \dots, 6$*, together with the sixth roots of unity* κ_j*,* $j = 1, \dots, 6$ *(right).*

(*iii*) There exists a global solution to the initial value problem (1.1)–(2.1): we suppose that $r_1(k) = 0$ for all $k \in [0, i]$, where $[0, i]$ is the vertical segment from 0 to i.

We emphasize that Assumption (*ii*) is generic. Let us also comment on Assumptions (*i*) and (*iii*):

- In [10], the long-time asymptotics for the solution to the initial value problem (1.1)–(2.1) is obtained in the sector $\frac{x}{t} \in (\frac{1}{\sqrt{3}}, 1)$ in the case when solitons are present. The solitonless assumption (*i*) is made here for simplicity; we believe that the case when solitons are present can be handled using similar ideas as in [10].
- The Boussinesq equation is ill-posed and admits solutions that blow up at any given positive time [9]. Assumption (*iii*) ensures that the solution belongs to the physically relevant class of global solutions. It follows from [9, Theorems 2.6 and 2.12] that this class is large. More precisely, associated to any functions $\mathsf{r}_1, \mathsf{r}_2$ satisfying Assumption (*iii*) and the conditions stated in [9, Theorem 2.3], there exist Schwartz class initial data u_0, v_0 such that $r_j(k; u_0, v_0) = \mathsf{r}_j(k)$, $j = 1, 2$, and such that the initial value problem (1.1)–(2.1) with $u_1 := \partial_x v_0$ admits a global Schwartz class solution.

2.3. **Statement of main result.** We now introduce the necessary material to present our main result. For easy comparison, we use the same notation as in [9]. Let $\zeta := x/t$ and assume that $\zeta \in (0, \frac{1}{\sqrt{3}})$. For $k \in \mathbb{C} \setminus \{0\}$, define $\Phi_{ij}(\zeta, k)$ for $1 \leq j < i \leq 3$ by

$$\Phi_{ij}(\zeta, k) = (l_i(k) - l_j(k))\zeta + (z_i(k) - z_j(k)). \tag{2.4}$$

The function $k \mapsto \Phi_{21}(\zeta, k)$ has four saddle points $\{k_j = k_j(\zeta)\}_{j=1}^4$ given by

$$k_2 = \frac{1}{4}\left(\zeta - \sqrt{8 + \zeta^2} - i\sqrt{2}\sqrt{4 - \zeta^2 + \zeta\sqrt{8 + \zeta^2}}\right), \tag{2.5a}$$

$$k_4 = \frac{1}{4}\left(\zeta + \sqrt{8 + \zeta^2} - i\sqrt{2}\sqrt{4 - \zeta^2 - \zeta\sqrt{8 + \zeta^2}}\right), \tag{2.5b}$$

$k_1 = \overline{k_2}$, and $k_3 = \overline{k_4}$. Moreover, $|k_2| = |k_4| = 1$, $\arg k_2 \in (-\frac{3\pi}{4}, -\frac{2\pi}{3})$, and $\arg k_4 \in (-\frac{\pi}{4}, -\frac{\pi}{6})$. Define also $\delta_j(\zeta, k)$, $j = 1, \dots, 5$, by

$$\delta_1(\zeta, k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega k_4}^{i} \frac{\ln(1 + r_1(\omega^2 s) r_2(\omega^2 s))}{s - k} ds\right\},$$

$$\delta_2(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_i^{\omega^2 k_2} \frac{\ln(1+r_1(s)r_2(s))}{s-k}ds\right\}, \quad \delta_3(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_i^{\omega^2 k_2} \frac{\ln f(s)}{s-k}ds\right\},$$
$$\delta_4(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega^2 k_2}^{\omega} \frac{\ln f(s)}{s-k}ds\right\}, \quad \delta_5(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega^2 k_2}^{\omega} \frac{\ln f(\omega^2 s)}{s-k}ds\right\}, \tag{2.6}$$

where the paths follow the unit circle in the counterclockwise direction, the principal branch is used for the logarithms, and

$$f(k) := 1 + r_1(k)r_2(k) + r_1(\tfrac{1}{\omega^2 k})r_2(\tfrac{1}{\omega^2 k}), \qquad k \in \partial\mathbb{D}. \tag{2.7}$$

Various properties of r_1, r_2, and f have been established in [9, Theorem 2.3 and Lemma 2.13]. In particular, r_1, r_2, f are well-defined on $\partial\mathbb{D}\setminus\hat{\mathcal{Q}}$ and the arguments of all logarithms appearing in the above definitions of $\{\delta_j(\zeta,k)\}_{j=1}^5$ are > 0. Define

$$\begin{aligned}
\mathcal{D}_1(k) = {}& \frac{\delta_1(\omega^2 k)\delta_1(\frac{1}{\omega k})^2\delta_1(\omega k)}{\delta_1(\frac{1}{k})\delta_1(\frac{1}{\omega^2 k})} \frac{\delta_2(k)\delta_2(\omega^2 k)\delta_2(\frac{1}{k})^2}{\delta_2(\omega k)^2\delta_2(\frac{1}{\omega^2 k})\delta_2(\frac{1}{\omega k})} \frac{\delta_3(\omega k)\delta_3(\omega^2 k)\delta_3(\frac{1}{\omega k})^2}{\delta_3(k)^2\delta_3(\frac{1}{k})\delta_3(\frac{1}{\omega^2 k})} \\
& \times \frac{\delta_4(\omega^2 k)^2\delta_4(\frac{1}{k})\delta_4(\frac{1}{\omega k})}{\delta_4(k)\delta_4(\omega k)\delta_4(\frac{1}{\omega^2 k})^2} \frac{\delta_5(\omega k)^2\delta_5(\frac{1}{\omega k})\delta_5(\frac{1}{\omega^2 k})}{\delta_5(k)\delta_5(\frac{1}{k})^2\delta_5(\omega^2 k)}, \\
\mathcal{D}_2(k) = {}& \frac{\delta_1(\omega^2 k)^2\delta_1(\frac{1}{\omega k})\delta_1(\frac{1}{\omega^2 k})}{\delta_1(k)\delta_1(\frac{1}{k})^2\delta_1(\omega k)} \frac{\delta_2(\frac{1}{k})\delta_2(\frac{1}{\omega k})}{\delta_2(\omega k)\delta_2(\frac{1}{\omega^2 k})^2\delta_2(\omega^2 k)} \frac{\delta_3(\omega^2 k)^2\delta_3(\frac{1}{\omega k})\delta_3(\frac{1}{\omega^2 k})}{\delta_3(\frac{1}{k})^2\delta_3(\omega k)} \\
& \times \frac{\delta_4(\omega^2 k)\delta_4(\frac{1}{\omega k})^2}{\delta_4(\omega k)^2\delta_4(\frac{1}{\omega^2 k})\delta_4(\frac{1}{k})} \frac{\delta_5(\omega k)\delta_5(\frac{1}{\omega^2 k})^2\delta_5(\omega^2 k)}{\delta_5(\frac{1}{k})\delta_5(\frac{1}{\omega k})}.
\end{aligned}$$

Define $\chi_j(\zeta,k)$, $j=1,2,3$, and $\tilde{\chi}_j(\zeta,k)$, $j=4,5$, by

$$\begin{aligned}
\chi_1(\zeta,k) &= \frac{1}{2\pi i}\int_{\omega k_4}^{i} \ln_s(k-s)\, d\ln(1+r_1(\omega^2 s)r_2(\omega^2 s)), \\
\chi_2(\zeta,k) &= \frac{1}{2\pi i}\int_{i}^{\omega^2 k_2} \ln_s(k-s)\, d\ln(1+r_1(s)r_2(s)), \\
\chi_3(\zeta,k) &= \frac{1}{2\pi i}\int_{i}^{\omega^2 k_2} \ln_s(k-s)\, d\ln(f(s)), \\
\tilde{\chi}_4(\zeta,k) &= \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \tilde{\ln}_s(k-s)\, d\ln(f(s)) \\
&:= \frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \tilde{\ln}_s(k-s)\, d\ln(f(s)) - \tilde{\ln}_\omega(k-\omega)\ln(f(e^{i(\frac{2\pi}{3}-\epsilon)}))\bigg), \\
\tilde{\chi}_5(\zeta,k) &= \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \tilde{\ln}_s(k-s)\, d\ln(f(\omega^2 s)) \\
&:= \frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \tilde{\ln}_s(k-s)\, d\ln(f(\omega^2 s)) - \tilde{\ln}_\omega(k-\omega)\ln(f(e^{-i\epsilon}))\bigg),
\end{aligned} \tag{2.8}$$

where the paths follow the unit circle in the counterclockwise direction. For $s \in \{e^{i\theta}\,|\,\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]\}$, $k \mapsto \ln_s(k-s) = \ln|k-s| + i\arg_s(k-s)$ has a cut along $\{e^{i\theta}\,|\,\theta\in[\frac{\pi}{2}, \arg s]\}\cup(i, i\infty)$ if $\arg s > \frac{\pi}{2}$, and a cut along $\{e^{i\theta}\,|\,\theta \in [\arg s, \frac{\pi}{2}]\}\cup(i,i\infty)$ if $\arg s < \frac{\pi}{2}$, and the branch is such that $\arg_s(1) = 2\pi$. Also, for $s \in \{e^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, $k \mapsto \tilde{\ln}_s(k-s) := \ln|k-s| + i\tilde{\arg}_s(k-s)$ has a cut along $\{e^{i\theta} : \theta \in [\arg s, \pi]\} \cup (-\infty, 0)$, and satisfies $\tilde{\arg}_s(1) = 0$. In the definitions of $\tilde{\chi}_4, \tilde{\chi}_5$, a regularized integral is needed because $f(1) = f(\omega) = 0$ (see [9, Lemma 2.13

(i)]). Let the functions $\{\nu_j\}_{j=1}^4$ and $\hat{\nu}_2$ be given by

$$\nu_1(k) = -\frac{1}{2\pi}\ln(1 + r_1(\omega k)r_2(\omega k)), \qquad \nu_2(k) = -\frac{1}{2\pi}\ln(1 + r_1(\omega^2 k)r_2(\omega^2 k)),$$
$$\nu_3(k) = -\frac{1}{2\pi}\ln(f(\omega k)), \qquad \nu_4(k) = -\frac{1}{2\pi}\ln(f(\omega^2 k)), \qquad \hat{\nu}_2(k) = \nu_2(k) + \nu_3(k) - \nu_4(k).$$

The following functions $\tilde{d}_{1,0}$ and $\tilde{d}_{2,0}$ appear in our final result:

$$\tilde{d}_{1,0}(\zeta, t) = e^{-4\pi\nu_1(\omega^2 k_4)} e^{2\chi_1(\zeta,\omega k_4)} e^{-2i\nu_3(\omega^2 i)\ln_i(\omega k_4 - i)} t^{-i\nu_1(\omega^2 k_4)} z_{1,\star}^{-2i\nu_1(\omega^2 k_4)} \mathcal{D}_1(\omega k_4), \tag{2.9}$$
$$\begin{aligned}\tilde{d}_{2,0}(\zeta, t) = {} & e^{-2\chi_2(\zeta,\omega^2 k_2)+\chi_3(\zeta,\omega^2 k_2)-\tilde{\chi}_4(\zeta,\omega^2 k_2)+2\tilde{\chi}_5(\zeta,\omega^2 k_2)} \\ & \times e^{i\nu_3(\omega^2 i)\ln_i(\omega^2 k_2 - i)} t^{i(\nu_4(k_2)-\nu_3(k_2)-\nu_2(k_2))} z_{2,\star}^{2i(\nu_4(k_2)-\nu_3(k_2)-\nu_2(k_2))} \mathcal{D}_2(\omega^2 k_2),\end{aligned} \tag{2.10}$$

where $z_{1,\star}$, $z_{2,\star}$ are given by

$$z_{1,\star} = z_{1,\star}(\zeta) := \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\omega \frac{4 - 3k_4\zeta - k_4^3\zeta}{4k_4^4}}, \qquad -i\omega k_4 z_{1,\star} > 0,$$
$$z_{2,\star} = z_{2,\star}(\zeta) := \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{-\omega^2 \frac{4 - 3k_2\zeta - k_2^3\zeta}{4k_2^4}}, \qquad -i\omega^2 k_2 z_{2,\star} > 0,$$

and the branches for the complex powers $z_{j,\star}^\alpha = e^{\alpha \ln z_{j,\star}}$ are fixed by

$$\ln z_{1,\star} = \ln|z_{1,\star}| + i\arg z_{1,\star} = \ln|z_{1,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega k_4)\big), \qquad \arg(\omega k_4) \in (\tfrac{\pi}{3}, \tfrac{\pi}{2}),$$
$$\ln z_{2,\star} = \ln|z_{2,\star}| + i\arg z_{2,\star} = \ln|z_{2,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega^2 k_2)\big), \qquad \arg(\omega^2 k_2) \in (\tfrac{\pi}{2}, \tfrac{2\pi}{3}).$$

Finally, define also

$$\tilde{q}_1 = |\tilde{r}(k_4)|^{\frac{1}{2}} r_1(k_4), \quad q_2 = \tilde{r}(\omega^2 k_2)^{\frac{1}{2}} r_1(\omega^2 k_2), \quad q_5 = |\tilde{r}(\omega k_2)|^{\frac{1}{2}} r_1(\omega k_2), \quad q_6 = |\tilde{r}(\tfrac{1}{k_2})|^{\frac{1}{2}} r_1(\tfrac{1}{k_2}),$$

where

$$\tilde{r}(k) := \frac{\omega^2 - k^2}{1 - \omega^2 k^2}, \qquad k \in \mathbb{C} \setminus \{\omega^2, -\omega^2\}. \tag{2.11}$$

We now state our main result, which establishes the long-time behavior of $u(x,t)$ in the sector $\zeta \in (0, 1/\sqrt{3})$. The statement involves the Gamma function $\Gamma(k)$, as well as the square roots of $\hat{\nu}_2(k_2)$ and $\nu_1(\omega^2 k_4)$. These square roots are well-defined and ≥ 0 thanks to the inequalities $\hat{\nu}_2(k_2) \geq 0$ and $\nu_1(\omega^2 k_4) \geq 0$ established in Lemma 6.3 below.

Theorem 2.1. *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ be real-valued and such that $\int_{\mathbb{R}} u_1 dx = 0$. Let $v_0(x) = \int_{-\infty}^x u_1(x')dx'$ and suppose that u_0, v_0 are such that Assumptions (i), (ii), and (iii) are fulfilled. Let $\mathcal{I}$ be a fixed compact subset of $(0, \frac{1}{\sqrt{3}})$. Then the global solution $u(x,t)$ of the initial value problem for (1.1) with initial data u_0, u_1 enjoys the following asymptotics as $t \to \infty$:*

$$u(x,t) = \frac{\tilde{A}_1(\zeta)}{\sqrt{t}} \cos\tilde{\alpha}_1(\zeta,t) + \frac{A_2(\zeta)}{\sqrt{t}} \cos\tilde{\alpha}_2(\zeta,t) + O\left(\frac{\ln t}{t}\right), \tag{2.12}$$

uniformly for $\zeta := \frac{x}{t} \in \mathcal{I}$, where

$$\tilde{A}_1(\zeta) := \frac{4\sqrt{3}\sqrt{\nu_1(\omega^2 k_4)}\,\mathrm{Im}\, k_4}{-i\omega k_4 z_{1,\star} |\tilde{r}(\frac{1}{k_4})|^{\frac{1}{2}}} \sin(\arg(\omega k_4)),$$
$$A_2(\zeta) := \frac{-4\sqrt{3}\sqrt{\hat{\nu}_2(k_2)}|\tilde{r}(\frac{1}{k_2})|^{\frac{1}{2}}\,\mathrm{Im}\, k_2}{-i\omega^2 k_2 z_{2,\star}} \sin(\arg(\omega^2 k_2)),$$
$$\tilde{\alpha}_1(\zeta,t) := \frac{3\pi}{4} - \arg\tilde{q}_1 + \arg\Gamma(i\nu_1(\omega^2 k_4)) + \arg\tilde{d}_{1,0} - t\,\mathrm{Im}\,\Phi_{31}(\zeta, \omega k_4),$$

$$\tilde{\alpha}_2(\zeta,t) := \frac{3\pi}{4} - \arg(q_6 - q_2 q_5) + \arg\Gamma(i\hat{\nu}_2(k_2)) + \arg\tilde{d}_{2,0} - t\,\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2 k_2).$$

2.4. **Notation.** We use the following notation throughout the paper.

- $C > 0$ and $c > 0$ denote generic constants that may change within a computation.
- $[A]_1$, $[A]_2$, and $[A]_3$ denote the first, second, and third columns of a 3×3 matrix A.
- If A is an $n \times m$ matrix, we define $|A| \geq 0$ by $|A|^2 = \Sigma_{i,j}|A_{ij}|^2$. For a piecewise smooth contour $\gamma \subset \mathbb{C}$ and $1 \leq p \leq \infty$, if $|A|$ belongs to $L^p(\gamma)$, we write $A \in L^p(\gamma)$ and define $\|A\|_{L^p(\gamma)} := \||A|\|_{L^p(\gamma)}$.
- $\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| < 1\}$ denotes the open unit disk and $\partial\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| = 1\}$ denotes the unit circle.
- $f^*(k) := \overline{f(\bar{k})}$ denotes the Schwarz conjugate of a function $f(k)$.
- $D_\epsilon(k)$ denotes the open disk of radius ϵ centered at a point $k \in \mathbb{C}$.
- $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of all smooth functions f on $\mathbb{R}$ such that f and all its derivatives have rapid decay as $x \to \pm\infty$.
- $\kappa_j = e^{\frac{\pi i(j-1)}{3}}$, $j = 1,\dots,6$, denote the sixth roots of unity, see Figure 1.
- We let $\mathcal{Q} := \{\kappa_j\}_{j=1}^6$ and $\hat{\mathcal{Q}} := \mathcal{Q} \cup \{0\}$.
- D_n, $n = 1,\dots,6$, denote the open subsets of the complex plane shown in Figure 1.
- $\Gamma = \cup_{j=1}^9 \Gamma_j$ denotes the contour shown and oriented as in Figure 1.
- $\hat{\Gamma}_j = \Gamma_j \cup \partial\mathbb{D}$ denotes the union of Γ_j and the unit circle.

3. The RH problem for n

The proof of Theorem 2.1 is based on a careful analysis of a RH problem derived in [9]. We now recall this RH problem, whose solution is denoted by n. Let $\theta_{ij}(x,t,k) = t\,\Phi_{ij}(\zeta,k)$. For $j = 1,\dots,6$, let us write $\Gamma_j = \Gamma_{j'} \cup \Gamma_{j''}$, where $\Gamma_{j'} = \Gamma_j \setminus \mathbb{D}$ and $\Gamma_{j''} := \Gamma_j \setminus \Gamma_{j'}$ with Γ_j as in Figure 1. Because $r_1(k) = 0$ for all $k \in [0,i]$ (by Assumption (iii)), the jump matrix $v(x,t,k)$ is given for $k \in \Gamma$ by

$$v_{1'} = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{1''} = \begin{pmatrix} 1 & 0 & 0 \\ r_1(\frac{1}{k})e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{2'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_2(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{2''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_2(\omega k)e^{\theta_{32}} & 1 \end{pmatrix}, \; v_{3'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_1(\omega^2 k)e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \; v_{3''} = \begin{pmatrix} 1 & 0 & r_1(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{4'} = \begin{pmatrix} 1 & -r_2(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{4''} = \begin{pmatrix} 1 & 0 & 0 \\ r_2(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{5'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{5''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_1(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \; v_{6'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_2(\frac{1}{\omega^2 k})e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \; v_{6''} = \begin{pmatrix} 1 & 0 & r_2(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7 = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & r_2(\omega^2 k)e^{-\theta_{31}} \\ -r_2(k)e^{\theta_{21}} & 1 + r_1(k)r_2(k) & \big(r_2(\frac{1}{\omega k}) - r_2(k)r_2(\omega^2 k)\big)e^{-\theta_{32}} \\ r_1(\omega^2 k)e^{\theta_{31}} & \big(r_1(\frac{1}{\omega k}) - r_1(k)r_1(\omega^2 k)\big)e^{\theta_{32}} & f(\omega^2 k) \end{pmatrix},$$

$$v_8 = \begin{pmatrix} f(k) & r_1(k)e^{-\theta_{21}} & \big(r_1(\frac{1}{\omega^2 k}) - r_1(k)r_1(\omega k)\big)e^{-\theta_{31}} \\ r_2(k)e^{\theta_{21}} & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ \big(r_2(\frac{1}{\omega^2 k}) - r_2(\omega k)r_2(k)\big)e^{\theta_{31}} & -r_2(\omega k)e^{\theta_{32}} & 1 + r_1(\omega k)r_2(\omega k) \end{pmatrix},$$

$$v_9 = \begin{pmatrix} 1 + r_1(\omega^2 k) r_2(\omega^2 k) & \big(r_2(\frac{1}{k}) - r_2(\omega k) r_2(\omega^2 k)\big) e^{-\theta_{21}} & -r_2(\omega^2 k) e^{-\theta_{31}} \\ \big(r_1(\frac{1}{k}) - r_1(\omega k) r_1(\omega^2 k)\big) e^{\theta_{21}} & f(\omega k) & r_1(\omega k) e^{-\theta_{32}} \\ -r_1(\omega^2 k) e^{\theta_{31}} & r_2(\omega k) e^{\theta_{32}} & 1 \end{pmatrix}, \tag{3.1}$$

where $v_j, v_{j'}, v_{j''}$ denote the restrictions of v to Γ_j, $\Gamma_{j'}$, and $\Gamma_{j''}$, respectively. Let $\Gamma_\star = \{i\kappa_j\}_{j=1}^6 \cup \{0\}$ denote the set of intersection points of Γ, and define

$$\mathcal{A} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The RH problem for n is as follows.

RH problem 3.1 (RH problem for n)**.** *Find a 1×3-row-vector valued function $n(x,t,k)$ with the following properties:*

(*a*) *$n(x,t,\cdot) : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{1\times 3}$ is analytic.*

(*b*) *The limits of $n(x,t,k)$ as k approaches $\Gamma \setminus \Gamma_\star$ from the left and right exist, are continuous on $\Gamma \setminus \Gamma_\star$, and are denoted by n_+ and n_-, respectively. Furthermore,*

$$n_+(x,t,k) = n_-(x,t,k) v(x,t,k) \qquad \textit{for} \quad k \in \Gamma \setminus \Gamma_\star. \tag{3.3}$$

(*c*) *$n(x,t,k) = O(1)$ as $k \to k_\star \in \Gamma_\star$.*

(*d*) *For $k \in \mathbb{C} \setminus \Gamma$, n obeys the symmetries*

$$n(x,t,k) = n(x,t,\omega k)\mathcal{A}^{-1} = n(x,t,k^{-1})\mathcal{B}. \tag{3.4}$$

(*e*) *$n(x,t,k) = (1,1,1) + O(k^{-1})$ as $k \to \infty$.*

Recall that u_0, v_0 are such that Assumptions (i), (ii), and (iii) hold, and that $u_1(x) := v_0'(x)$. Hence, it follows from [9, Theorems 2.6 and 2.12] that RH problem 3.1 has a unique solution $n(x,t,k)$ for each $(x,t) \in \mathbb{R} \times [0,\infty)$, that

$$n_3^{(1)}(x,t) := \lim_{k\to\infty} k(n_3(x,t,k) - 1)$$

is well-defined and smooth for $(x,t) \in \mathbb{R} \times [0,\infty)$, and that $u(x,t)$ defined by

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x} n_3^{(1)}(x,t) \tag{3.5}$$

is a Schwartz class solution of (1.1) on $\mathbb{R} \times [0,\infty)$ with initial data u_0, u_1.

Note that v in (3.1) obeys the symmetries

$$v(x,t,k) = \mathcal{A} v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} v(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma, \tag{3.6}$$

and that v depends on x,t only via the phase functions Φ_{21}, Φ_{31}, and Φ_{32}. As mentioned in Section 2, the saddle points of Φ_{21} are given by $\{k_j\}_{j=1}^4$. From the relations

$$\Phi_{31}(\zeta,k) = -\Phi_{21}(\zeta,\omega^2 k), \qquad \Phi_{32}(\zeta,k) = \Phi_{21}(\zeta,\omega k), \tag{3.7}$$

we then conclude that $\{\omega k_j\}_{j=1}^4$ are the saddle points of Φ_{31} and that $\{\omega^2 k_j\}_{j=1}^4$ are the saddle points of Φ_{32}. The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} are shown in Figure 2.

Our proof employs the Deift–Zhou [15] steepest descent method and involves a series of transformations $n \mapsto n^{(1)} \mapsto n^{(2)} \mapsto n^{(3)} \mapsto \hat{n}$. These transformations are invertible, meaning that the RH problems satisfied by $n^{(1)}, n^{(2)}, n^{(3)}, \hat{n}$ are equivalent to the original RH problem 3.1. In the sector $\zeta \in \mathcal{I}$ considered here[1], it turns out that the main contribution to the long-time asymptotics of $u(x,t)$ come from a global parametrix Δ and from twelve local parametrices near the saddle points $\{k_j, \omega k_j, \omega^2 k_j\}_{j=1}^4$.

[1]Recall that $\mathcal{I}$ denotes an arbitrary compact subset of $(0, \frac{1}{\sqrt{3}})$.

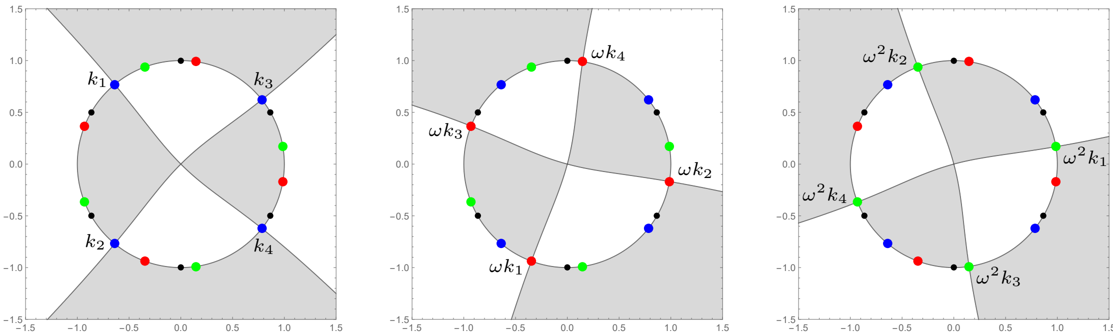

Figure 2. *From left to right: The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} for $\zeta = \frac{1}{2\sqrt{3}}$. The grey regions correspond to $\{k \,|\, \mathrm{Re}\,\Phi_{ij} > 0\}$ and the white regions to $\{k \,|\, \mathrm{Re}\,\Phi_{ij} < 0\}$. The saddle points k_1, k_2, k_3, k_4 of Φ_{21} are blue, the saddle points $\omega k_1, \omega k_2, \omega k_3, \omega k_4$ of Φ_{31} are red, and the saddle points $\omega^2 k_1, \omega^2 k_2, \omega^2 k_3, \omega^2 k_4$ of Φ_{32} are green. The black dots are the points $i\kappa_j$, $j = 1, \dots, 6$.*

The jump contours and jump matrices for the RH problems for $n^{(1)}, n^{(2)}, n^{(3)}, \hat{n}$ will be denoted by $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \hat{\Gamma}$ and $v^{(1)}, v^{(2)}, v^{(3)}, \hat{v}$, respectively. The symmetries (3.4) and (3.6) will be maintained after each transformation, so that, for $j = 1, 2, 3$,

$$v^{(j)}(x,t,k) = \mathcal{A} v^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} v^{(j)}(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma^{(j)}, \tag{3.8}$$

$$n^{(j)}(x,t,k) = n^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = n^{(j)}(x,t,k^{-1})\mathcal{B}, \qquad k \in \mathbb{C} \setminus \Gamma^{(j)}, \tag{3.9}$$

and similarly for $\hat{n}$ and $\hat{v}$. These symmetries will save us some efforts, because (i) we will only need to construct explicitly two local parametrices near ωk_4 and $\omega^2 k_2$, and (ii) we will only have to define the transformations $n^{(j)} \mapsto n^{(j+1)}$ and $n^{(3)} \mapsto \hat{n}$ in the sector $\mathsf{S} := \{k \in \mathbb{C} | \arg k \in [\frac{\pi}{3}, \frac{2\pi}{3}]\}$. In the rest of the complex plane, the local parametrices and the transformations will be defined using the $\mathcal{A}$- and $\mathcal{B}$-symmetries.

The analysis of the asymptotic sector $\zeta \in \mathcal{I}$ considered in this paper involves two main difficulties (which are not present for $\zeta > 1$): (i) the opening of the lenses can only be implemented by introducing poles in the jump matrices, which complicates considerably the constructions of the global and local parametrices, and (ii) the model RH problem used for the local parametrix near $\omega^2 k_2$, albeit using parabolic cylinder functions as in earlier works such as [22], involves jump matrices of a much more complicated nature than in the previous literature (see Lemma A.2).

4. The $n \to n^{(1)}$ transformation

The following properties of r_1 and r_2, which were proved in [9, Theorem 2.3], will be used throughout the steepest descent analysis: $r_1 \in C^\infty(\hat{\Gamma}_1)$, $r_2 \in C^\infty(\hat{\Gamma}_4 \setminus \{\omega^2, -\omega^2\})$, $r_1(\kappa_j) \neq 0$ for $j = 1, \dots, 6$, $r_2(k)$ has simple poles at $k = \pm\omega^2$, and simple zeros at $k = \pm\omega$, and r_1, r_2 are rapidly decreasing as $|k| \to \infty$. Furthermore,

$$r_1(\tfrac{1}{\omega k}) + r_2(\omega k) + r_1(\omega^2 k) r_2(\tfrac{1}{k}) = 0, \qquad k \in \partial\mathbb{D} \setminus \{\omega, -\omega\}, \tag{4.1}$$

$$r_2(k) = \tilde{r}(k)\overline{r_1(\bar{k}^{-1})}, \qquad \tilde{r}(k) := \frac{\omega^2 - k^2}{1 - \omega^2 k^2}, \qquad k \in \hat{\Gamma}_4 \setminus \{0, \omega^2, -\omega^2\}, \tag{4.2}$$

$$r_1(1) = r_1(-1) = 1, \qquad r_2(1) = r_2(-1) = -1. \tag{4.3}$$

The first transformation $n \to n^{(1)}$ consists in opening lenses around $\partial\mathbb{D} \setminus \mathcal{Q}$. For $j = 1, 2$, define

$$\hat{r}_j(k) := \frac{r_j(k)}{1 + r_1(k)r_2(k)}$$

and let

$$r_{j,1}(k) := r_j(k), \quad r_{j,2}(k) := \hat{r}_j(k), \quad r_{j,3}(k) := \frac{r_j(k) - r_j(\frac{1}{\omega k})r_j(\frac{1}{\omega^2 k})}{1 + r_1(\frac{1}{\omega k})r_2(\frac{1}{\omega k})},$$
$$r_{j,4}(k) := \frac{r_j(k)}{f(k)}, \quad r_{j,5}(k) = \frac{r_j(k) - r_j(\frac{1}{\omega k})r_j(\frac{1}{\omega^2 k})}{f(k)}.$$

Since the above spectral functions are (in general) not defined in a neighborhood of $\partial\mathbb{D}$, we need to decompose them into an analytic part and a small remainder. Let $M > 1$ and let $\{U_1^\ell = U_1^\ell(\zeta, M)\}_{\ell=1}^5$ be the open sets defined by (see Figure 3)

$$\begin{aligned}
U_1^1 &= \{k|\operatorname{Re}\Phi_{21} > 0\} \cap \big(\{k|\arg k \in (\tfrac{5\pi}{6}, \pi] \cup [-\pi, \arg(\omega^2 k_4)),\ M^{-1} < |k| < 1\} \\
&\qquad \cup \{k|\arg k \in (-\tfrac{\pi}{2}, \arg k_4),\ 1 < |k| < M\}\big), \\
U_1^2 &= \{k|\operatorname{Re}\Phi_{21} > 0\} \cap \big(\{k|\arg k \in (\arg k_4, -\tfrac{\pi}{6}),\ M^{-1} < |k| < 1\} \\
&\qquad \cup \{k|\arg k \in (\tfrac{\pi}{2}, \arg(\omega^2 k_2)),\ 1 < |k| < M\}\big), \\
U_1^3 &= \{k|\operatorname{Re}\Phi_{21} > 0\} \cap \{k|\arg k \in (\arg k_1, \tfrac{5\pi}{6}),\ M^{-1} < |k| < 1\}, \\
U_1^4 &= \{k|\operatorname{Re}\Phi_{21} > 0\} \cap \{k|\arg k \in (\arg(\omega k_2), \tfrac{\pi}{6}),\ M^{-1} < |k| < 1\}, \\
U_1^5 &= \{k|\operatorname{Re}\Phi_{21} > 0\} \cap \{k|\arg k \in (\arg(\omega^2 k_2), \arg k_1),\ 1 < |k| < M)\}.
\end{aligned}$$

It is known from [9, Lemma 2.13] that $1 + r_1(k)r_2(k) > 0$ for all $k \in \partial\mathbb{D}$ with $\arg k \in (\pi/3, \pi) \cup (-2\pi/3, 0)$, and that $f(k) = 0$ if and only if $k \in \{\pm 1, \pm\omega\}$. Hence, for each $\ell \in \{1, 2, 3\}$, $r_{1,\ell}$ is well-defined for $k \in \partial U_1^\ell \cap \partial\mathbb{D}$; $r_{1,4}$ is well-defined for $k \in (\partial U_1^4 \cap \partial\mathbb{D}) \setminus \{1\}$; and $r_{1,5}$ is well-defined for $k \in (\partial U_1^5 \cap \partial\mathbb{D}) \setminus \{\omega\}$. Let $n_1 \geq 0$ be the smallest integer such that $(k-1)^{n_1} r_{1,4}(k)$ remains bounded as $k \to 1$, $k \in \partial\mathbb{D}$, and let $n_\omega \geq 0$ be the smallest integer such that $(k-\omega)^{n_\omega} r_{1,5}(k)$ remains bounded as $k \to \omega$, $k \in \partial\mathbb{D}$.

The next lemma establishes decompositions of $\{r_{1,\ell}\}_{\ell=1}^5$; we will then obtain decompositions of $\{r_{2,\ell}\}_{\ell=1}^5$ using the symmetry (4.2). Let $N \geq 1$ be an integer.

Lemma 4.1 (Decomposition lemma)**.** *There exist $M > 1$ and decompositions*

$$r_{1,\ell}(k) = r_{1,\ell,a}(x,t,k) + r_{1,\ell,r}(x,t,k), \qquad k \in \partial U_1^\ell \cap \partial\mathbb{D},\ \ell = 1, \dots, 5, \tag{4.4}$$

such that the functions $\{r_{1,\ell,a}, r_{1,\ell,r}\}_{\ell=1}^5$ have the following properties:

(a) *For each $\zeta \in \mathcal{I}$, $t \geq 1$, and $\ell \in \{1, \dots, 5\}$, $r_{1,\ell,a}(x,t,k)$ is defined and continuous for $k \in \bar{U}_1^\ell$ and analytic for $k \in U_1^\ell$.*

(b) *For each $\zeta \in \mathcal{I}$, $t \geq 1$, and $\ell \in \{1, \dots, 5\}$, the function $r_{1,\ell,a}$ satisfies*

$$\Big| r_{1,\ell,a}(x,t,k) - \sum_{j=0}^{N} \frac{r_{1,\ell}^{(j)}(k_\star)}{j!}(k - k_\star)^j \Big| \leq C|k - k_\star|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|},\ k \in \bar{U}_1^\ell,\ k_\star \in \mathcal{R}_\ell,$$

where $\mathcal{R}_1 = \{e^{5\pi i/6}, \omega k_3, -1, \omega^2 k_4, -i, \omega^2 k_3, -\omega, k_4\}$, $\mathcal{R}_2 = \{i, \omega^2 k_2, k_4, e^{-\pi i/6}\}$, $\mathcal{R}_3 = \{k_1, e^{5\pi i/6}\}$, $\mathcal{R}_4 = \{\omega k_2, \omega^2 k_1, e^{\pi i/6}\}$, $\mathcal{R}_5 = \{\omega^2 k_2, k_1\}$, and the constant C is independent of ζ, t, k. Furthermore, for $\zeta \in \mathcal{I}$ and $t \geq 1$, we have

$$\Big| r_{1,4,a}(x,t,k) - \sum_{j=0}^{N+n_1} \frac{[(\cdot - 1)^{n_1} r_{1,4}(\cdot)]^{(j)}(1)}{j!}(k-1)^{j-n_1} \Big| \leq C|k-1|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|},$$

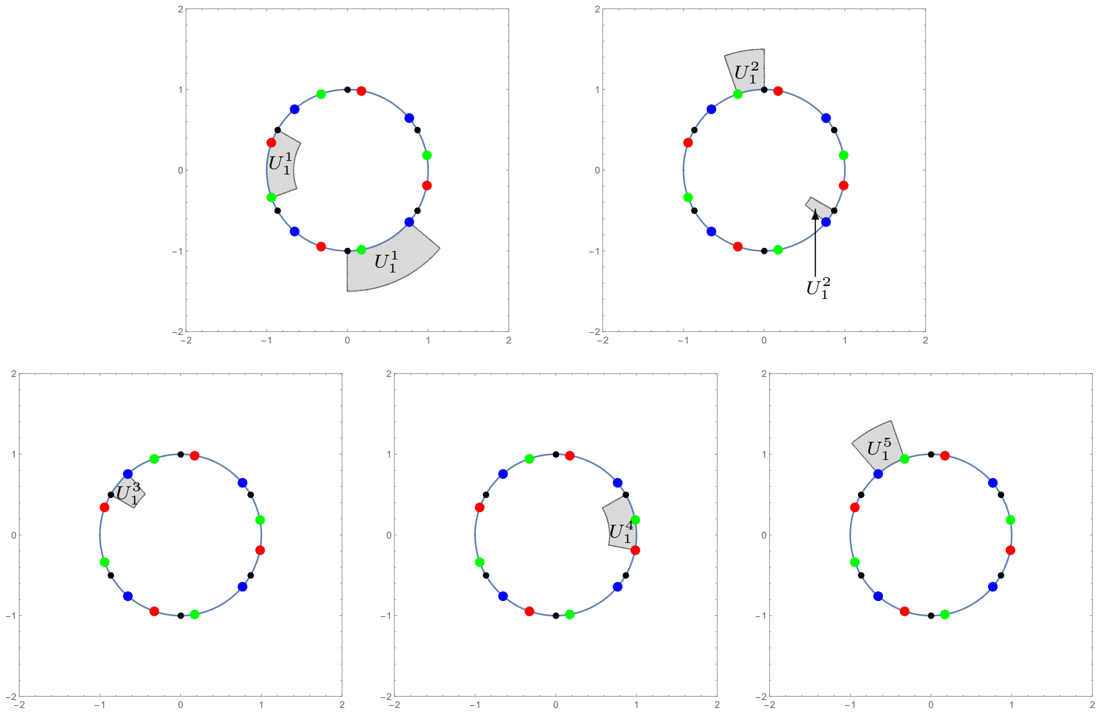

Figure 3. *The open subsets $\{U_1^\ell\}_{\ell=1}^5$ of the complex k-plane.*

$$\Big| r_{1,5,a}(x,t,k) - \sum_{j=0}^{N+n_\omega} \frac{[(\cdot - \omega)^{n_\omega} r_{1,5}(\cdot)]^{(j)}(\omega)}{j!}(k-\omega)^{j-n_\omega} \Big| \le C|k-\omega|^{N+1} e^{\frac{t}{4}|\mathrm{Re}\,\Phi_{21}(\zeta,k)|},$$

and these inequalities hold for $k \in \bar{U}_1^4$ and $k \in \bar{U}_1^5$, respectively.

(c) *For each $1 \le p \le \infty$ and $\ell \in \{1,\dots,5\}$, the L^p-norm of $r_{1,\ell,r}(x,t,\cdot)$ on $\partial\bar{U}_1^\ell \cap \partial\mathbb{D}$ is $O(t^{-N})$ uniformly for $\zeta \in \mathcal{I}$ as $t \to \infty$.*

Proof. For each $\ell \in \{1,\dots,5\}$, $\theta \mapsto \mathrm{Im}\,\Phi_{21}(\zeta, e^{i\theta}) = (\zeta - \cos\theta)\sin\theta$ is monotone on each connected component of $\partial U_1^\ell \cap \partial\mathbb{D}$. Hence the claim can be proved using the techniques of [15]. Since these techniques are rather standard by now, we omit the details. □

Note that $\tilde{r}(k) \in \mathbb{R}$ for $k \in \partial\mathbb{D} \setminus \{-\omega^2, \omega^2\}$ and that $\tilde{r}(k) = \tilde{r}(\frac{1}{\omega k})\tilde{r}(\frac{1}{\omega^2 k})$. By (4.2), we thus have $r_{2,\ell}(k) = \tilde{r}(k)\overline{r_{1,\ell}(\bar{k}^{-1})}$, $k \in \partial\mathbb{D} \cap U_1^\ell$, $\ell = 1,\dots,5$. Using this symmetry, for each $\ell \in \{1,\dots,5\}$ we let $U_2^\ell := \{k | \bar{k}^{-1} \in U_1^\ell\}$ and define a decomposition $r_{2,\ell} = r_{2,\ell,a} + r_{2,\ell,r}$ by

$$r_{2,\ell,a}(k) := \tilde{r}(k)\overline{r_{1,\ell,a}(\bar{k}^{-1})}, \quad k \in U_2^\ell, \qquad r_{2,\ell,r}(k) := \tilde{r}(k)\overline{r_{1,\ell,r}(\bar{k}^{-1})}, \quad k \in \partial U_2^\ell \cap \partial\mathbb{D}.$$

We now proceed with the first transformation $n \mapsto n^{(1)}$. As explained in Section 3, we will first define this transformation in the sector S, and then appeal to the symmetries (3.4) to extend it to the whole complex plane. Since we will need to open lenses differently on four subsets of $\partial\mathbb{D} \cap \mathsf{S}$, we find it convenient to introduce a new contour $\Gamma^{(0)}$. This contour coincides as a set with $\Gamma \cap \mathsf{S}$, but is oriented and labeled differently, see Figure 4. We let $\Gamma_j^{(0)}$ denote the subcontour of $\Gamma^{(0)}$ labeled by j in Figure 4. We emphasize that

$$\Gamma_{11}^{(0)} := \{e^{i\theta} \,|\, \theta \in (\arg(\omega^2 k_2), \tfrac{2\pi}{3})\}, \qquad \Gamma_2^{(0)} := \{e^{i\theta} \,|\, \theta \in (\tfrac{\pi}{3}, \arg(\omega k_4))\}.$$

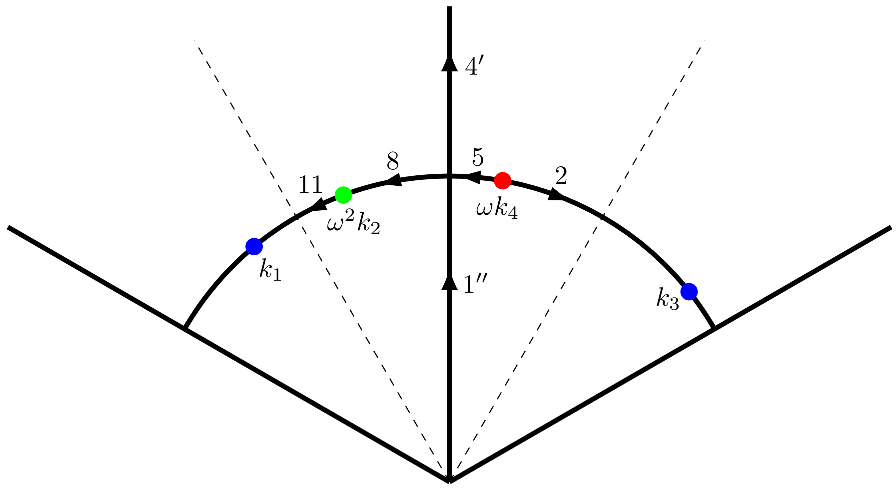

Figure 4. *The contour $\Gamma^{(0)}$ (solid), the boundary of S (dashed) and, from right to left, the saddle points k_3 (blue), ωk_4 (red), $\omega^2 k_2$ (green), and k_1 (blue).*

On $\Gamma_2^{(0)}$, we will use the factorization

$$v_9 = v_3^{(1)} v_2^{(1)} v_1^{(1)},$$

where

$$v_3^{(1)} = \begin{pmatrix} 1 & 0 & -r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ r_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & r_{1,a}(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_1^{(1)} = \begin{pmatrix} 1 & r_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ -r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{2,a}(\omega k)e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_2^{(1)} = I + v_{2,r}^{(1)}, \qquad v_{2,r}^{(1)} = \begin{pmatrix} r_{1,r}(\omega^2 k) r_{2,r}(\omega^2 k) & g_2(\omega k)e^{-\theta_{21}} & -r_{2,r}(\omega^2 k)e^{-\theta_{31}} \\ g_1(\omega k)e^{\theta_{21}} & g(\omega k) & h_1(\omega k)e^{-\theta_{32}} \\ -r_{1,r}(\omega^2 k)e^{\theta_{31}} & h_2(\omega k)e^{\theta_{32}} & 0 \end{pmatrix}, \tag{4.5}$$

and

$$\begin{aligned} h_1(k) &= r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k), \qquad h_2(k) = r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k}) r_{1,r}(\omega k), \\ g_1(k) &= r_{1,r}(\tfrac{1}{\omega^2 k}) - r_{1,r}(\omega k)\big(r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k)\big), \\ g_2(k) &= r_{2,r}(\tfrac{1}{\omega^2 k}) - r_{2,r}(\omega k)\big(r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k}) r_{1,r}(\omega k)\big), \\ g(k) &= r_{1,r}(k)\big(r_{1,r}(\omega k) r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) \\ &\quad + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k)\big(r_{1,r}(\omega k) r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) + r_{1,r}(\tfrac{1}{\omega^2 k}) r_{2,r}(\tfrac{1}{\omega^2 k}). \end{aligned}$$

On $\Gamma_5^{(0)}$, we will use the factorization

$$v_9^{-1} = v_4^{(1)} v_5^{(1)} v_6^{(1)}, \qquad v_5^{(1)} = \begin{pmatrix} \frac{1}{1+r_1(\omega^2 k) r_2(\omega^2 k)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + r_1(\omega^2 k) r_2(\omega^2 k) \end{pmatrix} + v_{5,r}^{(1)},$$

$$v_4^{(1)} = \begin{pmatrix} 1 & c_{12,a}e^{-\theta_{21}} & c_{13,a}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & c_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_6^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ c_{21,a}e^{\theta_{21}} & 1 & c_{23,a}e^{-\theta_{32}} \\ c_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \tag{4.6}$$

where

$$\begin{aligned} &c_{12} = -r_2(\tfrac{1}{k}), && c_{13} = \hat{r}_2(\omega^2 k), && c_{23} = r_2(\tfrac{1}{\omega k}), \\ &c_{21} = -r_1(\tfrac{1}{k}), && c_{31} = \hat{r}_1(\omega^2 k), && c_{32} = r_1(\tfrac{1}{\omega k}), \end{aligned}$$

and $c_{ij,a}, c_{ij,r}$ denote the analytic continuation and the remainder of c_{ij} from Lemma 4.1, i.e., $c_{12,a} = -r_{2,1,a}(\frac{1}{k})$, $c_{12,r} = -r_{2,1,r}(\frac{1}{k})$, $c_{13,a} = -r_{2,2,a}(\omega^2 k)$, etc.

On $\Gamma_8^{(0)}$, we will use the factorization

$$\begin{aligned}
&v_7 = v_7^{(1)} v_8^{(1)} v_9^{(1)}, \qquad v_8^{(1)} = \begin{pmatrix} \frac{1}{f(k)} & 0 & 0 \\ 0 & 1 + r_1(k) r_2(k) & 0 \\ 0 & 0 & \frac{f(k)}{1+r_1(k)r_2(k)} \end{pmatrix} + v_{8,r}^{(1)}, \\
&v_7^{(1)} = \begin{pmatrix} 1 & a_{12,a} e^{-\theta_{21}} & a_{13,a} e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & a_{32,a} e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_9^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ a_{21,a} e^{\theta_{21}} & 1 & a_{23,a} e^{-\theta_{32}} \\ a_{31,a} e^{\theta_{31}} & 0 & 1 \end{pmatrix},
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
&a_{12} = -\hat{r}_1(k), && a_{13} = -\frac{r_1(\frac{1}{\omega^2 k})}{f(k)}, && a_{23} = \frac{r_2(\frac{1}{\omega k}) - r_2(k) r_2(\omega^2 k)}{1 + r_1(k) r_2(k)}, \\
&a_{21} = -\hat{r}_2(k), && a_{31} = -\frac{r_2(\frac{1}{\omega^2 k})}{f(k)}, && a_{32} = \frac{r_1(\frac{1}{\omega k}) - r_1(k) r_1(\omega^2 k)}{1 + r_1(k) r_2(k)}.
\end{aligned}$$

and $a_{ij,a}, a_{ij,r}$ denote the analytic continuation and the remainder of a_{ij} from Lemma 4.1, e.g., $a_{13,a} = -r_{1,4,a}(\frac{1}{\omega^2 k})$ and $a_{23,a} = r_{2,3,a}(\frac{1}{\omega k})$.

Finally, on $\Gamma_{11}^{(0)}$, we will use the factorization

$$\begin{aligned}
&v_7 = v_{10}^{(1)} v_{11}^{(1)} v_{12}^{(1)}, \qquad v_{11}^{(1)} = \begin{pmatrix} \frac{1}{f(k)} & 0 & 0 \\ 0 & \frac{f(k)}{f(\omega^2 k)} & 0 \\ 0 & 0 & f(\omega^2 k) \end{pmatrix} + v_{11,r}^{(1)}, \\
&v_{10}^{(1)} = \begin{pmatrix} 1 & b_{12,a} e^{-\theta_{21}} & b_{13,a} e^{-\theta_{31}} \\ 0 & 1 & b_{23,a} e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_{12}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ b_{21,a} e^{\theta_{21}} & 1 & 0 \\ b_{31,a} e^{\theta_{31}} & b_{32,a} e^{\theta_{32}} & 1 \end{pmatrix},
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
&b_{12} = -\frac{r_1(k) - r_1(\frac{1}{\omega k}) r_1(\frac{1}{\omega^2 k})}{f(k)}, && b_{13} = \frac{r_2(\omega^2 k)}{f(\omega^2 k)}, && b_{23} = \frac{r_2(\frac{1}{\omega k}) - r_2(k) r_2(\omega^2 k)}{f(\omega^2 k)}, \\
&b_{21} = -\frac{r_2(k) - r_2(\frac{1}{\omega k}) r_2(\frac{1}{\omega^2 k})}{f(k)}, && b_{31} = \frac{r_1(\omega^2 k)}{f(\omega^2 k)}, && b_{32} = \frac{r_1(\frac{1}{\omega k}) - r_1(k) r_1(\omega^2 k)}{f(\omega^2 k)},
\end{aligned}$$

and $b_{ij,a}, b_{ij,r}$ denote the analytic continuation and the remainder of b_{ij} from Lemma 4.1, e.g., $b_{12,a} = -r_{1,5,a}(k)$ and $b_{23,a} = r_{2,5,a}(\frac{1}{\omega k})$. The long expressions for $v_{5,r}^{(1)}$, $v_{8,r}^{(1)}$, and $v_{11,r}^{(1)}$ are similar to $v_{2,r}^{(1)}$ and are omitted, but we mention that Lemma 4.1 ensures that

$$\|v_{j,r}^{(1)}\|_{(L^1 \cap L^\infty)(\Gamma_j^{(0)})} = O(t^{-1}), \qquad \text{as } t \to \infty, \ j = 2, 5, 8, 11.$$

Let $\Gamma^{(1)}$ be the contour shown in Figure 5. Define the piecewise analytic function $n^{(1)}$ by

$$n^{(1)}(x,t,k) = n(x,t,k) G^{(1)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(1)}, \tag{4.9}$$

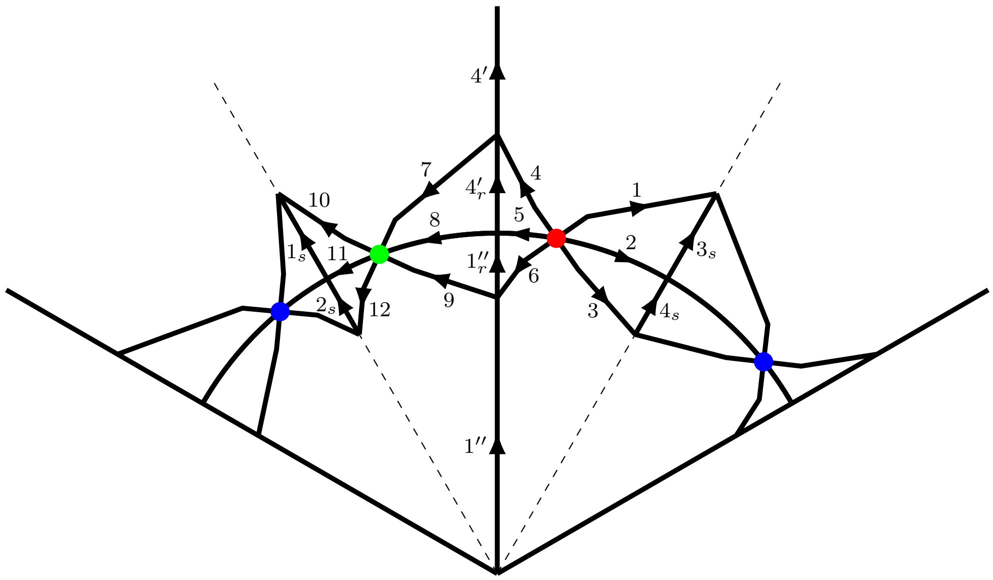

Figure 5. *The contour $\Gamma^{(1)}$ (solid), the boundary of S (dashed) and, from right to left, the saddle points k_3 (blue), ωk_4 (red), $\omega^2 k_2$ (green), and k_1 (blue).*

where $G^{(1)}$ is defined for $k \in \mathsf{S}$ by

$$G^{(1)}=\begin{cases} v_3^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_2^{(1)}, \\ (v_1^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_2^{(1)}, \\ v_4^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_5^{(1)}, \\ (v_6^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_5^{(1)}, \end{cases} \quad G^{(1)}=\begin{cases} v_7^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_8^{(1)}, \\ (v_9^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_8^{(1)}, \\ v_{10}^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_{11}^{(1)}, \\ (v_{12}^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_{11}^{(1)}, \\ I, & \text{otherwise}, \end{cases} \tag{4.10}$$

and $G^{(1)}$ is extended to $\mathbb{C} \setminus \Gamma^{(1)}$ by

$$G^{(1)}(x,t,k) = \mathcal{A}G^{(1)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}G^{(1)}(x,t,k^{-1})\mathcal{B}. \tag{4.11}$$

The functions $G^{(1)}$ and $n^{(1)}$ are analytic on $\mathbb{C}\setminus\Gamma^{(1)}$. Indeed, consider for example the small region inside the lens on the $-$ side of $\Gamma_{11}^{(1)}$; in this region $G^{(1)} = v_{10}^{(1)}$ involves the functions $r_{1,5,a}(k)$, $r_{2,5,a}(\frac{1}{\omega k})$, and $r_{2,4,a}(\omega^2 k)$, and these functions are all analytic in this region as a consequence of Lemma 4.1 and the definitions of U_1^5, U_2^5, and U_2^4.

The next lemma easily follows from Figure 2. Note that $v_{10}^{(1)}, v_{11}^{(1)}, v_{12}^{(1)}$ (and hence also $n^{(1)}(x,t,k)$) are singular at $k = \omega$, because $f(\omega) = f(1) = 0$. This is why the disks $D_\epsilon(\omega^j)$ have been excluded in Lemma 4.2.

Lemma 4.2. *For any $\epsilon > 0$, $G^{(1)}(x,t,k)$ and $G^{(1)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus (\Gamma^{(1)} \cup \cup_{j=0}^2 D_\epsilon(\omega^j))$, $t \geq 1$, and $\zeta \in \mathcal{I}$. Furthermore, $G^{(1)}(x,t,k) = I$ for all large enough $|k|$.*

By construction, $n_+^{(1)} = n_-^{(1)} v_j^{(1)}$ on $\Gamma_j^{(1)}$, where $v_j^{(1)}$ is given by (4.5) for $j = 1,2,3$, by (4.6) for $j = 4,5,6$, by (4.7) for $j = 7,8,9$, by (4.8) for $j = 10,11,12$, and

$$v_{1''}^{(1)} = v_{1''}, \quad v_{4'}^{(1)} = v_{4'}, \quad v_{1''_r}^{(1)} = v_6^{(1)} v_{1''} (v_9^{(1)})^{-1}, \quad v_{4'_r}^{(1)} = (v_4^{(1)})^{-1} v_{4'} v_7^{(1)},$$
$$v_{1_s}^{(1)} = G_-^{(1)}(k)^{-1} G_+^{(1)}(k) = v_{10}^{(1)}(k)^{-1} \mathcal{A}\mathcal{B} v_{12}^{(1)}(\tfrac{1}{\omega k})^{-1} \mathcal{B}\mathcal{A}^{-1}$$

$$= \begin{pmatrix} 1 & -(b_{12,a}(k) + b_{32,a}(\frac{1}{\omega k}))e^{-\theta_{21}} & (v_{1_s}^{(1)})_{13} \\ 0 & 1 & -(b_{21,a}(\frac{1}{\omega k}) + b_{23,a}(k))e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.12}$$

$$(v_{1_s}^{(1)})_{13} := \big(b_{12,a}(k)(b_{21,a}(\tfrac{1}{\omega k}) + b_{23,a}(k)) + b_{21,a}(\tfrac{1}{\omega k})b_{32,a}(\tfrac{1}{\omega k}) - b_{13,a}(k) - b_{31,a}(\tfrac{1}{\omega k})\big)e^{-\theta_{31}}.$$

The expressions for $v_{2_s}, v_{3_s}, v_{4_s}$ are similar to (4.12) and we do not write them down.

5. The $n^{(1)} \to n^{(2)}$ transformation

We now proceed with a second opening of lenses. We will use the following factorizations (the matrices with subscripts u and d will be used to deform contours up and down, respectively, see (5.3) below):

$$v_1^{(1)} = v_1^{(2)} v_{1u}^{(2)}, \quad v_1^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{1,a}(\omega^2 k)e^{\theta_{31}} & 0 & 1 \end{pmatrix},$$

$$v_{1u}^{(2)} = \begin{pmatrix} 1 & r_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & \big(r_{1,a}(\omega^2 k)r_{2,a}(\frac{1}{k}) + r_{2,a}(\omega k)\big)e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_3^{(1)} = v_{3d}^{(2)} v_3^{(2)}, \; v_{3d}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ r_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & \big(r_{1,a}(\omega k) + r_{1,a}(\frac{1}{k})r_{2,a}(\omega^2 k)\big)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_3^{(2)} = \begin{pmatrix} 1 & 0 & -r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_4^{(1)} = v_{4u}^{(2)} v_4^{(2)}, \quad v_{4u}^{(2)} = \begin{pmatrix} 1 & c_{12,a}e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & c_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \quad v_4^{(2)} = \begin{pmatrix} 1 & 0 & c_{13,a}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_6^{(1)} = v_6^{(2)} v_{6d}^{(2)}, \quad v_6^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \quad v_{6d}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ c_{21,a}e^{\theta_{21}} & 1 & c_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7^{(1)} = v_{7u}^{(2)} v_7^{(2)}, \; v_7^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \; v_{7u}^{(2)} = \begin{pmatrix} 1 & (a_{12,a} - a_{13,a}a_{32,a})e^{-\theta_{21}} & a_{13,a}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_9^{(1)} = v_9^{(2)} v_{9d}^{(2)}, \quad v_{9d}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ (a_{21,a} - a_{23,a}a_{31,a})e^{\theta_{21}} & 1 & 0 \\ a_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \quad v_9^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{10}^{(1)} = v_{10u}^{(2)} v_{10}^{(2)}, \; v_{10}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_{10u}^{(2)} = \begin{pmatrix} 1 & b_{12,a}e^{-\theta_{21}} & (b_{13,a} - b_{12,a}b_{23,a})e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{12}^{(1)} = v_{12}^{(2)} v_{12d}^{(2)}, \; v_{12d}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ b_{21,a}e^{\theta_{21}} & 1 & 0 \\ (b_{31,a} - b_{21,a}b_{32,a})e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \; v_{12}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}. \tag{5.1}$$

Let $\Gamma^{(2)}$ be the contour displayed in Figure 6, and let $\Gamma_j^{(2)}$ be the subcontour of $\Gamma^{(2)}$ labeled by j in Figure 6. Define the piecewise analytic function $n^{(2)}$ by

$$n^{(2)}(x,t,k) = n^{(1)}(x,t,k)G^{(2)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(2)}, \tag{5.2}$$

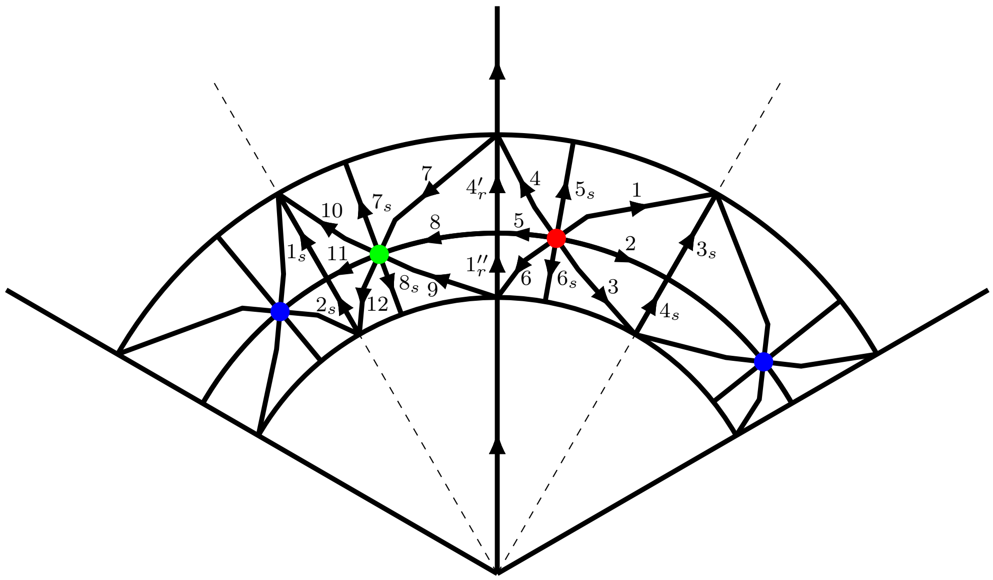

Figure 6. *The contour $\Gamma^{(2)}$ (solid), the boundary of S (dashed) and, from right to left, the saddle points k_3 (blue), ωk_4 (red), $\omega^2 k_2$ (green), and k_1 (blue).*

where $G^{(2)}$ is defined for $k \in \mathsf{S}$ by

$$G^{(2)}(x,t,k) = \begin{cases} (v_{1u}^{(2)})^{-1}, & k \text{ above } \Gamma_1^{(2)}, \\ v_{3d}^{(2)}, & k \text{ below } \Gamma_3^{(2)}, \\ v_{4u}^{(2)}, & k \text{ above } \Gamma_4^{(2)}, \\ (v_{6d}^{(2)})^{-1}, & k \text{ below } \Gamma_6^{(2)}, \end{cases} \quad G^{(2)}(x,t,k) = \begin{cases} v_{7u}^{(2)}, & k \text{ above } \Gamma_7^{(2)}, \\ (v_{9d}^{(2)})^{-1}, & k \text{ below } \Gamma_9^{(2)}, \\ v_{10u}^{(2)}, & k \text{ above } \Gamma_{10}^{(2)}, \\ (v_{12d}^{(2)})^{-1}, & k \text{ below } \Gamma_{12}^{(2)}, \\ I, & \text{otherwise}, \end{cases} \tag{5.3}$$

and $G^{(2)}$ is extended to $\mathbb{C} \setminus \Gamma^{(2)}$ using the $\mathcal{A}$- and $\mathcal{B}$-symmetries (as in (4.11)). The following lemma follows easily from Lemma 4.1 and Figure 2.

Lemma 5.1. *For any $\epsilon > 0$, $G^{(2)}(x,t,k)$ and $G^{(2)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus (\Gamma^{(2)} \cup \cup_{j=0}^{2} D_\epsilon(\omega^j))$, $t \geq 1$, and $\zeta \in \mathcal{I}$. Furthermore, $G^{(2)}(x,t,k) = I$ for large enough $|k|$.*

The jumps $v_j^{(2)}$ of $n^{(2)}$ are given for $j = 1, \dots, 12$ by (5.1) and by

$$\begin{aligned} &v_2^{(2)} = v_2^{(1)}, \quad v_5^{(2)} = v_5^{(1)}, \quad v_8^{(2)} = v_8^{(1)}, \quad v_{11}^{(2)} = v_{11}^{(1)}, \quad v_j^{(2)} = v_j^{(1)} \text{ for } j = 1_s, \dots, 4_s, \\ &v_{5_s}^{(2)} = v_{1u}^{(2)} v_{4u}^{(2)}, \quad v_{6_s}^{(2)} = v_{6d}^{(2)} v_{3d}^{(2)}, \quad v_{7_s}^{(2)} = (v_{7u}^{(2)})^{-1} v_{10u}^{(2)}, \quad v_{8_s}^{(2)} = v_{12d}^{(2)} (v_{9d}^{(2)})^{-1}. \end{aligned} \tag{5.4}$$

The jumps on the parts of $\Gamma^{(2)} \cap \mathsf{S}$ that are unnumbered in Figure 6 are small as $t \to \infty$, so we do not write them down. The matrix $v^{(2)}$ on $\Gamma^{(2)} \setminus \mathsf{S}$ can be obtained using (3.8). The next lemma is proved in the same way as [9, Lemma 8.5].

Lemma 5.2. *The L^∞-norm of $v^{(2)} - I$ on $\Gamma_{4'_r}^{(2)} \cup \Gamma_{1''_r}^{(2)}$ is $O(t^{-N})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.*

The next lemma shows that the jumps $v_j^{(2)}$, $j = 1_s, \dots, 8_s$, also are small for large t.

Lemma 5.3. *It is possible to choose the analytic approximations so that the L^∞-norm of $v_j^{(2)} - I$ on $\Gamma_j^{(2)}$, $j = 1_s, \dots, 8_s$, is $O(t^{-N})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.*

Proof. In the case of compactly supported data u_0, v_0, we can choose $r_{1,a} = r_1$, $r_{2,a} = r_2$ and $a_{ij,a} = a_{ij}$, $b_{ij,a} = b_{ij}$, $c_{ij,a} = c_{ij}$ for all $1 \leq i,j \leq 3$, and then long but straightforward calculations using (4.1) show that $v_{j_s}^{(2)} = I$ for $j = 1, \dots, 8$. In the general case of non-compactly supported data, the matrices $v_{j_s}^{(2)}$ are non-trivial, but it is still possible to choose decompositions such that they remain close to I as $t \to \infty$. The proof is similar to [9, Proof of Lemma 9.2] so we omit it. ☐

Thus $v^{(2)} - I$ is $O(t^{-N})$ as $t \to \infty$, uniformly for $k \in \Gamma^{(2)} \setminus \big(\cup_{j=0}^{2} \cup_{\ell=1}^{4} D_\epsilon(\omega^j k_\ell) \cup \partial \mathbb{D} \big)$. The next transformation $n^{(2)} \to n^{(3)}$ makes $v^{(3)} - I$ uniformly small as $t \to \infty$ for $k \in \partial \mathbb{D}$. To prepare ourselves for this transformation, we first construct a global parametrix.

6. Global parametrix

For each $\zeta \in \mathcal{I}$, let

$$\delta_1(\zeta, \cdot) : \mathbb{C} \setminus \Gamma_5^{(2)} \to \mathbb{C}, \quad \delta_2(\zeta, \cdot), \delta_3(\zeta, \cdot) : \mathbb{C} \setminus \Gamma_8^{(2)} \to \mathbb{C}, \quad \delta_4(\zeta, \cdot), \delta_5(\zeta, \cdot) : \mathbb{C} \setminus \Gamma_{11}^{(2)} \to \mathbb{C}$$

be analytic with continuous boundary values $\{\delta_{j,\pm}\}_{j=1}^5$ satisfying

$$\begin{aligned}
&\delta_{1,+}(\zeta,k) = \delta_{1,-}(\zeta,k)(1 + r_1(\omega^2 k) r_2(\omega^2 k)), && k \in \Gamma_5^{(2)}, \\
&\delta_{2,+}(\zeta,k) = \delta_{2,-}(\zeta,k)(1 + r_1(k) r_2(k)), && k \in \Gamma_8^{(2)}, \\
&\delta_{3,+}(\zeta,k) = \delta_{3,-}(\zeta,k) f(k), && k \in \Gamma_8^{(2)}, \\
&\delta_{4,+}(\zeta,k) = \delta_{4,-}(\zeta,k) f(k), && k \in \Gamma_{11}^{(2)}, \\
&\delta_{5,+}(\zeta,k) = \delta_{5,-}(\zeta,k) f(\omega^2 k), && k \in \Gamma_{11}^{(2)},
\end{aligned}$$

and such that

$$\delta_j(\zeta, k) = 1 + O(k^{-1}), \qquad k \to \infty, \ j = 1, \dots, 5. \tag{6.1}$$

By [9, Lemma 2.13],

$$1 + r_1(\omega^2 k) r_2(\omega^2 k) > 0 \text{ for all } k \in \overline{\Gamma_5^{(2)}}, \quad 1 + r_1(k) r_2(k) > 0 \text{ for all } k \in \overline{\Gamma_8^{(2)}},$$

$$f(k) > 0 \text{ for all } k \in \overline{\Gamma_8^{(2)}}, \ f(k) > 0 \text{ for all } k \in \overline{\Gamma_{11}^{(2)}} \setminus \{\omega\}, \ f(\omega^2 k) > 0 \text{ for all } k \in \overline{\Gamma_{11}^{(2)}} \setminus \{\omega\}.$$

Hence the solutions to the RH problems for $\delta_1, \delta_2, \delta_3$ are unique and given by (2.6). It also follows from [9, Lemma 2.13] that $f(\omega) = f(1) = 0$, and therefore the solutions to the above RH problems for δ_4, δ_5 are not unique. It turns out that the solutions relevant for us are those defined in (2.6).

The next lemma establishes several propreties of $\{\delta_j\}_{j=1}^5$. We omit the proof which consists of an analysis of the definitions (2.6) and uses the assumption that $r_1 = 0$ on $[0, i]$.

Lemma 6.1. *The functions $\delta_j(\zeta, k)$, $j = 1, \dots, 5$, have the following properties:*
(a) *The functions $\{\delta_j(\zeta, k)\}_{j=1}^5$ can be written as*

$$\begin{aligned}
&\delta_1(\zeta,k) = \exp\Big(-i\nu_1 \ln_{\omega k_4}(k - \omega k_4) + i\nu_3 \ln_i(k - i) - \chi_1(\zeta,k) \Big), \\
&\delta_2(\zeta,k) = \exp\Big(i\nu_2 \ln_{\omega^2 k_2}(k - \omega^2 k_2) - \chi_2(\zeta,k) \Big), \\
&\delta_3(\zeta,k) = \exp\Big(-i\nu_3 \ln_i(k - i) + i\nu_4 \ln_{\omega^2 k_2}(k - \omega^2 k_2) - \chi_3(\zeta,k) \Big), \\
&\delta_4(\zeta,k) = \exp\Big(-i\nu_4 \ln_{\omega^2 k_2}(k - \omega^2 k_2) - \chi_4(\zeta,k) \Big), \\
&\delta_5(\zeta,k) = \exp\Big(-i\nu_5 \ln_{\omega^2 k_2}(k - \omega^2 k_2) - \chi_5(\zeta,k) \Big),
\end{aligned}$$

where the branches for the logarithms are described below (2.8), the functions $\{\chi_j\}_{j=1}^3$ are defined in (2.8), the functions $\{\chi_j\}_{j=4}^5$ are defined in the same way as $\{\tilde{\chi}_j\}_{j=4}^5$ in (2.8) except that $\tilde{\ln}_s$ is replaced by $\ln_s$ and $\tilde{\ln}_\omega$ is replaced by $\ln_\omega$, and

$$\nu_1 := \nu_1(\omega^2 k_4) = -\frac{1}{2\pi}\ln(1 + r_1(k_4)r_2(k_4)),\ \nu_2 := \nu_2(k_2) = -\frac{1}{2\pi}\ln(1 + r_1(\omega^2 k_2)r_2(\omega^2 k_2)),$$
$$\nu_3 := \nu_3(\omega^2 i) = -\frac{1}{2\pi}\ln(f(i)) = -\frac{1}{2\pi}\ln(1 + r_1(\omega^2 i)r_2(\omega^2 i)),$$
$$\nu_4 := \nu_4(k_2) = -\frac{1}{2\pi}\ln(f(\omega^2 k_2)), \quad \nu_5 := \nu_3(k_2) = -\frac{1}{2\pi}\ln(f(\omega k_2)).$$

(b) *The functions $\{\delta_j(\zeta,k)\}_{j=2}^5$ can be written as*

$$\delta_2(\zeta,k) = \exp\Big(i\nu_2 \tilde{\ln}_{\omega^2 k_2}(k - \omega^2 k_2) - \tilde{\chi}_2(\zeta,k)\Big),$$
$$\delta_3(\zeta,k) = \exp\Big(- i\nu_3 \tilde{\ln}_i(k - i) + i\nu_4 \tilde{\ln}_{\omega^2 k_2}(k - \omega^2 k_2) - \tilde{\chi}_3(\zeta,k)\Big),$$
$$\delta_4(\zeta,k) = \exp\Big(- i\nu_4 \tilde{\ln}_{\omega^2 k_2}(k - \omega^2 k_2) - \tilde{\chi}_4(\zeta,k)\Big),$$
$$\delta_5(\zeta,k) = \exp\Big(- i\nu_5 \tilde{\ln}_{\omega^2 k_2}(k - \omega^2 k_2) - \tilde{\chi}_5(\zeta,k)\Big),$$

where the branches for the logarithms are described below (2.8), and $\tilde{\chi}_j(\zeta,k)$ is defined in the same way as $\chi_j(\zeta,k)$ except that $\ln_s$ is replaced by $\tilde{\ln}_s$ and $\ln_\omega$ is replaced by $\tilde{\ln}_\omega$.

(c) *For each $\zeta \in \mathcal{I}$ and $j \in \{1,\dots,5\}$, $\delta_j(\zeta,k)$ and $\delta_j(\zeta,k)^{-1}$ are analytic functions of k in their respective domains of definition. Furthermore, for any $\epsilon > 0$,*

$$\sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_5^{(2)}} |\delta_1(\zeta,k)^{\pm 1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_8^{(2)}} |\delta_2(\zeta,k)^{\pm 1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_8^{(2)}} |\delta_3(\zeta,k)^{\pm 1}| < \infty,$$
$$\sup_{\zeta\in\mathcal{I}} \sup_{\substack{k\in\mathbb{C}\setminus\Gamma_{11}^{(2)}\\ |k-\omega|>\epsilon}} |\delta_4(\zeta,k)^{\pm 1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{\substack{k\in\mathbb{C}\setminus\Gamma_{11}^{(2)}\\ |k-\omega|>\epsilon}} |\delta_5(\zeta,k)^{\pm 1}| < \infty. \tag{6.2}$$

(d) *As $k \to \omega k_4$ along a path which is nontangential to $\partial\mathbb{D}$, we have*

$$|\chi_1(\zeta,k) - \chi_1(\zeta,\omega k_4)| \le C|k - \omega k_4|(1 + |\ln|k - \omega k_4||), \tag{6.3}$$

and as $k \to \omega^2 k_2$ along a path which is nontangential to $\partial\mathbb{D}$, we have

$$|\chi_j(\zeta,k) - \chi_j(\zeta,\omega^2 k_2)| \le C|k - \omega^2 k_2|(1 + |\ln|k - \omega^2 k_2||), \qquad j = 2,3,4,5, \tag{6.4}$$

where C is independent of $\zeta \in \mathcal{I}$.

Remark 6.2. *The identity $f(i) = 1 + r_1(\omega^2 i)r_2(\omega^2 i)$, which is used in the above definition of ν_3, is a consequence of Assumption (iii) and the symmetries (4.1)–(4.2). Indeed, by Assumption (iii) we have $r_1(i) = 0$. This implies by (4.2) that $r_2(i) = 0$. We then find using (4.1) that $r_1(\omega^2 i) + r_2(\frac{1}{\omega^2 i}) = 0$ and $r_1(\frac{1}{\omega^2 i}) + r_2(\omega^2 i) = 0$. This implies in particular that $r_1(\frac{1}{\omega^2 i})r_2(\frac{1}{\omega^2 i}) = r_1(\omega^2 i)r_2(\omega^2 i)$, from which $f(i) = 1 + r_1(\omega^2 i)r_2(\omega^2 i)$ follows.*

The following inequalities will be important for us.

Lemma 6.3. *For all $\zeta \in (0, \frac{1}{\sqrt{3}})$, the following inequalities hold:*

$$\nu_1 \ge 0, \qquad \hat{\nu}_2 := \nu_2 + \nu_5 - \nu_4 \ge 0.$$

Proof. For $\zeta \in (0, \frac{1}{\sqrt{3}})$, we have $\arg k_4 \in (-\frac{\pi}{4}, -\frac{\pi}{6})$ and $\arg k_2 \in (-\frac{3\pi}{4}, -\frac{2\pi}{3})$. Hence, the inequality for ν_1 follows from (4.2) and the fact that $\tilde{r}(k) \le 0$ for $k \in \{e^{i\theta} : \theta \in (-\frac{\pi}{3}, \frac{\pi}{3})\}$, and the inequality for $\hat{\nu}_2$ follows from [9, Lemma 2.13 (iv)]. ∎

In what follows we drop the ζ-dependence of $\{\delta_j(\zeta,k)\}_{j=1}^5$ for conciseness. For $k \in \mathbb{C}\setminus\partial\mathbb{D}$, we define

$$\Delta_{33}(\zeta,k) = \frac{\delta_1(\omega^2 k)\delta_1(\frac{1}{\omega k})}{\delta_1(k)\delta_1(\frac{1}{k})}\frac{\delta_2(k)\delta_2(\frac{1}{k})}{\delta_2(\omega k)\delta_2(\frac{1}{\omega^2 k})}\frac{\delta_3(\omega^2 k)\delta_3(\frac{1}{\omega k})}{\delta_3(k)\delta_3(\frac{1}{k})}\frac{\delta_4(\omega^2 k)\delta_4(\frac{1}{\omega k})}{\delta_4(\omega k)\delta_4(\frac{1}{\omega^2 k})}\frac{\delta_5(\omega k)\delta_5(\frac{1}{\omega^2 k})}{\delta_5(k)\delta_5(\frac{1}{k})}.$$

This function satisfies $\Delta_{33}(\zeta,\frac{1}{k}) = \Delta_{33}(\zeta,k)$, and

$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k)\frac{1}{1+r_1(\omega^2 k)r_2(\omega^2 k)}, \qquad k \in \Gamma_5^{(2)}, \tag{6.5a}$$

$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k)\frac{1+r_1(k)r_2(k)}{f(k)}, \qquad k \in \Gamma_8^{(2)}, \tag{6.5b}$$

$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k)\frac{1}{f(\omega^2 k)}, \qquad k \in \Gamma_{11}^{(2)}. \tag{6.5c}$$

Define also $\Delta_{11}(\zeta,k) = \Delta_{33}(\zeta,\omega k)$ and $\Delta_{22}(\zeta,k) = \Delta_{33}(\zeta,\omega^2 k)$. We now define the inverse of the global parametrix by

$$\Delta(\zeta,k) = \begin{pmatrix} \Delta_{11}(\zeta,k) & 0 & 0 \\ 0 & \Delta_{22}(\zeta,k) & 0 \\ 0 & 0 & \Delta_{33}(\zeta,k) \end{pmatrix}, \quad \zeta \in \mathcal{I},\ k \in \mathbb{C}\setminus\partial\mathbb{D}. \tag{6.6}$$

The function Δ satisfies

$$\Delta(\zeta,k) = \mathcal{A}\Delta(\zeta,\omega k)\mathcal{A}^{-1} = \mathcal{B}\Delta(\zeta,\tfrac{1}{k})\mathcal{B},$$

and possesses the jumps

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)(1+r_1(\omega^2 k)r_2(\omega^2 k)), \qquad k \in \Gamma_5^{(2)}, \tag{6.7a}$$

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)f(k), \qquad k \in \Gamma_8^{(2)}, \tag{6.7b}$$

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)f(k), \qquad k \in \Gamma_{11}^{(2)}, \tag{6.7c}$$

and

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k), \qquad k \in \Gamma_5^{(2)}, \tag{6.8a}$$

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k)\frac{1}{1+r_1(k)r_2(k)}, \qquad k \in \Gamma_8^{(2)}, \tag{6.8b}$$

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k)\frac{f(\omega^2 k)}{f(k)}, \qquad k \in \Gamma_{11}^{(2)}. \tag{6.8c}$$

The next lemma follows from (6.6) and Lemma 6.1.

Lemma 6.4. *For any $\epsilon > 0$, $\Delta(\zeta,k)$ and $\Delta(\zeta,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C}\setminus(\cup_{j=0}^2 D_\epsilon(\omega^j) \cup \partial\mathbb{D})$ and $\zeta \in \mathcal{I}$. Furthermore, $\Delta(\zeta,k) = I + O(k^{-1})$ as $k \to \infty$.*

7. The $n^{(2)} \to n^{(3)}$ transformation

Let $n^{(3)}$ be the piecewise analytic function given by

$$n^{(3)}(x,t,k) = n^{(2)}(x,t,k)\Delta(\zeta,k), \qquad k \in \mathbb{C}\setminus\Gamma^{(3)}, \tag{7.1}$$

where $\Gamma^{(3)} = \Gamma^{(2)}$. It satisfies $n_+^{(3)} = n_-^{(3)}v^{(3)}$ on $\Gamma^{(3)}$, where $v^{(3)} = \Delta_-^{-1}v^{(2)}\Delta_+$.

Let $\mathcal{X}_1^\epsilon$ and $\mathcal{X}_2^\epsilon$ be small crosses centered at ωk_4 and $\omega^2 k_2$, respectively. More precisely, $\mathcal{X}_1^\epsilon := \cup_{j=1}^4 \mathcal{X}_{1,j}^\epsilon$, where

$$\mathcal{X}_{1,1}^\epsilon = \Gamma_1^{(3)} \cap D_\epsilon(\omega k_4), \qquad \mathcal{X}_{1,2}^\epsilon = \Gamma_4^{(3)} \cap D_\epsilon(\omega k_4),$$

$$\mathcal{X}_{1,3}^\epsilon = \Gamma_6^{(3)} \cap D_\epsilon(\omega k_4), \qquad \mathcal{X}_{1,4}^\epsilon = \Gamma_3^{(3)} \cap D_\epsilon(\omega k_4),$$

are oriented outwards from ωk_4, and $\mathcal{X}_2^\epsilon := \cup_{j=1}^4 \mathcal{X}_{2,j}^\epsilon$, where

$$\mathcal{X}_{2,1}^\epsilon = \Gamma_7^{(3)} \cap D_\epsilon(\omega^2 k_2), \qquad \mathcal{X}_{2,2}^\epsilon = \Gamma_{10}^{(3)} \cap D_\epsilon(\omega^2 k_2),$$
$$\mathcal{X}_{2,3}^\epsilon = \Gamma_{12}^{(3)} \cap D_\epsilon(\omega^2 k_2), \qquad \mathcal{X}_{2,4}^\epsilon = \Gamma_9^{(3)} \cap D_\epsilon(\omega^2 k_2),$$

are oriented outwards from $\omega^2 k_2$. Let $\hat{\mathcal{X}}^\epsilon = \bigcup_{j=1}^2 \big(\mathcal{X}_j^\epsilon \cup \omega\mathcal{X}_j^\epsilon \cup \omega^2\mathcal{X}_j^\epsilon \cup (\mathcal{X}_j^\epsilon)^{-1} \cup (\omega\mathcal{X}_j^\epsilon)^{-1} \cup (\omega^2\mathcal{X}_j^\epsilon)^{-1}\big)$ be the union of all the small crosses centered at the twelve saddle points.

Lemma 7.1. *$v^{(3)}$ converges to the identity matrix I as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$ and $k \in \Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon$. More precisely, for $\zeta \in \mathcal{I}$,*

$$\|v^{(3)} - I\|_{(L^1 \cap L^\infty)(\Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon)} \leq C t^{-1}. \tag{7.2}$$

Proof. From the expressions for $v_j^{(1)}$, $j = 2, 5, 8, 11$, in (4.5), (4.6), (4.7), and (4.8) together with (5.4), (6.5), (6.7), (6.8), we deduce that

$$v_j^{(3)} = \Delta_-^{-1} v_j^{(1)} \Delta_+ = I + \Delta_-^{-1} v_{j,r}^{(1)} \Delta_+, \quad j = 2, 5, 8, 11. \tag{7.3}$$

By Lemma 4.1 and Lemma 6.4, the matrices $\Delta_-^{-1} v_{j,r}^{(1)} \Delta_+$, $j = 2, 5, 8, 11$, are small as $t \to \infty$. Using also Lemmas 5.2 and 5.3, we infer that the L^1 and L^∞ norms of $v^{(3)} - I$ tend to 0 as $t \to \infty$, uniformly for $k \in \mathsf{S} \cap \Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon$. Then (7.2) follows from the symmetries (3.8). $\square$

To define the next transformation $n^{(3)} \to \hat{n}$, we need to construct local parametrices near $\{k_j, \omega k_j, \omega^2 k_j\}_{j=1}^4$. Thanks to the $\mathcal{A}$- and $\mathcal{B}$-symmetries, we can focus on the construction of the parametrices near ωk_4 and $\omega^2 k_2$. The construction of these two parametrices is the content of Sections 8 and 9. We will then show in Section 10 that these parametrices are good approximations of $n^{(3)}$ in $D_\epsilon(\omega k_4)$ and $D_\epsilon(\omega^2 k_2)$.

8. Local parametrix near ωk_4

As $k \to \omega k_4$, we have

$$-\left(\Phi_{31}(\zeta, k) - \Phi_{31}(\zeta, \omega k_4)\right) = \phi_{\omega k_4}(k - \omega k_4)^2 + O((k - \omega k_4)^3), \quad \phi_{\omega k_4} := \omega \frac{4 - 3k_4\zeta - k_4^3\zeta}{4k_4^4}.$$

Let $z_1 = z_1(\zeta, t, k)$ be given by

$$z_1 = z_{1,\star}\sqrt{t}(k - \omega k_4)\hat{z}_1, \qquad \hat{z}_1 = \sqrt{\frac{2i(\Phi_{31}(\zeta, \omega k_4) - \Phi_{31}(\zeta, k))}{z_{1,\star}^2 (k - \omega k_4)^2}},$$

where the principal branch is chosen for $\hat{z}_1 = \hat{z}_1(\zeta, k)$, and

$$z_{1,\star} = \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\phi_{\omega k_4}}, \qquad -i\omega k_4 z_{1,\star} > 0.$$

We have $\hat{z}_1(\zeta, \omega k_4) = 1$ and $t(\Phi_{31}(\zeta, k) - \Phi_{31}(\zeta, \omega k_4)) = \frac{iz_1^2}{2}$. Let $\epsilon > 0$ be fixed and small enough that the map z_1 is conformal from $D_\epsilon(\omega k_4)$ to a neighborhood of 0. The asymptotic behavior of z_1 as $k \to \omega k_4$ is given by

$$z_1 = z_{1,\star}\sqrt{t}(k - \omega k_4)(1 + O(k - \omega k_4)).$$

For all $k \in D_\epsilon(\omega k_4)$, the following identity holds:

$$\ln_{\omega k_4}(k - \omega k_4) = \ln[z_{1,\star}(k - \omega k_4)\hat{z}_1] - \ln \hat{z}_1 - \ln z_{1,\star} + 2\pi i,$$

where ln is the principal logarithm. Shrinking $\epsilon > 0$ if necessary, the function $k \mapsto \ln \hat{z}_1$ is analytic in $D_\epsilon(\omega k_4)$, $\ln \hat{z}_1 = O(k - \omega k_4)$ as $k \to \omega k_4$, and

$$\ln z_{1,\star} = \ln |z_{1,\star}| + i \arg z_{1,\star} = \ln |z_{1,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega k_4)\big),$$

where $\arg(\omega k_4) \in (\frac{\pi}{3}, \frac{\pi}{2})$. Using Lemma 6.1, we obtain

$$\delta_1(\zeta,k) = e^{-i\nu_1 \ln z_1} t^{\frac{i\nu_1}{2}} e^{i\nu_1 \ln \hat{z}_1} e^{i\nu_1 \ln z_{1,\star}} e^{i\nu_3 \ln_i(k-i)} e^{-\chi_1(\zeta,k)} e^{2\pi\nu_1}.$$

For conciseness, we from now on, for $a \in \mathbb{C}$, use the notation $z_1^a := e^{a \ln z_1}$, $z_{1,\star}^a := e^{a \ln z_{1,\star}}$, and $\hat{z}_1^a := e^{a \ln \hat{z}_1}$. Using (6.6) and the above expression for $\delta_1(\zeta,k)$, we get

$$\frac{\Delta_{33}(\zeta,k)}{\Delta_{11}(\zeta,k)} = \frac{\mathcal{D}_1(k)}{\delta_1(\zeta,k)^2} = \tilde{d}_{1,0}(\zeta,t) d_{1,1}(\zeta,k) z_1^{2i\nu_1}, \tag{8.1}$$

$$\tilde{d}_{1,0}(\zeta,t) = e^{-4\pi\nu_1} e^{2\chi_1(\zeta,\omega k_4)} e^{-2i\nu_3 \ln_i(\omega k_4 - i)} t^{-i\nu_1} z_{1,\star}^{-2i\nu_1} \mathcal{D}_1(\omega k_4),$$

$$d_{1,1}(\zeta,k) = e^{2\chi_1(\zeta,k) - 2\chi_1(\zeta,\omega k_4)} e^{-2i\nu_3 \left[\ln_i(k-i) - \ln_i(\omega k_4 - i)\right]} \hat{z}_1^{-2i\nu_1} \frac{\mathcal{D}_1(k)}{\mathcal{D}_1(\omega k_4)},$$

$$\begin{aligned}\mathcal{D}_1(k) = {} & \frac{\delta_1(\omega^2 k)\delta_1(\frac{1}{\omega k})^2 \delta_1(\omega k)}{\delta_1(\frac{1}{k})\delta_1(\frac{1}{\omega^2 k})} \frac{\delta_2(k)\delta_2(\omega^2 k)\delta_2(\frac{1}{k})^2}{\delta_2(\omega k)^2 \delta_2(\frac{1}{\omega^2 k})\delta_2(\frac{1}{\omega k})} \frac{\delta_3(\omega k)\delta_3(\omega^2 k)\delta_3(\frac{1}{\omega k})^2}{\delta_3(k)^2 \delta_3(\frac{1}{k})\delta_3(\frac{1}{\omega^2 k})} \\ & \times \frac{\delta_4(\omega^2 k)^2 \delta_4(\frac{1}{k})\delta_4(\frac{1}{\omega k})}{\delta_4(k)\delta_4(\omega k)\delta_4(\frac{1}{\omega^2 k})^2} \frac{\delta_5(\omega k)^2 \delta_5(\frac{1}{\omega k})\delta_5(\frac{1}{\omega^2 k})}{\delta_5(k)\delta_5(\frac{1}{k})^2 \delta_5(\omega^2 k)}.\end{aligned} \tag{8.2}$$

Define $Y_1(\zeta,t)$ by

$$Y_1(\zeta,t) = \tilde{d}_{1,0}(\zeta,t)^{\frac{\sigma}{2}} e^{-\frac{t}{2}\Phi_{31}(\zeta,\omega k_4)\sigma} \lambda_1^{\sigma},$$

where $\sigma = \operatorname{diag}(1,0,-1)$ and λ_1 is a free parameter that will be fixed later. Let

$$\tilde{n}(x,t,k) = n^{(3)}(x,t,k) Y_1(\zeta,t), \qquad k \in D_\epsilon(\omega k_4),$$

let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ be the restriction of $\tilde{v}$ to $\Gamma_j^{(3)} \cap D_\epsilon(\omega k_4)$. By (5.1), (7.1), and (8.1), we can write

$$\tilde{v}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d_{1,1}^{-1}\lambda_1^2 r_{1,a}(\omega^2 k) z_1^{-2i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix}, \quad \tilde{v}_3 = \begin{pmatrix} 1 & 0 & -d_{1,1}\lambda_1^{-2} r_{2,a}(\omega^2 k) z_1^{2i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_4 = \begin{pmatrix} 1 & 0 & d_{1,1}\lambda_1^{-2} \hat{r}_{2,a}(\omega^2 k) z_1^{2i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{v}_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d_{1,1}^{-1}\lambda_1^2 \hat{r}_{1,a}(\omega^2 k) z_1^{-2i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix}.$$

The matrices $\tilde{v}_j$, $j = 1,3,4,6$, suggest to approximate $\tilde{n}$ by $(1,1,1)Y_1\tilde{m}^{\omega k_4}$, where $\tilde{m}^{\omega k_4}(x,t,k)$ is the solution of the 3×3 RH problem with $\tilde{m}^{\omega k_4}(x,t,k) \to I$ as $t \to \infty$ uniformly for $k \in \partial D_\epsilon(\omega k_4)$ and whose jumps on $\mathcal{X}_1^\epsilon$ are

$$\begin{aligned} \tilde{v}^{\omega k_4}_{\mathcal{X}^\epsilon_{1,1}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda_1^2 r_1(\omega^2 k_\star) z_1^{-2i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix}, & \tilde{v}^{\omega k_4}_{\mathcal{X}^\epsilon_{1,2}} &= \begin{pmatrix} 1 & 0 & \lambda_1^{-2}\hat{r}_2(\omega^2 k_\star) z_1^{2i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \tilde{v}^{\omega k_4}_{\mathcal{X}^\epsilon_{1,3}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1^2 \hat{r}_1(\omega^2 k_\star) z_1^{-2i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix}, & \tilde{v}^{\omega k_4}_{\mathcal{X}^\epsilon_{1,4}} &= \begin{pmatrix} 1 & 0 & -\lambda_1^{-2} r_2(\omega^2 k_\star) z_1^{2i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{8.3}$$

where $k_\star = \omega k_4$. We fix the free parameter λ_1 as follows:

$$\lambda_1 = |\tilde{r}(k_4)|^{\frac{1}{4}} = |\tilde{r}(\tfrac{1}{k_4})|^{-\frac{1}{4}}. \tag{8.4}$$

Lemma 8.1. *The function $Y_1(\zeta,t)$ is uniformly bounded:*

$$\sup_{\zeta\in\mathcal{I}}\sup_{t\geq 2}|Y_1(\zeta,t)^{\pm 1}| \leq C. \tag{8.5}$$

Moreover, the functions $\tilde{d}_{1,0}(\zeta,t)$ and $d_{1,1}(\zeta,k)$ satisfy

$$|\tilde{d}_{1,0}(\zeta,t)| = 1, \qquad \zeta\in\mathcal{I},\ t\geq 2, \tag{8.6}$$

$$|d_{1,1}(\zeta,k)-1| \leq C|k-\omega k_4|(1+|\ln|k-\omega k_4||), \qquad \zeta\in\mathcal{I},\ k\in\mathcal{X}_1^\epsilon. \tag{8.7}$$

Proof. Standard estimates yield (8.5) and (8.7). Let us show (8.6). Observe that

$$|\tilde{d}_{1,0}(\zeta,t)| = e^{-4\pi\nu_1}e^{2\operatorname{Re}\chi_1(\zeta,\omega k_4)}e^{2\nu_3\arg_i(\omega k_4-i)}e^{2\nu_1\arg z_{1,\star}}|\mathcal{D}_1(\omega k_4)|. \tag{8.8}$$

We infer from Lemma 6.1 that, for $k\in\partial\mathbb{D}\setminus\{\arg k\in(\arg(\omega k_4),\pi/2)\}$,

$$|\delta_1(\zeta,k)| = e^{\nu_1\frac{\arg_i(k)+\arg(\omega k_4)+\pi}{2}-\nu_3\frac{\arg_i(k)+\arg(i)+\pi}{2}-\operatorname{Re}\chi_1(\zeta,k)} \tag{8.9}$$

where $\arg_i(k)\in(\pi/2,5\pi/2)$. It follows from the definition of χ_1 in (2.8) that the right-hand side of (8.9) is independent of $k\in\partial\mathbb{D}\setminus\{\arg k\in(\arg(\omega k_4),\pi/2)\}$. Similar calculations show that $|\delta_j(\zeta,k)|$, $j=2,3,4,5$, are independent of k for $k\in\partial\mathbb{D}\setminus\{\arg k\in(\pi/2,2\pi/3)\}$. Thus

$$|\mathcal{D}_1(\omega k_4)| = |\delta_1(\zeta,\omega k_4)|^2 = e^{2\nu_1\frac{2\arg(\omega k_4)+3\pi}{2}-2\nu_3\frac{\arg(\omega k_4)+\arg(i)+3\pi}{2}-2\operatorname{Re}\chi_1(\zeta,\omega k_4)}.$$

Substituting this expression for $|\mathcal{D}_1(\omega k_4)|$ as well as the expressions $\arg_i(\omega k_4 - i) = \frac{\arg(\omega k_4)+\arg(i)+3\pi}{2}$ and $\arg z_{1,\star} = \frac{\pi}{2}-\arg(\omega k_4)$ into (8.8), we obtain $|\tilde{d}_{1,0}(\zeta,t)|=1$. □

The local parametrix $m^{\omega k_4}$ is defined by

$$m^{\omega k_4}(x,t,k) = Y_1(\zeta,t)m^{X,(1)}(q,z_1(\zeta,t,k))Y_1(\zeta,t)^{-1}, \qquad k\in D_\epsilon(\omega k_4), \tag{8.10}$$

where $m^{X,(1)}$ is the solution to the model RH problem of Lemma A.1 with

$$q = |\tilde{r}(k_4)|^{\frac{1}{2}}r_1(k_4), \qquad \bar{q} = -|\tilde{r}(k_4)|^{-\frac{1}{2}}r_2(k_4).$$

Note that, for $\zeta\in\mathcal{I}$, we have $|q|<1$ and

$$q=\lambda_1^2 r_1(\omega^2 k_\star), \quad \frac{-\bar{q}}{1-|q|^2} = \lambda_1^{-2}\hat{r}_2(\omega^2 k_\star), \quad \bar{q} = -\lambda_1^{-2}r_2(\omega^2 k_\star), \quad \frac{q}{1-|q|^2} = \lambda_1^2\hat{r}_1(\omega^2 k_\star),$$

so that the jump matrices in Lemma A.1 agree with those in (8.3) if $z=z_1$ and $\nu=\nu_1$.

Lemma 8.2. *For each $t\geq 2$ and $\zeta\in\mathcal{I}$, $m^{\omega k_4}(x,t,k)$ defined in (8.10) is analytic for $k\in D_\epsilon(\omega k_4)\setminus\mathcal{X}_1^\epsilon$. Moreover, $m^{\omega k_4}(x,t,k)$ is uniformly bounded for $t\geq 2$, $\zeta\in\mathcal{I}$, and $k\in D_\epsilon(\omega k_4)\setminus\mathcal{X}_1^\epsilon$. On $\mathcal{X}_1^\epsilon$, $m^{\omega k_4}$ satisfies $m_+^{\omega k_4} = m_-^{\omega k_4}v^{\omega k_4}$, where the jump matrix $v^{\omega k_4}$ obeys*

$$\begin{cases}\|v^{(3)}-v^{\omega k_4}\|_{L^1(\mathcal{X}_1^\epsilon)} \leq Ct^{-1}\ln t,\\ \|v^{(3)}-v^{\omega k_4}\|_{L^\infty(\mathcal{X}_1^\epsilon)} \leq Ct^{-1/2}\ln t,\end{cases} \qquad \zeta\in\mathcal{I},\ t\geq 2. \tag{8.11}$$

Furthermore, as $t\to\infty$,

$$\|m^{\omega k_4}(x,t,\cdot)-I\|_{L^\infty(\partial D_\epsilon(\omega k_4))} = O(t^{-1/2}), \tag{8.12}$$

$$m^{\omega k_4}-I = \frac{Y_1(\zeta,t)m_1^{X,(1)}Y_1(\zeta,t)^{-1}}{z_{1,\star}\sqrt{t}(k-\omega k_4)\hat{z}_1(\zeta,k)} + O(t^{-1}) \tag{8.13}$$

uniformly for $\zeta\in\mathcal{I}$, and $m_1^{X,(1)} = m_1^{X,(1)}(q)$ is given by (A.3).

Proof. The proof is almost identical to [9, Lemma 9.10], so we omit it. □

9. Local parametrix near $\omega^2 k_2$

The construction of this local parametrix is similar to the one in [9, Section 9.6] (the difference is that the functions $\{\delta_j\}_{j=1}^5$, $d_{2,0}, d_{2,1}, \mathcal{D}_2$ appearing in [9, Section 9.6] need to be defined differently here). For convenience, we provide details of the construction. As $k \to \omega^2 k_2$, we have

$$-(\Phi_{32}(\zeta,k) - \Phi_{32}(\zeta,\omega^2 k_2)) = \phi_{\omega^2 k_2}(k-\omega^2 k_2)^2 + O((k-\omega^2 k_2)^3),$$
$$\phi_{\omega^2 k_2} := -\omega^2 \frac{4 - 3k_2\zeta - k_2^3\zeta}{4k_2^4}.$$

Let $z_2 = z_2(\zeta,t,k)$ be given by

$$z_2 = z_{2,\star}\sqrt{t}(k-\omega^2 k_2)\hat{z}_2, \qquad \hat{z}_2 = \sqrt{\frac{2i(\Phi_{32}(\zeta,\omega^2 k_2) - \Phi_{32}(\zeta,k))}{z_{2,\star}^2(k-\omega^2 k_2)^2}},$$

where the principal branch is chosen for $\hat{z}_2 = \hat{z}_2(\zeta,k)$ and

$$z_{2,\star} = \sqrt{2}e^{\frac{\pi i}{4}}\sqrt{\phi_{\omega^2 k_2}}, \qquad -i\omega^2 k_2 z_{2,\star} > 0.$$

We have $\hat{z}_2(\zeta,\omega^2 k_2) = 1$ and $t(\Phi_{32}(\zeta,k) - \Phi_{32}(\zeta,\omega^2 k_2)) = \frac{iz_2^2}{2}$. Let $\epsilon > 0$ be fixed and small enough that the map z_2 is conformal from $D_\epsilon(\omega^2 k_2)$ to a neighborhood of 0. As $k \to \omega^2 k_2$,

$$z_2 = z_{2,\star}\sqrt{t}(k-\omega^2 k_2)(1 + O(k-\omega^2 k_2)).$$

For all $k \in D_\epsilon(\omega^2 k_2)$, the following identities hold:

$$\ln_{\omega^2 k_2}(k-\omega^2 k_2) = \ln_0[z_{2,\star}(k-\omega^2 k_2)\hat{z}_2] - \ln\hat{z}_2 - \ln z_{2,\star},$$
$$\tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) = \ln[z_{2,\star}(k-\omega^2 k_2)\hat{z}_2] - \ln\hat{z}_2 - \ln z_{2,\star},$$

where $\ln_0(k) := \ln|k| + i\arg_0 k$, $\arg_0(k) \in (0,2\pi)$, and ln is the principal logarithm. Shrinking $\epsilon > 0$ if necessary, the function $k \mapsto \ln\hat{z}_2$ is analytic in $D_\epsilon(\omega^2 k_2)$, $\ln\hat{z}_2 = O(k-\omega^2 k_2)$ as $k \to \omega^2 k_2$, and

$$\ln z_{2,\star} = \ln|z_{2,\star}| + i\arg z_{2,\star} = \ln|z_{2,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega^2 k_2)\big),$$

where $\arg(\omega^2 k_2) \in (\frac{\pi}{2}, \frac{2\pi}{3})$. Using Lemma 6.1, we obtain

$$\delta_2(\zeta,k) = e^{-\chi_2(\zeta,k)}t^{-\frac{i\nu_2}{2}}z_{2,(0)}^{i\nu_2}\hat{z}_2^{-i\nu_2}z_{2,\star}^{-i\nu_2},$$
$$\delta_3(\zeta,k) = e^{-i\nu_3\ln_i(k-i)}e^{-\chi_3(\zeta,k)}t^{-\frac{i\nu_4}{2}}z_{2,(0)}^{i\nu_4}\hat{z}_2^{-i\nu_4}z_{2,\star}^{-i\nu_4},$$
$$\delta_4(\zeta,k) = t^{\frac{i\nu_4}{2}}z_2^{-i\nu_4}\hat{z}_2^{i\nu_4}z_{2,\star}^{i\nu_4}e^{-\tilde{\chi}_4(\zeta,k)},$$
$$\delta_5(\zeta,k) = t^{\frac{i\nu_5}{2}}z_2^{-i\nu_5}\hat{z}_2^{i\nu_5}z_{2,\star}^{i\nu_5}e^{-\tilde{\chi}_5(\zeta,k)},$$

where we have used the notation $z_{2,(0)}^a := e^{a\ln_0 z_2}$, $\hat{z}_2^a := e^{a\ln\hat{z}_2}$, $z_{2,\star}^a := e^{a\ln z_{2,\star}}$, $z_2^a := e^{a\ln z_2}$ for $a \in \mathbb{C}$. Using (6.6) and the above expressions for $\{\delta_j(\zeta,k)\}_{j=2}^5$, we get

$$\frac{\Delta_{33}}{\Delta_{22}} = \frac{\delta_2(k)^2\delta_4(k)}{\delta_3(k)\delta_5(k)^2}\mathcal{D}_2(k) = \tilde{d}_{2,0}(\zeta,t)d_{2,1}(\zeta,k)z_2^{i(2\nu_5-\nu_4)}z_{2,(0)}^{i(2\nu_2-\nu_4)}, \tag{9.1}$$
$$\tilde{d}_{2,0}(\zeta,t) = e^{-2\chi_2(\zeta,\omega^2 k_2)+\chi_3(\zeta,\omega^2 k_2)-\tilde{\chi}_4(\zeta,\omega^2 k_2)+2\tilde{\chi}_5(\zeta,\omega^2 k_2)}$$
$$\times e^{i\nu_3\ln_i(\omega^2 k_2 - i)}t^{i(\nu_4-\nu_5-\nu_2)}z_{2,\star}^{2i(\nu_4-\nu_5-\nu_2)}\mathcal{D}_2(\omega^2 k_2),$$
$$d_{2,1}(\zeta,k) = e^{-2\chi_2(\zeta,k)+2\chi_2(\zeta,\omega^2 k_2)+\chi_3(\zeta,k)-\chi_3(\zeta,\omega^2 k_2)-\tilde{\chi}_4(\zeta,k)+\tilde{\chi}_4(\zeta,\omega^2 k_2)+2\tilde{\chi}_5(\zeta,k)-2\tilde{\chi}_5(\zeta,\omega^2 k_2)}$$

$$\times\, e^{i\nu_3[\ln_i(k-i)-\ln_i(\omega^2k_2-i)]}\hat{z}_2^{2i(\nu_4-\nu_5-\nu_2)}\frac{\mathcal{D}_2(k)}{\mathcal{D}_2(\omega^2k_2)},$$

$$\begin{aligned}\mathcal{D}_2(k) = {} & \frac{\delta_1(\omega^2k)^2\delta_1(\frac{1}{\omega k})\delta_1(\frac{1}{\omega^2k})}{\delta_1(k)\delta_1(\frac{1}{k})^2\delta_1(\omega k)}\frac{\delta_2(\frac{1}{k})\delta_2(\frac{1}{\omega k})}{\delta_2(\omega k)\delta_2(\frac{1}{\omega^2k})^2\delta_2(\omega^2k)}\frac{\delta_3(\omega^2k)^2\delta_3(\frac{1}{\omega k})\delta_3(\frac{1}{\omega^2k})}{\delta_3(\frac{1}{k})^2\delta_3(\omega k)} \\ & \times \frac{\delta_4(\omega^2k)\delta_4(\frac{1}{\omega k})^2}{\delta_4(\omega k)^2\delta_4(\frac{1}{\omega^2k})\delta_4(\frac{1}{k})}\frac{\delta_5(\omega k)\delta_5(\frac{1}{\omega^2k})^2\delta_5(\omega^2k)}{\delta_5(\frac{1}{k})\delta_5(\frac{1}{\omega k})}. \end{aligned} \tag{9.2}$$

Define

$$Y_2(\zeta,t) = \tilde{d}_{2,0}(\zeta,t)^{\frac{\tilde{\sigma}}{2}}e^{-\frac{t}{2}\Phi_{32}(\zeta,\omega^2k_2)\tilde{\sigma}}\lambda_2^{\tilde{\sigma}},$$

where $\tilde{\sigma} = \mathrm{diag}\,(0,1,-1)$ and λ_2 is a free parameter that will be fixed later. Let

$$\tilde{n}(x,t,k) = n^{(3)}(x,t,k)Y_2(\zeta,t), \qquad k \in D_\epsilon(\omega^2k_2),$$

let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ be the restriction of $\tilde{v}$ to $\Gamma_j^{(3)} \cap D_\epsilon(\omega^2k_2)$. By (5.1), (7.1), and (9.1), we can write

$$\tilde{v}_7^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -d_{2,1}^{-1}\lambda_2^2\big(\frac{r_1(\frac{1}{\omega k})-r_1(k)r_1(\omega^2k)}{1+r_1(k)r_2(k)}\big)_a z_2^{-i(2\nu_5-\nu_4)}z_{2,(0)}^{-i(2\nu_2-\nu_4)}e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d_{2,1}\lambda_2^{-2}\big(\frac{r_2(\frac{1}{\omega k})-r_2(k)r_2(\omega^2k)}{f(\omega^2k)}\big)_a z_2^{i(2\nu_5-\nu_4)}z_{2,(0)}^{i(2\nu_2-\nu_4)}e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d_{2,1}^{-1}\lambda_2^2\big(\frac{r_1(\frac{1}{\omega k})-r_1(k)r_1(\omega^2k)}{f(\omega^2k)}\big)_a z_2^{-i(2\nu_5-\nu_4)}z_{2,(0)}^{-i(2\nu_2-\nu_4)}e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}_9^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d_{2,1}\lambda_2^{-2}\big(\frac{r_2(\frac{1}{\omega k})-r_2(k)r_2(\omega^2k)}{1+r_1(k)r_2(k)}\big)_a z_2^{i(2\nu_5-\nu_4)}z_{2,(0)}^{i(2\nu_2-\nu_4)}e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

The above expressions for the matrices $\tilde{v}_j$, $j = 7,9,10,12$, suggest that we approximate $\tilde{n}$ by $(1,1,1)Y_2\tilde{m}^{\omega^2k_2}$, where $\tilde{m}^{\omega^2k_2}(x,t,k)$ is the solution to the 3×3 RH problem with $\tilde{m}^{\omega^2k_2}(x,t,k) \to I$ as $t\to\infty$ uniformly for $k \in \partial\mathbb{D}(\omega^2k_2)$ and whose jumps on $\mathcal{X}_2^\epsilon$ are

$$\tilde{v}^{\omega^2k_2}_{\mathcal{X}^\epsilon_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\lambda_2^2\frac{r_1(\frac{1}{\omega k_\star})-r_1(k_\star)r_1(\omega^2k_\star)}{1+r_1(k_\star)r_2(k_\star)}z_2^{-i(2\nu_5-\nu_4)}z_{2,(0)}^{-i(2\nu_2-\nu_4)}e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}^{\omega^2k_2}_{\mathcal{X}^\epsilon_{2,2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda_2^{-2}\frac{r_2(\frac{1}{\omega k_\star})-r_2(k_\star)r_2(\omega^2k_\star)}{f(\omega^2k_\star)}z_2^{i(2\nu_5-\nu_4)}z_{2,(0)}^{i(2\nu_2-\nu_4)}e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}^{\omega^2k_2}_{\mathcal{X}^\epsilon_{2,3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_2^2\frac{r_1(\frac{1}{\omega k_\star})-r_1(k_\star)r_1(\omega^2k_\star)}{f(\omega^2k_\star)}z_2^{-i(2\nu_5-\nu_4)}z_{2,(0)}^{-i(2\nu_2-\nu_4)}e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}^{\omega^2 k_2}_{\mathcal{X}^\epsilon_{2,4}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\lambda_2^{-2}\frac{r_2(\frac{1}{\omega k_\star}) - r_2(k_\star) r_2(\omega^2 k_\star)}{1 + r_1(k_\star) r_2(k_\star)} z_2^{i(2\nu_5 - \nu_4)} z_{2,(0)}^{i(2\nu_2 - \nu_4)} e^{-\frac{i z_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.3}$$

where $k_\star = \omega^2 k_2$. We fix the free parameter λ_2 as follows:

$$\lambda_2 = |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{4}} = |\tilde{r}(\tfrac{1}{k_2})|^{\frac{1}{4}}. \tag{9.4}$$

The local parametrix $m^{\omega^2 k_2}$ is defined by

$$m^{\omega^2 k_2}(x,t,k) = Y_2(\zeta,t) m^{X,(2)}(q_2, q_4, q_5, q_6, z_2(\zeta,t,k)) Y_2(\zeta,t)^{-1}, \qquad k \in D_\epsilon(\omega^2 k_2), \tag{9.5}$$

where $m^{X,(2)}$ is the solution to the model RH problem of Lemma A.2 with

$$\begin{aligned}
q_2 &= \tilde{r}(k_\star)^{\frac{1}{2}} r_1(k_\star), & \bar{q}_2 &= \tilde{r}(k_\star)^{-\frac{1}{2}} r_2(k_\star) = \tilde{r}(k_\star)^{\frac{1}{2}} \overline{r_1(k_\star)}, \\
q_4 &= |\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{\frac{1}{2}} r_1(\tfrac{1}{\omega^2 k_\star}), & \bar{q}_4 &= -|\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{-\frac{1}{2}} r_2(\tfrac{1}{\omega^2 k_\star}) = |\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{\frac{1}{2}} \overline{r_1(\tfrac{1}{\omega^2 k_\star})}, \\
q_5 &= |\tilde{r}(\omega^2 k_\star)|^{\frac{1}{2}} r_1(\omega^2 k_\star), & \bar{q}_5 &= -|\tilde{r}(\omega^2 k_\star)|^{-\frac{1}{2}} r_2(\omega^2 k_\star) = |\tilde{r}(\omega^2 k_\star)|^{\frac{1}{2}} \overline{r_1(\omega^2 k_\star)}, \\
q_6 &= |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{2}} r_1(\tfrac{1}{\omega k_\star}), & \bar{q}_6 &= -|\tilde{r}(\tfrac{1}{\omega k_\star})|^{-\frac{1}{2}} r_2(\tfrac{1}{\omega k_\star}) = |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{2}} \overline{r_1(\tfrac{1}{\omega k_\star})}.
\end{aligned}$$

Using (4.1) and the relation

$$\big|\tilde{r}(\tfrac{1}{\omega^2 k})\big|^{-\frac{1}{2}} = \big|\tilde{r}(\omega^2 k)\big|^{\frac{1}{2}} = |\tilde{r}(k)|^{-\frac{1}{2}} \big|\tilde{r}(\tfrac{1}{\omega k})\big|^{\frac{1}{2}}, \qquad k \in \mathbb{C} \setminus \{-1, 1\},$$

we see that $q_4 - \bar{q}_5 - q_2 \bar{q}_6 = 0$, so that the condition (A.5) holds as it must. Moreover, for $\zeta \in \mathcal{I}$, we have $1 + |q_2|^2 - |q_4|^2 = f(k_\star) > 0$, $1 - |q_5|^2 - |q_6|^2 = f(\omega^2 k_\star) > 0$, and

$$\begin{aligned}
\tfrac{q_6 - q_2 q_5}{1 + |q_2|^2} &= \lambda_2^2 \tfrac{r_1(\frac{1}{\omega k_\star}) - r_1(k_\star) r_1(\omega^2 k_\star)}{1 + r_1(k_\star) r_2(k_\star)}, & -\tfrac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1 - |q_5|^2 - |q_6|^2} &= \lambda_2^{-2} \tfrac{r_2(\frac{1}{\omega k_\star}) - r_2(k_\star) r_2(\omega^2 k_\star)}{f(\omega^2 k_\star)}, \\
\tfrac{q_6 - q_2 q_5}{1 - |q_5|^2 - |q_6|^2} &= \lambda_2^2 \tfrac{r_1(\frac{1}{\omega k_\star}) - r_1(k_\star) r_1(\omega^2 k_\star)}{f(\omega^2 k_\star)}, & \tfrac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1 + |q_2|^2} &= -\lambda_2^{-2} \tfrac{r_2(\frac{1}{\omega k_\star}) - r_2(k_\star) r_2(\omega^2 k_\star)}{1 + r_1(k_\star) r_2(k_\star)},
\end{aligned}$$

so that the jump matrices in Lemma A.2 agree with those in (9.3) if $z = z_2$.

The next lemma is proved in the same way as [9, Lemma 9.9].

Lemma 9.1. *$Y_2(\zeta,t)$ satisfies $\sup_{\zeta \in \mathcal{I}} \sup_{t \geq 2} |Y_2(\zeta,t)^{\pm 1}| \leq C$. Furthermore,*

$$|\tilde{d}_{2,0}(\zeta,t)| = e^{\pi(2\nu_2 - \nu_4)}, \qquad \zeta \in \mathcal{I},\ t \geq 2, \tag{9.6}$$

$$|d_{2,1}(\zeta,k) - 1| \leq C|k - \omega^2 k_2|(1 + |\ln|k - \omega^2 k_2||), \qquad \zeta \in \mathcal{I},\ k \in \mathcal{X}_2^\epsilon. \tag{9.7}$$

The following lemma can be proved in the same way as [9, Lemmas 9.10 and 9.12].

Lemma 9.2. *For each $t \geq 2$ and $\zeta \in \mathcal{I}$, the function $m^{\omega^2 k_2}(x,t,k)$ defined in (9.5) is analytic for $k \in D_\epsilon(\omega^2 k_2) \setminus \mathcal{X}_2^\epsilon$. Moreover, $m^{\omega^2 k_2}(x,t,k)$ is uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}$, and $k \in D_\epsilon(\omega^2 k_2) \setminus \mathcal{X}_2^\epsilon$. On $\mathcal{X}_2^\epsilon$, $m^{\omega^2 k_2}$ satisfies $m_+^{\omega^2 k_2} = m_-^{\omega^2 k_2} v^{\omega^2 k_2}$, where the jump matrix $v^{\omega^2 k_2}$ obeys*

$$\begin{cases} \|v^{(3)} - v^{\omega^2 k_2}\|_{L^1(\mathcal{X}_2^\epsilon)} \leq C t^{-1} \ln t, \\ \|v^{(3)} - v^{\omega^2 k_2}\|_{L^\infty(\mathcal{X}_2^\epsilon)} \leq C t^{-1/2} \ln t, \end{cases} \qquad \zeta \in \mathcal{I},\ t \geq 2. \tag{9.8}$$

Furthermore, as $t \to \infty$,

$$\|m^{\omega^2 k_2}(x,t,\cdot) - I\|_{L^\infty(\partial D_\epsilon(\omega^2 k_2))} = O(t^{-1/2}), \tag{9.9}$$

$$m^{\omega^2 k_2} - I = \frac{Y_2(\zeta,t) m_1^{X,(2)} Y_2(\zeta,t)^{-1}}{z_{2,\star}\sqrt{t}(k - \omega^2 k_2)\hat{z}_2(\zeta,k)} + O(t^{-1}) \tag{9.10}$$

uniformly for $\zeta \in \mathcal{I}$, where $m_1^{X,(2)} = m_1^{X,(2)}(q_2, q_4, q_5, q_6)$ is given by (A.7).

10. The $n^{(3)} \to \hat{n}$ transformation

Using the relations

$$\begin{aligned}
m^{\omega k_4}(x,t,k) &= \mathcal{A} m^{\omega k_4}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} m^{\omega k_4}(x,t,k^{-1})\mathcal{B},\\
m^{\omega^2 k_2}(x,t,k) &= \mathcal{A} m^{\omega^2 k_2}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} m^{\omega^2 k_2}(x,t,k^{-1})\mathcal{B},
\end{aligned}$$

we extend the domain of definition of $m^{\omega k_4}$ and $m^{\omega^2 k_2}$ from $D_\epsilon(\omega k_4)$ and $D_\epsilon(\omega^2 k_2)$ to $\mathcal{D}_{\omega k_4}$ and $\mathcal{D}_{\omega^2 k_2}$ respectively, where

$$\begin{aligned}
\mathcal{D}_{\omega k_4} &= D_\epsilon(\omega k_4) \cup \omega D_\epsilon(\omega k_4) \cup \omega^2 D_\epsilon(\omega k_4) \cup D_\epsilon(\tfrac{1}{\omega k_4}) \cup \omega D_\epsilon(\tfrac{1}{\omega k_4}) \cup \omega^2 D_\epsilon(\tfrac{1}{\omega k_4}),\\
\mathcal{D}_{\omega^2 k_2} &= D_\epsilon(\omega^2 k_2) \cup \omega D_\epsilon(\omega^2 k_2) \cup \omega^2 D_\epsilon(\omega^2 k_2) \cup D_\epsilon(\tfrac{1}{\omega^2 k_2}) \cup \omega D_\epsilon(\tfrac{1}{\omega^2 k_2}) \cup \omega^2 D_\epsilon(\tfrac{1}{\omega^2 k_2}).
\end{aligned}$$

We will prove that

$$\hat{n} := \begin{cases} n^{(3)}(m^{\omega k_4})^{-1}, & k \in \mathcal{D}_{\omega k_4},\\ n^{(3)}(m^{\omega^2 k_2})^{-1}, & k \in \mathcal{D}_{\omega^2 k_2},\\ n^{(3)}, & \text{elsewhere}, \end{cases} \tag{10.1}$$

satisfies a small-norm RH problem as $t \to \infty$, $\zeta \in \mathcal{I}$. Let $\hat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ with $\mathcal{D} := \mathcal{D}_{\omega k_4} \cup \mathcal{D}_{\omega^2 k_2}$ be the contour represented in Figure 7. We orient the circles that are part of $\partial\mathcal{D}$ in the clockwise direction, and define $\hat{v}$ by

$$\hat{v} = \begin{cases} v^{(3)}, & k \in \hat{\Gamma} \setminus \bar{\mathcal{D}},\\ m^{\omega k_4}, & k \in \partial\mathcal{D}_{\omega k_4},\\ m^{\omega^2 k_2}, & k \in \partial\mathcal{D}_{\omega^2 k_2},\\ m_-^{\omega k_4} v^{(3)} (m_+^{\omega k_4})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}_{\omega k_4},\\ m_-^{\omega^2 k_2} v^{(3)} (m_+^{\omega^2 k_2})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}_{\omega^2 k_2}. \end{cases} \tag{10.2}$$

The function $\hat{n}$ satisfies the following RH problem: (a) $\hat{n} : \mathbb{C} \setminus \hat{\Gamma} \to \mathbb{C}^{1\times 3}$ is analytic, (b) $\hat{n}_+ = \hat{n}_- \hat{v}$ for $k \in \hat{\Gamma} \setminus \hat{\Gamma}_\star$, where $\hat{\Gamma}_\star$ is the set of self-intersection points of $\hat{\Gamma}$, (c) $\hat{n}(x,t,k) = (1,1,1) + O(k^{-1})$ as $k \to \infty$, and (d) $\hat{n}(x,t,k)$ remains bounded as $k \to k_\star \in \hat{\Gamma}_\star$.

Lemma 10.1. *Let $\hat{w} = \hat{v} - I$. Uniformly for $t \geq 2$ and $\zeta \in \mathcal{I}$, we have*

$$\|\hat{w}\|_{(L^1\cap L^\infty)(\hat{\Gamma}\setminus(\partial\mathcal{D}\cup\hat{\mathcal{X}}^\epsilon))} \leq Ct^{-1}, \tag{10.3a}$$

$$\|\hat{w}\|_{(L^1\cap L^\infty)(\partial\mathcal{D})} \leq Ct^{-1/2}, \tag{10.3b}$$

$$\|\hat{w}\|_{L^1(\hat{\mathcal{X}}^\epsilon)} \leq Ct^{-1}\ln t, \tag{10.3c}$$

$$\|\hat{w}\|_{L^\infty(\hat{\mathcal{X}}^\epsilon)} \leq Ct^{-1/2}\ln t, \tag{10.3d}$$

Proof. Since $\hat{\Gamma} \setminus (\partial\mathcal{D} \cup \hat{\mathcal{X}}^\epsilon) = \Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon$, (10.3a) is a consequence of (10.2), Lemma 7.1, and (for the part inside $\mathcal{D}$) the uniform boundedness of $m^{\omega k_4}$ and $m^{\omega^2 k_2}$. The estimate (10.3b) follows from (10.2), (8.12), and (9.9). Finally, (10.3c) and (10.3d) follow from (10.2), (8.11), (9.8), and the uniform boundedness of $m^{\omega k_4}$ and $m^{\omega^2 k_2}$. ☐

For a function h defined on $\hat{\Gamma}$, we define $\hat{\mathcal{C}}h$ by

$$(\hat{\mathcal{C}}h)(k) = \frac{1}{2\pi i}\int_{\hat{\Gamma}} \frac{h(k')dk'}{k'-k}, \qquad k \in \mathbb{C} \setminus \hat{\Gamma}.$$

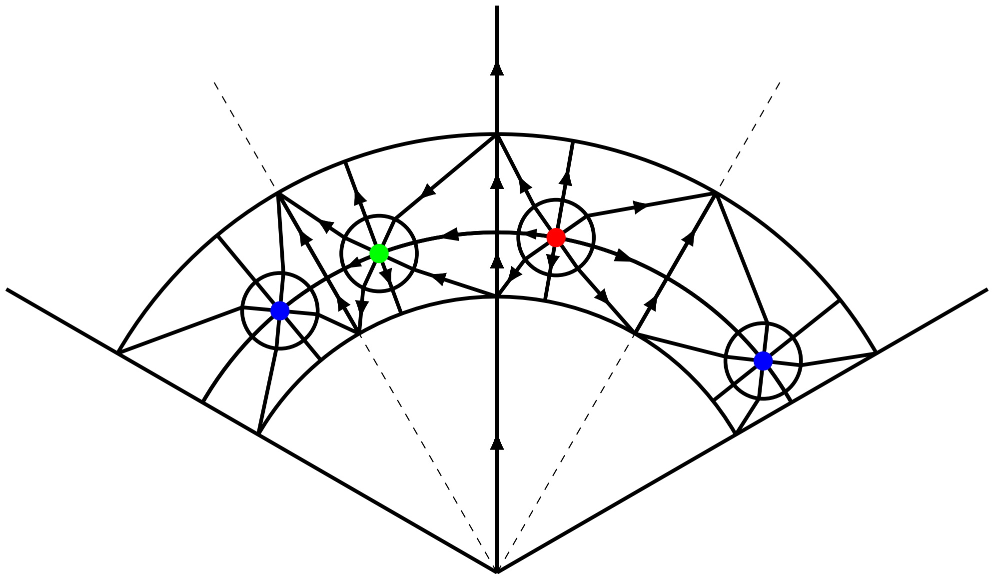

Figure 7. *The contour $\hat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ for $\arg k \in [\frac{\pi}{6}, \frac{5\pi}{6}]$ (solid), the boundary of S (dashed) and, from right to left, the saddle points k_3 (blue), ωk_4 (red), $\omega^2 k_2$ (green), and k_1 (blue).*

Lemma 10.1 implies that

$$\begin{cases} \|\hat{w}\|_{L^1(\hat{\Gamma})} \le Ct^{-1/2}, \\ \|\hat{w}\|_{L^\infty(\hat{\Gamma})} \le Ct^{-1/2}\ln t, \end{cases} \qquad t \ge 2, \ \zeta \in \mathcal{I}. \tag{10.4}$$

Therefore, using the estimate $\|f\|_{L^p} \le \|f\|_{L^1}^{1/p}\|f\|_{L^\infty}^{(p-1)/p}$, we find

$$\|\hat{w}\|_{L^p(\hat{\Gamma})} \le Ct^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}}, \qquad t \ge 2, \ \zeta \in \mathcal{I}, \tag{10.5}$$

for each $1 \le p \le \infty$. It follows from (10.5) that $\hat{w} \in L^2(\hat{\Gamma}) \cap L^\infty(\hat{\Gamma})$, and therefore $\hat{\mathcal{C}}_{\hat{w}}h := \hat{\mathcal{C}}_-(h\hat{w})$ is well-defined as an operator $\hat{\mathcal{C}}_{\hat{w}} = \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} : L^2(\hat{\Gamma}) + L^\infty(\hat{\Gamma}) \to L^2(\hat{\Gamma})$. It also follows from (10.5) that there exists a $T > 0$ such that $I - \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} \in \mathcal{B}(L^2(\hat{\Gamma}))$ is invertible whenever $t \ge T$ and $\zeta \in \mathcal{I}$, where $\mathcal{B}(L^2(\hat{\Gamma}))$ denotes the space of bounded linear operators on $L^2(\hat{\Gamma})$. By standard theory for small-norm RH problems, we have

$$\hat{n}(x,t,k) = (1,1,1) + \hat{\mathcal{C}}(\hat{\mu}\hat{w}) = (1,1,1) + \frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(x,t,s)\hat{w}(x,t,s)\frac{ds}{s-k} \tag{10.6}$$

for $t \ge T$ and $\zeta \in \mathcal{I}$, where

$$\hat{\mu} = (1,1,1) + (I - \hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}(1,1,1) \in (1,1,1) + L^2(\hat{\Gamma}). \tag{10.7}$$

Moreover, since $\|\hat{\mathcal{C}}_-\|_{\mathcal{B}(L^p(\hat{\Gamma}))} < \infty$ for $p \in (1,\infty)$, we infer from (10.5) and (10.7) that for any $p \in (1,\infty)$ there exists $C_p > 0$ such that

$$\|\hat{\mu} - (1,1,1)\|_{L^p(\hat{\Gamma})} \le C_p t^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}} \tag{10.8}$$

holds for all sufficiently large t and all $\zeta \in \mathcal{I}$. Now, we turn to the problem of finding asymptotics for $\hat{n}$. We introduce the following nontangential limit:

$$\hat{n}^{(1)}(x,t) := \lim_{k\to\infty}^{\angle} k(\hat{n}(x,t,k) - (1,1,1)) = -\frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(x,t,k)\hat{w}(x,t,k)dk.$$

Since

$$\hat{n}^{(1)}(x,t) = -\frac{(1,1,1)}{2\pi i}\int_{\partial \mathcal{D}} \hat{w}(x,t,k)dk - \frac{(1,1,1)}{2\pi i}\int_{\hat{\Gamma}\setminus \partial \mathcal{D}} \hat{w}(x,t,k)dk \\ - \frac{1}{2\pi i}\int_{\hat{\Gamma}} (\hat{\mu}(x,t,k) - (1,1,1))\hat{w}(x,t,k)dk,$$

we conclude from (10.8) and Lemma 10.1 that

$$\hat{n}^{(1)}(x,t) = -\frac{(1,1,1)}{2\pi i}\int_{\partial \mathcal{D}} \hat{w}(x,t,k)dk + O(t^{-1}\ln t) \qquad \text{as } t \to \infty, \tag{10.9}$$

uniformly for $\zeta \in \mathcal{I}$. Let us define $\{F_1^{(l)}, F_2^{(l)}\}_{l\in\mathbb{Z}}$ by

$$F_1^{(l)}(\zeta,t) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{l-1}(m^{\omega k_4} - I)dk,$$

$$F_2^{(l)}(\zeta,t) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega^2 k_2)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega^2 k_2)} k^{l-1}(m^{\omega^2 k_2} - I)dk.$$

Using (8.13), (9.10), and $\hat{z}_1(\zeta,\omega k_4) = 1 = \hat{z}_2(\zeta,\omega^2 k_2)$, and recalling that $\partial D_\epsilon(\omega k_4)$ and $\partial D_\epsilon(\omega^2 k_2)$ are oriented clockwise, we find

$$F_1^{(l)}(\zeta,t) = -i(\omega k_4)^l Z_1(\zeta,t) + O(t^{-1}) \qquad \text{as } t\to\infty, \tag{10.10}$$

$$F_2^{(l)}(\zeta,t) = -i(\omega^2 k_2)^l Z_2(\zeta,t) + O(t^{-1}) \qquad \text{as } t\to\infty, \tag{10.11}$$

uniformly for $\zeta \in \mathcal{I}$, where

$$Z_1(\zeta,t) = \frac{Y_1(\zeta,t) m_1^{X,(1)} Y_1(\zeta,t)^{-1}}{-i\omega k_4 z_{1,\star}\sqrt{t}} = \frac{t^{-\frac{1}{2}}}{-i\omega k_4 z_{1,\star}} \begin{pmatrix} 0 & 0 & \frac{\beta_{12}^{(1)}\tilde{d}_{1,0}\lambda_1^2}{e^{t\Phi_{31}(\zeta,\omega k_4)}} \\ 0 & 0 & 0 \\ \frac{\beta_{21}^{(1)} e^{t\Phi_{31}(\zeta,\omega k_4)}}{\tilde{d}_{1,0}\lambda_1^2} & 0 & 0 \end{pmatrix}, \tag{10.12a}$$

$$Z_2(\zeta,t) = \frac{Y_2(\zeta,t) m_1^{X,(2)} Y_2(\zeta,t)^{-1}}{-i\omega^2 k_2 z_{2,\star}\sqrt{t}} = \frac{t^{-\frac{1}{2}}}{-i\omega^2 k_2 z_{2,\star}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_{12}^{(2)}\tilde{d}_{2,0}\lambda_2^2}{e^{t\Phi_{32}(\zeta,\omega^2 k_2)}} \\ 0 & \frac{\beta_{21}^{(2)} e^{t\Phi_{32}(\zeta,\omega^2 k_2)}}{\tilde{d}_{2,0}\lambda_2^2} & 0 \end{pmatrix}, \tag{10.12b}$$

and, letting $\tilde{q}_1 := q = |\tilde{r}(k_4)|^{\frac{1}{2}} r_1(k_4)$,

$$\beta_{12}^{(1)} = \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\nu_1}{2}}\sqrt{2\pi}\bar{\tilde{q}}_1}{(e^{\pi\nu_1} - e^{-\pi\nu_1})\Gamma(-i\nu_1)}, \qquad \beta_{21}^{(1)} = \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\nu_1}{2}}\sqrt{2\pi}\tilde{q}_1}{(e^{\pi\nu_1} - e^{-\pi\nu_1})\Gamma(i\nu_1)}, \tag{10.13a}$$

$$\beta_{12}^{(2)} = \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi(\nu_4-\nu_2)}\sqrt{2\pi}(\bar{q}_6 - \bar{q}_2\bar{q}_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(-i\hat{\nu}_2)}, \qquad \beta_{21}^{(2)} = \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi\nu_2}\sqrt{2\pi}(q_6 - q_2 q_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(i\hat{\nu}_2)}, \tag{10.13b}$$

and ν_1, ν_2, ν_4, $\hat{\nu}_2$ have been defined in Lemmas 6.1 and 6.3.

Lemma 10.2. *For $l \in \mathbb{Z}$ and $j = 0,1,2$, we have*

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega k_4)} k^l \hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j} F_1^{(l+1)}(\zeta,t)\mathcal{A}^j, \tag{10.14}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega k_4)^{-1})} k^l \hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B} F_1^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j, \tag{10.15}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega^2 k_2)} k^l \hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j} F_2^{(l+1)}(\zeta,t)\mathcal{A}^j, \tag{10.16}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega^2 k_2)^{-1})} k^l \hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F_2^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j. \tag{10.17}$$

Proof. This follows from the symmetries $\hat{w}(x,t,k) = \mathcal{A}\hat{w}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\hat{w}(x,t,k^{-1})\mathcal{B}$ valid for $k \in \partial\mathcal{D}$. The proof is identical to [9, Proof of Lemma 9.10], so we omit further details here. □

Using Lemma 10.2, we obtain

$$\begin{aligned}\frac{-1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)dk &= \sum_{j=0}^{2}\frac{-1}{2\pi i}\int_{\omega^j\partial D_\epsilon(\omega k_4)}\hat{w}(x,t,k)dk - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon((\omega k_4)^{-1})}\hat{w}(x,t,k)dk \\ &\quad - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(\omega^2 k_2)}\hat{w}(x,t,k)dk - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon((\omega^2 k_2)^{-1})}\hat{w}(x,t,k)dk \\ &= \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\big(F_1^{(1)}(\zeta,t)+F_2^{(1)}(\zeta,t)\big)\mathcal{A}^j - \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}\big(F_1^{(-1)}(\zeta,t)+F_2^{(-1)}(\zeta,t)\big)\mathcal{B}\mathcal{A}^j.\end{aligned}$$

Hence, by (10.9), (10.10), and (10.11), we obtain $\hat{n}^{(1)} = (1,1,1)\hat{m}^{(1)}$, where

$$\begin{aligned}\hat{m}^{(1)}(x,t) = &\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}F_1^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}F_1^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j \\ &+ \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}F_2^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}F_2^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t) \\ &= -i(\omega k_4)^1\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}Z_1(\zeta,t)\mathcal{A}^j + i(\omega k_4)^{-1}\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}Z_1(\zeta,t)\mathcal{B}\mathcal{A}^j \\ &- i(\omega^2 k_2)^1\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}Z_2(\zeta,t)\mathcal{A}^j + i(\omega^2 k_2)^{-1}\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}Z_2(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t)\end{aligned} \tag{10.18}$$

as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.

11. Asymptotics of u

Recall from (3.5) that $u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}n_3^{(1)}(x,t)$, where $n_3(x,t,k) = 1 + n_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. Using (4.9), (5.2), (7.1), and (10.1) to invert the transformations $n \mapsto n^{(1)} \mapsto n^{(2)} \mapsto n^{(3)} \mapsto \hat{n}$, we obtain

$$n = \hat{n}\Delta^{-1}(G^{(2)})^{-1}(G^{(1)})^{-1}, \qquad k \in \mathbb{C}\setminus(\hat{\Gamma}\cup\bar{\mathcal{D}}),$$

where $G^{(1)}$, $G^{(2)}$, Δ are defined in (4.10), (5.3), and (6.6), respectively. Using that $\partial_x = t^{-1}\partial_\zeta$, it follows that

$$\begin{aligned}u(x,t) &= -i\sqrt{3}\frac{\partial}{\partial x}\Big(\hat{n}_3^{(1)}(x,t) + \lim^{\angle}_{k\to\infty} k(\Delta_{33}(\zeta,k)^{-1}-1)\Big) \\ &= -i\sqrt{3}\frac{\partial}{\partial x}\hat{n}_3^{(1)}(x,t) + O(t^{-1}), \qquad t \to \infty,\end{aligned} \tag{11.1}$$

where $\hat{n}_3(x,t,k) = 1 + \hat{n}_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. Utilizing (10.12) and (10.18), we obtain, as $t \to \infty$,

$$\hat{n}_3^{(1)} = Z_{1,13}(i(\omega k_4)^{-1} - i(\omega k_4)^1) + Z_{1,31}(i\omega^2(\omega k_4)^{-1} - i\omega^1(\omega k_4)^1)$$

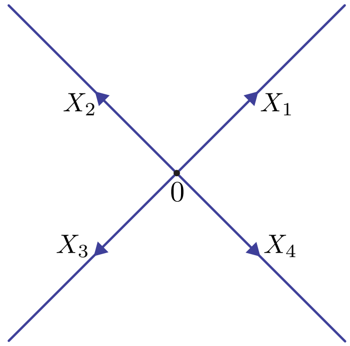

Figure 8. *The contour $X = X_1 \cup X_2 \cup X_3 \cup X_4$.*

$$+ Z_{2,23}(i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1) + Z_{2,32}(i\omega^1(\omega^2 k_2)^{-1} - i\omega^2(\omega^2 k_2)^1) + O(t^{-1}\ln t).$$

By Lemma 6.3, $\nu_1 \geq 0$ and $\hat{\nu}_2 \geq 0$. We also have $\nu_2, \nu_4 \in \mathbb{R}$. Hence (10.13) shows that $\overline{\beta_{12}^{(1)}} = \beta_{21}^{(1)}$ and $\overline{\beta_{12}^{(2)}} = e^{-2\pi(2\nu_2-\nu_4)}\beta_{21}^{(2)}$. Since $\lambda_1 = |\tilde{r}(k_4)|^{\frac{1}{4}} > 0$ and $\lambda_2 = |\tilde{r}(\frac{1}{k_2})|^{\frac{1}{4}} > 0$ by (8.4) and (9.4), we conclude with the help of (8.6) and (9.6) that $\overline{Z_{1,13}} = \lambda_1^4 Z_{1,31}$ and $\overline{Z_{2,23}} = \lambda_2^4 Z_{2,32}$. Using also that $\lambda_1^4 = -\tilde{r}(k_4)$ and $\lambda_2^4 = -\tilde{r}(\frac{1}{k_2})$, it follows that

$$\begin{aligned}\hat{n}_3^{(1)} &= 2i\,\mathrm{Im}\Big(Z_{1,13}(i(\omega k_4)^{-1} - i(\omega k_4)^1) + Z_{2,23}(i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1)\Big) + O(t^{-1}\ln t) \\ &= 2i\,\mathrm{Im}\left[\frac{\beta_{12}^{(1)}\tilde{d}_{1,0}(i(\omega k_4)^{-1} - i(\omega k_4)^1)}{-i\omega k_4 z_{1,\star}\sqrt{t}e^{t\Phi_{31}(\zeta,\omega k_4)}\big|\tilde{r}(k_4)\big|^{-\frac{1}{2}}} + \frac{\beta_{12}^{(2)}\tilde{d}_{2,0}(i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1)}{-i\omega^2 k_2 z_{2,\star}\sqrt{t}e^{t\Phi_{32}(\zeta,\omega^2 k_2)}\big|\tilde{r}(\frac{1}{k_2})\big|^{-\frac{1}{2}}}\right] + O\Big(\frac{\ln t}{t}\Big),\end{aligned}$$

as $t \to \infty$. Employing (10.13) and the relations

$$\begin{aligned}&\tilde{d}_{1,0}(\zeta,t) = e^{i\arg\tilde{d}_{1,0}(\zeta,t)}, \qquad \tilde{d}_{2,0}(\zeta,t) = e^{\pi(2\nu_2-\nu_4)}e^{i\arg\tilde{d}_{2,0}(\zeta,t)}, \\ &i(\omega k_4)^{-1} - i(\omega k_4)^1 = 2\sin(\arg(\omega k_4(\zeta))), \qquad i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1 = 2\sin(\arg(\omega^2 k_2(\zeta))), \\ &|\beta_{12}^{(1)}| = \sqrt{\nu_1}, \qquad |\beta_{12}^{(2)}|e^{\pi(2\nu_2-\nu_4)} = \sqrt{\hat{\nu}_2}, \qquad |\tilde{r}(k_4)|^{-\frac{1}{2}} = |\tilde{r}(\tfrac{1}{k_4})|^{\frac{1}{2}},\end{aligned}$$

we get

$$\begin{aligned}\hat{n}_3^{(1)} &= \frac{4i\sqrt{\nu_1}}{-i\omega k_4 z_{1,\star}\sqrt{t}|\tilde{r}(\frac{1}{k_4})|^{\frac{1}{2}}}\sin(\arg(\omega k_4))\sin(\tfrac{3\pi}{4} - \arg\tilde{q}_1 + \arg\Gamma(i\nu_1) + \arg\tilde{d}_{1,0} - t\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4)) \\ &+ \frac{4i\sqrt{\hat{\nu}_2}|\tilde{r}(\frac{1}{k_2})|^{\frac{1}{2}}}{-i\omega^2 k_2 z_{2,\star}\sqrt{t}}\sin(\arg(\omega^2 k_2))\sin(\tfrac{3\pi}{4} - \arg(q_6 - q_2 q_5) + \arg\Gamma(i\hat{\nu}_2) + \arg\tilde{d}_{2,0} - t\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2 k_2)) \\ &+ O(t^{-1}\ln t),\end{aligned}$$

as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$. Formula (2.12) now directly follows from (11.1) (as in [11], we can prove that the above error $O(t^{-1}\ln t)$ can be differentiated with respect to x without getting worse) and the fact that $\frac{d}{d\zeta}[\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4(\zeta))] = -\mathrm{Im}\,k_4$ and $\frac{d}{d\zeta}[\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2 k_2(\zeta))] = \mathrm{Im}\,k_2$. This finishes the proof of Theorem 2.1.

Appendix A. Model RH problems

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross defined by

$$\begin{aligned}&X_1 = \big\{se^{\frac{i\pi}{4}}\,\big|\,0 \leq s < \infty\big\}, \qquad X_2 = \big\{se^{\frac{3i\pi}{4}}\,\big|\,0 \leq s < \infty\big\}, \\ &X_3 = \big\{se^{-\frac{3i\pi}{4}}\,\big|\,0 \leq s < \infty\big\}, \qquad X_4 = \big\{se^{-\frac{i\pi}{4}}\,\big|\,0 \leq s < \infty\big\},\end{aligned} \tag{A.1}$$

and oriented away from the origin, see Figure 8.

Lemma A.1 (Model RH problem needed near $k = \omega k_4$). *Let $q \in \mathbb{D}$, and define*

$$\nu = -\tfrac{1}{2\pi} \ln(1 - |q|^2).$$

Define the jump matrix $v^{X,(1)}(z)$ for $z \in X$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -qz^{-2i\nu} e^{\frac{iz^2}{2}} & 0 & 1 \end{pmatrix} \text{ if } z \in X_1, \qquad \begin{pmatrix} 1 & 0 & \frac{-\bar{q}}{1-|q|^2} z^{2i\nu} e^{-\frac{iz^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_2,$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{q}{1-|q|^2} z^{-2i\nu} e^{\frac{iz^2}{2}} & 0 & 1 \end{pmatrix} \text{ if } z \in X_3, \qquad \begin{pmatrix} 1 & 0 & \bar{q} z^{2i\nu} e^{-\frac{iz^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_4, \tag{A.2}$$

where $z^{i\nu}$ has a branch cut along $(-\infty, 0]$, such that $z^{i\nu} = |z|^{i\nu} e^{-\nu \arg(z)}$, $\arg(z) \in (-\pi, \pi)$. *Then the RH problem*

(a) $m^{X,(1)}(\cdot) = m^{X,(1)}(q, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ is analytic;
(b) on $X \setminus \{0\}$, the boundary values of $m^{X,(1)}$ exist, are continuous, and satisfy $m_+^{X,(1)} = m_-^{X,(1)} v^{X,(1)}$;
(c) $m^{X,(1)}(z) = I + O(z^{-1})$ as $z \to \infty$, and $m^{X,(1)}(z) = O(1)$ as $z \to 0$;

has a unique solution $m^{X,(1)}(q, z)$. This solution satisfies

$$m^{X,(1)}(z) = I + \frac{m_1^{X,(1)}}{z} + O\bigg(\frac{1}{z^2}\bigg), \quad z \to \infty, \quad m_1^{X,(1)} := \begin{pmatrix} 0 & 0 & \beta_{12}^{(1)} \\ 0 & 0 & 0 \\ \beta_{21}^{(1)} & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where the error term is uniform with respect to $\arg z \in [-\pi, \pi]$ *and q in compact subsets of $\mathbb{D}$, and*

$$\beta_{12}^{(1)} := \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\nu}{2}} \sqrt{2\pi} \bar{q}}{(e^{\pi\nu} - e^{-\pi\nu})\Gamma(-i\nu)}, \qquad \beta_{21}^{(1)} := \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\nu}{2}} \sqrt{2\pi} q}{(e^{\pi\nu} - e^{-\pi\nu})\Gamma(i\nu)}. \tag{A.4}$$

(Note that $\beta_{12}^{(1)} \beta_{21}^{(1)} = \nu$ because $|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{\nu(e^{\pi\nu} - e^{-\pi\nu})}}$.)

Lemma A.2 below was also used in [9] for the local analysis near $k = \omega^2 k_2$ when $\frac{x}{t} \in (\frac{1}{\sqrt{3}}, 1)$. We provide here a proof of this lemma.

Lemma A.2 (Model RH problem needed near $k = \omega^2 k_2$). *Let $q_2, q_4, q_5, q_6 \in \mathbb{C}$ be such that $1 + |q_2|^2 - |q_4|^2 > 0$, $1 - |q_5|^2 - |q_6|^2 > 0$, and*

$$q_4 - \bar{q}_5 - q_2 \bar{q}_6 = 0. \tag{A.5}$$

Define $\nu_2, \nu_4, \nu_5 \in \mathbb{R}$ by

$$\nu_2 = -\tfrac{1}{2\pi} \ln(1 + |q_2|^2), \quad \nu_4 = -\tfrac{1}{2\pi} \ln(1 + |q_2|^2 - |q_4|^2), \quad \nu_5 = -\tfrac{1}{2\pi} \ln(1 - |q_5|^2 - |q_6|^2).$$

Define the jump matrix $v^{X,(2)}(z)$ for $z \in X$ by

$$v^{X,(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6 - q_2 q_5}{1+|q_2|^2} z^{-i(2\nu_5 - \nu_4)} z_{(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1-|q_5|^2-|q_6|^2} z^{i(2\nu_5 - \nu_4)} z_{(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{q_6 - q_2 q_5}{1-|q_5|^2-|q_6|^2} z^{-i(2\nu_5 - \nu_4)} z_{(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1+|q_2|^2} z^{i(2\nu_5 - \nu_4)} z_{(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_4, \end{cases} \tag{A.6}$$

where $z^{i\nu}$ has a branch cut along $(-\infty, 0]$ and $z_{(0)}^{i\nu}$ has a branch cut along $[0, +\infty)$ such that $z^{i\nu} = |z|^{i\nu} e^{-\nu \arg(z)}$, $\arg(z) \in (-\pi, \pi)$, and $z_{(0)}^{i\nu} = |z|^{i\nu} e^{-\nu \arg_0(z)}$, $\arg_0(z) \in (0, 2\pi)$. Then the RH problem

(a) *$m^{X,(2)}(\cdot) = m^{X,(2)}(q_2, q_4, q_5, q_6, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ is analytic;*

(b) *on $X \setminus \{0\}$, the boundary values of $m^{X,(2)}$ exist, are continuous, and satisfy $m_+^{X,(2)} = m_-^{X,(2)} v^{X,(2)}$;*

(c) *$m^{X,(2)}(z) = I + O(z^{-1})$ as $z \to \infty$, and $m^{X,(2)}(z) = O(1)$ as $z \to 0$;*

has a unique solution $m^{X,(2)}(z)$. This solution satisfies

$$m^{X,(2)}(z) = I + \frac{m_1^{X,(2)}}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad m_1^{X,(2)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{12}^{(2)} \\ 0 & \beta_{21}^{(2)} & 0 \end{pmatrix}, \tag{A.7}$$

where the error term is uniform with respect to $\arg z \in [-\pi, \pi]$ *and q_2, q_4, q_5, q_6 in compact subsets of $\{q_2, q_4, q_5, q_6 \in \mathbb{C} \,|\, 1 + |q_2|^2 - |q_4|^2 > 0, 1 - |q_5|^2 - |q_6|^2 > 0, q_4 - \bar{q}_5 - q_2\bar{q}_6 = 0\}$, and*

$$\beta_{12}^{(2)} := \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi \hat{\nu}_2}{2}} e^{2\pi(\nu_4 - \nu_2)} \sqrt{2\pi} (\bar{q}_6 - \bar{q}_2 \bar{q}_5)}{(e^{\pi \hat{\nu}_2} - e^{-\pi \hat{\nu}_2}) \Gamma(-i\hat{\nu}_2)}, \qquad \beta_{21}^{(2)} := \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi \hat{\nu}_2}{2}} e^{2\pi \nu_2} \sqrt{2\pi} (q_6 - q_2 q_5)}{(e^{\pi \hat{\nu}_2} - e^{-\pi \hat{\nu}_2}) \Gamma(i\hat{\nu}_2)}, \tag{A.8}$$

where $\hat{\nu}_2 := \nu_2 + \nu_5 - \nu_4$. (Note that $\beta_{12}^{(2)} \beta_{21}^{(2)} = \hat{\nu}_2$.)

Proof. Uniqueness of $m^{X,(2)}(z)$ follows from standard arguments. Let us temporarily assume that $m^{X,(2)}(z)$ exists. Define $m^{(X1)}(z) := m^{X,(2)}(z) z^{i(\nu_5 - \frac{\nu_4}{2})\tilde{\sigma}} z_{(0)}^{i(\nu_2 - \frac{\nu_4}{2})\tilde{\sigma}} e^{-\frac{iz^2}{4}\tilde{\sigma}}$ where $\tilde{\sigma} = \operatorname{diag}(0, 1, -1)$. We verify that

$$m_+^{(X1)}(z) = m_-^{(X1)}(z) v^{(X1)}(z) \quad \text{for} \quad z \in (X \cup \mathbb{R}) \setminus \{0\},$$

where $\mathbb{R}$ is oriented from left to right and $v^{(X1)}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6 - q_2 q_5}{1+|q_2|^2} & 1 \end{pmatrix} \text{ if } z \in X_1, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1-|q_5|^2-|q_6|^2} \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_2,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{q_6 - q_2 q_5}{1-|q_5|^2-|q_6|^2} & 1 \end{pmatrix} \text{ if } z \in X_3, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1+|q_2|^2} \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_4,$$
$$e^{\pi(\nu_4 - 2\nu_5)\tilde{\sigma}} \text{ if } z \in \mathbb{R}_-, \qquad e^{\pi(2\nu_2 - \nu_4)\tilde{\sigma}} \text{ if } z \in \mathbb{R}_+.$$

We next collapse the contours X_1 and X_4 onto $\mathbb{R}_+$ and the contours X_2 and X_3 onto $\mathbb{R}_-$. To this end we let

$$\psi(z) = m^{(X1)}(z) B(z), \tag{A.9}$$

where $B(z)$ is equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6 - q_2 q_5}{1+|q_2|^2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1-|q_5|^2-|q_6|^2} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{q_6 - q_2 q_5}{1-|q_5|^2-|q_6|^2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1+|q_2|^2} \\ 0 & 0 & 1 \end{pmatrix}$$

for $\arg z \in (0, \frac{\pi}{4})$, $\arg z \in (\frac{3\pi}{4}, \pi)$, $\arg z \in (-\pi, -\frac{3\pi}{4})$, $\arg z \in (-\frac{\pi}{4}, 0)$, respectively, and $B(z) = I$ otherwise. The function ψ satisfies $\psi_+(z) = \psi_-(z) v^\psi(z)$ for $z \in \mathbb{R} \setminus \{0\}$, where v^ψ is piecewise constant and defined by

$$\begin{aligned} v^\psi(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1+|q_2|^2} \\ 0 & 0 & 1 \end{pmatrix} e^{\pi(2\nu_2 - \nu_4)\tilde{\sigma}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6 - q_2 q_5}{1+|q_2|^2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi(2\nu_2 - \nu_4)} - e^{-\pi(2\nu_2 - \nu_4)} \frac{|q_6 - q_2 q_5|^2}{(1+|q_2|^2)^2} & e^{-\pi(2\nu_2 - \nu_4)} \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1+|q_2|^2} \\ 0 & -e^{-\pi(2\nu_2 - \nu_4)} \frac{q_6 - q_2 q_5}{1+|q_2|^2} & e^{-\pi(2\nu_2 - \nu_4)} \end{pmatrix}, \qquad z \in \mathbb{R}_+, \\ v^\psi(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6 - q_2 q_5}{1-|q_5|^2-|q_6|^2} & 1 \end{pmatrix} e^{\pi(\nu_4 - 2\nu_5)\tilde{\sigma}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1-|q_5|^2-|q_6|^2} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi(\nu_4 - 2\nu_5)} & e^{\pi(\nu_4 - 2\nu_5)} \frac{\bar{q}_6 - \bar{q}_2 \bar{q}_5}{1-|q_5|^2-|q_6|^2} \\ 0 & -e^{\pi(\nu_4 - 2\nu_5)} \frac{q_6 - q_2 q_5}{1-|q_5|^2-|q_6|^2} & e^{-\pi(\nu_4 - 2\nu_5)} - e^{\pi(\nu_4 - 2\nu_5)} \frac{|q_6 - q_2 q_5|^2}{(1-|q_5|^2-|q_6|^2)^2} \end{pmatrix}, \qquad z \in \mathbb{R}_-. \end{aligned}$$

Using (A.5), it is a simple calculation to verify that $v^\psi(z)$ is constant on the whole real line. This implies that $(\partial_z \psi)\psi^{-1}$ is an entire function. Note that

$$\psi(z) = (I + m_1^{X,(2)} z^{-1} + O(z^{-2})) z^{i(\nu_5 - \frac{\nu_4}{2})\tilde{\sigma}} z_{(0)}^{i(\nu_2 - \frac{\nu_4}{2})\tilde{\sigma}} e^{-\frac{iz^2}{4}\tilde{\sigma}} B(z) \qquad \text{as } z \to \infty. \tag{A.10}$$

Hence, assuming that the asymptotics can be differentiated termwise,

$$(\partial_z \psi)\psi^{-1} = -\frac{iz}{2}\tilde{\sigma} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\beta_{12}^{(2)} \\ 0 & -i\beta_{21}^{(2)} & 0 \end{pmatrix}, \tag{A.11}$$

for some $\beta_{12}^{(2)}, \beta_{21}^{(2)} \in \mathbb{C}$. Differentiating a second time we get

$$\begin{cases} \partial_z^2 \psi_{33} + (\frac{z^2}{4} - \frac{i}{2} - \beta_{12}^{(2)}\beta_{21}^{(2)})\psi_{33} = 0, \\ \partial_z^2 \psi_{22} + (\frac{z^2}{4} + \frac{i}{2} - \beta_{12}^{(2)}\beta_{21}^{(2)})\psi_{22} = 0, \end{cases} \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

By [26, Chapter 12],

$$\psi_{33}(z) = \begin{cases} c_1 D_{-i\nu}(e^{\frac{3\pi i}{4}} z) + c_2 D_{-i\nu}(e^{-\frac{\pi i}{4}} z), & \operatorname{Im} z > 0, \\ c_3 D_{-i\nu}(e^{\frac{3\pi i}{4}} z) + c_4 D_{-i\nu}(e^{-\frac{\pi i}{4}} z), & \operatorname{Im} z < 0, \end{cases} \tag{A.12a}$$

$$\psi_{22}(z) = \begin{cases} c_5 D_{i\nu}(e^{-\frac{3\pi i}{4}} z) + c_6 D_{i\nu}(e^{\frac{\pi i}{4}} z), & \operatorname{Im} z > 0, \\ c_7 D_{i\nu}(e^{-\frac{3\pi i}{4}} z) + c_8 D_{i\nu}(e^{\frac{\pi i}{4}} z), & \operatorname{Im} z < 0. \end{cases} \tag{A.12b}$$

for some constants $\{c_j\}_1^8$, where $\nu := \beta_{12}^{(2)}\beta_{21}^{(2)}$ and D_ν denotes the parabolic cylinder function. The 23 and 32 entries of ψ can be found directly using (A.11):

$$\psi_{23}(z) = \frac{i\psi_{33}'(z) + \frac{z}{2}\psi_{33}(z)}{\beta_{21}^{(2)}}, \qquad \psi_{32}(z) = \frac{-i\psi_{22}'(z) + \frac{z}{2}\psi_{22}(z)}{\beta_{12}^{(2)}},$$

where primes denote derivative with respect to z. Hence, we find

$$\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \psi_{22} & \frac{i\psi_{33}' + \frac{z}{2}\psi_{33}}{\beta_{21}^{(2)}} \\ 0 & \frac{-i\psi_{22}' + \frac{z}{2}\psi_{22}}{\beta_{12}^{(2)}} & \psi_{33} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{A.13}$$

From (A.10), we obtain

$$\psi_{33}(z) = z^{-i(\nu_5 - \frac{\nu_4}{2})} z_{(0)}^{-i(\nu_2 - \frac{\nu_4}{2})} e^{\frac{iz^2}{4}}(1 + O(z^{-1})) = \begin{cases} z^{-i(\nu_2+\nu_5-\nu_4)} e^{\frac{iz^2}{4}}(1 + O(z^{-1})), \\ e^{-\pi(\nu_4 - 2\nu_2)} z^{-i(\nu_2+\nu_5-\nu_4)} e^{\frac{iz^2}{4}}(1 + O(z^{-1})), \end{cases}$$

$$\psi_{22}(z) = z^{i(\nu_5 - \frac{\nu_4}{2})} z_{(0)}^{i(\nu_2 - \frac{\nu_4}{2})} e^{-\frac{iz^2}{4}}(1 + O(z^{-1})) = \begin{cases} z^{i(\nu_2+\nu_5-\nu_4)} e^{-\frac{iz^2}{4}}(1 + O(z^{-1})), \\ e^{\pi(\nu_4 - 2\nu_2)} z^{i(\nu_2+\nu_5-\nu_4)} e^{-\frac{iz^2}{4}}(1 + O(z^{-1})), \end{cases}$$

as $z \to \infty$, where in each bracket the first and second lines apply for $\arg z \in (\epsilon, \pi - \epsilon)$ and $\arg z \in (-\pi + \epsilon, -\epsilon)$, respectively (for any $\epsilon \in (0, \frac{\pi}{2})$). Since

$$D_\nu(z) = z^\nu e^{-\frac{z^2}{4}}(1 + O(z^{-2})),$$

as $z \to \infty$, $\arg z \in (-\frac{3\pi}{4} + \epsilon, \frac{3\pi}{4} - \epsilon)$, these asymptotic formulas are consistent with (A.12) provided that we choose $\nu = \nu_2 + \nu_5 - \nu_4$ and

$$\begin{aligned} &c_1 = 0, && c_2 = e^{\frac{\pi\nu}{4}}, && c_3 = e^{-\frac{3\pi\nu}{4}} e^{-\pi(\nu_4 - 2\nu_2)}, && c_4 = 0, \\ &c_5 = e^{-\frac{3\pi\nu}{4}}, && c_6 = 0, && c_7 = 0, && c_8 = e^{\frac{\pi\nu}{4}} e^{\pi(\nu_4 - 2\nu_2)}. \end{aligned}$$

With this choice, we get

$$\psi_{33}(z) = \begin{cases} e^{\frac{\pi\nu}{4}} D_{-i\nu}(e^{-\frac{\pi i}{4}} z), & \operatorname{Im} z > 0, \\ e^{-\frac{3\pi\nu}{4}} e^{-\pi(\nu_4 - 2\nu_2)} D_{-i\nu}(e^{\frac{3\pi i}{4}} z), & \operatorname{Im} z < 0, \end{cases} \tag{A.14a}$$

$$\psi_{22}(z) = \begin{cases} e^{-\frac{3\pi\nu}{4}} D_{i\nu}(e^{-\frac{3\pi i}{4}} z), & \operatorname{Im} z > 0, \\ e^{\frac{\pi\nu}{4}} e^{\pi(\nu_4 - 2\nu_2)} D_{i\nu}(e^{\frac{\pi i}{4}} z), & \operatorname{Im} z < 0. \end{cases} \tag{A.14b}$$

and evaluation of the jump relation $\psi_+ = \psi_- v^\psi$ at $z = 0$ then gives (A.8). The expressions in (A.14) lead to an explicit formula for $m^{X,(2)}$ in terms of parabolic cylinder functions. This expression was established under certain assumptions. However, it can now be verified directly that the constructed function $m^{X,(2)}(z)$ indeed satisfies the stated RH problem and the asymptotic formula (A.7). $\square$

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