The co-smash product as an intrinsic tensor product

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joint work-in-progress with Manfred Hartl

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Co-smash products in semi-abelian categories

[A Carboni & G Janelidze, 2003]

Any two objects X and Y in a semi-abelian category
induce a short exact sequence

$$X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

The kernel $X \diamond Y$ is called the **co-smash product** of X and Y.

Co-smash products may serve as (amongst other things)

1. formal (Higgins) commutators
   [S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013];
2. (intrinsic) tensor products
   (as shown for commutative rings by Carboni and Janelidze).

Our aim today is to explore 2.

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Problem: the Ganea term in the exact homology sequence

Theorem [T Ganea, 1968]

Any central extension of groups

$$0 \longrightarrow K \rightarrowtail B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

induces a six-term exact sequence

$$K \otimes_{\mathbb{Z}} \frac{B}{[B,B]} \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

- Versions of this are known in several other categories, but… until recently, we had no categorical-algebraic interpretation
- cf. long exact homology sequence [T Everaert, 2010]

The missing object is a co-smash product… but *where*?

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Where to take the co-smash product?

$\mathbb{X}$

$X \qquad X \diamond_{\mathbb{X}} Y \qquad Y$

In $\mathbb{X}$ itself, the co-smash product $X \diamond_{\mathbb{X}} Y$ is a formal commutator.

Definition

Given objects X and Y in a semi-abelian category $\mathbb{X}$,
their **bilinear product** $X \otimes_{\mathbb{X}} Y \in |\mathbf{Ab}(\mathbb{X})|$ is the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of $\mathbb{X}$.

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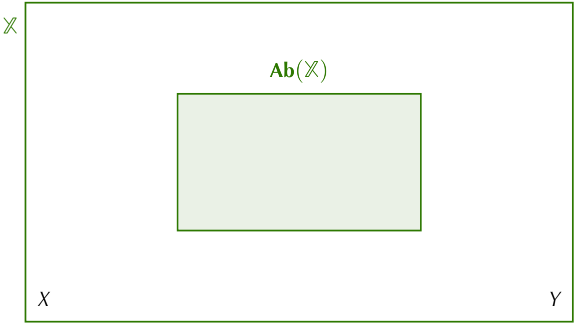

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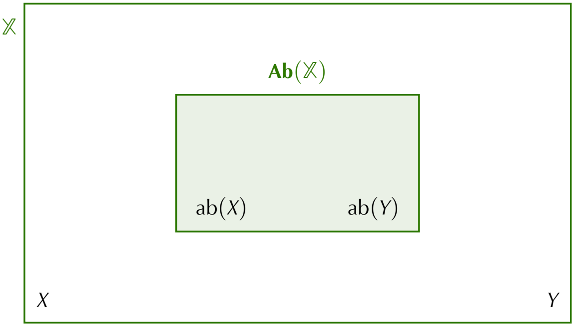

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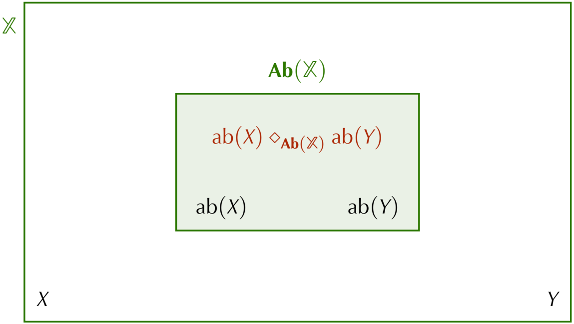

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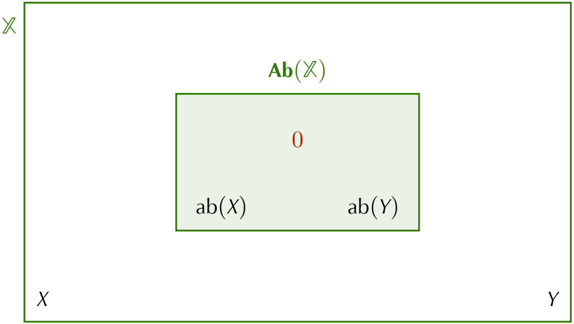

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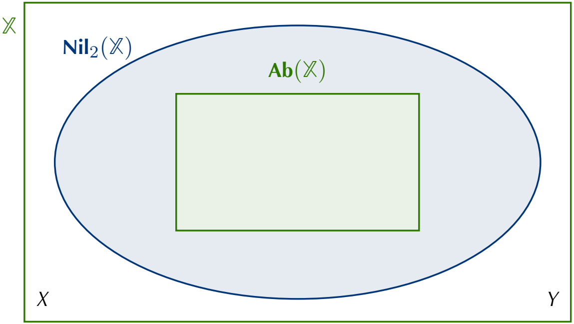

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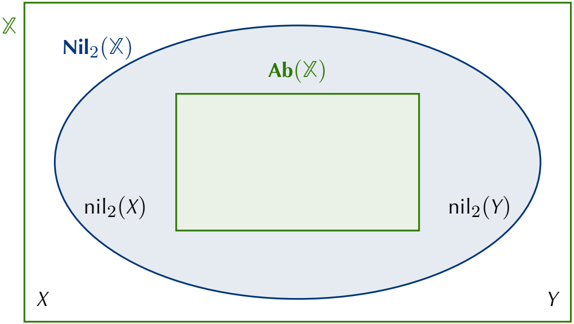

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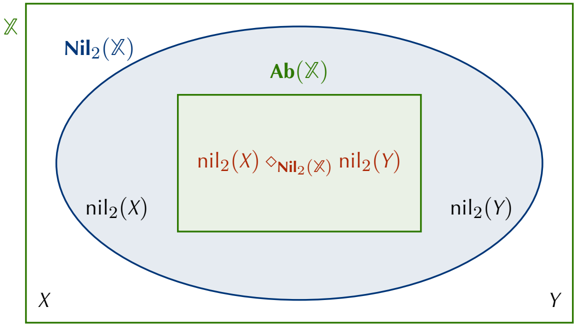

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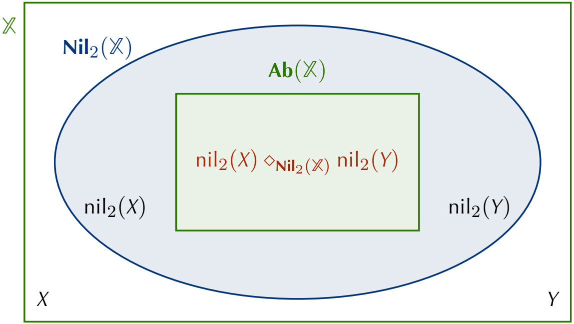

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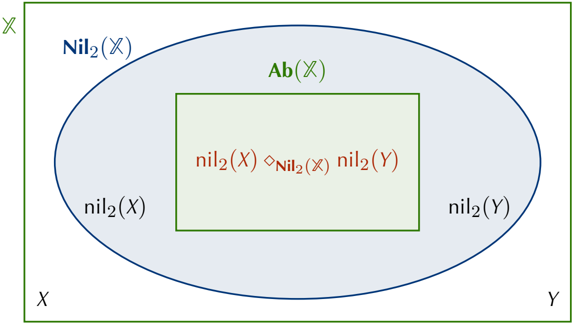

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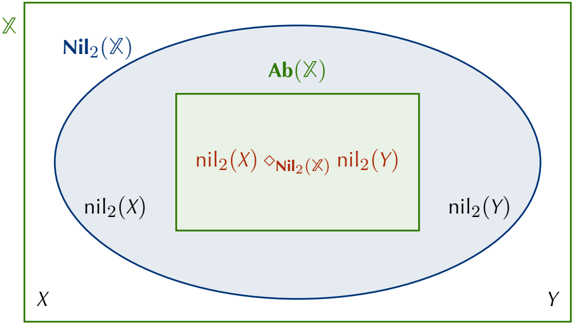

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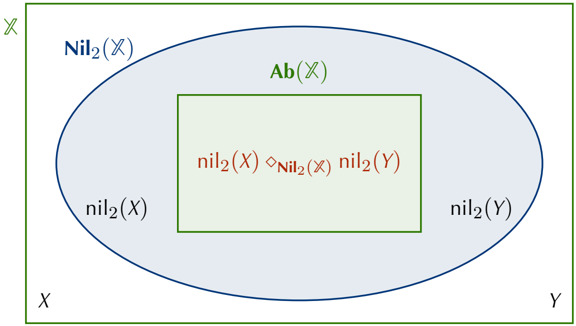

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Binary co-smash products and Higgins commutators

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Consider $K, L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

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- $[K, L] = 0$ if and only if $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004].
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Higher-order co-smash products and Higgins commutators

[P J Higgins, 1956] [A Carboni & G Janelidze, 2003] [M Hartl & TVdL, 2013]

The **higher-order co-smash product** $X_1 \diamond \cdots \diamond X_n$ defined as

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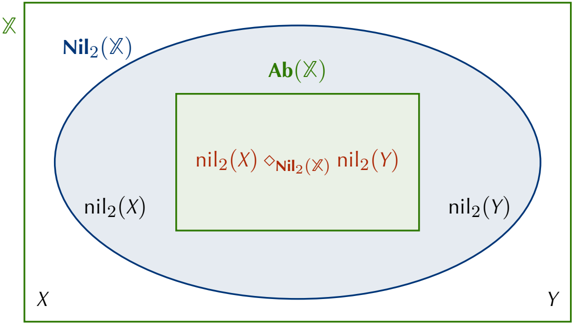
Where to take the co-smash product?



In $\mathbf{Nil}_2(\mathbb{X})$, the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ is abelian.

Definition

Given objects X and Y in a semi-abelian category $\mathbb{X}$,
their **bilinear product** $X \otimes_{\mathbb{X}} Y \in |\mathbf{Ab}(\mathbb{X})|$ is the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of $\mathbb{X}$.

Examples

For groups, we get $X \otimes_{\mathbf{Gp}} Y \cong \mathrm{ab}(X) \otimes_{\mathbb{Z}} \mathrm{ab}(Y)$.
As far as I know, this phenomenon was first described in
[T MacHenry, 1960], then rediscovered in [M Hartl & C Vespa, 2011].

Considered as an operation on an abelian category $\mathbb{A} = \mathbf{Ab}(\mathbb{X})$, the bilinear product depends on $\mathbb{X}$. We write $M \otimes_{\mathbb{X}} N$.

category $\mathbb{X}$	objects	abelian core $\mathbf{Ab}(\mathbb{X})$	$M \otimes_{\mathbb{X}} N$
$\mathbf{Gp}$	groups	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{Loop}$	loops	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{Nil}_2(\mathbf{Gp})$	two-nilpotent groups	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{CRng}$	commutative rings	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{Mod}_R$	R-modules	$\mathbf{Mod}_R$	0
$\mathbf{CAlg}_R$	commutative R-algebras	$\mathbf{Mod}_R$	$M \otimes_R N$
$\mathbf{Alg}_R$	R-algebras	$\mathbf{Mod}_R$	$(M \otimes_R N) \oplus (N \otimes_R M)$
$\mathbf{Lie}_R$	R-Lie algebras	$\mathbf{Mod}_R$	$M \otimes_R N$
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- Internal (pre)crossed modules (gives known concepts);
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Examples

R **commutative ring with unit**

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Basic properties of bilinear products

- bilinear products are symmetric: $X \otimes_{\mathbb{X}} Y \cong Y \otimes_{\mathbb{X}} X$
- bilinear products are **bilinear**: $X \otimes_{\mathbb{X}} 0 = 0$ and $X \otimes_{\mathbb{X}} (Y + Z) \cong (X \otimes_{\mathbb{X}} Y) + (X \otimes_{\mathbb{X}} Z)$
- $X \otimes_{\mathbb{X}} (-)$ is sequentially right exact; furthermore, it preserves *all* colimits when $\mathbb{X}$ is a variety
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- $X \otimes_{\mathbb{X}} Y \coloneqq \mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ may be computed in $\mathbb{X}$ by dividing

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Basic properties of bilinear products

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$$N_{X,Y} := \mathrm{im}(X \diamond X \diamond Y) \vee \mathrm{im}(X \diamond Y \diamond Y)$$

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- bilinear products are abelian, because $[X \diamond Y, X \diamond Y] \leqslant N_{X,Y}$ via

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{[X \diamond Y, X \diamond Y]} \arrow[rr, dotted] & & \mathrm{im}(X \diamond X \diamond Y)
\end{tikzcd}$$

Ganea's Theorem

Theorem

Any central extension

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

in a semi-abelian algebraically coherent category with enough projectives $\mathbb{X}$ induces a six-term exact sequence

$$K \otimes_{\mathbb{X}} B \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

- Algebraically coherent cats: [J R A Gray, A S Cigoli & TVdL, 2014]
- This captures the known results for groups, (pre)crossed modules, associative algebras, Lie and Leibniz algebras.
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Idea of proof

$$\begin{tikzcd} K \otimes_{\mathbb{X}} B \arrow[r, dotted, "?"'] & \mathrm{H}_2 B \arrow[r, "\mathrm{H}_2 f"'] & \mathrm{H}_2 A \end{tikzcd}$$

- P projective, $B \cong P/R$, $A \cong P/L$, and $K \cong L/R$
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$$\begin{tikzcd} & & P \arrow[d, two heads, "p"'] & & \\ 0 \arrow[r] & K \arrow[r, tail] & B \arrow[r, two heads, "f"'] & A \arrow[r] & 0 \end{tikzcd}$$

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\begin{tikzcd}
 & & R \arrow[r, tail] \arrow[d, tail] & L \arrow[d, tail] & \\
 & & P \arrow[r, equal] \arrow[d, "p"', two heads] & P \arrow[d, "f\circ p", two heads] & \\
0 \arrow[r] & K \arrow[r, tail] & B \arrow[r, "f"', two heads] & A \arrow[r] & 0
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0 \arrow[r] & K \arrow[r, tail] & B \arrow[r, two heads, "f"'] & A \arrow[r] & 0
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\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[P,P]\cap R}{[R,P]} \arrow[r] \arrow[d, equal] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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 & & {[P,P] \cap L} \arrow[d, two heads] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[P,P]\cap R}{[R,P]} \arrow[r] \arrow[d, equal] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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 & & {[P,P]\cap R} \arrow[r, tail] \arrow[dl, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] \arrow[dr, phantom, "\lrcorner", very near start] & {[P,P]\cap L} \arrow[d, -{Triangle[open]}] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[P,P]\cap R}{[R,P]} \arrow[r, -{Triangle[open]}] \arrow[d, equal] & \frac{[P,P]\cap R}{[L,P]} \arrow[r, tail] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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 & & & {[L,P]} \arrow[r, equal] \arrow[d, "{Triangle[open]}->"' description, phantom] \arrow[d, {Triangle[open]}->] & {[L,P]} \arrow[d, {Triangle[open]}->] \\
 & & & {[P,P]\cap R} \arrow[r, {Triangle[open]}->] \arrow[dl, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] & {[P,P]\cap L} \arrow[d, -{Triangle[open]}] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[L,P]}{[R,P]} \arrow[r, {Triangle[open]}->] & \frac{[P,P]\cap R}{[R,P]} \arrow[r, -{Triangle[open]}] \arrow[d, equal] & \frac{[P,P]\cap R}{[L,P]} \arrow[r, {Triangle[open]}->] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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L \otimes_{\mathbb{X}} P \arrow[d, two heads] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl, two heads] \arrow[d, two heads] & {[P,P] \cap L} \arrow[d, two heads] \\
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- under (AC), we have $[L, P, P] \leqslant [[L, P], P] \leqslant [R, P]$

Idea of proof

$$\begin{tikzcd}
L \diamond P \arrow[r] \arrow[d] & {[L,P]} \arrow[d] & & {[L,P]} \arrow[r, equal] \arrow[d, tail] & {[L,P]} \arrow[d, tail] \\
L \otimes_{\mathbb{X}} P \arrow[d] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl] \arrow[d] & {[P,P] \cap L} \arrow[d] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[L,P]}{[R,P]} \arrow[r, tail] & \frac{[P,P]\cap R}{[R,P]} \arrow[r] \arrow[d, equal] & \frac{[P,P]\cap R}{[L,P]} \arrow[r, tail] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
K \otimes_{\mathbb{X}} B \arrow[rr, dotted, "?"'] & & \mathrm{H}_2 B \arrow[rr, "\mathrm{H}_2 f"'] & & \mathrm{H}_2 A
\end{tikzcd}$$

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Ganea's Theorem

Theorem

Any central extension

$$0 \longrightarrow K \rightarrowtail^{k} B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

in a semi-abelian algebraically coherent category with enough projectives $\mathbb{X}$ *induces a six-term exact sequence*

$$K \otimes_{\mathbb{X}} B \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

- Algebraically coherent cats: [J R A Gray, A S Cigoli & TVdL, 2014]
- This captures the known results for groups, (pre)crossed modules, associative algebras, Lie and Leibniz algebras.
  [T Pirashvili, 2000] [J-M Casas & T Pirashvili, 2000]
  [D Arias & M Ladra, 2006]

Conclusion

Computed in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of a semi-abelian category $\mathbb{X}$, the co-smash product may play the role of intrinsic tensor product.

The Ganea term in the homology sequence of a central extension is such an intrinsic tensor product.

To do:

- Other applications of bilinear products?
- How does a bilinear product $\otimes$ on an abelian category $\mathbb{A}$ induce a semi-abelian category $\mathbb{X}$ such that $\mathbb{A} = \mathbf{Ab}(\mathbb{X})$ and $\otimes = \otimes_{\mathbb{X}}$?
  $\rightsquigarrow$ in the varietal case: algebras over an operad
- When is $\otimes_{\mathbb{X}}$ monoidal on $\mathbf{Ab}(\mathbb{X})$?
- How to capture the tensor product of associative algebras?
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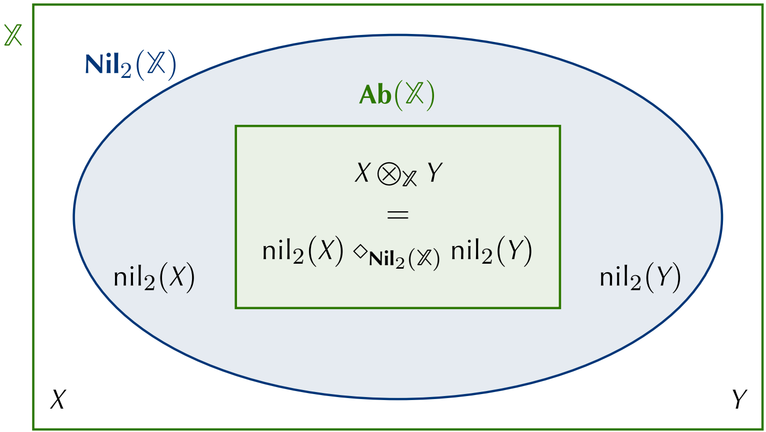
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X
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