



2017/37

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Alonso and the Scaling of Urban Profiles

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Keywords: Monocentric model, population density, scaling laws, agglomeration economies

Abstract

Urban characteristics scaling with total population has become an important urban research field since one needs to better understand the benefits and disadvantages of urban growth and further population concentration. Urban scaling research, however, is largely disconnected from the empirics and theory of intra-urban structure for it considers averaged attributes and ignores residential choice trade-offs between transport and housing costs within cities. Using this fundamental trade-off, the monocentric model of Alonso provides theory to urban density profiles. However, it is silent about how these profiles scale with population, thus preventing empirical scaling studies to anchor in a strong micro-economic theory. This paper fixes this gap by introducing power laws for land and for population density in the Alonso model. From an augmented model with land use, we derive the conditions at which equilibrium profiles match recent empirical findings about the scaling of urban land and population density profiles in European cities. We find that the Alonso model is theoretically compatible with the observed scaling of population density profiles and leads to a satisfactory representation of European cities. The conditions for this compatibility refine current understanding of wage and transport costs elasticities with population. Although they require a scaling power of the profile of the share of urbanised land that is different from what is observed, it is argued that alternatives specifications of transport cost functions could solve this issue. Thus our results call for revisiting theories about land development and housing processes as well as the empirics of agglomeration benefits and transport costs.

J. Delloye acknowledges funding by the Belgian "Fonds pour la Recherche en Sciences Humaines" (FNRS-FSR, FRESH Grant No 29315817). R. Lemoy and G. Caruso acknowledge funding from the "Fonds National de la Recherche" in Luxembourg (AFR PostDoc grant ALONSO-EU). The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

1 Introduction

It is theoretically elegant and empirically convenient to think of all the good and bad of cities simply in terms of their total population. We live in an increasingly urban World (UN-HABITAT, 2016) and liasing the social and environmental outcomes of cities to their size is definitely an important question today and for tomorrow. Yet, we know that many outcomes of cities depend crucially on their internal structure, especially on how densely citizens occupy the land they have developed. This occupation emerges from the location decisions of many people interacting in space and is often described or discussed in radial terms, that is how far reaching a city is (the urban fringe distance) and how flat/steep its density profile is. This is a key interest of theoretical and empirical urban economics (see Anas et al., 1998, for a reminder) and the favourite playground of urban planning. The long dispute between compactness or sprawl (e.g. Ewing et al., 2014, for a quick summary) just shows how much this internal structure matters and is worth being studied. Therefore, before summing-up a city as the outcome of a single termed function of population, one needs (i) first to make sure that the internal structure of cities is independent of population, or is at least independent of a simple (well-behaved) transformation of population, and (ii) second - particularly if desirable actions need to be made with potential social impacts - one needs to know if this internal structure responds to the same underlying decisional processes independent of size, in other terms that the same urban theory holds across the population distribution of cities.

Nordbeck (1971) provided an intuition to the first need, and opened up a literature strand on allometric urban growth by assuming that cities, similarly to biological objects, keep the same form across size. Lemoy and Caruso (2017) recently endorsed this idea and empirically identified the homothetic transformation¹ of density and land gradients with population for European cities. A logical extra step is then to address the second need described above and assess whether models that can generate observed urban gradients can also replicate their scaling with population. Finding a valid model that can be applied to any cities after simple rescaling would definitely bear powerful implications for understanding cities and identifying generic planning recipes independent of size. The Alonso-Muth-Mills monocentric framework (Alonso, 1964; Muth, 1969; Mills, 1972) is a perfect candidate because it issues micro-foundations to urban expansion limits and density gradients. It does so after fixing population in its closed equilibrium form, or after fixing its social outcome (utility) in its open form where equilibrium with other cities is then assumed and the population an output.

In this paper we assess the theoretical ability and conditions for the Alonso model to replicate the scaling behaviour of urban density and urban land profiles. Given that the Alonso model however assumes a fully urbanised disc, which is inconsistent with the presence of semi-natural land within cities and with a decreasing profile of urbanised land, our model exogeneously relaxes this assumption. We then test how the standard form of Alonso and its new land use form (named “Alonso-LU”) empirically perform in Europe.

2 Background

In the last few decades, and particularly since the advent of the complexity paradigm (Arthur et al., 1997; Vicsek, 2002; Batty, 2007; White et al., 2015), researchers have reinvested the question of scaling patterns for cities. Most of these investigations, conducted by economists, physicists and geographers, have been dedicated to systems of cities, i.e. the inter-urban scale, with particular attention on rank-size distributions and empirical testing of Zipf’s law through space and time (e.g. Pumain, 2004; Bettencourt et al., 2007; Shalizi, 2011; Batty, 2013; Louf and Barthelemy, 2014; Leitão et al., 2016; Cura et al., 2017). Theoretical grounds have been provided along dissipative systems analogies (Bettencourt, 2013) or Gibrat’s law of proportionate growth (Pumain, 1982; Gabaix, 1999), ruling out economics of agglomeration. These studies are essentially *a-spatial*, meaning that cities could be reshuffled anywhere (except Pumain SIMPOP2 where there is a distance decaying spatial interaction, see Pumain and Reuillon, 2017) and, most importantly in light of our objectives, meaning that their intra-urban structure is ignored.

Geographers and physicists have also explored intra-urban scaling, especially [Batty and Longley \(1994\)](#); [Frankhauser \(1994\)](#) have initiated research on fractal geometries and identified their resemblance with land urbanisation patterns. Most of this literature is devoted to identifying irregular urban boundaries (e.g. [Tannier et al., 2011](#)) and non-monocentric patterns ([Chen, 2013](#)). Apart from two noticeable exceptions by [Cavailhès et al. \(2004a, 2010\)](#), no link is explicitly drawn however in the fractal literature with the fundamental location trade-offs of the urban economics tradition. Even in these particular exceptions, though, densities and rents are output on top of an exogenous land pattern inspired by a Sierpinski carpet or multi-fractal. Furthermore, despite fractality implies repeating structures across scales, this literature does not relate to city size distribution and inter-urban research.

In urban economics, the set of monocentric models arising from Alonso-Muth-Mills explicitly aim at explaining land use patterns, densities and land/housing markets as a function of distance to an exogenous Central Business District (CBD) ([Fujita, 1989](#)) and a large theoretical literature has emerged ([Fujita and Thisse, 2013](#); [Duranton et al., 2015](#)). Some links have been drawn with inter-city research and the distribution of cities but without addressing population scaling as such. It is rather focussed on agglomeration effects and migration costs between cities (e.g. [Tabuchi et al., 2005](#)). Empirical studies are less numerous ([Cheshire and Mills, 1999](#); [Ahlfeldt, 2008](#); [Spivey, 2008](#)) and again hardly focus on scaling properties with respect to population size. A notable exception is [McGrath \(2005\)](#) who, following [Brueckner and Fansler \(1983\)](#), studied the evolution of city size (measured as the area or radius of urban regions) with different parameters, including population, using data from 33 U.S. cities over five decades. He observed that the sign of the variation of city size is statistically consistent with urban economic models, but did not say anything about the exact relationship nor the scaling properties of the land or density profiles.

Overall, population scaling in inter-urban research stays strongly disconnected from intra-urban empirics and theory. Scaling laws consider averaged attributes while ignoring the making of urban patterns and their effects on these attributes. They especially ignore the fundamental trade-off between transport and land/housing costs within cities as documented after Alonso, that gives rise to decreasing population and urban land density profiles with distance to the CBD. We attempt to bridge this theoretical gap by integrating recent empirical hints from [Lemoy and Caruso \(2017\)](#) about the scaling of urban profiles into the Alonso model.

[Lemoy and Caruso \(2017\)](#) carried out a radial analysis for over 300 European cities of more than 100 000 inhabitants as of 2006. They analysed the profile of the share of land devoted to housing with distance to the CBD and found that all profiles ideally superpose after their abscissae is rescaled with respect to urban population, thus following a two-dimensional (horizontal) homothetic scaling. Similarly, they analysed population density profiles and found these ideally superpose after a rescaling in abscissae and ordinates, thus following a three-dimensional homothetic scaling. Optimal rescaling is obtained numerically with the square root of population for land use profiles and the cube root of population for population density profiles. This yields the generic profiles shown on Figure 1, with $H_N(r)$ the share of housing land and $\rho_N(r)$ the population density as a function of distance r to the CBD. These are representative profiles which can be rescaled to describe any European city once its population N is given.

The validity of Figure 1 across city sizes cannot be explained by previous geographical research in scaling laws because they do not fit the monocentric framework. In order to be explained by the standard monocentric theory, one then needs to introduce scaling laws in the Alonso framework before assessing how it suits empirical evidence. By doing so we actually start bridging the gap between intra-urban and inter-urban theory.

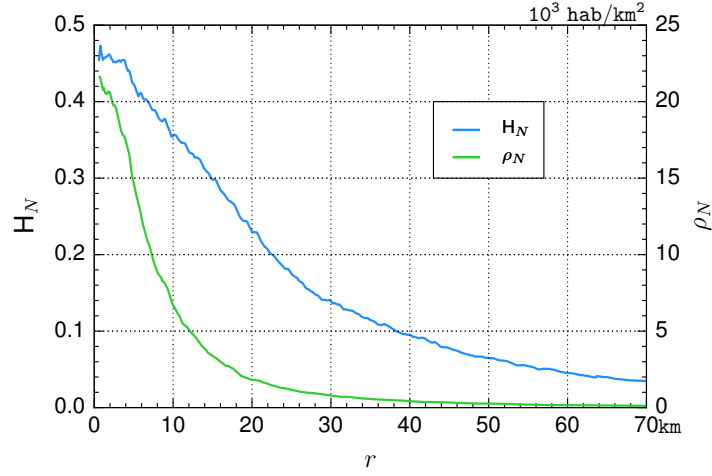


Figure 1: Average housing usage and population density profiles. These profiles have been rescaled without loss of generality to London's population (the largest European Larger Urban Zone in 2006), taken as $N = 1.21 \cdot 10^7$ (see [Lemoy and Caruso, 2017](#)).

In addition, we see from Figure 1 that the land used for housing is far from the 100% assumed by the Alonso model. In Europe, at the CBD land for housing is actually about half of the land and decreases to reach only 10% at 40 km of the CBD for a city like London. At this stage of the research and given our primary focus on scaling, we opt for an exogenous treatment of the housing land development process. We are aware of models that permit non-urbanised land (agricultural or semi-natural) land to be interspersed within the urban footprint because of spatial interactions with residents ([Cavallières et al., 2004b](#); [Caruso et al., 2007](#)) but leave their integration to future work.

We organize the remainder of the paper into a theoretical section and an empirical one. In section 3, we introduce power laws for density and for housing land profiles in a modified version of the Alonso model where housing do not necessarily fully occupy land around the CBD. We then derive conditions at which the equilibrium profiles match the scaling exponents of [Lemoy and Caruso \(2017\)](#). In section 4, we use their European data to calibrate the model, respectively its standard form with unitary occupation of land (Alonso) and a version with an exogenously given land profile function (Alonso-LU), thus leaving the model to produce densities within these constraints. We conclude in section 5.

3 Theory

First, we define the setting and introduce homothetic scaling in density and housing land profiles. Second, we define the decision making of households and introduce scaling within parameters of this choice. Third, we take an intra-urban perspective, and resolve the equilibrium for the closed form (given population, endogenous utility) of the Alonso model with log-linear utility. Total land, housing land and transport cost functions are kept general and conditions for the homothetic scaling of the population density profile are derived. Fourth, we open the city (given utility, endogenous population) and analyse whether the homothetic scaling is compatible with a system-of-cities where cities of different populations coexist at equilibrium with the same utility level. Finally, we operationalise the model with functional forms to prepare the empirical validation.

3.1 Alonso-LU and homothetic scaling profiles

The setting is a featureless plain except for a unique Central Business District (CBD), which concentrates all jobs on a point and is accessed by a radial transport system without congestion. Let r be the Euclidean distance to the CBD and $L(r)$ the exogenous land distribution around the CBD. In reality, $L(r)$ is not necessarily a circle of radius r typically because of water bodies (port cities). In our model, whatever the form of $L(r)$, we depart from Alonso by introducing $H(r)$, the share of $L(r)$ that can be used for housing, hence we have an urban land use augmented model, which we name “Alonso-LU”. In Alonso standard $H(r) = 1$, which obviously contrasts with the blue curve in Figure 1. In Alonso-LU, we impose $H(r)$ as a portion of $L(r)$ and provide its form exogenously. Densities emerge endogenously in Alonso-LU but are constrained by the available space $H(r)$ which we now is decreasing (Figure 1). $H(r)$ is only used for housing. Its complement $L(r)[1 - H(r)]$ cannot be used for housing (natural and semi-natural areas, transport networks, etc.).

We now introduce scalings. Following a monocentric approach, cities are two-dimensional circular objects whose symmetry can be exploited to describe them along a single horizontal dimension. Thus if one considers, for example, the population density profile with respect to the distance from the city center, it is actually a concise representation of a three dimensional cone with circular basis. The homothetic transformations considered in this paper are the projective transformations of this three dimensional cone with respect to the center of its basis and according to a constant dilation factor. Following [Lemoy and Caruso \(2017\)](#), population density cones of cities are assumed to be similar to an homothetic transformation whose dilation factor is a unique power law of their respective populations. This unique exponent is called the scaling power or scaling exponent. The abstract case of a city with a single inhabitant, which is useful as a theoretical reference case, is called the unitary city². Still following [Lemoy and Caruso \(2017\)](#), the cones of the share of land used for housing are assumed to be similar to a non-homothetic transformation, whose dilation factor in the vertical dimension is not the same as in the two horizontal dimensions. Hence the concepts of two-dimensional horizontal and (one-dimensional) vertical scaling exponents.

Let’s denote by $\rho_N(r)$ the population density profile and by $H_N(r)$ the profile of the share of urban land used for housing for a city of total population N . Following empirical evidence of [Lemoy and Caruso \(2017\)](#), we assume there exists α, γ such that population density profiles $\rho_N(r)$ scale homothetically in three dimensions with the power α of population N , and that housing land radial profiles $H_N(r)$ scale homothetically in the two horizontal dimensions with the power γ of population. These homothetic scalings can be formalized as:

$$\rho_N(r) = N^\alpha \rho_1\left(\frac{r}{N^\alpha}\right), \quad (1)$$

$$H_N(r) = H_1\left(\frac{r}{N^\gamma}\right), \quad (2)$$

where $\rho_1(r)$ and $H_1(r)$ are the population and land use radial profiles of the unitary city. According to [Lemoy and Caruso \(2017\)](#), European urban areas obey equations 1 and 2 (up to some fluctuations which are illustrated later in this work) with the exponents: $\alpha \simeq 1/3$ and $\gamma \simeq 1/2$.

3.2 Residential choice and scaling parameters

Each household in the model requires land s for housing, work in the CBD and consume a composite commodity z that is produced out of the region and imported at constant price. In that context, residential choice depends only on the distance r to the CBD.

Households are rational in the sense of [von Neumann and Morgenstern \(1944\)](#), see also [Myerson, 1997](#)) and their utility function U is

$$U(z, s) = (1 - \beta) \ln(z(r)) + \beta \ln(s(r)), \quad (3)$$

where $z(r)$ is the amount of composite good (including all consumption goods except housing surface) consumed at distance r from the CBD, $s(r)$ is the housing surface³ at the same distance r and $\beta \in]0, 1[$ is a parameter representing the share of income (net of transport expenses) devoted to housing, or the relative expenditure in housing. Note that β is assumed to remain constant across cities of different sizes, which is empirically supported (Davis and Ortalo-Magné, 2011).

Equation (3) is a log-linear utility function, i.e. the logarithmic transformation of the traditional Cobb-Douglas utility function (from Cobb and Douglas, 1928), and gives the same results in the present case since we work with an ordinal utility. We choose this form in accordance with urban economic literature since first, it matches the assumption of a *well-behaved utility function*⁴ (Fujita, 1989, p.12), which is central in the basic monocentric model and ensures that $U(z, s)$ is defined only for positive values of z and s . Second, it contains only a single parameter, β , which can be discussed empirically. This will crucially enhance the theoretical discussion (section 3.4) as well as the empirical potential (section 4) of the model. Third, β is independent of prices, as found in the empirical literature (Davis and Ortalo-Magné, 2011). Generalization to more general representations of preferences, such as utility functions with constant elasticity of substitution (CES), is left for further studies⁵.

We choose the composite commodity (z) as the numeraire (unit price) and the budget constraint of each household is binding since the monotonicity of households utility function includes no incentive to spend money otherwise. The budget constraint at distance r from the CBD is

$$z + R(r)s(r) = Y_N - T_N(r) , \quad (4)$$

where $R(r)$ is the housing rent at distance r , Y_N is the wage households earn, and $T_N(r)$ is the commuting cost at r .

We introduce important new scaling assumptions: wages and transport costs are assumed to depend on the total population N of the city. Their variations with city size will strive to reproduce the empirical radial profiles of small and large cities. The measure of agglomeration economies and costs through elasticities of wages and transport costs is well established in the empirical economic literature (Rosenthal and Strange, 2004; Combes et al., 2010, 2011, 2012). This literature implies power law functions, which are also most often used in urban scaling laws literature (Bettencourt et al., 2007; Shalizi, 2011; Bettencourt, 2013; Leitão et al., 2016).

Following both strands, we introduce power laws, such that $Y_N = N^\phi Y_1$, where Y_1 is the wage in a unitary city, and ϕ is the elasticity of wage with respect to urban population. Similarly, we assume that the transport cost function $T_N(r)$ is a scaling transformation (not necessarily homothetic) of $T_1(r)$, the transport cost function in a unitary city (assumed to be continuously increasing and differentiable in r). The exact form of this transformation will become clear in the next section (equation 9).

The households problem consists in maximizing their utility (3) such that the binding budget constraint (4) holds.

3.3 Intra-urban equilibrium

Consider a closed urban region with population N . Solving the maximisation problem of households (App. A.1) yields the bid rent function $\Psi(r, u)$, which is the maximal rent per unit of housing surface they are willing to pay for enjoying a utility level u (exogeneous) while residing at distance r .

Closing the model by linking the utility level u to the population size N yields two more conditions. The first one states that the fringe of the city is determined by a competition between urban land use (i.e. housing people) and agricultural land use, which is represented by an agricultural rent provided to the landowner. We will suppose here for mathematical convenience (see appendix A.2) that this agricultural rent is null. This assumption is common in urban economic theory (Fujita, 1989) and is empirically supported by the relatively low values of agricultural rents compared to housing rents (Chicoine, 1981). Consequently, the urban fringe f_N is the distance at which households spend their entire wage in commuting (and cannot afford more than a null rent):

$$Y_N = T_N(f_N) \Leftrightarrow f_N = T_N^{-1}(Y_N) . \quad (5)$$

The second closing condition expresses the fact that the quantity of land $L(r)$ at each commuting distance r is finite. In addition, summing the population density over the whole (finite) extent of the city, up to the fringe f_N , must yield the total population N . This writes (see appendix A.2):

$$e^{-u/\beta}(1-\beta)^{1/\beta-1} \int_0^{f_N} L(r)H_N(r) \left[Y_N - T_N(r) \right]^{1/\beta-1} dr = N. \quad (6)$$

Finding the unique equilibrium utility and urban fringe satisfying the equilibrium conditions⁶ yields the population density function $\rho_N(r)$ (appendix A.2). We note that in general, an analytical solution for the equilibrium utility u cannot be obtained without assuming that the agricultural land rent is null. However, with a null agricultural rent, we derive three conditions which ensure the homotheticity of the equilibrium population density function (appendix A.3),

$$\forall \lambda \in \mathbb{R} : \quad L(\lambda r) = \lambda L(r), \quad (7)$$

$$\gamma = \alpha, \quad (8)$$

$$\exists \theta \in \mathbb{R} : \quad T_N(r) = N^\theta T_1\left(\frac{r}{N^\alpha}\right). \quad (9)$$

Condition (7) is simply the linearity of L , which is clearly satisfied in a two-dimensional circular framework (where $L(r) = 2\pi r$). Condition (8) states that the horizontal scaling exponent of the housing usage profile (2) must be equal to the horizontal scaling exponent of the population density profile (1), and by extension to its homothetic scaling power. Finally, condition (9) actually refines the power-law form of the transport cost function by specifying that the transport cost function is at least (since θ can be zero) horizontally scaling with power α . Its consequences are discussed in the following inter-urban analysis. If these three assumptions hold, then the equilibrium population density function writes

$$\rho_N(r) = N^{1-2\alpha} H_1(r_1) \left[T_1(f_1) - T_1(r_1) \right]^{1/\beta-1} \left[\int_0^{f_1} L(r_1) H_1(r_1) \left[T_1(f_1) - T_1(r_1) \right]^{1/\beta-1} dr_1 \right]^{-1}, \quad (10)$$

where $r_1 = r/N^\alpha$ and $f_1 = f_N/N^\alpha$.

This population density profile follows the three-dimensional homothetic scaling (1) if and only if $\alpha = 1/3$, which coincidentally matches the empirical evidence of [Lemoy and Caruso \(2017\)](#). This means that cities of different sizes can be regarded as similar objects with respect to the fundamental households' trade-off between transport and housing costs. This results in a common population density profile, whose homothetic scaling across different urban sizes results then from the allometric scaling of land profile, wages and transport costs.

To sum up, in this section three new scaling laws have been introduced in Alonso's monocentric model with log-linear utility ([Alonso, 1964](#); [Fujita, 1989](#)): the scaling of the housing usage profile $H_N(r)$, of wages Y_N and of the transport cost function $T_N(r)$. The resulting population density profile $\rho_N(r)$ turned out to exhibit a three-dimensional homothetic scaling, under the condition that **(i)** land distribution $L(r)$ is linear, that **(ii)** the horizontal scaling powers of the housing usage and population density profiles are equal and that **(iii)** transport cost $T_N(r)$ is at least horizontally scaling with power α . In this case, the model concludes to a scaling power of $1/3$ that matches the empirical evidence of [Lemoy and Caruso \(2017\)](#). This provides us with an interesting scaling version of Alonso's model, where land use is more realistic. The main drawback is condition **(ii)** or (8) above, which requires that the scaling exponent of the housing usage profile in this model is $1/3$, instead of the value of $1/2$ (equation 2) found by [Lemoy and Caruso \(2017\)](#).

3.4 Inter-urban analysis

Up to now, a closed city of size N has been considered. Yet real-world urban agglomerations are not closed. Instead, they belong to an urban system where households may move from a city to another. Regarding the present study, this inter-urban viewpoint has two consequences. First, since cities of different population sizes coexist in real urban systems, the equilibrium of the model should be able to reproduce this fact. As a consequence, the benefits and costs of urban agglomeration should vary together when population size changes, so as to equilibrate whatever the size of the city. Otherwise, one force would dominate the other and the urban system's equilibrium would either collapse to a single giant city or spread into countless unitary cities. Second, since by definition households' location decisions are mutually consistent at equilibrium, the equilibrium utility level has to be the same whatever the city population N . Otherwise, households would have an incentive to move to larger or smaller cities.

Substituting the power-law expressions of the wage and transport cost function into the boundary rent and total population conditions, and accounting for the equality of equilibrium utilities across cities of different sizes yields the successive two equalities (see appendix A.4)

$$\phi = \theta = \frac{\beta}{3(1 - \beta)} . \quad (11)$$

The left-hand side equality in equation (11) implies that the elasticity of wages with respect to urban population equals the elasticity of transport cost. Following the approach of [Dixit \(1973\)](#), in the monocentric model this is equivalent to say that agglomeration economies are equal to agglomeration costs⁷, which is the condition for several cities of different population to coexist at equilibrium. This equality of population-elasticities of wage and transport cost function is supported by recent developments in the very limited empirical literature on agglomeration costs ([Combes et al., 2012](#)).

The right-hand side equality in equation (11) provides a relationship between the value of the population elasticity of transport cost θ (and of wages ϕ) and households' relative expenditure in housing β . This relation is increasing and suggests that a relative expenditure $\beta = 1/3$, which is in the range of empirically supported values ([Accardo and Bugeja, 2009](#); [Davis and Ortalo-Magné, 2011](#)), would be associated to elasticities $\phi = \theta = 1/6$ (Fig. 2). This value appears in urban studies based on dissipative systems ([Bettencourt, 2013](#); [Bettencourt and Lobo, 2016](#)). Consequently, equation (11) is reasonable and the proposed monocentric model is consistent with an inter-urban perspective. However, other authors consider this elasticity to range from 2% to 5% ([Combes et al., 2010, 2011](#)). Besides, considering measures of agglomeration economies that are not only based on wages, the elasticity of productivity with respect to city population is traditionally considered to be of maximum 3% to 8% ([Rosenthal and Strange, 2004](#)). This incompatibility deserves more research effort in the future, especially digging into the functional form of the transport cost function as discussed in the next section.

3.5 Functional form

We now propose an operational version of the previous model by selecting appropriate functional forms for the land distribution $L(r)$, the housing usage profile $H_N(r)$ and the transport cost function $T_N(r)$. The theoretical implications of those forms are discussed as well as their empirical supports. In brief, the functional model is specified by

$$L(r) = 2\pi r , \quad (12)$$

$$H_N(r) = b \exp\left(\frac{-r}{dN^{1/3}}\right) , \quad (13)$$

$$T_N(r) = cN^{\theta-\alpha}r , \quad (14)$$

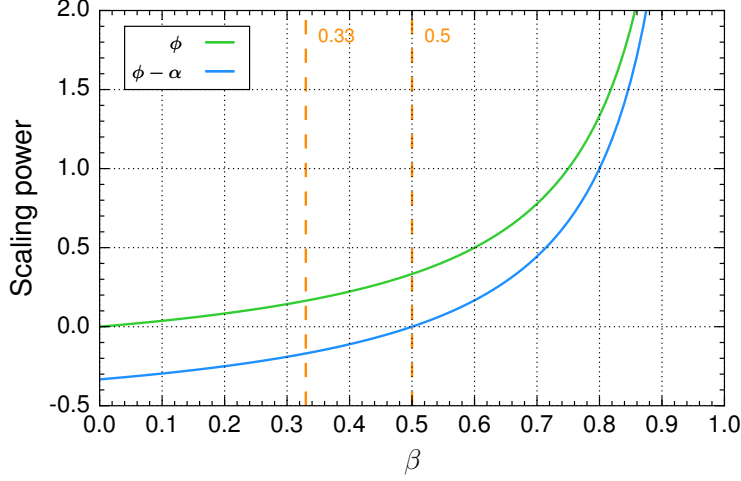


Figure 2: Population-elasticities of wage and transport cost with respect to housing relative expenditure. Dashed orange lines highlight values of reference discussed in the text.

where $\theta = \beta/(1-\beta)/3$ (equation 11), $\alpha = 1/3$, b is the housing usage at the CBD, d is the characteristic distance of the housing usage profile in a unitary city and c is the transport cost per unit distance in a unitary city. One can easily check that the functional form (12)-(14) follows the conditions for homotheticity (7)-(9), as well as for consistency with the inter-urban approach (11).

Firstly, regarding the land distribution, expression (12) is just the usual two-dimensional circular viewpoint, which is consistent with the empirical profiles of [Lemoy and Caruso \(2017\)](#). Secondly, the exponential form (13) of the housing usage profile has been chosen for its simplicity and goodness of fit, which is discussed in section 4. Thirdly, regarding the linear form (14) of the transport cost function, we note that the elasticity of unitary transport cost with respect to urban population $\theta - \alpha = (2\beta - 1)/(1 - \beta)/3$ is endogenous in this model (appendix A.5): it results from the conditions of homothetic scaling (9) and homogeneous utility across cities (11). Its expression suggests that for $\beta < 1/2$, which comprises the range of housing expenditure which is supported empirically, the unitary transport cost should decrease with urban population (Fig. 2). This is a clear drawback of our model, because the unitary transport cost is rather expected to increase with urban population in real urban systems, due to congestion among other reasons. This result is however a consequence of the linear form (14), which is most often used in urban economic theory but not very realistic, as real commuting costs are rather expected to be concave with respect to the distance r . Furthermore, we show that for realistic values of the housing relative expenditure, e.g. $\beta = 1/3$, a positive elasticity of unitary transport cost $\phi - \alpha$ appears for a power of r that is smaller than $1/2$ (appendix A.5) – in particular, changing (14) to $T_N(r) = c\sqrt{r}$ (that is, no scaling with population size N) would respect our conditions (9) and (11). However, we leave this as a note for now as we lack empirical input about the functional form and scaling properties of the radial transport cost function.

Finally, with the functional form (12)-(14), the equilibrium population density function (10) becomes (appendix A.6)

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} (f_1 - r_1)^{\frac{1}{\beta}-1} \left[\beta f_1^{1/\beta+1} - (\beta f_1 + d) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} dx \right]^{-1}, \quad (15)$$

where $r_1 = r/N^{1/3}$. This expression depends on the unitary urban fringe $f_1 = Y_1/c$, the share of households' wage spent on housing β and the characteristic distance of the unitary housing usage d . We illustrate it in the next section.

Table 1: Nonlinear least square results. Calibration are performed on European average profiles made up with 694 points, for a population of reference $\bar{N} = 7.03 \cdot 10^5$. Distances d and f_1 are expressed in kilometers. $C(b, d)$ is correlation between parameters. BIC is the Bayesian information criterion.

A. Housing usage			B. Population density				
	Alonso	Alonso-LU		Alonso		Alonso-LU	
b	0.167 (0.005)	0.523 (0.001)	β	0.02	0.34	0.02	0.34
d	∞	5.80 (0.02)	$f_{\bar{N}}$	172.9 (0.5)	12.94 (0.07)	409 (2)	23.1 (0.2)
$C(b, d)$		-0.74					
BIC	-2 808	-6 332	BIC	7 613	8 744	7 544	7 893

4 Empirics

In this section, the previous functional model is calibrated to the average housing usage and population density profiles of Fig. 1 using nonlinear least squares. The calibration procedure will be actually performed in two steps. First, the optimal value of d will be chosen by calibrating the housing usage function (13) to the average housing usage profile for a reference city of population $\bar{N}$. Second, the optimal value of d is substituted into the population density function (15), which in turn is calibrated to the average population density profile by optimizing the values of f_1 and β . A model with constrained values of β will be discussed as well and both are finally compared to the profiles of four individual cities.

4.1 Population of reference and housing usage profile

Let us consider first the calibration of the housing usage function (13) to the average housing usage profile (Figure 1) for a population of reference $\bar{N}$. This population can be chosen arbitrarily, yet because of the wrong scaling power required by the condition for homotheticity (8) – $1/3$ instead of $1/2$ – its choice influences the ability of the model to describe cities of any size. In order to minimize errors (appendix A.7), we choose a population of reference $\bar{N} = 7.03 \cdot 10^5$. For a city with this population size, the best fit suggests that $b = 52.3\%$ of land is dedicated to housing at the CBD, and that this share decreases exponentially with a characteristic distance of 5.8km (Table 1). In order to allow for comparison with the standard Alonso model, the optimal value for a constant profile has been computed as well (Table 1).

In order to discuss the error that is made when rescaling the best-fit profile, four cities of different sizes are chosen as illustrations, namely London (Ldn), the largest urban area of the dataset with a population of $N = 1.21 \cdot 10^7$ in 2006, Brussels (Bxl), the capital of Belgium with $N = 1.83 \cdot 10^6$, Luxembourg (Lux), capital of the country of the same name, with $N = 4.52 \cdot 10^5$ and Namur (Nam), the capital of Wallonia in Belgium, with $N = 1.39 \cdot 10^5$. Because of the wrong scaling exponent, the larger the difference between the population N of the considered city and the reference population $\bar{N}$, the larger is the error on housing usage (Figure 3). For $N > \bar{N}$, the housing usage is underestimated, and overestimated for $N < \bar{N}$. In the case of the four considered cities, the absolute error does not exceed 12%, which at this point represents a relative error of approximately 35% (Figure 3).

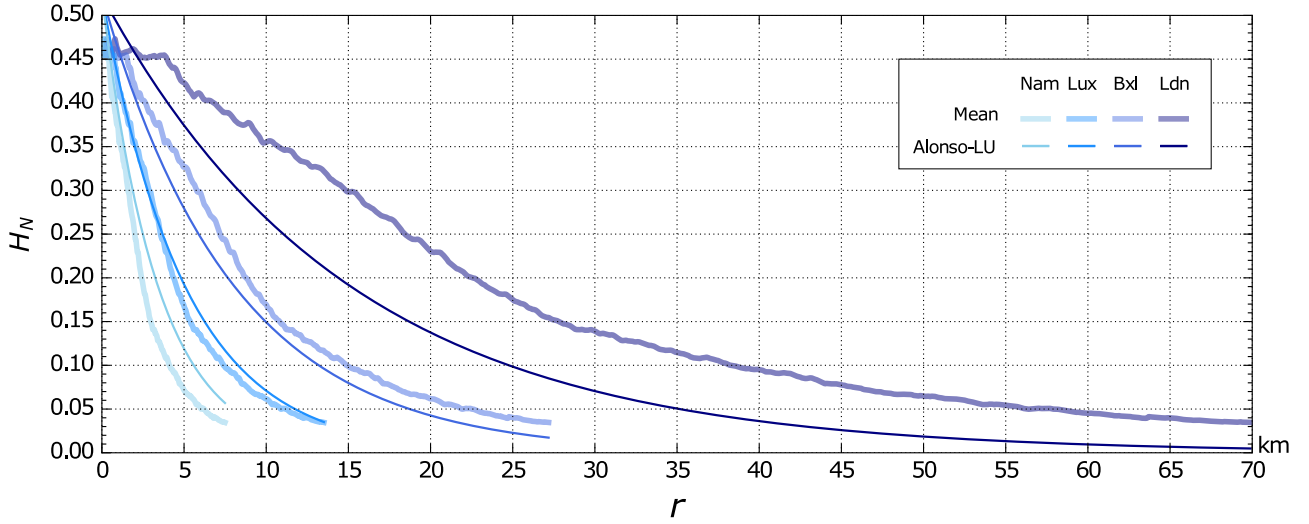


Figure 3: Calibration of the average housing usage profile. Mean and best-fit, rescaled to the four arbitrary cities.

4.2 Population density profile

We turn now to the calibration of the population density function (15), using the optimal value d (Table 1) obtained in the previous section, to the average population density profile (Figure 1). We focus on a city of size $\bar{N}$, which we choose without loss of generality this time, since the scaling of population density in the model is in agreement with empirical results. The optimal values of the urban fringe f_N and of the relative expenditure in housing β are negatively correlated, the best fit being for small values of β and high values of f_N (Figure 4). Hence the optimal model considered in the following has $\beta = 0.02$, the minimal (and unrealistically small) value we considered. On the opposite, a constrained model with a more realistic $\beta = 0.34$ is taken as a reference case (Figure 4).

Looking at the best-fit population density profile, we focus on the case of London (Figure 5), and smaller cities are obtained by homothetic rescaling (Figure 6). We observe that the Alonso-LU model outperforms the standard Alonso model, especially for realistic values of β , and (Figure 5) and (Figure 6) show furthermore that the Alonso-LU model gives a good description of population density profiles of the studied European cities, whatever their size. It is worth mentioning that on the semi-logarithmic plot of (Figure 5), the relative errors are exaggerated compared to (Figure 6). Four additional European urban areas are fitted in appendix B. We note also that both the Alonso and Alonso-LU models reproduce reasonably well the roughly exponentially decreasing population density of the empirical average profile. This is an interesting contribution to the urban economic literature, where different works have tried to obtain exponential population density functions (the model of [Clark, 1951](#)) within the standard urban economic framework. For instance, [Mills \(1972\)](#); [Brueckner \(1982\)](#) added a description of building construction in the model while [Anas et al. \(2000\)](#) had to use a specification of commuting costs which does not seem supported by empirical data. We reproduce the empirical curves here approximately, but without adding further ingredients to the standard model, in a parsimonious approach.

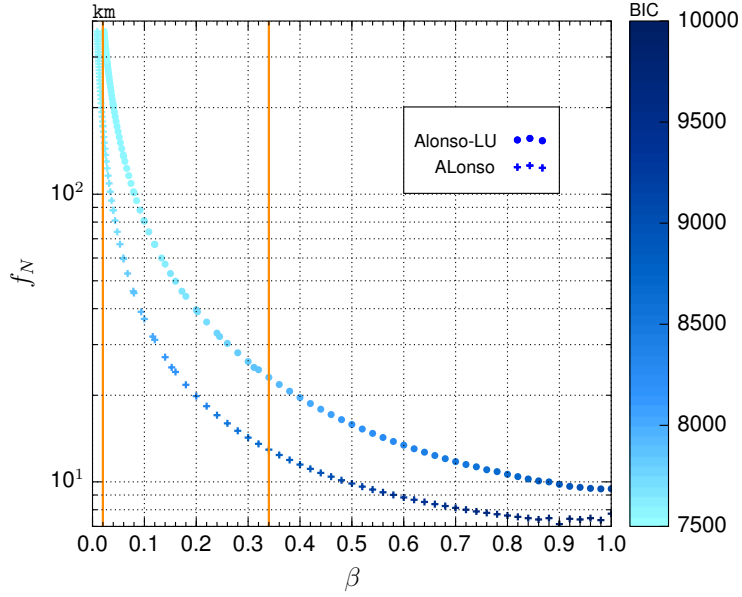


Figure 4: Best fit parameters for the average population density profile. The average profile has been rescaled without loss of generality to a reference city of size $\bar{N} = 7.03 \cdot 10^5$. Colours represent the Bayesian information criterion (BIC). Orange lines show parameter values of the optimal ($\beta = 0.02$) and constrained ($\beta = 0.34$) models.

Figure 5 sheds light on the results of the fitting process represented on Figure 4. On the semi-logarithmic plot of Figure 5, (the logarithm of) the empirical population density appears convex, while (the logarithms of) the models' density are concave, because the density is going to zero at $r = f_N$. As a consequence, the best fits of Figure 4 have very large values of f_N and very small ones of β , which stretches the density curves of the models and makes them almost exponential – and their logarithms almost linear. This reduces the concavity of the (logarithms of) the models' population density as much as possible, but it yields unrealistic values of the fitted parameters. This is a major result and issue of our calibration process, which questions the ability of monocentric models to describe real cities and their scaling. This issue, which we leave to further work, could probably be solved (at least partly) by including a more realistic description of commuting in our models. Indeed, we have chosen a linear transport cost (14), which is probably a reasonable description of the monetary cost of commuting by car, but a poor one of the time cost, which is more concave, and contributes of course to the global commuting cost. A more concave (and more realistic) transport cost in our models would increase the convexity of the population density curves (and of their logarithms). This is a perspective for further research.

Finally, visual comparison of the fitted model with individual data profiles for the four cities of reference (and four additional cities in appendix B, Figure 7) reveals that the error is mostly due to the deviations of individual data from the average profile, and less to deviations of the model from the average profile (Figure 6 and 7). Consequently, the model is regarded as successful with respect to the announced objectives and deviations from individual profiles are not further developed here.

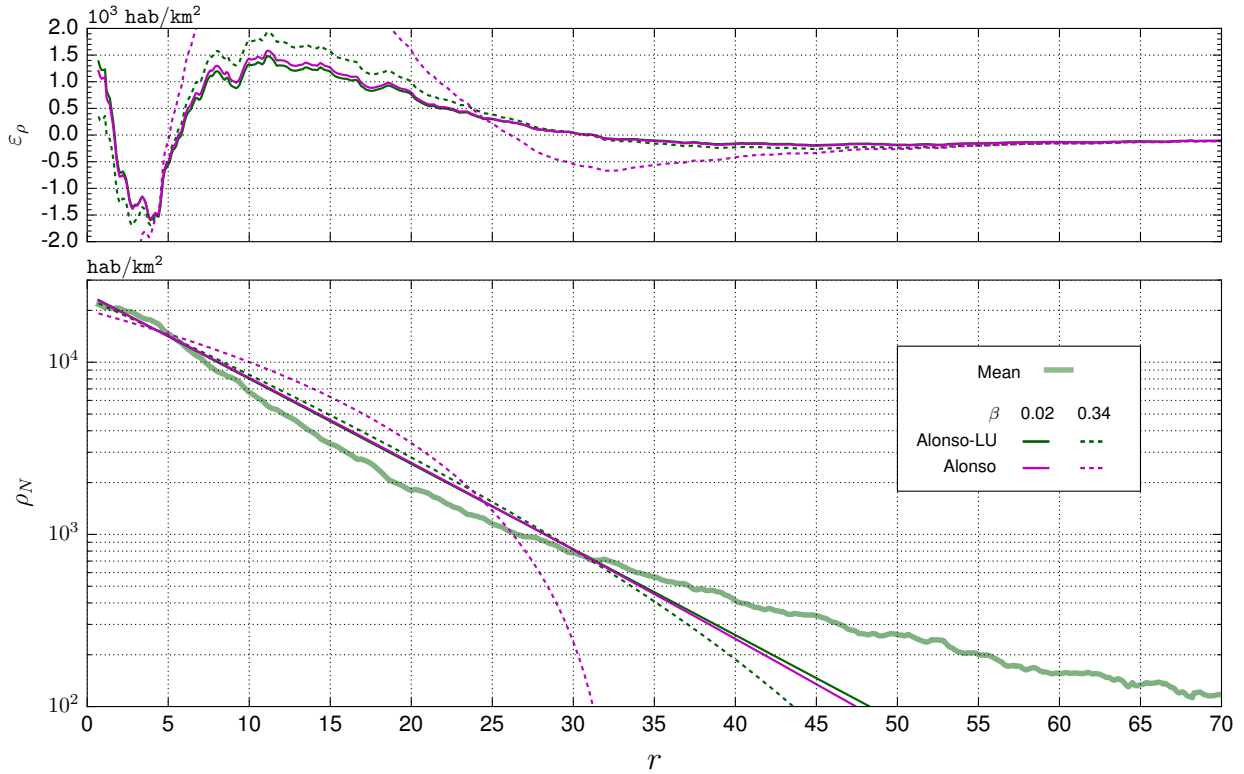


Figure 5: Calibration of the average population density profile Note that the semi-logarithmic scale of the bottom subplot visually exacerbates the error at larger distance from the CBD. On the top subplot, ε_ρ denotes residuals.

Let us note that we do not perform here a full calibration of the Alonso-LU model, because of our lack of price or rent data. Indeed, we do not fix the values of the income Y_1 and the unit distance transport cost c in a unitary city – they do not appear in the expression of the population density (15). Our calibration is only performed on land use and population density, and we leave to further work a simultaneous calibration of the model on real land prices or rents, which would be a more challenging test of the monocentric urban economic theory. However, we note that the scaling of prices (bid rents) in the model is the same three-dimensional homothetic scaling as for population density. In addition, the price curve is less steep than the population density curve due to a missing (exponential) $H_N(r)$ factor (A.18). This seems in agreement with the slower decrease of prices with distance to the center in real cities. The model would also predict that the profiles of land rent or prices, possibly adjusted for purchasing power parity, are (roughly) similar in cities of similar population size N , independently of all other characteristics, among which the country of location. This claim of the model needs to be tested against real data, as a perspective for further work.

5 Conclusion

In order to enhance our understanding of relationships between scaling properties of cities and their intra-urban structure, this paper has proposed an original, augmented version of Alonso’s monocentric model, called the “AlonsoLU” model, which considers an exogenous, scaling urban land profile. Further assuming the scalings of wages and transport costs with total urban population, it succeeded in reproducing the three dimensional homothetic scaling of the European population density profiles recently uncovered by [Lemoy and Caruso \(2017\)](#). Moreover, the model infers the empiric scaling power of $1/3$, and it is consistent with an inter-urban perspective. However, the inferred scaling power of the land use profile is significantly smaller than the empirical value of [Lemoy and Caruso \(2017\)](#). This may be corrected by modelling an endogenous housing production, in the way initially proposed by [Muth \(1969\)](#). This raises research opportunities for further improving the Alonso-LU model.

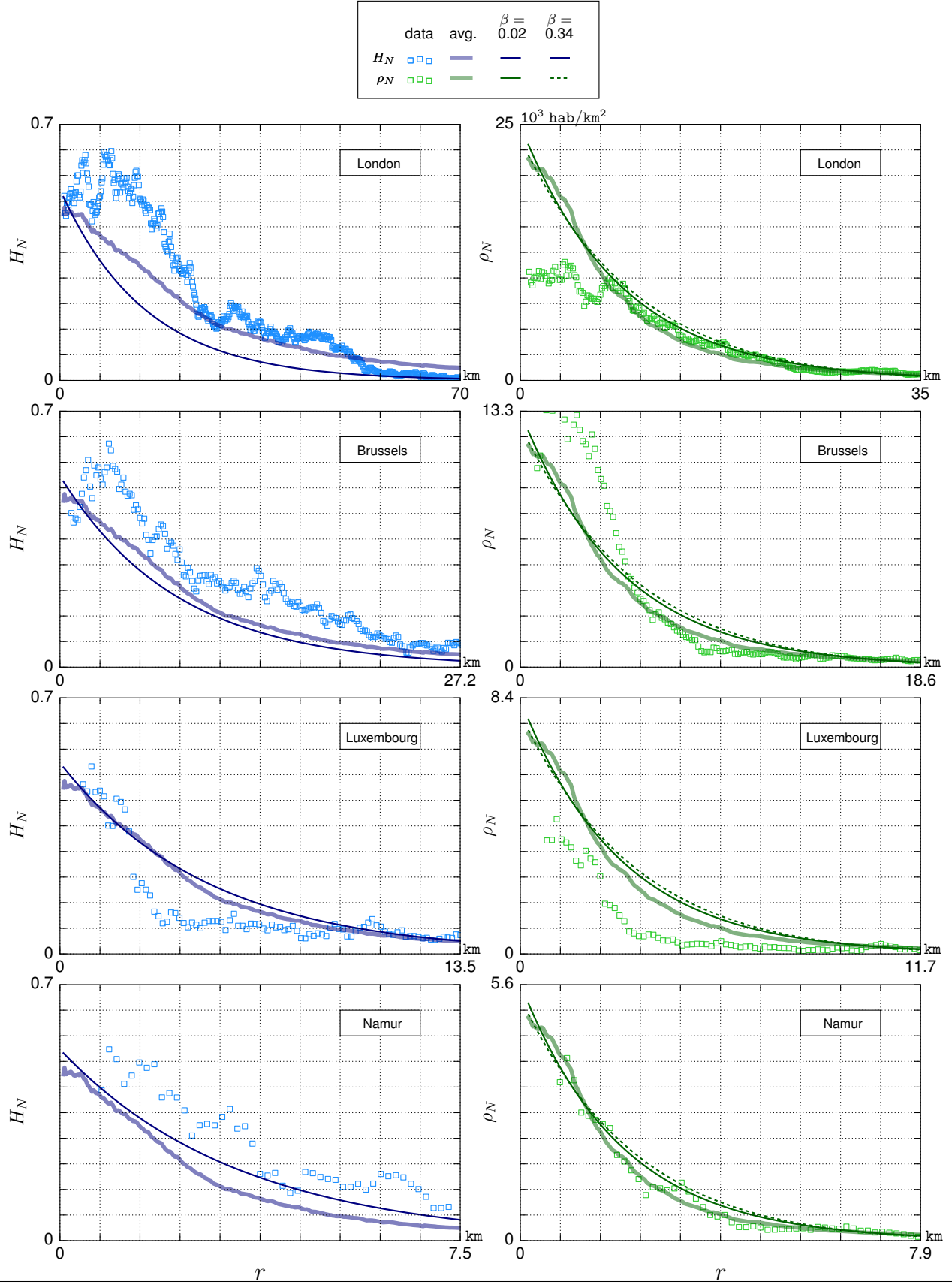


Figure 6: Summary plot of the results. Fitted average profiles compared to individual profiles. Left panel: housing share profile. Right panel: population density profile. Axes have been rescaled to maintain the average curves at the same position across subplots.

A functional form with linear transport cost has been proposed, which performs better than the original Alonso model in reproducing the empirical profiles of Figure 1. As a result, comparison with data from individual cities turned out to be surprisingly good providing that only the total urban population is used to model different European cities. However, the functional form challenges current empirical understanding of wage and transport costs elasticities with population. It has been shown that this issue can be addressed by using a non-linear transport cost function. This perspective constitutes another research agenda following on this paper.

Acknowledgements

The authors are grateful to Jean Cavailhès for insightful comments.

Notes

¹An homothetic transformation is a projective transformation of an affine space from a defined centre (Meserve, 2010).

²Throughout this paper, scaling properties will implicitly refer to scaling with respect to urban population N . Thus, indices “ N ” are used to indicate exogenous variables or functions that are assumed to vary with N . Accordingly, indices “1” are used to indicate the value of those variables for the unitary city.

³In the Alonso model, there is no development of land into housing commodities (land development was introduced into the monocentric theory by Muth, 1969). Hence the housing market is not distinguished from the land market. Throughout this paper, it is referred as the housing market in order to emphasize Alonso’s focus on households’ choice. Note also that the term “housing” is used in a broad sense without distinguishing, for example, gardens from built space.

⁴Formally, U must be twice continuously differentiable, strictly quasi-concave with decreasing marginal rates of substitution, positive marginal utilities and all goods must be essentials. See Fujita (1989, p.311).

⁵Actually, the log-linear utility is an homothetic function as well since it is the logarithmic transformation of the Cobb-Douglas utility, which is itself homogeneous. This corresponds to a representations of homothetic preferences (see for example Varian, 2011).

⁶To get more information on equilibrium conditions, existence and uniqueness, see Fujita (1989).

⁷The theoretical argument underpinning this interpretation is that in a competitive labour market, labour is paid to its marginal productivity, so that wage-elasticity is representative of labour productivity, which capitalises itself different effects of urban economies of agglomeration. Regarding the costs of agglomeration, the road congestions are the main element that is not considered, and it is assumed to be caught by the elasticity of the transport cost function.

A Mathematical appendices

A.1 Households consumption problem

Taking all the assumptions and notations from section 3 as given, households’ consumption problem in a city of population N is

$$\max \left\{ U(z, s) = (1 - \beta) \ln(z(r)) + \beta \ln(s(r)) \right\} \quad (\text{A.1})$$

$$\text{s.t. } z + R(r)s(r) = Y_N - T_N(r) . \quad (\text{A.2})$$

From the utility function, one computes the marginal rate of substitution

$$\frac{\partial U(z, s)}{\partial z} \left(\frac{\partial U(z, s)}{\partial s} \right)^{-1} = \frac{s(1 - \beta)}{z\beta} , \quad (\text{A.3})$$

which can be equalized to the ratio of prices in order to have the optimal choice equation, that is

$$\frac{s(1 - \beta)}{z\beta} = \frac{1}{R(r)} . \quad (\text{A.4})$$

Simultaneously solving the optimal choice equation (A.4) and the budget constraint (A.2) by appropriate substitutions yields the solution of the households consumption problem,

$$z(r) = (1 - \beta) [Y_N - T_N(r)] , \quad (\text{A.5})$$

$$s(r) = \frac{\beta [Y_N - T_N(r)]}{R(r)} . \quad (\text{A.6})$$

Substituting back the optimal consumptions (A.5) and (A.6) into the utility function (A.1) yields the indirect utility, which can be set to an arbitrary level u in order to express the bid rent function

$$\Psi(r, u) = e^{-u/\beta} \beta (1 - \beta)^{1/\beta-1} [Y_N - T_N(r)]^{1/\beta} , \quad (\text{A.7})$$

Finally substituting the bid rent (A.7) into the optimal housing consumption (A.6) yields the bid-max lot size

$$s(r, u) = e^{u/\beta} \left[(1 - \beta) [Y_N - T_N(r)] \right]^{1-1/\beta} . \quad (\text{A.8})$$

A.2 Equilibrium problem

Turning now to the urban equilibrium, let u denote the equilibrium utility level and let $n(r)$ be the population distribution (the population living between r and $r + dr$) at distance r from the CBD, which is a continuous and continuously differentiable function. Then the following equilibrium relationship states that land available for housing at a given commuting distance r within the city is finite and entirely occupied by households:

$$L(r) H_N(r) = n(r) s(r, u) , \quad (\text{A.9})$$

where $H_N(r)$ follows the horizontal scaling (2). From this follows the definition of the population density $\rho_N(r)$ in this model:

$$\rho_N(r) = n(r)/L(r) = H_N(r)/s(r, u) . \quad (\text{A.10})$$

We express now the two equilibrium conditions. The first one is the boundary rent condition

$$\Psi(f_N, u) = a , \quad (\text{A.11})$$

where a is the exogenous agricultural land rent. As traditionally in urban economic theory, the agricultural land use is no more than a default land use, that is why the agricultural sector is reduced to its most simple form, represented by a constant rent, although it is not really the case empirically (Chicoine, 1981; Colwell and Dillmore, 1999; Cavailhès et al., 2003). The second equilibrium condition is the population condition

$$\int_0^{f_N} n(r) dr = N . \quad (\text{A.12})$$

On the one hand, substituting the bid rent function (A.7) into the boundary rent condition (A.11) yields the equilibrium urban fringe

$$f_N = T_N^{-1} \left(Y_N - (1 - \beta)^{\beta-1} \beta^{-\beta} a^\beta e^u \right) . \quad (\text{A.13})$$

On the other hand, consecutively substituting the optimal housing consumption (A.8) into equation (A.9), and the resulting value of population distribution into the population condition (A.12) yields

$$e^{-u/\beta} (1 - \beta)^{1/\beta-1} \int_0^{f_N} L(r) H_N(r) [Y_N - T_N(r)]^{1/\beta-1} dr = N . \quad (\text{A.14})$$

In general, an analytical solution for the equilibrium utility u cannot be obtained by substituting the equilibrium urban fringe (A.13) into the expression of total population (A.14). However, with the assumption that the agricultural land rent is null ($a = 0$), the equilibrium urban fringe becomes

$$f_N = T_N^{-1}(Y_N) \Leftrightarrow Y_N = T_N(f_N) , \quad (\text{A.15})$$

which means that the urban fringe is the distance at which households spend their entire wage in commuting. Equation (A.15) is very powerful since it enables us to express the results with respect either to the urban fringe f_N or to the wage Y . It is also linking the two quantities in terms of scaling properties, which are discussed in section (3.4). Now, substituting the right-hand-side equation of (A.15) into the population constraint yields the equilibrium utility

$$e^{u/\beta} = N^{-1}(1 - \beta)^{1/\beta-1} \int_0^{f_N} L(r) H_N(r) [T_N(f_N) - T_N(r)]^{1/\beta-1} dr , \quad (\text{A.16})$$

which can be consecutively substituted into the optimal housing consumption (A.8) and into equation (A.10) in order to express the population density function

$$\rho_N(r) = N H_N(r) [T_N(f_N) - T_N(r)]^{1/\beta-1} \left[\int_0^{f_N} L(r) H_N(r) [T_N(f_N) - T_N(r)]^{1/\beta-1} dr \right]^{-1} . \quad (\text{A.17})$$

We note also that the bid rent $\psi_N(r)$ is given by $\psi_N(r) = \beta(T_N(f_N) - T_N(r))\rho_N(r)/H_N(r)$, that is

$$\psi_N(r) = N\beta [T_N(f_N) - T_N(r)]^{1/\beta} \left[\int_0^{f_N} L(r) H_N(r) [T_N(f_N) - T_N(r)]^{1/\beta-1} dr \right]^{-1} . \quad (\text{A.18})$$

A.3 Conditions of homothetic scaling

In order to derive conditions under which the population density function (A.17) respects the homothetic scaling (1), one first rescales distances accordingly. Formally, under the following change of variable

$$r_1 = \frac{r}{N^\alpha} , \quad (\text{A.19})$$

the population density function (A.17) rewrites

$$\rho_N(r) = \frac{N^{1-\alpha} H_N(r_1 N^\alpha) [T_N(f_1 N^\alpha) - T_N(r_1 N^\alpha)]^{1/\beta-1}}{\int_0^{f_1} L(r_1 N^\alpha) H_N(r_1 N^\alpha) [T_N(f_1 N^\alpha) - T_N(r_1 N^\alpha)]^{1/\beta-1} dr_1} , \quad (\text{A.20})$$

where we note that the urban fringe f_N has to be rescaled as well, following

$$f_1 = \frac{f_N}{N^\alpha} . \quad (\text{A.21})$$

This has, due to equation (A.15), important consequences on the scaling properties of Y_N and T_N .

Finally assume that $L(r)$ is linearly homogeneous, that γ (equation 2), the scaling power of H_N , is the same as α (equation 1), the scaling power of ρ_N , and that $T_N(r)$ is at least horizontally scaling. Formally,

$$\forall \lambda \in \mathbb{R} : \quad L(\lambda r) = \lambda L(r) , \quad (\text{A.22})$$

$$\gamma = \alpha , \quad (\text{A.23})$$

$$\exists \theta \in \mathbb{R} : \quad T_N(r) = N^\theta T_1\left(\frac{r}{N^\alpha}\right) . \quad (\text{A.24})$$

The first assumption will add a “ $-\alpha$ ” term to the power of N in the population density function (A.20). The second assumption implies that the horizontal scaling of the housing usage function (2) balances the effect of total population. The third assumption, equivalent to $T_N(rN^\alpha) = N^\theta T_1(r)$, will enable us to factorize $N^{(1/\beta-1)\theta}$ both in the numerator and the denominator, so that they cancel out. Altogether, this yields

$$\rho_N(r) = N^{1-2\alpha} H_1(r_1) [T_1(f_1) - T_1(r_1)]^{1/\beta-1} \left[\int_0^{f_1} L(r_1) H_1(r_1) [T_1(f_1) - T_1(r_1)]^{1/\beta-1} dr_1 \right]^{-1}, \quad (\text{A.25})$$

which is simply a power function of N . In order to finally get the homothetic scaling (1) of the population density function, one has to assume that $1 - 2\alpha = \alpha$ holds, resulting in

$$\alpha = \frac{1}{3}. \quad (\text{A.26})$$

The bid rent can be expressed accordingly as

$$\psi_N(r) = N^{1/3+\theta} \beta [T_1(f_1) - T_1(r_1)]^{1/\beta} \left[\int_0^{f_1} L(r_1) H_1(r_1) [T_1(f_1) - T_1(r_1)]^{1/\beta-1} dr_1 \right]^{-1}. \quad (\text{A.27})$$

A.4 Consistency with an inter-urban approach

Substituting the scaling of wages into the right-hand-side equation of relationship (A.15) yields

$$Y_1 N^\phi = T_N(f_N) = T_N(f_1 N^\alpha) = N^\theta T_1(f_1) = N^\theta Y_1, \quad (\text{A.28})$$

where we used also the scalings of the urban fringe (A.21) and of the transport cost function (A.24). This implies

$$\phi = \theta. \quad (\text{A.29})$$

Second, successively applying the two changes of variable (A.19) and (A.21) to the equilibrium utility (A.16), and substituting the conditions of homothetic scaling (A.22) and (A.23) yields

$$e^{u/\beta} = N^{(1/\beta-1)\theta-(1-2\alpha)} (1-\beta)^{1/\beta-1} \int_0^{f_1} L(r_1) H_1(r_1) [T_1(f_1) - T_1(r_1)]^{1/\beta-1} dr_1. \quad (\text{A.30})$$

Since at equilibrium households have no incentive to move to another city, equilibrium utility (A.30) should not change with N . Thus, equalizing the power of N in equation (A.30) to zero (the rest is independent of N) and substituting the value of $\alpha = 1/3$ (equation A.26) gives

$$\theta = \frac{\beta}{3(1-\beta)}. \quad (\text{A.31})$$

Finally, simultaneously solving equations (A.29) and (A.31) yields

$$\phi = \theta = \frac{\beta}{3(1-\beta)}. \quad (\text{A.32})$$

A.5 Functional transport cost function

Consider the following form of the transport cost function

$$T_N(r) = cN^\mu r^\sigma, \quad (\text{A.33})$$

where $\mu, \sigma \in \mathbb{R}^+$. Then the scaling condition (A.24) requires

$$\theta = \alpha\sigma + \mu, \quad (\text{A.34})$$

where the elasticity θ of the transport cost function has been broken into two parts. On the one hand, the nonlinear effect of distance contributes by $\alpha\sigma$ to the elasticity θ because of the horizontal scaling. On the other hand, the contribution of μ stands for the urban population effects. Further substituting (11) and $\alpha = 1/3$ into (A.34) yields

$$\mu = \frac{\beta - \sigma(1 - \beta)}{3(1 - \beta)}. \quad (\text{A.35})$$

On the one hand, assuming $\sigma = 1$ yields the linear case presented in the functional form (14). On the other hand, assuming $\beta = 1/3$ yields $\mu = 1/6 - \sigma/3$, which is zero for $\sigma = 1/2$.

A.6 Functional monocentric model

Substituting the functional form equations (12)-(14) into the equilibrium population density function (10) with $\alpha = 1/3$ yields

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} (f_1 - r_1)^{\frac{1}{\beta}-1} \left[\int_0^{f_1} r_1 e^{-r_1/d} (f_1 - r_1)^{1/\beta-1} \mathbf{d}r_1 \right]^{-1}, \quad (\text{A.36})$$

with $r_1 = r/N^{1/3}$. Now, under the change of variable $y = f_1 - r_1$ the integral in equation (A.36) becomes

$$f_1 e^{-f_1/d} \int_0^{f_1} e^{y/d} y^{1/\beta-1} \mathbf{d}y - e^{-f_1/d} \int_0^{f_1} e^{y/d} y^{1/\beta} \mathbf{d}y, \quad (\text{A.37})$$

that the second change of variable $x = y/d$ turns to

$$f_1 e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta-1} \mathbf{d}x - e^{-f_1/d} d^{1/\beta+1} \int_0^{f_1/d} e^x x^{1/\beta} \mathbf{d}x. \quad (\text{A.38})$$

The first integral in (A.38) can be integrated by parts using

$$x^{1/\beta-1} = \frac{\partial(\beta x^{1/\beta})}{\partial x}. \quad (\text{A.39})$$

After algebraic simplifications, this yields

$$\beta f_1^{1/\beta+1} - (\beta f_1 + d) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathbf{d}x, \quad (\text{A.40})$$

which can be finally substituted to the integral into equation (A.36) to give

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} (f_1 - r_1)^{\frac{1}{\beta}-1} \left[\beta f_1^{1/\beta+1} - (\beta f_1 + d) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathbf{d}x \right]^{-1}, \quad (\text{A.41})$$

and for the bid rent

$$\psi_N(r) = \frac{N^{1/3} + \theta}{2\pi} \beta (f_1 - r_1)^{\frac{1}{\beta}} \left[\beta f_1^{1/\beta+1} - (\beta f_1 + d) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathbf{d}x \right]^{-1}. \quad (\text{A.42})$$

A.7 Population of a reference city

From equations (2) and (13), the model of housing usage considered here is a negative exponential with a scaling characteristic distance. Considering the empirical exponents of [Lemoy and Caruso \(2017\)](#), the best model of housing usage is

$$H_N(r) = b \exp\left(\frac{-r}{gN^{1/2}}\right), \quad (\text{A.43})$$

whereas the approximate model is

$$H_N(r) = b \exp\left(\frac{-r}{dN^{1/3}}\right). \quad (\text{A.44})$$

The absolute error between the best model (A.43) and the approximate model (A.44) is given by

$$b \exp\left(\frac{-r}{dN^{1/3}}\right) - b \exp\left(\frac{-r}{gN^{1/2}}\right) = b \exp\left(-\frac{r}{gN^{1/2}}\right) \left[\exp\left(\frac{-r}{dN^{1/3}} - \frac{-r}{gN^{1/2}}\right) - 1 \right], \quad (\text{A.45})$$

where the relative error is the term between braces. By definition, $\bar{N}$ is a population size chosen arbitrarily, for which the two characteristic distances are equal, thus annihilating the relative error. That is,

$$d = g\bar{N}^{1/6}, \quad (\text{A.46})$$

such that the relative error rewrites

$$\exp\left(\left[\left(\frac{N}{\bar{N}}\right)^{1/6} - 1\right] \frac{-r}{gN^{1/2}}\right) - 1. \quad (\text{A.47})$$

It appears from (A.47) that for any European city with $N > \bar{N}$, the housing share is underestimated and *vice versa* (Fig. 3). The relative error is bigger, the bigger the difference between N and $\bar{N}$. Hence a first desirable property is that the relative error for the smallest city is the same as for the largest one. This is equivalent to minimizing the maximal relative error. However, this cannot be true for any value of r since the relative error is increasing in r . On the opposite, the absolute error (A.45) has a maximum value at

$$\bar{r} = -\frac{gN^{1/2}}{6} \ln\left(\frac{\bar{N}}{N}\right) \left[\left(\frac{N}{\bar{N}}\right)^{1/6} - 1\right]^{-1}, \quad (\text{A.48})$$

and at this distance the relative error is simply

$$\left(\frac{\bar{N}}{N}\right)^{\frac{1}{6}} - 1. \quad (\text{A.49})$$

Finally, the critical population $\bar{N}$ is chosen as the value for which the absolute value of the relative error at the critical distance $\bar{r}$ is the same for the smallest city in the database, Derry (UK, $1.03 \cdot 10^5$ hab), and for the largest, London. This yields

$$\bar{N} = \left(\frac{2}{(1.03 \cdot 10^5)^{-1/6} + (1.21 \cdot 10^7)^{-1/6}}\right)^6 \simeq 7.03 \cdot 10^5. \quad (\text{A.50})$$

B Further example cities

We illustrate on Fig. 7 the results of the Alonso-LU model on four additional European cities, in order to complement Fig. 6: Paris ($N = 1.14 \cdot 10^7$), the second biggest city of the database, Wrocław (Poland, $N = 1.03 \cdot 10^6$), Florence (Firenze, in Italy, $N = 6.81 \cdot 10^5$) and Varna (Bulgaria, $N = 3.48 \cdot 10^5$).

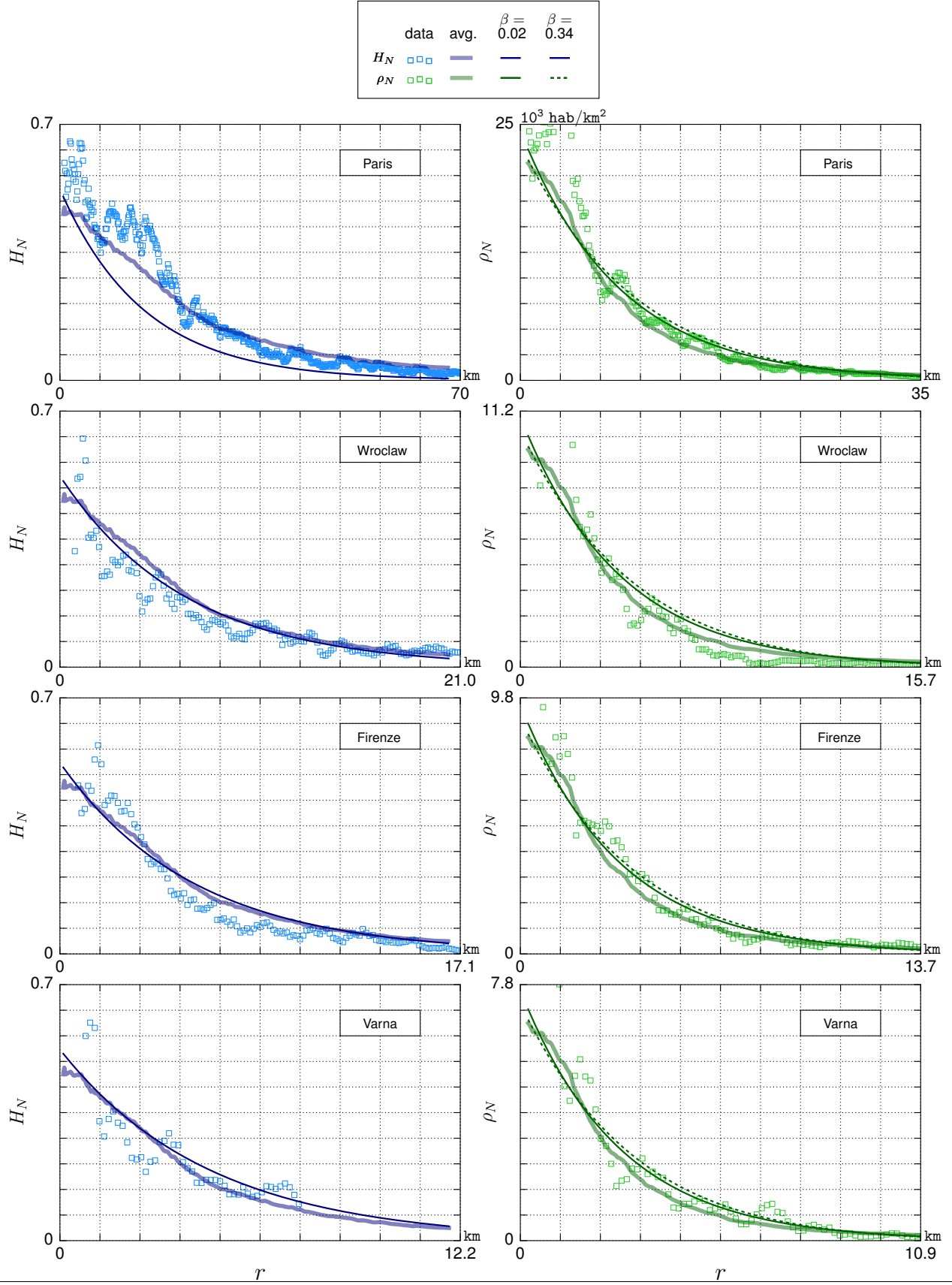


Figure 7: Summary plot of the results. Fitted average profiles compared to individual profiles. Left panel: housing share profile. Right panel: population density profile. Axes have been rescaled to maintain the average curves at the same position across subplots.

References

- Accardo, Jérôme and Fanny Bugeja (2009) “Le Poids des Dépenses de Logement depuis Vingt Ans,” in *Cinquante Ans de Consommation*, Paris, France: INSEE, pp. 33 – 47.
- Ahlfeldt, Gabriel M. (2008) “If Alonso Was Right: Residual Land Price, Accessibility and Urban Attraction,” SSRN Scholarly Paper ID 1305446, Social Science Research Network, Rochester, NY.
- Alonso, William (1964) *Location and land use*, Cambridge, United-States: Harvard University Press.
- Anas, Alex, Richard Arnott, and Kenneth A. Small (1998) “Urban spatial structure,” *Journal of Economic Literature*, Vol. 36, p. 1426.
- (2000) “The Panexponential Monocentric Model,” *Journal of Urban Economics*, Vol. 47, pp. 165–179.
- Arthur, W. Brian, Steven N. Durlauf, and David Lane eds. (1997) *The Economy as an Evolving Complex system II*, Reading, MA, United States: Addison-Wesley.
- Batty, Michael (2007) *Cities and Complexity: Understanding Cities with Cellular Automata, Agent-Based Models, and Fractals*: The MIT Press.
- (2013) *The New Science of Cities*, Cambridge, United-States: The MIT Press.
- Batty, Michael and Paul Longley (1994) *Fractal Cities. A Geometry of Form and Function*, London, United-Kingdom: Academic Press.
- Bettencourt, L. M. A. (2013) “The Origins of Scaling in Cities,” *Science*, Vol. 340, pp. 1438–1441.
- Bettencourt, Luís M. A. and José Lobo (2016) “Urban scaling in Europe,” *Journal of The Royal Society Interface*, Vol. 13, p. 20160005.
- Bettencourt, Luís M A, José Lobo, Dirk Helbing, Christian Kühnert, and Geoffrey B West (2007) “Growth, innovation, scaling, and the pace of life in cities,” *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 104, pp. 7301–7306.
- Brueckner, Jan K. (1982) “A note on sufficient conditions for negative exponential population densities,” *Journal of Regional Science*, Vol. 22, p. 353.
- Brueckner, Jan K. and David A. Fansler (1983) “The Economics of Urban Sprawl: Theory and Evidence on the Spatial Sizes of Cities,” *The Review of Economics and Statistics*, Vol. 65, pp. 479–482.
- Caruso, Geoffrey, Dominique Peeters, Jean Cavailhès, and Mark Rounsevell (2007) “Spatial configurations in a periurban city. A cellular automata-based microeconomic model,” *Regional Science and Urban Economics*, Vol. 37, pp. 542–567.
- Cavailhès, Jean, Pierre Frankhauser, Dominique Peeters, and Isabelle Thomas (2004a) “Where Alonso meets Sierpinski: an urban economic model of a fractal metropolitan area,” *Environment and Planning A*, Vol. 36, pp. 1471–1498.
- (2010) “Residential equilibrium in a multifractal metropolitan area,” *The Annals of Regional Science*, Vol. 45, pp. 681–704.
- Cavailhès, Jean, Dominique Peeters, Evangelos Sékeris, and Jacques-François Thisse (2003) “La ville périurbaine,” *Revue économique*, Vol. 54, p. 5.
- (2004b) “The periurban city: why to live between the suburbs and the countryside,” *Regional Science and Urban Economics*, Vol. 34, pp. 681–703.

- Chen, Yanguang (2013) “Fractal analytical approach of urban form based on spatial correlation function,” *Chaos, Solitons & Fractals*, Vol. 49, pp. 47–60.
- Cheshire, Paul and Edwin S. Mills eds. (1999) *Handbook of Regional and Urban Economics. Volume 3: Applied Urban Economics.*, Vol. 3, Amsterdam, The Netherlands: North-Holland.
- Chicoine, David L. (1981) “Farmland Values at the Urban Fringe: An Analysis of Sale Prices,” *Land Economics*, Vol. 57, pp. 353–362.
- Clark, Colin (1951) “Urban Population Densities,” *Journal of the Royal Statistical Society. Series A (General)*, Vol. 114, p. 490.
- Cobb, C. W. and P. H. Douglas (1928) “A theory of production,” *American Economic Review*, Vol. 18, pp. 139–165.
- Colwell, Peter F. and Gene Dillmore (1999) “Who Was First? An Examination of an Early Hedonic Study,” *Land Economics*, Vol. 75, pp. 620–626.
- Combes, P.-P., G. Duranton, and L. Gobillon (2011) “The identification of agglomeration economies,” *Journal of Economic Geography*, Vol. 11, pp. 253–266.
- Combes, Pierre-Philippe, Gilles Duranton, and Laurent Gobillon (2012) “The Costs of Agglomeration: Land Prices in French Cities,” Discussion Paper 7027, IZA Institute of Labor Economics, Bonn, Germany.
- Combes, Pierre-Philippe, Gilles Duranton, Laurent Gobillon, and Sébastien Roux (2010) “Estimating Agglomeration Economies with History, Geology, and Worker Effects,” in Edward Glaeser ed. *Agglomeration Economics*: University of Chicago Press, pp. 15–66.
- Cura, Robin, Clémentine Cottineau, Elfie Swerts, Cosmo Antonio Ignazzi, Anne Bretagnolle, Celine Vacchiani-Marcuzzo, and Denise Pumain (2017) “The Old and the New: Qualifying City Systems in the World with Classical Models and New Data,” *Geographical Analysis*, Vol. 49, pp. 363–386.
- Davis, Morris A. and François Ortalo-Magné (2011) “Household expenditures, wages, rents,” *Review of Economic Dynamics*, Vol. 14, pp. 248–261.
- Dixit, Avinash (1973) “The Optimum Factory Town,” *The Bell Journal of Economics and Management Science*, Vol. 4, p. 637.
- Duranton, Gilles, J. Vernon Henderson, and William C. Strange eds. (2015) *Handbook of Regional and Urban Economics.*, Vol. 5A, Amsterdam, The Netherlands: Elsevier.
- Ewing, Reid, Harry Richardson, Keith Bartholomew Burch, Arthur C Nelson, and Christine Bae (2014) “Compactness vs. sprawl revisited: Converging views.”
- Frankhauser, Pierre (1994) *La fractalité des structures urbaines*, Villes, Paris, France: Anthropos.
- Fujita, Masahisa (1989) *Urban economic theory: Land use and city size*, New York, United-States: Cambridge University Press.
- Fujita, Masahisa and Jacques-François Thisse (2013) *Economics of Agglomeration. Cities, Industrial Location and Globalization.*, Cambridge, United-Kingdom: Cambridge University Press, 2nd edition.
- Gabaix, Xavier (1999) “Zipf’s Law for Cities: An Explanation,” *The Quarterly Journal of Economics*, Vol. 114, pp. 739–767.
- Leitão, J. C., J. M. Miotto, M. Gerlach, and E. G. Altmann (2016) “Is this scaling nonlinear?” *Royal Society Open Science*, Vol. 3, p. 150649.

- Lemoy, Rémi and Geoffrey Caruso (2017) “Scaling evidence of the homothetic nature of cities,” *arXiv:1704.06508 [physics, q-fin]*, arXiv: 1704.06508.
- Louf, Rémi and Marc Barthélemy (2014) “Scaling: Lost in the Smog,” *Environment and Planning B: Planning and Design*, Vol. 41, pp. 767–769.
- McGrath, Daniel T. (2005) “More evidence on the spatial scale of cities,” *Journal of Urban Economics*, Vol. 58, pp. 1–10.
- Meserve, Bruce E. (2010) *Fundamental Concepts of Geometry*, New York: Dover Publications, revised edition.
- Mills, Edwin (1972) *Studies in the Structure of the Urban Economy*, Baltimore, United-States: The Johns Hopkins Press.
- Muth, Richard F. (1969) *Cities and housing : the spatial pattern of urban residential land use*, Chicago, United States: University of Chicago press.
- Myerson, Roger B. (1997) *Game Theory - Analysis of conflict*, Cambridge, United-States: Harvard University Press.
- von Neumann, John and O Morgenstern (1944) *Theory Of Games And Economic Behavior*: Princeton University Press.
- Nordbeck, Stig (1971) “Urban allometric growth,” *Geografiska Annaler. Series B, Human Geography*, Vol. 53, pp. 54–67.
- Pumain, Denise (1982) *La dynamique des villes*, Paris: Economica.
- (2004) “Scaling Laws and Urban Systems,” SFI Working Paper 2004-02-002, Santa Fe Institute, Santa-Fe, NM, United-States.
- Pumain, Denise and Romain Reuillon (2017) *Urban Dynamics and Simulation Models*, Lecture Notes in Morphogenesis, Cham: Springer International Publishing, DOI: 10.1007/978-3-319-46497-8.
- Rosenthal, Stuart S. and William C. Strange (2004) “Evidence on the nature and sources of agglomeration economies,” in *Handbook of Regional and Urban Economics*, Vol. 4, Amsterdam, The Netherlands: Elsevier, pp. 2119–2171.
- Shalizi, Cosma Rohilla (2011) “Scaling and Hierarchy in Urban Economies,” *arXiv:1102.4101 [physics, stat]*, arXiv: 1102.4101.
- Spivey, Christy (2008) “The Mills-Muth Model of Urban Spatial Structure: Surviving the Test of Time?” *Urban Studies*, Vol. 45, pp. 295–312.
- Tabuchi, T., J.-F. Thisse, and D.-Z. Zeng (2005) “On the number and size of cities,” *Journal of Economic Geography*, Vol. 5, pp. 423–448.
- Tannier, Cécile, Isabelle Thomas, Gilles Vuidel, and Pierre Frankhauser (2011) “A Fractal Approach to Identifying Urban Boundaries,” *Geographical Analysis*, Vol. 43, pp. 211–227.
- UN-HABITAT (2016) “World Cities Report : Urbanization and Development,” Technical report, United Nations.
- Varian, Hal R. (2011) *Introduction à la microéconomie*, Bruxelles: De Boeck, 7th edition.
- Vicsek, Tamas (2002) “Complexity: The bigger picture,” *Nature*, Vol. 418, pp. 131–131.
- White, Roger W., Guy Engelen, and Inge Uljee (2015) *Modeling Cities and Regions as Complex Systems*, Cambridge, United-States: The MIT Press.