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Author(s): Y. Felix and J. C. Thomas

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ON THE HOMOLOGY OF POSTNIKOV FIBRES

Y. FÉLIX AND J. C. THOMAS

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ABSTRACT. Let k be a field of positive characteristic and X be a simply connected space of the homotopy type of a finite type CW complex. The Postnikov fibre $X_{[n]}$ of X is defined as the homotopy fibre of the n -equivalence $f_n: X \rightarrow X_n$ coming from the Postnikov tower $\{X_n\}$ of X . We prove that if the Lusternik-Schnirelmann category of X is finite, then $H_*(X_{[n]}; k)$ contains a free module on a subalgebra K of $H_*(\Omega X_n; k)$ such that $H_*(\Omega X_n; k)$ is a finite-dimensional free K -module.

Let k be a field of positive characteristic p and X be a simply connected space which has the homotopy type of a finite type CW complex. The Postnikov tower of X consists of a sequence of principal fibrations

$$X_n \xrightarrow{p_n} X_{n-1} \rightarrow K(\pi_n(X), n+1)$$

and of n -equivalences $f_n: X \rightarrow X_n$ satisfying $p_n f_n = f_{n-1}$. The homotopy fibre of f_n is then denoted by $X_{[n]}$ and is called the n th Postnikov fibre of X ,

$$X_{[n]} \rightarrow X \xrightarrow{f_n} X_n.$$

The homotopy lifting property of the fibration f_n defines a natural action of $H_*(\Omega X_n; k)$ on $H_*(X_{[n]}; k)$ [8]. This action is called the *holonomy operation* and its behaviour in this context is the subject of this paper.

More generally we will consider a fibration of simply connected spaces

$$F \rightarrow E \xrightarrow{f} B$$

such that ΩB has a *stable r -stage Postnikov system* [7]. This means that ΩB can be obtained by a finite sequence of multiplicative fibrations

$$G_r \rightarrow G_{r-1} \rightarrow K_r, \quad r = 0, \dots, n, \quad \Omega B = G_n, \quad G_{-1} = \{*\},$$

with K_r a product of Eilenberg-Mac Lane spaces. The spaces ΩX_n are stable n -stable Postnikov systems. This happens also, for instance, when B has only a finite number of nonzero homotopy groups. We can now state our main theorems.

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The category of $f: E \rightarrow B$, $\text{cat } f$, is defined as the least $m \leq \infty$ such that E can be covered by $m + 1$ open sets U_i so that the restriction of f to each U_i is homotopic to zero [1]. Clearly $\text{cat } f \leq \text{cat } E$.

Theorem 1. *Let $F \rightarrow E \xrightarrow{f} B$ as before. We suppose that $\text{cat } f < \infty$. Then*
 (1) *there exists a nontrivial morphism of $H_*(\Omega B; k)$ -modules*

$$H_*(F; k) \rightarrow H_*(\Omega B; k),$$

(2) *there exists a subalgebra K of $H_*(\Omega B; k)$ such that $H_*(F; k)$ contains a free K -module and such that $H_*(\Omega B; k)$ is a finite-dimensional free K -module.*

Theorem 2. *Let $F \rightarrow E \rightarrow B$ as before. Suppose that E has the homotopy type of a finite CW complex and ΩB is a product of Eilenberg-Mac Lane spaces. Then the algebra $H_*(\Omega B; k)$ is isomorphic to $K \otimes G$ with G finite-dimensional and $H_*(F; K)$ is a free K -module.*

One can remark that if f is homotopically trivial, then F has the homotopy type of the product $\Omega B \times E$, and Theorem 1 is obviously true in this case. The point is that the homology of the fibre F is never very far from being free; this is the content of the results.

In [3] Halperin and the authors establish a relation between $\text{cat } f$ and some homological invariants of $H_*(F; k)$ as a module over $H_*(\Omega B; k)$. Let $G = \bigoplus_{i \geq 0} G_i$ be a graded Hopf algebra over the field k satisfying

$$\begin{aligned} G_0 &= k; \\ \dim G_i &< \infty \text{ for any } i; \\ G &\text{ is cocommutative.} \end{aligned}$$

The *grade* of a graded G -module M is the greatest n (or ∞) such that $\text{Ext}_G^n(M; G) \neq 0$. The *depth* of the Hopf algebra G is, by definition, the grade of the trivial module k . The main result of [3] reads as follows.

Theorem [3, Theorem A]. *With the above hypothesis, $\text{grade}(H_*(F; k)) \leq \text{cat } f$.*

Following Moore and Smith [7], we call a Hopf algebra G *p-solvable* if there exists a sequence of normal sub-Hopf algebras

$$k \subset G_{\langle -s \rangle} \subset G_{\langle -s+1 \rangle} \subset \cdots \subset G_{\langle 0 \rangle} = G$$

such that each quotient $G_{\langle t \rangle} // G_{\langle t-1 \rangle}$ is a commutative Hopf algebra with $x^p = 0$ for every x in $G_{\langle t \rangle} // G_{\langle t-1 \rangle}$, i.e., each quotient is a coprimitive Hopf algebra. The interest of *p-solvable* Hopf algebras in topology comes from the following result of Moore and Smith.

Theorem [7, Theorem 6.2]. *If ΩX is a stable r -stage Postnikov system, then the Hopf algebra $H_*(\Omega X; k)$ is p -solvable.*

Every finitely generated coprimitive Hopf algebra is finite dimensional. Thus from Lemma 1, every finitely generated *p-solvable* Hopf algebra is also finite dimensional. In particular, each element has finite height.

Lemma 1. *Let G be a finitely generated Hopf algebra and K be a normal sub-Hopf algebra such that the quotient $G//K$ is finite dimensional. Then K is also finitely generated.*

Proof. From the Hochschild-Serre spectral sequence associated to the short exact sequence of Hopf algebras

$$K \rightarrow G \rightarrow G//K,$$

we obtain an isomorphism

$$\mathrm{Tor}_G^1(k, k) \cong \mathrm{Tor}_{G//K}^1(k, k) \oplus (\mathrm{Tor}_{G//K}^0(k, \mathrm{Tor}_K^1(k, k)) / \mathrm{Im}(d_2)).$$

As $G//K$ is finite dimensional, the dimension of the vector space $\mathrm{Tor}_{G//K}^i(k, k)$ is finite for every i ; therefore, $\mathrm{Tor}_G^1(k, k)$ is finite dimensional if and only if $\mathrm{Tor}_K^1(k, k)$ is finite dimensional. $\square$

The next lemma is the main tool in the proof of Theorem 1. This is a generalization of [2, Proposition 3.1] with exactly the same proof.

Lemma 2. *Suppose $0 \neq \omega \in \mathrm{Ext}_G^m(M, G)$, G a Hopf algebra, and M a G -module. Then for some finitely generated sub-Hopf algebra $K \subset G$, ω restricts to a nonzero element of $\mathrm{Ext}_K^m(M, G)$.*

Lemma 3. *A module M of finite grade on a p -solvable Hopf algebra G has grade zero.*

Proof. We denote by ω a nonzero element in $\mathrm{Ext}_G^m(M, G)$. It then results from Lemma 2 that, for some finitely generated sub-Hopf algebra $H \subset G$, ω restricts to a nonzero element in $\mathrm{Ext}_H^m(M, G)$; therefore, $\mathrm{Ext}_H^m(M, H) \neq 0$.

Write $M = \varinjlim M^\alpha$ with each M^α a finitely generated A -module. From the canonical isomorphisms

$$\mathrm{Ext}_H^s(M, H) = \varprojlim \mathrm{Ext}_H^s(M^\alpha, H),$$

one can see that if $\mathrm{Ext}_H^m(M, H) \neq 0$, then $\mathrm{Ext}_H^m(M^\alpha, H) \neq 0$ for some finitely generated submodule M^α .

The Hopf algebra H is p -solvable and finitely generated; therefore, H is finite dimensional and, hence, elliptic in the sense of [5]. By [4, Lemma 3.10] $m = 0$. $\square$

Proof of Theorem 1. As the category of p is finite, the grade of the $H_*(\Omega B; k)$ -module $H_*(F; k)$ is finite; therefore, by Lemma 3 the grade is zero. This implies the existence of a nontrivial morphism of $H_*(\Omega B; k)$ -module $g: H_*(F; k) \rightarrow H_*(\Omega B; k)$.

For the sake of simplicity we denote $G = H_*(\Omega B; k)$ and $M = H_*(F; k)$. By hypothesis there exists a sequence of normal sub-Hopf algebras

$$k \subset A_{\langle -s \rangle} \subset A_{\langle -s+1 \rangle} \subset \cdots \subset A_{\langle 0 \rangle} = G$$

such that each quotient $A_{\langle t \rangle} / A_{\langle t-1 \rangle}$ is isomorphic as an algebra to

$$A_{\langle t \rangle} / A_{\langle t-1 \rangle} \cong \bigotimes_{i \in I_t} \Lambda x_i \otimes \bigotimes_{j \in J_t} k[y_j] / y_j^p.$$

For each degree q we denote by K_q the algebra generated by the x_i and the y_j of degree larger than q . By the previous decomposition, G is a free finitely generated K_q -module for each integer q .

Let m be an element of M such that $g(m) \neq 0$. The element $g(m)$ belongs to a finitely generated subalgebra H of G . Denote by t the maximal degree of

the homogeneous elements of H . As G is a K_t free module, $K_t \cdot g(m) \cong K_t$. This implies that the K_t -module generated by m in M is free. $\square$

Proof of Theorem 2. The homology Serre spectral sequence of the fibration

$$\Omega B \rightarrow F \rightarrow E$$

is a spectral sequence of $H_*(\Omega B; k)$ -modules. As $H_*(E; k)$ is finite dimensional, there is only a finite number of nonzero differentials d_r in this spectral sequence. We will prove by induction on r that each E_r is a free finitely generated module over some subalgebra H_r of $G = H_*(\Omega B; k)$ such that G is a free finitely generated H_r -module.

This is true for E_2 . The algebra $H_*(\Omega B; k)$ is a tensor product $\bigotimes_i \Lambda x_i \otimes \bigotimes_j k[y_j]/y_j^p$. The integer q will be defined by

$$q = (\text{the maximal degree of the homogeneous elements of } H_*(E; k)) + 2.$$

Denote by H_3 the tensor product of the components Λx_i and $k[y_j]/y_j^p$ with x_i and y_j of degree greater than q . G is then the tensor product $G = H_3 \otimes R_3$ with R_3 finite-dimensional. The E_2 -term of the Serre spectral sequence, (E_2, d_2) , is then isomorphic to $(H_3, 0) \otimes (H_*(E; k) \otimes R_3, d_2)$ as an H_3 -module. Its homology E_3 is therefore isomorphic to $H_3 \otimes H(H_*(E; k) \otimes R_3, d_2)$ and is a free finitely generated H_3 -module.

We proceed in exactly the same way for the general case. At each stage H_r will be the intersection of H_{r-1} and the tensor product of the components Λx_i and $k[y_j]/y_j^p$ with x_i and y_j of degree greater than $2 +$ (the maximal degree of the generators of E_{r-1} as an H_{r-1} -module). $\square$

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DEPARTEMENTE MATHÉMATIQUE, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, BELGIUM

DEPARTEMENTE MATHÉMATIQUE, UNIVERSITÉ DES SCIENCES ET TECHNIQUES DE LILLE, FRANCE