Asymptotics of Hankel determinants with a one-cut regular potential and Fisher-Hartwig singularities

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Abstract

We obtain asymptotics of large Hankel determinants whose weight depends on a one-cut regular potential and any number of Fisher-Hartwig singularities. This generalises two results: 1) a result of Berestycki, Webb and Wong [5] for root-type singularities, and 2) a result of Its and Krasovsky [37] for a Gaussian weight with a single jump-type singularity. We show that when we apply a piecewise constant thinning on the eigenvalues of a random Hermitian matrix drawn from a one-cut regular ensemble, the gap probability in the thinned spectrum, as well as correlations of the characteristic polynomial of the associated conditional point process, can be expressed in terms of these determinants.

1 Introduction and main results

We are interested in the large n asymptotics of the Hankel determinant

$$D_n(\vec{\alpha},\vec{\beta},V,W) = \det\left(\int_{\mathbb{R}} x^{j+k-2} w(x)dx\right)_{j,k=1,\dots,n}, \tag{1.1}$$

where the weight w is defined by

$$w(x) = e^{-nV(x)}e^{W(x)}\omega(x), \qquad \omega(x) = \prod_{j=1}^{m} \omega_{\alpha_j}(x)\omega_{\beta_j}(x), \qquad m \in \mathbb{N}, \tag{1.2}$$

and for each $k \in \{1,...,m\}$, we have

$$\omega_{\alpha_k}(x) = |x-t_k|^{\alpha_k}, \qquad \omega_{\beta_k}(x) = \begin{cases} e^{i\pi\beta_k}, & \text{if } x < t_k, \\ e^{-i\pi\beta_k}, & \text{if } x > t_k. \end{cases} \tag{1.3}$$

The number of Fisher-Hartwig (FH) singularities is denoted by $m \in \mathbb{N}$, and $t_1,...,t_m \in \mathbb{R}$ are the locations of these singularities. For each $k \in \{1,...,m\}$, the root-type singularity and the jump-type singularity located at t_k are parametrized by $\alpha_k \in \{z \in \mathbb{C} : \Re z > -1\}$ and $\beta_k \in \mathbb{C}$ respectively. Since for any $n_0 \in \mathbb{Z}$, we have $\omega_{\beta_k+n_0} = (-1)^{n_0}\omega_{\beta_k}$, we can assume without loss of generality that $\Re\beta_k \in (\frac{-1}{2},\frac{1}{2}]$. The potential $V : \mathbb{R} \to \mathbb{R}$ is real analytic, the function $W : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and they satisfy

$$\lim_{x\to\pm\infty} V(x)/\log|x| = +\infty \qquad \text{and} \qquad W(x) = \mathcal{O}(V(x)), \text{ as } |x| \to \infty.$$

The $\mathcal{O}$ notation means that there exist two constants $c_1, c_2 > 0$ such that $|W(x)| \leq c_1|V(x)|$ for all $|x| > c_2$. Thus w is integrable on $\mathbb{R}$ and has finite moments for $n > c_1$. By Heine's formula, $D_n(\vec{\alpha},\vec{\beta},V,W)$ admits the following n-fold integral representation:

$$D_n(\vec{\alpha},\vec{\beta},V,W) = \frac{1}{n!}\int_{\mathbb{R}^n} \prod_{1\leq j<k\leq n} (x_k-x_j)^2 \prod_{j=1}^{n} w(x_j)dx_j. \tag{1.4}$$

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This determinant depends on several parameters. To simplify the notation, we denoted it by $D_n(\vec{\alpha},\vec{\beta},V,W)$, where the dependence in $t_1,...,t_m$ is omitted and where we write $\vec{\alpha}$ and $\vec{\beta}$ for $(\alpha_1,...,\alpha_m)$ and $(\beta_1,...,\beta_m)$ respectively.

The analogous situation for Toeplitz determinants (i.e. determinants constant along the diagonals) with a weight defined on the circle with FH singularities has been studied by many authors (e.g. [32, 51, 3, 4, 13, 30]), and the most general results can be found in [23, 24]. Asymptotics for large Hankel determinants associated to a weight of the form

$$e^{W(x)}\omega(x)\chi_{[-1,1]}(x), \tag{1.5}$$

where $\chi_{[-1,1]}$ is the characteristic function of $[-1,1]$, were also obtained in [23] by using a relation between these Hankel determinants and Toeplitz determinants with FH singularities. Note that the weight (1.5) is not a particular case of (1.2) (and vice-versa).

Hankel determinants of the form (1.4) appear naturally in random matrix theory. Consider the set of $n\times n$ Hermitian matrices M endowed with the probability distribution

$$\frac{1}{\widehat{Z}_n}e^{-n\mathrm{Tr}V(M)}dM, \qquad dM=\prod_{i=1}^{n}dM_{ii}\prod_{1\leq i<j\leq n}d\Re M_{ij}d\Im M_{ij}, \tag{1.6}$$

where $\widehat{Z}_n$ is the normalisation constant, also called the partition function. This distribution of matrices is invariant under unitary conjugations and induces a probability distribution on the eigenvalues $x_1,...,x_n$ of M which is of the form

$$\frac{1}{n!Z_n}\prod_{1\leq j<k\leq n}(x_k-x_j)^2\prod_{j=1}^{n}e^{-nV(x_j)}dx_j, \qquad (x_1,...,x_n)\in\mathbb{R}^n, \tag{1.7}$$

where the partition function Z_n can be rewritten as

$$Z_n=D_n(\vec{0},\vec{0},V,0). \tag{1.8}$$

Probably the most studied random matrix ensemble is the Gaussian Unitary Ensemble (GUE) [1, 2, 45], and corresponds to the case $V(x)=2x^2$. In that instance, the partition function is explicitly known (see e.g. [45])

$$\begin{aligned} D_n(\vec{0},\vec{0},2x^2,0) &= (2\pi)^{\frac{n}{2}}2^{-n^2}n^{-\frac{n^2}{2}}\prod_{j=1}^{n-1}j! \\ &= 2^{-n^2}e^{-\frac{3}{4}n^2}(2\pi)^n n^{-\frac{1}{12}}e^{\zeta'(-1)}\left(1+\mathcal{O}(n^{-1})\right), \qquad \text{as } n\to\infty, \end{aligned} \tag{1.9}$$

where ζ is Riemann's zeta-function.

The potentials V we are interested in are described in terms of properties of the equilibrium measure μ_V, which is the unique minimizer of the functional

$$\iint \log|x-y|^{-1}d\mu(x)d\mu(y)+\int V(x)d\mu(x) \tag{1.10}$$

among all Borel probability measures μ on $\mathbb{R}$. It is known (see e.g. [47]) that this measure and its support (denoted $\mathcal{S}$) are completely characterized by the Euler-Lagrange variational conditions

$$2\int_{\mathcal{S}}\log|x-s|d\mu_V(s)=V(x)-\ell, \qquad \text{for } x\in\mathcal{S}, \tag{1.11}$$

$$2\int_{\mathcal{S}}\log|x-s|d\mu_V(s)\leq V(x)-\ell, \qquad \text{for } x\in\mathbb{R}\setminus\mathcal{S}. \tag{1.12}$$

A potential V is called one-cut regular if it satisfies the following conditions (see e.g. [7, 18, 5]):

1. $V : \mathbb{R} \to \mathbb{R}$ is analytic.
2. $\lim_{x\to\pm\infty} V(x)/\log|x| = +\infty$.
3. The inequality (1.12) is strict.
4. The equilibrium measure is supported on $\mathcal{S} = [a, b]$ and is of the form $d\mu_V(x) = \psi(x)\sqrt{(b-x)(x-a)}dx$, where ψ is positive on $[a, b]$.

The above conditions imply that ψ is also real analytic. This is a consequence of [25, Theorem 1.38] (for more details, see the beginning of Section 4). In the present work, we restrict ourselves to the class of one-cut regular potentials whose equilibrium measure is supported on $[-1, 1]$ instead of $[a, b]$. This is without loss of generality, as can be easily seen from a change of variables in (1.4), (1.11) and (1.12). For the reader's convenience, we show that explicitly in Remark 1.4 below. An example of a one-cut regular potential is the Gaussian potential $V(x) = 2x^2$. In this case, it is well-known [47] that $\ell = 1 + 2\log 2$ and $\psi(x) = \frac{2}{\pi}$.

A lot of results are available in the literature for large n asymptotics of $D_n(\vec{\alpha}, \vec{\beta}, V, W)$ for particular values of the parameters, and we briefly discuss them here.

We start with Hankel determinants without singularities. If V is a polynomial, is one-cut regular and such that all zeros of $\psi(x)$ are nonreal, Johansson [38] obtained rigorously the asymptotics of $\frac{D_n(\vec{0},\vec{0},V,W)}{D_n(\vec{0},\vec{0},V,0)}$ (see [38, Theorem 2.4] for the precise assumptions on W). In particular, this implies a central limit theorem for the linear statistics, i.e. it gives information about the global fluctuation properties of the spectrum around the equilibrium measure. For a polynomial one-cut regular potential V, large n asymptotics for the partition function $D_n(\vec{0}, \vec{0}, V, 0)$ have been obtained via the Riemann-Hilbert method in [31] (under certain assumptions on the coefficients of V, see [31, Theorem 1.1 and above]), and via deformation equations in [7] (under further technical assumptions on V, see [7, equation (1.5) and remark (2) below]). There are also results available for partition functions in more general situations, see e.g. [18] when the equilibrium measure is supported on two intervals.

Hankel determinants with root-type singularities are related to the statistical properties of the characteristic polynomial $p_n(t) = \prod_{j=1}^{n}(t - x_j)$ of (1.7). There is numerical evidence and conjectures of links between $p_n(t)$ and the behaviour of the Riemann ζ-function along the critical line (see e.g. [41]). In [42], Krasovsky studied the correlations between $|p_n(t_1)|$,...,$|p_n(t_m)|$, and powers of these quantities, when $-1 < t_1 < ... < t_m < 1$ are fixed in the case of GUE. The large n asymptotics of these correlations are given by

$$\mathbb{E}_{\mathrm{GUE}}\left(\prod_{k=1}^{m}|p_n(t_k)|^{\alpha_k}\right) = \frac{D_n(\vec{\alpha},\vec{0},2x^2,0)}{D_n(\vec{0},\vec{0},2x^2,0)} = 2^{-\mathcal{A}n}\prod_{1\leq j<k\leq m}(2|t_j - t_k|)^{-\frac{\alpha_j\alpha_k}{2}}$$
$$\times\prod_{j=1}^{m}\frac{G\left(1+\frac{\alpha_j}{2}\right)^2}{G(1+\alpha_j)}\left(2\sqrt{1-t_j^2}n\right)^{\frac{\alpha_j^2}{4}}\exp\left(\frac{\alpha_j n}{2}(2t_j^2-1)\right)\left(1+\mathcal{O}\left(\frac{\log n}{n}\right)\right), \quad (1.13)$$

where $\Re\alpha_k > -1$ for $k = 1, ..., m$, $\mathcal{A} = \sum_{j=1}^{m}\alpha_j$ and G is Barnes' G-function.

This result was recently generalized for the class of one-cut regular potentials by Berestycki, Webb and Wong. In [5, Theorem 1.1], they proved that a sufficiently small power of the absolute value of the characteristic polynomial $p_n(t)$ of a one-cut regular ensemble converges weakly in distribution to a Gaussian multiplicative chaos measure. It was crucial in their analysis to obtain the large n asymptotics of

$$\mathbb{E}_V\left(\prod_{j=1}^{n}e^{W(x_j)}\prod_{k=1}^{m}|p_n(t_k)|^{\alpha_k}\right) = \frac{D_n(\vec{\alpha},\vec{0},V,W)}{D_n(\vec{0},\vec{0},V,0)}, \quad (1.14)$$

where $\alpha_k \in (-1,\infty)$ for $k = 1, ..., m$ and $W \in C^\infty(\mathbb{R})$ is a compactly supported function, analytic in a neighbourhood of $[-1, 1]$. In particular, they proved two conjectures of Forrester and Frankel [34] concerning asymptotics of Hankel determinants with root-type singularities.

Only limited results are available concerning Hankel determinants with jump discontinuities. Such determinants allow to generalize the correlations (1.14). Let us define the argument of the characteristic polynomial as follows:

$$\arg p_n(t) = \sum_{j=1}^{n} \arg(t - x_j), \qquad \text{where} \qquad \arg(t - x_j) = \begin{cases} 0, & \text{if } x_j < t, \\ -\pi, & \text{if } x_j > t. \end{cases} \tag{1.15}$$

We have

$$\mathbb{E}_V\left(\prod_{j=1}^{n} e^{W(x_j)} \prod_{k=1}^{m} |p_n(t_k)|^{\alpha_k} e^{2i\beta_k \arg p_n(t_k)} \right) = \frac{D_n(\vec{\alpha}, \vec{\beta}, V, W)}{D_n(\vec{0}, \vec{0}, V, 0)} \prod_{k=1}^{m} e^{-i\pi\beta_k}. \tag{1.16}$$

Hankel determinants with jump singularities appear also when we thin the eigenvalues of a random matrix. Consider the point process of the n eigenvalues of an $n \times n$ random GUE matrix. An independent and constant thinning consists of removing each of these eigenvalues with a certain probability $s \in [0,1]$. The remaining (thinned) eigenvalues are denoted $y_1, ..., y_N$, where N is a random variable following the binomial distribution $\mathrm{Bin}(n, 1-s)$. The gap probability in the thinned spectrum can be expressed in terms of a Hankel determinant with a single jump discontinuity (i.e. $m = 1$) as follows: let $t_1 \in \mathbb{R}$ and consider the partition of $\mathbb{R}^n$

$$A_k = \big\{(x_1, ..., x_n) \in \mathbb{R}^n : \sharp\{x_i : x_i < t_1\} = k\big\}, \qquad \bigsqcup_{k=0}^{n} A_k = \mathbb{R}^n. \tag{1.17}$$

Since we thin the eigenvalues independently of each other and independently of their positions, we have

$$\mathbb{P}_{\mathrm{GUE}}\big(\sharp\{y_i : y_i < t_1\} = 0 | \sharp\{x_i : x_i < t_1\} = k\big) = s^k. \tag{1.18}$$

Therefore, using the integral representation for Hankel determinants (1.4), one has

$$\mathbb{P}_{\mathrm{GUE}}\big(\sharp\{y_i : y_i < t_1\} = 0\big) = \sum_{k=0}^{n} s^k \mathbb{P}_{\mathrm{GUE}}\big(\sharp\{x_i : x_i < t_1\} = k\big) \tag{1.19}$$

$$= \frac{1}{n! Z_n} \sum_{k=0}^{n} \int_{A_k} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{n} e^{-2nx_j^2} \left\{ \begin{array}{ll} s, & \text{if } x_j < t_1 \\ 1, & \text{if } x_j > t_1 \end{array} \right\} dx_j = e^{in\pi\beta_1} \frac{D_n(\vec{0}, \vec{\beta}, 2x^2, 0)}{D_n(\vec{0}, \vec{0}, 2x^2, 0)},$$

where in the last equality $\vec{\beta} = \beta_1 = \frac{\log s}{2\pi i} \in i\mathbb{R}^+$ if $s > 0$. The large n asymptotics of this ratio of Hankel determinants were studied by Its and Krasovsky in [37] for a fixed $t_1 \in (-1, 1)$. Their asymptotic formula is valid not only for purely imaginary β_1, but as long as $\Re\beta_1 \in \big(\frac{-1}{4}, \frac{1}{4}\big)$. Note that the quantity in (1.19) is a polynomial in s, but admits an expression in terms of a Hankel determinant with a Fisher-Hartwig singularity only if $s \neq 0$. If $s = 0$, there is no thinning, and the probability $\mathbb{P}_{\mathrm{GUE}}\big(\sharp\{y_i : y_i < t_1\} = 0\big) = \mathbb{P}_{\mathrm{GUE}}\big(\sharp\{x_i : x_i < t_1\} = 0\big)$ can be expressed in terms of a Hankel determinant associated to the weight $\chi_{[t_1,\infty)}(x) e^{-2nx^2}$. We do not study such Hankel determinants in the present paper. Their asymptotics as $n \to \infty$ deserve a separate analysis, see e.g. [21, 16] (and see [50, 14, 15] for analogous situations for Toeplitz determinants).

The operation of thinning is general and can be applied on any point process [20]. It was introduced in random matrix theory by Bohigas and Pato [8, 9]. The point process (1.7) can be rewritten as a determinant, and is called determinantal (see e.g. [39, 48] for the definition and properties of determinantal point processes). If we apply an independent and constant thinning to a determinantal point process, the thinned point process is still determinantal [40]. This provides an efficient way to interpolate between the original point process (when $s = 0$) and an uncorrelated Poisson process (when $s \to 1$ at a certain speed). Several such transitions have been studied, see e.g. [11, 12, 14, 15, 6, 44, 10]. Hankel determinants with several jump discontinuities are related to piecewise constant (and independent) thinning. We start from the point process of (1.7) for a general potential V, and consider $\mathcal{K} \subseteq \{1, ..., m+1\}$, possibly empty. For $k \in \mathcal{K}$, each eigenvalue on (t_{k-1}, t_k) is removed with a probability $s_k \in (0, 1]$ (in this setting $t_0 = -\infty$ and $t_{m+1} = +\infty$). The thinned point process is again denoted $y_1, ..., y_N$, but N is no longer binomial. In the same spirit as (1.19), one can show that

$$\mathbb{P}_V\Big(\sharp\{y_j \in \bigcup_{k\in\mathcal{K}}(t_{k-1}, t_k)\} = 0\Big) = \frac{1}{n! Z_n}\int_{\mathbb{R}^n} \prod_{1\le j<k\le n}(x_k - x_j)^2 \prod_{j=1}^{n} e^{-nV(x_j)} \prod_{k\in\mathcal{K}} \begin{cases} s_k, & \text{if } x_j \in (t_{k-1}, t_k) \\ 1, & \text{if } x_j \notin (t_{k-1}, t_k) \end{cases} dx_j. \tag{1.20}$$

The piecewise constant factor in (1.20) can be rewritten in the form

$$\prod_{k\in\mathcal{K}} \left\{ \begin{array}{ll} s_k, & \text{if } x \in (t_{k-1}, t_k) \\ 1, & \text{if } x \notin (t_{k-1}, t_k) \end{array} \right\} = \prod_{k\in\mathcal{K}} s_k^{1/2} \prod_{j=1}^{m} \omega_{\widetilde{\beta}_j}(x), \qquad 2i\pi\widetilde{\beta}_j = \log\left(\frac{\widetilde{s}_j}{\widetilde{s}_{j+1}}\right), \tag{1.21}$$

where $\widetilde{s}_j = s_j$ if $j \in \mathcal{K}$, and $\widetilde{s}_j = 1$ if $j \notin \mathcal{K}$. Thus, the gap probability in the thinned point process is given by

$$\mathbb{P}_V\Big(\sharp\{y_j \in \bigcup_{k\in\mathcal{K}}(t_{k-1}, t_k)\} = 0\Big) = \frac{D_n(\vec{0}, \vec{\beta}_\star, V, 0)}{D_n(\vec{0}, \vec{0}, V, 0)} \prod_{k\in\mathcal{K}} s_k^{1/2}, \tag{1.22}$$

and $\vec{\beta}_\star = (\widetilde{\beta}_1, ..., \widetilde{\beta}_m)$ is defined as in (1.21).

Following [15], instead of studying the thinned spectrum, we can consider a situation where we have information about the thinned spectrum (in the language of probability, we observe an event A), and from there, we try to obtain information about the initial/complete spectrum. The initial point process $x_1, ..., x_n$ is called *conditional* to A. Let us assume that A is the event of $\sharp\{y_j \in \bigcup_{k\in\mathcal{K}}(t_{k-1}, t_k)\} = 0$, i.e. we observe a gap in the thinned spectrum. Let us also assume that n is known, as well as the thinning parameters $t_1, ..., t_m, \widetilde{s}_1, ..., \widetilde{s}_m$. From Bayes' formula for conditional probabilities, the conditional point process with respect to A follows the distribution

$$\frac{1}{n!\widetilde{Z}_n} \prod_{1\le i<j\le n} (x_j - x_i)^2 \prod_{j=1}^{n} \widetilde{w}(x_j) dx_j, \qquad \widetilde{Z}_n = D_n(\vec{0}, \vec{\beta}_\star, V, 0), \tag{1.23}$$

where $\widetilde{w}(x) = e^{-nV(x)} \prod_{j=1}^{m} \omega_{\widetilde{\beta}_j}(x)$. Thus, the generalized correlations of the characteristic polynomial of the conditional point process with respect to A is expressed as a ratio of Hankel determinants with general FH singularities

$$\mathbb{E}_{V,\mathrm{cond}}\left(\prod_{j=1}^{n} e^{W(x_j)} \prod_{k=1}^{m} |p(t_k)|^{\alpha_k} e^{2i\beta_k \arg p_n(t_k)}\right) = \frac{D_n(\vec{\alpha}, \vec{\beta} + \vec{\beta}_\star, V, W)}{D_n(\vec{0}, \vec{\beta}_\star, V, 0)} \prod_{k=1}^{m} e^{-i\pi\beta_k}. \tag{1.24}$$

The contribution of this paper is to obtain large n asymptotics for $D_n(\vec{\alpha}, \vec{\beta}, V, W)$ for one-cut regular potentials, up to the constant term. The case of general FH singularities (even for $m = 1$) and of pure jump-type singularities (for $m \ge 2$) have not been obtained rigorously, even for the Gaussian potential $V(x) = 2x^2$. In particular, asymptotics for the partition functions $\widetilde{Z}_n$ and for the correlations (1.16) and (1.24) are not known.

Theorem 1.1 *Let $m \in \mathbb{N}$, and let t_j, α_j and β_j be such that $t_j \in (-1,1)$, $t_j \neq t_k$ for $1 \leq j \neq k \leq m$, $\Re\alpha_j > -1$ and $\Re\beta_j \in (\frac{-1}{4}, \frac{1}{4})$, for $j = 1,...,m$. Let V be a one-cut regular potential whose equilibrium measure is supported on $[-1,1]$ with density $\psi(x)\sqrt{1-x^2}$, and let $W : \mathbb{R} \to \mathbb{R}$ be analytic in a neighbourhood of $[-1,1]$, locally Hölder-continuous on $\mathbb{R}$ and such that $W(x) = \mathcal{O}(V(x))$, as $|x| \to \infty$. As $n \to \infty$, we have*

$$D_n(\vec{\alpha}, \vec{\beta}, V, W) = \exp\left(C_1 n^2 + C_2 n + C_3 \log n + C_4 + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big)\right), \tag{1.25}$$

with $\beta_{\max} = \max\{|\Re\beta_1|, ..., |\Re\beta_m|\}$ and

$$C_1 = -\log 2 - \frac{3}{4} - \frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}(V(x) - 2x^2)\left(\frac{2}{\pi} + \psi(x)\right)dx, \tag{1.26}$$

$$\begin{aligned} C_2 = {}& \log(2\pi) - \mathcal{A}\log 2 - \frac{\mathcal{A}}{2\pi}\int_{-1}^{1} \frac{V(x) - 2x^2}{\sqrt{1-x^2}}dx + \int_{-1}^{1} \psi(x)\sqrt{1-x^2}W(x)dx \\ & + \sum_{j=1}^{m}\left(\frac{\alpha_j}{2}(V(t_j) - 1) + \pi i \beta_j \left(1 - 2\int_{t_j}^{1} \psi(x)\sqrt{1-x^2}dx\right)\right), \end{aligned} \tag{1.27}$$

$$C_3 = -\frac{1}{12} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right), \tag{1.28}$$

$$\begin{aligned} C_4 = {}& \zeta'(-1) + \frac{\mathcal{A}}{2\pi}\int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}}dx - \frac{1}{4\pi^2}\int_{-1}^{1} \frac{W(y)}{\sqrt{1-y^2}}\left(⨍_{-1}^{1} \frac{W'(x)\sqrt{1-x^2}}{x-y}dx\right)dy \\ & - \frac{1}{24}\log\left(\frac{\pi^2}{4}\psi(1)\psi(-1)\right) + \sum_{1 \leq j < k \leq m} \log\left(\frac{\left(1 - t_j t_k - \sqrt{(1-t_j^2)(1-t_k^2)}\right)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j - t_k|^{\frac{\alpha_j\alpha_k}{2} + 2\beta_j\beta_k}}\right) \\ & + \sum_{j=1}^{m}\left(i\mathcal{A}\beta_j \arcsin t_j - \frac{i\pi}{2}\beta_j\mathcal{A}_j + \log\frac{G(1 + \frac{\alpha_j}{2} + \beta_j)G(1 + \frac{\alpha_j}{2} - \beta_j)}{G(1+\alpha_j)}\right) \\ & + \sum_{j=1}^{m}\left(\left(\frac{\alpha_j^2}{4} - \beta_j^2\right)\log\left(\frac{\pi}{2}\psi(t_j)\right) - \frac{\alpha_j}{2}W(t_j) + i\frac{\beta_j}{\pi}\sqrt{1-t_j^2}⨍_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}(t_j - x)}dx\right) \\ & + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - 3\beta_j^2\right)\log\left(2\sqrt{1-t_j^2}\right), \end{aligned} \tag{1.29}$$

where G is Barnes' G-function, ζ is Riemann's zeta-function and where we use the notations

$$\mathcal{A} = \sum_{j=1}^{m} \alpha_j, \qquad \mathcal{A}_j = \sum_{l=1}^{j-1} \alpha_l - \sum_{l=j+1}^{m} \alpha_l. \tag{1.30}$$

Furthermore, the error term in (1.25) is uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \Re z > -1\}$, for all β_k in compact subsets of $\{z \in \mathbb{C} : \Re z \in \left(\frac{-1}{4}, \frac{1}{4}\right)\}$, and uniform in $t_1, ..., t_m$, as long as there exists $\delta > 0$ independent of n such that

$$\min_{j \neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{1.31}$$

Remark 1.2 *The notation $⨍$ stands for the Cauchy principal value integral.*

Remark 1.3 *If some parameters among $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m, t_1, ..., t_m$ tend to the boundaries of their domains at a sufficiently slow speed as $n \to \infty$, the asymptotic formula (1.25) is still expected to hold, but with a worse error term. If the speed is faster, critical transitions should take place and (1.25) is no longer valid. We don't analyse these cases here. Such transitions have been studied in [14, 15] for Toeplitz determinants with a single jump-type singularity with $\beta_1 \to \pm i\infty$, in [17] for Hankel determinants with two merging root-type singularities, in [19] for*

Toeplitz determinants with two merging FH singularities and in [52] for Hankel determinants with a FH singularity approaching the edge (i.e. approaching 1 *or* -1*).*

The assumption $\Re\beta_j \in (\frac{-1}{4}, \frac{1}{4})$*,* $j = 1, ..., m$ *makes it easier to handle some technicalities in our analysis. The methods presented here allow with extra efforts to treat the more general situation* $\Re\beta_j \in (\frac{-1}{2}, \frac{1}{2})$ *and to compute more terms in the asymptotics. A similar technical problem was first observed in [37] for large* n *asymptotics of* $D_n(\vec{0}, \vec{\beta}, 2x^2, 0)$ *with* $m = 1$ *(i.e.* $\vec{\beta} = \beta_1$*). For the reader interested to extend Theorem 1.1 to the most general situation* $\Re\beta_j \in (\frac{-1}{2}, \frac{1}{2}]$*, we recommend to read [24, Section 5.2] and [23, Remark 1.22], where analogous situations have been studied for Toeplitz determinants with FH singularities and for Hankel determinants associated to a weight of the form (1.5).*

Remark 1.4 *The assumption that the support of the equilibrium measure is* $[-1, 1]$ *is without loss of generality. Suppose that* $\widetilde{V}$ *is a one-cut regular potential whose equilibrium measure is supported on* $[a, b]$*. Then, the change of variables* $y_j = \left(x_j - \frac{a+b}{2}\right) / \left(\frac{b-a}{2}\right)$*,* $j = 1, ..., n$ *in (1.4) shows that*

$$D_n(\vec{\alpha}, \vec{\beta}, \widetilde{V}, \widetilde{W}; \vec{t}_\star) = \left(\frac{b-a}{2}\right)^{n^2 + n\mathcal{A}} D_n(\vec{\alpha}, \vec{\beta}, V, W; \vec{t}), \tag{1.32}$$

where we have explicitly written the dependence in $\vec{t}_\star = (\widetilde{t}_1, ..., \widetilde{t}_m)$*, where* $\vec{t} = (t_1, ..., t_m)$ *is given by* $t_j = \left(\widetilde{t}_j - \frac{a+b}{2}\right) / \left(\frac{b-a}{2}\right)$*,* $j = 1, ..., m$*, and where*

$$V(x) = \widetilde{V}\left(\frac{a+b}{2} + x\frac{b-a}{2}\right), \quad W(x) = \widetilde{W}\left(\frac{a+b}{2} + x\frac{b-a}{2}\right).$$

Also, if $\widetilde{\ell}$ *and* $\widetilde{\psi}(x)\sqrt{(b-x)(x-a)}$ *are respectively the Euler-Lagrange constant and the density of the equilibrium measure of* $\widetilde{V}$*, the change of variables* $u = \left(s - \frac{a+b}{2}\right) / \left(\frac{b-a}{2}\right)$ *in (1.11) and (1.12) implies that the equilibrium measure of* V *is supported on* $[-1, 1]$ *and can be written as* $\psi(x)\sqrt{1 - x^2}$*. Furthermore,* ψ *and the Euler-Lagrange constant* ℓ *of* V *are given by*

$$\ell = \widetilde{\ell} + 2\log\left(\frac{b-a}{2}\right), \qquad \psi(x) = \left(\frac{b-a}{2}\right)^2 \widetilde{\psi}\left(\frac{a+b}{2} + x\frac{b-a}{2}\right). \tag{1.33}$$

1.1 Outline

We will compute the asymptotics for $D_n(\vec{\alpha}, \vec{\beta}, V, W)$ in three steps which can be schematized as

$$D_n(\vec{\alpha}, \vec{0}, 2x^2, 0) \mapsto D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0) \mapsto D_n(\vec{\alpha}, \vec{\beta}, V, 0) \mapsto D_n(\vec{\alpha}, \vec{\beta}, V, W). \tag{1.34}$$

Each of these steps is subdivided into three parts: 1) a differential identity for $\log D_n(\vec{\alpha}, \vec{\beta}, V, W)$, 2) an asymptotic analysis of a Riemann-Hilbert (RH) problem, and 3) the integration of the differential identity.

As mentioned in the introduction, asymptotics for $D_n(\vec{\alpha}, \vec{0}, 2x^2, 0)$ are already known (see (1.9) and (1.13)). In the first step, for each β_k, $k \in \{1, ..., m\}$, we will obtain an identity which expresses $\partial_{\beta_k} \log D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0)$ in terms of orthogonal polynomials (OPs). These differential identities will be similar to the ones presented in [37, 42], but some extra care due to the presence of general FH singularities is needed. We integrate these differential identities in Section 5.

In the second step, we follow [5] and introduce a deformation parameter $s \in [0, 1]$ for the potential,

$$V_s(x) = (1 - s)2x^2 + sV(x). \tag{1.35}$$

This potential interpolates between $2x^2$ (for $s = 0$) and $V(x)$ (for $s = 1$). From (1.11) and (1.12), it is easily checked that V_s is a one-cut regular potential for every $s \in [0, 1]$, and the associated density $\psi_s(x)$ and Euler-Lagrange constant ℓ_s are simply given by

$$\psi_s(x) = (1 - s)\frac{2}{\pi} + s\psi(x), \qquad \ell_s = (1 - s)(1 + 2\log 2) + s\ell. \tag{1.36}$$

We present an identity for $\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0)$ in Section 3, and its integration in Section 6.

In the last step, the potential V is fixed and W is analytic in a neighbourhood of $[-1,1]$ and locally Hölder continuous on $\mathbb{R}$. For $t \in [0,1]$, we define

$$W_t(z) = \log\left(1 - t + te^{W(z)}\right), \tag{1.37}$$

where the principal branch of the log is taken. For every $t \in [0,1]$, W_t is analytic on a neighbourhood of $[-1,1]$ independent of t and is still Hölder continuous on $\mathbb{R}$. This deformation is the same as the one used in [24, 5]. Then, we proceed similarly as in the two previous steps, with an identity for $\partial_t \log D_n(\vec{\alpha}, \vec{\beta}, V, W_t)$ in Section 3, and its integration in Section 7.

The differential identities for $\log D_n(\vec{\alpha}, \vec{\beta}, V, W)$ are expressed in terms of OPs, which are orthogonal with respect to a weight which depends on the step:

$$\begin{aligned} &w(x) = e^{-2nx^2}\omega(x), && \text{in step 1,} \\ &w_s(x) = e^{-nV_s(x)}\omega(x), && \text{in step 2,} \\ &w_t(x) = e^{-nV(x)}e^{W_t(x)}\omega(x), && \text{in step 3.} \end{aligned}$$

All these weights belong to the class of weights presented in (1.2) (which we also denote by w). In Section 2, we introduce the family of OPs with respect to w and the associated RH problem found by Fokas, Its and Kitaev [33]. We perform an asymptotic analysis of this RH problem using the Deift/Zhou [28, 29, 26, 27] steepest descent method in Section 4.

2 Orthogonal polynomials and a Riemann-Hilbert problem

The orthonormal polynomials $p_k(x) = p_k^{(n)}(x; \vec{\alpha}, \vec{\beta}, V, W)$ of degree k associated to the weight w given in (1.2) are defined through the following orthogonality conditions:

$$\int_{\mathbb{R}} p_k(x)p_j(x)w(x)dx = \delta_{jk}, \qquad j = 0, 1, ..., k. \tag{2.1}$$

We consider $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W) = \det(w_{j+l-2})_{j,l=1,...,k}$, where $w_j = w_j^{(n)}(\vec{\alpha}, \vec{\beta}, V, W) = \int_{-\infty}^{\infty} x^j w(x)dx$ is the j-th moment of w. When there is no confusion, we will simply write $D_k^{(n)}$ instead of $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)$, and we set $D_0^{(n)} = 1$. With this notation, we have $D_n^{(n)} = D_n$, where D_n is defined in (1.1). The existence of the system of orthogonal polynomials depends if one (or several) of these determinants vanishes. It is well-known (see e.g. [49, 22]) that if

$$D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W) \neq 0 \quad \text{ and } \quad D_{k+1}^{(n)}(\vec{\alpha}, \vec{\beta}, V, W) \neq 0, \tag{2.2}$$

then the orthogonal polynomial p_k exists and is given by

$$p_k(x) = \frac{\begin{vmatrix} w_0 & \cdots & w_{k-1} & w_k \\ \vdots & \ddots & \vdots & \vdots \\ w_{k-1} & \cdots & w_{2k-2} & w_{2k-1} \\ 1 & \cdots & x^{k-1} & x^k \end{vmatrix}}{\sqrt{D_{k+1}^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)}\sqrt{D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)}} = \kappa_k x^k + ..., \tag{2.3}$$

where $\kappa_k = \kappa_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)$ is the leading coefficient of p_k, and is given by

$$\kappa_k = \frac{\sqrt{D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)}}{\sqrt{D_{k+1}^{(n)}(\vec{\alpha}, \vec{\beta}, V, W)}} \neq 0, \qquad \text{and where} \qquad \arg\sqrt{D_k^{(n)}}, \arg\sqrt{D_{k+1}^{(n)}} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{2.4}$$

Note that (2.1) alone does not uniquely define the orthonormal polynomials p_k and the leading coefficients κ_k (they are unique up to multiplicative factors of -1). We fixed this with the (arbitrary) choice of the arguments in (2.4). However, in the present paper we will only use κ_k^2, $\kappa_k p_k$, as well as the monic orthogonal polynomials $\kappa_k^{-1} p_k$, which are always uniquely defined by (2.1) (if they exist), i.e. they are independent of the choice of the arguments in (2.4). We will also use the subleading coefficients of the monic orthogonal polynomials:

$$\kappa_k^{-1} p_k(x) = x^k + \eta_k x^{k-1} + \gamma_k x^{k-2} + ..., \qquad k \in \mathbb{N},\ k \geq 2. \tag{2.5}$$

If $D_k^{(n)} \neq 0$ for every $k = 0, 1, ..., n$, (2.4) implies the following well-known formula

$$D_n(\vec{\alpha}, \vec{\beta}, V, W) = \prod_{j=0}^{n-1} \kappa_k^{-2}. \tag{2.6}$$

Note that if $\alpha_k \in (-1, \infty)$ and $\beta_k \in i\mathbb{R}$ for all $k = 1, ..., m$, then w is positive and it follows from Heine's integral representation for Hankel determinants that $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W) > 0$ for all k. As a consequence, the sequence of orthogonal polynomials $(p_k)_{k\in\mathbb{N}}$ exists. Nevertheless, for general values of the parameters $\vec{\alpha}$ and $\vec{\beta}$, the weight w is complex and there is no guarantee of existence for the orthogonal polynomials (one of several determinants $D_k^{(n)}$ may vanish). This will cause some extra difficulties in the analysis. We discuss this in more detail at the beginning of Section 3.

For sufficiently large n, we will prove existence and obtain explicit asymptotics of p_n via the Deift-Zhou steepest descent method applied on a Riemann-Hilbert problem. Consider the matrix valued function $Y(z) = Y^{(n)}(z; \vec{\alpha}, \vec{\beta}, V, W)$, defined by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{\kappa_n^{-1}}{2\pi i} \int_{\mathbb{R}} \frac{p_n(x) w(x)}{x - z} dx \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(x) w(x)}{x - z} dx \end{pmatrix}. \tag{2.7}$$

Note that Y is expressed only in terms of $\kappa_n^{-1} p_n$, $\kappa_{n-1} p_{n-1}$ and their Cauchy transforms. It is known [33] that Y can be characterized as the solution to a boundary value problem for analytic functions, called the RH problem for Y. Furthermore, Y defined by (2.7) (as well as the solution to the RH problem for Y), exists if and only if $D_{n-1}^{(n)} \neq 0$, $D_n^{(n)} \neq 0$ and $D_{n+1}^{(n)} \neq 0$, and is always unique.

RH problem for Y

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The limits of $Y(z)$ as z approaches $\mathbb{R} \setminus \{t_1, ..., t_m\}$ from above and below exist, are continuous on $\mathbb{R} \setminus \{t_1, ..., t_m\}$ and are denoted by Y_+ and Y_- respectively. Furthermore they are related by

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R} \setminus \{t_1, ..., t_m\}. \tag{2.8}$$

(c) As $z \to \infty$, we have $Y(z) = \left(I + \mathcal{O}(z^{-1})\right) z^{n\sigma_3}$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) As z tends to t_k, $k \in \{1, ..., m\}$, the behaviour of Y is

$$\begin{aligned} Y(z) &= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - t_k)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - t_k)) \end{pmatrix}, && \text{if } \Re\alpha_k = 0, \\ Y(z) &= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_k)^{\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_k)^{\alpha_k}) \end{pmatrix}, && \text{if } \Re\alpha_k \neq 0. \end{aligned} \tag{2.9}$$

The above condition (b) follows from the Sokhotski formula and relies on the assumption that W is locally Hölder continuous on $\mathbb{R}$ (see e.g. [36]).

3 Differential identities

In this section, we derive several identities which will be useful to prove Theorem 1.1. These identities will be valid only when all orthogonal polynomials p_k for $k = 0, ..., n$ exist. We give here an overview of how we will handle these technicalities in this paper, following [42, 37, 23]. By applying Lebesgue's dominated convergence theorem on the moments w_j, we can prove that

(a) the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, 2x^2, 0)$ are analytic functions of $(\vec{\alpha}, \vec{\beta}) \in \mathcal{P}_\alpha \times \mathbb{C}^m$, where $\mathcal{P}_\alpha = \{\vec{\alpha} \in \mathbb{C}^m : \Re\alpha_j > -1, \text{ for all } j = 1, ..., m\}$,

(b) the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, 0)$ are analytic functions of $(\vec{\alpha}, \vec{\beta}, s) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times (0, 1)$,

(c) the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W_t)$ are analytic functions of $(\vec{\alpha}, \vec{\beta}, t) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times [0, 1]$.

The property (b) deserves further explanation. If $V(x)/x^2 \to +\infty$ as $|x| \to +\infty$, the moments are not well-defined if $\Re s < 0$, since $V_s(x) \to -\infty$ as $|x| \to +\infty$. In this case, the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, 0)$ are not analytic as functions of s in a neighbourhood of $s = 0$. Similarly, if $V(x) = o(x^2)$ as $|x| \to +\infty$, the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, 0)$ are not analytic as functions of s in a neighbourhood of $s = 1$. Note also from (a), (b), (c) above and from (2.7), (2.3) and (2.4), that $Y^{(n)}(z; \vec{\alpha}, \vec{\beta}, 2x^2, 0)$, $Y^{(n)}(z; \vec{\alpha}, \vec{\beta}, V_s, 0)$ and $Y^{(n)}(z; \vec{\alpha}, \vec{\beta}, V, W_t)$ are meromorphic functions of $(\vec{\alpha}, \vec{\beta}) \in \mathcal{P}_\alpha \times \mathbb{C}^m$, $(\vec{\alpha}, \vec{\beta}, s) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times (0, 1)$ and $(\vec{\alpha}, \vec{\beta}, t) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times [0, 1]$ respectively.

We will find identities of the forms (see equations (3.26), (3.29) and (3.30) below)

$$\partial_{\beta_k} \log D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0) = F_{1,n}(\vec{\alpha}, \vec{\beta}), \tag{3.1}$$

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = F_{2,n}(\vec{\alpha}, \vec{\beta}, s), \tag{3.2}$$

$$\partial_t \log D_n(\vec{\alpha}, \vec{\beta}, V, W_t) = F_{3,n}(\vec{\alpha}, \vec{\beta}, t), \tag{3.3}$$

where the functions $F_{1,n}$, $F_{2,n}$ and $F_{3,n}$ (we omit the dependence of $F_{1,n}$ in k) are expressed only in terms of Y. The identities (3.1), (3.2) and (3.3) are valid only for $(\vec{\alpha}, \vec{\beta}) \in \widetilde{\mathcal{P}}_1^{(n)}$, for $(\vec{\alpha}, \vec{\beta}, s) \in \widetilde{\mathcal{P}}_2^{(n)}$ and for $(\vec{\alpha}, \vec{\beta}, t) \in \widetilde{\mathcal{P}}_3^{(n)}$ respectively, where

$$\widetilde{\mathcal{P}}_1^{(n)} = \left\{(\vec{\alpha}, \vec{\beta}) \in \mathcal{P}_\alpha \times \mathbb{C}^m : D_j^{(n)}(\vec{\alpha}, \vec{\beta}, 2x^2, 0) \neq 0 \text{ for all } j = 1, ..., n+1\right\},$$
$$\widetilde{\mathcal{P}}_2^{(n)} = \left\{(\vec{\alpha}, \vec{\beta}, s) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times (0,1) : D_j^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, 0) \neq 0 \text{ for all } j = 1, ..., n+1\right\},$$
$$\widetilde{\mathcal{P}}_3^{(n)} = \left\{(\vec{\alpha}, \vec{\beta}, t) \in \mathcal{P}_\alpha \times \mathbb{C}^m \times [0,1] : D_j^{(n)}(\vec{\alpha}, \vec{\beta}, V, W_t) \neq 0 \text{ for all } j = 1, ..., n+1\right\}.$$

In Section 4, we will perform an asymptotic analysis on the RH problem for Y as $n \to +\infty$. Let Ω be a compact subset of $\mathcal{P}_\alpha \times \mathcal{P}_\beta^{(\frac{1}{4})}$, where

$$\mathcal{P}_\beta^{(\frac{1}{4})} = \{\vec{\beta} \in \mathbb{C}^m : \Re\beta_j \in (-\tfrac{1}{4}, \tfrac{1}{4}), \text{ for all } j = 1, ..., m\}.$$

The analysis of Section 4 will imply the existence of $F_{1,n}(\vec{\alpha}, \vec{\beta})$, $F_{2,n}(\vec{\alpha}, \vec{\beta}, s)$ and $F_{3,n}(\vec{\alpha}, \vec{\beta}, t)$ for *all* $n \geq n_\star$, where $n_\star$ depends only on Ω, for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$ and for *all* $s, t \in [0, 1]$. Furthermore, their large n asymptotics will be explicitly computed up to the constant term in Section 5, 6 and 7, and will be valid uniformly for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$ and for *all* $s, t \in [0, 1]$.

Thus, we will prove that the r.h.s. of (3.1) exists for *all* $n \geq n_\star$ and for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$, but the identity (3.1) itself is valid only for

$$(\vec{\alpha}, \vec{\beta}) \in \Omega \cap \widetilde{\mathcal{P}}_1^{(n)} = \Omega \setminus \widetilde{\Omega}_1^{(n)}, \tag{3.4}$$

where $\widetilde{\Omega}_1^{(n)}$ consists of at most a finite number of points (see (a) above). The analyticity in β_k of the determinant $D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0)$ will allow us to extend this differential identity from $\Omega \setminus \widetilde{\Omega}_1^{(n)}$ to Ω. We will show that explicitly when integrating the differential identity in Section 5.

Likewise, the subset of $\Omega \times (0,1)$ for which the differential identity (3.2) is valid is

$$\big(\Omega \times (0,1)\big) \cap \widetilde{\mathcal{P}}_2^{(n)} = \big(\Omega \times (0,1)\big) \setminus \widetilde{\Omega}_2^{(n)}. \tag{3.5}$$

However, since the determinants $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, 0)$ are analytic as functions of $s \in (0,1)$, but not analytic at $s = 0$ and $s = 1$ (see (b) above), $\widetilde{\Omega}_2^{(n)}$ is locally finite on its interior but may have accumulation points on its boundary. More precisely, if $\epsilon \in (0, \frac{1}{2})$, then $\widetilde{\Omega}_{2,\epsilon}^{(n)} = \{(\vec{\alpha}, \vec{\beta}, s) \in \widetilde{\Omega}_2^{(n)} : s \in (\epsilon, 1-\epsilon)\}$ consists of at most a finite number of points. This extra difficulty does not appear in [42, 37, 23], but does not change drastically the idea. We provide the details in Section 6.

Since $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V, W_t)$ are analytic as functions of $t \in [0,1]$ (see (c) above), the subset of $\Omega \times [0,1]$ for which the identity (3.3) is valid is $(\Omega \times [0,1]) \cap \widetilde{\mathcal{P}}_3^{(n)} = (\Omega \times [0,1]) \setminus \widetilde{\Omega}_3^{(n)}$, where $\widetilde{\Omega}_3^{(n)}$ consists of at most a finite number of points. We can extend this identity for *all* $(\vec{\alpha}, \vec{\beta}, t) \in \Omega \times [0,1]$ in a similar way as for the identity (3.1).

Once the differential identities (3.1), (3.2) and (3.3) are extended from $\Omega \setminus \widetilde{\Omega}_1^{(n)}$ to Ω, from $\big(\Omega \times (0,1)\big) \setminus \widetilde{\Omega}_2^{(n)}$ to $\Omega \times (0,1)$ and from $(\Omega \times [0,1]) \setminus \widetilde{\Omega}_3^{(n)}$ to $\Omega \times [0,1]$ respectively, we prove Theorem 1.1 by integrating them.

3.1 Differential identity with respect to β_k, $k \in \{1, ..., m\}$

As mentioned in the outline, we will first compute the asymptotics of $D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0)$, i.e. when the weight has the form $w(x) = e^{-2nx^2}\omega(x)$. For this we will need a differential identity with respect to β_k, for each $k \in \{1, ..., m\}$. Suppose f is a smooth and integrable function on $\mathbb{R}$ with sufficient decay at $\pm\infty$. The integral of $\frac{f(x)\omega(x)}{x-t_k}$ on $\mathbb{R}$ is not well-defined if $\Re\alpha_k \leq 0$, even in the sense of principal value, because of the jump caused by β_k. Therefore, we define a regularized integral by

$$\mathrm{Reg}_k\,(f) = \lim_{\epsilon \to 0_+} \left[\alpha_k \int_{\mathbb{R}\setminus[t_k-\epsilon, t_k+\epsilon]} \frac{f(x)\omega(x)}{x-t_k} dx - f(t_k)\omega_k(t_k)(e^{i\pi\beta_k} - e^{-i\pi\beta_k})\epsilon^{\alpha_k}\right], \tag{3.6}$$

where

$$\omega_k(z) = \prod_{j \neq k} \omega_{\alpha_j}(z)\omega_{\beta_j}(z). \tag{3.7}$$

It is easy to observe that the limit in (3.6) converges. We have the following property.

Proposition 3.1 *(adapted from [42]). The regularised integral* (3.6) *satisfies*

$$\mathrm{Reg}_k(f) = \lim_{z \to t_k} \left(\alpha_k \int_{\mathbb{R}} \frac{f(x)\omega(x)}{x-z} dx - \mathcal{J}_k(z)\right), \tag{3.8}$$

where the limit is taken along a non-tangential to the real line path in $\{z \in \mathbb{C} : \Im z > 0\}$*, and*

$$\mathcal{J}_k(z) = \begin{cases} \dfrac{\pi\alpha_k}{\sin(\pi\alpha_k)} f(t_k)\omega_k(t_k)(e^{i\pi\beta_k} - e^{-i\pi\alpha_k}e^{-i\pi\beta_k})(z-t_k)^{\alpha_k}, & \text{if } \Re\alpha_k \leq 0, \alpha_k \neq 0, \\ f(t_k)\omega_k(t_k)(e^{i\pi\beta_k} - e^{-i\pi\beta_k}), & \text{if } \alpha_k = 0, \\ 0, & \text{if } \Re\alpha_k > 0. \end{cases} \tag{3.9}$$

Proof. In [42, equation (29) and below], the author proved that if $\alpha_k \neq 0$, we have as $\epsilon \to 0_+$

that

$$\int_{-\infty}^{t_k-\epsilon} \frac{f(x)\omega_k(x)|x-t_k|^{\alpha_k}}{x-t_k}dx - \int_{-\infty}^{t_k-\epsilon} \frac{f(x)\omega_k(x)|x-t_k|^{\alpha_k}}{x-z}dx = \frac{-\pi f(z)\omega_k(z)}{\sin(\pi\alpha_k)}(z-t_k)^{\alpha_k} + \frac{\epsilon^{\alpha_k}}{\alpha_k}f(t_k)\omega_k(t_k) + \frac{G_1(z)(z-t_k)+\mathcal{O}(\epsilon^{\Re\alpha_k+1})}{2i\sin(\pi\alpha_k)}, \tag{3.10}$$

$$\int_{t_k+\epsilon}^{\infty} \frac{f(x)\omega_k(x)|x-t_k|^{\alpha_k}}{x-t_k}dx - \int_{t_k+\epsilon}^{\infty} \frac{f(x)\omega_k(x)|x-t_k|^{\alpha_k}}{x-z}dx = \frac{\pi e^{-i\pi\alpha_k} f(z)\omega_k(z)}{\sin(\pi\alpha_k)}(z-t_k)^{\alpha_k} - \frac{\epsilon^{\alpha_k}}{\alpha_k}f(t_k)\omega_k(t_k) - \frac{G_2(z)(z-t_k)+\mathcal{O}(\epsilon^{\Re\alpha_k+1})}{2i\sin(\pi\alpha_k)}, \tag{3.11}$$

where G_1 and G_2 are analytic in a neighbourhood of t_k. Multiplying (3.10) by $\alpha_k e^{i\pi\beta_k}$ and (3.11) by $\alpha_k e^{-i\pi\beta_k}$, and summing them we obtain as $\epsilon \to 0_+$

$$\alpha_k \int_{\mathbb{R}\setminus[t_k-\epsilon,t_k+\epsilon]} \frac{f(x)\omega(x)}{x-t_k}dx - f(t_k)\omega_k(t_k)(e^{i\pi\beta_k}-e^{-i\pi\beta_k})\epsilon^{\alpha_k} = \alpha_k \int_{\mathbb{R}} \frac{f(x)\omega(x)}{x-z}dx$$
$$-\frac{\pi\alpha_k f(z)\omega_k(z)}{\sin(\pi\alpha_k)}(z-t_k)^{\alpha_k}(e^{i\pi\beta_k}-e^{-i\pi\alpha_k}e^{-i\pi\beta_k}) + \frac{(G_1(z)e^{i\pi\beta_k}-G_2(z)e^{-i\pi\beta_k})(z-t_k)+\mathcal{O}(\epsilon^{\Re\alpha_k+1})}{2i\sin\pi\alpha_k}\alpha_k.$$

Taking first the limit $\epsilon \to 0_+$ and then $z \to t_k$, we find the claim for $\alpha_k \neq 0$. The claim for $\alpha_k = 0$ is straightforward. □

The following developments are similar to those done in [37, 42], but the presence of both root-type and jump-type singularities requires some changes, and we provide the details below. In this subsection we assume that $(\vec{\alpha},\vec{\beta}) \in \widetilde{\mathcal{P}}_1^{(n)}$, such that the formula (2.6) holds. We start with a general differential identity obtained in [42], which we apply to the parameter β_k, for a $k \in \{1,...,m\}$:

$$\partial_{\beta_k} \log D_n = -n\partial_{\beta_k}\log\kappa_{n-1} + \frac{\kappa_{n-1}}{\kappa_n}(I_1 - I_2), \tag{3.12}$$

where

$$I_1 = \int_{-\infty}^{\infty} p_{n-1}'(x)\partial_{\beta_k}p_n(x)w(x)dx, \qquad I_2 = \int_{-\infty}^{\infty} p_n'(x)\partial_{\beta_k}p_{n-1}(x)w(x)dx. \tag{3.13}$$

We will simplify I_1 and I_2 for our particular weight $w(x) = e^{-2nx^2}\omega(x)$. Let $\epsilon > 0$ be sufficiently small such that $\min_{i\neq j}|t_i - t_j| > 3\epsilon$, I_1 can be split as

$$I_1 = \sum_{l=1}^{m+1} \int_{t_{l-1}+\epsilon}^{t_l-\epsilon} p_{n-1}'(x)\partial_{\beta_k}p_n(x)w(x)dx + \sum_{l=1}^{m} \int_{t_l-\epsilon}^{t_l+\epsilon} p_{n-1}'(x)\partial_{\beta_k}p_n(x)w(x)dx, \tag{3.14}$$

where $t_0 = -\infty$ and $t_{m+1} = \infty$. Noting that

$$w'(x) = \Big(-4nx + \sum_{j=1}^m \frac{\alpha_j}{x-t_j}\Big)w(x), \qquad x \in \mathbb{R}\setminus\{t_1,...,t_m\}, \tag{3.15}$$

we have, by integration by parts, for $l \in \{1,...,m+1\}$

$$\int_{t_{l-1}+\epsilon}^{t_l-\epsilon} p_{n-1}'(x)\partial_{\beta_k}p_n(x)w(x)dx = p_{n-1}(t_l-\epsilon)\partial_{\beta_k}p_n(t_l-\epsilon)w(t_l-\epsilon) - p_{n-1}(t_{l-1}+\epsilon)\partial_{\beta_k}p_n(t_{l-1}+\epsilon)w(t_{l-1}+\epsilon)$$
$$-\int_{t_{l-1}+\epsilon}^{t_l-\epsilon} p_{n-1}(x)\Big[\partial_{\beta_k}p_n'(x) + \partial_{\beta_k}p_n(x)\Big(-4nx + \sum_{j=1}^m \frac{\alpha_j}{x-t_j}\Big)\Big]w(x)dx. \tag{3.16}$$

In the above expression, the quantities $p_{n-1}(t_{m+1}-\epsilon)\partial_{\beta_k}p_n(t_{m+1}-\epsilon)w(t_{m+1}-\epsilon)$ and $p_{n-1}(t_0+\epsilon)\partial_{\beta_k}p_n(t_0+\epsilon)w(t_0+\epsilon)$ have to be understood as equal to 0. On the other hand, for $l \in \{1,...,m\}$,

$$\int_{t_l-\epsilon}^{t_l+\epsilon} p_{n-1}'(x)\partial_{\beta_k}p_n(x)w(x)dx = \mathcal{O}\big(\epsilon^{1+\Re\alpha_l}\big), \qquad \text{as } \epsilon \to 0. \tag{3.17}$$

Therefore, summing (3.16) for $l = 1,...,m+1$, and taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned} I_1 = &-\int_{-\infty}^{\infty} p_{n-1}(x)\partial_{\beta_k}p_n'(x)w(x)dx + 4n\int_{-\infty}^{\infty} p_{n-1}(x)\partial_{\beta_k}p_n(x)xw(x)dx \\ &-\sum_{j=1}^m \alpha_j \int_{-\infty}^{\infty} p_{n-1}(x)\frac{\partial_{\beta_k}p_n(x)-\partial_{\beta_k}p_n(t_j)}{x-t_j}w(x)dx - \sum_{j=1}^m \partial_{\beta_k}p_n(t_j)\mathrm{Reg}_j\big(p_{n-1}(x)e^{-2nx^2}\big). \end{aligned} \tag{3.18}$$

By orthogonality,

$$\int_{-\infty}^{\infty} p_{n-1}(x)\partial_{\beta_k}p_n'(x)w(x)dx = n\frac{\partial_{\beta_k}\kappa_n}{\kappa_{n-1}}, \qquad \int_{-\infty}^{\infty} p_{n-1}(x)\frac{\partial_{\beta_k}p_n(x)-\partial_{\beta_k}p_n(t_j)}{x-t_j}w(x)dx = \frac{\partial_{\beta_k}\kappa_n}{\kappa_{n-1}}.$$

The next integral is a bit more involved and was done in [42, equation (21)],

$$\int_{-\infty}^{\infty} p_{n-1}(x)\partial_{\beta_k}p_n(x)xw(x)dx = \frac{\kappa_n}{\kappa_{n-1}}\left[\frac{\partial_{\beta_k}\kappa_n}{\kappa_n}\left(\frac{\kappa_{n-1}}{\kappa_n}\right)^2 + \partial_{\beta_k}\gamma_n - \eta_n\partial_{\beta_k}\eta_n\right]. \tag{3.19}$$

Therefore, one can rewrite I_1 as

$$\begin{aligned} I_1 = -(n+\mathcal{A})\frac{\partial_{\beta_k}\kappa_n}{\kappa_{n-1}} + 4n\frac{\kappa_n}{\kappa_{n-1}}&\left[\frac{\partial_{\beta_k}\kappa_n}{\kappa_n}\left(\frac{\kappa_{n-1}}{\kappa_n}\right)^2 + \partial_{\beta_k}\gamma_n - \eta_n\partial_{\beta_k}\eta_n\right] \\ &- \sum_{j=1}^m \partial_{\beta_k}p_n(t_j)\mathrm{Reg}_j\big(p_{n-1}(x)e^{-2nx^2}\big), \end{aligned} \tag{3.20}$$

where $\mathcal{A} = \sum_{j=1}^m \alpha_j$. A similar and simpler calculation for I_2 leads to

$$I_2 = 4n\frac{\partial_{\beta_k}\kappa_{n-1}}{\kappa_n} - \sum_{j=1}^m \partial_{\beta_k}p_{n-1}(t_j)\mathrm{Reg}_j\big(p_n(x)e^{-2nx^2}\big). \tag{3.21}$$

For $l \in \{1,...,m\}$, we define $\widetilde{Y}(t_l)$ as

$$\widetilde{Y}(t_l) = \begin{pmatrix} Y_{11}(t_l) & \mathrm{Reg}_l\left(\frac{1}{2\pi i}Y_{11}(x)e^{-2nx^2}\right) \\ Y_{21}(t_l) & \mathrm{Reg}_l\left(\frac{1}{2\pi i}Y_{21}(x)e^{-2nx^2}\right) \end{pmatrix}. \tag{3.22}$$

Note that from Proposition 3.1, for $l \in \{1,...,m\}$,

$$\widetilde{Y}_{j2}(t_l) = \lim_{z\to t_l} \alpha_l Y_{j2}(z) - c_l Y_{j1}(t_l)(z-t_l)^{\alpha_l}, \qquad j = 1,2, \tag{3.23}$$

where the limit is taken along a non-tangential to the real line path and

$$c_l = \frac{\pi\alpha_l}{\sin(\pi\alpha_l)}\frac{e^{-2nt_l^2}}{2\pi i}\omega_l(t_l)(e^{i\pi\beta_l} - e^{-i\pi\alpha_l}e^{-i\pi\beta_l}). \tag{3.24}$$

Thus the matrix $\widetilde{Y}(t_l)$ has not determinant 1, but

$$\alpha_l = \alpha_l \det Y(t_l) = \det \widetilde{Y}(t_l). \tag{3.25}$$

Putting together (3.12), (3.20) and (3.21), we obtain

$$\begin{aligned} \partial_{\beta_k}\log D_n(\vec{\alpha},\vec{\beta},2x^2,0) = &-(n+\mathcal{A})\partial_{\beta_k}\log(\kappa_n\kappa_{n-1}) - 2n\partial_{\beta_k}\left(\frac{\kappa_{n-1}^2}{\kappa_n^2}\right) + 4n\partial_{\beta_k}\left(\gamma_n - \frac{\eta_n^2}{2}\right) \\ &+\sum_{j=1}^m\left[\widetilde{Y}_{22}(t_j)\partial_{\beta_k}Y_{11}(t_j) - \widetilde{Y}_{12}(t_j)\partial_{\beta_k}Y_{21}(t_j) + Y_{11}(t_j)\widetilde{Y}_{22}(t_j)\partial_{\beta_k}\log(\kappa_n\kappa_{n-1})\right]. \end{aligned} \tag{3.26}$$

3.2 A general differential identity

Let γ be a parameter of the weight w whose dependence is smooth. From the well-known [49, 22] relation $D_n(\vec{\alpha},\vec{\beta},V,W)=\prod_{j=0}^{n-1}\kappa_j^{-2}$ (which we assume is valid), taking the log and then differentiating with respect to γ, we have

$$\partial_\gamma \log D_n(\vec{\alpha},\vec{\beta},V,W) = -\int_{\mathbb{R}} \partial_\gamma\left(\sum_{j=0}^{n-1} p_j^2(x)\right) w(x)dx = \int_{\mathbb{R}} \left(\sum_{j=0}^{n-1} p_j^2(x)\right)\partial_\gamma w(x)dx, \tag{3.27}$$

where the last equality comes from the orthogonality relations, since $\int_{\mathbb{R}}\sum_{j=0}^{n-1}p_j^2(x)w(x)dx=n$. By the Christoffel-Darboux formula (see e.g. [49, 22]) and (2.7), the equation (3.27) can be rewritten as

$$\partial_\gamma \log D_n(\vec{\alpha},\vec{\beta},V,W) = \frac{1}{2\pi i}\int_{\mathbb{R}} [Y^{-1}(x)Y'(x)]_{21}\partial_\gamma w(x)dx. \tag{3.28}$$

We will use this differential identity for the steps 2 and 3, which correspond respectively to w_s and w_t, as described in the outline. In these cases, (3.28) becomes

$$\partial_s \log D_n(\vec{\alpha},\vec{\beta},V_s,0) = \frac{1}{2\pi i}\int_{\mathbb{R}} [Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x)dx, \tag{3.29}$$

$$\partial_t \log D_n(\vec{\alpha},\vec{\beta},V,W_t) = \frac{1}{2\pi i}\int_{\mathbb{R}} [Y^{-1}(x)Y'(x)]_{21}\partial_t w_t(x)dx, \tag{3.30}$$

where (3.29) is valid for $(\vec{\alpha},\vec{\beta},s)\in\widetilde{\mathcal{P}}_2^{(n)}$, and where (3.30) is valid for $(\vec{\alpha},\vec{\beta},t)\in\widetilde{\mathcal{P}}_3^{(n)}$ (see the discussion at the beginning of Section 3).

4 Steepest descent analysis

In this section, we will perform an asymptotic analysis on the RH problem for Y as $n\to\infty$. Our analysis is based on the Deift/Zhou steepest descent method [28, 29].

4.1 Equilibrium measure and g-function

We denote by U_V the maximal open neighbourhood of $\mathbb{R}$ in which V is analytic, and by U_W an open neighbourhood of $\mathcal{S}=[-1,1]$ in which W is analytic, sufficiently small such that $U_W\subset U_V$. From [25, Theorem 1.38 (i), equations (1.39), (1.43), (1.44) and (1.45)], we have

$$\psi(x)\sqrt{1-x^2} = \Re\left(i\frac{V'(x)}{2\pi}+\frac{h(x)}{2\pi}\sqrt{1-x^2}\right), \tag{4.1}$$

where

$$h(z) = -\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{V'(y)}{\sqrt{y^2-1}}\frac{dy}{y-z}, \tag{4.2}$$

and $\mathcal{C}$ is a closed loop oriented in the clockwise direction encircling z and $[-1,1]$. Clearly, h is analytic in U_V. Also, since V is real, (4.1) reduces to

$$\psi(x) = \frac{h(x)}{2\pi} = \frac{1}{2\pi^2}\fint_{-1}^{1}\frac{V'(y)}{\sqrt{1-y^2}}\frac{dy}{y-x}. \tag{4.3}$$

Thus, we have that ψ is analytic in U_V and $\psi(x)\in\mathbb{R}$ if $x\in\mathbb{R}$.
We define the g-function, which is useful in order to normalize the RH problem at ∞, by

$$g(z) = \int_{\mathcal{S}}\log(z-s)\rho(s)ds, \qquad \rho(x)=\psi(x)\sqrt{1-x^2}, \tag{4.4}$$

where the principal branch cut is chosen for the logarithm. The g-function is analytic in $\mathbb{C} \setminus (-\infty, 1]$ and possesses the following properties

$$g_+(x) + g_-(x) = 2\int_S \log|x-s|\rho(s)ds, \qquad x \in \mathbb{R}, \tag{4.5}$$

$$g_+(x) - g_-(x) = 2\pi i, \qquad x < -1, \tag{4.6}$$

$$g_+(x) - g_-(x) = 2\pi i \int_x^1 \rho(s)ds, \qquad x \in \mathcal{S}, \tag{4.7}$$

$$g_+(x) - g_-(x) = 0, \qquad 1 < x. \tag{4.8}$$

Consider the function

$$\xi(z) = -\pi \int_1^z \tilde{\rho}(s)ds, \tag{4.9}$$

where the path of integration lies in $U_V \setminus (-\infty, 1)$ and $\tilde{\rho}(s) = \psi(s)\sqrt{s^2-1}$ is analytic in $U_V \setminus \mathcal{S}$. Since $\tilde{\rho}_\pm(s) = \pm i\rho(s)$ for $s \in \mathcal{S}$, we have

$$2\xi_\pm(x) = g_\pm(x) - g_\mp(x) = 2g_\pm(x) + \ell - V(x), \qquad x \in \mathcal{S}, \tag{4.10}$$

where we have used (1.11) together with (4.5). Analytically continuing $\xi(z) - g(z)$ in (4.10), we have

$$\xi(z) = g(z) + \frac{\ell}{2} - \frac{V(z)}{2}, \qquad \text{for all } z \in U_V \setminus (-\infty, 1). \tag{4.11}$$

The function $\xi(z)$ possesses certain properties which will be useful later to show that the jump matrices of a RH problem converge to the identity matrix as $n \to \infty$. First, note from (4.11) that the variational inequality (1.12) (which we recall is strict) can be rewritten as

$$\xi_+(x) + \xi_-(x) < 0, \qquad \text{for } |x| > 1,\ x \in \mathbb{R}. \tag{4.12}$$

Also, since $V(x)/\log|x| \to +\infty$ and $g(x) = \mathcal{O}(\log|x|)$ as $|x| \to +\infty$, $x \in \mathbb{R}$, we have

$$\big(\xi_+(x) + \xi_-(x)\big)/V(x) \to -1, \qquad \text{as } |x| \to +\infty,\ x \in \mathbb{R}. \tag{4.13}$$

Finally, from the assumption that ψ is positive on $\mathcal{S}$, a direct analysis (see e.g. [5, Lemma 4.4]) of the integral (4.9) shows that for z in a small enough neighbourhood of $(-1, 1)$,

$$\Re\xi(z) > 0, \qquad \text{if } \Im z \neq 0. \tag{4.14}$$

We will also need later large z asymptotics of $e^{ng(z)}$ for the case $V(x) = 2x^2$:

$$e^{ng(z)} = z^n \left(1 - \frac{n}{8z^2} + \mathcal{O}(z^{-4})\right), \qquad \text{as } z \to \infty. \tag{4.15}$$

4.2 First transformation: $Y \mapsto T$

The first step consists of normalizing the RH problem at ∞, which can be done with the following transformation

$$T(z) = e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3}. \tag{4.16}$$

T satisfies the following RH problem.

RH problem for T

(a) $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Using (1.11), (4.5), (4.6), (4.7), (4.8) and (4.10), a direct computation shows that T has the following jumps:

$$T_+(x) = T_-(x) J_T(x), \qquad \text{for } x \in \mathbb{R} \setminus \{t_1, ..., t_m\}, \tag{4.17}$$

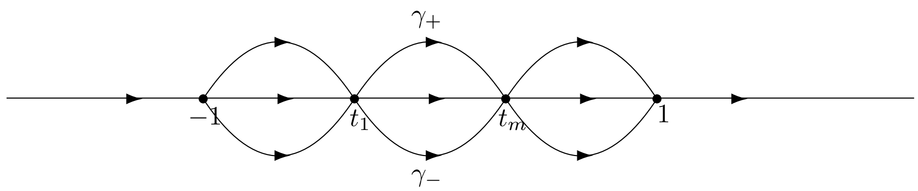

Figure 1: Jump contours for the RH problem for S with $m = 2$. The lens contours are labelled $\gamma_+ \subset U_W$ and $\gamma_- \subset U_W$, and they lie in the upper and lower half plane respectively.

where

$$J_T(x) = \begin{cases} \begin{pmatrix} 1 & e^{W(x)}\omega(x)e^{n(\xi_+(x)+\xi_-(x))} \\ 0 & 1 \end{pmatrix}, & \text{if } x < -1, \\ \begin{pmatrix} e^{-2n\xi_+(x)} & e^{W(x)}\omega(x) \\ 0 & e^{2n\xi_+(x)} \end{pmatrix}, & \text{if } x \in (-1,1) \setminus \{t_1, ..., t_m\}, \\ \begin{pmatrix} 1 & e^{W(x)}\omega(x)e^{2n\xi(x)} \\ 0 & 1 \end{pmatrix}, & \text{if } 1 < x. \end{cases} \tag{4.18}$$

(c) As $z \to \infty$, we have $T(z) = I + \mathcal{O}(z^{-1})$.

(d) As z tends to t_k for a certain $k \in \{1, ..., m\}$, the behaviour of T is

$$\begin{aligned} T(z) &= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - t_k)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - t_k)) \end{pmatrix}, & \text{if } \Re\alpha_k = 0, \\ T(z) &= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_k)^{\Re\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_k)^{\Re\alpha_k}) \end{pmatrix}, & \text{if } \Re\alpha_k \neq 0. \end{aligned} \tag{4.19}$$

As z tends to -1 or 1, we have $T(z) = \mathcal{O}(1)$.

4.3 Second transformation: $T \mapsto S$

We will use the following factorization of $J_T(x)$ for $x \in \mathcal{S} \setminus \{t_1, ..., t_m\}$:

$$\begin{aligned} \begin{pmatrix} e^{-2n\xi_+(x)} & e^{W(x)}\omega(x) \\ 0 & e^{-2n\xi_-(x)} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ e^{-W(x)}\omega(x)^{-1}e^{-2n\xi_-(x)} & 1 \end{pmatrix} \\ &\times \begin{pmatrix} 0 & e^{W(x)}\omega(x) \\ -e^{-W(x)}\omega(x)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-W(x)}\omega(x)^{-1}e^{-2n\xi_+(x)} & 1 \end{pmatrix}. \end{aligned} \tag{4.20}$$

The functions ω_{α_k} and ω_{β_k} (see (1.3)) can be analytically continued as follows:

$$\omega_{\alpha_k}(z) = \begin{cases} (t_k - z)^{\alpha_k}, & \text{if } \Re z < t_k, \\ (z - t_k)^{\alpha_k}, & \text{if } \Re z > t_k, \end{cases} \qquad \omega_{\beta_k}(z) = \begin{cases} e^{i\pi\beta_k}, & \text{if } \Re z < t_k, \\ e^{-i\pi\beta_k}, & \text{if } \Re z > t_k. \end{cases} \tag{4.21}$$

We open the lenses γ_+ and γ_- around S as illustrated in Figure 1, such that they are inside U_W and we define

$$S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } z \text{ is inside the lenses}, \Im z > 0, \\ \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } z \text{ is inside the lenses}, \Im z < 0, \\ I, & \text{if } z \text{ is outside the lenses}. \end{cases} \tag{4.22}$$

S satisfies the following RH problem

RH problem for S

(a) $S : \mathbb{C} \setminus (\mathbb{R} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) S has the following jumps:

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{W(z)}\omega(z)e^{n(\xi_+(z)+\xi_-(z))} \\ 0 & 1 \end{pmatrix}, \qquad \text{if } z < -1, \tag{4.23}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{W(z)}\omega(z)e^{2n\xi(z)} \\ 0 & 1 \end{pmatrix}, \qquad \text{if } 1 < z, \tag{4.24}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \qquad \text{if } z \in (-1,1) \setminus \{t_1,...,t_m\}, \tag{4.25}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, \qquad \text{if } z \in \gamma_+ \cup \gamma_-. \tag{4.26}$$

(c) As $z \to \infty$, we have $S(z) = I + \mathcal{O}(z^{-1})$.

(d) As z tends to t_k for a certain $k \in \{1,...,m\}$, we have

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z-t_k)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z-t_k)) \end{pmatrix}, & \text{if } z \text{ is outside the lenses,} \\ \begin{pmatrix} \mathcal{O}(\log(z-t_k)) & \mathcal{O}(\log(z-t_k)) \\ \mathcal{O}(\log(z-t_k)) & \mathcal{O}(\log(z-t_k)) \end{pmatrix}, & \text{if } z \text{ is inside the lenses,} \end{cases}, \qquad \text{if } \Re\alpha_k = 0,$$

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \text{if } z \text{ is outside the lenses,} \\ \begin{pmatrix} \mathcal{O}((z-t_k)^{-\Re\alpha_k}) & \mathcal{O}(1) \\ \mathcal{O}((z-t_k)^{-\Re\alpha_k}) & \mathcal{O}(1) \end{pmatrix}, & \text{if } z \text{ is inside the lenses,} \end{cases}, \qquad \text{if } \Re\alpha_k > 0,$$

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}((z-t_k)^{\Re\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}((z-t_k)^{\Re\alpha_k}) \end{pmatrix}, \qquad \text{if } \Re\alpha_k < 0.$$

As z tends to -1 or 1, we have $S(z) = \mathcal{O}(1)$.

From (4.12) and (4.14), the jumps for S on $(\gamma_+ \cup \gamma_- \cup \mathbb{R}) \setminus \mathcal{S}$ are exponentially close to the identity matrix as $n \to \infty$. Nevertheless, this is only a pointwise convergence and it breaks down as z approaches t_1,...,t_m, -1 and 1. This simply follows from

$$\begin{aligned} &\xi_+(x) + \xi_-(x) \to \xi_+(\pm 1) + \xi_-(\pm 1) = 0, && \text{as } x \to \pm 1, \ x \in \mathbb{R}, \\ &\Re\xi_\pm(x) = 0, && \text{for } x \in \mathcal{S}. \end{aligned}$$

Therefore, for z outside of a neighbourhood of $\mathcal{S}$, from (4.13) and the assumption $W(x) = \mathcal{O}(V(x))$ as $|x| \to \infty$, the jumps for S are uniformly exponentially close to I as $n \to \infty$. If we ignore these jumps, we are left with a simpler RH problem (called the global parametrix) whose solution is a good approximation of S away from these points. For z in small neighbourhoods of t_1,...,t_m, -1 and 1, the global parametrix is no longer a good approximation for S, and we will construct local parametrices around these points.

4.4 Global parametrix

Ignoring the exponentially small terms as $n \to \infty$ in the jumps of S and small neighbourhoods of -1, $t_1, ..., t_m$ and 1, we are left with the following RH problem, whose solution $P^{(\infty)}$ is a good approximation of S away from neighbourhoods of -1, $t_1, ..., t_m$ and 1:

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus [-1,1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ has the following jumps:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \quad \text{if } z \in (-1,1) \setminus \{t_1, ..., t_m\}. \tag{4.27}$$

(c) As $z \to \infty$, we have $P^{(\infty)}(z) = I + P_1^{(\infty)} z^{-1} + P_2^{(\infty)} z^{-2} + \mathcal{O}(z^{-3})$.
As z tends to -1, we have $P^{(\infty)}(z) = \mathcal{O}((z+1)^{-1/4})$.
As z tends to 1, we have $P^{(\infty)}(z) = \mathcal{O}((z-1)^{-1/4})$.
As z tends to t_k, $k \in \{1, ..., m\}$, we have $P^{(\infty)}(z) = \mathcal{O}(1)(z-t_k)^{-(\frac{\alpha_k}{2}+\beta_k)\sigma_3}$.

The unique solution $P^{(\infty)}$ of the above RH problem can be constructed similarly as in [43]. Define $a(z) = \sqrt[4]{\frac{z+1}{z-1}}$, analytic on $\mathbb{C} \setminus [-1,1]$ and such that $a(z) \sim 1$ as $z \to \infty$. It can be checked that $P^{(\infty)}$ is given by

$$P^{(\infty)}(z) = D_\infty^{\sigma_3} \begin{pmatrix} \frac{1}{2}(a(z) + a^{-1}(z)) & \frac{1}{-2i}(a(z) - a^{-1}(z)) \\ \frac{1}{2i}(a(z) - a^{-1}(z)) & \frac{1}{2}(a(z) + a^{-1}(z)) \end{pmatrix} D(z)^{-\sigma_3}, \tag{4.28}$$

with $D(z) = D_W(z)D_\alpha(z)D_\beta(z)$, and $D_W(z)$, $D_\alpha(z)$ and $D_\beta(z)$ are the three Szegő functions defined by

$$D_W(z) = \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right), \tag{4.29}$$

$$D_\alpha(z) = \prod_{j=1}^{m} \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log \omega_{\alpha_j}(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right) = (z + \sqrt{z^2-1})^{-\frac{\mathcal{A}}{2}} \prod_{j=1}^{m} (z-t_j)^{\frac{\alpha_j}{2}}, \tag{4.30}$$

$$D_\beta(z) = \prod_{j=1}^{m} \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log \omega_{\beta_j}(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right) = e^{\frac{i\pi\mathcal{B}}{2}} \prod_{j=1}^{m} \left(\frac{zt_j - 1 - i\sqrt{(z^2-1)(1-t_j^2)}}{z-t_j} \right)^{\beta_j}, \tag{4.31}$$

where $\mathcal{B} = \sum_{j=1}^{m} \beta_j$, and we recall $\mathcal{A} = \sum_{j=1}^{m} \alpha_j$. A proof of the simplified form of (4.30) and (4.31) can be found in [42, equation (51)] and [37, equation (4.14)] respectively. Also, the function D has a limit at ∞ given by

$$D_\infty = \lim_{z\to\infty} D(z) = D_{W,\infty} D_{\alpha,\infty} D_{\beta,\infty}, \qquad D_{W,\infty} = \exp\left(\frac{1}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}} dx \right), \tag{4.32}$$

$$D_{\alpha,\infty} = 2^{-\frac{\mathcal{A}}{2}}, \qquad D_{\beta,\infty} = \exp\left(i \sum_{j=1}^{m} \beta_j \arcsin t_j \right).$$

We will use later the following expansions of $D(z)$: as $z \to t_k$, $\Im z > 0$, we have

$$D_\alpha(z) = e^{-i\frac{\mathcal{A}}{2}\arccos t_k} \left(\prod_{j\neq k} |t_k - t_j|^{\frac{\alpha_j}{2}} \prod_{j=k+1}^{m} e^{\frac{i\pi\alpha_j}{2}} \right) (z-t_k)^{\frac{\alpha_k}{2}} (1 + \mathcal{O}(z-t_k)), \tag{4.33}$$

$$D_\beta(z) = e^{-\frac{i\pi}{2}(\mathcal{B}_k + \beta_k)} \left(\prod_{j\neq k} T_{kj}^{\beta_j} \right) (1-t_k^2)^{-\beta_k} 2^{-\beta_k} (z-t_k)^{\beta_k} (1 + \mathcal{O}(z-t_k)), \tag{4.34}$$

where

$$\mathcal{B}_k = \sum_{j=1}^{k-1} \beta_j - \sum_{j=k+1}^{m} \beta_j, \qquad T_{kj} = \frac{1 - t_k t_j - \sqrt{(1-t_k^2)(1-t_j^2)}}{|t_k - t_j|}.$$

The expansions of $D(z)$ near 1 and -1 are nicely expressed in terms of

$$\widetilde{\mathcal{B}}_1 = 2i\sum_{j=1}^m \sqrt{\frac{1+t_j}{1-t_j}}\beta_j, \qquad \widetilde{\mathcal{B}}_{-1} = 2i\sum_{j=1}^m \sqrt{\frac{1-t_j}{1+t_j}}\beta_j.$$

As $z \to 1$, $\Im z > 0$, we have

$$D_\alpha^2(z)\prod_{j=1}^m (z-t_j)^{-\alpha_j} = 1 - \sqrt{2}\mathcal{A}\sqrt{z-1} + \mathcal{A}^2(z-1) + \mathcal{O}((z-1)^{3/2}), \tag{4.35}$$

$$D_\beta^2(z)e^{i\pi\mathcal{B}} = 1 + \sqrt{2}\widetilde{\mathcal{B}}_1\sqrt{z-1} + \widetilde{\mathcal{B}}_1^2(z-1) + \mathcal{O}((z-1)^{3/2}). \tag{4.36}$$

As $z \to -1$, $\pm\Im z > 0$, we have

$$D_\alpha^2(z)\prod_{j=1}^m (t_j-z)^{-\alpha_j} = 1 \pm i\sqrt{2}\mathcal{A}\sqrt{z+1} - \mathcal{A}^2(z+1) + \mathcal{O}((z+1)^{3/2}), \tag{4.37}$$

$$D_\beta^2(z)e^{-i\pi\mathcal{B}} = 1 \pm i\sqrt{2}\widetilde{\mathcal{B}}_{-1}\sqrt{z+1} - \widetilde{\mathcal{B}}_{-1}^2(z+1) + \mathcal{O}((z+1)^{3/2}). \tag{4.38}$$

The first term of the expansion of (4.28) as $z \to \infty$ with $W \equiv 0$ is given by

$$P_1^{(\infty)} = \begin{pmatrix} \sum_{j=1}^m \left(\frac{\alpha_j t_j}{2} + i\sqrt{1-t_j^2}\beta_j\right) & \frac{i}{2}D_\infty^2 \\ -\frac{i}{2}D_\infty^{-2} & -\sum_{j=1}^m \left(\frac{\alpha_j t_j}{2} + i\sqrt{1-t_j^2}\beta_j\right) \end{pmatrix}. \tag{4.39}$$

The next term requires more calculations, but we will only use the $1,1$ entry:

$$P_{2,11}^{(\infty)} = \frac{1}{8} + \frac{1}{2}\left(\sum_{j=1}^m \left(\frac{\alpha_j t_j}{2} + i\sqrt{1-t_j^2}\beta_j\right)\right)^2 - \frac{1}{8}\sum_{j=1}^m \alpha_j(1-2t_j^2) + \frac{i}{2}\sum_{j=1}^m t_j\sqrt{1-t_j^2}\beta_j. \tag{4.40}$$

4.5 Local parametrix near t_k

We assume that there exists $\delta > 0$ independent of n such that

$$\min_{j\neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{4.41}$$

Consider a fixed disk $\mathcal{D}_{t_k}$, centered at t_k, of radius smaller than $\delta/3$ and such that $\mathcal{D}_{t_k} \subset U_W$. Inside $\mathcal{D}_{t_k}$, the local parametrix $P^{(t_k)}$ is a good approximation of S and must satisfy the following RH problem.

RH problem for $P^{(t_k)}$

(a) $P^{(t_k)} : \mathcal{D}_{t_k} \setminus (\mathbb{R} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(t_k)}$ has the following jumps:

$$P_+^{(t_k)}(z) = P_-^{(t_k)}(z)\begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \quad \text{if } z \in (\mathbb{R}\setminus\{t_k\}) \cap \mathcal{D}_{t_k}, \tag{4.42}$$

$$P_+^{(t_k)}(z) = P_-^{(t_k)}(z)\begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, \quad \text{if } z \in (\gamma_+ \cup \gamma_-) \cap \mathcal{D}_{t_k}. \tag{4.43}$$

(c) As $n \to \infty$, we have $P^{(t_k)}(z) = (I + \mathcal{O}(n^{-1+2|\Re\beta_k|}))P^{(\infty)}(z)$ uniformly for $z \in \partial\mathcal{D}_{t_k}$.

(d) As z tends to t_k, we have $S(z)P^{(t_k)}(z)^{-1} = \mathcal{O}(1)$.

We follow [37, 35, 23] to construct the solution of the above RH problem, and we define the function f_{t_k} by

$$f_{t_k}(z) = -2\left\{\begin{array}{ll} \xi(z) - \xi_+(t_k), & \Im z > 0, \\ -\xi(z) - \xi_+(t_k), & \Im z < 0, \end{array}\right. = 2\pi i \int_{t_k}^{z} \psi(s)\sqrt{1-s^2}ds. \tag{4.44}$$

This a conformal map from $\mathcal{D}_{t_k}$ to a neighbourhood of 0, as shown from the expansion of $f_{t_k}(z)$ near t_k

$$f_{t_k}(z) = 2\pi i\psi(t_k)\sqrt{1-t_k^2}(z-t_k)\left(1+\mathcal{O}((z-t_k))\right), \qquad \text{as } z \to t_k. \tag{4.45}$$

Now, we use the freedom we had in the choice of the lenses by requiring that

$$f_{t_k}(\gamma_+ \cap \mathcal{D}_{t_k}) \subset (\Gamma_4 \cup \Gamma_2), \qquad f_{t_k}(\gamma_- \cap \mathcal{D}_{t_k}) \subset (\Gamma_6 \cup \Gamma_8), \tag{4.46}$$

where the contour $\Gamma_4, \Gamma_2, \Gamma_6$ and Γ_8 are shown in Figure 4. We define

$$\widetilde{W}_k(z) = \left\{\begin{array}{ll} (z-t_k)^{\frac{\alpha_k}{2}} e^{-\frac{i\pi\alpha_k}{2}}, & z \in Q_{+,k}^R, \\ (t_k-z)^{\frac{\alpha_k}{2}} e^{\frac{i\pi\alpha_k}{2}}, & z \in Q_{+,k}^L, \\ (t_k-z)^{\frac{\alpha_k}{2}} e^{-\frac{i\pi\alpha_k}{2}}, & z \in Q_{-,k}^L, \\ (z-t_k)^{\frac{\alpha_k}{2}} e^{\frac{i\pi\alpha_k}{2}}, & z \in Q_{-,k}^R, \end{array}\right. \tag{4.47}$$

and where $Q_{+,k}^R$, $Q_{+,k}^L$, $Q_{-,k}^L$ and $Q_{-,k}^R$ are the following four quadrant centred at t_k:

$$Q_{\pm,k}^R = \{z \in \mathcal{D}_{t_k} : \mp\Re f_{t_k}(z) > 0, \Im f_{t_k}(z) > 0\}, \qquad Q_{\pm,k}^L = \{z \in \mathcal{D}_{t_k} : \mp\Re f_{t_k}(z) > 0, \Im f_{t_k}(z) < 0\}.$$

The solution of the above RH problem is given in terms of the confluent hypergeometric model RH problem, denoted $\Phi_{\mathrm{HG}}(z;\alpha_k,\beta_k)$ (see Section 8 for the definition and properties of $\Phi_{\mathrm{HG}}(z;\alpha_k,\beta_k)$). We obtain

$$P^{(t_k)}(z) = E_{t_k}(z)\Phi_{\mathrm{HG}}(nf_{t_k}(z);\alpha_k,\beta_k)\widetilde{W}_k(z)^{-\sigma_3}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}\omega_k(z)^{-\frac{\sigma_3}{2}}, \tag{4.48}$$

where E_{t_k} is given by

$$E_{t_k}(z) = P^{(\infty)}(z)\omega_k(z)^{\frac{\sigma_3}{2}}e^{\frac{W(z)}{2}\sigma_3}\widetilde{W}_k(z)^{\sigma_3}\left\{\begin{array}{ll} e^{\frac{i\pi\alpha_k}{4}\sigma_3}e^{-i\pi\beta_k\sigma_3}, & z \in Q_{+,k}^R \\ e^{-\frac{i\pi\alpha_k}{4}\sigma_3}e^{-i\pi\beta_k\sigma_3}, & z \in Q_{+,k}^L \\ e^{\frac{i\pi\alpha_k}{4}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q_{-,k}^L \\ e^{-\frac{i\pi\alpha_k}{4}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q_{-,k}^R \end{array}\right\}e^{n\xi_+(t_k)\sigma_3}(nf_{t_k}(z))^{\beta_k\sigma_3}. \tag{4.49}$$

From the jumps of $P^{(\infty)}$ (4.27) and the definition of $\widetilde{W}_k$ (4.47), we can check that E_{t_k} has no jumps at all in $\mathcal{D}_{t_k}$. Furthermore, since $P^{(\infty)}(z) = \mathcal{O}(1)(z-t_k)^{-(\frac{\alpha_k}{2}+\beta_k)\sigma_3}$ as $z \to t_k$, we conclude from (4.49) that $E_{t_k}(z) = \mathcal{O}(1)$ as $z \to t_k$. As a result, E_{t_k} is analytic in the disk $\mathcal{D}_{t_k}$. Note also from (4.49) that $E_{t_k}(z) = \mathcal{O}(n^{|\Re\beta_k|})$ as $n \to \infty$, uniformly for $z \in \mathcal{D}_{t_k}$. Since $P^{(t_k)}$ and S have exactly the same jumps on $(\mathbb{R} \cup \gamma_+ \cup \gamma_-) \cap \mathcal{D}_{t_k}$, $S(z)P^{(t_k)}(z)^{-1}$ is analytic in $\mathcal{D}_{t_k} \setminus \{t_k\}$. As $z \to t_k$, z outside the lenses, by condition (d) in the RH problem for S, by (8.10) and by (4.48), $S(z)P^{(t_k)}(z)^{-1}$ behaves as $\mathcal{O}(\log(z-t_k))$ if $\Re\alpha_k = 0$, as $\mathcal{O}(1)$ if $\Re\alpha_k > 0$, and as $\mathcal{O}((z-t_k)^{\Re\alpha_k})$ if $\Re\alpha_k < 0$. Thus, in all the cases, the singularity of $S(z)P^{(t_k)}(z)^{-1}$ at $z = t_k$ is removable and condition (d) of the RH problem for $P^{(t_k)}$ is verified.

The value of $E_{t_k}(t_k)$ can be obtained by taking the limit $z \to t_k$ in (4.49). Using (4.45), (4.33) and (4.34), it gives

$$E_{t_k}(t_k) = \frac{D_\infty^{\sigma_3}}{2\sqrt[4]{1-t_k^2}}\begin{pmatrix} e^{-\frac{\pi i}{4}}\sqrt{1+t_k} + e^{\frac{\pi i}{4}}\sqrt{1-t_k} & i\left(e^{-\frac{\pi i}{4}}\sqrt{1+t_k} - e^{\frac{\pi i}{4}}\sqrt{1-t_k}\right) \\ -i\left(e^{-\frac{\pi i}{4}}\sqrt{1+t_k} - e^{\frac{\pi i}{4}}\sqrt{1-t_k}\right) & e^{-\frac{\pi i}{4}}\sqrt{1+t_k} + e^{\frac{\pi i}{4}}\sqrt{1-t_k} \end{pmatrix}\Lambda_k^{\sigma_3},$$

where

$$\Lambda_k = e^{\frac{W(t_k)}{2}} D_{W,+}(t_k)^{-1} e^{i\frac{\lambda_k}{2}} (4\pi\psi(t_k)n(1-t_k^2)^{3/2})^{\beta_k} \prod_{j\neq k} T_{kj}^{-\beta_j}, \tag{4.50}$$

and

$$\lambda_k = \mathcal{A}\arccos t_k - \frac{\pi}{2}\alpha_k - \sum_{j=k+1}^{m} \pi\alpha_j + 2\pi n \int_{t_k}^{1} \rho(s)ds. \tag{4.51}$$

A direct calculation, using equation (8.7) in the appendix, shows that as $n\to\infty$, uniformly for $z\in\partial\mathcal{D}_{t_k}$, $P^{(t_k)}(z)P^{(\infty)}(z)^{-1}$ admits an expansion in the inverse power of n multiplied by $n^{2|\Re\beta_k|}$. We will need explicitly the n^{-1} term:

$$P^{(t_k)}(z)P^{(\infty)}(z)^{-1} = I + \frac{v_k}{nf_{t_k}(z)} E_{t_k}(z) \begin{pmatrix} -1 & \tau(\alpha_k,\beta_k) \\ -\tau(\alpha_k,-\beta_k) & 1 \end{pmatrix} E_{t_k}(z)^{-1} + \mathcal{O}(n^{-2+2|\Re\beta_k|}), \tag{4.52}$$

where $v_k = \beta_k^2 - \frac{\alpha_k^2}{4}$, and $\tau(\alpha_k,\beta_k)$ is defined in equation (8.8).

4.6 Local parametrix near 1

Consider a fixed disk $\mathcal{D}_1$, centered at 1, of radius smaller than $\delta/3$ and such that $\mathcal{D}_1\subset U_W$. Inside $\mathcal{D}_1$, the local parametrix $P^{(1)}$ is a good approximation of S and must satisfy the following RH problem.

RH problem for $P^{(1)}$

(a) $P^{(1)}: \mathcal{D}_1\setminus(\mathbb{R}\cup\gamma_+\cup\gamma_-)\to\mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(1)}$ has the following jumps:

$$P_+^{(1)}(z) = P_-^{(1)}(z)\begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \qquad \text{if } z\in(-\infty,1)\cap\mathcal{D}_1, \tag{4.53}$$

$$P_+^{(1)}(z) = P_-^{(1)}(z)\begin{pmatrix} 1 & e^{W(z)}\omega(z)e^{2n\xi(z)} \\ 0 & 1 \end{pmatrix}, \qquad \text{if } z\in(1,\infty)\cap\mathcal{D}_1, \tag{4.54}$$

$$P_+^{(1)}(z) = P_-^{(1)}(z)\begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, \qquad \text{if } z\in(\gamma_+\cup\gamma_-)\cap\mathcal{D}_1. \tag{4.55}$$

(c) As $n\to\infty$, we have $P^{(1)}(z) = (I+\mathcal{O}(n^{-1}))P^{(\infty)}(z)$ uniformly for $z\in\partial\mathcal{D}_1$.

(d) As z tends to 1, we have $P^{(1)}(z) = \mathcal{O}(1)$.

The construction of this local parametrix is now standard (see e.g. [27]). We denote

$$f_1(z) = \left(-\frac{3}{2}\xi(z)\right)^{2/3} = \left(\frac{3\pi}{2}\int_1^z \psi(s)\sqrt{s^2-1}ds\right)^{2/3}. \tag{4.56}$$

This is a conformal map from $\mathcal{D}_1$ to a neighbourhood of 0 and

$$f_1(z) = (\sqrt{2}\pi\psi(1))^{2/3}(z-1)\left(1+\frac{1}{10}\left(1+4\frac{\psi'(1)}{\psi(1)}\right)(z-1)+\mathcal{O}((z-1)^2)\right), \qquad \text{as } z\to 1. \tag{4.57}$$

We choose the lenses such that $f_1(\gamma_+\cap\mathcal{D}_1)\subset e^{\frac{2\pi i}{3}}\mathbb{R}^+$ and $f_1(\gamma_-\cap\mathcal{D}_1)\subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+$. The solution of the above RH problem is given by

$$P^{(1)}(z) = E_1(z)\Phi_{\mathrm{Ai}}(n^{2/3}f_1(z))\omega(z)^{-\frac{\sigma_3}{2}}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}, \tag{4.58}$$

where

$$E_1(z) = P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}N^{-1}f_1(z)^{\frac{\sigma_3}{4}}n^{\frac{\sigma_3}{6}}, \qquad N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{4.59}$$

and $\Phi_{\mathrm{Ai}}(z)$ is the solution to the Airy model RH problem presented in the Appendix, see Section 8. One can check that E_1 has no jumps in $\mathcal{D}_1$ and is bounded as $z \to 1$, and thus E_1 is analytic in $\mathcal{D}_1$. A direct calculation, using (8.2), shows that as $n \to \infty$, uniformly for $z \in \partial\mathcal{D}_1$, $P^{(1)}(z)P^{(\infty)}(z)^{-1}$ admits an expansion in the inverse power of n. The explicit forms of the first terms are given in the following expression

$$P^{(1)}(z)P^{(\infty)}(z)^{-1} = I+\frac{P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}}{8nf_1(z)^{3/2}}\begin{pmatrix}\frac{1}{6} & i\\ i & -\frac{1}{6}\end{pmatrix}\omega(z)^{-\frac{\sigma_3}{2}}e^{-\frac{W(z)}{2}\sigma_3}P^{(\infty)}(z)^{-1}+\mathcal{O}(n^{-2}). \tag{4.60}$$

4.7 Local parametrix near -1

Consider a fixed disk $\mathcal{D}_{-1}$, centered at -1, of radius smaller than $\delta/3$ and such that $\mathcal{D}_{-1} \subset U_W$. Inside $\mathcal{D}_{-1}$, the local parametrix $P^{(-1)}$ is a good approximation of S and must satisfy the following RH problem.

RH problem for $P^{(-1)}$

(a) $P^{(-1)} : \mathcal{D}_{-1} \setminus (\mathbb{R} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(-1)}$ has the following jumps:

$$P_+^{(-1)}(z) = P_-^{(-1)}(z)\begin{pmatrix}1 & e^{W(z)}\omega(z)e^{n(\xi_+(z)+\xi_-(z))}\\ 0 & 1\end{pmatrix}, \quad \text{if } z \in (-\infty,-1)\cap\mathcal{D}_{-1}, \tag{4.61}$$

$$P_+^{(-1)}(z) = P_-^{(-1)}(z)\begin{pmatrix}0 & e^{W(z)}\omega(z)\\ -e^{-W(z)}\omega(z)^{-1} & 0\end{pmatrix}, \quad \text{if } z \in (-1,\infty)\cap\mathcal{D}_{-1}, \tag{4.62}$$

$$P_+^{(-1)}(z) = P_-^{(-1)}(z)\begin{pmatrix}1 & 0\\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1\end{pmatrix}, \quad \text{if } z \in (\gamma_+\cup\gamma_-)\cap\mathcal{D}_{-1}. \tag{4.63}$$

(c) As $n \to \infty$, we have $P^{(-1)}(z) = (I + \mathcal{O}(n^{-1}))P^{(\infty)}(z)$ uniformly for $z \in \partial\mathcal{D}_{-1}$.

(d) As z tends to -1, we have $P^{(-1)}(z) = \mathcal{O}(1)$.

The construction of this local parametrix around -1 is very similar to the construction done in subsection 4.6 for the local parametrix around 1. We denote

$$f_{-1}(z) = -\left(-\frac{3}{2}\left\{\begin{array}{ll}\xi(z)-\xi_+(-1), & \mathrm{Im} z>0,\\ \xi(z)-\xi_-(-1), & \mathrm{Im} z<0,\end{array}\right\}\right)^{2/3} = \left(\frac{3\pi}{2}\int_{-1}^{z}\psi(s)\sqrt{1-s^2}ds\right)^{2/3}. \tag{4.64}$$

This is a conformal map from $\mathcal{D}_{-1}$ to a neighbourhood of 0 and

$$f_{-1}(z) = (\sqrt{2}\pi\psi(-1))^{2/3}(z+1)\left(1-\frac{1}{10}\left(1-4\frac{\psi'(-1)}{\psi(-1)}\right)(z+1)+\mathcal{O}((z+1)^2)\right), \quad \text{as } z\to -1. \tag{4.65}$$

We choose the lenses such that $-f_{-1}(\gamma_+\cap\mathcal{D}_{-1}) \subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+$ and $-f_{-1}(\gamma_-\cap\mathcal{D}_{-1}) \subset e^{\frac{2\pi i}{3}}\mathbb{R}^+$. The solution of the above RH problem is given by

$$P^{(-1)}(z) = E_{-1}(z)\sigma_3\Phi_{\mathrm{Ai}}(-n^{2/3}f_{-1}(z))\sigma_3\omega(z)^{-\frac{\sigma_3}{2}}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}, \tag{4.66}$$

where E_{-1} is analytic in $\mathcal{D}_{-1}$ and is given by

$$E_{-1}(z) = (-1)^n P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}N(-f_{-1}(z))^{\frac{\sigma_3}{4}}n^{\frac{\sigma_3}{6}}. \tag{4.67}$$

Similarly to (4.60), one shows with a direct calculation together with (8.2), that as $n\to\infty$, uniformly for $z \in \partial\mathcal{D}_{-1}$, $P^{(-1)}(z)P^{(\infty)}(z)^{-1}$ admits an expression in the inverse power of n. The first terms are given by

$$P^{(-1)}(z)P^{(\infty)}(z)^{-1} = I+\frac{P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}}{8n(-f_{-1}(z))^{3/2}}\begin{pmatrix}\frac{1}{6} & -i\\ -i & -\frac{1}{6}\end{pmatrix}\omega(z)^{-\frac{\sigma_3}{2}}e^{-\frac{W(z)}{2}\sigma_3}P^{(\infty)}(z)^{-1}+\mathcal{O}(n^{-2}). \tag{4.68}$$

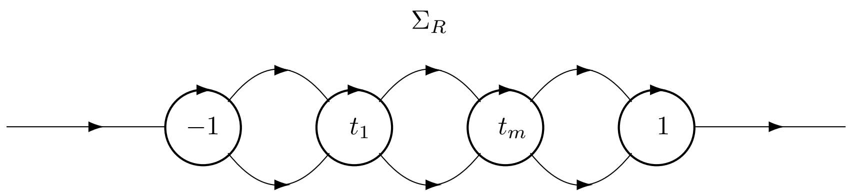

Figure 2: Jump contours for the RH problem for R with $m = 2$. The circles are oriented in the clockwise direction.

4.8 Small norm RH problem

Let $\mathcal{D} = \mathcal{D}_{-1} \cup \mathcal{D}_1 \cup \bigcup_{j=1}^{m} \mathcal{D}_{t_j}$ be the union of the disks, whose boundaries are oriented in the clockwise direction as shown in Figure 2, and let P be defined on $\mathcal{D}$ such that $P|_{\mathcal{D}_a} = P^{(a)}$, where $a = -1, t_1, ..., t_m, 1$. We define

$$R(z) = R(z; \vec{\alpha}, \vec{\beta}, V, W) = \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \mathcal{D}, \\ S(z)P(z)^{-1}, & z \in \mathcal{D}. \end{cases} \tag{4.69}$$

Since P and S have exactly the same jumps inside $\mathcal{D}$, R is analytic in $\mathcal{D} \setminus \{-1, t_1, ..., t_m, 1\}$. Also, from the behaviours of P and S near -1, t_1, ..., t_m and 1, these points are removable singularities of R. Thus, R satisfies the following RH problem:

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_R is shown in Figure 2.

(b) R has the jumps $R_+(z) = R_-(z)J_R(z)$ for $z \in \Sigma_R$, where $J_R(z)$ is given in (4.52), (4.60) and (4.68) for z on respectively $\partial\mathcal{D}_{t_k}$, $\partial\mathcal{D}_1$, and $\partial\mathcal{D}_{-1}$. As $n \to \infty$ we have

$$J_R(z) = I + \mathcal{O}(e^{-cn}), \qquad \text{uniformly for } z \in (\gamma_+ \cup \gamma_- \cup \mathbb{R}) \setminus (\mathcal{S} \cup \mathcal{D}), \tag{4.70}$$
$$J_R(z) = I + \mathcal{O}(n^{-1}), \qquad \text{uniformly for } z \in \partial\mathcal{D}_{-1} \cup \partial\mathcal{D}_1, \tag{4.71}$$
$$J_R(z) = I + \mathcal{O}(n^{-1+2|\Re\beta_k|}), \qquad \text{uniformly for } z \in \partial\mathcal{D}_{t_k}, k = 1, ..., m, \tag{4.72}$$

where $c > 0$ is a constant.

(c) As $z \to \infty$, we have $R(z) = I + \mathcal{O}(z^{-1})$.

Let us define

$$\mathcal{P}_\beta^{(\frac{1}{2})} = \{\vec{\beta} \in \mathbb{C}^m : \Re\beta_j \in (-\tfrac{1}{2}, \tfrac{1}{2}), \text{ for all } j = 1, ..., m\}. \tag{4.73}$$

From the standard theory for small-norm RH problems [26, 27], if Ω is a compact subset of $\mathcal{P}_\alpha \times \mathcal{P}_\beta^{(\frac{1}{2})}$, there exists $n_\star = n_\star(\Omega)$ such that R exists for *all* $n \geq n_\star$, for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$ and for *all* $z \in \mathbb{C} \setminus \Sigma_R$. Furthermore, for any $r \in \mathbb{N}$, as $n \to \infty$ we have

$$R(z) = \sum_{j=0}^{r} R^{(j)}(z)n^{-j} + R_R^{(r+1)}(z)n^{-r-1}, \qquad R^{(0)}(z) \equiv I, \tag{4.74}$$

$$R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}}) \quad R^{(j)}(z)' = \mathcal{O}(n^{2\beta_{\max}}), \quad R_R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{\max}}) \quad R_R^{(r+1)}(z)' = \mathcal{O}(n^{2\beta_{\max}}),$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, uniformly for $(\vec{\alpha}, \vec{\beta}) \in \Omega$ and uniformly in $t_1, ..., t_m$, as long as there exists $\delta > 0$ independent of n such that

$$\min_{j \neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{4.75}$$

We can find explicit expressions for $R^{(1)}$, $R^{(2)}$,..., $R^{(r)}$ by induction. The factor $n^{\beta_k\sigma_3}$ in the expression (4.49) for E_{t_k} induces factors of the forms $n^{\pm 2\beta_k}$ in the entries of $J_R^{(j)}(z)$, for $j \geq 1$ and $z \in \partial\mathcal{D}_{t_k}$, as can be seen in (4.52). Thus, from (4.70), (4.52), (4.60) and (4.68), the jumps of R admit an expansion as $n \to \infty$ in powers of n^{-1} multiplied by terms of order $n^{2\beta_{\max}}$. As $n \to \infty$, we have

$$J_R(z) = \sum_{j=0}^{r} J_R^{(j)}(z) n^{-j} + \mathcal{O}(n^{-r-1+2\beta_{\max}}), \qquad \text{uniformly for } z \in \partial\mathcal{D}, \tag{4.76}$$

where $J_R^{(0)}(z) \equiv I$ and for $j \geq 1$, $J_R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}})$ as $n \to \infty$. From a perturbative analysis of the small norm RH problem for R, for every $j \geq 1$, $R^{(j)}(z)$ is analytic on $\mathbb{C} \setminus \partial\mathcal{D}$ and satisfies

$$R_+^{(j)}(z) = R_-^{(j)}(z) + \sum_{\ell=1}^{j} R_-^{(j-\ell)}(z) J_R^{(\ell)}(z), \qquad z \in \partial\mathcal{D}, \tag{4.77}$$

$$R^{(j)}(z) = \mathcal{O}(z^{-1}), \qquad \text{as } z \to \infty. \tag{4.78}$$

Therefore, by the Sokhotsky-Plemelj formula, $R^{(j)}$ is simply given by

$$R^{(j)}(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\sum_{\ell=1}^{j} R_-^{(j-\ell)}(s) J_R^{(\ell)}(s)}{s-z} ds, \qquad j \geq 1, \tag{4.79}$$

where we recall that the orientation on the disks is clockwise, as shown in Figure 2.

Since $R(z)$ exists for *all* $n \geq n_\star$ and for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$, by inverting the transformations $Y \mapsto T \mapsto S \mapsto R$, we have constructed a solution for the RH problem for Y that exists for *all* $n \geq n_\star$ and for *all* $(\vec{\alpha}, \vec{\beta}) \in \Omega$. From (2.7) and the determinantal representation for orthogonal polynomials (see (2.3)), this implies that the polynomials $\kappa_n^{-1} p_n$ and $\kappa_{n-1} p_{n-1}$ are analytic functions of $(\vec{\alpha}, \vec{\beta}) \in \Omega$ (since they exist, they can not have poles). Therefore, $R(z)$ is also an analytic function of $(\vec{\alpha}, \vec{\beta}) \in \Omega$ (some extra calculations are required to verify this at $z = t_1, ..., t_m$, which are similar to those in Subsection 5.1, and we omit them here). Furthermore, by (4.79), and since the asymptotic series of the confluent hypergeometric function that appears in the jumps for R (4.52) (see also (8.7)) is differentiable in α_k's and β_k's, the matrices $R^{(j)}$, $j = 1, ..., r$, are also analytic functions of $(\vec{\alpha}, \vec{\beta}) \in \Omega$. Hence, the terms in (4.74) are both uniform and analytic in $(\vec{\alpha}, \vec{\beta}) \in \Omega$. Furthermore, since factors of the forms $n^{\pm 2\beta_k}$, $k = 1, ..., m$, appear in the entries of $J_R^{(j)}$ for $j \geq 1$ (see (4.76) and the comment above), as $n \to \infty$ we have

$$\partial_{\beta_k} R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}} \log n), \qquad \partial_{\beta_k} R_R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{\max}} \log n), \tag{4.80}$$

and these asymptotics are also uniform in z, $\vec{\alpha}$, $\vec{\beta}$ and t_1,...,t_m as in (4.74).

Remark 4.1 *We will need the following further properties of R for particular values of the parameters V and W.*

(a) If V is replaced by V_s (see (1.35) for the definition of V_s) and W by 0, the asymptotics (4.74) also hold uniformly for $s \in [0,1]$. This follows from standard arguments for small-norm Riemann-Hilbert problems and a detailed analysis of the Cauchy operator associated to R, see e.g. [5, Lemma 4.35] for a similar situation. Furthermore, the determinantal representation for orthogonal polynomials implies that R is an analytic function of $s \in (0,1)$.

(b) If W is replaced by W_t (see (1.37) for the definition of W_t), the asymptotics (4.74) also hold uniformly for $t \in [0,1]$, see e.g. [5, Lemma 4.35] for a similar situation. Furthermore, the determinantal representation for orthogonal polynomials implies that R is analytic for $t \in [0,1]$.

The goal for the rest of this section is to compute $R^{(1)}(z)$ from (4.79). Note that the expression for $J_R^{(1)}(z)$ for $z \in \partial\mathcal{D}$ can be analytically continued on $\overline{\mathcal{D}}$, except at $-1, t_1, ..., t_m$ and 1, where the expressions in (4.52), (4.60) and (4.68) admit poles. These poles are of order 2 at -1 and 1, and of order 1 at t_k, $k = 1, ..., m$. Therefore, for z outside the disks, $R^{(1)}(z)$ is given by

$$\begin{aligned} R^{(1)}(z) \quad = \quad & \sum_{j=1}^{m} \frac{1}{z-t_j} \mathrm{Res}\big(J_R^{(1)}(s), s = t_j\big) \\ & + \frac{1}{z-1} \mathrm{Res}\big(J_R^{(1)}(s), s=1\big) + \frac{1}{(z-1)^2} \mathrm{Res}\big((s-1)J_R^{(1)}(s), s=1\big) \\ & + \frac{1}{z+1} \mathrm{Res}\big(J_R^{(1)}(s), s=-1\big) + \frac{1}{(z+1)^2} \mathrm{Res}\big((s+1)J_R^{(1)}(s), s=-1\big). \end{aligned} \tag{4.81}$$

We can compute these residues explicitly in the case $W \equiv 0$. For the residues at t_k, $k \in \{1, ..., m\}$, using (4.52), (4.45), (4.33) and (4.34), we have

$$\mathrm{Res}\big(J_R^{(1)}(z), z = t_k\big) = \frac{v_k D_\infty^{\sigma_3}}{2\pi\psi(t_k)(1-t_k^2)} \begin{pmatrix} t_k + \widetilde{\Lambda}_{I,k} & -i - i\widetilde{\Lambda}_{R,2,k} \\ -i + i\widetilde{\Lambda}_{R,1,k} & -t_k - \widetilde{\Lambda}_{I,k} \end{pmatrix} D_\infty^{-\sigma_3}, \tag{4.82}$$

where

$$\widetilde{\Lambda}_{I,k} = \frac{\tau(\alpha_k, \beta_k)\Lambda_k^2 - \tau(\alpha_k, -\beta_k)\Lambda_k^{-2}}{2i}, \tag{4.83}$$

$$\widetilde{\Lambda}_{R,1,k} = \frac{\tau(\alpha_k, \beta_k)\Lambda_k^2 e^{i \arcsin t_k} + \tau(\alpha_k, -\beta_k)\Lambda_k^{-2} e^{-i\arcsin t_k}}{2}, \tag{4.84}$$

$$\widetilde{\Lambda}_{R,2,k} = \frac{\tau(\alpha_k, \beta_k)\Lambda_k^2 e^{-i \arcsin t_k} + \tau(\alpha_k, -\beta_k)\Lambda_k^{-2} e^{i\arcsin t_k}}{2}. \tag{4.85}$$

Note the relation

$$\widetilde{\Lambda}_{R,1,k} - \widetilde{\Lambda}_{R,2,k} = -2t_k \widetilde{\Lambda}_{I,k}. \tag{4.86}$$

For the two residues at 1, using (4.60), (4.57), (4.35) and (4.36), we obtain

$$\mathrm{Res}\big((z-1)J_R^{(1)}(z), z=1\big) = \frac{5}{2^5 3\pi\psi(1)} D_\infty^{\sigma_3} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} D_\infty^{-\sigma_3}, \tag{4.87}$$

and

$$\mathrm{Res}\big(J_R^{(1)}(z), z=1\big) = \frac{D_\infty^{\sigma_3}}{2^5\pi\psi(1)} \times$$

$$\begin{pmatrix} -2(\mathcal{A}-\widetilde{\mathcal{B}_1})^2 + 1 + \frac{\psi'(1)}{\psi(1)} & 2i\left((\mathcal{A}-\widetilde{\mathcal{B}_1})^2 + 2(\mathcal{A}-\widetilde{\mathcal{B}_1}) + \frac{2}{3} - \frac{1}{2}\frac{\psi'(1)}{\psi(1)}\right) \\ 2i\left((\mathcal{A}-\widetilde{\mathcal{B}_1})^2 - 2(\mathcal{A}-\widetilde{\mathcal{B}_1}) + \frac{2}{3} - \frac{1}{2}\frac{\psi'(1)}{\psi(1)}\right) & 2(\mathcal{A}-\widetilde{\mathcal{B}_1})^2 - 1 - \frac{\psi'(1)}{\psi(1)} \end{pmatrix} D_\infty^{-\sigma_3}.$$

Finally, for the two residues at -1, using (4.68), (4.65), (4.37) and (4.38), we obtain

$$\mathrm{Res}\big((z+1)J_R^{(1)}(z), z=-1\big) = \frac{5}{2^5 3\pi\psi(-1)} D_\infty^{\sigma_3} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} D_\infty^{-\sigma_3}, \tag{4.88}$$

and

$$\mathrm{Res}\big(J_R^{(1)}(z), z=-1\big) = \frac{D_\infty^{\sigma_3}}{2^5\pi\psi(-1)} \times$$

$$\begin{pmatrix} 2(\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2 - 1 + \frac{\psi'(-1)}{\psi(-1)} & 2i\left((\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2 + 2(\mathcal{A}+\widetilde{\mathcal{B}}_{-1}) + \frac{2}{3} + \frac{1}{2}\frac{\psi'(-1)}{\psi(-1)}\right) \\ 2i\left((\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2 - 2(\mathcal{A}+\widetilde{\mathcal{B}}_{-1}) + \frac{2}{3} + \frac{1}{2}\frac{\psi'(-1)}{\psi(-1)}\right) & -2(\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2 + 1 - \frac{\psi'(-1)}{\psi(-1)} \end{pmatrix} D_\infty^{-\sigma_3}.$$

5 Integration in $\beta_1, ..., \beta_m$

In this section we use the differential identity (3.26) and the RH analysis of Section 4 only for $W \equiv 0$ and $V(x) = 2x^2$.

5.1 Computation of $\widetilde{Y}(t_k)$

For $z \in \mathcal{D}_{t_k}$, $k \in \{1, ..., m\}$, z outside the lenses and $z \in Q^R_{+,k}$, we invert the transformations $Y \mapsto T \mapsto S \mapsto R$ in order to express Y in terms of R and $P^{(t_k)}$,

$$Y(z) = e^{-\frac{n\ell}{2}\sigma_3} R(z) P^{(t_k)}(z) e^{ng(z)\sigma_3} e^{\frac{n\ell}{2}\sigma_3}, \tag{5.1}$$

with

$$P^{(t_k)}(z) = E_{t_k}(z) \Phi_{\mathrm{HG}}(nf_{t_k}(z); \alpha_k, \beta_k) e^{\frac{\pi i \alpha_k}{2}\sigma_3} (z - t_k)^{-\frac{\alpha_k}{2}\sigma_3} e^{-n\xi(z)\sigma_3} \omega_k(z)^{-\frac{\sigma_3}{2}}. \tag{5.2}$$

Note that for $z \in Q^R_{+,k}$, z outside the lenses, $nf_{t_k}(z) \in II$ (see Figure 4) and we can use (8.11)

$$\Phi_{\mathrm{HG}}(nf_{t_k}(z)) e^{\frac{i\pi\alpha_k}{4}\sigma_3} =$$
$$\begin{pmatrix} \frac{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)}{\Gamma(1+\alpha_k)} G(\frac{\alpha_k}{2} + \beta_k, \alpha_k; nf_{t_k}(z)) e^{-\frac{i\pi\alpha_k}{2}} & -\frac{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)}{\Gamma(\frac{\alpha_k}{2}+\beta_k)} H(1 + \frac{\alpha_k}{2} - \beta_k, \alpha_k; nf_{t_k}(z) e^{-\pi i}) \\ \frac{\Gamma(1+\frac{\alpha_k}{2}+\beta_k)}{\Gamma(1+\alpha_k)} G(1 + \frac{\alpha_k}{2} + \beta_k, \alpha_k; nf_{t_k}(z)) e^{-\frac{i\pi\alpha_k}{2}} & H(\frac{\alpha_k}{2} - \beta_k, \alpha_k; nf_{t_k}(z) e^{-\pi i}) \end{pmatrix}.$$

In [46, Section 13.14(iii)], using the relations (8.12), we can find asymptotics of these functions near the origin:

$$G(a, \alpha_k; z) = z^{\frac{\alpha_k}{2}}(1 + \mathcal{O}(z)), \qquad \text{as } z \to 0, \tag{5.3}$$

and, if $\alpha_k \neq 0$ and $a - \frac{\alpha_k}{2} \pm \frac{\alpha_k}{2} \neq 0, -1, -2, ...$,

$$H(a, \alpha_k; z) = \begin{cases} \frac{\Gamma(\alpha_k)}{\Gamma(a)} z^{-\frac{\alpha_k}{2}} + \mathcal{O}(z^{1-\frac{\Re\alpha_k}{2}}) + \mathcal{O}(z^{\frac{\Re\alpha_k}{2}}), & \text{if } \Re\alpha_k > 0, \\ \frac{\Gamma(-\alpha_k)}{\Gamma(a-\alpha_k)} z^{\frac{\alpha_k}{2}} + \frac{\Gamma(\alpha_k)}{\Gamma(a)} z^{-\frac{\alpha_k}{2}} + \mathcal{O}(z^{1+\frac{\Re\alpha_k}{2}}), & \text{if } -1 < \Re\alpha_k \leq 0. \end{cases} \tag{5.4}$$

Note that conditions $a - \frac{\alpha_k}{2} \pm \frac{\alpha_k}{2} \neq 0, -1, -2, ...$ are always satisfied for $a = \frac{\alpha_k}{2} - \beta_k$ and $a = 1 + \frac{\alpha_k}{2} - \beta_k$, since $\Re\beta_k \in \left(\frac{-1}{4}, \frac{1}{4}\right)$ and $\Re\alpha_k > -1$. Therefore, using (4.45), the leading terms in $\Phi_{\mathrm{HG}}(nf_{t_k}(z)) e^{\frac{\pi i \alpha_k}{2}\sigma_3} (z - t_k)^{-\frac{\alpha_k}{2}\sigma_3} \omega_k(z)^{-\frac{\sigma_3}{2}}$ as $z \to t_k$ for $-1 < \Re\alpha_k \leq 0$, $\alpha_k \neq 0$ are given by

$$\begin{pmatrix} \Phi_{k,11} & \alpha_k^{-1}\left(\Phi_{k,12} + \widetilde{c}_k \Phi_{k,11}(z - t_k)^{\alpha_k}\right) \\ \Phi_{k,21} & \alpha_k^{-1}\left(\Phi_{k,22} + \widetilde{c}_k \Phi_{k,21}(z - t_k)^{\alpha_k}\right) \end{pmatrix}, \qquad \widetilde{c}_k = \frac{\alpha_k \Gamma(1 + \alpha_k)\Gamma(-\alpha_k) e^{-\frac{\pi i \alpha_k}{2}} \omega_k(t_k)}{\Gamma(-\frac{\alpha_k}{2} - \beta_k)\Gamma(1 + \frac{\alpha_k}{2} + \beta_k)}, \tag{5.5}$$

and

$$\Phi_{k,11} = \frac{\Gamma(1 + \frac{\alpha_k}{2} - \beta_k)}{\Gamma(1 + \alpha_k)} \left(4n\sqrt{1 - t_k^2}\right)^{\frac{\alpha_k}{2}} \omega_k(t_k)^{-\frac{1}{2}}, \quad \Phi_{k,12} = \frac{-\alpha_k \Gamma(\alpha_k)}{\Gamma(\frac{\alpha_k}{2} + \beta_k)} \left(4n\sqrt{1 - t_k^2}\right)^{-\frac{\alpha_k}{2}} \omega_k(t_k)^{\frac{1}{2}},$$
$$\Phi_{k,21} = \frac{\Gamma(1 + \frac{\alpha_k}{2} + \beta_k)}{\Gamma(1 + \alpha_k)} \left(4n\sqrt{1 - t_k^2}\right)^{\frac{\alpha_k}{2}} \omega_k(t_k)^{-\frac{1}{2}}, \quad \Phi_{k,22} = \frac{\alpha_k \Gamma(\alpha_k)}{\Gamma(\frac{\alpha_k}{2} - \beta_k)} \left(4n\sqrt{1 - t_k^2}\right)^{-\frac{\alpha_k}{2}} \omega_k(t_k)^{\frac{1}{2}}. \tag{5.6}$$

By the connection formula $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$, we can rewrite $\widetilde{c}_k$ as

$$\widetilde{c}_k = \frac{\pi\alpha_k}{\sin(\pi\alpha_k)} \frac{e^{i\pi\beta_k} - e^{-i\pi\alpha_k} e^{-i\pi\beta_k}}{2\pi i} \omega_k(t_k). \tag{5.7}$$

If $\Re\alpha_k > 0$, the same expression (5.5) is valid, except that $\widetilde{c}_k$ has to be replaced by 0. Therefore, by (3.8), (3.9), (3.22), (4.11), (4.74) and (5.1), we obtain as $n \to \infty$, for $\alpha_k \neq 0$,

$$\widetilde{Y}(t_k) = e^{-\frac{n\ell}{2}\sigma_3} \left(I + \mathcal{O}(n^{-1+2\beta_{\max}})\right) E_{t_k}(t_k) \begin{pmatrix} \Phi_{k,11} & \Phi_{k,12} \\ \Phi_{k,21} & \Phi_{k,22} \end{pmatrix} e^{nt_k^2\sigma_3}. \tag{5.8}$$

The case of $\alpha_k = 0$ is simpler. We can obtain $\widetilde{Y}(t_k)$ directly from (3.22) and (3.6). It suffices to take the limit $\alpha_k \to 0$ in (5.8) and to note that $\lim_{x\to 0} x\Gamma(x) = 1$ (see e.g. [46, Chapter 5]). The goal for the rest of this section is to prove the following.

Proposition 5.1 *As $n \to \infty$,*

$$\begin{aligned}\log \frac{D_n(\vec{\alpha},\vec{\beta},2x^2,0)}{D_n(\vec{\alpha},\vec{0},2x^2,0)} \quad = \quad & 2in\sum_{j=1}^m \left(\arcsin t_j + t_j\sqrt{1-t_j^2}\right)\beta_j + i\mathcal{A}\sum_{j=1}^m \beta_j \arcsin t_j \\ & -\frac{i\pi}{2}\sum_{j=1}^m \beta_j \mathcal{A}_j - \sum_{j=1}^m \beta_j^2 \log\left(8n(1-t_j^2)^{3/2}\right) + \sum_{1\le j<k\le m} \log T_{jk}^{2\beta_j\beta_k} \\ & + \sum_{j=1}^m \log\left(\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\frac{\alpha_j}{2})^2}\right) + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right),\end{aligned}$$

where $\mathcal{A}_k = \sum_{j=1}^{k-1}\alpha_j - \sum_{j=k+1}^m \alpha_j$.

By (2.5) and (2.7), we have

$$\kappa_{n-1}^2 = \lim_{z\to\infty} \frac{iY_{21}(z)}{2\pi z^{n-1}}, \quad \eta_n = \lim_{z\to\infty}\frac{Y_{11}(z)-z^n}{z^{n-1}}, \quad \gamma_n = \lim_{n\to\infty}\frac{Y_{11}(z)-z^n-\eta_n z^{n-1}}{z^{n-2}}. \tag{5.9}$$

For z outside the lenses and outside the disks, by inverting the transformations, we can express Y in terms of R and $P^{(\infty)}$, one has

$$Y(z) = e^{-\frac{n\ell}{2}\sigma_3}R(z)P^{(\infty)}(z)e^{ng(z)\sigma_3}e^{\frac{n\ell}{2}\sigma_3}. \tag{5.10}$$

From (4.15), (4.39), (4.74) and (4.81), we obtain

$$\kappa_{n-1}^2 = e^n 2^{2(n-1)+\mathcal{A}}\pi^{-1}\exp\left(-2i\sum_{j=1}^m \beta_j\arcsin t_j\right)\left(1+\frac{R_{1,21}^{(1)}}{nP_{1,21}^{(\infty)}} + \mathcal{O}(n^{-2+2\beta_{\max}})\right), \tag{5.11}$$

where the notation $R_{k,ij}^{(1)}$ refers to the z^{-k} coefficient in the large z asymptotics of $R_{ij}^{(1)}$. More explicitly, combining (4.39) and (4.81), we find

$$\frac{R_{1,21}^{(1)}}{P_{1,21}^{(\infty)}} = \sum_{j=1}^m \frac{v_j(1-\widetilde{\Lambda}_{R,1,j})}{2(1-t_j^2)} - \frac{1}{8}\left(\frac{2}{3} - 2\mathcal{A} + \mathcal{A}^2 + (1-\mathcal{A})(\widetilde{\mathcal{B}}_1 - \widetilde{\mathcal{B}}_{-1}) + \frac{\widetilde{\mathcal{B}}_1^2 + \widetilde{\mathcal{B}}_{-1}^2}{2}\right). \tag{5.12}$$

The asymptotics for κ_n^2 are a bit more delicate to obtain than simply replace n by $n+1$ in (5.11), because the weight w depends also on n. By writing explicitly the dependence of the weight w in the parameters n, t_1,...,t_m, we have the relation

$$w\left(\sqrt{\frac{n+1}{n}}x; n, t_1, ..., t_m\right) = \left(\frac{n+1}{n}\right)^{\frac{\mathcal{A}}{2}} w\left(x; n+1, \sqrt{\frac{n}{n+1}}t_1, ..., \sqrt{\frac{n}{n+1}}t_m\right). \tag{5.13}$$

From (5.13) and with a change of variable in the orthogonality conditions (2.1), we see that to get large n asymptotics for κ_n^2, first we have to replace n by $n+1$ in (5.11), then we multiply the result by $\left(\frac{n}{n+1}\right)^{n+\frac{1+\mathcal{A}}{2}}$ and finally we replace each t_k by $\sqrt{\frac{n}{n+1}}t_k$. We find

$$\kappa_n^2 = e^n 2^{2n+\mathcal{A}}\pi^{-1}\exp\left(-2i\sum_{j=1}^m \beta_j\arcsin t_j\right)\left(1+\frac{R_{1,21}^{(1)}}{nP_{1,21}^{(\infty)}} - \frac{\mathcal{A}}{2n} + \sum_{j=1}^m \frac{it_j\beta_j}{\sqrt{1-t_j^2}n} + \mathcal{O}(n^{-2+2\beta_{\max}})\right).$$

By (5.9), (5.10), (4.15), (4.39) and (4.81), we have as $n \to \infty$

$$\eta_n = P_{1,11}^{(\infty)} + \frac{R_{1,11}^{(1)}}{n} + \mathcal{O}(n^{-2+2\beta_{\max}}). \tag{5.14}$$

From the same equations, and with increasing effort, we obtain large n asymptotics for γ_n,

$$\gamma_n = -\frac{n}{8} + P^{(\infty)}_{2,11} + \frac{1}{n}\left[P^{(\infty)}_{1,11}R^{(1)}_{1,11} + R^{(1)}_{2,11} + P^{(\infty)}_{1,21}R^{(1)}_{1,12}\right] + \mathcal{O}(n^{-2+2\beta_{\max}}). \tag{5.15}$$

Taking into account that $R^{(1)}_{1,11} = \mathcal{O}(n^{2\beta_{\max}})$ as $n \to \infty$, we see that

$$\gamma_n - \frac{\eta_n^2}{2} = -\frac{n}{8} + P^{(\infty)}_{2,11} - \frac{(P^{(\infty)}_{1,11})^2}{2} + \frac{1}{n}\left[R^{(1)}_{2,11} + P^{(\infty)}_{1,21}R^{(1)}_{1,12}\right] + \mathcal{O}(n^{-2+4\beta_{\max}}), \qquad \text{as } n \to \infty. \tag{5.16}$$

These terms are made explicit using (4.39), (4.40) and (4.81):

$$\begin{aligned}
P^{(\infty)}_{2,11} - \frac{(P^{(\infty)}_{1,11})^2}{2} &= \frac{1}{8} - \frac{1}{8}\sum_{j=1}^{m}\alpha_j(1-2t_j^2) + \frac{i}{2}\sum_{j=1}^{m} t_j\sqrt{1-t_j^2}\,\beta_j,\\
R^{(1)}_{2,11} &= \sum_{j=1}^{m}\frac{t_j v_j(t_j + \widetilde{\Lambda}_{I,j})}{4(1-t_j^2)} - \frac{1}{2^5}\left(\frac{2}{3} + (\mathcal{A} - \widetilde{\mathcal{B}}_1)^2 + (\mathcal{A} + \widetilde{\mathcal{B}}_{-1})^2\right),\\
P^{(\infty)}_{1,21}R^{(1)}_{1,12} &= -\sum_{j=1}^{m}\frac{v_j(1 + \widetilde{\Lambda}_{R,2,j})}{8(1-t_j^2)} + \frac{1}{2^5}\left(\frac{2}{3} + 2\mathcal{A} + \mathcal{A}^2 - (1+\mathcal{A})(\widetilde{\mathcal{B}}_1 - \widetilde{\mathcal{B}}_{-1}) + \frac{\widetilde{\mathcal{B}}_1^2 + \widetilde{\mathcal{B}}_{-1}^2}{2}\right).
\end{aligned}$$

We are now able to compute the differential identity (3.26). After some calculations (there is a lot of cancellations), using (4.86) (and (4.80) for the $o(1)$ term), we obtain

$$\begin{aligned}
-(n+\mathcal{A})\partial_{\beta_k}\log(\kappa_n\kappa_{n-1}) - 2n\partial_{\beta_k}\left(\frac{\kappa_{n-1}^2}{\kappa_n^2}\right) + 4n\partial_{\beta_k}\left(\gamma_n - \frac{\eta_n^2}{2}\right) &=\\
2in\left(\arcsin t_k + t_k\sqrt{1-t_k^2}\right) + 2i\mathcal{A}\arcsin t_k - 2\beta_k + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right)&.
\end{aligned} \tag{5.17}$$

The second part of the differential identity is obtained using (5.8):

$$\begin{aligned}
&\sum_{j=1}^{m}\left[\widetilde{Y}_{22}(t_j)\partial_{\beta_k}Y_{11}(t_j) - \widetilde{Y}_{12}(t_j)\partial_{\beta_k}Y_{21}(t_j) + Y_{11}(t_j)\widetilde{Y}_{22}(t_j)\partial_{\beta_k}\log(\kappa_n\kappa_{n-1})\right] = \qquad (5.18)\\
&= -\mathcal{A}\partial_{\beta_k}\log D_\infty + \sum_{j=1}^{m}\left(\partial_{\beta_k}\Phi_{j,11}\Phi_{j,22} - \Phi_{j,12}\partial_{\beta_k}\Phi_{j,21} - 2\beta_j\partial_{\beta_k}\log\Lambda_j\right) + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right).
\end{aligned}$$

These terms have more explicit forms using (4.32), (4.50) and (5.6):

$$-2\sum_{j=1}^{m}\beta_j\partial_{\beta_k}\log\Lambda_j = -2\beta_k\log(8n(1-t_k^2)^{3/2}) + \sum_{j\neq k}\log T_{jk}^{2\beta_j}, \tag{5.19}$$

$$\begin{aligned}
\sum_{j=1}^{m}\left(\partial_{\beta_k}\Phi_{j,11}\Phi_{j,22} - \Phi_{j,12}\partial_{\beta_k}\Phi_{j,21}\right) &= -\frac{i\pi}{2}\mathcal{A}_j \qquad (5.20)\\
&+ \beta_k\partial_{\beta_k}\log\frac{\Gamma(1+\frac{\alpha_k}{2}+\beta_k)}{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)} + \frac{\alpha_k}{2}\partial_{\beta_k}\log\left(\Gamma(1+\frac{\alpha_k}{2}+\beta_k)\Gamma(1+\frac{\alpha_k}{2}-\beta_k)\right),
\end{aligned}$$

and $\mathcal{A}_j$ is defined in Proposition 5.1. Finally, summing (5.17) and (5.18), we obtain

$$\begin{aligned}
\partial_{\beta_k}\log D_n(\vec{\alpha},\vec{\beta},2x^2,0) &= 2in\Big(\arcsin t_k + t_k\sqrt{1-t_k^2}\Big) + i\mathcal{A}\arcsin t_k - \frac{i\pi}{2}\mathcal{A}_k - 2\beta_k + \sum_{j\neq k}\log T_{jk}^{2\beta_j}\\
&- 2\beta_k\log\left(8n(1-t_k^2)^{3/2}\right) + \frac{\alpha_k}{2}\partial_{\beta_k}\log\left(\Gamma(1+\frac{\alpha_k}{2}+\beta_k)\Gamma(1+\frac{\alpha_k}{2}-\beta_k)\right)\\
&+ \beta_k\partial_{\beta_k}\log\frac{\Gamma(1+\frac{\alpha_k}{2}+\beta_k)}{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)} + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right).
\end{aligned} \tag{5.21}$$

For convenience, we denote $F_{1,n}(\vec{\alpha},\vec{\beta})$ for the r.h.s. of (3.26), which can be expressed in terms of R and the parametrices. From Subsection 4.8, if Ω is a compact subset of $\mathcal{P}_\alpha \times \mathcal{P}_\beta^{(\frac{1}{4})}$, there exists $n_\star$ depending only on Ω, such that $F_{1,n}(\vec{\alpha},\vec{\beta})$ exists for *all* $n \geq n_\star$ and for *all* $(\vec{\alpha},\vec{\beta}) \in \Omega$, and is analytic for $(\vec{\alpha},\vec{\beta}) \in \Omega$. Furthermore, the large n asymptotics for $F_{1,n}(\vec{\alpha},\vec{\beta})$ given by the r.h.s. of (5.21) are uniform for *all* $(\vec{\alpha},\vec{\beta}) \in \Omega$. Nevertheless, the identity (5.21) itself is valid only for $(\vec{\alpha},\vec{\beta}) \in \Omega \setminus \widetilde{\Omega}_1^{(n)}$, where $\widetilde{\Omega}_1^{(n)}$ consists of at most a finite number of points (see the beginning of Section 3).

Now, we show how to extend this differential identity for *all* $(\vec{\alpha},\vec{\beta}) \in \Omega$, following [42, 37, 23]. First, we use the identity (5.21) with $k = 1$, $\beta_2 = ... = \beta_m = 0$ with $\vec{\alpha}$ fixed. We assume that $(\vec{\alpha},\vec{0}) \in \Omega$, we write $\vec{\beta}_1 = (\beta_1, 0, ..., 0)$ and we define

$$H(\beta_1) = D_n(\vec{\alpha},\vec{\beta}_1, 2x^2, 0) \exp\bigg(- \int_0^{\beta_1} F_{1,n}(\vec{\alpha},(s_1,0,...,0))ds_1\bigg). \tag{5.22}$$

Equation (5.21) is the statement that $H'(\beta_1) = 0$ for all β_1 such that $(\vec{\alpha},\vec{\beta}_1) \in \Omega \setminus \widetilde{\Omega}_1^{(n)}$. Since the large n asymptotics for $F_{1,n}(\vec{\alpha},\vec{\beta})$ given by (5.21) hold uniformly and continuously for all β_1 such that $(\vec{\alpha},\vec{\beta}_1) \in \Omega$, H is continuously differentiable for all β_1 such that $(\vec{\alpha},\vec{\beta}_1) \in \Omega$ (for $n \geq n_\star$). Thus by continuity, since $\widetilde{\Omega}_1^{(n)}$ consists of at most a finite number of points, one has in fact that $H'(\beta_1) = 0$ for all β_1 such that $(\vec{\alpha},\vec{\beta}_1) \in \Omega$ (for $n \geq n_\star$). Moreover, as $H(0) = D_n(\vec{\alpha},\vec{0}, 2x^2, 0) \neq 0$ (if n is chosen sufficiently large, see (1.9) and (1.13)), the determinant $D_n(\vec{\alpha},\vec{\beta}_1, 2x^2, 0)$ is never zero. Hence the identity (5.21) for $k = 1$ holds for all β_1 such that $(\vec{\alpha},\vec{\beta}_1) \in \Omega$ (for sufficiently large n). Integrating this identity in β_1 gives, as $n \to \infty$

$$\begin{aligned}\log \frac{D_n(\vec{\alpha},(\beta_1,0,...,0),2x^2,0)}{D_n(\vec{\alpha},(0,0,...,0),2x^2,0)} &= 2in\Big(\arcsin t_1 + t_1\sqrt{1-t_1^2}\Big)\beta_1 + i\mathcal{A}\beta_1 \arcsin t_1 - \frac{i\pi}{2}\mathcal{A}_1\beta_1 \\ &- \beta_1^2 \log\big(8n(1-t_1^2)^{3/2}\big) - \beta_1^2 + \frac{\alpha_1}{2}\log\frac{\Gamma(1+\frac{\alpha_1}{2}+\beta_1)\Gamma(1+\frac{\alpha_1}{2}-\beta_1)}{\Gamma(1+\frac{\alpha_1}{2})^2} \\ &+ \int_0^{\beta_1} x\partial_x \log\frac{\Gamma(1+\frac{\alpha_1}{2}+x)}{\Gamma(1+\frac{\alpha_1}{2}-x)}dx + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right).\end{aligned} \tag{5.23}$$

Since Ω is arbitrary, (5.23) is valid for any β_1 such that $\Re\beta_1 \in (\frac{-1}{4}, \frac{1}{4})$. This equation can be simplified using the known formula (see [46, formula 5.17.4])

$$\int_0^z \log\Gamma(1+x)dx = \frac{z}{2}\log 2\pi - \frac{z(z+1)}{2} + z\log\Gamma(z+1) - \log G(z+1), \tag{5.24}$$

where G is Barnes' G-function. After integrations by parts, one has

$$\begin{aligned}\int_0^{\beta_1} x\partial_x \log\frac{\Gamma(1+\frac{\alpha_1}{2}+x)}{\Gamma(1+\frac{\alpha_1}{2}-x)}dx &= \beta_1^2 - \frac{\alpha_1}{2}\log\frac{\Gamma(1+\frac{\alpha_1}{2}+\beta_1)\Gamma(1+\frac{\alpha_1}{2}-\beta_1)}{\Gamma(1+\frac{\alpha_1}{2})^2} \\ &+ \log\frac{G(1+\frac{\alpha_1}{2}+\beta_1)G(1+\frac{\alpha_1}{2}-\beta_1)}{G(1+\frac{\alpha_1}{2})^2}.\end{aligned} \tag{5.25}$$

Therefore (5.23) can be rewritten as

$$\begin{aligned}\log \frac{D_n(\alpha,(\beta_1,0,...,0),2x^2,0)}{D_n(\alpha,(0,0,...,0),2x^2,0)} &= 2in\Big(\arcsin t_1 + t_1\sqrt{1-t_1^2}\Big)\beta_1 + i\mathcal{A}\beta_1 \arcsin t_1 - \frac{i\pi}{2}\mathcal{A}_1\beta_1 \\ - \beta_1^2 \log\big(8n(1-t_1^2)^{3/2}\big) &+ \log\frac{G(1+\frac{\alpha_1}{2}+\beta_1)G(1+\frac{\alpha_1}{2}-\beta_1)}{G(1+\frac{\alpha_1}{2})^2} + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right).\end{aligned} \tag{5.26}$$

Now, we use (5.21) for $k = 2$, where we fix β_1 and $\vec{\alpha}$, and we set $\beta_3 = ... = \beta_m = 0$. For convenience, we write $\vec{\beta}_2 = (\beta_1, \beta_2, 0, ..., 0)$. Equation (5.21) is valid for all β_2 such that

$(\vec{\alpha}, \vec{\beta}_2) \in \Omega \setminus \Omega_1^{(n)}$. We can extend it to all β_2 such that $(\vec{\alpha}, \vec{\beta}_2) \in \Omega$ with a similar argument to the one done below equation (5.21). After an integration in β_2, we obtain a similar formula to (5.26) but with an extra term

$$\log \frac{D_n(\alpha, (\beta_1, \beta_2, 0, ..., 0), 2x^2, 0)}{D_n(\alpha, (\beta_1, 0, ..., 0), 2x^2, 0)} = 2in\Big(\arcsin t_2 + t_2\sqrt{1-t_2^2}\Big)\beta_2 + i\mathcal{A}\beta_2 \arcsin t_2 + \log T_{12}^{2\beta_1\beta_2}$$
$$- \frac{i\pi}{2}\mathcal{A}_2\beta_2 - \beta_2^2 \log\left(8n(1-t_2^2)^{3/2}\right) + \log \frac{G(1+\frac{\alpha_2}{2}+\beta_2)G(1+\frac{\alpha_2}{2}-\beta_2)}{G(1+\frac{\alpha_2}{2})^2} + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right).$$

Again, by freedom in the choice of Ω, the above expansion is valid for any β_2 such that $\Re\beta_2 \in (\frac{-1}{4}, \frac{1}{4})$. We can proceed in the same way recursively for each β_k, $k = 1, ..., m$. After m integrations, it suffices to sum these identities to obtain Proposition 5.1.

6 Integration in V

In this section, we will use the RH analysis done in Section 4 with $W \equiv 0$ and where V is replaced by V_s, defined in (1.35) and we will make explicit the dependence of the weight in s:

$$w_s(x) = e^{-nV_s(x)}\omega_\alpha(x)\omega_\beta(x). \tag{6.1}$$

In the RH analysis, we also have to replace the Euler-Lagrange constant ℓ by ℓ_s and the function ψ by ψ_s, see (1.36). We will also use the differential identity

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \frac{1}{2\pi i}\int_{\mathbb{R}} [Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x)dx, \tag{6.2}$$

which was obtained in (3.29), and is valid only when $D_k^{(n)}(\vec{\alpha}, \vec{\beta}, V_s, W) \neq 0$ for all $k = 1, 2, ..., n+1$, i.e. for $(\vec{\alpha}, \vec{\beta}, s) \in \widetilde{\mathcal{P}}_2^{(n)}$ (see also the discussion at the beginning of Section 3). Some calculations in this section and the next one are similar to those done in [5], in which the authors put great effort in showing all the details, so we will sometimes refer to equations and lemmas in their paper. Our goal is to prove the following.

Proposition 6.1 *As $n \to \infty$, we have*

$$\log \frac{D_n(\vec{\alpha}, \vec{\beta}, V, 0)}{D_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0)} = -\frac{n^2}{2}\int_{-1}^{1}\sqrt{1-x^2}\big(V(x)-2x^2\big)\Big(\frac{2}{\pi}+\psi(x)\Big)dx + n\sum_{j=1}^{m}\frac{\alpha_j}{2}\left(V(t_j)-2t_j^2\right)$$
$$- \frac{n\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x)-2x^2}{\sqrt{1-x^2}}dx - 2\pi n\sum_{j=1}^{m} i\beta_j\int_{t_j}^{1}\Big(\psi(x)-\frac{2}{\pi}\Big)\sqrt{1-x^2}dx$$
$$- \sum_{j=1}^{m}\left(\beta_j^2 - \frac{\alpha_j^2}{4}\right)\log\Big(\frac{\pi}{2}\psi(t_j)\Big) - \frac{1}{24}\log\left(\frac{\pi^2}{4}\psi(1)\psi(-1)\right) + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big).$$

Using the jump relations (2.8) of Y, the differential identity (6.2) becomes

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \int_{\mathbb{R}\setminus[-1-\epsilon, 1+\epsilon]} [Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x)\frac{dx}{2\pi i} - \frac{1}{2\pi i}\int_{\mathcal{C}} [Y^{-1}(z)Y'(z)]_{11}\partial_s \log w_s(z)dz, \tag{6.3}$$

where $\epsilon > 0$ is fixed and $\mathcal{C}$ is a closed curve surrounding $[-1, 1]$ and the lenses $\gamma_+ \cup \gamma_-$, is oriented clockwise, and passes through $-1-\epsilon$ and $1+\epsilon$.
For z outside the lenses, similarly to (5.10), we have

$$Y(z) = e^{-\frac{n\ell_s}{2}\sigma_3}R(z)P^{(\infty)}(z)e^{ng(z)\sigma_3}e^{\frac{n\ell_s}{2}\sigma_3}, \tag{6.4}$$

and thus, by (1.12) and (4.5), one has

$$[Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x) = \mathcal{O}(e^{-cn}), \qquad \text{as } n \to \infty, \tag{6.5}$$

uniformly for $x \in \mathbb{R} \setminus [-1-\epsilon, 1+\epsilon]$ and $s \in [0,1]$ (see Remark 4.1 (a)), where $c > 0$ is a fixed constant. We can explicitly compute $[Y^{-1}(z)Y'(z)]_{11}$ using (6.4). Thus, equation (6.3) becomes as $n \to \infty$

$$\begin{aligned} \partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0) &= I_{1,s} + I_{2,s} + I_{3,s} + \mathcal{O}(e^{-cn}), \\ I_{1,s} &= \frac{-n}{2\pi i} \int_{\mathcal{C}} g'(z) \partial_s \log w_s(z) dz, \\ I_{2,s} &= \frac{-1}{2\pi i} \int_{\mathcal{C}} [P^{(\infty)}(z)^{-1} P^{(\infty)}(z)']_{11} \partial_s \log w_s(z) dz, \\ I_{3,s} &= \frac{-1}{2\pi i} \int_{\mathcal{C}} [P^{(\infty)}(z)^{-1} R^{-1}(z) R'(z) P^{(\infty)}(z)]_{11} \partial_s \log w_s(z) dz. \end{aligned} \tag{6.6}$$

Since $\partial_s \log w(z) = -n \partial_s V_s(z)$, by (4.74), we have that

$$I_{1,s} = \mathcal{O}(n^2), \qquad I_{2,s} = \mathcal{O}(n), \qquad I_{3,s} = \mathcal{O}(n^{2\beta_{\max}}), \qquad \text{as } n \to \infty. \tag{6.7}$$

More detailed calculations will later show that we actually have $\int_0^1 I_{3,s} ds = \mathcal{O}(1)$ as $n \to \infty$, due to the fact that $I_{3,s}$ is highly oscillatory in s as $n \to \infty$. By (4.7), a direct calculation shows that

$$I_{1,s} = -n^2 \int_{-1}^{1} \big(V(x) - 2x^2\big) \psi_s(x) \sqrt{1-x^2} dx, \tag{6.8}$$

and therefore

$$\int_0^1 I_{1,s} ds = -\frac{n^2}{2} \int_{-1}^{1} \sqrt{1-x^2} \big(V(x) - 2x^2\big) \Big(\frac{2}{\pi} + \psi(x)\Big) dx. \tag{6.9}$$

From (4.28), we have $[P^{(\infty)}(z)^{-1} P^{(\infty)}(z)']_{11} = -\partial_z \log D(z)$, and thus $I_{2,s} = I_{2,s,\alpha} + I_{2,s,\beta}$, with

$$I_{2,s,\alpha} = \frac{-n}{2\pi i} \int_{\mathcal{C}} \partial_z \log D_\alpha(z) \partial_s V_s(z) dz, \qquad I_{2,s,\beta} = \frac{-n}{2\pi i} \int_{\mathcal{C}} \partial_z \log D_\beta(z) \partial_s V_s(z) dz. \tag{6.10}$$

Note that since $\partial_s V_s(x) = V(x) - 2x^2$, $I_{2,s}$ is in fact independent of s. We can compute $\partial_z \log D(z)$ using the explicit form of (4.30) and (4.31). This gives

$$\partial_z \log D_\alpha(z) = -\frac{\mathcal{A}}{2} \frac{1}{\sqrt{z^2-1}} + \sum_{j=1}^{m} \frac{\alpha_j}{2} \frac{1}{z - t_j}, \qquad \partial_z \log D_\beta(z) = \sum_{j=1}^{m} \frac{i\beta_j \sqrt{1-t_j^2}}{\sqrt{z^2-1}(z-t_j)}. \tag{6.11}$$

By a contour deformation of $\mathcal{C}$, we obtain

$$I_{2,s,\alpha} = n \sum_{j=1}^{m} \frac{\alpha_j}{2} \big(V(t_j) - 2t_j^2\big) - \frac{n\mathcal{A}}{2\pi} \int_{-1}^{1} \frac{V(x) - 2x^2}{\sqrt{1-x^2}} dx, \tag{6.12}$$

and

$$I_{2,s,\beta} = n \sum_{j=1}^{m} \frac{i\beta_j}{\pi} \sqrt{1-t_j^2} \fint_{-1}^{1} \frac{V(x) - 2x^2}{\sqrt{1-x^2}(x-t_j)} dx. \tag{6.13}$$

By [5, equation (5.17)], this principal value integral can be a bit simplified:

$$I_{2,s,\beta} = -2\pi n \sum_{j=1}^{m} i\beta_j \int_{t_j}^{1} \Big(\psi(x) - \frac{2}{\pi}\Big) \sqrt{1-x^2} dx. \tag{6.14}$$

Integrating (6.12) and (6.14) in s, we get

$$\begin{aligned} \int_0^1 I_{2,s} ds &= n \sum_{j=1}^{m} \frac{\alpha_j}{2} \big(V(t_j) - 2t_j^2\big) - \frac{n\mathcal{A}}{2\pi} \int_{-1}^{1} \frac{V(x) - 2x^2}{\sqrt{1-x^2}} dx \\ &\quad -2\pi n \sum_{j=1}^{m} i\beta_j \int_{t_j}^{1} \Big(\psi(x) - \frac{2}{\pi}\Big) \sqrt{1-x^2} dx. \end{aligned} \tag{6.15}$$

An explicit form of the asymptotics for $I_{3,s}$ requires more work, as it involves $R^{(1)}$. By (4.74) and Remark 4.1 (a), $I_{3,s}$ can be rewritten as

$$I_{3,s}=\frac{1}{2\pi i}\int_{\mathcal{C}}[P^{(\infty)}(z)^{-1}R^{(1)}(z)'P^{(\infty)}(z)]_{11}\partial_s V_s(z)dz+\mathcal{O}\big(n^{-1+4\beta_{\max}}\big),\qquad \text{as } n\to\infty, \tag{6.16}$$

uniformly for $(\vec{\alpha},\vec{\beta},s)\in\Omega\times[0,1]$, where Ω is an arbitrary compact subset of $\mathcal{P}_\alpha\times\mathcal{P}_\beta^{(\frac{1}{4})}$. Using (4.28), it becomes, as $n\to\infty$

$$\begin{aligned}I_{3,s}=\frac{1}{2\pi i}\int_{\mathcal{C}}&\bigg(\frac{a(z)^2+a(z)^{-2}}{4}[R_{11}^{(1)}(z)'-R_{22}^{(1)}(z)']+\frac{1}{2}[R_{11}^{(1)}(z)'+R_{22}^{(1)}(z)']\\&+i\frac{a(z)^{-2}-a(z)^2}{4}[R_{12}^{(1)}(z)'D_\infty^{-2}+R_{21}^{(1)}(z)'D_\infty^{2}]\bigg)\big(V(z)-2z^2\big)dz+\mathcal{O}\big(n^{-1+4\beta_{\max}}\big).\end{aligned}\tag{6.17}$$

From (4.81) and below, we have

$$\begin{aligned}&R_{11}^{(1)\prime}(z)-R_{22}^{(1)\prime}(z)=\sum_{j=1}^m\frac{1}{(z-t_j)^2}\frac{v_j(-2t_j-2\widetilde{\Lambda}_{I,j})}{2\pi\psi_s(t_j)(1-t_j^2)}+\frac{1}{(z-1)^3}\frac{5}{2^3 3\pi\psi_s(1)}\\&+\frac{1}{(z-1)^2}\frac{(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2-\frac{1}{2}-\frac{1}{2}\frac{\psi_s'(1)}{\psi_s(1)}}{2^3\pi\psi_s(1)}+\frac{1}{(z+1)^2}\frac{-(\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2+\frac{1}{2}-\frac{1}{2}\frac{\psi_s'(-1)}{\psi_s(-1)}}{2^3\pi\psi_s(-1)}+\frac{1}{(z+1)^3}\frac{5}{2^3 3\pi\psi_s(-1)},\\&R_{11}^{(1)\prime}(z)+R_{22}^{(1)\prime}(z)=0,\\&i[R_{12}^{(1)\prime}(z)D_\infty^{-2}+R_{21}^{(1)\prime}(z)D_\infty^{2}]=\sum_{j=1}^m\frac{1}{(z-t_j)^2}\frac{v_j(-2+\widetilde{\Lambda}_{R,1,j}-\widetilde{\Lambda}_{R,2,j})}{2\pi\psi_s(t_j)(1-t_j^2)}+\frac{1}{(z-1)^3}\frac{5}{2^3 3\pi\psi_s(1)}\\&+\frac{1}{(z-1)^2}\frac{(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2+\frac{2}{3}-\frac{1}{2}\frac{\psi_s'(1)}{\psi_s(1)}}{2^3\pi\psi_s(1)}+\frac{1}{(z+1)^2}\frac{(\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2+\frac{2}{3}+\frac{1}{2}\frac{\psi_s'(-1)}{\psi_s(-1)}}{2^3\pi\psi_s(-1)}+\frac{1}{(z+1)^3}\frac{-5}{2^3 3\pi\psi_s(-1)}.\end{aligned}$$

Plugging these equations in (6.17), we can split $I_{3,s}$ into $m+2$ integrals and an error term:

$$I_{3,s}=\sum_{j=1}^m I_{3,s,t_j}+I_{3,s,1}+I_{3,s,-1}+\mathcal{O}\big(n^{-1+4\beta_{\max}}\big),\qquad \text{as } n\to\infty, \tag{6.18}$$

where

$$\begin{aligned}&I_{3,s,t_k}=\frac{-v_k}{8\pi^2\psi_s(t_k)\sqrt{1-t_k^2}}\int_{\mathcal{C}}\left[\frac{a_+^2(t_k)}{a^2(z)}+\frac{a^2(z)}{a_+^2(t_k)}+\widetilde{\Lambda}_{I,k}\left(\frac{a_+^2(t_k)}{a^2(z)}-\frac{a^2(z)}{a_+^2(t_k)}\right)\right]\frac{\partial_s V_s(z)}{(z-t_k)^2}dz,\\&I_{3,s,1}=\int_{\mathcal{C}}\left[\frac{a^{-2}(z)}{4\pi\psi_s(1)}\left(\frac{2(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2+\frac{1}{6}-\frac{\psi_s'(1)}{\psi_s(1)}}{2^3(z-1)^2}+\frac{5}{2^2 3(z-1)^3}\right)+\frac{a^2(z)}{4(z-1)^2}\frac{-\frac{7}{6}}{2^3\pi\psi_s(1)}\right]\partial_s V_s(z)\frac{dz}{2\pi i},\\&I_{3,s,-1}=\int_{\mathcal{C}}\left[\frac{a^2(z)}{4\pi\psi_s(-1)}\left(\frac{-2(\mathcal{A}+\widetilde{\mathcal{B}}_{-1})^2-\frac{1}{6}-\frac{\psi_s'(-1)}{\psi_s(-1)}}{2^3(z+1)^2}+\frac{5}{2^2 3(z+1)^3}\right)+\frac{a^{-2}(z)}{4(z+1)^2}\frac{\frac{7}{6}}{2^3\pi\psi_s(-1)}\right]\partial_s V_s(z)\frac{dz}{2\pi i}.\end{aligned}$$

From [5, equations (5.16), (5.17), (5.22), (5.23)], we have

$$\int_{\mathcal{C}}\left(\frac{a_+^2(t_k)}{a^2(z)}+\frac{a^2(z)}{a_+^2(t_k)}\right)\frac{\partial_s V_s(z)}{(z-t_k)^2}dz=8\pi^2\sqrt{1-t_k^2}\Big(\psi(t_k)-\frac{2}{\pi}\Big), \tag{6.19}$$

$$\int_{\mathcal{C}}\left(\frac{a_+^2(t_k)}{a^2(z)}-\frac{a^2(z)}{a_+^2(t_k)}\right)\frac{\partial_s V_s(z)}{(z-t_k)^2}dz=\frac{8\pi^2}{1-t_k^2}\int_{t_k}^1\Big(\psi(x)-\frac{2}{\pi}\Big)\sqrt{1-x^2}dx. \tag{6.20}$$

This allows us to evaluate I_{3,s,t_k} more explicitly:

$$I_{3,s,t_k}=-\frac{v_k}{\psi_s(t_k)}\Big(\psi(t_k)-\frac{2}{\pi}\Big)-\frac{v_k\widetilde{\Lambda}_{I,k}}{(1-t_k^2)^{3/2}\psi_s(t_k)}\int_{t_k}^1\Big(\psi(x)-\frac{2}{\pi}\Big)\sqrt{1-x^2}dx. \tag{6.21}$$

Note that Λ_k (and therefore $\widetilde{\Lambda}_{I,k}$) depends on s. Therefore, integrating it in s, we see that

$$\int_0^1 I_{3,s,t_k} ds = -v_k \log\left(\frac{\pi}{2}\psi(t_k)\right) - \frac{v_k}{(1-t_k^2)^{3/2}} \int_{t_k}^1 \left(\psi(x) - \frac{2}{\pi}\right)\sqrt{1-x^2}dx \int_0^1 \frac{\widetilde{\Lambda}_{I,k}}{\psi_s(t_k)} ds. \tag{6.22}$$

From (4.50), one has $\Lambda_k^{\pm 2} = \mathcal{O}(n^{\pm 2\beta_k})$ as $n \to \infty$, but it is also highly oscillatory in s. We can write

$$\Lambda_k^2 = g_{k,n} n^{2\beta_k} \psi_s(t_k)^{2\beta_k} e^{2\pi i n s \int_{t_k}^1 \left(\psi(x) - \frac{2}{\pi}\right)\sqrt{1-x^2}dx}, \tag{6.23}$$

where $g_{k,n}$ is independent of s and $g_{k,n} = \mathcal{O}(1)$ as $n \to \infty$. Thus,

$$\int_{t_k}^1 \left(\psi(x) - \frac{2}{\pi}\right)\sqrt{1-x^2}dx \int_0^1 \frac{\Lambda_k^{\pm 2}}{\psi_s(t_k)} ds = \frac{\pm g_{k,n}^{\pm 1}}{n^{1\mp 2\beta_k}} \int_0^1 \frac{\partial_s \left(e^{\pm 2\pi i n s \int_{t_k}^1 \left(\psi(x) - \frac{2}{\pi}\right)\sqrt{1-x^2}dx}\right)}{\psi_s(t_k)^{1\mp 2\beta_k}} \frac{ds}{2\pi i}. \tag{6.24}$$

From integration by parts of the right-hand side of (6.24), it follows that this integral is $\mathcal{O}(n^{-1\pm 2\beta_k})$ as $n \to \infty$, and thus

$$\int_0^1 I_{3,s,t_k} ds = -v_k \log\left(\frac{\pi}{2}\psi(t_k)\right) + \mathcal{O}(n^{-1+2|\Re\beta_k|}), \qquad \text{as } n \to \infty. \tag{6.25}$$

We now turn to the computation of $I_{3,s,1}$. From [5, equations (5.27), (5.28) and (5.29)], we have $\int_{\mathcal{C}} \frac{a^{-2}(z)}{(z-1)^2} \partial_s V_s(z) dz = 0$ and

$$\int_{\mathcal{C}} \frac{a^{-2}(z)}{(z-1)^3} \partial_s V_s(z) dz = -\frac{8\pi^2 i}{3}\left(\psi(1) - \frac{2}{\pi}\right), \quad \int_{\mathcal{C}} \frac{a^{2}(z)}{(z-1)^2} \partial_s V_s(z) dz = -\frac{16\pi^2 i}{3}\left(\psi(1) - \frac{2}{\pi}\right).$$

Therefore, we obtain $I_{3,s,1} = \frac{-\left(\psi(1) - \frac{2}{\pi}\right)}{24\psi_s(1)}$. The computation of $I_{3,s,-1}$ is similar, and gives $I_{3,s,-1} = \frac{-\left(\psi(-1) - \frac{2}{\pi}\right)}{24\psi_s(-1)}$. By integrating it in s from 0 to 1, it gives

$$\int_0^1 I_{3,s,1} ds = -\frac{1}{24}\log\left(\frac{\pi}{2}\psi(1)\right). \tag{6.26}$$

The computation of $I_{3,s,-1}$ is similar, and gives $\int_0^1 I_{3,s,-1} ds = -\frac{1}{24}\log\left(\frac{\pi}{2}\psi(-1)\right)$. Since the $\mathcal{O}$ term in (6.18) is uniform in $s \in [0,1]$ (see Remark 4.1 (a)), we obtain, as $n \to \infty$

$$\int_0^1 I_{3,s} ds = -\sum_{j=1}^m v_j \log\left(\frac{\pi}{2}\psi(t_j)\right) - \frac{1}{24}\log\left(\frac{\pi^2}{4}\psi(1)\psi(-1)\right) + \mathcal{O}(n^{-1+4\beta_{\max}}). \tag{6.27}$$

From the above calculations, as $n \to \infty$, we have

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = I_{1,s} + I_{2,s,\alpha} + I_{2,s,\beta} + \sum_{j=1}^m I_{3,s,t_j} + I_{3,s,1} + I_{3,s,-1} + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big), \tag{6.28}$$

where the quantities $I_{1,s}$, $I_{2,s,\alpha}$,..., as well as the integrals $\int_0^1 I_{1,s}ds$, $\int_0^1 I_{2,s,\alpha}ds$,... have already been computed explicitly. For convenience, we denote $F_{2,n}(\vec{\alpha}, \vec{\beta}, s)$ for the r.h.s. of (6.3), which can be expressed in terms of R and the parametrices. We recall that (see the beginning of Section 3 and Subsection 4.8), if Ω is a compact subset of $\mathcal{P}_\alpha \times \mathcal{P}_\beta^{(\frac{1}{4})}$, there exists $n_\star$ depending only on Ω, such that $F_{2,n}(\vec{\alpha}, \vec{\beta}, s)$ exists for *all* $n \geq n_\star$ and for *all* $(\vec{\alpha}, \vec{\beta}, s) \in \Omega \times [0,1]$, and is analytic for $s \in (0,1)$. Furthermore, the large n asymptotics for $F_{2,n}(\vec{\alpha}, \vec{\beta}, s)$ given by the r.h.s. of (6.28) is uniform for *all* $(\vec{\alpha}, \vec{\beta}, s) \in \Omega \times [0,1]$. Nevertheless, the identity (6.28) itself is valid only for $(\vec{\alpha}, \vec{\beta}, s) \in (\Omega \times (0,1)) \setminus \widetilde{\Omega}_2^{(n)}$, where $\widetilde{\Omega}_2^{(n)}$ is such that, for any $\epsilon \in (0, \frac{1}{2})$, $\widetilde{\Omega}_{2,\epsilon}^{(n)} = \{(\vec{\alpha}, \vec{\beta}, s) \in \Omega_2^{(n)} : s \in (\epsilon, 1-\epsilon)\}$ consists of at most a finite number of points.

Now, we show how to extend this differential identity for *all* $(\vec{\alpha},\vec{\beta},s) \in \Omega \times (0,1)$, by adapting slightly the argument presented in Section 5, below equation (5.21). We fix $\vec{\alpha}$ and $\vec{\beta}$ such that $(\vec{\alpha},\vec{\beta}) \in \Omega$, and we define $H(s) = D_n(\vec{\alpha},\vec{\beta},V_s,0)\exp(-\int_0^s F_{2,n}(\vec{\alpha},\vec{\beta},\tilde{s})d\tilde{s})$. The equation (6.28) is the statement that $H'(s) = 0$ for all $s \in (0,1)$ such that $(\vec{\alpha},\vec{\beta},s) \in (\Omega \times (0,1)) \setminus \widetilde{\Omega}_2^{(n)}$. Since the large n asymptotics of $F_{2,n}(\vec{\alpha},\vec{\beta},s)$ given by the r.h.s. of (6.28) hold uniformly for $s \in [0,1]$ and continuously (even analytically) for $s \in (0,1)$, H is continuously differentiable for all $s \in (0,1)$ (for $n \geq n_\star$). Thus, by continuity, one has $H'(s) = 0$ for all $s \in (\epsilon, 1-\epsilon)$. Since $\epsilon > 0$ can be chosen arbitrarily small, one has in fact $H'(s) = 0$ for all $s \in (0,1)$ (for $n \geq n_\star$). Also, since $D_n(\vec{\alpha},\vec{\beta},V_s,0)$ is a continuous function of $s \in [0,1]$ (this can be proved by applying Lebesgue's dominated convergence theorem on the associated moments $w_j(\vec{\alpha},\vec{\beta},V_s,0)$), $H(s)$ is continuous (and thus constant) for $s \in [0,1]$. Moreover, as $H(0) = D_n(\vec{\alpha},\vec{\beta},2x^2,0) \neq 0$ (if n is chosen sufficiently large, see (1.9), (1.13) and Proposition 5.1), the determinant $D_n(\vec{\alpha},\vec{\beta},V_s,0)$ is never zero. Hence, the identity (6.28) holds for *all* $s \in (0,1)$. By integrating (6.28) in s from 0 to 1, and by using the above calculations for the quantities $\int_0^1 I_{1,s}ds$,..., this finishes the proof of Proposition 6.1 (since Ω is arbitrary).

7 Integration in W

In this section, we will use the RH analysis done in Section 4, where the potential V is one-cut regular and independent of any parameter, and where W is replaced by W_t defined in (1.37). The weight depends on t and is denoted by

$$w_t(x) = e^{-nV(x)}e^{W_t(x)}\omega(x). \tag{7.1}$$

We will use the differential identity

$$\partial_t \log D_n(\vec{\alpha},\vec{\beta},V,W_t) = \frac{1}{2\pi i}\int_{\mathbb{R}} [Y^{-1}(x)Y'(x)]_{21}\partial_t w_t(x)dx, \tag{7.2}$$

which was obtained in (3.30), and which is valid only for $(\vec{\alpha},\vec{\beta},t) \in \widetilde{\mathcal{P}}_3^{(n)}$ (see the discussion at the beginning of Section 3). The main result of this section is the following.

Proposition 7.1 *As $n \to \infty$,*

$$\log \frac{D_n(\vec{\alpha},\vec{\beta},V,W)}{D_n(\vec{\alpha},\vec{\beta},V,0)} = n\int_{-1}^1 \psi(x)\sqrt{1-x^2}W(x)dx - \frac{1}{4\pi^2}\int_{-1}^1 \frac{W(y)}{\sqrt{1-y^2}}\left(\fint_{-1}^1 \frac{W'(x)\sqrt{1-x^2}}{x-y}dx\right)dy$$
$$+\frac{\mathcal{A}}{2\pi}\int_{-1}^1 \frac{W(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^m \frac{\alpha_j}{2}W(t_j) + \sum_{j=1}^m \frac{i\beta_j}{\pi}\sqrt{1-t_j^2}\fint_{-1}^1 \frac{W(x)}{\sqrt{1-x^2}(t_j-x)}dx + \mathcal{O}\big(n^{-1+2\beta_{\max}}\big).$$

By a very similar calculation to the one done at the beginning of section 6, we obtain exactly the same equation as (6.6), except that s is replaced by t and $\partial_s \log w_s(z)$ is replaced by $\partial_t \log w_t(z) = \partial_t W_t(z)$:

$$\begin{aligned}
\partial_t \log D_n(\vec{\alpha},\vec{\beta},V,W_t) &= I_{1,t} + I_{2,t} + I_{3,t} + \mathcal{O}(e^{-cn}), \\
I_{1,t} &= \frac{-n}{2\pi i}\int_{\mathcal{C}} g'(z)\partial_t \log w_t(z)dz, \\
I_{2,t} &= \frac{-1}{2\pi i}\int_{\mathcal{C}} [P^{(\infty)}(z)^{-1}P^{(\infty)}(z)']_{11}\partial_t \log w_t(z)dz, \\
I_{3,t} &= \frac{-1}{2\pi i}\int_{\mathcal{C}} [P^{(\infty)}(z)^{-1}R^{-1}(z)R'(z)P^{(\infty)}(z)]_{11}\partial_t \log w_t(z)dz.
\end{aligned} \tag{7.3}$$

We thus have, as $n \to \infty$,

$$I_{1,t} = \mathcal{O}(n), \qquad I_{2,t} = \mathcal{O}(1), \qquad I_{3,t} = \mathcal{O}(n^{-1+2\beta_{\max}}), \tag{7.4}$$

and therefore

$$\partial_t \log D_n(\vec{\alpha},\vec{\beta},V,W_t) = I_{1,t} + I_{2,t} + \mathcal{O}(n^{-1+2\beta_{\max}}). \tag{7.5}$$

Let us denote $F_{3,n}(\vec{\alpha},\vec{\beta},t)$ for the r.h.s. of (7.2). We recall that (see the beginning of Section 3 and Subsection 4.8), if Ω is a compact subset of $\mathcal{P}_\alpha \times \mathcal{P}_\beta^{(\frac{1}{4})}$, there exists $n_\star = n_\star(\Omega)$ such that $F_{3,n}(\vec{\alpha},\vec{\beta},t)$ exists for *all* $(\vec{\alpha},\vec{\beta},t) \in \Omega \times [0,1]$. Furthermore, the large n asymptotics of $F_{3,n}(\vec{\alpha},\vec{\beta},t)$ given by the r.h.s. of (7.5) (more explicit expressions for $I_{1,t}$ and $I_{2,t}$ will be computed below) are valid uniformly for *all* $(\vec{\alpha},\vec{\beta},t) \in \Omega \times [0,1]$, but the identity (7.5) itself is valid only for $(\vec{\alpha},\vec{\beta},t) \in (\Omega \times [0,1]) \setminus \widetilde{\Omega}_3^{(n)}$, where $\widetilde{\Omega}_3^{(n)}$ consists of at most a finite number of points. For sufficiently large n, we can extend the identity (7.5) from $(\Omega \times [0,1]) \setminus \widetilde{\Omega}_3^{(n)}$ to $\Omega \times [0,1]$. The argument is similar to the one presented in Section 5, below equation (5.21), and we omit the discussion here. In the rest of this section we compute more explicitly $I_{1,t}$ and $I_{2,t}$. From (4.7), we have

$$I_{1,t} = n \int_{-1}^{1} \psi(x)\sqrt{1-x^2}\,\partial_t W_t(x)dx, \tag{7.6}$$

and therefore

$$\int_0^1 I_{1,t}dt = n \int_{-1}^{1} \psi(x)\sqrt{1-x^2}\,W(x)dx. \tag{7.7}$$

Also, from (4.28), we have $[P^{(\infty)}(z)^{-1}P^{(\infty)}(z)']_{11} = -\partial_z \log D(z)$, and thus

$$I_{2,t} = I_{2,t,W} + I_{2,t,\alpha} + I_{2,t,\beta}, \qquad \text{where} \qquad I_{2,t,W} = \frac{1}{2\pi i}\int_{\mathcal{C}} \partial_z \log D_{W_t}(z)\partial_t W_t(z)dz,$$

$$I_{2,t,\alpha} = \frac{1}{2\pi i}\int_{\mathcal{C}} \partial_z \log D_{\alpha}(z)\partial_t W_t(z)dz, \qquad I_{2,t,\beta} = \frac{1}{2\pi i}\int_{\mathcal{C}} \partial_z \log D_{\beta}(z)\partial_t W_t(z)dz.$$

From [5, equation (5.14) and Lemma 5.4], we have

$$\int_0^1 I_{2,t,W}dt = -\frac{1}{4\pi^2}\int_{-1}^{1} \frac{W(y)}{\sqrt{1-y^2}}\left(\fint_{-1}^{1} \frac{W'(x)\sqrt{1-x^2}}{x-y}dx\right)dy, \tag{7.8}$$

$$\int_0^1 I_{2,t,\alpha}dt = \frac{\mathcal{A}}{2\pi}\int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^{m} \frac{\alpha_j}{2}W(t_j). \tag{7.9}$$

What remains to be done is to evaluate more explicitly $I_{2,t,\beta}$ and integrate it in t. This can be achieved from (6.11) and a contour deformation of $\mathcal{C}$, and gives

$$\int_0^1 I_{2,t,\beta}dt = \sum_{j=1}^{m} \frac{i\beta_j}{\pi}\sqrt{1-t_j^2}\fint_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}(t_j-x)}dx. \tag{7.10}$$

Since the $\mathcal{O}$ terms in (4.74) are uniform for $t \in [0,1]$ (see Remark 4.1 (b)), we have $\int_0^1 I_{3,t}dt = \mathcal{O}\big(n^{-1+2\beta_{\max}}\big)$ as $n \to \infty$, which finishes the proof.

8 Appendix

In this section we recall some well-known model RH problems: 1) the Airy model RH problem, whose solution is denoted Φ_{Ai} and 2) the confluent hypergeometric model RH problem, whose solution is denoted $\Phi_{\mathrm{HG}}(z) = \Phi_{\mathrm{HG}}(z;\alpha,\beta)$. We will only consider the cases when α and β are such that $\Re\alpha > -1$ and $\Re\beta \in (-\frac{1}{4},\frac{1}{4})$.

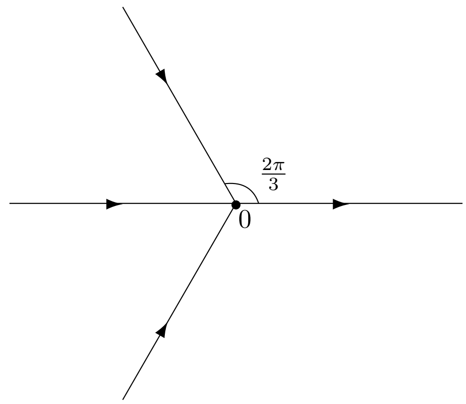

Figure 3: The jump contour Σ_A for Φ_{Ai}.

8.1 Airy model RH problem

(a) $\Phi_{\mathrm{Ai}} : \mathbb{C} \setminus \Sigma_A \to \mathbb{C}^{2\times 2}$ is analytic, and Σ_A is shown in Figure 3.

(b) Φ_{Ai} has the jump relations

$$\begin{aligned}
\Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad && \text{on } \mathbb{R}^-, \\
\Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad && \text{on } \mathbb{R}^+, \\
\Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad && \text{on } e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\
\Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad && \text{on } e^{-\frac{2\pi i}{3}}\mathbb{R}^+.
\end{aligned} \tag{8.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_A$, we have

$$\Phi_{\mathrm{Ai}}(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{Ai,k}}}{z^{3k/2}} \right) e^{-\frac{2}{3}z^{3/2}\sigma_3}, \tag{8.2}$$

where $N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $\Phi_{\mathrm{Ai},1} = \frac{1}{8}\begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$.

As $z \to 0$, we have

$$\Phi_{\mathrm{Ai}}(z) = \mathcal{O}(1). \tag{8.3}$$

The Airy model RH problem was introduced and solved in [27] (see also [27, equation (7.30)], where explicit forms for the constant matrices $\Phi_{\mathrm{Ai,k}}$ can be found). We have

$$\Phi_{\mathrm{Ai}}(z) := M_A \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2\mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2\mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2\mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2\mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{8.4}$$

with $\omega = e^{\frac{2\pi i}{3}}$, Ai the Airy function and

$$M_A := \sqrt{2\pi}e^{\frac{\pi i}{6}}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{8.5}$$

8.2 Confluent hypergeometric model RH problem

(a) $\Phi_{\mathrm{HG}} : \mathbb{C} \setminus \Sigma_{\mathrm{HG}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{HG} is shown in Figure 4.

(b) For $z \in \Gamma_k$ (see Figure 4), $k = 1, ..., 8$, Φ_{HG} has the jump relations

$$\Phi_{\mathrm{HG},+}(z) = \Phi_{\mathrm{HG},-}(z) J_k, \tag{8.6}$$

where

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}, \quad J_3 = J_7 = \begin{pmatrix} e^{\frac{i\pi\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\pi\alpha}{2}} \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}, J_4 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, J_6 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, J_8 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}.$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{HG}}$, we have

$$\Phi_{\mathrm{HG}}(z) = \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{HG},k}}{z^k} \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} M^{-1}(z), \tag{8.7}$$

where

$$\Phi_{\mathrm{HG},1} = \left(\beta^2 - \frac{\alpha^2}{4}\right) \begin{pmatrix} -1 & \tau(\alpha,\beta) \\ -\tau(\alpha,-\beta) & 1 \end{pmatrix}, \qquad \tau(\alpha,\beta) = \frac{-\Gamma\left(\frac{\alpha}{2} - \beta\right)}{\Gamma\left(\frac{\alpha}{2} + \beta + 1\right)}, \tag{8.8}$$

and

$$M(z) = \begin{cases} e^{\frac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \dfrac{\pi}{2} < \arg z < \pi, \\ e^{-\frac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \pi < \arg z < \dfrac{3\pi}{2}, \\ e^{\frac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < 0, \\ e^{-\frac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 0 < \arg z < \frac{\pi}{2}. \end{cases} \tag{8.9}$$

In (8.7), $z^{-\beta}$ has a cut along $i\mathbb{R}^-$, such that $z^{-\beta} \in \mathbb{R}$ as $z \in \mathbb{R}^+$.

As $z \to 0$, we have

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in II \cup III \cup VI \cup VII, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in I \cup IV \cup V \cup VIII, \end{cases}, \qquad \text{if } \Re\alpha = 0,$$

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) \end{pmatrix}, & \text{if } z \in II \cup III \cup VI \cup VII, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) \end{pmatrix}, & \text{if } z \in I \cup IV \cup V \cup VIII, \end{cases}, \qquad \text{if } \Re\alpha > 0,$$

$$\Phi_{\mathrm{HG}}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{\frac{\Re\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{\frac{\Re\alpha}{2}}) \end{pmatrix}, \qquad \text{if } \Re\alpha < 0. \tag{8.10}$$

This model RH problem was first introduced and solved explicitly in [37] for the case $\alpha = 0$, and then in [35] and [23] for the general case. The constant matrices $\Phi_{\mathrm{HG},k}$ depend analytically on α and β (they can be found explicitly e.g. in [35, equation (56)]). Consider the matrix

$$\widehat{\Phi}_{\mathrm{HG}}(z) = \begin{pmatrix} \frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(1+\alpha)} G(\frac{\alpha}{2} + \beta, \alpha; z) e^{-\frac{i\pi\alpha}{2}} & -\frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(\frac{\alpha}{2}+\beta)} H(1 + \frac{\alpha}{2} - \beta, \alpha; z e^{-i\pi}) \\ \frac{\Gamma(1+\frac{\alpha}{2}+\beta)}{\Gamma(1+\alpha)} G(1 + \frac{\alpha}{2} + \beta, \alpha; z) e^{-\frac{i\pi\alpha}{2}} & H(\frac{\alpha}{2} - \beta, \alpha; z e^{-i\pi}) \end{pmatrix} e^{-\frac{i\pi\alpha}{4}\sigma_3}, \tag{8.11}$$

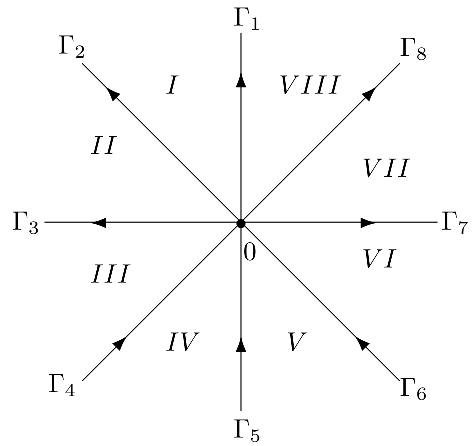

Figure 4: The jump contour Σ_{HG} for $\Phi_{\mathrm{HG}}(z)$. The ray Γ_k is oriented from 0 to ∞, and forms an angle with $\mathbb{R}^+$ which is a multiple of $\frac{\pi}{4}$.

where G and H are related to the Whittaker functions:

$$G(a,\alpha;z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \quad H(a,\alpha;z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}}, \quad \mu = \frac{\alpha}{2}, \quad \kappa = \frac{1}{2} + \frac{\alpha}{2} - a. \tag{8.12}$$

The solution Φ_{HG} is given by

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1}, & \text{for } z \in I, \\ \widehat{\Phi}_{\mathrm{HG}}(z), & \text{for } z \in II, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_3, & \text{for } z \in III, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_3 J_4^{-1}, & \text{for } z \in IV, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1} J_6, & \text{for } z \in V, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1}, & \text{for } z \in VI, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1}, & \text{for } z \in VII, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1}, & \text{for } z \in VIII. \end{cases} \tag{8.13}$$

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References

[1] G. Akemann, J. Baik and P.D. Francesco, *The Oxford Handbook of Random Matrix Theory*, Oxford Univ. Press, Oxford (2011).

[2] G.W. Anderson, A. Guionnet and O. Zeitouni, *An introduction to random matrices*, Cambridge studies in advanced mathematics **118** (2010).

[3] E. Basor, Asymptotic formulas for Toeplitz determinants, *Trans. Amer. Math. Soc.* **239** (1978), 33–65.

[4] E. Basor, A localization theorem for Toeplitz determinants, *Indiana Univ. Math. J.* **28** (1979), 975–983.

[5] N. Berestycki, C. Webb and M.D. Wong, Random Hermitian Matrices and Gaussian Multiplicative Chaos, arXiv:1701.03289.

[6] T. Berggren and M. Duits, Mesoscopic fluctuations for the thinned Circular Unitary Ensemble, *Math. Phys. Anal. Geom.*, **20** (2017), 40 pp.

[7] P. Bleher and A. Its, Asymptotics of the partition function of a random matrix model, *Ann. Inst. Fourier* **55** (2005), 1943–2000.

[8] O. Bohigas and M.P. Pato, Missing levels in correlated spectra, *Phys. Lett.* **B595** (2004), 171–176.

[9] O. Bohigas and M.P. Pato, Randomly incomplete spectra and intermediate statistics, *Phys. Rev. E* (3) **74** (2006).

[10] T. Bothner and R. Buckingham, Large deformations of the Tracy-Widom distribution I. Non-oscillatory asymptotics, arXiv:1702.04462.

[11] T. Bothner, P. Deift, A. Its and I. Krasovsky, On the asymptotic behavior of a log gas in the bulk scaling limit in the presence of a varying external potential I, *Comm. Math. Phys.* **337** (2015), 1397–1463.

[12] T. Bothner, P. Deift, A. Its and I. Krasovsky, *On the asymptotic behavior of a log gas in the bulk scaling limit in the presence of a varying external potential II*, Large Truncated Toeplitz Matrices, Toeplitz Operators, and Related Topics (2017), 213–234, Springer.

[13] A. Böttcher and B. Silbermann, Toeplitz operators and determinants generated by symbols with one Fisher-Hartwig singularity, *Math. Nachr.* **127** (1986), 95–123.

[14] C. Charlier and T. Claeys, Asymptotics for Toeplitz determinants: perturbation of symbols with a gap, *J. Math. Phys.* **56** (2015), 022705.

[15] C. Charlier and T. Claeys, Thinning and conditioning of the Circular Unitary Ensemble, *Random Matrices Theory Appl.* **6** (2017), 51 pp.

[16] C. Charlier and A. Deaño, Asymptotics for Hankel determinants associated to a Hermite weight with a varying discontinuity, arXiv:1708.02519.

[17] T. Claeys and B. Fahs, Random matrices with merging singularities and the Painlev V equation, *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016), 44 pp.

[18] T. Claeys, T. Grava and K. T-R McLaughlin, Asymptotics for the partition function in two-cut random matrix models, *Comm. Math. Phys.* **339** (2015), 513–587.

[19] T. Claeys and I. Krasovsky, Toeplitz determinants with merging singularities, *Duke Math. J.* **164** (2015), 2897–2987.

[20] D.J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes: Volume II: General Theory and structure. Second edition.* Probability and its Applications (2008), Springer, New York.

[21] A. Deaño and N.J. Simm, On the probability of positive-definiteness in the gGUE via semi-classical Laguerre polynomials, *J. Approx. Theory* **220** (2017), 4459.

[22] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Amer. Math. Soc. **3** (2000).

[23] P. Deift, A. Its and I. Krasovky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities, *Ann. Math.* **174** (2011), 1243–1299.

[24] P. Deift, A. Its and I. Krasovsky, On the asymptotics of a Toeplitz determinant with singularities, *MSRI Publications* **65** (2014), Cambridge University Press.

[25] P. Deift, T. Kriecherbauer and K.T-R McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field, *J. Approx. Theory* **95** (1998), 388–475.

[26] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.

[27] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.

[28] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems, *Bull. Amer. Math. Soc. (N.S.)* **26** (1992), 119–123.

[29] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* **137** (1993), 295–368.

[30] T. Ehrhardt, A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities, *Operator Theory: Adv. Appl.* **124** (2001), 217–241.

[31] N.M. Ercolani and K.T-R McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration, *Int. Math. Res. Not.* **14** (2003), 755–820.

[32] M. E. Fisher and R. E. Hartwig, Toeplitz determinants: some applications, theorems, and conjectures, *Advan. Chem. Phys.* **15** (1968), 333–353.

[33] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* **147** (1992), 395–430.

[34] P.J. Forrester and N.E. Frankel, Applications and generalizations of Fisher-Hartwig asymptotics, *J. Math. Phys.* **45** (2004), 2003–2028.

[35] A. Foulquié Moreno, A. Martinez-Finkelshtein and V.L. Sousa, On a conjecture of A. Magnus concerning the asymptotic behavior of the recurrence coefficients of the generalized Jacobi polynomials, *J. Approx. Theory*, **162**(2010), 807–831.

[36] F. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford (1966). Reprinted by Dover Publications, New York (1990).

[37] A. Its and I. Krasovsky, Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump, *Contemporary Mathematics* **458** (2008), 215–248.

[38] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, *Duke Math. J.* **91** (1998), 151-204.

[39] K. Johansson, *Random matrices and determinantal processes*, Mathematical statistical physics (2006), Elsevier B.V., Amsterdam, 1–55.

[40] O. Kallenberg, A limit theorem for thinning of point processes, *Inst. Stat. Mimeo Ser.* **908** (1974).

[41] J.P. Keating and N.C. Snaith, Random matrix theory and $\zeta(1/2+it)$, *Comm. Math. Phys.* **214** (2001), 57–89.

[42] I. Krasovsky, Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant, *Duke Math J.* **139** (2007), 581–619.

[43] A.B.J. Kuijlaars, K. T-R McLaughlin, W. Van Assche and M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$, *Adv. Math.* **188** (2004), 337–398

[44] G. Lambert, Incomplete determinantal processes: from random matrix to Poisson statistics, *arXiv:1612.00806.*

[45] M.L. Mehta, *Random matrices, Third Edition*, Pure and Applied Mathematics Series **142** (2004), Elsevier Academic Press.

[46] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, *NIST handbook of mathematical functions* (2010), Cambridge University Press.

[47] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag (1997).

[48] A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys* **55** (2000), 923–975.

[49] G. Szegő, *Orthogonal polynomials*, AMS Colloquium Publ. **23** (1959), New York: AMS.

[50] H. Widom, The strong Szegő limit theorem for circular arcs, *Indiana Univ. Math. J.* **21** (1971), 277–283.

[51] H. Widom. Toeplitz determinants with singular generating functions. *Amer. J. Math.* **95** (1973), 333–383.

[52] X-B Wu, S-X Xu and Y-Q Zhao, Gaussian unitary ensemble with boundary spectrum singularity and σ-form of the Painlev II equation, arXiv:1706.03174.