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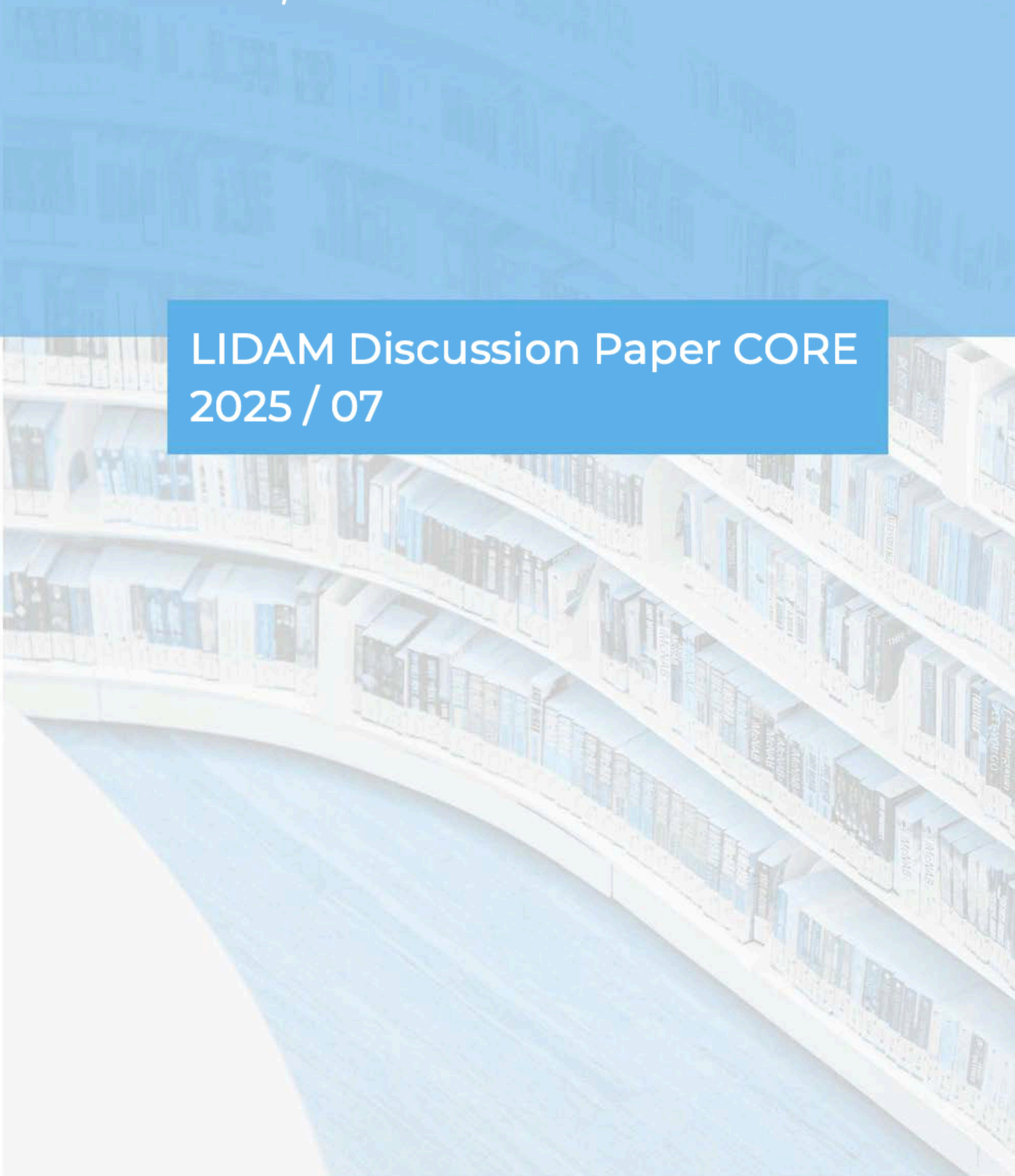

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School Choice with Unobservable Matchings

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Abstract

We consider priority-based school choice problems where students can form two types of matches: public matches observed by everyone and private matches generally not observed by others. We introduce the notion of rationalizable conjectural stable (RCS) matching, which generalizes Gale and Shapley (1962)'s stability notion to accommodate private matches. We partially characterize RCS matchings and we show that the Efficiency-Adjusted Deferred Acceptance (EADA) matching is RCS when private and public matches are perfect substitutes. Finally, we extend the definition of RCS matching to school choice problems when private and public matches are imperfect substitutes.

Keywords: School choice; Private information; Private matchings; Stability; Rationalizability.
JEL classification: C70, C78, D82.

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1 Introduction

Students typically have incomplete information about other students' school matches. Expectation about these unobserved school matches might then influence a student's school choice. Despite this fact, a large literature studies school choice problems with complete information. It assumes students know not only their own preferences over possible matchings, but also all other students' preferences. In addition, it assumes that students perfectly observe all other students' matches. Both assumptions are often made for tractability and may be reasonable for some applications. However, students may not be fully informed of other students' preferences and their applications to schools, but they may be better informed about their peers through social networks. Ignoring both types of incomplete information matters when one investigates the stability of possible matchings.

This paper provides a framework for studying priority-based school choice problems where students can form public as well as private matches. While public matches are observed by everyone, private matches are generally not observed by others. We introduce the notion of rationalizable conjectural stable (RCS) matching, which generalizes the stability notion of Gale and Shapley (1962) to incomplete information about the matching and accommodates for private matches. Our framework provides the foundation to model and understand richer matching problems when agents can form different types of matches that vary in visibility and potentially in payoff consequences.

We introduce the notion of conjectural stability for school choice problems with private and public matches. The possibility of private matches requires the introduction of beliefs of the students about other students' matches. Students revise their matches based on their beliefs about other students' matches. A tuple of a matching and beliefs is conjectural stable (CS) if: (i) each student is at least as well off than being assigned to the null school (individual rationality), (ii) for each student who would like to be matched to another school she holds a consistent conjecture about other students' matches such that this school is full and all students assigned to this school have a higher priority than her (non-wastefulness and no justified envy), and (iii) students' beliefs are consistent with their private signals.[1]

[1]Beliefs are consistent with students' private signals similar to Battigalli and Guaitoli (1997) notion of conjectural equilibrium and Foerster, Mauleon and Vannetelbosch (2021) notion of conjectural pairwise stability for networks.

Conjectural stability is weak since it does not impose any restrictions on beliefs beyond consistency. Based on Rubinstein and Wolinsky (1994) and Foerster, Mauleon and Vannetelbosch (2021), we refine CS by requiring common knowledge of rationality.[2] A set of matchings is rationalizable if for each matching in the set, there exist beliefs such that the tuple of matching and beliefs is CS, and the support of these beliefs belongs to this set. A matching is rationalizable conjectural stable (RCS) if this matching belongs to a rationalizable set.

When public and private matches are perfect substitutes, any fully private matching where each match belongs to some matching that is stable under complete information is RCS. More interestingly, each matching with only private matches is RCS if each student who would have incentives to deviate under complete information believes she is in fact facing another matching that is stable under complete information.

Two prominent mechanisms used for priority-based matching are the Gale and Shapley's (1962) Deferred Acceptance (DA) mechanism and the Shapley and Scarf's (1974) Top Trading Cycles (TTC) mechanism.[3] On the one hand, the TTC mechanism is Pareto efficient while the DA mechanism may select an inefficient matching. On the other hand, the DA mechanism is stable under complete information while the TTC mechanism may select an unstable matching.[4] Obviously, the DA matching is RCS. While the TTC matching is CS, it is not RCS. Kesten (2010) introduces the Efficiency-Adjusted Deferred Acceptance (EADA) mechanism and shows that the EADA matching is Pareto efficient when all students consent. We show that the EADA matching is RCS when public and private matches are perfect substitutes. This result even holds if not all students consent. Thus, the EADA mechanism leads to a matching that is stable under incomplete information. Hence, privacy of school assignments may improve efficiency while disclosing them may lead to Pareto inefficient assignments.

[2]Rubinstein and Wolinsky (1994) analyze non-cooperative games where players get imperfect signals about others' strategies. At equilibrium, each player's strategy is optimal given her signal and that it is common knowledge that all players maximize utility given their signals. See also Bernheim (1984), Pearce (1984) for related rationalizability concepts.

[3]We refer to Abdulkadiroğlu and Andersson (2023) for an overview of school choice markets. Ehlers and Morrill (2020) analyze legal assignments in school choice.

[4]Atay, Mauleon and Vannetelbosch (2025) show that the TTC matching is stabilized when students become farsighted.

Finally, we extend the definition of rationalizable conjectural stable matching to school choice problems with imperfect substitutes, i.e. each student is not indifferent between being assigned privately or publicly to a school. We show that the set of matchings that are RCS under imperfect substitutes is included in the set of matchings that are RCS when each student ranks schools based on her best match's type for each school. If all students prefer to be matched privately to every school, then both sets coincide.

The literature on matching markets with incomplete information only deals with incomplete information about preferences. On the theoretical side, Liu, Mailath, Postlewaite and Samuelson (2014) study stable matchings for two-sided markets where agents on one side of the market are privately informed of their payoff relevant attributes, matching outcomes are observed, and agents consider deviating from a matching outcome based on their updated information and inference. They propose a stability concept derived from an iterated elimination procedure. Liu (2020) goes further by proposing a criterion of stability with a formulation of Bayesian consistency of prior beliefs, on-path stable beliefs, and off-path stable beliefs. Bikhchandani (2017) extends the analysis of Liu, Mailath, Postlewaite and Samuelson (2014) to markets without transferable utility. Pomatto (2022) provides an epistemic foundation for Liu, Mailath, Postlewaite and Samuelson (2014) notion of incomplete-information stability.[5]

Another part of the literature looks at the stability of centralized mechanisms when there is incomplete information about preferences. Fernandez, Rudov and Yariv (2022) analyze the impact of incomplete information on the DA mechanism, and show that many complete information results are not robust to a small amount of uncertainty about others' preferences.[6]

On the experimental side, Agranov, Dianat, Samuelson and Yariv (2023) experimentally study decentralized one-to-one matching markets with transfers. They find that, in the presence of transfers, a substantial fraction of matching outcomes remains efficient, while fewer matching outcomes are stable.

To the best of our knowledge, we are the first to study the strategic choice of students to form private or public matches and to adapt the rationalizability concept

[5]Chen and Hu (2023) provide a framework for studying two-sided matching markets with two-sided incomplete information. See Liu (2024) for a survey on solution concepts for matching markets with incomplete information about preferences.

[6]See also Roth (1989), Ehlers and Massó (2015), among others.

of Rubinstein and Wolinsky (1994) to school choice problems. Students have the opportunity to disclose or not their match, but they do not take into account the reaction of others to disclosing or hiding their match.

The paper is organized as follows. In Section 2, we introduce priority-based school choice problems. In Section 3, we propose and characterize the notion of rationalizable conjectural stability (RCS) for school choice problems. In Section 4, we show that the EADA matching is RCS when private and public matches are perfect substitutes. In Section 5, we show that the TTC matching is CS but not RCS. In Section 6, we study school choice problems where private and public matches are imperfect substitutes. In Section 7, we conclude.

2 School choice problems

A school choice problem is a list $\langle I, S, q, P, F \rangle$ where

(i) $I = \{i_1, ..., i_n\}$ is the set of students,

(ii) $S = \{s_1, ..., s_m\}$ is the set of schools,

(iii) $q = (q_{s_1}, ..., q_{s_m})$ is the quota vector where q_s is the number of available seats at school s,

(iv) $P = (P_{i_1}, ..., P_{i_n})$ is the preference profile where P_i is the strict preference of student i over the set of schools S and her outside option, the null school s_0 with enough capacity $q_{s_0} = n$.[7]

(v) $F = (F_{s_1}, ..., F_{s_m})$ is the strict priority structure of the schools over the students.

Let i be a generic student and s be a generic school. Let $S^* = \{s_0\} \cup S$. The preference P_i of student i is a linear order over S^*. Student i prefers school s to school s' if sP_is'. School s is acceptable to student i if sP_is_0. We often write $P_i = s, s', s''$ meaning that student i's most preferred school is s, her second best is

[7]It is the same as a matching model that allows the student to self-match: being matched to the null school would revert to being matched to herself.

s', her third best is s'' and any other school is unacceptable for her. Let R_i be the weak preference relation associated with the strict preference relation P_i.[8]

The priority F_s of school s is a linear order over I. That is, F_s assigns ranks to students according to their priority for school s. The rank of student i for school s is denoted $F_s(i)$ and $F_s(i) < F_s(j)$ means that student i has higher priority for school s than student j. For $s \in S, i \in I$, let $\Phi(s,i) = \{j \in I \mid F_s(j) < F_s(i)\}$ be the set of students who have higher priority than student i for school s.

An outcome of a school choice problem is a matching matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu_{i_1}} \\ \boldsymbol{\mu_{i_2}} \\ \vdots \\ \boldsymbol{\mu_{i_n}} \end{pmatrix} = \begin{pmatrix} \mu_{i_1,s_0} & \mu_{i_1,s_1} & \cdots & \mu_{i_1,s_m} \\ \mu_{i_2,s_0} & \mu_{i_2,s_1} & \cdots & \mu_{i_2,s_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i_n,s_0} & \mu_{i_n,s_1} & \cdots & \mu_{i_n,s_m} \end{pmatrix}$$

such that, for $i \in I, s \in S^*$,

(i) $\mu_{i,s} \in \{0,1,2\}$, where $\mu_{i,s} = 1$ refers to a public match, $\mu_{i,s} = 2$ refers to a private match between i and s, and $\mu_{i,s} = 0$ means that i is not assigned to s;

(ii) $\mu_{i,s} \neq 0 \Leftrightarrow \mu_{i,s'} = 0$ for all $s' \neq s$;

(iii) $\#\{i \in I \mid \mu_{i,s} \neq 0\} \leq q_s$.

Condition (i) means that student i is publicly (privately) assigned a seat at school $s \in S$ under $\boldsymbol{\mu}$ if $\mu_{i,s} = 1$ (2) and is publicly (privately) unassigned under $\boldsymbol{\mu}$ if $\mu_{i,s_0} = 1$ (2). Condition (ii) means that each student is assigned to only one school $s \in S$ or is assigned to the null school. Condition (iii) requires that no school exceeds its quota under $\boldsymbol{\mu}$. The set of all matching matrices is denoted $\mathbf{M}$.[9]

Let $\sigma(i, \boldsymbol{\mu})$ be the school $s \in S^*$ to which student i is assigned under $\boldsymbol{\mu}$. For instance,

$$\boldsymbol{\mu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\mu'} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

[8] Haeringer and Klijn (2009) investigate constrained school choice problems where students can only rank a fixed number of schools.

[9] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subset$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

are, respectively, the matching where student i_1 is publicly assigned to school s_2, student i_2 is publicly assigned to school s_1, student i_3 is privately assigned to school s_1, and student i_4 is publicly assigned to the null school s_0, and the matching where student i_1 is privately assigned to school s_1, student i_2 is privately assigned to school s_2, student i_3 is privately assigned to the null school s_0, and student i_4 is publicly assigned to school s_1. Thus, $\sigma(i_1, \boldsymbol{\mu}) = s_2$, $\sigma(i_2, \boldsymbol{\mu}) = s_1$, $\sigma(i_3, \boldsymbol{\mu}) = s_1$, and $\sigma(i_4, \boldsymbol{\mu}) = s_0$, and $\sigma(i_1, \boldsymbol{\mu'}) = s_1$, $\sigma(i_2, \boldsymbol{\mu'}) = s_2$, $\sigma(i_3, \boldsymbol{\mu'}) = s_0$, and $\sigma(i_4, \boldsymbol{\mu'}) = s_1$. In Figure 1 we depict the graphs associated to where private (public) matches are indicated by light gray (black) lines.

For convenience, we often write such matchings as a collection of matches $\mu = \{(i, s, x) \in I \times S^* \times \{1, 2\} \mid \mu_{is} = x \neq 0 \text{ in } \boldsymbol{\mu}\}$. In the above example, we have:

$$\mu = \{(i_1, s_2, 1), (i_2, s_1, 1), (i_3, s_1, 2), (i_4, s_0, 1)\},$$
$$\mu' = \{(i_1, s_1, 2), (i_2, s_2, 2), (i_3, s_0, 2), (i_4, s_1, 1)\}.$$

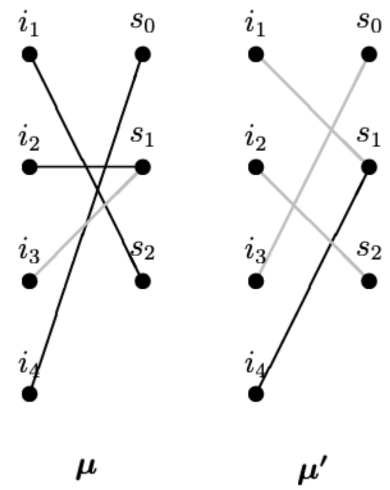


Figure 1: Private (public) matches are indicated by light gray (black) lines.

Let $\mathbf{M}_{|\boldsymbol{\mu_i}} = \{\boldsymbol{\mu'} \in \mathbf{M} \mid \boldsymbol{\mu'_i} = \boldsymbol{\mu_i}\}$ be the set of all matchings such that student i is matched to the same partner and in the same way (i.e. publicly or privately) as in $\boldsymbol{\mu}$. Let $\mathbf{M}_{|\mathbf{0},\mathbf{1}} = \{\boldsymbol{\mu} \in \mathbf{M} \mid \mu_{i,s} \neq 2, i \in I, s \in S^*\}$ be the set of all matchings when only public matches are feasible, and let $\mathbf{M}_{|\mathbf{0},\mathbf{2}} = \{\boldsymbol{\mu} \in \mathbf{M} \mid \mu_{i,s} \neq 1, i \in I, s \in S^*\}$ be the set of all matchings when only private matches are feasible.

Perfect substitutes Public matches and private matches are perfect substitutes and hence only differ in visibility. Students' preferences $(P_i)_{i \in I}$ over schools can be straightforwardly extended to preferences $(\succ_i)_{i \in I}$ over matchings. We say that student i prefers the matching $\boldsymbol{\mu}$ to the matching $\boldsymbol{\mu'}$ if she prefers her assignment under $\boldsymbol{\mu}$ to her assignment under $\boldsymbol{\mu'}$. Formally,

$$\boldsymbol{\mu} \succ_i \boldsymbol{\mu'} \text{ if and only if } \sigma(i, \boldsymbol{\mu}) P_i \sigma(i, \boldsymbol{\mu'}), \text{ and}$$
$$\boldsymbol{\mu} \succeq_i \boldsymbol{\mu'} \text{ if and only if } \neg(\boldsymbol{\mu'} \succ_i \boldsymbol{\mu}).$$

Student i knows her own match $\boldsymbol{\mu_i}$ (i.e. to which school she is assigned) and receives a private signal $\tau_i(\boldsymbol{\mu})$ that may contain information about other students' matches. Students observe all public matches of other students, while private matches of other students are not observed. Let $(\tau_i(\boldsymbol{\mu}))_{j,s}$ be the information of student i about student j's match at school s. Private signals of student i about student j's match are such that:

(i) If $\mu_{j,k} = 1$ for some $k \in S^*$ then $(\tau_i(\boldsymbol{\mu}))_{j,s} = \{\mu_{j,s}\}$ for all $s \in S^*$,

(ii) If $\mu_{j,k} = 2$ for some $k \in S^*$ then

$$\begin{aligned} &\text{either } (\tau_i(\boldsymbol{\mu}))_{j,k} = \{0,2\},\ (\tau_i(\boldsymbol{\mu}))_{j,s'} = \{0,2\} \text{ for some } s' \in S^*, s' \neq k, \\ &\qquad \text{and } (\tau_i(\boldsymbol{\mu}))_{j,s} \in \{\{0\},\{0,2\}\} \text{ for all } s \in S^*, s \neq s', k, \\ &\quad \text{or } (\tau_i(\boldsymbol{\mu}))_{j,k} = \{2\} \text{ and } (\tau_i(\boldsymbol{\mu}))_{j,s} = \{0\} \text{ for all } s \in S^*, s \neq k. \end{aligned}$$

Private signals are said to be fully not informative if each student only knows her own assignment and all public assignments.

Let $\Delta(\mathbf{M}_{|\boldsymbol{\mu_i}})$ denote the set of all probability distributions on $\mathbf{M}_{|\boldsymbol{\mu_i}}$ and $c_i \in \Delta(\mathbf{M}_{|\boldsymbol{\mu_i}})$ denote the belief of student i with matching vector $\boldsymbol{\mu_i}$ about the assignments of all students. We require the students' beliefs not to contradict their signals.[10]

Consistent beliefs Students' beliefs $(c_i)_{i\in I}$ are said to be consistent with the private signals $(\tau_i(\boldsymbol{\mu}))_{i\in I}$ obtained in $\boldsymbol{\mu} \in \mathbf{M}$ if $\tau_i(\boldsymbol{\mu'}) = \tau_i(\boldsymbol{\mu})$ for all $i \in I$ and $\boldsymbol{\mu'} \in \mathbf{M}_{|\boldsymbol{\mu_i}}$ such that $c_i(\boldsymbol{\mu'}) > 0$.

Remember that $\mathbf{M}_{|\mathbf{0,1}}$ is the set of all matching matrices when only public matches are feasible. Given a school choice problem $\langle I, S, q, P, F\rangle$, a matching $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0,1}}$ is stable under complete information if:

[10]Lipnowski and Sadler (2019) study non-cooperative games with an exogenously given network and assume that players have correct conjectures about the strategies of their neighbors. See McBride (2006), Battigalli, Panebianco and Pin (2023) for related contributions.

(i) for all $i \in I$ we have $\boldsymbol{\mu} \succeq_i \boldsymbol{\mu'}$ where $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$ is such that $\sigma(i, \boldsymbol{\mu'}) = s_0$ (individual rationality),

(ii) for all $i \in I$ and all $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$, if $\boldsymbol{\mu} \prec_i \boldsymbol{\mu'}$ where $\sigma(i, \boldsymbol{\mu'}) = s$ then $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu}) = s\} = q_s$ (non-wastefulness),

(iii) for all $i, j \in I$ with $\sigma(j, \boldsymbol{\mu}) = s$, if $\boldsymbol{\mu} \prec_i \boldsymbol{\mu'}$ where $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$ and $\sigma(i, \boldsymbol{\mu'}) = s$ then $j \in \Phi(s, i)$ (no justified envy).

Let $\boldsymbol{\Sigma}$ be the set of stable matching under complete information.

3 Conjectural stability and rationalizability

We now propose a notion of stability for school choice problems with private and public matches. The possibility of private matches requires the introduction of beliefs of the students about other students' matches.[11] Students revise their matches based on their beliefs about other students' matches. Beliefs are consistent with students' private signals similar to Battigalli and Guaitoli (1997) notion of conjectural equilibrium and Foerster, Mauleon and Vannetelbosch (2021) notion of conjectural pairwise stability for networks.

Definition 1. Given a school choice problem $\langle I, S, q, P, F \rangle$, the tuple $(\boldsymbol{\mu}, (c_i)_{i \in I})$ is conjectural stable (CS) with respect to $(\succ_i)_{i \in I}$ and $(\tau_i)_{i \in I}$ if:

(i) for all $i \in I$, we have $\boldsymbol{\mu} \succeq_i \boldsymbol{\mu'}$ if $\sigma(i, \boldsymbol{\mu'}) = s_0$;

(ii) for all $i \in I$, for all $\boldsymbol{\mu'}$ such that $\boldsymbol{\mu} \prec_i \boldsymbol{\mu'}$, then for each $\boldsymbol{\mu''} \in \mathbf{M}_{|\boldsymbol{\mu_i}}$ such that $c_i(\boldsymbol{\mu''}) > 0$, we have $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu''}) = \sigma(i, \boldsymbol{\mu'})\} = q_{\sigma(i, \boldsymbol{\mu'})}$ and $i \notin \Phi(\sigma(i, \boldsymbol{\mu'}), j)$ for all j such that $\sigma(j, \boldsymbol{\mu''}) = \sigma(i, \boldsymbol{\mu'})$;

(iii) $(c_i)_{i \in I})$ are consistent with $(\tau_i(\boldsymbol{\mu}))_{i \in I}$.

Moreover, $\boldsymbol{\mu}$ is CS under correct beliefs if additionally $c_i(\boldsymbol{\mu}) = 1$ for all $i \in I$.

Thus, a tuple of a matching $\boldsymbol{\mu}$ and beliefs $(c_i)_{i \in I}$ is conjectural stable (CS) if: (i) each student i is at least as well off than being assigned to the null school (individual rationality); (ii) for each student i who would like to be assigned to another school

[11] Only students act strategically while schools act mechanically. It implies that students cannot infer information from their interactions with schools.

$s \neq \sigma(i, \boldsymbol{\mu})$ she holds a consistent conjecture about other students' assignments such that school s is full and all students assigned to school s have a higher priority than her (non-wastefulness and no justified envy); and (iii) students' beliefs are consistent with their private signals.

Remark 1. Let $\langle I, S, q, P, F \rangle$ be a school choice problem. The matching $\boldsymbol{\mu} \in \mathbf{M}$ is CS with respect to $(\succ_i)_{i \in I}$ and completely informative private signals $(\tau_i)_{i \in I}$ if and only if $\widetilde{\boldsymbol{\mu}}$ is stable where

$$\widetilde{\mu_{i,s}} = \begin{cases} 1 & \text{if } \mu_{i,s} \neq 0 \\ \\ 0 & \text{if } \mu_{i,s} = 0 \end{cases} \qquad \text{for all } i \in I, s \in S^*.$$

Conjectural stability is weak since it does not impose any restrictions on beliefs beyond consistency. Based on Rubinstein and Wolinsky (1994) and Foerster, Mauleon and Vannetelbosch (2021), we refine CS by requiring common knowledge of rationality.

Definition 2. Given a school choice problem $\langle I, S, q, P, F \rangle$, the set of matchings $M \subseteq \mathbf{M}$ is rationalizable with respect to $(\succ_i)_{i \in I}$ and $(\tau_i)_{i \in I}$ if for all $\boldsymbol{\mu} \in M$ there exists beliefs $(c_i)_{i \in I}$ such that:

(i) the tuple $(\boldsymbol{\mu}, (c_i)_{i \in I})$ is CS, and

(ii) for all $i \in I$ and all $\boldsymbol{\mu'} \in \mathbf{M}_{|\boldsymbol{\mu_i}}$ such that $c_i(\boldsymbol{\mu'}) > 0$, $\boldsymbol{\mu'} \in M$.

The matching $\boldsymbol{\mu} \in \mathbf{M}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i)_{i \in I}$ and $(\tau_i)_{i \in I}$ if there exists a rationalizable set $M \subseteq \mathbf{M}$ such that $\boldsymbol{\mu} \in M$.

A set of matchings is rationalizable if for each matching in the set, there exist beliefs such that the tuple of matching and beliefs is CS, and the support of these beliefs belongs to this set. A matching is rationalizable conjectural stable (RCS) if this matching belongs to a rationalizable set. Let $\boldsymbol{\Lambda}$ be the set of matchings that are RCS under perfect substitutes.

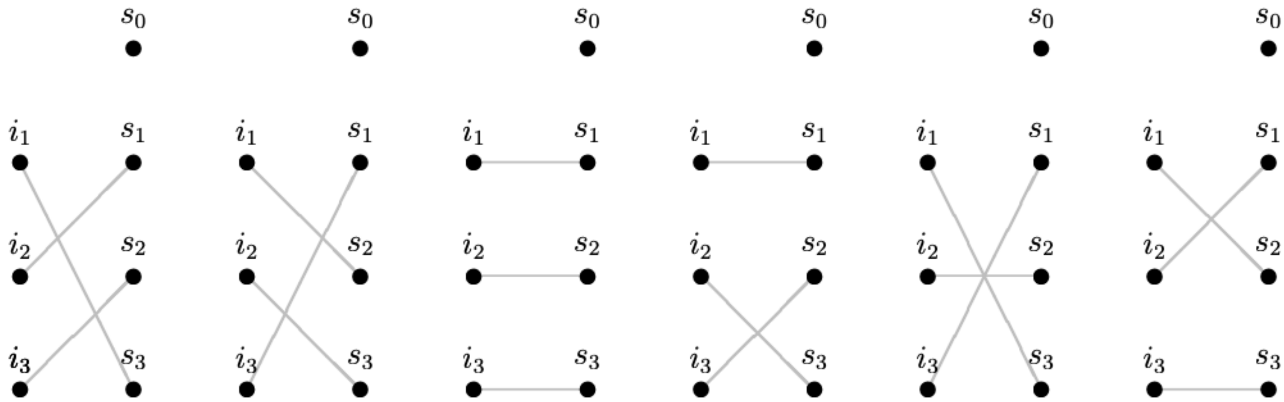
We now provide en example to illustrate our concept and to show that a matching may be RCS although it would not be stable under complete information.

Example 1. Consider a school choice problem $\langle I, S, q, P, F \rangle$ with $I = \{i_1, i_2, i_3, \}$ and $S = \{s_1, s_2, s_3\}$. Students' preferences and schools' priorities and capacities are as follows.

Students				Schools		
P_{i_1}	P_{i_2}	P_{i_3}		F_{s_1}	F_{s_2}	F_{s_3}
			q_s	1	1	1
s_3	s_1	s_2		i_3	i_1	i_2
s_1	s_2	s_3		i_1	i_2	i_3
s_2	s_3	s_1		i_2	i_3	i_1
s_0	s_0	s_0				

$$\boldsymbol{\mu^1} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \boldsymbol{\mu^2} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \boldsymbol{\mu^3} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$\boldsymbol{\mu^4} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \boldsymbol{\mu^5} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \boldsymbol{\mu^6} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$



$\boldsymbol{\mu^1} = \boldsymbol{\mu}^{\text{DA}}$ $\boldsymbol{\mu^2} = \boldsymbol{\mu}^{\text{DA}*}$ $\boldsymbol{\mu^3}$ $\boldsymbol{\mu^4}$ $\boldsymbol{\mu^5}$ $\boldsymbol{\mu^6}$

Figure 2: Rationalizable conjectural stable matchings.

In Example 1 the matchings $\boldsymbol{\mu^1} = \boldsymbol{\mu}^{\text{DA}}$, $\boldsymbol{\mu^2} = \boldsymbol{\mu}^{\text{DA}*}$ and $\boldsymbol{\mu^3}$ are stable under complete information.[12] In the presence of private matches, the set $\{\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}, \boldsymbol{\mu^4}, \boldsymbol{\mu^5}, \boldsymbol{\mu^6}\}$ is rationalizable and each $\boldsymbol{\mu} \in \{\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}, \boldsymbol{\mu^4}, \boldsymbol{\mu^5}, \boldsymbol{\mu^6}\}$ is RCS. For instance, $\boldsymbol{\mu^4}$ is RCS when student i_1 believes that she is in $\boldsymbol{\mu^3}$, student i_2 believes that she is in $\boldsymbol{\mu^2}$ and student i_3 believes that she is in $\boldsymbol{\mu^1}$. That is, $\boldsymbol{\mu^4}$ is RCS under $c_{i_1}(\boldsymbol{\mu^3}) = 1$, $c_{i_2}(\boldsymbol{\mu^2}) = 1$ and $c_{i_3}(\boldsymbol{\mu^1}) = 1$. Those beliefs are rationalizable since they put weight

[12] $\boldsymbol{\mu}^{\text{DA}}$ is the DA matching when students propose while $\boldsymbol{\mu}^{\text{DA}*}$ is the DA matching when schools propose.

on matchings that are stable under complete information. Similarly, $\boldsymbol{\mu^5}$ is RCS under $c_{i_1}(\boldsymbol{\mu^1}) = 1$, $c_{i_2}(\boldsymbol{\mu^3}) = 1$ and $c_{i_3}(\boldsymbol{\mu^2}) = 1$ and $\boldsymbol{\mu^6}$ is RCS under $c_{i_1}(\boldsymbol{\mu^2}) = 1$, $c_{i_2}(\boldsymbol{\mu^1}) = 1$ and $c_{i_3}(\boldsymbol{\mu^3}) = 1$. In Figure 2 we depict those six matchings with private matches.

Let $\boldsymbol{\Sigma^{sh}} = \{\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'} \in \boldsymbol{\Sigma} \text{ and } \mu_{i,s} = 2 \Leftrightarrow \mu'_{i,s} = 1 \text{ for } i \in I, s \in S^*\}$. That is, $\boldsymbol{\Sigma^{sh}}$ is the set of matchings with only private matches that are stable when those private matches are public ones. In other words, those are the stable matchings under complete information where each public match is replaced by a private match. Let $\boldsymbol{\Sigma_i^{sh}} = \{\boldsymbol{\mu_i} \mid \boldsymbol{\mu} \in \boldsymbol{\Sigma^{sh}}\}$ be the set of matching vectors of student $i \in I$ that are part of stable matchings.

The next proposition shows that any private matching where each match belongs to some matching that is stable under complete information is RCS.

Proposition 1. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem. The matching $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0,2}}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i)_{i\in I}$ and $(\tau_i)_{i\in I}$ if*

(i) $\boldsymbol{\mu} \in \{\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'_i} \in \boldsymbol{\Sigma_i^{sh}},\ i \in I\}$,

(ii) *for all $i \in I$, $c_i(\boldsymbol{\mu'}) = 1$ for some $\boldsymbol{\mu'} \in \boldsymbol{\Sigma^{sh}}$ such that $\boldsymbol{\mu_i} = \boldsymbol{\mu'_i}$,*

(iii) *for all $i \in I$, $\tau_i(\boldsymbol{\mu})$ is fully not informative.*

Proof. For each $\boldsymbol{\mu} \in \{\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'_i} \in \boldsymbol{\Sigma_i^{sh}},\ i \in I\}$,

(i) each student $i \in I$ believes with probability one to be in a matching with only private matches that is stable if all private matches were public matches (no incentive to deviate to another assignment);

(ii) students' beliefs are consistent with fully not informative private signals and put weight only on matchings belonging to $\{\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'_i} \in \boldsymbol{\Sigma_i^{sh}},\ i \in I\}$.

Hence, $\{\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'_i} \in \boldsymbol{\Sigma_i^{sh}},\ i \in I\}$ is rationalizable and each $\boldsymbol{\mu} \in \{\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0,2}} \mid \boldsymbol{\mu'_i} \in \boldsymbol{\Sigma_i^{sh}},\ i \in I\}$ are RCS. □

Example 2. Consider a school choice problem $\langle I, S, q, P, F\rangle$ with $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2, s_3\}$. Students' preferences and schools' priorities and capacities are as follows.

Students

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_1	s_1	s_2	s_1
s_2	s_2	s_1	s_3
s_3	s_3	s_3	s_2
s_0	s_0	s_0	s_0

Schools

	F_{s_1}	F_{s_2}	F_{s_3}
q_s	2	1	1
	i_1	i_1	i_2
	i_3	i_2	i_3
	i_4	i_4	i_4
	i_2	i_3	i_1

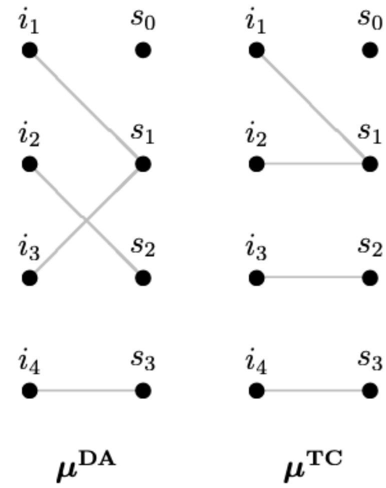


Figure 3: DA and TTC matchings with only private matches (light gray lines).

Let $\boldsymbol{\mu}^{\mathbf{DA}}$ be the Deferred Acceptance (DA) matching and $\boldsymbol{\mu}^{\mathbf{TC}}$ be the Top Trading Cycles (TTC) matching. Suppose that all matches are private. In Example 2 the set $\{\boldsymbol{\mu}^{\mathbf{DA}}, \boldsymbol{\mu}^{\mathbf{TC}}\}$ is rationalizable and both $\boldsymbol{\mu}^{\mathbf{DA}}$ and $\boldsymbol{\mu}^{\mathbf{TC}}$ are RCS. In Figure 3 we depict $\boldsymbol{\mu}^{\mathbf{DA}}$ and $\boldsymbol{\mu}^{\mathbf{TC}}$ with private matches. Notice that $\boldsymbol{\mu}^{\mathbf{DA}}$ is stable while $\boldsymbol{\mu}^{\mathbf{TC}}$ is not stable under complete information. However, $\boldsymbol{\mu}^{\mathbf{TC}}$ is RCS when all matches are private. Students i_1 and i_4 hold the belief that they are facing the DA matching $\boldsymbol{\mu}^{\mathbf{DA}}$ while the two other students hold the belief that they are in the TTC matching $\boldsymbol{\mu}^{\mathbf{TC}}$. That is, $c_{i_1}(\boldsymbol{\mu}^{\mathbf{DA}}) = c_{i_4}(\boldsymbol{\mu}^{\mathbf{DA}}) = 1$ and $c_i(\boldsymbol{\mu}^{\mathbf{TC}}) = 1$ for $i \neq i_1, i_4$. Given their beliefs that are consistent and rationalizable, all of them have no incentive to deviate from their assignments. Thus, this is an example of a situation where matches (i_2, s_1) and (i_3, s_2) that are not stable under complete information are now stabilized in the presence of private matches. Notice that the Boston matching obtained from the Immediate Acceptance (IA) algorithm cannot be rationalizable in this example.

Proposition 2 tells us that, each matching with only private matches is RCS if each student who would have incentives to deviate under complete information believes she is in fact facing another matching that is stable under complete information. Given $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$, let $\boldsymbol{\mu}^{\mathbf{c}}$ be such that $\mu^c_{i,s} = 1 \Leftrightarrow \mu_{i,s} = 2$ for $i \in I$, $s \in S^*$.

Proposition 2. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem. The matching $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i)_{i\in I}$ and $(\tau_i)_{i\in I}$ if for each student i such that, for some $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$, either*

(i) $\boldsymbol{\mu}^{\mathbf{c}} \prec_i \boldsymbol{\mu'}$ *where* $\sigma(i, \boldsymbol{\mu'}) = s$ *and* $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu}^{\mathbf{c}}) = s\} < q_s$*, or*

(ii) $\boldsymbol{\mu}^{\mathbf{c}} \prec_i \boldsymbol{\mu'}$ *where* $\sigma(i, \boldsymbol{\mu'}) = s$, $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu}^{\mathbf{c}}) = s\} = q_s$ *and* $i \in \Phi(s, j)$ *for some* j *such that* $\sigma(j, \boldsymbol{\mu'}) = s$*,*

then there is $\boldsymbol{\mu''} \in \mathbf{M}_{|\boldsymbol{\mu_i}} \cap \boldsymbol{\Sigma}$*.*

Proof.

- Let $J \subseteq I$ be the set of students such that each student $i \in J$ prefers some $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$ to $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ and she could deviate to $\boldsymbol{\mu'}$ under complete information (either the school she would be assigned to under $\boldsymbol{\mu'}$ has not full capacity at $\boldsymbol{\mu}$ or she has higher priority than some student assigned to this school under $\boldsymbol{\mu}$). Suppose student $i \in J$ holds the belief $c_i(\boldsymbol{\mu''}) = 1$ where $\boldsymbol{\mu''} \in \mathbf{M}_{|\boldsymbol{\mu_i}} \cap \boldsymbol{\Sigma}$. Since $\boldsymbol{\mu''} \in \mathbf{M}_{|\boldsymbol{\mu_i}} \cap \boldsymbol{\Sigma}$ is stable under complete information, she has no incentive to deviate.

- Take any student $i \notin J$. Either she does not prefer any $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$ to $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ or she prefers some $\boldsymbol{\mu'} \in \mathbf{M}_{|\mathbf{0},\mathbf{1}}$ to $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ but the school she would be assigned to under $\boldsymbol{\mu'}$ has full capacity at $\boldsymbol{\mu}$ and she has lower priority than any student assigned to this school under $\boldsymbol{\mu}$. Suppose she holds the correct belief $c_i(\boldsymbol{\mu}) = 1$ then she has no incentive to deviate.

- Beliefs are consistent with fully not informative private signals.

- Thus, the matching $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ is CS. The matching $\boldsymbol{\mu} \in \mathbf{M}_{|\mathbf{0},\mathbf{2}}$ is RCS since the set formed by the matching $\boldsymbol{\mu}$ and each matching $\boldsymbol{\mu''}$ such that $c_i(\boldsymbol{\mu''}) = 1$ for some $i \in J$ is rationalizable (notice that each $\boldsymbol{\mu''}$ is CS under correct beliefs).

□

Example 2 illustrates Proposition 2. Student i_4 is the only student who has incentives to deviate from the TTC matching $\boldsymbol{\mu}^{\mathbf{TC}}$ under complete information since she prefers s_1 to s_3 and she has higher priority than i_2 at s_1. In the DA matching $\boldsymbol{\mu}^{\mathbf{DA}}$ she is matched to s_3 but $\boldsymbol{\mu}^{\mathbf{DA}}$ is stable under complete information (i_3 is now matched to s_1 and has higher priority than i_4). Thus, $\boldsymbol{\mu}^{\mathbf{TC}}$ is RCS if student i_4 believes she is in $\boldsymbol{\mu}^{\mathbf{DA}}$ while the three other students correctly believe that they are in $\boldsymbol{\mu}^{\mathbf{TC}}$. Notice that in Example 2 the TTC matching $\boldsymbol{\mu}^{\mathbf{TC}}$ coincides with the Efficiency-Adjusted Deferred Acceptance (EADA) matching $\boldsymbol{\mu}^{\mathbf{EA}}$.

We next show that, in general, the EADA matching $\boldsymbol{\mu}^{\mathbf{EA}}$ is rationalizable conjectural stable (RCS) in the presence of private matches.

4 The EADA algorithm

Tan and Yu (2014) propose the simplified Efficiency-Adjusted Deferred Acceptance (EADA) mechanism which is outcome equivalent to Kesten's (2010) EADA mechanism for school choice problems with consent.[13] Cerrone, Hermstrüwer and Kesten (2023) run an experiment on school choice with consent. The EADA mechanism outperforms the DA mechanism both in terms of efficiency and truth-telling rates, even though the EADA mechanism is not strategy proof. When all students consent, efficiency further increases, while truth-telling rates decrease relative to the EADA mechanism where some of the students do not consent.[14]

Definition 3. Given a school choice problem $\langle I, S, q, P, F\rangle$, a school $s \in S^*$ is under-demanded at $\boldsymbol{\mu}$ if no student $i \in I$ prefers s to her assignment under $\boldsymbol{\mu}$.

That is, $s \in S^*$ is under-demanded at $\boldsymbol{\mu}$ if $\nexists\, i \in I$ such that $sP_i\sigma(i, \boldsymbol{\mu})$. Notice that s is under-demanded at the DA matching $\boldsymbol{\mu}^{\mathbf{DA}}$ if and only if s never rejects any student throughout the DA algorithm.

The Efficiency-Adjusted Deferred Acceptance (EADA) mechanism when all students consent finds a matching by means of the following algorithm.

[13] Each student is asked whether she consents to give up her priorities at desired schools if doing so does not hurt her own assignment but potentially improves the assignments of other students.

[14] Dŏgan and Ehlers (2021) investigate efficient and minimally unstable Pareto improvements over the DA mechanism and they show that the EADA mechanism provides an assignment that is minimally unstable among efficient assignments.

Round 0. Run the Deferred Acceptance (DA) algorithm for the school choice problem $\langle I, S, q, P, F \rangle$. Let $\widehat{\boldsymbol{\mu}}^0 = \boldsymbol{\mu}^{\mathbf{DA}}(I, S, q, P, F)$, where $\boldsymbol{\mu}^{\mathbf{DA}}(I, S, q, P, F)$ is the outcome of the DA algorithm. Let $\underline{S}^0$ be the set of schools that are under-demanded at round 0.[15] Let $\underline{I}^0$ be the set of students matched under $\widehat{\boldsymbol{\mu}}^0$ to some school $s \in \underline{S}^0$.

Round k, $k \geq 1$. Let

$$I^k = I \setminus \bigcup_{l=0}^{k-1} \underline{I}^l, \; S^k = S \setminus \bigcup_{l=0}^{k-1} \underline{S}^l,$$
$$q^k = (q_s)_{s \in S^k}, \; P^k = (P_i)_{i \in I^k}, \; F^k = (F_s)_{s \in S^k}.$$

Run the Deferred Acceptance (DA) algorithm for the school choice problem $\langle I^k, S^k, q^k, P^k, F^k \rangle$. Let $\boldsymbol{\mu}^{\mathbf{DA}}(I^k, S^k, q^k, P^k, F^k)$ be the outcome of the DA algorithm at round k. The outcome of round k of the EADA mechanism is the matching matrix $\widehat{\boldsymbol{\mu}}^k$ such that

$$\widehat{\mu}^k_{i,s} = \begin{cases} \widehat{\mu}^{k-1}_{i,s} & \text{for } i \in \bigcup_{l=0}^{k-1} \underline{I}^l, \\ \\ \mu^{\mathbf{DA}}_{i,s}(I^k, S^k, q^k, P^k, F^k) & \text{for } i \in I^k. \end{cases}$$

End. Stop when all schools are removed. Let $\bar{k}$ be the round at which the algorithm stops. Let $\boldsymbol{\mu}^{\mathbf{EA}}$ denote the matching obtained from the EADA mechanism and it is given by $\widehat{\boldsymbol{\mu}}^{\bar{k}}$.

Kesten (2010) shows that the EADA matching is Pareto-efficient when all students consent.[16] In addition, Tang and Yu (2014) show that, for each $k \geq 1$, the round-k DA matching of the EADA weakly Pareto dominates that of round-$(k-1)$.

Let $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ be such that $\widetilde{\mu}^{EA}_{i,s} = 2 \Leftrightarrow \mu^{EA}_{i,s} = 1$ for $i \in I, s \in S^*$. That is, $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ is simply the outcome of the EADA mechanism when all matches are private. Let $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}$ be such that $\widetilde{\mu}^k_{i,s} = 2 \Leftrightarrow \widehat{\mu}^k_{i,s} = 1$ for $i \in I, s \in S^*$. That is, $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}$ is the outcome of round k of the EADA mechanism when all matches are private. Let $k(i)$ be the

[15]The null school s_0 is always under-demanded at round 0.

[16]Reny (2012) introduces the Priority-Efficient (PE) mechanism that always selects a Pareto efficient matching that dominates the DA stable matching, but PE is not strategy-proof. The EADA mechanism always yields the unique priority-efficient matching.

round k at which student i is removed during the process of the EADA mechanism, i.e. $i \in \underline{I}^{k(i)}$. Theorem 1 shows that the EADA matching is RCS.[17]

Theorem 1. *Let $\langle I, S, q, P, F \rangle$ be a school choice problem. The EADA matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i)_{i \in I}$ if*

(i) *for $k = 0, \ldots, \bar{k}$, $c_i(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}) = 1$ for all $i \in \underline{I}^k$,*

(ii) *for all $i \in I$, private signals $(\tau_i)_{i \in I}$ are such that*

$$(\tau_i(\boldsymbol{\mu}))_{j,s} = \begin{cases} \{\mu_{j,s}\} & \text{if} \quad j \in \bigcup_{l=0}^{k(i)} \underline{I}^l, \\ \{\mu_{j,s}\} & \text{if} \quad \mu_{j,t} = 1 \text{ for some } t \in S^*, \\ \{0, 2\} & \text{if} \quad \mu_{j,t} = 2 \text{ for some } t \in S^* \text{ and } j \notin \bigcup_{l=0}^{k(i)} \underline{I}^l, \end{cases}$$

for all j, s $(j \neq i)$.

The set $\{\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}, \widetilde{\boldsymbol{\mu}}^{\mathbf{1}}, \ldots, \widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}-\mathbf{1}}, \widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}\}$ is rationalizable.

Proof. The set $\{\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}, \widetilde{\boldsymbol{\mu}}^{\mathbf{1}}, \ldots, \widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}-\mathbf{1}}, \widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}\}$ is rationalizable since

0. $\widetilde{\mu}^0 = \widetilde{\mu}^{\mathbf{DA}}$ is CS under $c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}) = 1$ for all $i \in I$. Each student $i \in I$ has no incentive to deviate since $\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}$ is the DA matching and is stable.

1. $\widetilde{\mu}^1$ is CS under $c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}) = 1$ for all $i \in \underline{I}^0$ and $c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{1}}) = 1$ for all $i \in I^1 = I \setminus \underline{I}^0$. Each student $i \in \underline{I}^0$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}$ and each student $i \in I \setminus \underline{I}^0$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\mathbf{1}}$.

$\vdots$

k. $\widetilde{\mu}^k$ is CS under

$$\begin{aligned} c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}) &= 1 \text{ for all } i \in \underline{I}^0, \\ c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{1}}) &= 1 \text{ for all } i \in \underline{I}^1, \\ &\vdots \\ c_i(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}-\mathbf{1}}) &= 1 \text{ for all } i \in \underline{I}^{k-1}, \\ c_i(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}) &= 1 \text{ for all } i \in I^k = I \setminus \cup_{l=0}^{k-1} \underline{I}^l, \end{aligned}$$

[17] Doğan and Ehlers (2025) show that the EADA matching forms a singleton myopic-farsighted stable set when one side is farsighted and the other is myopic. See Mauleon, Vannetelbosch and Vergote (2009), Herings, Mauleon and Vannetelbosch (2020, 2025), for related contributions on farsightedness in matching problems.

Each student $i \in \underline{I}^0$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}$, each student $i \in \underline{I}^1$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\mathbf{1}}$, ..., each student $i \in I^k = I \setminus \cup_{l=0}^{k-1} \underline{I}^l$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}$.

⋮

$\bar{k}$. $\widetilde{\mu}^{\bar{k}} = \widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ is CS under

$$
\begin{aligned}
c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}) &= 1 \text{ for all } i \in \underline{I}^0, \\
c_i(\widetilde{\boldsymbol{\mu}}^{\mathbf{1}}) &= 1 \text{ for all } i \in \underline{I}^1, \\
&\vdots \\
c_i(\widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}\mathbf{-1}}) &= 1 \text{ for all } i \in \underline{I}^{\bar{k}-1}, \\
c_i(\widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}}) &= 1 \text{ for all } i \in I^{\bar{k}} = \underline{I}^{\bar{k}} = I \setminus \cup_{l=0}^{\bar{k}-1} \underline{I}^l,
\end{aligned}
$$

Each student $i \in \underline{I}^0$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\mathbf{0}}$, ..., each student $i \in \underline{I}^{\bar{k}-1}$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}\mathbf{-1}}$, each student $i \in I^{\bar{k}} = \underline{I}^k = I \setminus \cup_{l=0}^{\bar{k}-1} \underline{I}^l$ has no incentive to deviate since she believes that all students are matched as in $\widetilde{\boldsymbol{\mu}}^{\bar{\boldsymbol{k}}} = \widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$.

Beliefs are consistent with private signals since each student $i \in \underline{I}^l$ only observes private matches $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{l}}_{\boldsymbol{j}}$ for $j \in \bigcup_{l'=0}^{l} \underline{I}^{l'}$, for $l = 0, \ldots, \bar{k}$. □

Remark 2. Let $\langle I, S, q, P, F \rangle$ be a school choice problem. The EADA matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i)_{i \in I}$ if

(i) for $k = 0, \ldots, \bar{k}$, $c_i(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{k}}) = 1$ for all $i \in \underline{I}^k$,

(ii) for all $i \in I$, private signals $(\tau_i)_{i \in I}$ are fully not informative.

Example 3 (Kesten, 2010; Tang and Yu, 2014)**.** Consider a school choice problem $\langle I, S, q, P, F \rangle$ with $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$. Students' preferences and schools' priorities and capacities are as follows.

Students

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_3	s_1	s_1	s_4
s_1	s_1	s_4	s_2	s_5	s_1
s_3	s_5	s_2	s_4	s_2	s_3
s_0	s_0	s_0	s_0	s_0	s_2
					s_5
					s_0

Schools

	F_{s_1}	F_{s_2}	F_{s_3}	F_{s_4}	F_{s_5}
q_s	1	1	1	1	2
	i_2	i_3	i_1	i_4	i_1
	i_1	i_6	i_6	i_3	i_2
	i_5	i_4	i_2	i_6	i_3
	i_6	i_1	i_3	i_2	i_4
	i_4	i_2	i_5	i_5	i_5
	i_3	i_5	i_4	i_1	i_6

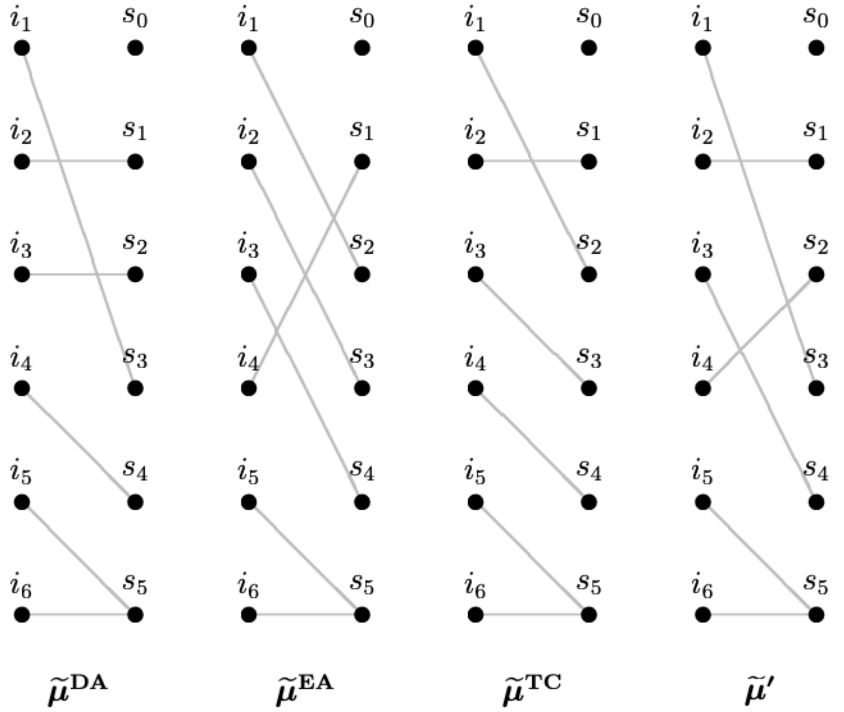


Figure 4: DA, EADA and TTC matchings in Example 3.

Example 3 illustrates Theorem 1. The outcomes of the various algorithms are:

$$\mu^{\mathbf{DA}} = \{(i_1, s_3, 2), (i_2, s_1, 2), (i_3, s_2, 2), (i_4, s_4, 2), (i_5, s_5, 2), (i_6, s_5, 2)\},$$
$$\mu^{\mathbf{EA}} = \{(i_1, s_2, 2), (i_2, s_3, 2), (i_3, s_4, 2), (i_4, s_1, 2), (i_5, s_5, 2), (i_6, s_5, 2)\},$$
$$\mu^{\mathbf{TC}} = \{(i_1, s_2, 2), (i_2, s_1, 2), (i_3, s_3, 2), (i_4, s_4, 2), (i_5, s_5, 2), (i_6, s_5, 2)\}.$$

Notice that $\boldsymbol{\mu}^{\mathbf{DA}}$ is the unique stable matching under complete information. The matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ is RCS given that students i_1, i_2, i_3 and i_4 believe with probability one that they face the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ while students i_5 and i_6 believe with probability one that they face the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{DA}}$. Given those beliefs, none of them has an incentive to deviate. The matchings $\widetilde{\boldsymbol{\mu}}^{\mathbf{EA}}$ and $\widetilde{\boldsymbol{\mu}}^{\mathbf{DA}}$ are not the only ones that are RCS. For instance, take the matching $\widetilde{\boldsymbol{\mu}}'$ with only private matches and given by

$$\mu' = \{(i_1, s_3, 2), (i_2, s_1, 2), (i_3, s_4, 2), (i_4, s_2, 2), (i_5, s_5, 2), (i_6, s_5, 2)\}.$$

This matching $\widetilde{\boldsymbol{\mu}}'$ is RCS given that students i_1, i_2, i_3, i_4 and i_5 believe with probability one that they face the matching $\widetilde{\boldsymbol{\mu}}'$ while student i_6 believes with probability one that she faces the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{DA}}$. In addition, the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ is RCS given that students i_1, i_2, i_3 and i_5 believe with probability one that they face the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ while students i_4 and i_6 believe with probability one that they face the matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{DA}}$. Given those beliefs, none of them has an incentive to deviate and the set $\{\widetilde{\boldsymbol{\mu}}^{\mathbf{DA}}, \widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}\}$ is rationalizable. However, the TTC matching may not be RCS as we show in the next section.

Notice that Theorem 1 holds in the case where not all students consent in the EADA mechanism. Let $I^- \subseteq I$ be the set of students who do not consent. At each round k of the algorithm, preferences of the remaining students are adapted as in Tang and Yu (2014). That is, for each removed student $i \in \underline{I}^k \cap I^-$ who does not consent, each remaining school $s \in S^k$ that student i desires (i.e. $sP_i\sigma(i, \widehat{\boldsymbol{\mu}}^{k-1})$) and each remaining student $j \in I^k$ such that i is ranked above j at s (i.e. $i \in \Phi(s, j)$), we remove s from j's preference P_j (or we simply downgrade s such that $s_0 P_j s$).

5 The TTC algorithm

Abdulkadiroğlu and Sönmez (2003) introduce the Top Trading Cycles (TTC) mechanism for selecting a matching for each school problem. The TTC mechanism finds a matching by means of the following TTC algorithm.

Round 0. Set $q_s^1 = q_s$ for all $s \in S$ where q_s^1 is equal to the initial capacity of school s at round 0. Each student $i \in I$ points to the school that is ranked first in P_i. If student i points to the null school s_0, she forms a self-cycle and she is removed and assigned to s_0. Each school $s \in S$ points to the student that has the highest priority in F_s. Since the number of students and schools are finite, there is at least one cycle. A cycle ψ^0 is an ordered list of distinct schools and distinct students $(s^1, i^1, s^2, ..., s^l, i^l)$ where s^1 points to i^1 (denoted $s^1 \mapsto i^1$), i^1 points to s^2 ($i^1 \mapsto s^2$), s^l points to i^l ($s^l \mapsto i^l$) and i^l points to s^1 ($i^l \mapsto s^1$). Each school (student) can be part of at most one cycle. Every student in a cycle is assigned a seat at the school she points to and she is removed.[18] Let Ψ^0 be the set of cycles in round 0. Let $I^0 = \{j \in \psi^0 \mid \psi^0 \in \Psi^0\}$ be the set of students who are assigned to some school at round 0. For each $i \in I^0$, let $\boldsymbol{\mu_i^{\mathbf{TC}}}$ be such that

$$\mu_{i,s}^{\mathbf{TC}} = \begin{cases} 1 & \text{if } i \mapsto s \\ 0 & \text{otherwise,} \end{cases}$$

and let $\boldsymbol{\mu_i^{-\mathbf{TC}}}$ be such that

$$\mu_{i,s}^{-\mathbf{TC}} = \begin{cases} 1 & \text{if } s \mapsto i \\ 0 & \text{otherwise.} \end{cases}$$

Round k, $k \geq 1$. Notice that q_s^k keeps track of how many seats are still available at the school at round k of the TTC algorithm. Each remaining student $i \in I \setminus \cup_{l=1}^{k-1} I^l$ points to the school s that is ranked first in P_i such that $q_s^k \geq 1$. If student i points to the null school s_0, she forms a self-cycle and she is removed and assigned to s_0. Each school $s \in S$ such that $q_s^k \geq 1$ points to the student $j \in I \setminus \cup_{l=1}^{k-1} I^l$ that has the highest priority in F_s. There is at least one cycle. Every student in a cycle ψ^k is assigned a seat at the school she points to and she is removed. If a school s is part of a cycle, then its remaining capacity q_s^{k+1} is equal to $q_s^k - 1$. Let Ψ^k be the set of cycles in round k. Let $I^k = \{j \in \psi^k \mid \psi^k \in \Psi^k\}$ be the set of students who are assigned to some school at round k. For each $i \in I^k$, let $\boldsymbol{\mu_i^{\mathbf{TC}}}$ be such that

$$\mu_{i,s}^{\mathbf{TC}} = \begin{cases} 1 & \text{if } i \mapsto s \\ 0 & \text{otherwise,} \end{cases}$$

and let $\boldsymbol{\mu_i^{-\mathbf{TC}}}$ be such that

$$\mu_{i,s}^{-\mathbf{TC}} = \begin{cases} 1 & \text{if } s \mapsto i \\ 0 & \text{otherwise,} \end{cases}$$

End. The algorithm stops when all students have been removed. Let $\bar{k}$ be the step at which the algorithm stops. The matching obtained from the Top Trading

[18] If a school s is part of a cycle, then its remaining capacity q_s^2 is equal to $q_s^1 - 1$. If a school s is not part of any cycle, then its remaining capacity q_s^2 remains equal to q_s^1. If $q_s^2 = 0$, then school s is removed.

Cycles mechanism is given by

$$\boldsymbol{\mu}^{\mathbf{TC}} = \begin{pmatrix} \boldsymbol{\mu}^{\mathbf{TC}}_{\boldsymbol{i_1}} \\ \boldsymbol{\mu}^{\mathbf{TC}}_{\boldsymbol{i_2}} \\ \vdots \\ \boldsymbol{\mu}^{\mathbf{TC}}_{\boldsymbol{i_n}} \end{pmatrix}.$$

Abdulkadiroğlu and Sönmez (2003) show that the TTC mechanism is Pareto efficient and strategy-proof. TTC is also individually rational and non-wasteful, but it is not stable.[19]

Let $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ be such that $\widetilde{\mu}^{\mathbf{TC}}_{i,s} = 2 \Leftrightarrow \mu^{\mathbf{TC}}_{i,s} = 1$ for $i \in I, s \in S^*$. That is, $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ is simply the outcome of the TTC when all matches are private. Let $\widetilde{\boldsymbol{\mu}}^{-\mathbf{TC}}$ be such that $\widetilde{\mu}^{-\mathbf{TC}}_{i,s} = 2 \Leftrightarrow \mu^{-\mathbf{TC}}_{i,s} = 1$ for $i \in I, s \in S^*$. That is, $\widetilde{\boldsymbol{\mu}}^{-\mathbf{TC}}$ is the matching where each student is privately matched to the school she trades her priority in the TTC mechanism (or to the null school).

Remark 3. Let $\langle I, S, q, P, F \rangle$ be a school choice problem. The matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ is CS with respect to $(\succ_i)_{i \in I}$ but is not RCS.

The matching $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ is conjectural stable (CS) if, for $k = 0, \ldots, \bar{k}$, each $i \in I^k$ believes that each $j \notin I^k$ is matched under $\widetilde{\boldsymbol{\mu}}^{-\mathbf{TC}}_{\boldsymbol{j}}$. However, $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ is not rationalizable conjectural stable (RCS) since $\widetilde{\boldsymbol{\mu}}^{-\mathbf{TC}}$ is not CS.

Example 4 (Morrill, 2015)**.** Consider a school choice problem $\langle I, S, q, P, F \rangle$ with $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2\}$. Students' preferences and schools' priorities and capacities are as follows.

Students

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_2
s_1	s_2	s_1
s_0	s_0	s_0

Schools

	F_{s_1}	F_{s_2}
q_s	2	1
	i_1	i_2
	i_2	i_3
	i_3	i_1

[19] Pais and Pintér (2008) run an experimental study to compare the IA, the DA, and the TTC mechanisms in different informational settings. The TTC mechanism outperforms both the IA and the DA mechanisms in terms of efficiency and it is less sensitive to the amount of information that participants hold.

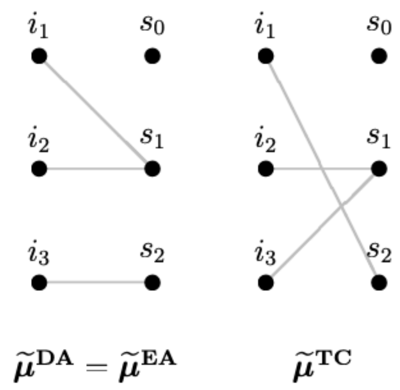


Figure 5: DA (EADA) and TTC matchings with only private matches.

In Example 4 the TTC matching is not RCS. Under complete information, there is a unique stable matching, namely $\boldsymbol{\mu}^{\mathbf{DA}}$, and it coincides with the EADA matching. The only possibility for sustaining $\widetilde{\boldsymbol{\mu}}^{\mathbf{TC}}$ as CS is that student i_3 believes that she faces the matching $\mu' = \{(i_1, s_1, 2), (i_2, s_2, 2), (i_3, s_1, 2)\}$. But, this matching μ' cannot be rationalized since student i_2 knows for sure that students i_1 and i_3 are assigned to school s_1 and i_2 has a higher priority for s_1 compared to i_3.

6 Imperfect substitutes

We now consider school choice problems where public and private matches are imperfect substitutes, i.e. each student is not indifferent between being assigned privately or publicly to a school.

Imperfect substitutes Let $P^* = (P^*_{i_1}, ..., P^*_{i_n})$ be the preference profile where P^*_i is the strict preference of student i over $S^* \times \{1, 2\}$, i.e. over the set of schools S^* (including the null school s_0) and the type of match (public or private). That is, $(s, 2)P^*_i(s', 1)$ means that student i prefers being privately assigned to school s than being publicly assigned to school s'.

Students' preferences $(P^*_i)_{i \in I}$ over schools and types of match can be straightforwardly extended to preferences $(\succ^*_i)_{i \in I}$ over matchings. We say that student i prefers the matching $\boldsymbol{\mu}$ to the matching $\boldsymbol{\mu'}$ if she prefers her assignment under $\boldsymbol{\mu}$ to her assignment under $\boldsymbol{\mu'}$. Formally,

$$\boldsymbol{\mu} \succ^*_i \boldsymbol{\mu'} \text{ if and only if } (\sigma(i, \boldsymbol{\mu}), \mu_{i,\sigma(i,\boldsymbol{\mu})}) P^*_i (\sigma(i, \boldsymbol{\mu'}), \mu'_{i,\sigma(i,\boldsymbol{\mu'})}), \text{ and}$$
$$\boldsymbol{\mu} \succeq^*_i \boldsymbol{\mu'} \text{ if and only if } \neg(\boldsymbol{\mu'} \succ^*_i \boldsymbol{\mu}).$$

We now extend the definition of rationalizable conjectural stable matching to school choice problems with imperfect substitutes.

Definition 4. Given a school choice problem $\langle I, S, q, P^*, F\rangle$ with imperfect substitutes, the tuple $(\boldsymbol{\mu}, (c_i)_{i\in I})$ is conjectural stable (CS) with respect to $(\succ_i^*)_{i\in I}$ and $(\tau_i)_{i\in I}$ if

(ia) for all $i \in I$, we have $\boldsymbol{\mu} \succeq_i^* \boldsymbol{\mu'}$ if $\sigma(i, \boldsymbol{\mu'}) = s_0$;

(ib) for all $i \in I$, we have $\boldsymbol{\mu} \succeq_i^* \boldsymbol{\mu'}$ if $\sigma(i, \boldsymbol{\mu}) = \sigma(i, \boldsymbol{\mu'})$;

(ii) for all $i \in I$, for all $\boldsymbol{\mu'}$ such that $\boldsymbol{\mu} \prec_i^* \boldsymbol{\mu'}$ and $\sigma(i, \boldsymbol{\mu}) \neq \sigma(i, \boldsymbol{\mu'})$, then for each $\boldsymbol{\mu''} \in \mathbf{M}_{|\boldsymbol{\mu_i}}$ such that $c_i(\boldsymbol{\mu''}) > 0$, we have $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu''}) = \sigma(i, \boldsymbol{\mu'})\} = q_{\sigma(i,\boldsymbol{\mu'})}$ and $i \notin \Phi(\sigma(i, \boldsymbol{\mu'}), j)$ for all j such that $\sigma(j, \boldsymbol{\mu''}) = \sigma(i, \boldsymbol{\mu'})$;

(iii) $(c_i)_{i\in I})$ are consistent with $(\tau_i(\boldsymbol{\mu}))_{i\in I}$.

Moreover, $\boldsymbol{\mu}$ is CS under correct beliefs if additionally $c_i(\boldsymbol{\mu}) = 1$ for all $i \in I$.

A tuple of a matching $\boldsymbol{\mu}$ and beliefs $(c_i)_{i\in I}$ is now conjectural stable (CS) if: (ia) each student i is at least as well off than being assigned to the null school; (ib) each student i is at least as well off than at any matching where she is assigned to the same school; (ii) for each student i who would like to be assigned to another school $s \neq \sigma(i, \boldsymbol{\mu})$ she holds a consistent conjecture about other students' assignments such that school s is full and all students assigned to school s have a higher priority than her; and (iii) students' beliefs are consistent with their private signals. There is only one main difference with respect to Definition 1: individual rationality now requires that each student also does not want to switch the type of her match without changing of school.

Again, the set of matchings $M \subseteq \mathbf{M}$ is rationalizable if for all $\boldsymbol{\mu} \in M$ there exists beliefs $(c_i)_{i\in I}$ such that (i) the tuple $(\boldsymbol{\mu}, (c_i)_{i\in I})$ is CS, and (ii) for all $i \in I$ and all $\boldsymbol{\mu'} \in \mathbf{M}_{|\boldsymbol{\mu_i}}$ such that $c_i(\boldsymbol{\mu'}) > 0$, $\boldsymbol{\mu'} \in M$. The matching $\boldsymbol{\mu} \in \mathbf{M}$ is rationalizable conjectural stable (RCS) with respect to $(\succ_i^*)_{i\in I}$ and $(\tau_i)_{i\in I}$ if there exists a rationalizable set $M \subseteq \mathbf{M}$ such that $\boldsymbol{\mu} \in M$. Let $\boldsymbol{\Lambda}^*$ be the set of matchings that are RCS under imperfect substitutes.

Let $\widetilde{P}$ be students' preferences over schools that are derived from students' preferences P^* over schools and type of match as follows. Students' preferences $\widetilde{P}$ are such that $s\widetilde{P}_i s'$ if and only if either $(s, 1)P_i^*(s', 1)$ and $(s, 1)P_i^*(s', 2)$, or $(s, 2)P_i^*(s', 1)$

and $(s,2)P_i^*(s',2)$. That is, $\widetilde{P}_i$ is student's i rank over schools based on her best match's type for each school. Let $\mathbf{\Sigma}(\widetilde{P})$ be the set of matchings that are stable under complete information when students' preferences are given by $\widetilde{P}$. Let

$$\mathbf{\Sigma}(\widetilde{P}^*) = \left\{ \boldsymbol{\mu} \in \mathbf{M} \,\middle|\, \boldsymbol{\mu'} \in \mathbf{\Sigma}(\widetilde{P}) \text{ and } \mu_{i,s} = \begin{cases} 0 & \text{if } \mu'_{i,s} = 0 \\ 1 & \text{if } \mu'_{i,s} \neq 0 \text{ and } (s,1)P_i^*(s,2) \\ 2 & \text{if } \mu'_{i,s} \neq 0 \text{ and } (s,2)P_i^*(s,1) \end{cases} ,i \in I, s \in S^* \right\}$$

be the set of matchings obtained from the set of stable matchings under complete information $\mathbf{\Sigma}(\widetilde{P})$ when public matches are replaced by private ones if they are preferred by students. Let $\mathbf{\Lambda}(\widetilde{P})$ be the set of matchings that are RCS when students' preferences are given by $\widetilde{P}$.

Example 5. Consider a school choice problem $\langle I, S, q, P^*, F\rangle$ with $I = \{i_1, i_2, i_3, \}$ and $S = \{s_1, s_2, s_3\}$ and where public and private matches are imperfect substitutes. Students' preferences and schools' priorities and capacities are as follows.

Students		
$P_{i_1}^*$	$P_{i_2}^*$	$P_{i_3}^*$
$(s_3,2)$	$(s_1,2)$	$(s_2,1)$
$(s_1,1)$	$(s_2,2)$	$(s_3,2)$
$(s_1,2)$	$(s_1,1)$	$(s_2,2)$
$(s_3,1)$	$(s_3,1)$	$(s_3,1)$
$(s_2,2)$	$(s_3,2)$	$(s_1,2)$
$(s_2,1)$	$(s_2,1)$	$(s_1,1)$
$(s_0,1)$	$(s_0,1)$	$(s_0,1)$
$(s_0,2)$	$(s_0,2)$	$(s_0,2)$

Students		
$\widetilde{P}_{i_1}$	$\widetilde{P}_{i_2}$	$\widetilde{P}_{i_3}$
s_3	s_1	s_2
s_1	s_2	s_3
s_2	s_3	s_1
s_0	s_0	s_0

Schools			
	F_{s_1}	F_{s_2}	F_{s_3}
q_s	1	1	1
	i_3	i_1	i_2
	i_1	i_2	i_3
	i_2	i_3	i_1

In Example 5 private and public matches are not perfect substitutes. Student i_1 prefers a private assignment to a public assignment for schools s_3 and s_2 while she prefers a public assignment to a private assignment for school s_1. Hence, a matching cannot be CS if either $\mu_{i_1,s_1} = 2$, $\mu_{i_1,s_2} = 1$ or $\mu_{i_1,s_3} = 1$. Similarly, a matching cannot be CS if either $\mu_{i_2,s_1} = 1$, $\mu_{i_2,s_2} = 1$, $\mu_{i_2,s_3} = 2$, $\mu_{i_3,s_1} = 1$, $\mu_{i_3,s_2} = 2$ or $\mu_{i_3,s_3} = 1$. So, $\widetilde{P}_{i_1} = s_3, s_1, s_2$, $\widetilde{P}_{i_2} = s_1, s_2, s_3$, $\widetilde{P}_{i_3} = s_2, s_3, s_1$, and it reverts to the same students' ranking over schools as in Example 1. In Figure 6 we depict those six matchings that were RCS under perfect substitutes. Obviouly, $\boldsymbol{\mu^1}$, $\boldsymbol{\mu^2}$ and $\boldsymbol{\mu^3}$ are RCS under correct beliefs: $\mathbf{\Sigma}(\widetilde{P}^*) = \{\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}\}$. Take $\boldsymbol{\mu^4}$ where the assignment

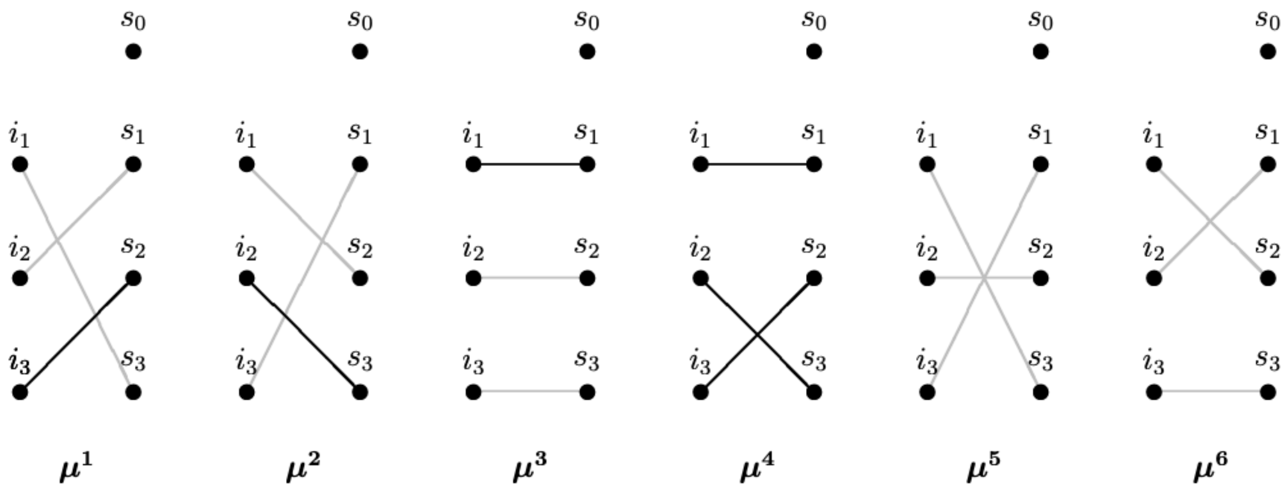


Figure 6: Only $\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}$ are RCS with imperfect substitutes.

of student i_1 to school s_1 is public. So, student i_2 (i_3) knows that student i_3 (i_2) is assigned to school s_2 (s_3). Hence, $\boldsymbol{\mu^4}$ is not CS since student i_2 has incentives to match to school s_2 where she has priority over student i_3. Take $\boldsymbol{\mu^5}$ where student i_3 is privately assigned to school s_1. She knows that student i_2 is not assigned to school s_3, otherwise student i_2's match would be public. Hence, she knows that students i_1 and i_2 are, respectively assigned to school s_3 and s_2. So, $\boldsymbol{\mu^5}$ is not CS since student i_3 has incentives to match to school s_3 where she has priority over student i_1. Take $\boldsymbol{\mu^6}$ where student i_1 is privately assigned to school s_2. She knows that student i_2 is not assigned to school s_3, otherwise student i_2's match would be public. Hence, she knows that students i_2 and i_3 are, respectively assigned to school s_1 and s_3. So, $\boldsymbol{\mu^6}$ is not CS since student i_1 has incentives to match to school s_1 where she has priority over student i_2. Thus, $\boldsymbol{\Lambda}^* = \{\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}\}$.

In Example 5 we observe that $\boldsymbol{\Sigma}(\widetilde{P}^*) = \{\boldsymbol{\mu^1}, \boldsymbol{\mu^2}, \boldsymbol{\mu^3}\} = \boldsymbol{\Lambda}^* \subseteq \boldsymbol{\Lambda}(\widetilde{P})$. Indeed, $\boldsymbol{\Lambda}(\widetilde{P})$ includes $\boldsymbol{\mu^1}$, $\boldsymbol{\mu^2}$, $\boldsymbol{\mu^3}$, the six matchings of Figure 6 but with only private matches, as well as the matchings that only differ in terms of their visibility with respect to $\boldsymbol{\mu^1}$, $\boldsymbol{\mu^2}$ and $\boldsymbol{\mu^3}$. In fact, this relationship holds in general.

Theorem 2. *Let $\langle I, S, q, P^*, F\rangle$ be a school choice problem where public and private matches are imperfect substitutes. We have $\boldsymbol{\Sigma}(\widetilde{P}^*) \subseteq \boldsymbol{\Lambda^*} \subseteq \boldsymbol{\Lambda}(\widetilde{P})$.*

Proof. We first show that $\boldsymbol{\Sigma}(\widetilde{P}^*) \subseteq \boldsymbol{\Lambda^*}$. Take some $\boldsymbol{\mu} \in \boldsymbol{\Sigma}(\widetilde{P}^*)$. We have that each student i is privately or publicly assigned under $\boldsymbol{\mu}$ to a school and she has no incentive to switch the type of her match remaining assigned to the same school. In addition, each student i is at least as well off under $\boldsymbol{\mu}$ than being privately or publicly assigned to the null school. Hence, $\boldsymbol{\mu}$ satisfies under P^* conditions (ia) and (ib) of Definition 4. Moreover, if some student i prefers some $\boldsymbol{\mu}'$ to $\boldsymbol{\mu}$ where $\sigma(i, \boldsymbol{\mu}') = s \neq \sigma(i, \boldsymbol{\mu})$ then $\#\{j \in I \mid \sigma(j, \boldsymbol{\mu}) = s\} = q_s$ and $i \notin \Phi(\sigma(i, \boldsymbol{\mu}'), j)$ for all j such that $\sigma(j, \boldsymbol{\mu}') = s$. Hence, $\boldsymbol{\mu}$ satisfies under P^* condition (ii) of Definition 4 when all students hold correct beliefs, i.e. $c_i(\boldsymbol{\mu}) = 1$ for all $i \in I$. Finally, for each $i \in I$, let her private signal τ_i be such that $(\tau_i(\boldsymbol{\mu}))_{j,s} = \{\mu_{j,s}\}$ for all j, s $(j \neq i)$. Correct beliefs are consistent with private signals and condition (iii) of Definition 4 is satisfied. Thus, we have that $(\boldsymbol{\mu}, (c_i)_{i\in I})$ is CS under P^* and since all students hold correct beliefs, $\boldsymbol{\mu}$ is RCS and so $\boldsymbol{\mu} \in \boldsymbol{\Lambda}^*$.

We next show that $\boldsymbol{\Lambda}^* \subseteq \boldsymbol{\Lambda}(\widetilde{P})$. Take some $\boldsymbol{\mu}$ and $(c_i)_{i\in I}$ so that $(\boldsymbol{\mu}, (c_i)_{i\in I})$ is CS under P^* and $\boldsymbol{\mu} \in \boldsymbol{\Lambda}^*$ under $(c_i)_{i\in I}$. Conditions (ia) and (ib) of Definition 4 imply that each student $i \in I$ is at least as well off under $\boldsymbol{\mu}$ than being privately or publicly assigned to the null school. Hence, $\boldsymbol{\mu}$ satisfies condition (i) of Definition 1 under $\widetilde{P}$. In addition, condition (ib) implies that each student $i \in I$ does not want to switch the type of her assignment under $\boldsymbol{\mu}$ without being assigned to another school. Condition (ii) of Definition 4 implies that, given students' beliefs $(c_i)_{i\in I}$, if student i would prefer to switch her school s for another school s' (possibly with another type of match) then she believes that school s' is full and that she has a lower priority than any other student already matched to school s'. Hence, $\boldsymbol{\mu}$ satisfies condition (ii) of Definition 1 under $\widetilde{P}$. Thus, we have that $(\boldsymbol{\mu}, (c_i)_{i\in I})$ is CS under $\widetilde{P}$ and $\boldsymbol{\mu}$ is RCS under $(c_i)_{i\in I}$ and so $\boldsymbol{\mu} \in \boldsymbol{\Lambda}(\widetilde{\boldsymbol{P}})$.

□

Notice that if students' preferences P^* are such that $(s, 2)P_i^*(s, 1)$ for each student $i \in I$ and each school $s \in S^*$ then $\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}(\widetilde{P})$. On the contrary, if students' preferences P^* are such that $(s, 1)P_i^*(s, 2)$ for each student $i \in I$ and each school $s \in S^*$ then $\boldsymbol{\Lambda}^* = \boldsymbol{\Sigma}(\widetilde{P})$.

7 Conclusion

We have considered priority-based school choice problems where students can form two types of matches: public matches observed by everyone and private matches generally not observed by others. We have introduced the notion of rationalizable conjectural stable (RCS) matching, which generalizes Gale and Shapley (1962)'s stability notion to accommodate private matches. We have partially characterized RCS matchings and we have shown that the Efficiency-Adjusted Deferred Acceptance (EADA) matching is RCS when private and public matches are perfect substitutes. So, hiding matches may improve efficiency while disclosing private matches may lead to Pareto inefficient assignments. When students are not indifferent between being assigned privately or publicly to a school, we have shown that the set of matchings that are RCS under imperfect substitutes is included in the set of matchings that are RCS when each student ranks schools based on her best match's type for each school. If all students prefer to be matched privately to every school, then both sets coincide. Finally, our approach provides the foundation to model and understand richer matching problems when agents can form different types of matches that vary in visibility and potentially in payoff consequences.

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