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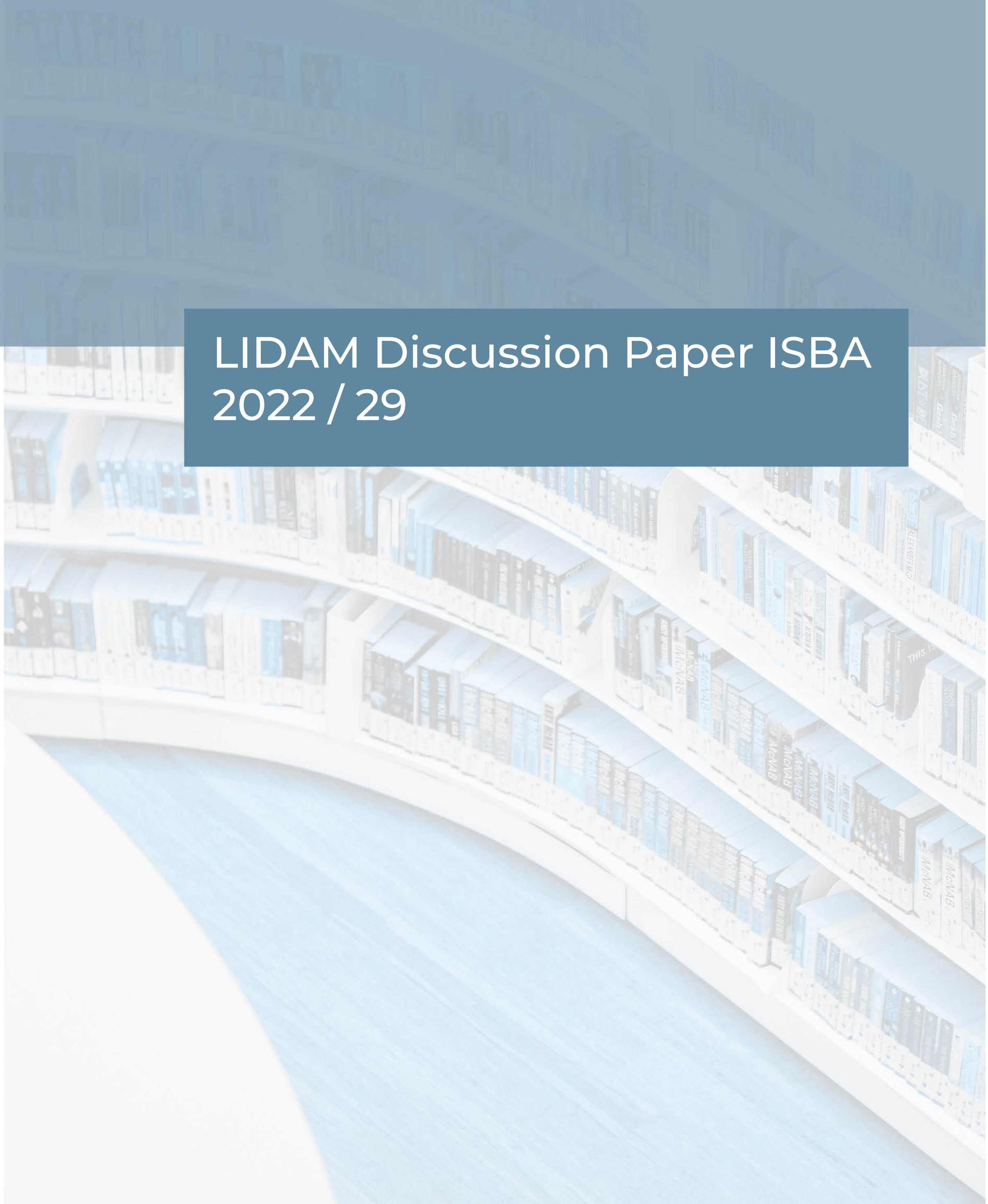

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ALLOCATION OF BENEFITS IN MUTUAL AID AND SURVIVOR FUNDS

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Abstract

Consider a group of individuals contributing in advance (ex ante) a fixed amount to a pool constituted over a reference period to mutually protect against the financial consequences of the occurrence of a predefined event such as death, survival or being diagnosed with a critical illness for instance. They agree that the sum of the contributions is shared in arrear (ex post) among those participants having experienced the predefined event. The allocation can be uniform among claiming participants, each one receiving an equal share of total contributions, or participants may be offered the choice to select in advance an expected amount of benefit, corresponding to their desired protection level. As an application, the paper considers mutual aid funds and survivor funds. The proposed systems are fully funded since contributions are paid in advance. The benefits in case the event occurs are therefore random but the volatility of the terminal payouts turns out to be limited when the number of participants gets large enough. Insurance at fair price is recovered at the limit, within infinitely large pools. The link with takaful insurance is discussed, as well as minimum guarantees to make the system more attractive.

Keywords: Risk pooling, Actuarial fairness, Mutual inheritance, Insurance at fair price, Takaful.

1 Introduction and motivation

This paper considers two closely-related risk sharing mechanisms: mutual aid funds and survivor funds that are precisely described hereafter. In both cases, each member of a pool contributes in advance (ex ante) a fixed amount to mutually protect against the financial consequences of the occurrence of a predefined event during a fixed period of time. Events under consideration include death, survival or being diagnosed with a critical illness for instance. They agree that the sum of the contributions is shared in arrear (ex post) among those participants having experienced the predefined event, providing them with financial support in adverse situations. The proposed systems are fully funded since contributions are received in advance. Thus, amounts of contributions are known and paid with certainty, avoiding any counter-party risk whereas benefits remain unknown until the end of the coverage period.

In organization theory, mutual aid is a voluntary reciprocal exchange of resources and services for mutual benefit. The mutual aid funds considered in this paper are inspired from Asian mutual aid platforms. The latter are community-based online platforms connecting individuals exposed to given perils. Participants typically share the risk of being diagnosed with severe diseases, covering many critical illnesses, including cancer, heart attack, and acute myocardial infarction for example. These platforms fill a gap in health systems, between the public cover with limited benefits and private insurance charging expensive premiums. Mutual aid platforms thus target low- and middle-income households needing cheap healthcare protection. We refer the reader e.g. to Abdikerimova and Feng (2022) and to Chen et al. (2020) for a more extensive presentation of these systems.

In most mutual aid platforms operated in Asia, participants contribute the same amount ex post, which is the total benefits divided by the number of participants (supplemented with commissions and subscription fees). The amount of contribution paid in arrear is thus random and subject to counterparty risk. While participants generally contribute equally to the total benefits, some platforms let participants' contributions vary with age to recognize different loss probabilities. In this paper, we replace the payments in arrear with contributions in advance, so that the system is fully funded and sponsors do not assume any commitment. This comes at the cost of volatility in the ex-post benefits, but it will be seen that this volatility stays within an acceptable range for moderately large numbers of participants and that insurance at fair price is recovered at the limit. Moreover, a partnering (re-)insurer may propose additional guarantees to participants, ensuring that the ex-post benefits never fall below some minimum threshold for instance. This articulates mutual aid funds to commercial insurance, offering an integrated risk management solution to participants.

The second risk-sharing vehicle considered in this paper is the survivor fund where participants contribute a fixed amount ex ante and agree that the accumulated contributions are shared among survivors, only. It can be seen as a particular case of mutual aid where the predefined event is survival until the end of the period. Survivor funds are named after Forman and Sabin (2016). Several variants exist and they are also known under various appellations, such as individual tontine accounts (Fullmer and Sabin, 2019) or pooled-survival fund (Newfield, 2014), for instance.

A survivor fund refers to a fixed-term fund that boosts the yield of investments thanks to a mutual inheritance rule among participants. According to this rule agreed at inception, the contributions are lost in case of death during the reference period and the accumulated value of the fund is distributed among survivors, only. The mechanism proposed in this paper allows for heterogeneity in death probabilities so that large communities can be built and the volatility in payouts can be reduced to acceptable limits. In Denuit et al. (2022), mortality credits are awarded to all participants, those who survive as well as those who die (participants designate beneficiaries in case of death, who can be family members or a charity, for instance) whereas here, only survivors get a share of the total ex-ante contributions.

Several ex-post sharing rules of the total contributions paid in advance among claiming participants are considered. In the first system, total contributions are shared equally among all participants having experienced the predefined event. If nobody experienced the event then contributions are reimbursed to participants. The latter situation is relevant for small communities but becomes unlikely for larger pools. The amount of initial contribution is then obtained from actuarial fairness and targeted benefit per claiming participant as specified by the sponsor. This amount depends on claiming probabilities since heterogeneity is explicitly allowed for.

Imposing equal benefits to all claiming participants may be regarded as being restrictive in some applications. For this reason, a second system where benefits paid to claiming participants are proportional to the amounts they initially contributed is considered next. It is shown that this sharing rule conflicts with actuarial fairness, in general. This is why a third system is designed where each participant is free to subscribe one or several units of protection from the mutual aid fund. Total benefits are shared ex post, equally among all claiming units. The sponsor then specifies a targeted amount of benefit per claiming unit and participants can select the appropriate number of units to reach their desired level of protection.

The contributions of this paper are as follows. Fully funded systems eliminating all default risk are proposed to allow individuals to mutually protect against a given peril. Heterogeneity is allowed for among participants who may differ in their occurrence probabilities and targeted expected benefits. Since heterogeneity is taken into account, large pools can be constituted and limiting results are derived. The latter describe how diversification operates for large numbers of participants, as well as limiting values for individual contributions. It is shown that insurance at fair price can be recoved at the limit, under mild regularity conditions, including mutual independence.

The remainder of this paper is organized as follows. Section 2 describes the individual model considered in this paper. Sections 3-5 study different ways to allocate ex post total contributions among claiming participants. The first one discussed in Section 3 divides total contributions equally among all members who experienced the predefined event. Amounts of contributions reflect occurrence probabilities and the targeted amount of benefits per participant. The other ones discussed in Sections 4-5 let the benefits vary among participants, each of them selecting his or her own targeted amount of benefits. In the first case considered in Section 4, total contribution is shared ex post in proportion of individual contributions. It is shown that this allocation rule conflicts with actuarial fairness, in general. For this reason, a third system is proposed in Section 5, where participants buy a number of protection units from the mutual aid fund, each unit providing the claiming participant with the same amount of benefit. The limiting case where the number of participants tends to infinity is considered for the different systems, under mutual independence. Section 6 compares the proposed system with takaful, the Sharia-compliant alternative to conventional insurance. The final Section 7 briefly summarizes the results obtained in this paper and discusses some possible applications and extensions. All proofs are gathered in Appendix.

2 Risk-sharing in the individual life model

Consider a group of individuals wishing to mutually protect against the financial consequences of the occurrence of a pre-defined event such as survival, death or critical illness for instance. Assume that individual i contributes an amount c_i at time 0, $i = 1, 2, \ldots, n$. This contribution may be supplemented with fees covering running expenses (that are not considered in our analysis). The amount c_i may be invested and accumulate to a_i at the end of the reference period (time 1, say). Here, we work under zero interest rate so that $a_i = c_i$. Formulas are easily extended to the case $a_i \neq c_i$ accounting for (deterministic) financial return, if needed.

Let $b = \sum_{j=1}^{n} c_j$ be the total contributions paid in advance by participants. The aim of the paper is to propose sound ways to allocate the total budget b at time 1 among those participants who experienced the predefined event. The allocation rule must be as simple and transparent as possible, to ease its acceptance by participants. At the same time, it must be actuarially fair, to avoid systematic transfers among members of the pool.

The respective probabilities to experience the event under consideration are denoted as $q_1, q_2, \dots, q_n$ for participants $1, 2, \dots, n$. These values are assumed to be known and all participants agree about them. They are only used to compute individual contributions so that any error in $q_1, q_2, \dots, q_n$ does not impact on solvency, the proposed system being fully funded and void of any default risk.

Let I_i denote the loss indicator

$$I_i = \begin{cases} 1 \text{ if individual } i \text{ experiences the predefined event before time 1,} \\ \\ 0 \text{ otherwise,} \end{cases}$$

with $\mathrm{P}[I_i = 1] = q_i = 1 - \mathrm{P}[I_i = 0]$. Also, let

$$M_n = \sum_{j=1}^{n} I_j$$

be the number of participants who experienced the event during the reference period (0,1). Depending on the situation, M_n can be the number of deaths recorded among the n participants during time interval (0,1), the number of survivors or the number of individuals diagnosed with a critical illness, for instance.

3 Uniform sharing rule

3.1 Terminal payout

Assume that participants decide that the total budget must be evenly distributed at time 1 among those M_n individuals who experienced the predefined event. Initial contributions under this uniform allocation rule are henceforth denoted as $c_{i,n}^{\text{uni}}$, with sum $b_n^{\text{uni}} = \sum_{i=1}^{n} c_{i,n}^{\text{uni}}$. The terminal payout $T_{i,n}^{\text{uni}}$ for participant i is then given by

$$T_{i,n}^{\text{uni}} = \begin{cases} \frac{1}{1+\sum_{j\neq i} I_j} b_n^{\text{uni}} \text{ if individual } i \text{ experiences the pre-defined event before time 1,} \\ \\ c_{i,n}^{\text{uni}} \text{ if nobody experiences the pre-defined event before time 1, that is, } M_n = 0, \\ \\ 0 \text{ otherwise,} \end{cases}$$

or equivalently

$$T_{i,n}^{\text{uni}} = \frac{I_i}{M_n}\mathrm{I}[M_n \geq 1]b_n^{\text{uni}} + c_{i,n}^{\text{uni}}\mathrm{I}[M_n = 0],$$

where $\mathrm{I}[\cdot]$ denotes the indicator function (equal to 1 if the event appearing between the brackets is realized and to 0 otherwise). Notice that participants just get their initial contributions back when nobody experienced the event under consideration, that is, if $I_i = 0$ for all i then each participant i just gets $c_{i,n}^{\text{uni}}$ back at time 1.

3.2 Initial contribution

To avoid systematic transfers among participants, initial contributions $c_{i,n}^{\text{uni}}$ should reflect individual expected benefits. This is expressed by actuarial fairness which requires that no participant gains nor loses by joining the pool, on average. The initial contribution must then be given by

$$c_{i,n}^{\text{uni}} = \mathrm{E}[T_{i,n}^{\text{uni}}] = \mathrm{E}\left[\frac{I_i}{M_n}\mathrm{I}[M_n \geq 1]\right] b_n^{\text{uni}} + c_{i,n}^{\text{uni}}\mathrm{P}[M_n = 0], \quad i = 1, \ldots, n,$$

which reduces to

$$c_{i,n}^{\text{uni}} = \frac{b_n^{\text{uni}}}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}\left[\sum_{j \neq i} I_j = k-1, I_i = 1\right], \quad i = 1, \ldots, n. \tag{3.1}$$

The initial contributions $c_{i,n}^{\text{uni}}$ in (3.1) are only defined up to a constant factor. In mutual aid funds, the sponsor can for instance determine the targeted average amount of benefit per claiming participant and then set b_n^{uni} to reach this target. More specifically, if the targeted average benefit per participant is β then b_n^{uni} can be obtained from

$$b_n^{\text{uni}}\mathrm{E}\left[\frac{1}{M_n}\middle| M_n \geq 1\right] = \beta \Leftrightarrow b_n^{\text{uni}} = \frac{\beta \mathrm{P}[M_n \geq 1]}{\sum_{k=1}^{n} k^{-1}\mathrm{P}[M_n = k]}. \tag{3.2}$$

Inserting this value in (3.1) gives the contributions

$$c_{i,n}^{\text{uni}} = \beta \frac{\sum_{k=1}^{n} k^{-1}\mathrm{P}\left[M_n = k, I_i = 1\right]}{\sum_{k=1}^{n} k^{-1}\mathrm{P}\left[M_n = k\right]}, \quad i = 1, \ldots, n. \tag{3.3}$$

Notice that the expected benefit for participant i is not exactly equal to β and depends on the joint distribution of the loss indicators $I_1, I_2, \ldots, I_n$.

3.3 Independent case

In the particular case where the loss indicators $I_1, I_2, \ldots, I_n$ are mutually independent, we have

$$c_{i,n}^{\mathrm{uni}} = \frac{b_n^{\mathrm{uni}} q_i}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}\left[\sum_{j \neq i} I_j = k-1\right]. \tag{3.4}$$

With the targeted benefit β in (3.2), (3.4) can be rewritten as

$$c_{i,n}^{\mathrm{uni}} = q_i \beta \frac{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[\sum_{j \neq i} I_j = k-1\right]}{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[M_n = k\right]}.$$

Example 3.1. *Assume that $I_1, I_2, \ldots$ are mutually independent. Figure 1 displays individual contributions $c_{i,n}^{\mathrm{uni}}$ when 50% of the participants (Group 1) have a probability 0.01 of experiencing the event under consideration, 30% of the participants (Group 2) have an occurrence probability 0.05 and the remaining 20% (Group 3) have an occurrence probability of 0.1. The results are given according to the number n of participants with $n \in \{50, 100, \ldots, 1000\}$, for targeted benefits $\beta = 1$. Figure 1 shows that individual contributions stabilize at level q_i as $n \to \infty$. The standard deviation of b_n^{uni}/M_n given $M_n \geq 1$ is displayed in Figure 2. We can see there its rapid decrease with n so that volatility in actual benefits becomes acceptable when the number of participants is high enough.*

Example 3.1 suggests that $c_{i,n}^{\mathrm{uni}}$ stablizes at level βq_i as $n \to \infty$ (remember that $\beta = 1$ there). This is indeed the case in general under mild regularity conditions, as shown in the next result.

Proposition 3.2. *Assume that $I_1, I_2, \ldots$ are mutually independent, that $q_{\min} = \min_{i \geq 1} q_i > 0$ and that there exists $0 < \overline{q} < 1$ such that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} q_i = \overline{q}$. Then,*

$$\lim_{n \to \infty} c_{i,n}^{\mathrm{uni}} = \beta q_i \text{ and } T_{i,n}^{\mathrm{uni}} \to I_i \beta \text{ with probability 1.}$$

The proof of Proposition 3.2 can be found in Appendix A. This result shows that within large pools, contributions correspond to pure premiums, or fair prices for an insurance contract providing the policyholder with a lump sum benefit β with probability q_i. The terminal payout tends to the amount paid by an insurer for that contract. Also, the composition of the pool (that is, the loss probabilities q_j for $j \neq i$) becomes irrelevant when $n \to \infty$ and only q_i matters.

Although the terminal payout $T_{i,n}^{\mathrm{uni}}$ for participant i in case he or she experiences the predefined event before time 1 is random, its variance decreases to 0 at a rate proportional to

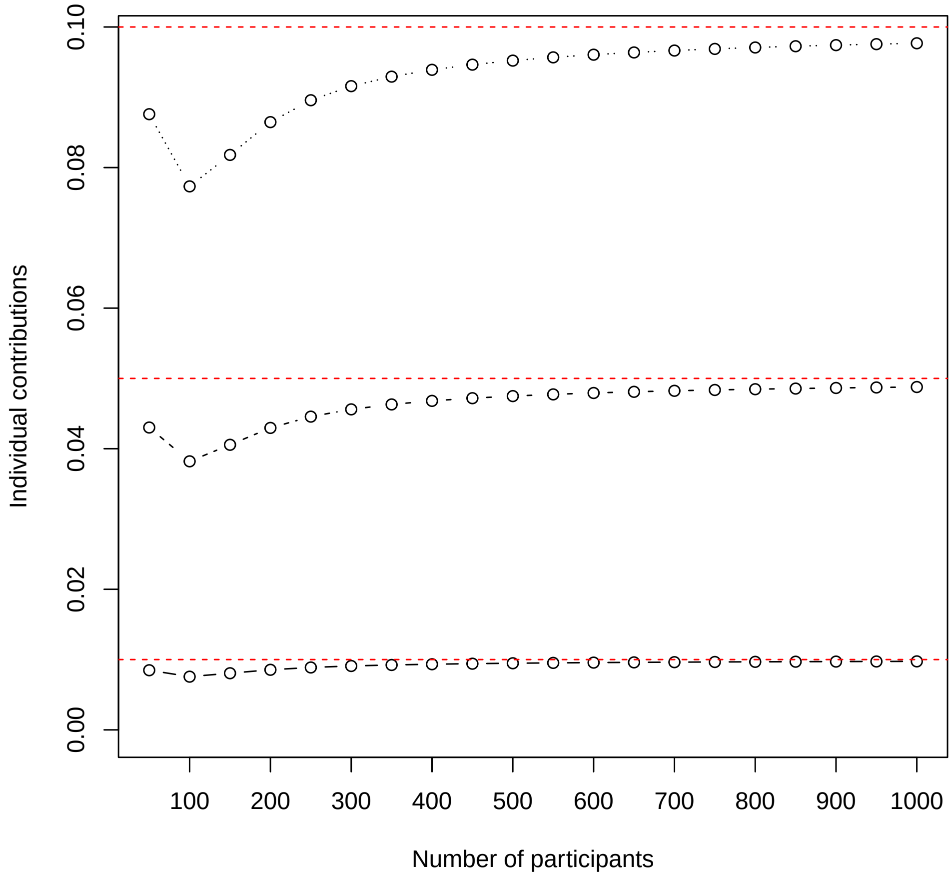

Figure 1: Individual contributions in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$), red horizontal lines at occurrence probabilities.

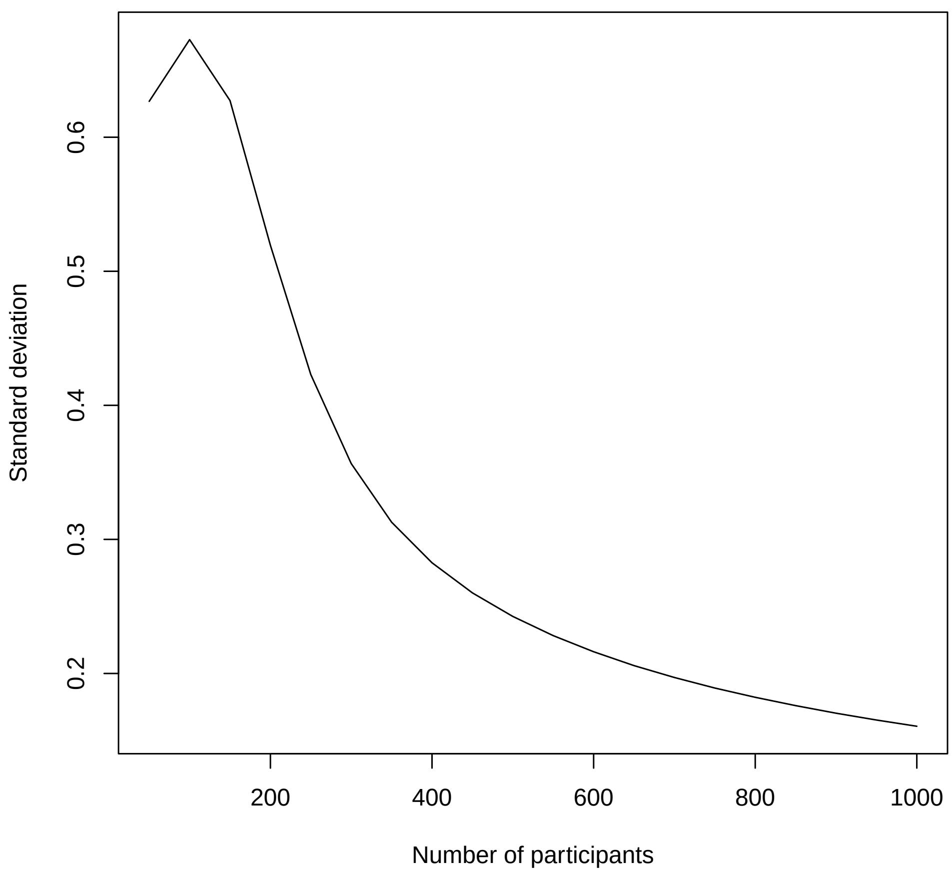

Figure 2: Standard deviation of b_n^{uni}/M_n given $M_n \geq 1$ according to the number n of participants.

the inverse of the number of participants. This allows the minimum number of participants to be set to give individuals confidence in the variability of benefits.

Proposition 3.3. *Assume that $I_1, I_2, \ldots$ are mutually independent, that $q_{\min} = \min_{i\geq 1} q_i > 0$ and that there exists $0 < \overline{q} < 1$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n q_i = \overline{q}$. Then,*

$$n\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}}\middle| I_i = 1\right] \underset{n\to\infty}{\to} 2\beta^2 \frac{\overline{q(1-q)}}{(\overline{q})^2}$$

where $\overline{q(1-q)} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n q_i(1-q_i)$.

The proof of Proposition 3.3 can be found in Appendix B. Coming back to Example 3.1, we get

$$\overline{q} = 0.5 \times 0.01 + 0.3 \times 0.05 + 0.2 \times 0.1 = 0.04$$

and

$$\overline{q(1-q)} = 0.5 \times 0.01 \times 0.99 + 0.3 \times 0.05 \times 0.95 + 0.2 \times 0.1 \times 0.9 = 0.0372$$

so that the rate of decrease established in Proposition 3.3 is equal to 46.5. The standard deviation of the terminal payout of a participant is then approximately equal to 0.215 when $n = 1000$ and to 0.068 when $n = 10000$.

When occurrence probabilities are small, Poisson approximations should appear as appealing tools. The idea is to approximate the distribution of M_n with the Poisson distribution with mean $\lambda = \sum_{j=1}^n q_j$ and the distribution of $\sum_{j\neq i} I_j$ with the Poisson distribution with mean $\lambda_i = \sum_{j=1}^n q_j - q_i$. This is the classical approximation used to derive the collective counterpart to the individual risk model. Then,

$$c_{i,n}^{\mathrm{uni}} \approx q_i \beta \frac{\sum_{k=1}^n k^{-1} \exp(-\lambda_i) \frac{\lambda_i^{k-1}}{(k-1)!}}{\sum_{k=1}^n k^{-1} \exp(-\lambda) \frac{\lambda^k}{k!}}.$$

The next example suggests that the accuracy of the Poisson approximation may be enough for practical applications.

Example 3.4 (Poisson approximation)**.** *Consider again the pool described in Example 3.1 except that occurrence probabilities are 10 times smaller. Specifically, participants in Group 1 have a probability 0.001 of experiencing the event under consideration, participants in Group 2 have an occurrence probability 0.005 and participants in Group 3 have an occurrence probability of 0.01. Figure 3 displays the relative differences between individual contributions $c_{i,n}^{\mathrm{uni}}$ and the values obtained under the Poisson approximation. The results are given according to the number n of participants with $n \in \{50, 100, \ldots, 1000\}$. We can see there that the*

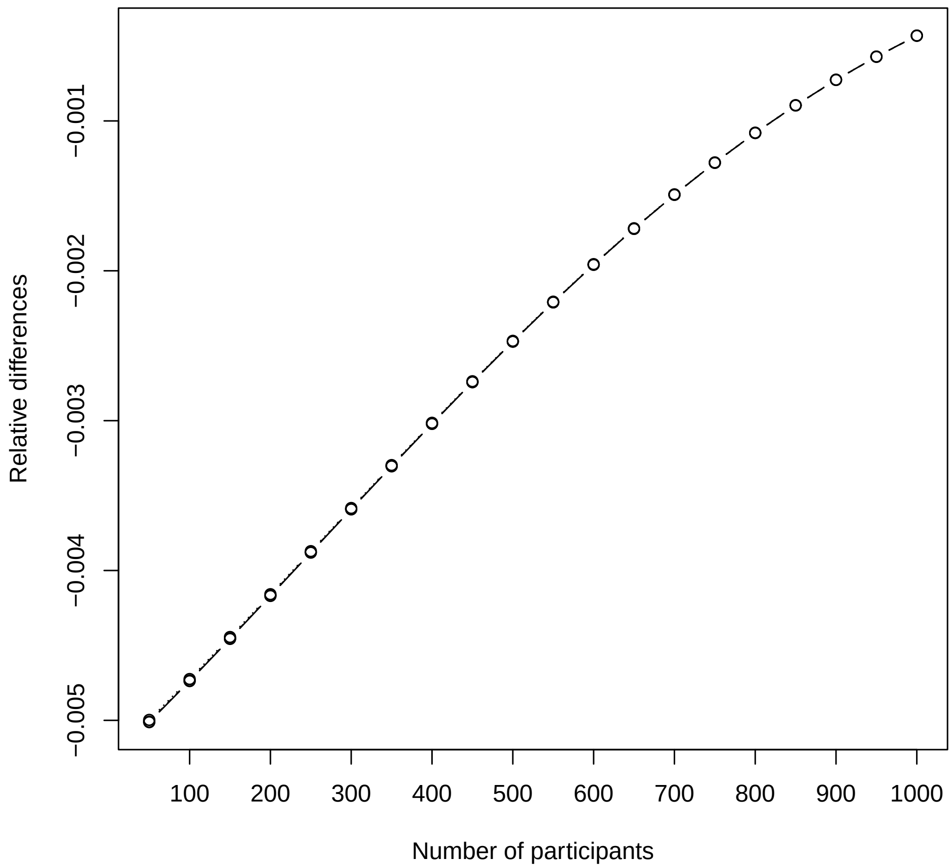

Figure 3: Relative differences for participants in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$).

relative differences are very small and almost identical across the three groups (variations between groups are so small that they are not visible from Figure 3, in accordance with Proposition 3.5 below).

The next result provides us with a sufficient condition and a uniform approximation for large pool sizes.

Proposition 3.5. *Assume that* $I_1, I_2, \ldots$ *are mutually independent, that* $\sup_{1\le i\le n} q_i \to 0$ *as* $n \to \infty$*, and that* $\lim_{n\to\infty} \sum_{i=1}^{n} q_i = \lambda > 0$*. Then,*

$$\lim_{n\to\infty} c_{i,n}^{\text{uni}} = \beta q_i \frac{\mathrm{E}\left[\frac{1}{1+N}\right]}{\mathrm{E}\left[\frac{1}{N}\middle|\, N \ge 1\right]},$$

where the random variable N *obeys the Poisson distribution with parameter* λ*.*

The proof of Proposition 3.5 can be found in Appendix C. This explains why the relative differences in Example 3.4 are so close for the three groups under consideration.

4 Proportional sharing rule

4.1 Terminal payout

Under the uniform sharing rule and as the pool size is large, all participants get on average the targeted amount β of benefit in case the event occurs. The initial contributions reflect the values of occurrence probabilities. Let us now investigate the situation where each participant is willing to get a desired level of compensation proportional to their initial contribution in case the predefined event occurs. Let $c_{i,n}^{\text{prop}}$ denote the amount of individual contribution for participant i and let $b_n^{\text{prop}} = \sum_{i=1}^{n} c_{i,n}^{\text{prop}}$ be the total contribution to be shared among the claiming participants. In this case, b_n^{prop} is no more evenly distributed among all participants who experienced the event under consideration but the sharing rule must account for the individual targeted benefits reflected in their respective initial contributions $c_{i,n}^{\text{prop}}$.

Assuming that b_n^{prop} is distributed among those participants who experienced the predefined event in proportion to their initial contributions $c_{i,n}^{\text{prop}}$, the terminal payout $T_{i,n}^{\text{prop}}$ for participant i is then given by

$$T_{i,n}^{\text{prop}} = \begin{cases} \frac{c_{i,n}^{\text{prop}}}{c_{i,n}^{\text{prop}} + \sum_{j \neq i} c_{j,n}^{\text{prop}} I_j} b_n^{\text{prop}} \text{ if individual } i \text{ experiences the pre-defined event before time 1,} \\ \\ c_{i,n}^{\text{prop}} \text{ if nobody experiences the pre-defined event before time 1, that is, } M_n = 0, \\ \\ 0 \text{ otherwise,} \end{cases}$$

or equivalently

$$T_{i,n}^{\text{prop}} = \frac{c_{i,n}^{\text{prop}} I_i}{c_{i,n}^{\text{prop}} I_i + \sum_{j \neq i} c_{j,n}^{\text{prop}} I_j} \mathrm{I}[M_n \geq 1] b_n^{\text{prop}} + c_{i,n}^{\text{prop}} \mathrm{I}[M_n = 0].$$

As in the uniform sharing rule, participants just get their initial contributions back when nobody experienced the event under consideration.

4.2 Initial contribution

The condition for actuarial fairness which requires that no participant gains nor loses by joining the pool, on average, is now written

$$c_{i,n}^{\text{prop}} = \mathrm{E}[T_{i,n}^{\text{prop}}] = q_i \mathrm{E}\left[\frac{c_{i,n}^{\text{prop}}}{\sum_{j=1}^n c_{j,n}^{\text{prop}} I_j}\Big| I_i = 1\right] b_n^{\text{prop}} + c_{i,n}^{\text{prop}} \mathrm{P}[M_n = 0], \quad i = 1, \ldots, n,$$

or equivalently

$$\mathrm{P}[M_n \geq 1] = q_i h_{i,n}((c_{1,n}^{\text{prop}}, \ldots, c_{n,n}^{\text{prop}})), \quad i = 1, \ldots, n, \tag{4.1}$$

where, for $\boldsymbol{c}_n = (c_1, \ldots, c_n)$, the functions $h_{i,n} : \mathbb{R}_+^n \mapsto \mathbb{R}$ are defined as

$$h_{i,n}(\boldsymbol{c}_n) = \mathrm{E}\left[\frac{\sum_{j=1}^n c_j}{\sum_{j=1}^n c_j I_j}\Big| I_i = 1\right], \quad i = 1, \ldots, n.$$

The initial contributions $c_{i,n}^{\text{prop}}$ are still only defined up to a constant factor. Notice however that the system of equations (4.1) imposes n different constraints on the set of initial contributions $\left(c_{i,n}^{\text{prop}}\right)_{i=1,\ldots,n}$. The conditional expectations appearing in the system of equations defining individual contributions can be computed as follows (see Appendix D for a proof).

Property 4.1. *The homogeneous function of order* 0*,* $g_{i,n}$*, defined as*

$$g_{i,n}(\boldsymbol{c}_n) = \mathrm{E}\left[\frac{c_i I_i}{\sum_{j=1}^n c_j I_j}\Big| I_i = 1\right]$$

can be computed from

$$g_{i,n}(\boldsymbol{c}_n) = c_i \int_0^\infty \mathrm{e}^{-tc_i} \mathrm{E}\big[\mathrm{e}^{-tM_{n,\boldsymbol{c}}^{(\setminus i)}}\big] \mathrm{d}t,$$

where $M_{n,\boldsymbol{c}}^{(\setminus i)}$ *is distributed as* $\sum_{j \neq i} c_j I_j$ *given* $I_i = 1$*. If* $I_1, I_2, \ldots, I_n$ *are mutually independent then*

$$g_{i,n}(\boldsymbol{c}) = c_i \int_0^\infty \mathrm{e}^{-tc_i} \prod_{j \neq i} \big(1 - q_j + q_j \mathrm{e}^{-c_j t}\big) \mathrm{d}t.$$

Unfortunately, (4.1) does not necessarily possess a solution. This is formally shown next when $I_1, I_2, \ldots, I_n$ are mutually independent and when the number of participants gets large. To this end, we need the following result giving the limits of the functions $h_{i,n}$ as $n \to \infty$.

Proposition 4.2. *Assume that* $I_1, I_2, \ldots$ *are mutually independent, that* $q_{\min} = \min_{i \geq 1} q_i > 0$ *and that there exists* $0 < \overline{q} < 1$ *such that* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n q_i = \overline{q}$*. Let* $\boldsymbol{c} = (c_1, c_2, \ldots)$ *be such that there exists a positive constant* C *satisfying* $sup_{i \geq 1} c_i \leq C$ *and* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n c_i = \overline{c} > 0$*.*

Then

$$\lim_{n\to\infty} h_{i,n}(\boldsymbol{c}_n) = \frac{\overline{c}}{\overline{cq}}$$

where $\overline{cq} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_i q_i$.

The proof of Proposition 4.2 can be found in Appendix E. As $n \to \infty$, the equations in (4.1) become

$$1 = q_i \frac{\overline{c}}{\overline{cq}}, \quad i = 1, \ldots$$

and we clearly see that they cannot be satisfied in general. The actuarial fairness condition can be satisfied only if $q_i = q$ for all i.

Since actuarial fairness is generally considered as an important requirement, we propose in the next section another allocation rule fulfilling this property and allowing for individualized protection levels. The results derived in this section will nevertheless be relevant for the comparison with takaful insurance in Section 6.

5 Uniform sharing rule with varying protection levels

5.1 Terminal payout

In order to offer each participant the opportunity to select the level of protection he or she desires, an actuarially fair system generalizing the uniform sharing rule is designed in this section. Now, each participant is free to subscribe one or several protection units from the mutual aid fund and total contributions are divided equally among all claiming units. By claiming unit we mean a protection unit subscribed by a participant who experienced the predefined event. The sponsor determines the expected benefit per claiming unit so that participants can select the number of units they need to reach the desired level of protection.

Since no assumption has been made about the dependence structure of the loss indicators $I_1, I_2, \ldots, I_n$ in Section 3, the system considered here corresponds to the situation where I_i is repeated a number of times equal to the number of protection units subscribed by participant i, for each $i = 1, 2, \ldots, n$. We can nevertheless obtain more explicit formulas and exploit mutual independence between individuals, when it holds.

Let w_i be the number of protection units subscribed by participant i. We assume that $w_1, w_2, \ldots, w_n$ are positive integers and we denote as $w_\bullet = \sum_{i=1}^{n} w_i$ the total number of protection units for the entire fund. The number of claiming units is

$$O_n = \sum_{i=1}^{n} w_i I_i.$$

Total contributions are now evenly distributed at time 1 among those O_n units held by individuals who experienced the predefined event. Initial contributions per unit under this sharing rule are henceforth denoted as $c^{\mathrm{w}}_{i,n}$, with sum

$$b^{\mathrm{w}}_n = \sum_{i=1}^{n} w_i c^{\mathrm{w}}_{i,n}.$$

As previously, we assume that participants just get their initial contributions back when nobody experienced the event under consideration. The terminal payout $T^{\mathrm{w}}_{i,n}$ per unit subscribed by participant i is then given by

$$T^{\mathrm{w}}_{i,n} = \frac{I_i}{O_n}\mathrm{I}[O_n \geq 1] b^{\mathrm{w}}_n + c^{\mathrm{w}}_{i,n}\mathrm{I}[O_n = 0].$$

The total payout for participant i is $w_i T^{\mathrm{w}}_{i,n}$.

5.2 Initial contribution

Imposing actuarial fairness, the initial contribution per unit must satisfy

$$c^{\mathrm{w}}_{i,n} = \mathrm{E}[T^{\mathrm{w}}_{i,n}] = \mathrm{E}\left[\frac{I_i}{O_n}\mathrm{I}[O_n \geq 1]\right] b^{\mathrm{w}}_n + c^{\mathrm{w}}_{i,n}\mathrm{P}[O_n = 0], \quad i = 1, \ldots, n,$$

which reduces to

$$c^{\mathrm{w}}_{i,n} = \frac{b^{\mathrm{w}}_n}{\mathrm{P}[O_n \geq 1]} \sum_{k=1}^{w_\bullet} \frac{1}{k}\mathrm{P}\left[O_n = k, I_i = 1\right], \quad i = 1, \ldots, n, \tag{5.1}$$

where

$$\mathrm{P}\left[O_n = k, I_i = 1\right] = \mathrm{P}\left[O_n - w_i I_i = k - w_i, I_i = 1\right] = \mathrm{P}\left[\sum_{j \neq i} w_j I_j = k - w_i, I_i = 1\right].$$

The initial contributions $c^{\mathrm{w}}_{i,n}$ in (5.1) are only defined up to a constant factor. Denoting as β_{w} the targeted average benefit per claiming unit specified by the sponsor, b^{w}_n can be obtained from

$$b^{\mathrm{w}}_n \mathrm{E}\left[\frac{1}{O_n}\middle| O_n \geq 1\right] = \beta_{\mathrm{w}} \Leftrightarrow b^{\mathrm{w}}_n = \frac{\beta_{\mathrm{w}}\mathrm{P}[O_n \geq 1]}{\sum_{k=1}^{w_\bullet} k^{-1}\mathrm{P}[O_n = k]}. \tag{5.2}$$

Inserting β_{w} appearing in (5.2) in (5.1) gives the amount of contribution per unit subscribed by participant i

$$c_{i,n}^{\mathrm{w}} = \beta_{\mathrm{w}} \frac{\sum_{k=1}^{w_\bullet} k^{-1}\mathrm{P}\left[O_n = k, I_i = 1\right]}{\sum_{k=1}^{w_\bullet} k^{-1}\mathrm{P}\left[O_n = k\right]}, \quad i = 1, \ldots, n. \tag{5.3}$$

5.3 Independent case

In the particular case where the loss indicators $I_1, I_2, \ldots, I_n$ are mutually independent, we have

$$c_{i,n}^{\mathrm{w}} = \frac{b_n^{\mathrm{w}} q_i}{\mathrm{P}[O_n \geq 1]} \sum_{k=1}^{w_\bullet} \frac{1}{k} \mathrm{P}\left[\sum_{j \neq i} w_j I_j = k - w_i\right]. \tag{5.4}$$

With the targeted benefit β_{w} in (5.2), (5.4) can be rewritten as

$$c_{i,n}^{\mathrm{w}} = q_i \beta_{\mathrm{w}} \frac{\sum_{k=1}^{w_\bullet} k^{-1}\mathrm{P}\left[\sum_{j \neq i} w_j I_j = k - w_i\right]}{\sum_{k=1}^{w_\bullet} k^{-1}\mathrm{P}\left[O_n = k\right]}.$$

Example 5.1. *Consider again the pool described in Example 3.1 with participants partitioned into three groups: occurrence probability equal to 0.01 in Group 1 gathering 50% of the participants, occurrence probability equal to 0.05 in Group 2 gathering 30% of the participants, and occurrence probability equal to 0.1 in Group 3 gathering 20% of the participants. In each group, let us now assume that half of the participants select $w_i = 1$ and the other ones $w_i = 2$. Figure 4 displays individual contributions $c_{i,n}^{\mathrm{w}}$ per unit. The results are given according to the number n of participants, for targeted benefits $\beta_{\mathrm{w}} = 1$ per unit. We can see there that individual contributions again stabilize at level q_i as $n \to \infty$.*

Example 5.1 suggests that $c_{i,n}^{\mathrm{w}}$ stablizes at level $\beta_{\mathrm{w}} q_i$ as $n \to \infty$ (remember that $\beta_{\mathrm{w}} = 1$ there). This is indeed the case in general under mild regularity conditions, as shown in the next result.

Proposition 5.2. *Assume that $I_1, I_2, \ldots$ are mutually independent, that $q_{\min} = \min_{i \geq 1} q_i > 0$, that there exists a positive constant W satisfying $sup_{i \geq 1} w_i \leq W$ and that there exists $0 < \overline{wq} < 1$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} w_i q_i = \overline{wq}$. Then,*

$$\lim_{n\to\infty} c_{i,n}^{\mathrm{w}} = \beta_{\mathrm{w}} q_i \text{ and } T_{i,n}^{\mathrm{w}} \to I_i \beta_{\mathrm{w}} \text{ with probability 1.}$$

The proof of Proposition 5.2 is given in Appendix F. We can see that, at the limit, participant i pays an amount corresponding to the pure premium, or fair price of an insurance policy with lump sum benefit $\beta_{\mathrm{w}} w_i$ in case the predefined event occurs. The terminal payout tends to the payout of this contract and at the limit, only (w_i, q_i) matters and not (w_j, q_j) for $j \neq i$ anymore.

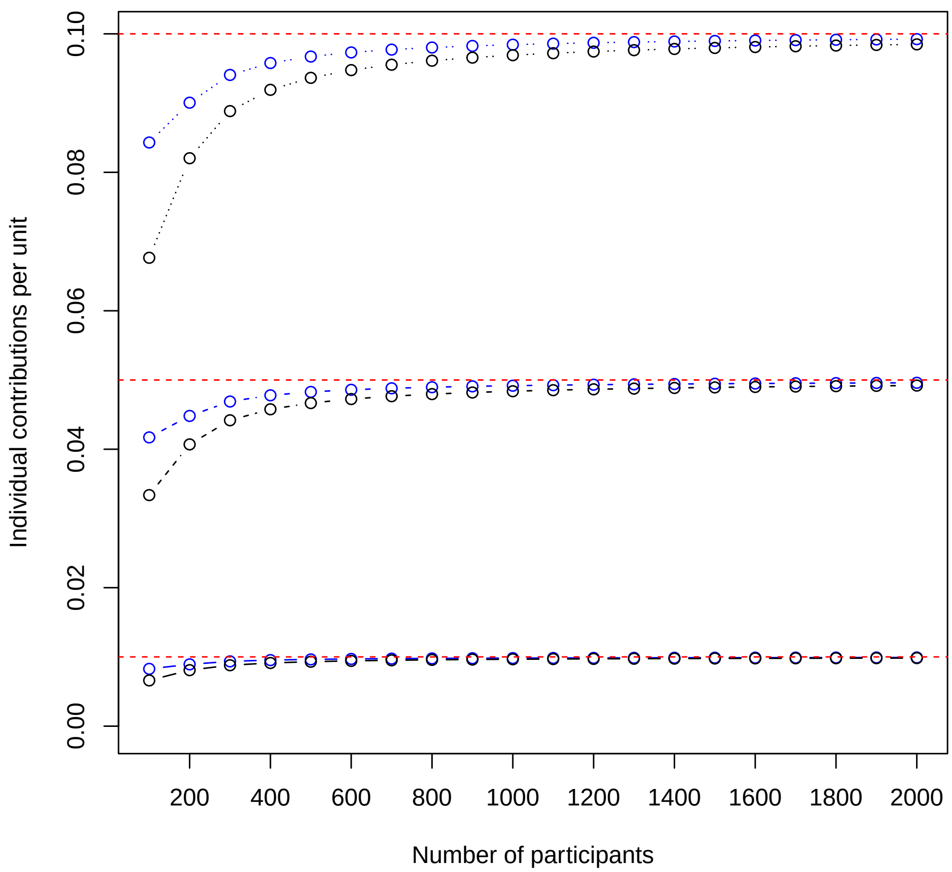

Figure 4: Individual contributions per unit in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$), blue curves correspond to participants with $w_i = 1$ and black curves to participants with $w_i = 2$, red horizontal lines at occurrence probabilities.

6 Comparison with takaful

The results derived in this paper appear to be useful in the context of takaful, the Sharia-compliant alternative to conventional insurance. Takaful can be translated as solidarity or mutual guarantee. We refer the reader to Malik and Ullah (2019) or Billah (2019) for an introduction to the topic. In a nutshell, conventional insurance has elements of riba (usury), maysir (gambling) and gharar (excessive uncertainty), which make it non-permissible from the Sharia perspective. Conventional insurance involves excessive uncertainty (gharar) because the transaction between the policyhoder and the insurer is uncertain and possibly unbalanced as the insurance contract may constitute a disproportionate loss in favour of one participant at the expense of the other.

Gharar being prohibited, pure risk transfer is impossible and takaful is therefore characterized by risk sharing. Islam indeed includes among its principles cooperation and mutual aid as well as the fair sharing of risks and benefits. To exclude gharar, the idea developed by jurists in Muslim law is to bring the insurance contribution closer to tabarru, which in Arabic means a donation. A donation does not require any counterparty and therefore cannot be associated with an uncertain return. When one of the participants is not a claimant, his or her contribution is considered as a donation since it benefits the claimants and is therefore similar to tabarru.

The fact that the contribution in takaful is considered as a donation in order to eliminate gharar constitutes the main distinctive feature compared to mutual aid and survivor funds considered in the preceding sections. In particular, the fact that participants to mutual aid funds get their initial contributions back when nobody experienced the event under consideration (that is, $I_i = 0$ for all i) is not in line with the irrevocable donation required by takaful which imposes that if $M_n = 0$ then all contributions are donated to a charity selected by the participants.

If contributions $c_{i,n}^{\text{taka}}$ are donated to a charity when $M_n = 0$ then actuarial fairness is no more possible, as shown next. Let $b_n^{\text{taka}} = \sum_{i=1}^{n} c_{i,n}^{\text{taka}}$ denote total contributions. If b_n^{taka} is shared evenly among claiming participants then the terminal payout is given by

$$T_{i,n}^{\text{taka}} = \begin{cases} \frac{1}{1+\sum_{j\neq i} I_j} b_n^{\text{taka}} \text{ if } I_i = 1, \\ \\ 0 \text{ otherwise.} \end{cases}$$

Equating $c_{i,n}^{\text{taka}}$ to $\mathrm{E}[T_{i,n}^{\text{taka}}]$ then leads to an impossible relationship because

$$\sum_{i=1}^{n} \mathrm{E}[T_{i,n}^{\text{taka}}] = b_n^{\text{taka}} \mathrm{P}[M_n \geq 1]$$

cannot be equal to b_n^{taka} unless $\mathrm{P}[M_n \geq 1] = 1$. This is inherent to takaful since contributions are donated by participants who can only get poorer on average when joining the pool. A practical way to solve this issue is to consider that participants indeed get $c_{i,n}^{\text{taka}}$ back but engage to donate it to a charity. Since the probability $\mathrm{P}[M_n = 0]$ rapidly tends to 0 when n increases, this has no material effect in practice within sufficiently large pools.

Let us now illustrate the design of a family takaful product where the insured peril corresponds to participant's death. We assume that the system obeys the following rules:

Rule 1 survivors' contributions proportionally increase those of the beneficiaries of the deceased at time 1; survivors' contributions are related to the probabilities of death by a function defined ex ante;

Rule 2 if all participants die before time 1 then the beneficiaries of each participant receive a share of the fund corresponding to the proportion of the contributions paid at time 0;

Rule 3 if no one dies before time 1, no participant benefits from the distribution of the common fund which is entirely transferred to a charity.

Rules 1-2 implicitely assume that no reserves are made for the survivors and that the contributions have an irrevocable character to beneficiaries, in accordance with the gift character of tabarru. Rules 1-3 imply that the beneficiaries of participant i receive the terminal payout

$$T_{i,n}^{\text{prop-taka}} = \begin{cases} \frac{c_i^{\text{prop-taka}}}{c_i^{\text{prop-taka}} + \sum_{j \neq i} c_j^{\text{prop-taka}} I_j} b_n^{\text{prop-taka}} \text{ if } I_i = 1, \\ \\ 0 \text{ otherwise,} \end{cases}$$

where $c_i^{\text{prop-taka}} = h(q_i)$, with h a positive function that all participants agree, and $b_n^{\text{prop-taka}} = \sum_{j=1}^{n} c_j^{\text{prop-taka}}$. If no participant dies then $T_{i,n}^{\text{prop-taka}} = 0$ for all i and total contributions $b_n^{\text{prop-taka}}$ are transferred to a charity according to Rule 3.

For large pools, we deduce from Proposition 4.2 that the ratio of the expected beneficiaries of participant i's terminal payout over participant i's contribution converges to a positive constant given by

$$\lim_{n\to\infty}\frac{\mathrm{E}[T_{i,n}^{\text{prop}-\text{taka}}]}{c_i^{\text{prop}-\text{taka}}}=\frac{q_i\overline{h\left(q\right)}}{\overline{qh\left(q\right)}},$$

where $\overline{h\left(q\right)}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}h\left(q_i\right)$ and $\overline{qh\left(q\right)}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}q_ih\left(q_i\right)$. If

$$q_i\geq\frac{\overline{qh\left(q\right)}}{\overline{h\left(q\right)}}:=\gamma,$$

beneficiaries of participant i's receive more on average than participant i contribute. Participants with the highest probabilities of death therefore give more to their beneficiaries relative to their donation.

Let us now consider the following two examples of interest and compare them with conventional insurance.

Case 1: In term life insurance, the pure premium is proportional to the insured's death probability. It is therefore natural to specify that participants' initial contributions are proportional to their death probabilities. Therefore, let us assume that $c_i^{\text{prop}-\text{taka}}=q_i\omega$ for $i=1,\ldots,n$, for some positive factor ω, i.e. $h\left(q_i\right)=q_i\omega$. Then we obtain

$$\gamma=\frac{\overline{q^2}}{\overline{q}}.$$

In the case of a term life insurance contract with unit death benefit, policyholder pays a pure premium equal to q_i and receives nothing in case of survival while the beneficiary receives 1 in case of death. For the takaful product, participant i contributes q_i, and receives nothing in case of survival while the beneficiary receives on average $\overline{q}q_i/\overline{q^2}$ in case of death (for large pools).

Case 2: Another relevant choice is to require that the contributions to the beneficiaries are all equal. Specifically, let us assume that $c_i^{\text{prop}-\text{taka}}=\omega$ for $i=1,\ldots,n$, for some positive factor ω, i.e. $h\left(q_i\right)=\omega$. Then we obtain

$$\gamma=\overline{q}.$$

In the case of a term life insurance contract with unit death benefit, policyholder pays a pure premium equal to 1 and receives nothing in case of survival while the beneficiary receives $1/q_i$ in case of death. For the takaful product, participant i contributes 1, and receives nothing in case of survival while the beneficiary receives on average $1/\overline{q}$ in case of death (for large pools).

7 Discussion

In this paper, we have considered different ways to allocate ex post total contributions among claiming participants. The first one divides total contributions equally among all members who experienced the predefined event while the other ones let the benefits vary among participants. Some allocation rules can be fair but other ones generally induce systematic transfers among participants. The limiting case where the number of participants tends to infinity is considered for the different systems. Under mutual independence, insurance at fair price is recovered at the limit under mild technical conditions. These systems have been compared with takaful insurance. The fundamental difference is that contributions are considered as irrevocable donations under takaful, to comply with Sharia precepts. Therefore, contributions cannot be returned back to participants if no one experiences the predefined event during the coverage period. As a consequence, actuarial fairness cannot hold (even if the difference is not material within large pools). As an illustration, we describe a family takaful scheme providing participants with coverage in case of death.

There are many applications of the mechanisms described in this paper. For instance, survivor funds could be sponsored by states to borrow at low, or even negative interest rate given the attractive returns corresponding to mortality credits at older ages. The idea is to set up a tontine scheme. Tontine operations are still permitted throughout EU. Le Conservateur in France is an insurer proposing this kind of product (registered as "société à forme tontinière" in France). In Belgium, tontines can be sold by an insurer registered under insurance Branch 25 "opérations tontinières/tontineverrichtingen".

Retired people could lend money to the state on a voluntary basis, entering a survivor scheme in the sense that the debt would be repaid to survivors, only. The return on this operation for survivors corresponds to mortality credits. The latter are the amounts invested by those participants who died before maturity, which are inherited by survivors. These inheritance gains constitute the product's return, making the operation very attractive at older ages even without extra interest rate. The state sponsors the system, making it operational and risk-free. In practice, a national register can be used to identify beneficiaries and the system would be costless to operate. The proposed approach avoids that public debt is traded on financial markets and may thus serve speculative purposes.

The sponsor may guarantee that the benefit paid to participant i never falls below a given threshold τ_i. Since this guarantee is risky, it can only be covered by an insurance company partnering with the fund. The cost of this guarantee must then be paid by participant i on the top of his or her initial contribution.

Consider for instance the uniform sharing rule. If participant i wishes a terminal payout at least equal to τ_i in case the predefined event occurs then his or her terminal payout $T_{i,n}^{\text{uni}}$ is supplemented with

$$T_{i,n}^{\text{uni},+} = I_i \left(\tau_i - \frac{b_n^{\text{uni}}}{M_n} \right)_+ \mathrm{I}[M_n \geq 1].$$

Then,

$$T_{i,n}^{\text{uni}} + T_{i,n}^{\text{uni},+} = I_i \max \left\{ \frac{b_n^{\text{uni}}}{M_n}, \tau_i \right\} \mathrm{I}[M_n \geq 1] + c_{i,n}^{\text{uni}} \mathrm{I}[M_n = 0].$$

The additional payout $T_{i,n}^{\text{uni},+}$ is then covered by an insurance policy sold by the partnering insurer.

Finally, let us mention that the extension of the mechanisms proposed in this paper to more general losses than just binary risks may be a promising future research avenue.

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Appendix

A Proof of Proposition 3.2

The initial contribution of participant i is given by

$$c_{i,n}^{\text{uni}} = q_i \beta \frac{\sum_{k=0}^{n-1} \frac{n}{k+1} \mathrm{P}\left[M_n^{(\setminus i)} = k\right]}{\sum_{k=1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right]}$$

where $M_n^{(\setminus i)} = M_n - I_i = \sum_{j \neq i} I_j$. Let us begin with the sum

$$S_n = \sum_{k=1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right],$$

appearing in the denominator. Denote the expectation of M_n by $E_n = \sum_{i=1}^{n} q_i$ and let $\beta_n = \lfloor n^\beta \rfloor$ for $1/2 < \beta < 1$. Since $\lim_{n\to\infty} E_n/n = \overline{q}$, we have, for n large enough,

$$\begin{aligned} S_n &= \sum_{k=1}^{\lfloor E_n \rfloor - \beta_n} \frac{n}{k} \mathrm{P}\left[M_n = k\right] + \sum_{k=\lfloor E_n \rfloor - \beta_n + 1}^{\lfloor E_n \rfloor + \beta_n} \frac{n}{k} \mathrm{P}\left[M_n = k\right] + \sum_{k=\lfloor E_n \rfloor + \beta_n + 1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right] \\ &= S_{1,n} + S_{2,n} + S_{3,n}. \end{aligned}$$

Let us now recall Hoeffding's inequality for the sequence $(I_i)_{i \geq 1}$: for every $t > 0$

$$\mathrm{P}\left[|M_n - E_n| > t\right] < 2 \exp\left(-2 \frac{t^2}{n}\right).$$

Using this inequality, we deduce that

$$\begin{aligned} S_{1,n} &\leq n \mathrm{P}\left[M_n < \lfloor E_n \rfloor - \beta_n + 1\right] \\ &\leq n \mathrm{P}\left[M_n - E_n < -(E_n - \lfloor E_n \rfloor + \beta_n - 1)\right] \\ &< 2n \exp\left(-2 \frac{(\beta_n - 1)^2}{n}\right) \to_{n\to\infty} 0 \end{aligned}$$

and that

$$
\begin{aligned}
S_{3,n} &\leq n\mathrm{P}\left[M_n > \lfloor E_n \rfloor - \beta_n + 1\right] \\
&\leq n\mathrm{P}\left[M_n - E_n > \lfloor E_n \rfloor - E_n + \beta_n)\right] \\
&< 2n \exp\left(-2\frac{(\beta_n - 1)^2}{n}\right) \to_{n\to\infty} 0.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
S_{2,n} &= \sum_{j=-\beta_n+1}^{\beta_n} \frac{n}{\lfloor E_n \rfloor + j} \mathrm{P}\left[M_n = \lfloor E_n \rfloor + j\right] \\
&= \sum_{j=1}^{2\beta_n} \frac{1}{\frac{\lfloor E_n \rfloor}{n} + \frac{j}{n}} \mathrm{P}\left[M_n = \lfloor E_n \rfloor - \beta_n + j\right] \\
&= \frac{1}{\frac{\lfloor E_n \rfloor}{n}} \sum_{j=1}^{2\beta_n} \frac{1}{1 + \frac{j}{\lfloor E_n \rfloor}} \mathrm{P}\left[M_n = \lfloor E_n \rfloor - \beta_n + j\right] \\
&= \frac{1}{\frac{\lfloor E_n \rfloor}{n}} \sum_{j=1}^{2\beta_n} (1 + o(1))\, \mathrm{P}\left[M_n = \lfloor E_n \rfloor - \beta_n + j\right] \\
&= \frac{1}{\frac{\lfloor E_n \rfloor}{n}} (1 + o(1))\, \mathrm{P}\left[\lfloor E_n \rfloor - \beta_n + 1 \leq M_n \leq \lfloor E_n \rfloor + \beta_n\right].
\end{aligned}
$$

Using once again Hoeffding's inequality, we can derive that

$$
S_{2,n} \to_{n\to\infty} \frac{1}{q}.
$$

From the same approach, it is also possible to prove that

$$
\sum_{k=0}^{n-1} \frac{n}{k+1} \mathrm{P}\left[M_n^{(\setminus i)} = k\right] \to_{n\to\infty} \frac{1}{q}
$$

and therefore that

$$
c_{i,n}^{\mathrm{uni}} \to_{n\to\infty} q_i \beta.
$$

Let us now establish that

$$
T_{i,n}^{\mathrm{uni}} \to_{n\to\infty} I_i \beta \text{ with probability 1.}
$$

We have

$$T_{i,n}^{\text{uni}} = \beta I_i \frac{n}{M_n}(1 - \mathrm{I}[M_n = 0]) \frac{\mathrm{P}[M_n \geq 1]}{\sum_{k=1}^{n} \frac{n}{k} \mathrm{P}[M_n = k]} + c_{i,n}^{\text{uni}} \mathrm{I}[M_n = 0].$$

From the previous ligns, we know that

$$c_{i,n}^{\text{uni}} \to_{n\to\infty} q_i \beta \text{ and } \sum_{k=1}^{n} \frac{n}{k} \mathrm{P}[M_n = k] \to_{n\to\infty} \frac{1}{\overline{q}}.$$

Since $\mathrm{P}[M_n = 0] = \prod_{i=1}^{n}(1 - q_i) \leq (1 - q_{\min})^n$, we have that $\mathrm{I}[M_n = 0] \overset{a.s.}{\to}_{n\to\infty} 0$. Since

$$\sum_{i=1}^{\infty} \frac{q_i (1 - q_i)}{i^2} < \infty,$$

we deduce from Kolmogorov's strong law of large numbers that

$$\frac{M_n}{n} \overset{a.s.}{\to}_{n\to\infty} \overline{q}.$$

It follows that

$$T_{i,n}^{\text{uni}} \overset{a.s.}{\to}_{n\to\infty} I_i \beta.$$

B Proof of Proposition 3.3

From the definition of $T_{i,n}^{\text{uni}}$, we have

$$\mathrm{Var}\left[T_{i,n}^{\text{uni}} \middle| I_i = 1\right] = \mathrm{Var}\left[\frac{1}{(1 + \sum_{j\neq i} I_j)/n} \frac{b_n^{\text{uni}}}{n}\right] = \left(\frac{b_n^{\text{uni}}}{n}\right)^2 \mathrm{Var}\left[\frac{1}{(1 + \sum_{j\neq i} I_j)/n}\right]$$

where, by Proposition 3.2,

$$\frac{b_n^{\text{uni}}}{n} = \frac{\beta \mathrm{P}[M_n \geq 1]}{\sum_{k=1}^{n} k^{-1} \mathrm{P}[M_n = k]} = \frac{\beta}{\mathrm{E}\left[\frac{n}{M_n} \middle| M_n \geq 1\right]} \underset{n\to\infty}{\to} \beta \overline{q}.$$

Moreover Property 4.1 gives us that

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{(1 + \sum_{j\neq i} I_j)/n}\right] &= \int_0^{\infty} \mathrm{e}^{-s/n} \prod_{j\neq i} \left(1 + q_j \left(\mathrm{e}^{-s/n} - 1\right)\right) \mathrm{d}s \\
&= \int_0^{\infty} \mathrm{e}^{-s/n} \exp\left(\sum_{j\neq i} \ln\left(1 + q_j \left(\mathrm{e}^{-s/n} - 1\right)\right)\right) \mathrm{d}s.
\end{aligned}$$

We deduce from the following power series expansions

$$
\begin{aligned}
\ln(1+u) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} u^k, \quad 0<u<1, \\
\mathrm{e}^{-u}-1 &= \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k!} u^k, \quad u>0,
\end{aligned}
$$

that

$$
\begin{aligned}
\sum_{j\neq i} \ln\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right) &= \sum_{j\neq i}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} q_j^k \left(\sum_{l=1}^{\infty} \frac{(-1)^{l}}{l!}\left(\frac{s}{n}\right)^l\right)^k \\
&= -\frac{1}{n}\sum_{j\neq i} q_j s + \frac{1}{2}\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\frac{s^2}{n} + \sum_{k=3}^{\infty} d_{i,k}\frac{s^k}{n^{k-1}}
\end{aligned}
$$

where $|d_{i,k}|<1$. Then we have

$$
\begin{aligned}
&\mathrm{e}^{-s/n}\prod_{j\neq i}\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right) \\
=\;& \left(1-\frac{s}{n}+\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\left(\frac{s}{n}\right)^k\right)\exp\left(-\frac{1}{n}\sum_{j\neq i} q_j s + \frac{1}{2}\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\frac{s^2}{n} + \sum_{k=3}^{\infty} d_{i,k}\frac{s^k}{n^{k-1}}\right) \\
=\;& \exp\left(-\left[\frac{1}{n}\sum_{j\neq i} q_j\right]s\right)\left(1-\frac{s}{n}+\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\left(\frac{s}{n}\right)^k\right) \\
&\times\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left[\frac{1}{2}\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\frac{s^2}{n}+\sum_{l=3}^{\infty} d_{i,l}\frac{s^l}{n^{l-1}}\right]^k\right) \\
=\;& \exp\left(-\left[\frac{1}{n}\sum_{j\neq i} q_j\right]s\right)\left(1+\frac{1}{n}\left[-s+\left[\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right]s^2\right]+\sum_{k=2}^{\infty} P_{i,k,n}(s)\frac{1}{n^k}\right)
\end{aligned}
$$

where $P_{i,k,n}(s)$ are polynomials of order at most $2k$. It follows that

$$
\begin{aligned}
&\mathrm{E}\left[\frac{1}{(1+\sum_{j\neq i} I_j)/n}\right] \\
=\;& \frac{1}{\frac{1}{n}\sum_{j\neq i} q_j}+\frac{1}{n}\left[-\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^2}+2\left[\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right]\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3}\right]+o\left(\frac{1}{n}\right)
\end{aligned}
$$

By Property 4.1, we also have

$$\mathrm{E}\left[\frac{1}{\left[(1+\sum_{j\neq i} I_j)/n\right]^2}\right] = \int_0^\infty s\mathrm{e}^{-s/n}\prod_{j\neq i}\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right)\mathrm{d}s$$

In the same way

$$\begin{aligned}&\mathrm{E}\left[\frac{1}{\left[(1+\sum_{j\neq i} I_j)/n\right]^2}\right]\\ =\;& \frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^2}+\frac{1}{n}\left[-\frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3}+6\left[\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right]\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4}\right]+o\left(\frac{1}{n}\right).\end{aligned}$$

We therefore deduce that

$$\begin{aligned} n\mathrm{Var}\left[\frac{1}{(1+\sum_{j\neq i} I_j)/n}\right] &= n\left(\mathrm{E}\left[\frac{1}{\left[(1+\sum_{j\neq i} I_j)/n\right]^2}\right]-\left(\mathrm{E}\left[\frac{1}{(1+\sum_{j\neq i} I_j)/n}\right]\right)^2\right)\\ &= -\frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3}+6\left[\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right]\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4}\\ &\quad+\frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3}-4\left[\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right]\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4}+o\left(1\right)\\ &= 2\frac{\overline{q(1-q)}}{\left(\overline{q}\right)^4}+o\left(1\right)\end{aligned}$$

and finally that

$$n\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}}\middle| I_i=1\right]\underset{n\to\infty}{\to} 2\beta^2\frac{\overline{q(1-q)}}{\left(\overline{q}\right)^2}.$$

C Proof of Proposition 3.5

The Laplace transform of M_n is given by

$$\mathrm{L}_{M_n}(t)=\mathrm{E}[\exp(-tM_n)]=\prod_{i=1}^n \mathrm{L}_{X_i}(t)=\prod_{i=1}^n\left(1+q_i\left(\mathrm{e}^{-t}-1\right)\right).$$

As $n \to \infty$, since $\sup_{1 \leq i \leq n} q_i \to 0$ and $\lim_{n\to\infty} \sum_{i=1}^{n} q_i = \lambda > 0$, we have

$$\ln \mathrm{L}_{M_n}(t) = \sum_{i=1}^{n} \ln\left(1 + q_i\left(\mathrm{e}^{-t} - 1\right)\right) = (1 + o(1)) \sum_{i=1}^{n} q_i\left(\mathrm{e}^{-t} - 1\right) \to \lambda\left(\mathrm{e}^{-t} - 1\right),$$

and we conclude that M_n converges in distribution to a Poisson random variable N with parameter λ. Since, for all n and $\varepsilon > 0$,

$$\mathrm{E}\left[\left(\frac{1}{M_n}\right)^{1+\varepsilon} \middle| M_n \geq 1\right] \leq 1,$$

the sequence $(M_n^{-1})_{n\geq 1}$ is uniformly integrable and we deduce that

$$\mathrm{E}\left[\frac{1}{M_n} \middle| M_n \geq 1\right] \to_{n\to\infty} \mathrm{E}\left[\frac{1}{N} \middle| N \geq 1\right].$$

With the same arguments, we can prove that $M_n^{(\backslash i)} = \sum_{j\neq i} I_j$ also converges in distribution to a Poisson random variable N with parameter λ and that

$$\mathrm{E}\left[\frac{1}{1 + M_n^{(\backslash i)}}\right] \to_{n\to\infty} \mathrm{E}\left[\frac{1}{1+N}\right].$$

D Proof of Property 4.1

The following result that can be found in Fuchs et al. (2002) appears to be useful in the context of our study. Following the convention made by these authors, we set $\frac{0}{0} = 0$.

Property D.1. *Let α be a positive integer.*

(i) *If Y and Z are non-negative random variables with joint Laplace transform $\mathrm{L}_{(Y,Z)}(t_1, t_2) = \mathrm{E}[\exp(-t_1 Y - t_2 Z)]$ then*

$$\mathrm{E}\left[\frac{Y^{k_1} Z^{k_2}}{(Y+Z)^{\alpha}}\right] = (-1)^{k_1+k_2} \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \left.\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} \mathrm{L}_{(Y,Z)}(t_1, t_2)\right|_{t_1=t_2=t} \mathrm{d}t$$

for any non-negative integers k_1 and k_2 such that $k_1 + k_2 \geq 1$.

(ii) *If $X_1, X_2, \ldots, X_n$ are independent non-negative random variables with respective Laplace transforms $\mathrm{L}_{X_i}(t) = \mathrm{E}[\exp(-tX_i)]$, $i = 1, 2, \ldots, n$, then*

$$\mathrm{E}\left[\frac{X_1^k}{(X_1 + X_2 + \ldots + X_n)^{\alpha}}\right] = (-1)^k \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathrm{L}_{X_1}(t) \prod_{i=2}^{n} \mathrm{L}_{X_i}(t) \mathrm{d}t$$

for any non-negative integer $k \geq 1$.

Taking $\alpha = 1$, $Y = c_i$ and $Z = \sum_{j\neq i} c_j I_j$ in Property D.1, we get

$$\begin{aligned} g_{i,n}(\boldsymbol{c}_n) &= -\int_0^{\infty} \frac{\partial}{\partial t_1} \mathrm{E}\left[\exp\left(-t_1 c_i - t_2 \sum_{j\neq i} c_j I_j\right) \Big| I_i = 1\right]\Bigg|_{t_1=t_2=t} \mathrm{d}t \\ &= c_i \int_0^{\infty} \mathrm{E}\left[\exp\left(-tc_i - t\sum_{j\neq i} c_j I_j\right) \Big| I_i = 1\right] \mathrm{d}t \end{aligned}$$

which gives the announced formula.

The result in the independent case directly follows from the general expression derived in Property D.1 since the expectation of the product over participants factors out into the product of expectations, noticing that $\mathrm{E}\left[\exp\left(-tI_j\right)\right] = 1 - q_j + q_j \mathrm{e}^{-t}$. This ends the proof.

E Proof of Proposition 4.2

We have

$$h_{i,n}(\boldsymbol{c}_n) = \frac{1}{n}\left(\sum_{j=1}^{n} c_j\right) \mathrm{E}\left[\frac{n}{c_i + \sum_{j\neq i} c_j I_j}\right], \quad i = 1, \ldots, n.$$

Since $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_i = \overline{c} > 0$, we only consider the following quantity

$$\mathrm{E}\left[\frac{1}{\frac{c_i}{n} + \frac{1}{n}\sum_{j\neq i} J_j}\right]$$

where the random variables $J_j = c_j I_j$ are independent, bounded ($\sup_{i\geq 1} c_i \leq C$) and with mean $c_j q_j$. Hoeffding's inequality for $\sum_{j\neq i} J_j$ states that

$$\mathrm{P}\left[\left|\sum_{j\neq i} J_j - \sum_{j\neq i} J_j c_j q_j\right| > t\right] < 2\exp\left(-2\frac{t^2}{\sum_{j\neq i} c_j^2}\right) \leq 2\exp\left(-2\frac{t^2}{(n-1)C^2}\right).$$

Therefore we can use the same approach as in the proof of Proposition 3.2 to get

$$\mathrm{E}\left[\frac{1}{\frac{c_i}{n} + \frac{1}{n}\sum_{j\neq i} J_j}\right] \to_{n\to\infty} \frac{1}{\overline{cq}}$$

where $\overline{cq} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_i q_i$.

F Proof of Proposition 5.2

The lines of this proof follow those of the proof of Proposition 3.2 where M_n and $M_n^{(\backslash i)}$ have to be replaced by O_n and $\sum_{j \neq i} w_j I_j$, respectively, and by noting that Hoeffding's inequality becomes:

$$\mathrm{P}\left[\left|O_n - \sum_{i=1}^{n} w_i q_i\right| > t\right] < 2\exp\left(-2\frac{t^2}{\sum_{i=1}^{n} w_i^2}\right) \leq 2\exp\left(-2\frac{t^2}{(n-1)W^2}\right), \quad t > 0.$$