

The importance of stepping up in the excursion set approach

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ABSTRACT

There is a simple analytic approximation for the first crossing distribution associated with random walks having correlated steps, which is very accurate in the limit of few steps. The approximation is accurate for the wide range of barrier shapes of interest in excursion set studies of cosmological structure formation. For example, it provides a useful fitting formula for the high-mass end of the dark halo mass function. The approximation is based on the requirement that, in addition to having the right height, the walk must cross the barrier going upwards. Therefore, it only requires knowledge of the bivariate distribution of the walk height and slope. However, it diverges at lower masses. We show how to cure this divergence by using a formulation which requires knowledge of just one other variable. While our analysis is general, we use examples based on Gaussian initial conditions to illustrate our results. Our formulation is simple and fast, and yields excellent agreement, even at very low masses and for a wide variety of moving barriers, with considerably more computationally expensive Monte Carlo solution of the first crossing distribution.

Key words: large-scale structure of Universe.

1 INTRODUCTION

Simulations of hierarchical gravitational clustering suggest that the abundance and clustering of gravitationally bound objects in the Universe can be a powerful tool for constraining the nature of the initial fluctuation field. Since simulations are expensive, there is considerable interest in models which can provide a better understanding of how cluster abundances and clustering depend on cosmological parameters. The excursion set approach (Bond et al. 1991) is perhaps the most developed of these: motivated by the seminal work of Press & Schechter (1974) it provides an analytical framework which relates the statistics of gravitationally bound dark matter haloes to fluctuations in the primordial density field, and the subsequent expansion history.

In this approach, at a given (randomly chosen) position in space one looks at the overdensity field smoothed on some scale R : plotting this smoothed δ as a function of (the inverse of) R resembles a random trajectory, the steps of which are, in general, correlated. The nature of the correlations depends on the smoothing filter (e.g. tophat, Gaussian), and on the nature of the initial fluctuation field (Gaussian or non-Gaussian). Repeating this for every position in space gives an ensemble of trajectories, each one of which starts from $\delta(R = \infty) = 0$ (the universe is homogeneous on large smoothing scales). For each trajectory, one searches for the largest R (if

any) for which the value of the smoothed density field lies above some threshold value (which may itself depend on R), the value of which is determined by the expansion history of the background cosmology. An object of mass $M \sim R^3$ is then associated with that trajectory.

If dn/dM denotes the comoving number density of haloes of mass M , then the mass fraction in such haloes is $(M/\bar{\rho}) dn/dM$, where $\bar{\rho}$ is the comoving background density. The excursion set approach assumes that this halo mass fraction equals the fraction of walks which cross the threshold (the ‘barrier’) for the first time when the smoothing scale is R :

$$f(R) dR = (M/\bar{\rho}) (dn/dM) dM. \quad (1)$$

Although recent work has focused on the shortcomings of this ansatz (Paranjape & Sheth 2012), not to mention the fact that variables other than the overdensity affect halo formation, and this is not evident in the simplest version of the approach outlined above (Sheth, Mo & Tormen 2001), the first crossing distribution is nevertheless expected to provide substantial insight into the dependence of dn/dM on cosmological parameters.

In practice, one works not with $f(R)$ but with $f(s)$, where

$$s(R) \equiv \int \frac{dk}{k} \frac{k^3 P(k)}{2\pi^2} W^2(kR) \quad (2)$$

denotes the variance in the fluctuation field when smoothed on scale R with a filter of shape W . In hierarchical models, s is a monotonic function of R , so $f(R)dR = f(s)ds$. Working with s has the advantage

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of removing most of the dependence on the shape of the power spectrum: $P(k)$ mainly matters only through the dependence of s on R .

To solve the first crossing problem, we must be able to identify the fraction of trajectories for which $\delta = b(s)$ and $\delta < b(S)$ for all $S < s$. While this is rather straightforward to implement numerically (Bond et al. 1991), the latter condition is hard to deal with analytically. However, a good approximation to $f(s)$ can be computed from considering the joint probability $p(\delta, v; s)$ that a walk reaches δ at scale s with velocity $v \equiv d\delta/ds$. This is because, if $\delta = b(s)$, then one also wants $v \geq db/ds$, so

$$f(s) \simeq p(b) \int_{b'}^{\infty} dv (v - b') p(v|b) \equiv f_{\text{up}}(s) \quad (3)$$

(Musso & Sheth 2012). (This expression generalizes equation 3.14a of Bond et al. 1991 which assumed a constant barrier $b = \delta_c$ for which $b' = 0$.)

For Gaussian statistics, $p(b)$ is a Gaussian with zero mean and variance s , and $p(v|b)$ is Gaussian with mean $\langle v|b \rangle = b \langle \delta v \rangle / \langle \delta \delta \rangle = b/2s$ and variance $\langle vv \rangle - \langle \delta v \rangle^2 / \langle \delta \delta \rangle = s \langle (\delta/\sqrt{s})^2 \rangle$; this makes

$$sf_{\text{up}}(s) = sf_{\text{PS}}(s) \left[\frac{1 + \text{erf}(x/\sqrt{2})}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}x} \right], \quad (4)$$

where

$$x \equiv -\frac{(b/\sqrt{s})'}{\langle (\delta/\sqrt{s})^2 \rangle^{1/2}} = -2\Gamma s \frac{d(b/\sqrt{s})}{ds}, \quad (5)$$

with $\Gamma^2 = \gamma^2/(1 - \gamma^2)$ and $\gamma^2 \equiv \langle \delta v \rangle^2 / \langle \delta \delta \rangle \langle vv \rangle = 1/4s \langle v^2 \rangle$, and

$$f_{\text{PS}}(s) = -\frac{d(b/\sqrt{s})}{ds} \frac{e^{-b^2/2s}}{\sqrt{2\pi}} \quad (6)$$

is the old approximation of Press & Schechter (1974) (who really only considered a constant barrier $b = \delta_c$).

For future reference, we note that one can also write $f_{\text{PS}}(s) = \langle (v - b')|b \rangle p(b)$, as can be seen assuming $b' \ll \langle v|b \rangle$ in equation (3). For a constant barrier $b = \delta_c$, if we define $v \equiv \delta_c/\sqrt{s}$, then $d(b/\sqrt{s})/ds = -v/2s$ and $x = \Gamma v$, so that one gets the familiar

$$sf_{\text{PS}}(s) = \frac{v}{2} \frac{e^{-v^2/2}}{\sqrt{2\pi}}. \quad (7)$$

This highlights the fact that equation (4) can be thought of as providing a correction to the Press–Schechter result; a correction which only matters when $x \lesssim 1$ (or $v \lesssim 1/\Gamma$).

This cannot be the full story of course, because this procedure fails to discard those walks that were above threshold at $S < s_1 < s$, but diffused back below the barrier at s_1 to cross again at s , i.e. $\delta > b(S)$ but $\delta < b(s_1)$. However, the fraction of such walks should be tiny at small s , since the correlations between the steps make sharp turns very unlikely. Musso & Sheth (2012) showed that equation (3) indeed works very well for the small values of s (i.e. large R and hence large M) which are of most interest in cosmology. This accuracy is remarkable, since it only requires knowledge of (a judiciously chosen!) bivariate distribution, rather than of the full n -point distribution which describes the walk heights at each $S \leq s$ (and note that $n \rightarrow \infty$).

The success and simplicity of the approach motivates the search for an approximation that allows even greater accuracy. Since the integral of equation (3) over all s diverges, it is clear that something new is needed at larger s (i.e. smaller m). This is not surprising, since equation (3) fails to discard walks with additional crossings

at $S < s$, and these can be a large fraction as s increases. Therefore, we would like to find a scheme that removes this divergence, and to do it by keeping track of as few additional correlated variables as possible. We show how to do this in Section 2, where we treat the case of a constant barrier in some detail, before generalizing to moving barriers. A final section summarizes our results. For many aspects of this problem, fully analytic results can be obtained from Gaussian smoothing, i.e. with $W(kR) = \exp(-k^2 R^2/2)$, of a Gaussian field with power spectrum $P(k) \propto k^{-1}$; these are collected in Appendix A. Appendix B gives details of the numerical back-substitution algorithm we use to derive our results.

2 BACK SUBSTITUTION WITH CORRELATED STEPS

The starting point of the analysis is that one can write the first crossing distribution as

$$f(s) = \int_{b'}^{\infty} dv (v - b') p(b, v, \text{first } s); \quad (8)$$

dropping the ‘first s ’ constraint, while keeping the upcrossing requirement $v > b'$, leads to equation (3). This actually works very well down to $\Gamma v \gtrsim 0.1$ for a wide variety of barrier shapes (Musso & Sheth 2012). Earlier crossings may be accounted for by means of recursive corrections (see appendix C of Musso & Sheth 2013, which generalizes appendix A of Bond et al. 1991); these introduce two additional variables for each crossing.

One may instead consider the probability $p_{\times}(\delta)$ that a walk reaches δ at scale s having crossed the barrier at least once at larger scale. Following the same approach, and classifying all walks by the scale S on which they first crossed, this can be written – exactly – as

$$p_{\times}(\delta, s) = \int_0^s dS \int_{B'}^{\infty} dV (V - B') p(\delta, B, V, \text{first } S), \quad (9)$$

where B and $V \equiv d\delta/dS$ denote the barrier height and the slope of the walk on scale S . Since all walks that are above the barrier on scale s must have crossed it at some larger scale $S < s$, then $p_{\times}(\delta \geq b) = p(\delta \geq b)$ necessarily. Multiplying and dividing the integrand by $f(S)$ leads to

$$p(\delta \geq b, s) = \int_0^s dS f(S) p(\delta \geq b, s | \text{first } S), \quad (10)$$

where (considering for simplicity a barrier of constant height $b = B = \delta_c$, for which $B' = 0$) we can formally identify

$$p(\delta \geq b, s | \text{first } S) \equiv \int_0^{\infty} dV V \frac{p(\delta \geq b, V, B, \text{first } S)}{f(S)} \quad (11)$$

as the conditional probability of a walk ending above threshold at s given that it crossed for the first time at S .

For a Gaussian field the left-hand side of equation (10) is just $\text{erfc}(b/\sqrt{2s})/2$; therefore, if one can come up with a good approximation for $p(\delta \geq b, s | \text{first } S)$, then $f(S)$ can be solved for numerically by simple back substitution. Note however that this logic is general, applying as well to both Gaussian and non-Gaussian walks. Since one usually knows the quantity on the left-hand side, the problem reduces to guessing $p(\delta \geq b, s | \text{first } S)$. Of course, the devil is in this last detail.

2.1 Separability

If $p(\delta \geq b, s|\text{first } S)$ is a separable function of s and S , then the expression above can be manipulated to obtain a simple expression for $f(s)$. Namely, if

$$p(\delta \geq b, s|\text{first } S) = g(S)h(s), \quad (12)$$

then

$$p(\delta \geq b, s) = h(s) \int_0^s dS f(S) g(S), \quad (13)$$

making

$$f(s) = \frac{1}{g(s)} \frac{d}{ds} \left[\frac{p(\delta \geq b, s)}{h(s)} \right]. \quad (14)$$

For walks with completely correlated or completely uncorrelated steps, and a constant barrier δ_c , the product $g(S)h(s)$ equals 1 and $1/2$, respectively (Bond et al. 1991; Paranjape, Lam & Sheth 2012). Since this is independent of both s and S , both limits yield particularly simple relations between the first crossing distribution and the probability distribution of δ on the first crossing scale: these limits yield the expressions for $f(s)$ derived by Press & Schechter (1974) and Bond et al. (1991), respectively.

In general, of course, $p(\delta \geq b, s|\text{first } S)$ is more complicated. However, this simple case is still very instructive because it shows that the back-substitution method can interpolate between these two regimes. It is therefore more general than the approach which leads to equation (3), which, while accurate when the correlation between steps is large, diverges in the limit of uncorrelated steps.

2.2 Normalization

Before we study the general case, it is worth noting that this approach is also attractive because it provides an easy way to see that the result will be correctly normalized. While this is trivial to see geometrically, the argument relies on using the correct expression for $p(\delta \geq b, s|\text{first } S)$.

For a constant barrier, there is equal probability of ending up on either side of it if one waits long enough: evaluating equation (10) at $s = \infty$ one has therefore

$$\frac{1}{2} = \int_0^\infty dS f(S) p(\delta \geq b, \infty|\text{first } S). \quad (15)$$

For the same reason, it must also be true that $p(\delta \geq b, \infty|\text{first } S) = 1/2$, which shows that the exact solution has $\int_0^\infty dS f(S) = 1$, i.e. $f(S)$ is correctly normalized to unity (except for walks with completely correlated steps, where the integral gives $1/2$, as it should).

More importantly, this relation also ensures that, if the approximate $p(\delta \geq b, s|\text{first } S)$ that we choose has the same large- s limit, then the approximate $f(S)$ that we get from equation (10) is correctly normalized to unity (for constant barriers; we will come back to this point in the case of moving barriers). Our goal will be to find such an approximation for the conditional distribution.

2.3 The simplest approximation

We remarked earlier that the performance of this approach depends critically on having a good approximation for $p(\delta \geq b, s|\text{first } S)$. If we require that the walk had height B on scale S , but drop the

constraint that the barrier had not been crossed prior to this, then one might approximate

$$p(\delta \geq b, s|\text{first } S) \approx p(\delta \geq b, s|B, S); \quad (16)$$

the result of inserting this in equation (10) can be used to obtain an approximation for $f(S)$ by numerical back substitution.

For a Gaussian process and a barrier of constant height

$$p(\delta \geq \delta_c, s|\delta_c, S) = \frac{1}{2} \operatorname{erfc} \left[\frac{\delta_c(1 - S_\times/S)}{\sqrt{2}(s - S_\times^2/S)^{1/2}} \right], \quad (17)$$

where $S_\times \equiv \langle \delta \Delta \rangle$, and Δ denotes the walk height on scale S . To see that the resulting estimate for $f(s)$ will be correctly normalized, recall that equation (15) shows that we must look at the $s \rightarrow \infty$ (i.e. the $r \rightarrow 0$) limit of this expression. For the filters of interest here $S_\times/S$ is finite, so this limit is $\operatorname{erfc}(0)/2 = 1/2$, indicating that the estimate for $f(s)$ which results from approximation (16) will be normalized to unity.

Although this approach (equation 16 in 10) has been followed in the past (e.g. Jedamzik 1995; Nagashima 2001; Lapi, Salucci & Danese 2013), all previous work has failed to appreciate that equation (16) is actually only an approximation, and, for most filters of interest, a bad one at that. To see how bad, we show in the appendix that this problem can be solved exactly, analytically, for Gaussian smoothing of a Gaussian field with $P(k) \propto k^{-1}$, and that the resulting expression for the first crossing distribution is much further from the correct answer than is f_{up} .

For walks with uncorrelated steps, on the other hand, this approach is exact. For a constant barrier it returns equation (14) with $g = 1/2$ and $h = 1$. For a moving barrier, equation (17) would depend on $(b - B)/\sqrt{(s - S)}$, while the left-hand side of equation (10) would be $(1/2)\operatorname{erfc}(b/\sqrt{2}s)$.

2.4 The two-step approximation

How might one improve on this approximation? One obvious possibility is to add the additional constraint that the walk height at one other (larger) scale was less than δ_c :

$$p(\delta \geq \delta_c, s|\text{first } S) \approx \frac{\int_{-\infty}^{\delta_c} d\Delta_1 \int_{\delta_c}^{\infty} d\delta p(\delta, s, \Delta_1, S_1|\delta_c, S)}{\int_{-\infty}^{\delta_c} d\Delta_1 p(\Delta_1, S_1|\delta_c, S)} \quad (18)$$

for some $S_1 \leq S$.

To see that there is little point in having $S_1 \ll S$, we first write $p(\delta, \Delta_1|\delta_c, S) = p(\Delta_1|\delta_c, S)p(\delta|\Delta_1, \delta_c)$. These are all Gaussian distributions with different means and variances. The distribution $p(\delta|\Delta_1, \delta_c)$ only depends on Δ_1 and δ_c through its mean, which is given by $\langle \delta|\Delta_1, \delta_c \rangle = r_{c\delta}\delta_c + (r_{1\delta} - r_{c\delta}r_{1c})(\Delta_1 - r_{1c}\delta_c)/(1 - r_{1c}^2)$. If $S_1 \ll S$ then $r_{1c} \ll 1$, and since $r_{1\delta} \leq r_{1c} \ll 1$ $\langle \delta|\Delta_1, \delta_c \rangle \rightarrow r_{c\delta}\delta_c$. Thus, in this limit, $p(\delta|\Delta_1, \delta_c) \rightarrow p(\delta|\delta_c)$. As a result, the integrals in the numerator become separable, and one of them is cancelled by the integral in the denominator, leaving $p(\delta \geq \delta_c, s|\delta_c, S)$, i.e. when $S_1 \ll S$ then this approximation reduces to equation (16). Thus, this additional constraint only matters when the correlation coefficient between the two scales is close to unity: i.e. when $S_1 \sim S$. The $S_1 \rightarrow S$ limit is, essentially, the Musso–Sheth approximation, which we know works very well, especially at large ν .

2.5 The importance of stepping up

The success of $f_{\text{up}}(s)$ in approximating $f(s)$ suggests that further improvement may be obtained by relaxing the ‘first S ’ constraint in

$p(\delta \geq b, s | \text{first } S)$ into the simpler requirement of merely upcrossing at S . Therefore, equation (10) may be approximated by

$$p(\delta \geq b, s) \approx \int_0^s dS f(S) p(\delta \geq b, s | \text{up } S), \quad (19)$$

where the conditional probability can be obtained dropping ‘first S ’ in equation (11), and therefore replacing $f(S)$ with $f_{\text{up}}(S) = \int_0^\infty dV V p(V, B)$; for $b = B = \delta_c$ this leads to

$$p(\delta \geq \delta_c, s | \text{up } S) = \frac{\int_0^\infty dV V p(\delta \geq \delta_c, V | B)}{\int_0^\infty dV V p(V | B)}, \quad (20)$$

and the upcrossing is guaranteed by the fact that the walk must have positive slope at S . For small enough S (meaning $S \ll \delta_c^2$), most walks of height δ_c will have positive slope, so this constraint should matter little, but at large S it may begin to play a role. Finally, for $s \gg S$ the probability of having $\delta > \delta_c$ is $1/2$ regardless of V , which ensures that $f(s)$ derived from equation (19) is correctly normalized.

We have already seen that $p(\delta \geq \delta_c, s | \text{up } S) \simeq 1$ implies that one recovers $f(s) \approx f_{\text{PS}}$ from equation (14). This is in fact what happens with correlated steps at small s , when, if the walk reaches the barrier at $S < s$, it is almost certainly crossing upwards and thus remains above threshold at s . In this regime, both numerator and denominator in equation (20) are approximately equal to $\langle V | B \rangle = \int_{-\infty}^{+\infty} dV V p(V | B)$.

Similarly, it is instructive to see under what conditions one can recover $f(s) \approx f_{\text{up}}(s)$. For instance, this is the case if only the numerator equals $\langle V | B \rangle$: then, because the denominator does not depend on s , the resulting expression will be the product of $h(s) = 1$ and $g(S) = f_{\text{PS}}(S)/f_{\text{up}}(S)$, which in equation (14) will yield $f(S) \approx f_{\text{up}}(S)$. In fact, this picture is correct at larger s , when the fraction of walks crossing δ_c downwards becomes relevant; the numerator includes values of $\delta > \delta_c$ for which earlier crossings are less and less likely, and thus on average it remains close to $\langle V | B \rangle$ for longer than the denominator. We provide an analysis of the $n = -1$ case showing this explicitly in the appendix.

To recover the approximation of equation (16), suppose (wrongly!) that $p(\delta, V | B) = p(\delta | B) p(V | B)$. Then the integral over V equals the term in the denominator, so only the integral over $p(\delta | B)$ remains, and the resulting expression reduces to equation (16). This shows that equation (16) is the result of assuming that the walk height δ on scale s is only correlated with the slope $d\Delta/dS$ of the walk on a larger scale S because both δ and $d\Delta/dS$ are correlated with Δ .

To see the effect of the δ - V correlation, let us compute equation (20) explicitly. This is most easily done by using the rescaled barriers $v \equiv b/\sqrt{s}$ and $\eta \equiv B/\sqrt{S}$, and the unit variance variables that factorize $p(B, V)$, that is $\Delta/\sqrt{S}$ and $u \equiv 2\Gamma S(\Delta/\sqrt{S})'$, with $\Gamma = 1/\sqrt{4S\langle\Delta^2\rangle} - 1$. This yields

$$p(\delta \geq \delta_c, s | \text{up } S) = \frac{\int_{-X}^{\infty} du (u + X) p(u) \text{erfc}(\mu_{v\eta u})}{2 \int_{-X}^{\infty} du (u + X) p(u)}, \quad (21)$$

where $X \equiv \Gamma\eta$ and, after defining $\xi \equiv \langle (\delta/\sqrt{s})(\Delta/\sqrt{S}) \rangle$ and $\Sigma \equiv \langle (\delta/\sqrt{s})u \rangle$,

$$\mu_{v\eta u} = \frac{v - \xi\eta - \Sigma u}{\sqrt{2(1 - \xi^2 - \Sigma^2)}}. \quad (22)$$

In particular, if $\Sigma \simeq 0$ then $p(\delta \geq \delta_c, s | \text{up } S)$ reduces to the simpler form given by equation (16). Moreover, all v, ξ and Σ vanish when $s \rightarrow \infty$; then in this limit $\text{erfc}(\mu_{v\eta u}) \rightarrow 1$, which confirms that the approximation for $f(s)$ that one gets from equation (19) is normalized to unity.

We will show shortly that solving (19) yields an accurate enough approximation to the true first crossing distribution that it can be considered the solution to the original excursion set problem posed by Bond et al. (1991), of walks with correlated steps crossing a threshold of constant height. Although it is a numerical solution, the fact that $f \sim f_{\text{up}}$ at least down to $\Gamma v \sim 1$ suggests that analytic progress may be made writing $f = f_{\text{up}} + (f - f_{\text{up}})$ in equation (19). Differentiating with respect to s and rearranging the terms yields

$$f(s) = f_{\text{up}}(s) + \frac{d}{ds} \int_{\delta_c}^{\infty} d\delta \left[p(\delta, s) - p_{\times}^{\text{up}}(\delta, s) \right] + \int_0^s dS [f(S) - f_{\text{up}}(S)] \frac{d p(\geq \delta_c, s | \text{up } S)}{ds}, \quad (23)$$

where

$$p_{\times}^{\text{up}}(\delta, s) \equiv \int_0^s dS f_{\text{up}}(S) p(\delta, s | \text{up } S) \quad (24)$$

corresponds to equation (9) if one drops the requirement that the walk never crossed before S , and thus at small s provides a good approximation to $p(\delta, s) = p_{\times}(\delta, s)$ for all $\delta > \delta_c$. Note however that, unlike equation (19), this formulation is not valid for uncorrelated steps, since in this case $p(\delta \geq \delta_c, s | \text{up } S)$ with $S = s$ does not equal 1 but $1/2$.

The first term in equation (23) provides a correction to f_{up} that depends on the deviation of $p_{\times}^{\text{up}}(\geq \delta_c, s)$ from $p(\geq \delta_c, s)$, which is small at small s . The last term contains the correction to f_{up} integrated over S , and therefore is expected to kick in only after the first one becomes relevant, as a second order correction. However, this term is crucial to preserve the normalization of the solution, which being a constraint on the integral of f is non-perturbative in s . Inserting equation (20) in the back-substitution expression (equation 10) allows one to go beyond the first order correction, keeping the approximation under control also in the large- s regime. We will discuss an approximation that boils down to ignoring this last term in Section 2.7.

2.5.1 Gaussian filters

In the case of a power law $P(k)$ with a Gaussian filter one has

$$\frac{\Sigma}{\Gamma\xi} = \frac{1 - (r/R)^2}{1 + (r/R)^2}, \quad (25)$$

where r and R are the smoothing scales associated with s and S . It is a simple matter to check that $\Sigma/\sqrt{1 - \xi^2} \rightarrow 1 - \epsilon^2(1 + \epsilon)(1 + \Gamma^2)/16$ as $\epsilon \equiv 1 - (r/R)^2 \rightarrow 0$. (Use of this limiting case helps numerical stability in the $r \rightarrow R$ limit.)

Fig. 1 compares our back-substitution estimate of $sf(s)$ with that based on equation (16), and with the simpler f_{up} approximation for walks associated with a range of power-law power spectra $P(k) \propto k^n$ crossing a barrier of constant height δ_c . Symbols show the ‘exact’ result obtained by Monte Carlo methods (following Bond et al. 1991). (Details of our implementation of the back-substitution algorithm are provided in Appendix B.) We have chosen $n = +1$ and -2 to compare directly with the results in Bond et al. (1991) and $n = -1.2$ (rather than $n = -1$) to compare with Musso & Sheth (2012).

Our Monte Carlos are in excellent agreement with those shown in fig. 9 of Bond et al. (1991). For example, for $n = (+1, -1, -2)$, their Monte Carlos cross the curve for walks with uncorrelated steps at $\log_{10}(\delta_c^2/s) = (-1.6, -1.3, -1.1)$. The approach based on the back substitution in equation (19) is now in excellent quantitative agreement with the actual first crossing distribution obtained by

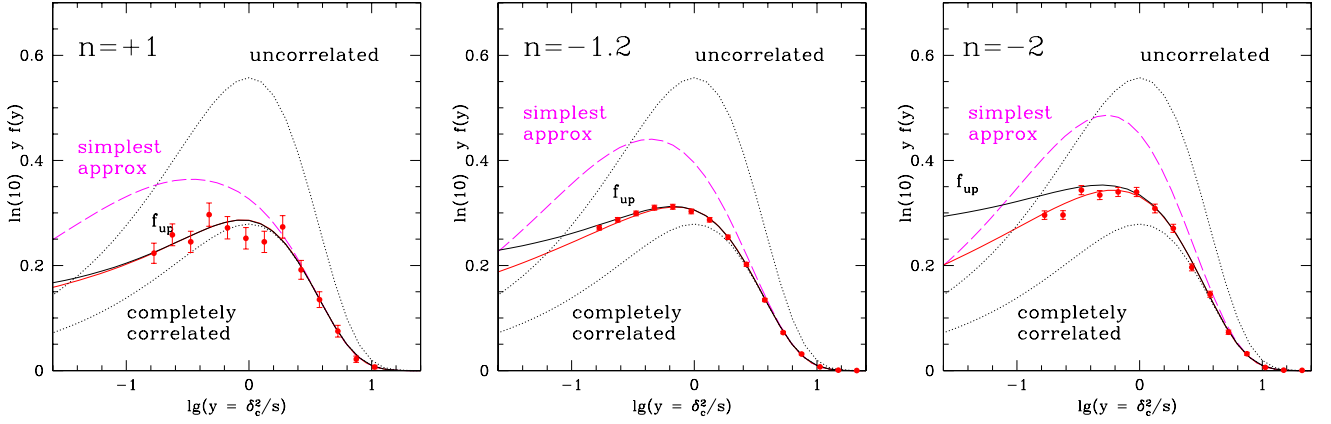


Figure 1. Comparison of various approximations for the distribution of the scale $y = \delta_c^2/s$ on which walks associated with Gaussian smoothing of a power spectrum with $P(k) \propto k^n$ first cross a barrier of constant height δ_c . In each panel, the long dashed curve shows the result of inserting the simple approximation (16) in equation (10); thin solid curve shows the fully analytic approximation of f_{up} with the appropriate values of Γ^2 (i.e. 2, 0.9 and 0.5); and thick solid curve shows the better approximation which follows from using equation (20) in equation (10). This last provides an excellent description of the exact solution obtained by Monte Carlo methods (symbols with error bars).

direct Monte Carlo simulation of the (Gaussian smoothed) walks at all s . In particular, our approach brings a small correction to f_{up} , but one which cures the divergence at small ν . On the scales shown, the correction to f_{up} is significant for $n = -2$ and -1 , but still small for $n = +1$. And f_{up} itself is a smaller correction to the completely correlated limit as n increases. Both these are consistent with the expectation that $\Gamma\nu \sim 1$ sets the new scale.

2.5.2 Tophat smoothing

We have also studied walks for which the correlations between steps arise from a tophat rather than a Gaussian smoothing filter. This is an interesting test since, from the point of view of the f_{up} approximation, the two filters are rather different: for $P(k) \propto k^n$, Gaussian smoothing has $\Gamma^2 = (n+3)/2$ whereas tophat smoothing has $\Gamma^2 = (n+1)(n+3)/(n-3)$ (and is constrained to $-3 < n < -1$), i.e. Γ^2 is monotonic with n for Gaussian smoothing, but it has a maximum at $n = 3 - 2\sqrt{6} \approx -1.9$ for the tophat. Moreover Γ^2 for a tophat is always smaller than for a Gaussian, so we might expect f_{up} to be a poorer approximation for tophat smoothing than for Gaussian (recall that $\Gamma\nu \sim 1$ sets the scale below which we expect the approximation to begin to fail). For example, when $n = -2$, then Γ^2 is $1/2$ and $1/5$, so we expect the approximation to fail at $\nu^2 \leq 2$ for Gaussian smoothing, but $\nu^2 \leq 5$ for tophat. Fig. 2 shows that this is indeed the case; f_{up} vastly overpredicts the first crossing rate for small $\nu < 1$. The actual rate is less dependent on the smoothing filter, and is very well matched by our back-substitution algorithm for which the relevant parameters are

$$\nu = \frac{1}{\sqrt{6}}, \quad \xi = \sqrt{\frac{S}{s}} \frac{5 - (S/s)^2}{4} \quad \text{and} \quad \frac{\Sigma}{\Gamma\xi} = \frac{1 - (S/s)^2}{1 - (S/s)^2/5}$$

for $S \leq s$.

Tophat filters exhibit an additional complication, since $\langle \nu^2 \rangle$ for these filters diverges for power spectra with enough small-scale power. For power-law spectra, this means $n < -1$ is required for $\langle \nu^2 \rangle$ to be convergent, whereas for $n \geq +1$ even $\langle \delta^2 \rangle$ diverges. There are two ways to treat such filters; one is to assume a hard cut-off on $P(k)$ above some k_{max} ; the other smooths the sharp corners of the tophat with a Gaussian, by setting the Fourier transform of the smoothing filter to $W_{TH}(kR) \exp(-a^2 k^2 R^2/2)$. In this second case, $\langle \delta^2 \rangle$ is within a per cent or so of the pure tophat value (at least for

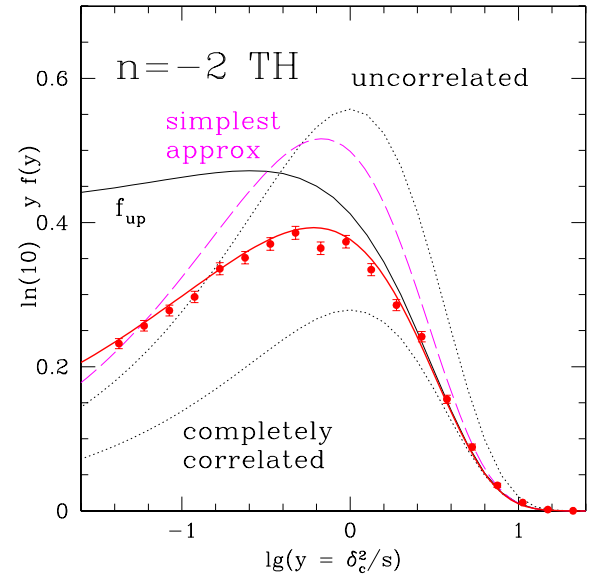


Figure 2. Same as previous figure, but now for tophat smoothing of $P(k) \propto k^{-2}$ for which $\Gamma^2 = 1/5$. Note that, compared to Gaussian smoothing (for which $\Gamma^2 = 1/2$), f_{up} is a rather bad approximation for a much larger range of ν .

$n \leq 0$) provided $a \ll 0.01$. When $a = 0.01$, then $\Gamma^2 = (0.2, 0.14, 0.027, 0.0037)$ for $n = (-2, -1, 0, +1)$. As one might expect, these small values of Γ^2 mean that f_{up} vastly overestimates the true $f(s)$ over a wide range of s . However, our back-substitution algorithm continues to work well. Indeed, it correctly gets the $a \rightarrow 0$ limit of the $n = 0$ case, being $f(s) = 2f_{PS}(s)$. This $n = 0$ case illustrates nicely that, at least for first crossing distributions, Gaussian and tophat smoothing filters can be very different.

2.6 Moving barriers

Although equation (10) did not make any assumption about the barrier shape, the approximations we discussed and the examples we have shown so far all assumed a barrier of constant height. However, dealing with moving barriers – barriers that depend on

s – is straightforward. The appropriate generalization of equation (20) is

$$p(\delta \geq b(s)|\text{up } S) \equiv \frac{\int_b^\infty d\delta \int_{B'}^\infty dV (V - B') p(\delta, V|B)}{\int_{B'}^\infty dV (V - B') p(V|B)}, \quad (26)$$

which inserted into equation (10), with $b = b(s)$ on the left-hand side, allows solving for f . Computing this expression adds no further technical difficulty; in fact, it yields exactly the same result as equation (21), upon redefining $X \rightarrow -2\Gamma S(B/\sqrt{S})'$. The result is a correction to f_{up} that cures the divergence at small s , if present. Nonetheless, as we now argue, for many moving barriers of current interest, and particularly for Gaussian smoothing filters for which Γ is not negligible, f_{up} may already be sufficiently accurate.

To make the discussion easier, it may help to consider barriers of the form $b(s) = \delta_c + \alpha s^\omega$, and write

$$\lim_{s \rightarrow \infty} \int_0^s dS f(S) \frac{p(\delta \geq b(s)|\text{up } S)}{p(\delta \geq b(s))} \approx 1, \quad (27)$$

which is the generalized version of equation (15). Insight on the normalization of the solution can then be obtained by studying the limit for $s \rightarrow \infty$ of the ratio $\text{erfc}(\mu_{v\eta u})/\text{erfc}(v/\sqrt{2})$, with $\mu_{v\eta u}$ given by equation (22). The denominator tends to 1 for $\omega < 1/2$ and to $\text{erfc}(\alpha/\sqrt{2})$ for $\omega = 1/2$, while for $\omega > 1/2$ the limit is either 2 (if $\alpha < 0$) or 0 (if $\alpha > 0$). The same is true for the numerator: since both ξ and Σ become typically proportional to $\sqrt{S/s}$ when $s \gg S$, the presence of η and u (which are numbers of order unity) could be reabsorbed rescaling δ_c , and this does not affect the limit. Thus, the normalization of $f(s)$ obtained from equation (26) is always unity, except if $\alpha > 0$ and $\omega > 1/2$, in which case no conclusion can be drawn a priori.

Regardless of the sign of α (the direction of the barrier), the value $\omega = 1/2$ discriminates between barriers that are moving slower or faster than the rms of the distribution. For faster barriers, virtually all crossings must happen when $s \lesssim \delta_c$: at smaller scales diffusion simply cannot keep up with the barrier, and $f(s)$ is suppressed like $\exp(-b^2(s)/s)$. For these barriers, f_{up} is expected to be all that is needed for cosmology. On the other hand, sizeable corrections can arise at large s for slower barriers. We may thus introduce a classification of moving barriers in three categories which are as follows.

(i) Fast decreasing barriers ($\alpha < 0$ and $\omega > 1/2$), whose first crossing distribution is normalized to 1. The probability that δ lies above $b(s)$ given that it crossed $b(S)$ approaches unity, simply because $b(s) \ll b(S)$ even for $S \lesssim s$. At small s , this makes equation (14) with $g(s) = h(s) = 1$ a good approximation (the ‘completely correlated’ limit, which corresponds to setting the term in square brackets in equation 4 equal to unity). Although this was noted by Paranjape et al. (2012), our equation (10) is an easy way to see why it works so well. That said, numerical tests show that the full f_{up} is even more accurate, yielding $\int_0^\infty ds f_{\text{up}}(s) \simeq 1$ at less than per cent level for linear barriers, and improved accuracy for steeper barriers. Therefore, no further correction is needed. Examples of such barriers include the excursion set model of the non-linear probability distribution function (Sheth 1998; Lam & Sheth 2008) or the non-linear collapse along one axis (Shen et al. 2006) (i.e. excursion set models of large-scale filaments).

(ii) Slowly moving barriers ($\omega \leq 1/2$), whose first crossing distribution is not normalized to 1. In this case, $p(\delta \geq b(s)|\text{first } S)$ is significantly less than 1, making equation (14) a bad approximation [something Paranjape et al. (2012) also saw]. On the other hand, equation (3) remains a good approximation down to scales of the

order of δ_c and beyond. However, the integral of f_{up} is divergent, signalling that the normalized solution obtained by inserting equation (26) in equation (10) provides an important correction at very large s . Barriers with $\omega = 1/2$ (the limit of this category) are often found in the literature as simple approximations to ellipsoidal collapse (see for instance Lam & Sheth 2009; Moreno, Giocoli & Sheth 2009; Paranjape et al. 2012; Lapi et al. 2013).

(iii) Fast increasing barriers ($\alpha > 0$ and $\omega > 1/2$). In this case, the barrier increases too steeply and not all walks eventually cross, so the normalization is less than unity (in fact, this is already true even for f_{up} , which overestimates the result). Although calculating this factor becomes complicated, the accuracy of our approach is not jeopardized: equation (10) is always exact and equation (26) is an excellent approximation for $s \gtrsim S$ because – just as for slowly moving barriers – much of the error due to the inclusion of walks with earlier crossings cancels out in the ratio. For example, the actual effective barrier for ellipsoidal collapse has $\omega = 0.615$ (Sheth et al. 2001). For even larger values of ω , most crossings will happen at small s , where we know that f_{up} is accurate enough: in this case, the additional corrections presented in this paper bring only minor improvement. For barriers that increase linearly with s , the accuracy of f_{up} was already known (Musso & Sheth 2012). Fig. 3 shows that f_{up} remains extremely accurate even when $\omega = 2$.

The example above used Gaussian smoothing of $P(k) \propto k^{-1}$, for which $\Gamma^2 = 1$. To study how much of the discussion above remains true as Γ decreases, Fig. 4 shows an analysis of walks first crossing the linear barrier $B(s) = \delta_c(1 + s/\delta_c^2)$ when the smoothing filter is tophat and $P(k) \propto k^{-2}$. In this case, Γ^2 is only $1/5$, and, for a constant barrier, f_{up} vastly overestimates the correct answer. For

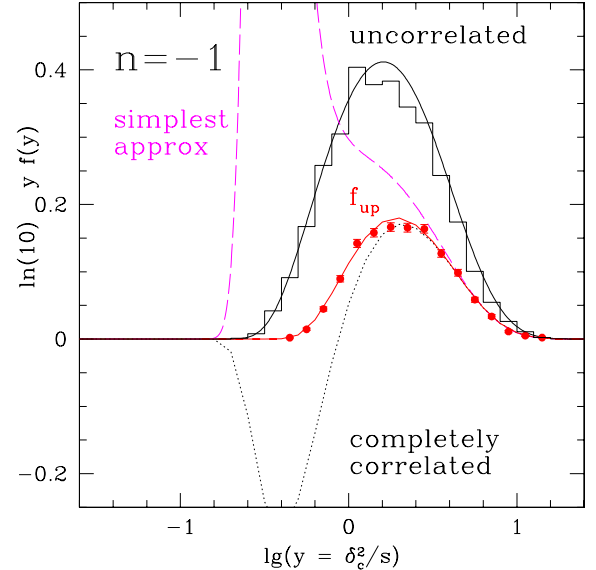


Figure 3. First crossing distribution $yf(y) = sf(s)$ of a steeply increasing barrier of height $\delta_c[1 + (s/\delta_c^2)^2/4]$, by walks with uncorrelated steps (histogram) and steps having correlations which come from Gaussian smoothing of $P(k) \propto 1/k$ (symbols with error bars). Upper solid curve shows the result of inserting equation (16) in equation (10) for the uncorrelated steps, which should be exact. Dashed curve shows the same calculation but now for correlated steps; it lies far from the symbols. Dotted curve shows that f_{PS} is also a poor description, since it even goes negative (signalling that there are more walks crossing downwards than upwards). Lower solid curve shows f_{up} , which provides an excellent description of the first crossing distribution. For this process and barrier, it is indistinguishable from the solution obtained by numerical back substitution with equation (26).

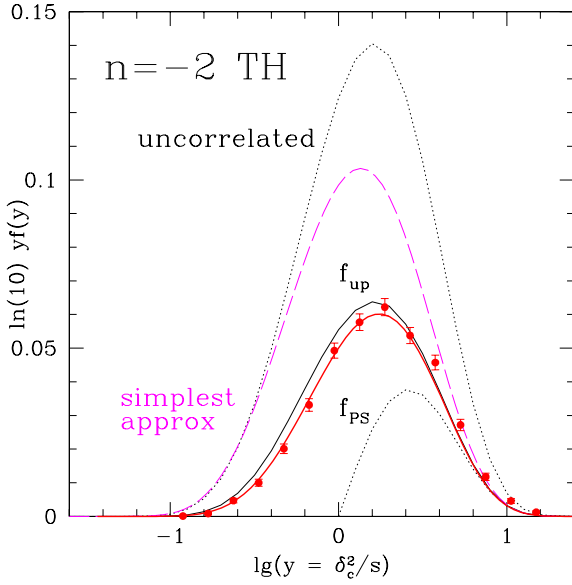


Figure 4. First crossing distribution $yf(y) = sf(s)$ of a linearly increasing barrier of height $\delta_c[1 + s/\delta_c^2]$, by walks with steps having correlations which come from tophat smoothing of $P(k) \propto k^{-2}$ (symbols with error bars). Upper dotted curve shows the expected (inverse Gaussian) distribution which results from inserting equation (16) in equation (10) and assuming uncorrelated steps. Dashed curve shows the same calculation but now for correlated steps; it lies far from the symbols. Lower dotted curve shows that f_{PS} is also a poor description, since it even goes negative (signalling that there are more walks crossing downwards than upwards). Upper solid curve shows that f_{up} (with $\Gamma^2 = 0.2$) slightly overestimates the first crossing distribution, whereas lower solid curve which goes through the symbols shows our back-substitution solution.

the linearly increasing barrier, f_{up} is closer to the correct answer than for a constant barrier, but it is not as good an approximation as it was for Gaussian smoothing. We have checked (but do not show here) that the agreement degrades as Γ^2 decreases further: for tophat smoothed $P(k) \propto k^{-1.2}$ walks, $\Gamma^2 = 0.08$, and f_{up} for this linear barrier lies well above the correct answer which is not very different from when $\Gamma^2 = 1/5$.

The arguments above are general – they remain true for non-Gaussian processes – as long as the rms of the distribution grows like $\sqrt{s}$. We conclude therefore that our approach introduces a correction that matters at large s , for any mildly scale dependent barrier. This correction guarantees the correct normalization of the solution, and, being based on an integral constraint, is non-perturbative in s . For steeper (increasing or decreasing) barriers, this approach is also correct, but, if Γ is large, then it provides only a small correction to f_{up} before the exponential cutoff dominates the solution. In practice, this means that for Gaussian smoothing filters, the added complexity is not necessary; one only needs our full back-substitution algorithm when Γ becomes sufficiently small [e.g. for tophat smoothing of sufficiently steep $P(k)$].

2.7 Other related approximations

An equivalent formulation of the first crossing rate is

$$f(s) = -\frac{d}{ds} \int_{-\infty}^b d\delta [p(\delta, s) - p_{\times}(\delta, s)], \quad (28)$$

with $p_{\times}(\delta)$ given by equation (9). Using $\int_{-\infty}^b = \int_{-\infty}^{\infty} - \int_b^{\infty}$, and the fact that by construction one has $p_{\times}(\delta, s) = p(\delta, s)$ for $\delta > b$, one can

in fact check that from this expression one recovers equation (8). Note however that here we are looking at $p_{\times}(\delta, s)$ for $\delta < b$: for barriers of constant height, equation (28) is the generalization to correlated steps of the symmetry argument used by Bond et al. (1991) for walks with uncorrelated steps [in which case $p_{\times}(\delta)$ is a Gaussian with mean $2b$ and variance s].

In the spirit of Musso & Sheth (2012), we can approximate $p_{\times}(\delta)$ in equation (28) with equation (24), obtaining

$$f(s) \simeq f_{PS}(s) + \int_0^s dS \int_0^{\infty} dV \frac{dp(\delta \leq b, B, V)}{ds}, \quad (29)$$

where we have brought d/ds inside the integral over S since its action on s in the integration limit gives zero. Evaluating the derivative via the Fokker–Planck equation always involves one derivative with respect to δ , so that the integral over δ is trivial. The integral over V can thus be computed analytically in full generality, leaving only the integral over S (for which we could find no analytical result) to be computed numerically.

This is apparently an expansion in which $f_{PS}(s)$ is the leading order term; however, using again $\int_{-\infty}^b = \int_{-\infty}^{\infty} - \int_b^{\infty}$, the same can be recast as

$$f(s) \simeq f_{up}(s) + \frac{d}{ds} \int_b^{\infty} d\delta [p(\delta, s) - p_{\times}^{up}(\delta, s)], \quad (30)$$

where now the second term no longer vanishes. Therefore, the deviation of $p_{\times}^{up}$ from $p(\delta, s)$ provides at small s the first order correction to the approximation $f(s) \simeq f_{up}(s)$. This deviation is small in this regime, which is why f_{up} works so well. However, the expression above corresponds to ignoring the final term in equation (23), which as we have discussed becomes important at large s .

To illustrate this approximation, we ran walks whose velocities (not walk heights!) are Markovian. These can be tuned, following methods given in Musso & Sheth (2013b), to mimic to remarkable accuracy the results of the actual excursion set walks for a given power spectrum and filter; nevertheless, their Monte Carlo simulation benefits from the many appealing properties of a Markovian process, which makes it easy to reach very large values of s . The relevant parameters for the back-substitution algorithm are

$$\gamma = \frac{1}{2}, \quad \xi = \sqrt{\frac{S}{s}} \frac{3 - S/s}{2} \quad \text{and} \quad \frac{\Sigma}{\Gamma\xi} = \frac{1 - S/s}{1 - (S/s)/3} \quad (30)$$

for $S \leq s$.

This correlation structure, which is quite different from that for Gaussian smoothing, is interesting for a few reasons. First, it is not too different from that for tophat smoothing of $P(k) \propto k^{-2}$, which we studied in the previous section. In this respect, it is closer to the halo counts problem which the excursion set approach was originally intended to model, since these are defined using tophat smoothing. In addition, $\gamma \sim 1/2$ is a reasonable approximation to tophat smoothed Λ cold dark matter (Λ CDM; see fig. 4 in Paranjape et al. 2012). And finally, as we show shortly, the first crossing distribution associated with this model can be scaled to provide a very good description of dark matter halo abundances.

Fig. 5 compares the Monte Carlo first crossing distribution with the full back-substitution solution (equation 20), and with its first order term (equation 29), which for these walks reads

$$sf(s) \approx sf_{PS}(s) + \int_0^1 \frac{dy}{y} \frac{1}{1-y} \int_0^{\infty} dw w^2 \times \frac{e^{-w^2/2y}}{\sqrt{2\pi y}} \frac{e^{-[w-2(b/\sqrt{s})]^2/6y}}{\sqrt{6\pi y}} \frac{e^{-w^2/2(1-y)}}{\sqrt{2\pi(1-y)}} \quad (31)$$

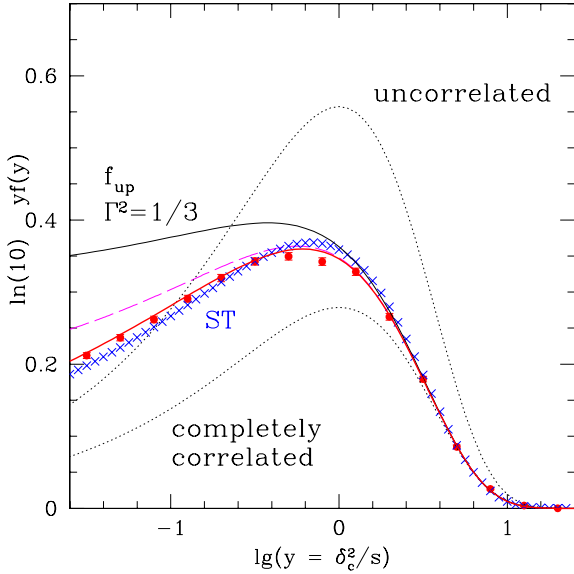


Figure 5. First crossing distribution $yf(y) = sf(s)$ of a constant barrier of height δ_c , by walks with Markovian velocities which have $\gamma = 1/2$ (i.e. $\Gamma^2 = 1/3$). Filled circles show the Monte Carlo distribution; thick solid curve shows the full back-substitution expression; dashed curve shows equation (31); and thinner solid curve shows the simpler approximation of Musso & Sheth (2012). The two dotted curves show the approximation of Press & Schechter (1974) and twice this value. Crosses show the fitting formula of Sheth & Tormen (1999) with $\sqrt{0.7}\delta_c$ in their formula replaced with δ_c . The similarity between the crosses and circles indicates that our Markovian velocities model, with either $\delta_c \rightarrow \sqrt{0.7}\delta_c$ or $s \rightarrow s/0.7$ will provide a good description of the number density of dark matter haloes in simulations.

(see Musso & Sheth 2013b for details). Clearly, equation (31), which boils down to ignoring the final term in equation (23), is a better approximation than is f_{up} , and the full back-substitution expression is even more accurate.

The crosses in Fig. 5 show the fitting formula of Sheth & Tormen (1999),

$$yf_{\text{ST}}(y) = 0.322 [1 + (qy)^{-0.3}] \sqrt{qy/2\pi} \exp(-qy/2), \quad (32)$$

with $q = 1$. This formula, with $q = 0.7$, provides a very good description of halo counts in simulations. Therefore, the similarity between the crosses and circles indicates that our Markovian velocities model, with either $\delta_c \rightarrow \sqrt{0.7}\delta_c$ or $s \rightarrow s/0.7$ will provide a good description of the number density of dark matter haloes.

Finally, we note that an alternative formulation of our integral equation approach can be obtained by replacing the left-hand side of equation (10) with the fraction of walks that are crossing the barrier b upwards, regardless of whether or not they had crossed before: that is, with $f_{\text{up}}(s)$. Then consider a lower barrier $c = b - b_0$, for some constant $b_0 > 0$. Since each of the walks that cross b at s must have first crossed $C \equiv c(s)$ at some $S < s$ one could replace equation (10) with

$$f_{\text{up}}(s) = \int_0^s dS f(S) f_{\text{up}}(s|\text{first } S), \quad (33)$$

where $f(S)$ is the first crossing distribution for the lower barrier c , and $f_{\text{up}}(s|\text{first } S)$ is the fraction of walks that cross the barrier b at s going upwards (whether or not for the first time), and that had crossed the lower barrier c for the first time at $S < s$. We may then

approximate

$$f_{\text{up}}(s|\text{first } S) \approx \int_{b'}^\infty dv v p(b, v|C) \equiv f_{\text{up}}(s|C), \quad (34)$$

where the final expression is the conditional distribution of Musso, Paranjape & Sheth (2012, see their equations 28 and fig. 2). This approximation also requires knowledge of only a trivariate distribution, but this time for (b, v, C) rather than (b, B, V) , and it too can be solved by back substitution.

3 DISCUSSION

Previous work on the first crossing distribution has shown the power of including the constraint that walks must cross upwards. The simplest version of this approach uses the joint distribution of the walk height and its slope on a given scale, and reduces the problem from one involving an infinite number of correlated variables to only two: $f_{\text{up}}(s)$ of equation (3). This works well down to scales where a large fraction of walks may have zig-zagged their way around the barrier more than once. Although one can account for walks with more zigs and zags order by order, each order in this expansion requires the introduction of two additional (correlated) variables, and even the first correction contains integrals that cannot be done analytically (Musso & Sheth 2013).

Our approach here is different: we started from an integral equation, equation (10), and showed how to use the same upcrossing requirement to account for the effects of walks with multiple crossings (equation 20). Although the solution still cannot be computed analytically, it requires knowledge of just one additional variable. Overall, the approach we presented uses the joint distribution of three variables: the walk height at the scale of interest, and its height and slope on a larger scale (we showed that not including the slope is a bad approximation.) The resulting first crossing distribution, obtained by back substitution, is a very good approximation to the exact solution at all scales (Fig. 1), and is guaranteed to be correctly normalized: it integrates to one for all barriers that do not grow too fast with scale (for very steep barriers the normalization is less than one, as expected).

Over the range of scales that is typically of interest in cosmology – the large mass regime – our result yields only a small correction to equation (3). However, the integral equation that we solve is non-perturbative in s and this keeps the solution under control even in the very large s regime (i.e. for very small masses). Comparison with Monte Carlo simulation of the walks bears this out (Fig. 1). Our procedure is equally valid for walks generated by those filters and power spectra for which equation (3) performs less well (e.g. Gaussian smoothing filters on power spectra with more negative spectral indices, or generally when Γ , the parameter defined in the text below equation 5 is small), and even for walks with uncorrelated steps, for which our approach returns the exact result whereas equation (3) diverges.

Although we illustrated our results using walks from Gaussian smoothing of a Gaussian field with power-law power spectrum crossing a barrier of constant height, we found excellent agreement for walks with tophat smoothing (Fig. 2). We also studied walks with a rather different correlation structure (Fig. 5), which mimics the tophat smoothing of a Λ CDM power spectrum. These walks have Markovian velocities (rather than Markovian steps) and they are particularly interesting because the shape of the first crossing distribution for these walks can be scaled to make it remarkably like the shape of the mass fraction in haloes seen in simulations. If the fundamental ansatz (equation 1) were correct, then this Markovian

velocities model would have a number of interesting applications which we explore elsewhere. For the moment, we feel we have demonstrated that our approach has effectively solved the original excursion set problem posed by Bond et al. (1991): that of walks with correlated steps crossing a barrier of constant height, and it has done so by making use of just three correlated variables. We believe that this spectacular agreement stems from applying the upcrossing approximation which leads to equation (3) to both numerator and denominator of equation (20), with the error that appears at small masses in both largely cancelling out from their ratio.

In addition, we argued that our formulation (like that of Musso & Sheth 2013) applies equally well to moving barriers (simply replace equation 20 with equation 26) and to non-Gaussian walks. For Gaussian smoothing filters, we showed that the simpler analytical approximation of equation (3) is already accurate enough at all scales for steeply increasing barriers (Fig. 3). The same is true for steeply decreasing barriers, in which case the normalization of f_{up} is very close to unity. In fact, the steeper the barrier, the greater the accuracy of f_{up} . Hence, for Gaussian smoothing, most of the complications described here only matter in the intermediate regime in which the barrier to be crossed is a weaker function of smoothing scale s than (or comparable with) the rms of the underlying distribution. This is indeed typical for many barriers of interest in studies of cosmological structure formation. For tophat smoothing filters, our fuller back-substitution algorithm is usually required if the barrier does not decrease sufficiently steeply with s , and $P(k)$ is such that $\Gamma \ll 1$ (e.g. Fig. 4 and related discussion). For tophat smoothed Λ CDM, Γ is typically large enough (see fig. 4 in Paranjape et al. 2012) that f_{up} is a good approximation over the range of mass scales of current interest ($\nu \geq 0.1$).

Solution of our integral equation is trivial and fast, so although we provided an analysis of the leading order corrections to equation (3) that it yields (Fig. 5 indicates these are indeed more accurate than f_{up} , though not as accurate as the full solution of course), as well as an example of another way in which one might have set-up the back-substitution calculation (equation 33), we have not bothered to search for an analytic approximation to the full solution.

There is however the additional concern that the fundamental excursion set assumption of equating the first crossing distribution to the halo mass function, equation (1), is incorrect (Appel & Jones 1990; Manrique et al. 1998; Nagashima 2001). A better treatment of the problem, which accounts for the fact that haloes collapse around special positions in the initial field is necessary (Sheth et al. 2001), such as local maxima of the initial random field (Bardeen et al. 1986; Bond & Myers 1996). When added to the excursion set ansatz, this constraint effectively selects a subset of the walks that have steeper slopes (Musso & Sheth 2012; Paranjape & Sheth 2012), resulting in an additional weighting term on the ensemble of walks (Musso & Sheth 2012; Castorina & Sheth 2013) that the analysis here does not include. We expect this weight to simply enter as an additional factor of $f(V)/(2\pi R^2)^{3/2}$ in both the numerator and denominator of equation (26). If this is indeed all there is to it, then because this additional weight selects steeper walks, its inclusion will matter less than when this weight is not included, since the requirement of steeper walks itself already fixes some of the divergence at large s : i.e. any correction to equation (20) (once the additional weight on the distribution of ν has been included) will be small.

An alternative, more phenomenological, way to obtain better agreement with halo counts in N -body simulations relies on an ad hoc rescaling $\delta_c \rightarrow \sqrt{0.7\delta_c}$ of the threshold for spherical collapse (Bond & Myers 1996; Sheth & Tormen 1999). This approach was outlined by Sheth et al. (2001) and Sheth & Tormen (2002) and

has been recently adopted, with some variants, by Maggiore & Riotto (2010), De Simone, Maggiore & Riotto (2011), Corasaniti & Achitouv (2011) and Lapi et al. (2013) (who, however, use a technically much more involved expansion than the approximation of correlated steps). Although this rescaling does not follow from first principles, and can be seen as a mere parametrization of our ignorance of the process of gravitational collapse, it does provide a rather effective description of what is seen in simulations (Achitouv et al. 2013; Castorina & Sheth 2013). Even in this case, however, accuracy in the small mass regime will require the back-substitution technique described in this paper.

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REFERENCES

- Achitouv I., Rasera Y., Sheth R., Corasaniti P., 2013, *Phys. Rev. Lett.*, 111, 231303
 Appel L., Jones B., 1990, *MNRAS*, 245, 522
 Bardeen J. M., Bond J. R., Kaiser N., Szalay A. S., 1986, *ApJ*, 304, 15
 Bond J., Myers S., 1996, *ApJS*, 103, 1
 Bond J., Cole S., Efstathiou G., Kaiser N., 1991, *ApJ*, 379, 440
 Castorina E., Sheth R. K., 2013, *MNRAS*, 433, 1529
 Corasaniti P., Achitouv I., 2011, *Phys. Rev. Lett.*, 106, 241302
 De Simone A., Maggiore M., Riotto A., 2011, *MNRAS*, 412, 2587
 Jedamzik K., 1995, *ApJ*, 448, 1
 Lam T. Y., Sheth R. K., 2008, *MNRAS*, 386, 407
 Lam T. Y., Sheth R. K., 2009, *MNRAS*, 398, 2143
 Lapi A., Salucci P., Danese L., 2013, *ApJ*, 772, 85
 Maggiore M., Riotto A., 2010, *ApJ*, 717, 515
 Manrique A., Raig A., Solanes J. M., Gonzalez G., Stein P., Salvador-Sol   E., 1998, *ApJ*, 499, 548
 Moreno J., Giocoli C., Sheth R. K., 2009, *MNRAS*, 397, 397
 Musso M., Sheth R. K., 2012, *MNRAS*, 423, L102
 Musso M., Sheth R. K., 2013, preprint (arXiv:1305.0724)
 Musso M., Paranjape A., Sheth R. K., 2012, *MNRAS*, 427, 3145
 Nagashima M., 2001, *ApJ*, 562, 7
 Paranjape A., Sheth R. K., 2012, *MNRAS*, 426, 2789
 Paranjape A., Lam T. Y., Sheth R. K., 2012, *MNRAS*, 420, 1429
 Press W. H., Schechter P., 1974, *ApJ*, 187, 425
 Shen J., Abel T., Mo H., Sheth R. K., 2006, *ApJ*, 645, 783
 Sheth R. K., 1998, *MNRAS*, 300, 1057
 Sheth R. K., Tormen G., 1999, *MNRAS*, 308, 119
 Sheth R. K., Tormen G., 2002, *MNRAS*, 329, 61
 Sheth R. K., Mo H., Tormen G., 2001, *MNRAS*, 323, 1

APPENDIX A: GAUSSIAN SMOOTHING OF A POWER-LAW SPECTRUM

We remarked in the main text that Gaussian smoothing of $P(k) \propto k^{-1}$ allows for some clean analytic results. These are collected in this appendix.

We first consider the simple approximation, equation (16). For Gaussian smoothing of a Gaussian field with $P(k) \propto k^n$ with $n = -1$, the cross-correlation of the field at two different scales is

$$\frac{S_{\times}}{S} \equiv \frac{\langle \delta \Delta \rangle}{\langle \Delta^2 \rangle} = \sqrt{\frac{s}{S}} \xi = \frac{2}{1 + S/s}. \quad (\text{A1})$$

This makes the term in square brackets of equation (17) particularly simple:

$$\frac{\delta_c(1 - S_\times/S)}{\sqrt{s - S_\times^2/S}} = -\frac{\delta_c}{\sqrt{s}}. \quad (\text{A2})$$

Since it is independent of S , $p(\delta \geq \delta_c, s|\delta_c, S)$ can be written as $g(S)h(s)$, with $g(S) = 1$ and $h(s) = \text{erfc}(-\delta_c/\sqrt{2s})/2$. Hence, this is the first non-trivial case in which equation (14) can be used, yielding

$$sf(s) = \frac{d}{d \ln s} \frac{\text{erfc}(\delta_c/\sqrt{2s})}{\text{erfc}(-\delta_c/\sqrt{2s})} = \frac{sf_{\text{PS}}(s)}{[1 + \text{erf}(v/\sqrt{2})]^2/4}. \quad (\text{A3})$$

At $v \gg 1$, this result correctly tends to $sf_{\text{PS}}(s)$, but the leading order multiplicative correction factor of $[1 + (2/v)e^{-v^2/2}/\sqrt{2\pi}]$ is larger than for $sf_{\text{up}}(s)$, for which the correction is $1 + e^{-\Gamma^2 v^2/2}/\sqrt{2\pi}(\Gamma v)^3$ (remember $\Gamma^2 = (n+3)/2$ for Gaussian smoothing, and we are considering $n = -1$, for which $\Gamma^2 = 1$). Since f_{up} is rather close to the correct answer, this indicates that approximation (16) predicts more crossings at large s than actually occur: it is not a very good approximation for the special case of $n = -1$. Since there is nothing particularly special about $n = -1$, other than that it could be solved analytically, we expect equation (16) to remain a bad approximation in general.

The poor performance of equation (A3) is confirmed by the results shown in Fig. 1 of the main text (the figure actually shows results for $n = -1.2$, but $n = -1$ is very similar). This approximation is therefore not a useful correction to f_{up} (it actually makes it worse). Still, it provides a non-trivial fully analytic solution to equation (10), and is thus a very useful test to check the accuracy of the numerical back-substitution algorithm that we adopt to solve also the more general case. We used it for this purpose when developing the code.

For linear barriers, $b(s) = \delta_c(1 + \alpha s/\delta_c^2)$, the term in square brackets of equation (17) becomes

$$\frac{b(s) - (S_\times/S)B(S)}{\sqrt{s - S_\times^2/S}} = -\frac{\delta_c}{\sqrt{s}} + \alpha \frac{\sqrt{s}}{\delta_c}, \quad (\text{A4})$$

which is still independent of S . This means that $sf(s)$ can be obtained analytically. If we define $\beta(s) \equiv b/\sqrt{s}$, then the term above is $2 d\beta/d \ln s = 2s\beta'$, and so

$$f(s) = f_{\text{PS}}(s) \frac{e^{2\alpha} \beta \text{erfc}(\beta/\sqrt{2}) - (2s\beta')\text{erfc}(\sqrt{2s}\beta')}{-s\beta' [\text{erfc}(\sqrt{2s}\beta')]^2}. \quad (\text{A5})$$

When $\alpha = 0$ then $\beta = v$ and $d\beta/d \ln s = -v/2$, so this reduces to equation (A3). Additionally, the fact that for linear barriers equation (A4) does not depend on S means that it can be used to predict the normalization of the solution, as done in equation (15) for constant barriers. Taking the $s \rightarrow \infty$ limit of the ratio of $\text{erfc}(\beta)$ to equation (17) returns $\exp(-2\alpha)$. This coincides with the normalization of $f(s)$ for linearly increasing barriers and uncorrelated steps (which can also be derived using equation 17 for a check).

More insight into the general case comes not so much from this expression for $sf(s)$, as from equation (A4). For $\beta < 0$ (linearly decreasing barriers) this will make the erfc of equation (17) tend to unity – the value associated with completely correlated steps – at both large and small s . This is an explicit demonstration of the statement in the main text about the accuracy of the completely correlated limit for barriers which decrease with s . On the other hand, $\beta > 0$ will have $\text{erfc} \rightarrow 0$ for large s , which is why equation (16) does not work well for increasing barriers.

We now move on to consider the approximation that fared substantially better, equation (20). We remarked that there too, at least

in some regimes, the full expression could be reduced to the form of equation (12). This nearly happens for Gaussian smoothing of $n = -1$. To see it, we write $p(\delta, V|B)$ in equation (20) as $p(\delta|B)p(V|\delta, B)$ and do the integral over V first. Using the same dimensionless variables as in equation (21), one gets

$$p(\tilde{v}|\eta) \frac{\int_{-X}^{\infty} du (u+X) p(u|\tilde{v}, \eta)}{\int_{-X}^{\infty} du (u+X) p(u)}, \quad (\text{A6})$$

where the conditional distribution $p(u|\tilde{v}, \eta)$ is a Gaussian with mean $(\tilde{v} - \xi\eta)\Sigma/(1 - \xi^2)$ and variance $1 - [\Sigma^2/(1 - \xi^2)]$.

This expression, once integrated over $\tilde{v} \geq \beta = b/\sqrt{s}$, returns equation (20). When the mean is separated from $-X$ by more than the square root of the variance, that is when

$$\tilde{X} \equiv \sqrt{\frac{1 - \xi^2}{1 - \xi^2 - \Sigma^2}} \left(X + \frac{\tilde{v} - \xi\eta}{1 - \xi^2} \Sigma \right) \gg 1, \quad (\text{A7})$$

then the numerator of equation (A6) is approximated by $Xp(\tilde{v}|\eta) - \Sigma(\partial/\partial \tilde{v})p(\tilde{v}|\eta)$. In this case, the integral over $\tilde{v}$ can be done analytically.

For a constant barrier, $X = \Gamma\eta$ and $\tilde{v} \geq v$. Also, when $n = -1$, then $\Gamma = 1$ and $\Sigma = \xi\sqrt{1 - \xi^2}$, so

$$\tilde{X} = \frac{\eta + \xi(\tilde{v} - \xi\eta)/\sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \geq \frac{\eta - \xi v}{\sqrt{1 - \xi^2}} = \eta = X, \quad (\text{A8})$$

where we used equation (A2) to set $v - \xi\eta = -v\sqrt{1 - \xi^2}$, and we then simplified $\eta - \xi v$ similarly. When $\tilde{X} \gg 1$ for most $\tilde{v} \geq v$, then integrating over $\tilde{v}$ yields

$$\frac{\text{erfc}(-v/\sqrt{2})/2 + (\xi v/\eta)e^{-v^2/2}/\sqrt{2\pi}v}{\text{erfc}(-\eta/\sqrt{2})/2 + e^{-\eta^2/2}/\sqrt{2\pi}\eta}; \quad (\text{A9})$$

since $\tilde{X} > X$, the condition $\tilde{X} \gg 1$ remains true for most $\tilde{v} \geq v$ up to larger S than $X \gg 1$, when also the denominator becomes unity.

Notice that this expression only depends on S through $\xi v/\eta = 2(S/s)/(1 + S/s)$. In the limit where $\xi v/\eta \rightarrow 0$, this term will be the same as for our simplest approximation. As a result, the expression for f from this approximation will be equation (A3), with f_{PS} replaced by f_{up} . Since the net effect of this term will be to provide a correction to f_{up} rather than to f_{PS} , and the correction to f_{PS} is already too large, we know this will overestimate the true f .

The other limit, $\xi v/\eta \rightarrow 1$, is more interesting, since then this term becomes $f_{\text{up}}(s)/f_{\text{PS}}(s)$ which is independent of S . This separability makes

$$sf(s) = sf_{\text{up}}(s) \frac{1 + \text{erfc}(v/\sqrt{2})/2v^2}{[f_{\text{up}}(s)/f_{\text{PS}}(s)]^2}; \quad (\text{A10})$$

the term that multiplies $sf_{\text{up}}(s)$ is unity at $v \gg 1$, and gently decreases to a minimum of 0.9 at $v \sim 1$ before the approximation breaks down and this term starts to exceed unity again at $v < 0.8$. This is the $sf(s) \approx sf_{\text{up}}(s)$ behaviour we promised in the main text.

We could have gone further by setting $\xi v/\eta \rightarrow S/s$, which has the right limiting behaviour at both large and small S/s . The result is messier, as we must now take two derivatives with respect to s to get an analytic expression for f , but the result is still analytic. Clearly, one could expand $\xi v/\eta$ in powers of S/s , each one of which will require another derivative with respect to s to obtain f . We have not pursued this further, since we are in any case assuming that the term in square brackets in equation (3) is unity, and this will break down as s increases. One could also imagine expanding this term in powers of X when integrating over $\delta \geq \delta_c$ but we have not done so

because the other approximation we provide in the main text (e.g. equation 23) is more efficient and general.

APPENDIX B: BACK SUBSTITUTION

We solve equation (10) numerically as follows. The first step is to write the integral on the right-hand side as a discrete sum:

$$p_j = \sum_{i \leq j} F_i P_{ji}, \quad (\text{B1})$$

where we have absorbed the step size into the definition of F_i . Recall that p and P are known, and the problem is to find F . The logic of the solution is set by noting that $p_1 = F_1 P_{11}$, yields an expression for F_1 , which can be used in the expression for $p_2 = F_1 P_{12} + F_2 P_{22}$ to yield an expression for F_2 , and so on. The general case reads

$$F_j = \frac{p_j - \sum_{i \leq j} F_i P_{ji}}{P_{jj}}. \quad (\text{B2})$$

Implementing this is particularly simple in languages that have been optimized for matrix manipulations. As a result, most of the hard

work is in pre-computing the matrix elements P_{ji} , since these require numerical integrals (e.g. equation 20).

The only remaining questions are the step size and the range in S over which to step. Since we know the solution will have exponential tails, it makes sense to take steps that are spaced equally in logarithmic rather than linear intervals. Moreover, the speed of the algorithm is set by the number of steps, so using log steps yields significant speed up, because it allows us to cover a large dynamical range in S with few steps. In effect, this means that we treat F_i in the expression above as being $(\Delta \ln S) S f(S)$, for some constant $\Delta \ln S$, i.e. the quantity we care about, $S f(S)$, is obtained by dividing the F_i by the step size $\Delta \ln S$. The accuracy of the algorithm depends on this step size: tests with the known analytic solutions discussed in the main text have shown that $\Delta \ln S \leq 0.1$ yields sufficient accuracy. The results in the main text were obtained with $\ln(S/\delta_c^2)$ running over $[-5, 5]$ and $\Delta \ln S = 0.1$. This required 100 steps, which took about 1 s on a standard laptop; neither one of us is an accomplished programmer.

This paper has been typeset from a $\text{\LaTeX}$ file prepared by the author.