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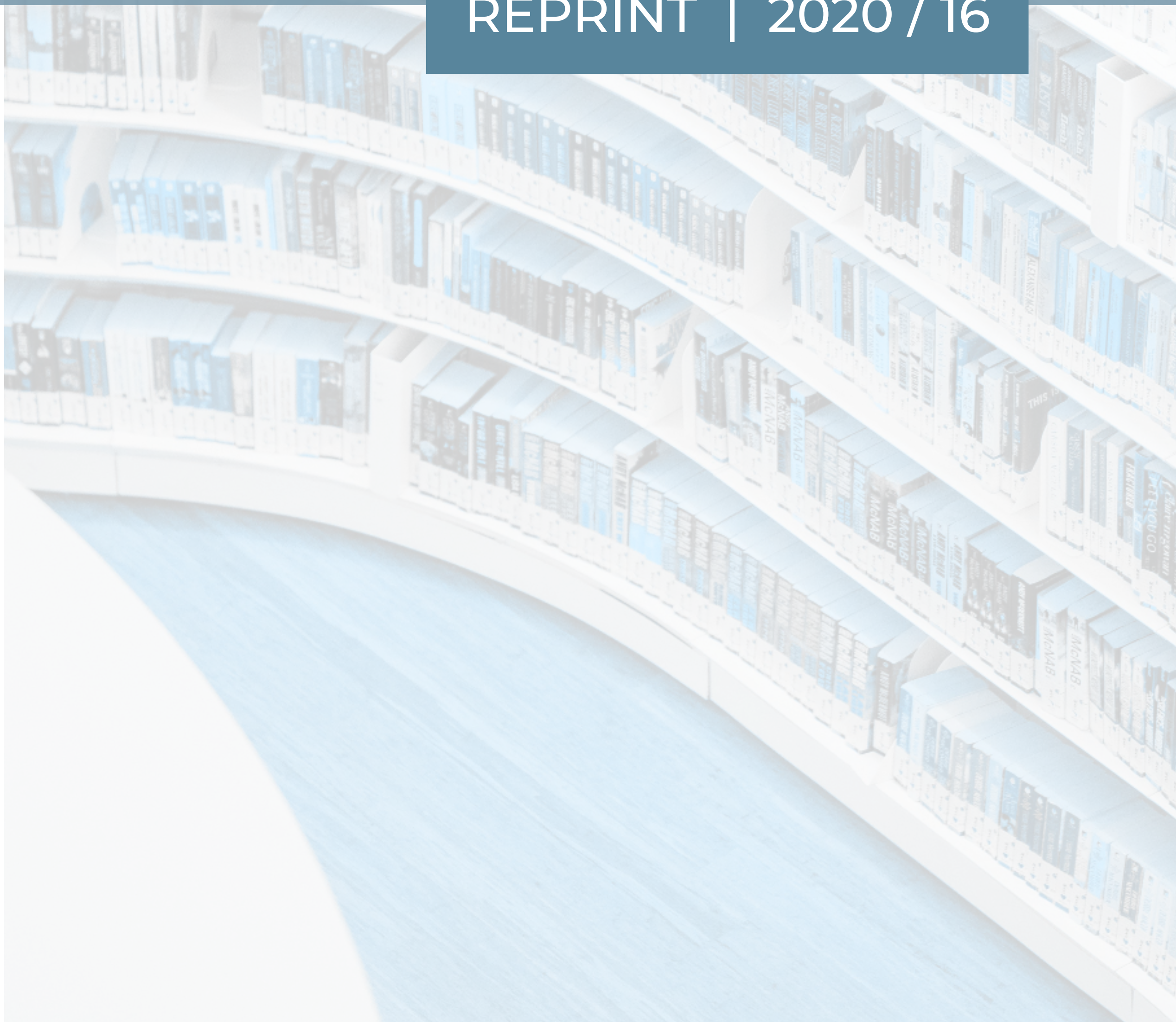

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Between DB and DC: optimal hybrid PAYG pension schemes

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Abstract In classical pension design, there are essentially two kinds of pension schemes: Defined Benefit (DB) or Defined Contribution (DC) plans. Each corresponds to a different philosophy of spreading risk between the stakeholders: in a DB the main risks are taken by the sponsor of the plan while in a DC the active members must bear all the risks. Especially when applied to social security pension systems, this traditional view can in both cases lead to unfair intergenerational equilibrium.

The purpose of this paper, which focuses on social security, is twofold. First, we present alternative architectures based on a mix between DB and DC in order to achieve both financial sustainability and social adequacy. An example of this approach is the so-called Musgrave rule, but other risk-sharing approaches will be developed in a pay-as-you-go philosophy. More precisely, we build convex and log-convex families of hybrid pension schemes whose extremal points correspond to DB and DC. Second, we study these new architectures in a stochastic environment, and present different rules to select the most efficient ones. To do so, we search for optimality from a risk-sharing point of view among the new pension plans[1].

Keywords pension · pay-as-you-go · risk-sharing

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[1] A short version of this paper has been presented to the EAJ2018 conference in Leuven on September 11$^{\text{th}}$.

1 Introduction

The social security pension systems of a large number of European countries are under deep pressure. The Belgian first pillar of pension is one of them: its pay-as-you-go mechanism and Defined Benefit architecture make it strongly vulnerable to demographic and macroeconomic shocks. Over the last decades, the social security has thus been intensely affected by the ageing and the "papy boom" phenomena on one hand, and the high level of unemployment in Belgium on the other hand (see e.g. [3]).

For this reason, the long term sustainability and the social fairness of the pension system are not guaranteed anymore, and there is a crucial need for structural reforms, different from the parametric changes that have been implemented over the few last years. This stands in contrast with the situation of other European countries, where global reforms have been put into effect. We could for example mention the Swedish Notional Defined Contribution ([8, 10, 4]) and the German sustainability factor ([1]).

However, the foundations of a structural reform of the Belgian first pillar have been laid by a group of academic experts, formed in 2013 at the instigation of the government. Among other things, the report published by this commission ([2, 9]) discusses ideas to address a major problem of the current system: the risk sharing between the active and retiree populations. In a classical DB architecture, the retirees are not affected by demographic shocks, which have to be totally absorbed by the contributors. On the contrary, in a DC architecture, the retirees bear all the risk, and the contributors are not affected at all.

The solution to this imbalance that has been proposed by the commission is an automatic adjustment mechanism, a topic already discussed in the scientific literature (see among others [11, 5]). In particular, the Musgrave rule (first introduced in [7]) consists in defining a relationship between the contributions and the benefits that has to be satisfied at each instant, in such a way that these two quantities move together. Our contribution is to generalize Musgrave by introducing different families of hybrid pension schemes between DB and DC.

The aim of this paper is twofold. In the first part (Section 2), we study these hybrid pension schemes in a deterministic environment. In the second part (Section 3), we consider their behaviour under demographic uncertainty. More precisely, we investigate the optimality of some Musgrave-like systems with respect to the difference between the living standards of retirees and active workers. We finally comment our results and conclude.

2 Hybrid pension plans

2.1 The Musgrave rule

In order to introduce a first form of risk sharing, let us first set up a simple deterministic two-period framework in which the Musgrave rule can be easily defined. Consider an initial stable situation ($t = 0$), where each active (resp. retiree) individual receives a homogeneous salary S (resp. a homogeneous pension P) based on a replacement rate $\delta_0 \in (0,1)$ and a contribution rate $\pi_0 \in (0,1)$. If the ratio between the number of retirees and the number of contributors (i.e. the dependence ratio) is denoted by D_0, the pension system is characterized by the following equations:

$$\begin{aligned} D_0\ P &= \pi_0\ S \qquad \text{(budget equation)},\\ P &= \delta_0\ S \qquad \text{(pension equation)}, \end{aligned}$$

leading to the condition

$$\pi_0 = D_0\ \delta_0. \tag{1}$$

Consider now that a demographic shock occurs at a second instant ($t = 1$), modifying D_0 to D_1 (with $D_1 > D_0$, i.e. in the situation of an ageing population). The impact of this shock on δ_1 and π_1 depends of course of the pension architecture. In a pure DB scheme, the replacement rate is constant and the active population bears the whole burden of the demographic shock through an increase of the contribution rate:

$$\begin{aligned} \delta_1 &= \delta_0,\\ \pi_1 &= \pi_0\ \frac{D_1}{D_0}. \end{aligned}$$

On the contrary, in a pure DC scheme, the contribution rate is constant and the retiree population bears the whole burden of the demographic shock through a decrease of the benefits:

$$\begin{aligned} \pi_1 &= \pi_0,\\ \delta_1 &= \delta_0\ \frac{D_0}{D_1}. \end{aligned}$$

In order to obtain a form of sharing of the risk between the two generations, Musgrave has defined (in [7]) the quantity which is now known as the Musgrave ratio:

$$M_0 = \frac{P}{S(1-\pi_0)} = \frac{\delta_0}{1-\pi_0}.$$

It corresponds to the ratio between the pension and the salary net of pension contributions.

Table 1 Contribution rate and replacement rate formulas in DB, DC and DM.

	Contribution rate	Replacement rate
DB	$\pi_1 = \pi_0 \dfrac{D_1}{D_0}$	$\delta_1 = \delta_0$
DC	$\pi_1 = \pi_0$	$\delta_1 = \delta_0 \dfrac{D_0}{D_1}$
DM	$\pi_1 = \pi_0 \dfrac{D_1}{D_0 + \pi_0(D_1 - D_0)}$	$\delta_1 = \delta_0 \dfrac{1}{1 + \delta_0(D_1 - D_0)}$

First note that this ratio is impacted by the evolution of the dependence ratio in standard pension architectures. In DB, it increases when D increases:

$$M_1 = \frac{\delta_1}{1 - \pi_1} = \frac{\delta_0}{1 - \pi_0 \frac{D_1}{D_0}} > M_0,$$

while in DC, it decreases in this case:

$$M_1 = \frac{\delta_1}{1 - \pi_1} = \frac{\delta_0}{1 - \pi_0} \frac{D_0}{D_1} < M_0.$$

Let us now introduce a new pension architecture, which we will call DM, *Defined Musgrave*. Its idea is simply to fix the Musgrave ratio:

$$M_1 = M_0 = M = \frac{\delta_0}{1 - \pi_0},$$

i.e. to adapt at time $t = 1$ the values of δ and π in such a way that this ratio remains constant. The evolution of the replacement and contribution rates is easily obtained from this relation and (1):

$$M = \frac{\delta_1}{1 - \pi_1} = \frac{\delta_1}{1 - D_1 \delta_1},$$

so that

$$\delta_1 = \delta_0 \ \frac{1}{1 + \delta_0(D_1 - D_0)} \quad \text{and} \quad \pi_1 = \pi_0 \ \frac{D_1}{D_0 + \pi_0(D_1 - D_0)}.$$

Table 1 summarises the evolution of the contribution and replacement rates in DB, DC and DM.

The DM architecture can be seen as an intermediary scheme between DB and DC. Indeed, one can easily show that in the context of an ageing population,

$$\delta_1^{DB} > \delta_1^{DM} > \delta_1^{DC} \quad \text{and} \quad \pi_1^{DC} < \pi_1^{DM} < \pi_1^{DB}. \tag{2}$$

Remark that if the population is not ageing but getting younger (i.e. $D_1 < D_0$), the inequalities in (2) are reversed.

2.2 The risk-sharing coefficient

In previous section, we have established that the DM architecture is an intermediate scheme between DB and DC by comparing the evolution of the replacement rate and the contribution rate in the three mechanisms. We will now present a new metric related to risk-sharing.

Let us first introduce the parameters λ_1^π and λ_1^δ, which represent the relative changes in the rates π and δ from $t = 0$ to $t = 1$:

$$\pi_1 = \pi_0(1 + \lambda_1^\pi), \tag{3}$$

$$\delta_1 = \delta_0(1 - \lambda_1^\delta). \tag{4}$$

Using the equilibrium equation at time $t = 1$:

$$\pi_1 = D_1\ \delta_1, \tag{5}$$

we obtain

$$(1 + \lambda_1^\pi) = \frac{D_1}{D_0}(1 - \lambda_1^\delta). \tag{6}$$

The risk-sharing coefficient can then be defined as the ratio between the proportional efforts supported by the contributors and the ones supported by the retirees:

$$\kappa_1 = \frac{\lambda_1^\pi}{\lambda_1^\delta}.$$

If this coefficient is equal to 1, the risk-sharing is perfect for this metric, i.e. the risk is equally borne by the two generations.

It should be stressed that this coefficient induces a *relative* risk-sharing mechanism, i.e. it is about relative (as opposed to absolute) changes of the contribution and replacement rates. For example, in the case we call *perfect risk-sharing* (when $\kappa = 1$), a 1% increase of the contribution rate is linked to a 1% decrease of the replacement rate. In terms of absolute changes, however, one could argue that the risk-sharing is not perfect, as 1% $\pi_0 S$ (the amount the active workers "lose") is less than 1% $\delta_0 S$ (the amount the retirees "lose") for realistic values of π_0 and δ_0.

We can now compare the DB, DC and DM schemes using this metric. We obtain from equations (3)-(6):

$$\kappa_1 = \begin{cases} +\infty & \text{in DB,} \\ 0 & \text{in DC,} \\ \frac{1}{\pi_0} - 1 & \text{in DM.} \end{cases} \tag{7}$$

For usual values of the contribution rate ($\pi_0 < 0.5$), the sharing coefficient in the DM case is higher than 1, showing that, in such a scheme, contributors make a bigger effort than the retirees (i.e. this architecture is closer to DB than DC).

Let us remark that in DM this coefficient depends on the contribution rate. Therefore, successive applications of this rule will change the value of the risk-sharing coefficient (along with the change in the contribution rate).

2.3 The "convex" family of hybrid schemes

In order to generate other kinds of hybrid schemes, let us observe that following the Musgrave rule is equivalent to ensuring that the a convex combination of these rates remains constant:

$$M = \frac{\delta_t}{1-\pi_t} \quad \text{for } t = 0,1 \;\Leftrightarrow\; \alpha\ \delta_t + (1-\alpha)\ \pi_t = 1-\alpha \quad \text{for } t = 0,1$$

if we define

$$\alpha = 1/(1+M). \tag{8}$$

Expressing the Musgrave rule as the invariance of a convex combination suggests that the DM scheme is only one member of a whole family of hybrid pension schemes. This family is indexed by $\alpha \in [0,1]$: any value of this parameter yields a particular pension scheme (called *Defined Convex Combination*, DCC), whose contribution and replacement rates are characterized by

$$\begin{cases} D_t\ \delta_t = \pi_t \\ \alpha\ \delta_t + (1-\alpha)\ \pi_t = C_\alpha \end{cases} \qquad \text{for } t = 0,1, \tag{9}$$

where $C_\alpha \in (0,1)$ is a constant (which is equal to the initial value of the convex combination, $\alpha\ \delta_0 + (1-\alpha)\ \pi_0$). The schemes corresponding to the "lower" and "upper" boundaries of the $[0,1]$ interval in which α lies are respectively DC (for $\alpha = 0$) and DB (for $\alpha = 1$).

For example, we could choose to consider the pension plan which is situated on the midpoint of the $[0,1]$ interval, i.e. the one which is associated to $\alpha = 1/2$. In this case, the evolution of the contribution and replacement rates is governed by the equation

$$\frac{1}{2}\ \delta_1 + \frac{1}{2}\ \pi_1 = C_{\frac{1}{2}} = \frac{1}{2}\ \delta_0 + \frac{1}{2}\ \pi_0,$$

meaning that the sum of the contribution rate and the replacement rate is constant.

It is possible to solve equations (9) for π and δ: we obtain straightforwardly

$$\begin{cases} \delta_1 = \dfrac{C_\alpha}{\alpha + (1-\alpha)D_1}, \\[2ex] \pi_1 = \dfrac{D_1\ C_\alpha}{\alpha + (1-\alpha)D_1}. \end{cases} \tag{10}$$

These expressions naturally lead to the following values of the sharing parameters:

$$\begin{cases} \lambda_1^\delta = \dfrac{(1-\alpha)(D_1 - D_0)}{\alpha + (1-\alpha)D_1}, \\[2ex] \lambda_1^\pi = \dfrac{1}{D_0}\dfrac{\alpha(D_1 - D_0)}{\alpha + (1-\alpha)D_1}. \end{cases}$$

We therefore obtain the following risk-sharing coefficient:

$$\kappa_1 = \frac{\lambda_1^\pi}{\lambda_1^\delta} = \frac{1}{D_0}\frac{\alpha}{1-\alpha}.$$

This result shows that if the dependence ratio is changing over time, and we apply successively the automatic adjustment rule, the two quantities α and κ cannot remain simultaneously constant, except in the two limit cases of DB or DC. For instance, a DM scheme is characterized by a constant $\alpha = 1/(1+M)$ (see Equation (8)), but its risk-sharing coefficient is not constant (see Equation (7)).

2.4 The "log-convex" family of hybrid schemes

We can go further in the generalization of the Musgrave rule by remarking that the convex combination in Equation (9) could be replaced by any family of functions $f_\alpha : [0,1]^2 \to [0,1]$ for $\alpha \in [0,1]$. Choosing f_α leads to a pension scheme defined by the equation

$$f_\alpha(\delta_t, \pi_t) = C_\alpha \quad \text{for } t = 0, 1,$$

where $C_\alpha = f_\alpha(\delta_0, \pi_0)$ is a constant. The considered functions f_α should be mildly regular, but the class of possible choices is very wide, as e.g. piecewise continuous functions could lead to meaningful plans. If moreover $f_0(x, y) = y$ and $f_1(x, y) = x$, we obtain a family of pension schemes "joining" DC ($\alpha = 0$) and DB ($\alpha = 1$). We of course recover the plans defined in (9) by choosing

$$f_\alpha(x, y) = \alpha x + (1-\alpha)y.$$

Another natural choice is

$$f_\alpha(x, y) = x^\alpha \; y^{1-\alpha},$$

leading to the pension schemes defined by

$$\begin{cases} \delta_t \; D_t = \pi_t \\ \delta_t^\alpha \; \pi_t^{1-\alpha} = C_\alpha \end{cases} \quad \text{for } t = 0, 1. \tag{11}$$

Note that the second equation of this system can be expressed as

$$\alpha \log \delta_t + (1-\alpha) \log \pi_t = \log C_\alpha.$$

It therefore defines a convex combination of the log-rates instead of the rates themselves, the reason why we call it the *Defined Log-Convex Combination* (DLCC) family of schemes. Note that the business sense of this second family of pension plans is made obvious in the next section of this paper.

As the first family of hybrid schemes (see (10)), this second family yields closed expressions for π and δ: from (11) we obtain

$$\begin{cases} \pi_1 = C_\alpha \; D_1^\alpha, \\ \delta_1 = C_\alpha \; D_1^{-(1-\alpha)}. \end{cases} \tag{12}$$

2.5 Generalisation to multi-period discrete time models

The simple two-period frameworks developed in the previous sections can be straightforwardly generalized in a multi-period discrete time model, using recursive relations. These relations will give us a better interpretation of the two families introduced in sections 2.3 and 2.4.

2.5.1 The multi-period convex family

The DCC family of pension schemes presented in Section 2.3 can be extended to a multi-period framework and is defined by the two equations

$$\begin{cases} D_t\delta_t = \pi_t \\ \alpha\delta_t + (1-\alpha)\pi_t = C_\alpha \end{cases} \qquad (t = 0, 1, 2, \dots), \tag{13}$$

where the value of the constant C_α is fixed at time $t = 0$, i.e. $C_\alpha = \alpha\delta_0 + (1-\alpha)\pi_0$. The solution of these equations, generalizing Formula (10), is given by

$$\delta_t = \frac{C_\alpha}{\alpha + (1-\alpha)D_t}, \tag{14}$$

$$\pi_t = \frac{D_t C_\alpha}{\alpha + (1-\alpha)D_t}. \tag{15}$$

In order to analyze the evolution of the replacement rate and the contribution rate, we can obtain recursive relations that reflect automatic adjustments of these parameters.

First, we will obtain recursive relations expressing how the contribution rate and the replacement rate are adjusted in accordance to the dependence ratio. In general, if $(f_t)_t$ is a discrete time process, we will denote its "absolute variation" by $\Delta(f_t) = f_t - f_{t-1}$, and, when $f_{t-1} > 0$, its "percentage change" by $\rho(f_t) = \frac{f_t}{f_{t-1}}$.

Then, using equations (14), we get respectively

1. For the replacement rate:

$$\Delta\left(\frac{1}{\delta_t}\right) = \frac{1-\alpha}{C_\alpha}\Delta\left(D_t\right), \tag{16}$$

2. For the contribution rate:

$$\Delta\left(\frac{1}{\pi_t}\right) = \frac{\alpha}{C_\alpha}\Delta\left(\frac{1}{D_t}\right). \tag{17}$$

Relations (16) and (17) suggest that a proportion α of the demographic shock influences the contribution rate, while a proportion $1-\alpha$ influences the replacement rate.

We can also obtain a link between the variation of the replacement rate and the variation of the contribution rate. By direct calculation using (14) and

(15), we get the following expression for the variation of the replacement rate and of the contribution rate:

$$\Delta(\delta_t) = \frac{-C_\alpha(1-\alpha)\Delta(D_t)}{(\alpha+(1-\alpha)D_t)(\alpha+(1-\alpha)D_{t-1})} \tag{18}$$

and

$$\Delta(\pi_t) = \frac{C_\alpha \alpha \Delta(D_t)}{(\alpha+(1-\alpha)D_t)(\alpha+(1-\alpha)D_{t-1})}. \tag{19}$$

Combining (18) and (19), we get

$$\Delta(\pi_t) = -\frac{\alpha}{1-\alpha}\Delta(\delta_t) \quad \Leftrightarrow \quad \alpha\Delta(\delta_t) + (1-\alpha)\Delta(\pi_t) = 0. \tag{20}$$

Therefore, in the defined convex family, there is a (negative) proportional relation between the variation of the contribution rate and the variation of the replacement rate, the proportionality factor being equal to $-\frac{\alpha}{1-\alpha}$. When α is close to 0 (resp. close to 1), the variation of the contribution rate is thus very small (resp. large) with respect to the variation of the replacement rate. In the case $\alpha = 0.5$, the two variations are equal.

2.5.2 The multi-period log-convex family

The DLCC family of pension schemes presented in Section 2.4 can be extended in a multi-period framework and is defined by the two equations:

$$\begin{cases} D_t\delta_t = \pi_t \\ \alpha \log \delta_t + (1-\alpha)\log \pi_t = \log C_\alpha \end{cases} \qquad (t = 0, 1, 2, \dots), \tag{21}$$

where the value of the constant C_α is fixed at time $t = 0$, i.e. $C_\alpha = \delta_0^\alpha \cdot \pi_0^{1-\alpha}$. The solution of these equations, generalizing Formula (12), is given by:

$$\begin{aligned} \delta_t &= C_\alpha \cdot \left(\frac{1}{D_t}\right)^{1-\alpha}, \\ \pi_t &= C_\alpha \cdot (D_t)^\alpha. \end{aligned} \tag{22}$$

Instead of considering absolute variations as in the convex family, we will now look at recursive relations expressed in terms of percentage changes.

First, we establish a recursive link between the replacement rate / contribution rate and the dependence ratio. We get from Relation (22)

1. For the replacement rate:

$$\rho(\delta_t) = \rho(D_t)^{-(1-\alpha)}, \tag{23}$$

2. For the contribution rate:

$$\rho(\pi_t) = \rho(D_t)^\alpha. \tag{24}$$

Combining relations (23) and (24), we can get a relation between the percentage change of the replacement rate and the percentage change of the contribution rate:

$$\rho\left(\pi_t\right)=\rho\left(\delta_t\right)^{-\frac{\alpha}{1-\alpha}} \quad \Leftrightarrow \quad \rho\left(\delta_t\right)^{\alpha}\rho\left(\pi_t\right)^{1-\alpha}=1. \tag{25}$$

We thus observe, in the defined log-convex family case, that the percentage change of the contribution rate is a power function of the percentage change of the replacement rate, the exponent begin equal to $-\frac{\alpha}{1-\alpha}$. When α is close to 0 (resp. close to 1), the variation of the contribution rate is thus very small (resp. large) with respect to the variation of the replacement rate. In the case $\alpha=0.5$, the two variations are equal.

3 Optimal risk sharing based on living standards spread

In preceding sections, we have defined continuous families of pension schemes, whose members correspond to hybrid architecture, giving compromises between DB and DC. A natural question now arises: how can we choose between all these plans? In other words, can we define criteria to select the "best" plan? This section is devoted to proposing answers to these questions in a realistic framework, i.e. in a stochastic demographic environment.

Note that the definitions (13) and (21) imply that the two extreme plans for both families (i.e. for $\alpha=0$ and 1), are "pure" DC and DB: the value of π_t (in DC) or δ_t (in DB) is once and for all fixed and never changes. This is not what we observe in real world pension plans, as adjustments are made from time to time so that π_t is sometimes adapted in DC, and δ_t in DB. This reality illustrates the importance of hybridation of pension plans that can be implicit or explicit and the interest in looking at proper adjustment mechanisms from a theoretical point of view.

3.1 Models for the dependence ratio

In order to model the demographic uncertainty, we assume that the dependence ratio is a stochastic process which is log-normally distributed, i.e. that

$$D_t \sim \mathcal{LN}\left(m(t)\,;\,s(t)^2\right). \tag{26}$$

for some time-functions m and s (whose dependence on t will be dropped when the context is clear). We derive part of our results in this general setting.

However, in order to obtain numerical results, we have to make an actual choice regarding m and s. We consider two specific models. The first one corresponds to the standard Geometric Brownian motion (Model 1):

$$\mathrm{d}D_t = \mu D_t\ \mathrm{d}t+\sigma D_t\ \mathrm{d}W_t \quad \Rightarrow \quad \begin{cases} m(t)=\log D_0+\left(\mu-\frac{\sigma^2}{2}\right)t \\ s(t)^2=\sigma^2\ t \end{cases} \tag{27}$$

Table 2 Calibrated values of the parameters μ, σ, k, θ and τ.

Model 1		Model 2		
μ	σ	k	θ	τ
0.008	0.010	0.014	−0.848	0.009

for some parameters $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, where W is a standard Brownian motion.

The second one is the exponential Vasicek model (Model 2):

$$D_t = e^{E_t} \quad \text{with} \quad \mathrm{d}E_t = k\,(\theta - E_t)\ \mathrm{d}t + \tau\ \mathrm{d}W_t$$

$$\Rightarrow \quad \begin{cases} m(t) = \log D_0\ \exp(-kt) + \theta\,(1 - \exp(-kt)) \\ \\ s(t)^2 = \frac{\tau^2}{2k}\,(1 - \exp(-2kt)) \end{cases} \tag{28}$$

for some parameters $\theta \in \mathbb{R}$, $k, \tau \in \mathbb{R}^+$.

3.2 Models rationales and calibration

The rationales of these modelling choices are as follows. First note that, at our best knowledge, the literature is silent about stochastic models for the dependence ratio. Second, given the nature of this demographic quantity, the chosen models should ensure that the stochastic process is always positive. Third, we have chosen, among the distributions supported on $(0, +\infty)$, the log-normal distribution for analytical tractability reasons (that will become clear later in this section). Finally, we have selected two models whose main difference consists in mean reversion: Model 1 is not mean reverting, while Model 2 is. This choice reflects the two viewpoints about mean reversion that are discussed for instance in the stochastic mortality literature ([12, 6]), a framework that shares several characteristics with our demographic modelling framework.

The parameters of both models have been calibrated using historical data from the `OECD.Stat` website, consisting of Belgian population annual counts from 1950 to 2016. The maximum likelihood method has been applied for this purpose. Table 2 gives the calibrated values of the parameters μ, σ, k, θ and τ. In Figure 1, the historical time series is compared to the trends of the stochastic processes corresponding to the calibrations of models 1 and 2 we obtained.

The long-term behaviour of the dependence ratio is of course different in the two considered models. In Model 1, the trend of the process is an increasing exponential function, and is thus unbounded when $t \to \infty$. In Model 2, on the contrary, the trend approaches the target $\exp(\theta)$, whose calibrated value is equal to 0.428, as t becomes large.

Remark that bringing continuous demographic models in a discrete pension framework is not a problem, as pension adjustments are generally made on a yearly basis, while populations evolve continuously.

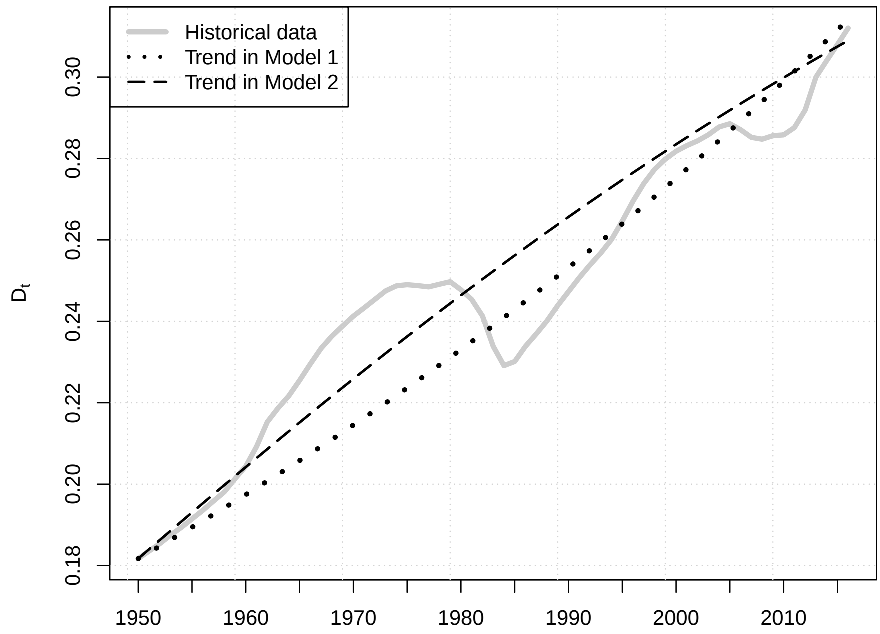

Fig. 1 Comparison of the historical data for the dependence ratio D_t and the means of the stochastic processes corresponding to the calibrations of models 1 and 2 we obtained.

3.3 Living standards

In a pay-as-you-go pension scheme, the living standard of the population is equal to

$$\begin{cases} (1-\pi)S & \text{for workers,} \\ \delta S & \text{for retirees,} \end{cases}$$

where S is the salary (in the following, we assume the salary is equal to 1), π is the contribution rate and δ is the replacement rate.

We are here interested in the comparison between those two living standards, or more precisely in the variability of some kind of "spread" between them. Considering successively the two families of pension plans defined in sections 2.3, 2.4 and 2.5, we will search for the optimal plan with respect to this spread measure. More formally, and using the notations introduced in Section 2.4, we study the following minimization problem: for a family of schemes defined by f_α, $\alpha \in [0,1]$, we search for

$$\min_{\alpha\in[0,1]} \left(\text{variability}\left[\text{spread between } 1-\pi_t \text{ and } \delta_t\right]\right)$$
$$\text{under the constraint that } f_\alpha(\pi_t,\delta_t) = C_\alpha \text{ for all } t = 0,1,2,\dots$$

The variability metric we have chosen to consider is the variance of the corresponding spread process. Unlike other metrics, variance is symmetric: an upward shock in the spread gives the same variance difference than a downward shock. The corollary of this feature is that we give the same importance to an improvement of retirees' living standards (at the expense of workers) as to an improvement of workers' living standards (at the expense of retirees). This property of variance is thus desirable in our context, in contrast to some other modelling frameworks, e.g. when considering the variability of financial returns.

3.4 Difference of the living standards

In this first approach, we consider the variance of the difference between those two living standards, denoted by:

$$\mathrm{Var}\left[\delta_t - (1-\pi_t)\right] = \mathrm{Var}\left[\delta_t + \pi_t\right].$$

3.4.1 Defined convex plans

We first consider plans defined by the following equations (see relations (13)):

$$\begin{cases} \pi_t = D_t \delta_t \\ \alpha\delta_t + (1-\alpha)\pi_t = C_\alpha = \alpha\delta_0 + (1-\alpha)\pi_0 \end{cases}$$

and we consider the minimization problem

$$\min_{\alpha\in[0,1]} \mathrm{Var}\left[\delta_t + \pi_t\right].$$

The solution of this problem is straightforward, and does not depend on the model specified for the dependence ratio:

$$\mathrm{Var}\left[\delta_t + \pi_t\right] = 0 \text{ if } \delta_t + \pi_t \text{ is constant } \Leftrightarrow \frac{1}{2}\delta_t + \frac{1}{2}\pi_t \text{ is constant, i.e. if } \alpha = \frac{1}{2}.$$

Using the model given for D_t, we can assess other plans' (including Defined Musgrave plan's) variance: using equations (10) we obtain

$$\mathrm{Var}\left[\delta_t + \pi_t\right] = C_\alpha^2\ \mathrm{Var}\left[\frac{1+D_t}{\alpha + (1-\alpha)D_t}\right]$$

which can be computed numerically, as shown in Figure 2 for $t = 40$.

In the two extreme cases (DC and DB), a closed formula can be obtained for the variance thanks to the log-normality assumption (26):

$$\begin{cases} \text{DC: } \mathrm{Var}\left[\delta_t + \pi_t\right] = \mathrm{Var}\left[\delta_t + \pi_0\right] = \mathrm{Var}\left[\delta_t\right] = \mathrm{Var}\left[\frac{\pi_0}{D_t}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \pi_0^2\ e^{-2m(t)+s(t)^2}\ \left(e^{s(t)^2} - 1\right) \\ \\ \text{DB: } \mathrm{Var}\left[\delta_t + \pi_t\right] = \mathrm{Var}\left[\delta_0 + \pi_t\right] = \mathrm{Var}\left[\pi_t\right] = \mathrm{Var}\left[\delta_0 D_t\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \delta_0^2\ e^{2m(t)+s(t)^2}\ \left(e^{s(t)^2} - 1\right) \end{cases}$$

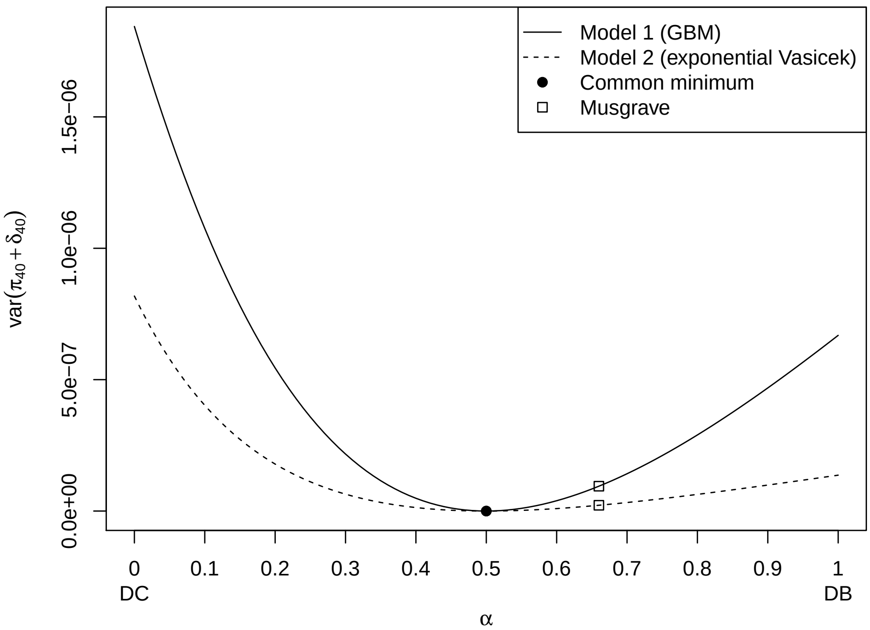

Fig. 2 Variance of the sum in the case of Defined Convex plans (for $t = 40$).

where $m(t)$ and $s(t)$ are given by (27) for Model 1 and by (28) for Model 2. These two variances can be compared:

$$\mathrm{Var}\left[\pi_t + \delta_t\right](DB) < \; \mathrm{Var}\left[\pi_t + \delta_t\right](DC) \quad \Leftrightarrow \quad e^{2m(t)} < D_0.$$

For Model 1, this inequality reads

$$D_0 \exp\left(2\left(\mu - \frac{1}{2}\sigma^2\right)t\right) < 1$$

which is always true with realistic values of the parameters. We cannot draw this kind of conclusion in Model 2, even if this inequality holds for the parameter values we have calibrated, as one can observe on Figure 2.

In both models, DB and DC are extreme situations of the curve and clearly not optimal. DC plans seem to be the worst strategy. The Musgrave rule is better than DB and DC, but also not optimal. The best hybrid pension plan with respect to this criterion corresponds to $\alpha = 1/2$.

3.4.2 Defined log-convex plans

In the case of Defined Log-convex plans, we obtain from (see relations (21))

$$\begin{cases} \pi_t = D_t \delta_t \\ \delta_t^{\alpha} \;\cdot\; \pi_t^{1-\alpha} = C_\alpha = \delta_0^{\alpha} \;\cdot\; \pi_0^{1-\alpha} \end{cases}$$

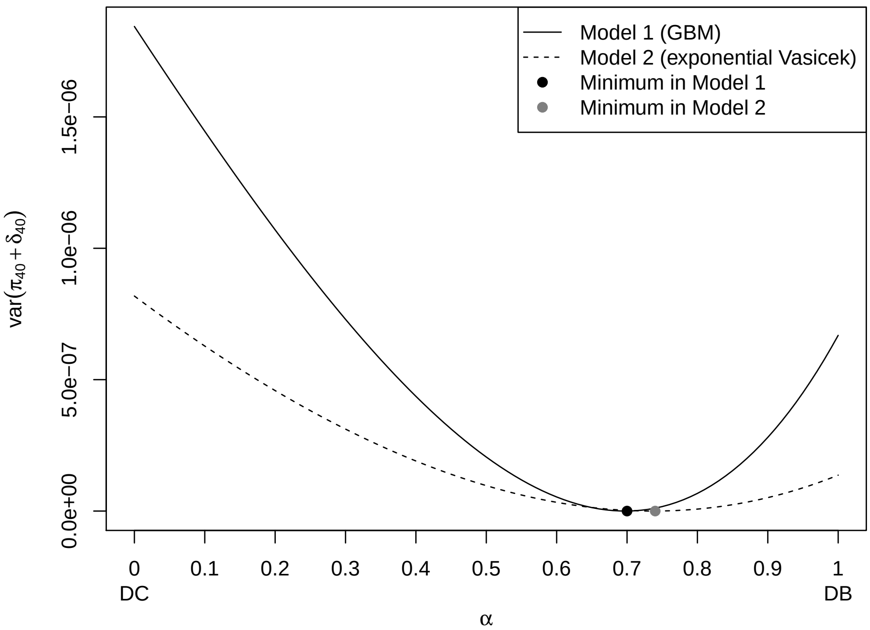

Fig. 3 Variance of the sum in the case of Defined Log-convex plans (for $t = 40$).

the expressions

$$\begin{cases} \delta_t = C_\alpha D_t^{\alpha-1} \sim \mathcal{LN}\Big(\log C_\alpha + (\alpha-1)m(t)\,;\,(\alpha-1)^2 s(t)^2\Big), \\ \\ \pi_t = C_\alpha D_t^{\alpha} \sim \mathcal{LN}\Big(\log C_\alpha + \alpha m(t)\,;\,\alpha^2 s(t)^2\Big). \end{cases}$$

The variance of the sum of the contribution and replacement rates can then be explicitly obtained as a function of the parameter α, as shown in Appendix A.1. It is pictured in Figure 3 for both models for $t = 40$. Altough they are close from each other, we see on this graph that the optimal plan obtained with Model 1 is not equal to the optimal plan obtained with Model 2.

We obtain similar conclusions as for DCC plans: DB and DC are extreme, with DC the worst one. The optimal situation corresponds now to an higher α than $1/2$ and depends on the model for the dependence rate. We did not represent here the Musgrave rule because this strategy is not a member of the DLCC family.

3.5 Ratio of the living standards

In this second section, we consider the variance of the ratio of these two living standards:

$$\mathrm{Var}\left[\frac{1-\pi_t}{\delta_t}\right] = \mathrm{Var}\left[\frac{1}{\delta_t} - \frac{\pi_t}{\delta_t}\right] = \mathrm{Var}\left[\frac{1}{\delta_t} - D_t\right].$$

Remark that we deliberately choose to work with $(1-\pi_t)/\delta_t$ instead of its inverse $\delta_t/(1-\pi_t)$ for technical reasons (mathematical expressions are easier to handle this way, as the split of the ratio into two terms in the equation here above shows).

3.5.1 Defined convex plans

When we consider Defined convex plans, we have

$$\alpha\delta_t + (1-\alpha)\pi_t = C_\alpha \qquad \Leftrightarrow \qquad \frac{1-\pi_t}{\delta_t} = \frac{\alpha}{1-\alpha} + \frac{1-\frac{C_\alpha}{1-\alpha}}{\delta_t}. \tag{29}$$

This expression is constant if and only if

$$1-\alpha = C_\alpha \quad \Leftrightarrow \quad 1-\alpha = \alpha\delta_0 + (1-\alpha)\pi_0 \quad \Leftrightarrow \quad \alpha = \frac{1-\pi_0}{1+\delta_0-\pi_0}$$

i.e. if and only if the plan is Defined Musgrave. This result is trivially obtained from the fact that the Defined Musgrave plan is defined as the one such that $\delta_t/(1-\pi_t)$ is constant, which is of course equivalent to saying that $(1-\pi_t)/\delta_t$ is constant.

In the other cases, we can compute the variance, as shown in Appendix A.2. Figure 4 shows how this function behaves for $t = 40$. Models 1 and 2 do not lead to different optimal plans, which is not surprising when we consider the expression we have obtained for $\mathrm{Var}\left[\frac{1-\pi_t}{\delta_t}\right]$: it can be expressed as the product of a function depending only on δ_0, π_0 and α and a function depending on the model for D_t.

Figure 4 suggests once again that DC plans lead to the bigger dispersion when comparing active and retirees.

3.5.2 Defined log-convex plans

In the case of a Defined log-convex plan, we obtain a closed formula which is detailed in Appendix A.3. Figure 5 shows how the total variance behaves for $t = 40$. The optimal plan we obtain using Model 1 is slightly different from the optimal plan we obtain using Model 2.

The results are summarized in Table 3.

3.6 General comments

From figures 2 to 5, the following conclusions can be drawn:

- In all the situations, DC appears to be the worst solution in terms of dispersion of the spread of living standards (higher variance whatever the model, the family of plans or the spread measurement),

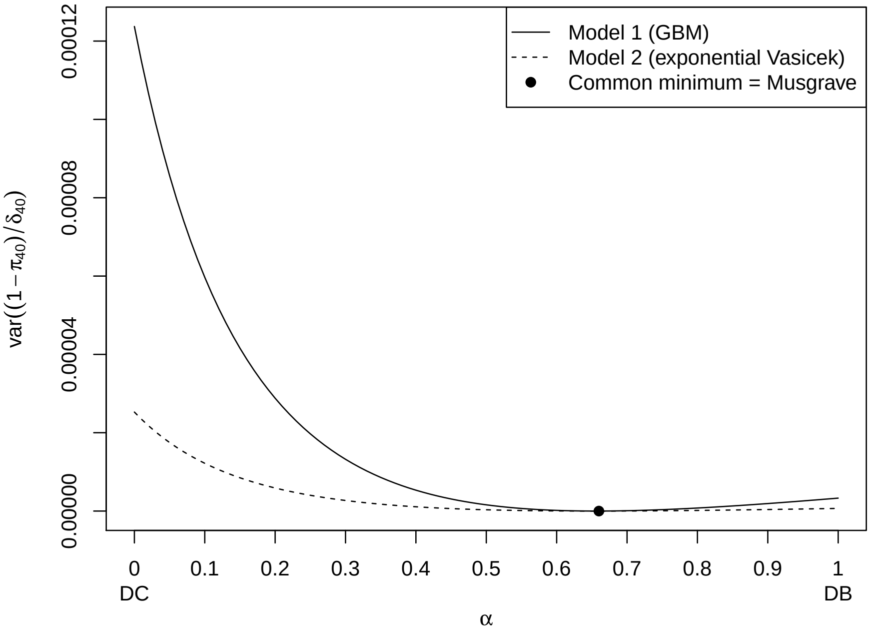

Fig. 4 Variance of the ratio in the case of Defined Convex plans (for $t = 40$).

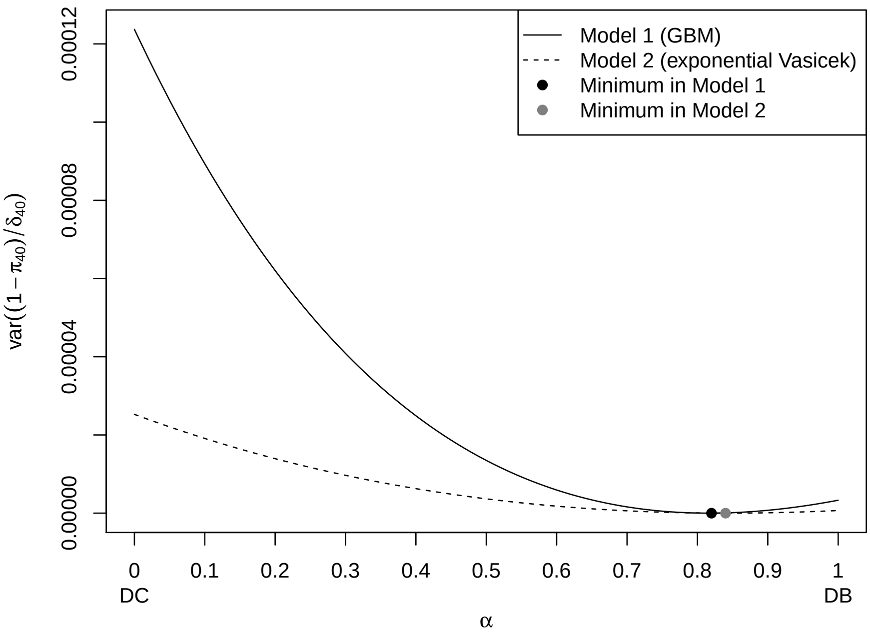

Fig. 5 Variance of the ratio in the case of Defined Log-convex plans (for $t = 40$).

Table 3 Summary of our results.

		Difference	Quotient
DCC family	Figure	2	4
	Model 1	$\alpha = 0.50$	$\alpha = 0.66$ (DM)
	Model 2	$\alpha = 0.50$	$\alpha = 0.66$ (DM)
DLCC family	Figure	3	5
	Model 1	$\alpha \approx 0.70$	$\alpha \approx 0.81$
	Model 2	$\alpha \approx 0.74$	$\alpha \approx 0.84$

- In general (except Figure 2) the optimal value of the risk sharing parameter is strictly greater than 0.5, suggesting that in this metric, optimal plans are closer to DB than to DC,
- The Musgrave rule corresponds to one of the optimal solutions (constant ratio).

Conclusion

Classical pension schemes in PAYG are usually DB (classical social security arrangements) or DC (notional accounts systems). Neither of these two solutions is optimal in terms of risk sharing between generations. In fact, we can consider them as extreme cases. In this paper, we introduce various families of hybrid plans between DB and DC; we generate therefore an infinite number of possible schemes. In order to choose one optimal solution, we propose to consider as optimal criterion for the splitting of the demographic risk between active workers and retirees, the minimization of the variance of a spread between their respective living standards. We obtain optimal solutions inside families of hybrid plans. In particular, we proved that the Musgrave rule corresponds to one of the possible optimal cases and that pure fDC plans are always the worst solution in terms of risk sharing between generations.

A Appendix: Technical derivations

A.1 Variance of the difference in DLCC

In the case we consider Defined Log Convex Combination schemes and the difference of the rates, we have:

$$\begin{aligned}\mathrm{Var}\left[\delta_t + \pi_t\right] &= C_\alpha^2 \ \mathrm{Var}\left[D_t^{\alpha-1} + D_t^{\alpha}\right] \\ &= C_\alpha^2 \Big(\mathrm{Var}\left[D_t^{\alpha-1}\right] + \mathrm{Var}\left[D_t^{\alpha}\right] + 2\left(\mathbb{E}\left[D_t^{\alpha-1} D_t^{\alpha}\right] - \mathbb{E}\left[D_t^{\alpha-1}\right]\mathbb{E}\left[D_t^{\alpha}\right]\right)\Big).\end{aligned}$$

Let us compute each of these terms (using the expressions of the expectation and variance of log-normal random variables):

$$\begin{aligned}
\mathrm{Var}\left[D_t^{\alpha-1}\right] &= \exp\left(2(\alpha-1)m(t)+(\alpha-1)^2 s(t)^2\right)\cdot\left(\exp\left((\alpha-1)^2 s(t)^2\right)-1\right)\\
\mathrm{Var}\left[D_t^{\alpha}\right] &= \exp\left(2\alpha m(t)+\alpha^2 s(t)^2\right)\left(\exp\left(\alpha^2 s(t)^2\right)-1\right)\\
\mathbb{E}\left[D_t^{2\alpha-1}\right] &= \exp\left((2\alpha-1)m(t)+\frac{1}{2}(2\alpha-1)^2 s(t)^2\right)\\
\mathbb{E}\left[D_t^{\alpha-1}\right]\mathbb{E}\left[D_t^{\alpha}\right] &= \exp\left((\alpha-1)m(t)+\frac{1}{2}(\alpha-1)^2 s(t)^2\right)\cdot\exp\left(\alpha m(t)+\frac{1}{2}\alpha^2 s(t)^2\right)\\
&= \exp\left((2\alpha-1)m(t)+(\alpha^2-\alpha+\frac{1}{2})s(t)^2\right).
\end{aligned}$$

A.2 Variance of the quotient in DCC

In the case we consider Defined Convex Combination schemes and the quotient of the rates, we have: Using (29) and (14), we compute:

$$\begin{aligned}
\mathrm{Var}\left[\frac{1-\pi_t}{\delta_t}\right] &= \mathrm{Var}\left[\frac{\alpha}{1-\alpha}+\frac{1-\frac{C_\alpha}{1-\alpha}}{\delta_t}\right]\\
&= \left(1-\frac{C_\alpha}{1-\alpha}\right)^2 \mathrm{Var}\left[\frac{1}{\delta_t}\right]\\
&= \left(1-\frac{C_\alpha}{1-\alpha}\right)^2\left(\frac{(1-\alpha)}{C_\alpha}\right)^2 \mathrm{Var}\left[D_t\right]\\
&= \left(\frac{(1-\alpha)}{\alpha\delta_0+(1-\alpha)\pi_0}-1\right)^2 e^{2m(t)+s(t)^2}\left(e^{s(t)^2}-1\right).
\end{aligned}$$

A.3 Variance of the quotient in DLCC

In the case we consider Defined Log Convex Combination schemes and the quotient of the rates, we have:

$$\begin{aligned}
\mathrm{Var}\left[\frac{1-\pi_t}{\delta_t}\right] &= \mathrm{Var}\left[\frac{1}{\delta_t}-D_t\right]\\
&= \frac{1}{C_\alpha^2}\mathrm{Var}\left[D_t^{1-\alpha}\right]+\mathrm{Var}\left[D_t\right]-\frac{2}{C_\alpha}\mathrm{Cov}\left(D_t^{1-\alpha},D_t\right).
\end{aligned}$$

Each term can then be computed:

$$\begin{aligned}
\mathrm{Var}\left[D_t^{1-\alpha}\right] &= \exp\left(2(1-\alpha)m(t)+(1-\alpha)^2 s(t)^2\right)\cdot\left(\exp\left((1-\alpha)^2 s(t)^2\right)-1\right)\\
\mathrm{Var}\left[D_t\right] &= \exp\left(2m(t)+s(t)^2\right)\ \left(\exp\left(s(t)^2\right)-1\right)\\
\mathrm{Cov}\left(D_t^{1-\alpha},D_t\right) &= \mathbb{E}\left[D_t^{2-\alpha}\right]-\mathbb{E}\left[D_t^{1-\alpha}\right]\mathbb{E}\left[D_t\right]\\
&= \exp\left((2-\alpha)m(t)\right)\\
&\quad\cdot\left\{\exp\left(\frac{1}{2}(2-\alpha)^2 s(t)^2\right)-\exp\left(\frac{1}{2}((1-\alpha)^2+1)s(t)^2\right)\right\}.
\end{aligned}$$

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