Two-Step Nilpotent Leibniz Algebras

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- Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space L over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : L \times L \to L$ which satisfies the *left Leibniz identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \;\; \forall x,y,z \in L.$$

- We can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]] \;\; \forall x,y,z \in L.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.

- We assume that $\dim_{\mathbb{F}} L < \infty$.

- The subalgebra

$$\mathrm{Leib}(L) = \mathrm{Span}_{\mathbb{F}} \{[x, x] \,|\, x \in L\},$$

is the smallest bilateral ideal of L such that $L/\,\mathrm{Leib}(L)$ is a Lie algebra.

- The *left* and the *right center* of L are

$$\mathrm{Z}_l(L) = \{x \in L \mid [x, L] = 0\}, \;\; \mathrm{Z}_r(L) = \{x \in L \mid [L, x] = 0\}.$$

- The *center of L* is $\mathrm{Z}(L) = \mathrm{Z}_l(L) \cap \mathrm{Z}_r(L)$.
- If L is symmetric, $\mathrm{Z}_l(L)$ and $\mathrm{Z}_r(L)$ are bilateral ideals.

Definition

Let L be a left Leibniz algebra and let

$$L^{(0)} = L, \ L^{(k+1)} = [L, L^{(k)}], \ \forall k \geq 0,$$

be the *lower central series of* L. L is $n-$*step nilpotent* if $L^{(n-1)} \neq 0$ and $L^{(n)} = 0$.

Proposition

If L is a left two-step nilpotent Leibniz algebra, then $L^{(1)} = [L, L] \subseteq Z(L)$ *and L is symmetric.*

Proposition

If L is a left nilpotent Leibniz algebra with $\dim_{\mathbb{F}} L^{(1)} = 1$, *then* $L^{(1)} \subseteq Z(L)$ *and L is symmetric.*

- Let L be a two-step nilpotent Leibniz algebra with $[L, L] = Span_{\mathbb{F}}\{z_1, \ldots, z_t\}$, then

$$[x, y] = \phi_1(x, y)z_1 + \cdots + \phi_t(x, y)z_t, \ \ \forall x, y \in L$$

where $\phi_i : L \times L \to \mathbb{F}$ is a bilinear form, $\forall i = 1, \ldots, t$.

- Our purpose is to reduce such a given bilinear form to a suitable canonical representation.

- Let L be a nilpotent Leibniz algebra with $[L, L] = \mathrm{Span}_{\mathbb{F}}\{z\}$ and $[x, y] = \phi(x, y)z,\ \forall x, y \in L$.

- If L is not a Lie algebra, then $0 \neq \mathrm{Leib}(L) = [L, L] \subseteq \mathrm{Z}(L)$.

- $\phi = \phi^- + \phi^+$, where

$$\phi^- = \frac{\phi - \phi^t}{2}, \quad \phi^+ = \frac{\phi + \phi^t}{2}.$$

- We use the simultaneous reduction to canonical form of a pair of bilinear forms.

Definition

A Kronecker module is a quadruple (V_1, V_2, f_1, f_2), where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 : V_1 \to V_2$ are linear maps.

- Given a pair of bilinear forms $\phi^-, \phi^+ : L \times L \to \mathbb{F}$ as above, we define the Kronecker module $\left(L, L^*, \overline{\phi^-}, \overline{\phi^+}\right)$, where $\overline{\phi^-}(x)$ and $\overline{\phi^+}(y)$ are defined by

$$\overline{\phi^-}(x) : z \mapsto \phi^-(x,z), \quad \overline{\phi^+}(y) : z \mapsto \phi^+(y,z), \ \forall z \in L.$$

- L is *decomposable* if $L = U \oplus U^\perp$.
- We can assume that L is indecomposable.

The indecomposable modules $\left(L, L^*, \overline{\phi^-}, \overline{\phi^+}\right)$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \tag{3}$$

where $A \in \mathrm{M}_n(\mathbb{F})$, and J_1, J_2 are the matrices associated with the linear maps $F_1, F_2 : \mathbb{F}^n \to \mathbb{F}^{n+1}$ defined by

$$F_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0),$$

$$F_2(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n).$$

Proposition

Let L_1 and L_2 be Leibniz algebras of dimension n with one-dimensional commutator ideals $[L_1, L_1] = \mathbb{F}z_1$ and $[L_2, L_2] = \mathbb{F}z_2$, and let ϕ_1 and ϕ_2 be the bilinear forms associated with L_1 and L_2 respectively. One can fix bases of L_1 and L_2 such that, if Φ_1 and Φ_2 are the matrices of ϕ_1 and ϕ_2 respectively, then L_1 is isomorphic to $L_2 \Longleftrightarrow \Phi_1$ is congruent to Φ_2.

If $X \in \mathrm{GL}_n(\mathbb{F})$, then

$$\begin{pmatrix} X & 0 \\ 0 & \left(X^{-1}\right)^t \end{pmatrix}$$

induces a change of basis that transforms (1) and (2) respectively in

$$\left(\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & XAX^{-1} \\ \left(XAX^{-1}\right)^t & 0 \end{pmatrix}\right),$$

$$\left(\begin{pmatrix} 0 & XAX^{-1} \\ -\left(XAX^{-1}\right)^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\right),$$

so we can always reduce A in its rational canonical form.

- If $A \in \mathrm{GL}_n(\mathbb{F})$, then the second canonical pair is equivalent to the first one:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & A^{-1} \\ (A^{-1})^t & 0 \end{pmatrix}.$$

- If $A \notin \mathrm{GL}_n(\mathbb{F})$, then $A \sim J_0$ and we have a unique Kronecker module of type (2) up to isomorphisms.

Definition

Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $n \in \mathbb{N}$ and let $A = C(f(x)^n)$. The *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^{A}$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (1).

- If $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ and $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{A}$, then the non-trivial commutators are

$$[e_i, f_j] = (\delta_{ij} + a_{ij})\, h, \ \ [f_j, e_i] = (-\delta_{ij} + a_{ij})\, h, \ \ \forall i, j = 1, \dots, n,$$

- If $(a_{ij}) = 0$, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition

Let $n \in \mathbb{N}$ and let $A = C(x^n)$. The *Kronecker* Leibniz algebra $\mathfrak{k}_n$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (2).

Definition

The *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ is the $(2n+2)-$dimensional Leibniz algebra with associated Kronecker module of type (3).

The case $\mathbb{F} = \mathbb{C}$

Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$. Then

$$C((x-a)^n) \sim J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}$$

Thus $\mathfrak{l}_{2n+1}^{A} \cong \mathfrak{l}_{2n+1}^{J_a}$ and the non-trivial Leibniz brackets are

$$[e_i, f_i] = (1+a)h, \ [f_i, e_i] = (-1+a)h \ \forall i = 1, \dots, n$$

$$[e_i, f_{i-1}] = h, \ [f_{i-1}, e_i] = h, \ \forall i = 2, \dots, n$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{J_a}$.

Proposition

Let $a \in \mathbb{C}$. The Heinseberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ and $\mathfrak{l}_{2n+1}^{J_{-a}}$ are isomorphic.

Proof.

- $\mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{-J_a^t}$ via the linear map φ defined by

$$\varphi(e_i) = f_i', \;\; \varphi(f_i) = e_i', \;\; \varphi(h) = -h', \;\; \forall i = 1, \cdots, n$$

- $-J_a^t \sim J_{-a} \Rightarrow \mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{J_{-a}}$.

□

Proposition

Let $a, a' \in \mathbb{C}$. Then $\mathfrak{l}_3^a \cong \mathfrak{l}_3^{a'} \iff a' = \pm a$.

Thank you for your attention!

Current and Future Work:

- Find the isomorphism classes of the complex and real Heisenberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$, with $n > 1$.
- Classification of derivations and biderivations of the two-step nilpotent Leibniz algebras.

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