

Rational solutions to the Pfaff lattice and Jack polynomials

M. Adler* V. B. Kuznetsov† P. van Moerbeke‡

February 15, 2002

To the memory of Jürgen Moser

Abstract

The finite Pfaff lattice is given by commuting Lax pairs involving a finite matrix L (zero above the first subdiagonal) and a projection onto $Sp(N)$. The lattice admits solutions such that the entries of the matrix L are rational in the time parameters $t_1, t_2, \dots$, after conjugation by a diagonal matrix. The sequence of polynomial τ -functions, solving the problem, belongs to an intriguing chain of subspaces of Schur polynomials, associated to Young diagrams, dual with respect to a finite chain of rectangles. Also, this sequence of τ -functions is given inductively by the action of a fixed vertex operator.

As examples, one such sequence is given by Jack polynomials for rectangular Young diagrams, while another chain starts with any two-column Jack polynomial.

*Department of Mathematics, Brandeis University, Waltham, Mass 02454, USA. E-mail: adler@brandeis.edu. The support of a National Science Foundation grant # DMS-01-00782 is gratefully acknowledged.

†Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT. E-mail: V.B.Kuznetsov@leeds.ac.uk. The support of a FSR grant, Universite de Louvain, Belgium, and from the EPSRC, UK, is gratefully acknowledged. This work was done, while visiting the University of Louvain and Brandeis University.

‡Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA. E-mail: vanmoerbeke@geom.ucl.ac.be and @brandeis.edu. The support of a National Science Foundation grant # DMS-01-00782, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged.

Contents

1	Introduction	2
2	The vector fields $\partial m / \partial t_k = \Lambda^k m + m \Lambda^{\top k}$ and the finite Pfaff lattice	12
3	The solution to the Pfaff lattice with anti-diagonal skew-symmetric initial condition	17
4	Proof of Theorem 1.1	25
5	Vertex operators	32
6	Duality	43
7	Examples	46

1 Introduction

Self-dual partitions: For positive integer n and $n|k$, define the following sets of partitions,

$$\begin{aligned}
 \mathbb{Y} &= \{\lambda = (\lambda_1, \lambda_2, \dots), \lambda_1 \geq \lambda_2 \geq \dots \geq 0\} \\
 \mathbb{Y}_k &= \{\lambda \in \mathbb{Y}, |\lambda| = \sum \lambda_i = k\} \\
 \mathbb{Y}_k^{(n)} &= \left\{ \begin{array}{ll} \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Y}_k, & \hat{\lambda}_1 \leq n, \\ \lambda_i + \lambda_{n+1-i} = \frac{2k}{n}, & 1 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil \end{array} \right\},
 \end{aligned}$$

with

$$\#\mathbb{Y}_k^{(n)} = \binom{\left\lceil \frac{n}{2} + \frac{k}{n} \right\rceil}{\left\lceil \frac{n+1}{2} \right\rceil}.$$

These are a few examples:

$$\mathbb{Y}_8^{(4)} = \left\{ \begin{array}{c} \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \right\}$$

$$\mathbb{Y}_6^{(3)} = \left\{ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right\}.$$

Let $\mathbf{s}_\lambda(t) := \det(s_{\lambda_i - i + j}(t))_{1 \leq i, j}$ be the Schur polynomials corresponding to λ , with $\mathbf{s}_i(t)$ being the elementary Schur polynomials, defined by

$$e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{i \geq 0} \mathbf{s}_i(t) z^i \quad \text{with} \quad \mathbf{s}_i(t) = 0 \text{ for } i < 0.$$

The linear space

$$\mathbb{L}_k^{(n)} := \left\{ \sum_{\lambda \in \mathbb{Y}_k^{(n)}} a_\lambda \mathbf{s}_\lambda \mid a_\lambda \in \mathbb{C} \right\}$$

will play an ubiquitous role in this work.

The finite Pfaff Lattice: The $N \times N$ skew-symmetric matrices,

$$\begin{aligned} J &= \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & 0 \\ & \ddots & \\ & 0 & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \end{pmatrix}, \quad \text{for } \underline{N \text{ even}} \\ &= \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & 0 \\ & \ddots & \\ & 0 & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & \\ & & & 0 \end{pmatrix}, \quad \text{for } \underline{N \text{ odd}}, \end{aligned} \quad (1.0.1)$$

satisfy

$$J^2 = \begin{cases} -I_N, & \text{for } N \text{ even} \\ \begin{pmatrix} -I_{N-1} & O \\ O & 0 \end{pmatrix}, & \text{for } N \text{ odd.} \end{cases} \quad (1.0.2)$$

Also consider the Lie algebra $\mathfrak{k}$ of lower-triangular matrices of the form

$$\mathfrak{k} = \left\{ \begin{pmatrix} \boxed{\begin{smallmatrix} a_1 & 0 \\ 0 & a_1 \end{smallmatrix}} & & & O \\ & \ddots & & \\ & & * & \boxed{\begin{smallmatrix} a_{[N/2]} & 0 \\ 0 & a_{[N/2]} \end{smallmatrix}} \end{pmatrix}, \text{ for } N \text{ even} \right. \\ \left. \begin{pmatrix} \boxed{\begin{smallmatrix} a_1 & 0 \\ 0 & a_1 \end{smallmatrix}} & & & O \\ & \ddots & & \\ & & \boxed{\begin{smallmatrix} a_{[N/2]} & 0 \\ 0 & a_{[N/2]} \end{smallmatrix}} & \\ & & * & \boxed{a_{(N+1)/2}} \end{pmatrix}, \text{ for } N \text{ odd.} \right\} \quad (1.0.3)$$

For each $a \in gl(N)$, consider the decomposition¹

$$\begin{aligned} a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\ &= \pi_{\mathfrak{k}} a + \pi_{\mathfrak{n}} a \\ &= \left((a_- - J(a_+)^\top J) + \frac{1}{2}(a_0 - J(a_0)^\top J) \right) \\ &\quad + \left((a_+ + J(a_+)^\top J) + \frac{1}{2}(a_0 + J(a_0)^\top J) \right). \end{aligned} \quad (1.0.4)$$

For N even, this corresponds to a Lie algebra splitting, given by

$$gl(N) = \mathfrak{k} + \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (1.0.3)}\} \\ \mathfrak{n} = sp(N) = \{a \text{ such that } Ja^\top J = a\}, \end{cases}$$

¹ $a_{\pm}$ refers to projection onto strictly upper (strictly lower) triangular matrices, with all 2×2 diagonal blocks equal zero. a_0 refers to projection onto the “diagonal”, consisting of 2×2 blocks.

$$(1.0.5)$$

For N odd, this is merely a vector space splitting

$$gl(N) = \mathfrak{k} + \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (1.0.3)}\} \\ \mathfrak{n} = \text{span}\{\pi_{\mathfrak{n}}(a) \text{ with } a \in gl(N)\}. \end{cases} \quad (1.0.6)$$

The Pfaff lattice is defined on $N \times N$ matrices L of the form

$$L = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & -d_1 & a_1 & & & & & & & O \\ & & d_1 & 1 & & & & & & \\ & & & -d_2 & a_2 & & & & & \\ & & & & d_2 & & & & & \\ & * & & & & \ddots & a_{\frac{N-2}{2}} & & & \\ & & & & & & -d_{\frac{N-2}{2}} & 1 & & \\ & & & & & & & 0 & & \end{pmatrix}, \text{ for } N \text{ even}$$

$$= \begin{pmatrix} 0 & 1 & & & & & & & & O \\ & -d_1 & a_1 & & & & & & & \\ & & d_1 & 1 & & & & & & \\ & & & -d_2 & a_2 & & & & & \\ & & & & d_2 & & & & & \\ & * & & & & \ddots & 1 & & & \\ & & & & & & -d_{\frac{N-1}{2}} & a_{\frac{N-1}{2}} & & \\ & & & & & & & d_{\frac{N-1}{2}} & & \end{pmatrix}, \text{ for } N \text{ odd.} \quad (1.0.7)$$

namely,

$$\frac{\partial L}{\partial t_i} = [-(L^i)_{\mathfrak{k}}, L]. \quad \textbf{(Pfaff lattice)} \quad (1.0.8)$$

Given arbitrary, but fixed parameters

$$b_0, \dots, b_{[\frac{N-2}{2}]} \in \mathbb{C}, \quad (1.0.9)$$

consider the skew-symmetric antidiagonal initial condition,

$$\begin{aligned}
m_N(0) &= \begin{pmatrix} & O & & b_{\frac{N-2}{2}} \\ & & \diagup & \\ & & b_0 & \\ & -b_0 & & \\ \diagup & & & \\ -b_{\frac{N-2}{2}} & & & O \end{pmatrix}, \text{ for } N \text{ even,} \\
&= \begin{pmatrix} & O & & b_{\frac{N-3}{2}} \\ & & \diagup & \\ & & b_0 & \\ & 0 & & \\ \diagup & & & \\ -b_0 & & & \\ \diagup & & & \\ -b_{\frac{N-3}{2}} & & & O \end{pmatrix}, \text{ for } N \text{ odd}
\end{aligned} \tag{1.0.10}$$

and its time evolution (respecting the skew-symmetry),

$$m_\ell(t) = E_{\ell,N}(t) m_N(0) E_{\ell,N}^\top(t), \tag{1.0.11}$$

where²

$$E_{\ell,N}(t) := \left(e^{\sum_1^\infty t_i \Lambda^i} \right)_{\substack{1, \dots, \ell \\ 1, \dots, N}}. \tag{1.0.12}$$

The Pfaffian $pf \ m_\ell(t)$ of the skew-symmetric matrix $m_\ell(t)$ will play an important role in this paper.

Rational solutions to the Pfaff Lattice:

Theorem 1.1 *Modulo conjugation by a $N \times N$ diagonal matrix $D(t)$ (see remark below), the finite Pfaff lattice*

$$\frac{\partial L}{\partial t_i} = [-(L^i)_\mathfrak{k}, L]. \quad \textbf{(Pfaff lattice)}$$

has rational solutions in $t_1, t_2, \dots$; i.e., the matrix

$$D^{-1}(t) L(t) D(t) = \tilde{Q}(t) \Lambda \tilde{Q}(t)^{-1} \tag{1.0.13}$$

² Λ is the finite shift matrix $\Lambda := (\delta_{i,j-1})_{1 \leq i,j \leq N}$ and $(A)_{\substack{1, \dots, \ell \\ 1, \dots, N}}$ denotes the matrix formed by the first ℓ rows and first N columns of A .

is rational in $t_1, t_2, \dots$, with $\tilde{Q}(t)$ a lower-triangular $N \times N$ matrix with rational entries, obtained by Taylor expanding $\tau_{2n}(t - [z^{-1}])$ in z^{-1} , with $\tau_0 = 1$,

$$\begin{aligned}\tilde{q}_{2n}(t; z) &:= \sum_{j=0}^{2n} \tilde{Q}_{2n+1, j+1}(t) z^j = z^{2n} \tau_{2n}(t - [z^{-1}]) \quad \text{with } 0 \leq n \leq \left\lfloor \frac{N-1}{2} \right\rfloor \\ \tilde{q}_{2n+1}(t; z) &:= \sum_{j=0}^{2n+1} \tilde{Q}_{2n+2, j+1}(t) z^j = z^{2n} \left(z + \frac{\partial}{\partial t_1} \right) \tau_{2n}(t - [z^{-1}]),\end{aligned}\tag{1.0.14}$$

with (see the definition of the $\mathbb{L}$ -space in the beginning of this section)

$$\begin{aligned}\tau_\ell(t) &= pf \left(E_{\ell, N}(t) m_N(0) E_{\ell, N}^\top(t) \right) \\ &= \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [\frac{N+1}{2}]} \right) \mathbf{s}_\lambda(t), \quad \text{for } \begin{cases} 0 \leq \ell \leq N-1 \\ \ell \text{ even} \end{cases} \\ &\in \mathbb{L}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}.\end{aligned}\tag{1.0.15}$$

The polynomials $q_k = D_k \tilde{q}_k$ (in z) of degree $0 \leq k \leq N-1$ are “skew-orthonormal” with respect to the skew inner-product $\langle z^i, z^j \rangle = m_{ij}(t)$, i.e.,

$$\langle q_i, q_j \rangle = J_{ij},\tag{1.0.16}$$

and the N -vector $(q_0, \dots, q_{N-1})^\top$ is an eigenvector for the matrix L , with modified boundary conditions. The fact that $Q_{2n, 2n-1} = 0$ defines the skew-orthogonal polynomials in a unique way, up to ± 1 .

Example: For $\ell = 2$, we have

$$\begin{aligned}\tau_2(t) &= \sum_{i=0}^{\frac{N-2}{2}} b_i \mathbf{s}_{\frac{N-2}{2}+i, \frac{N-2}{2}-i}(t), \quad \text{for } N \text{ even,} \\ &= \sum_{i=0}^{\frac{N-3}{2}} b_i \mathbf{s}_{\frac{N-1}{2}+i, \frac{N-3}{2}-i}(t), \quad \text{for } N \text{ odd.}\end{aligned}\tag{1.0.17}$$

Remark:

$D(t)$

$$\begin{aligned} &= \text{diag} \left(\frac{1}{\sqrt{\tau_0 \tau_2}}, \frac{1}{\sqrt{\tau_0 \tau_2}}, \frac{1}{\sqrt{\tau_2 \tau_4}}, \frac{1}{\sqrt{\tau_2 \tau_4}}, \dots, \frac{1}{\sqrt{\tau_{N-2} \tau_N}}, \frac{1}{\sqrt{\tau_{N-2} \tau_N}} \right) \quad \text{for } N \text{ even} \\ &= \text{diag} \left(\frac{1}{\sqrt{\tau_0 \tau_2}}, \frac{1}{\sqrt{\tau_0 \tau_2}}, \dots, \frac{1}{\sqrt{\tau_{N-3} \tau_{N-1}}}, \frac{1}{\sqrt{\tau_{N-3} \tau_{N-1}}}, \frac{1}{\sqrt{\tau_{N-1}}} \right) \quad \text{for } N \text{ odd.} \end{aligned}$$

Duality: For the case of odd N , we can even define $\tau_\ell(t)$ for odd ℓ , by slightly deforming the initial moment matrix $m_N(0)$. In section 6, we prove a duality between these τ_k 's for k even and odd, as follows

$$\tilde{\tau}_\ell(t) = (-1)^{\ell(N-\ell)/2} \left(\prod_0^{\frac{N-3}{2}} b_i \right) \left(\tau_{N-\ell}(-t)|_{b_i \rightarrow b_i^{-1}} \right), \quad \text{for } \ell \text{ odd.}$$

Fay identities:

Theorem 1.2 *The sequence of functions*

$$\tau_\ell(t) = \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [\frac{N+1}{2}]} \right) \mathbf{s}_\lambda(t), \quad \begin{array}{l} 0 \leq \ell \leq N-1 \\ \ell \text{ even} \end{array}, \quad (1.0.18)$$

together with the “boundary condition”

$$\tau_0 = 1 \quad \text{and} \quad \begin{cases} \tau_N = \prod_0^{\frac{N-2}{2}} b_i, & \text{for even } N \\ \tau_{N+1} = 0, & \text{for odd } N, \end{cases} \quad (1.0.19)$$

satisfies the the “differential Fay identity”³:

$$\begin{aligned} &\{\tau_{2n}(t - [u]), \tau_{2n}(t - [v])\} \\ &\quad + (u^{-1} - v^{-1})(\tau_{2n}(t - [u])\tau_{2n}(t - [v]) - \tau_{2n}(t)\tau_{2n}(t - [u] - [v])) \\ &\quad = uv(u - v)\tau_{2n-2}(t - [u] - [v])\tau_{2n+2}(t). \end{aligned} \quad (1.0.20)$$

³Define the Wronskian $\{f, g\} = \frac{\partial f}{\partial t_1} g - \frac{\partial g}{\partial t_1} f$.

Vertex operator constructions of the rational solutions: Consider the vertex operator acting on functions $f(t)$ of $t = (t_1, t_2, \dots) \in \mathbb{C}^\infty$, namely

$$X(t; z) = e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}}; \quad (1.0.21)$$

and the vector vertex operator

$$\mathbb{X}(t; z) = \Lambda^\top e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}} \chi(z), \quad (1.0.22)$$

acting on vectors of functions $F = (f_0(t), f_1(t), \dots)$, with $\chi(z) := (z^i)_{i \geq 0}$. Then the composition $\mathbb{X}(t; \lambda) \mathbb{X}(t; \mu)$ is a vertex operator for the Pfaff lattice, i.e., for any τ -vector $= (\tau_0, \tau_2, \tau_4, \dots)$ of the Pfaff lattice,

$$\tau(t) + a \mathbb{X}(t; \mu) \mathbb{X}(t; \lambda) \tau(t), \quad a \in \mathbb{C}$$

is again a τ -vector of the Pfaff lattice, or coordinatewise

$$\tau_{2n} + a \left(1 - \frac{\lambda}{\mu}\right) \mu^{2n-1} \lambda^{2n-2} e^{\sum t_i (\lambda^i + \mu^i)} \tau_{2n-2}(t - [\lambda^{-1}] - [\mu^{-1}])$$

provides a new sequence of Pfaff τ -functions.

In terms of the distributional weight, with the b_i as in (1.0.9),

$$\rho_b(x) := \begin{cases} \rho_b^{(e)}(x) = \sum_{i \geq 0} b_i (x^{-i-1} - x^i), & \text{for } N \text{ even,} \\ \rho_b^{(0)}(x) = x^{-1/2} \sum_{i \geq 0} b_i (x^{-i-1} - x^{i+1}), & \text{for } N \text{ odd.} \end{cases}$$

and

$$\beta := \frac{N}{2} - \ell + 1, \quad (1.0.23)$$

we define the *integrated vertex operator*, in terms of the vertex operator (1.0.21),

$$Y_\beta(t) := \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} X(t; y) X(t; z) \frac{\rho_b(y/z) dy dz}{z^2 (yz)^\beta}$$

and the *integrated vector vertex operator*, in terms of the (1.0.22),

$$\mathbb{Y}_N(t) = \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \mathbb{X}(t; y) \mathbb{X}(t; z) \frac{\rho_b(y/z) dy dz}{2(yz)^{N/2} z}. \quad (1.0.24)$$

In both cases, the double integral around two contours about ∞ amounts to computing the coefficient of $1/yz$.

Theorem 1.3 *For a given set of b_i , the sequence of τ -functions $\tau_0, \tau_2, \tau_4, \dots$, defined in (1.0.15), is generated by the vertex operators Y_p ; to be precise, inductively*

$$Y_{\frac{N}{2}-\ell+1} \tau_{\ell-2} = \ell \tau_{\ell}.$$

Corollary 1.4 *The vector of τ -functions*

$$I = (I_0, I_2, I_4, \dots), \text{ with } I_{\ell} = \left(\frac{\ell}{2}\right)! \tau_{\ell}$$

is a fixed point for the vertex operator $\mathbb{Y}_N$, namely

$$(\mathbb{Y}_N I)_{\ell} = I_{\ell}, \text{ for even } \ell.$$

The rational solutions to the Pfaff lattice can be q -deformed; this will be reported on at a later stage.

Example 1: Rectangular Jack polynomials

Jack polynomials are symmetric polynomials in the variables x_i , which are orthogonal with respect to the inner-product

$$\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda\mu} (1^{m_1} 2^{m_2} \dots) m_1! m_2! \dots \alpha^{\lambda_1^{\top}},$$

where $m_i = m_i(\lambda)$ is the number of times that i appears in the partition λ and where

$$p_{\lambda}(x_1, x_2, \dots) := p_{\lambda_1} p_{\lambda_2} \dots = \sum_i x_i^{\lambda_1} \sum_i x_i^{\lambda_2} \dots$$

Precise definitions and properties of Jack polynomials can be found in [8, 9, 4, 6, 7].

Proposition 1.5 *When*

$$\begin{aligned} b_i &= 2i + 1 \text{ for } N \text{ even} \\ &= 2i + 2 \text{ for } N \text{ odd,} \end{aligned}$$

then the $\tau_{2n}(t)$'s are Jack polynomials for rectangular partitions

$$\begin{aligned} \tau_{2n}(t) &= \sum_{\lambda \in \mathbb{Y}_{n(N-2n)}^{(2n)}} \prod_1^n (k_i - k_{2n+1-i}) \mathbf{s}_\lambda(t), \text{ where } \begin{cases} k_i = \lambda_i - i + 2n \\ 0 \leq 2n \leq N, \end{cases} \\ &= pf \ m_{2n}(t) \\ &= \frac{1}{n!} \int_{\mathbb{R}^n} \Delta(z)^4 \prod_{k=1}^n e^{2 \sum_1^\infty t_i z_k^i} \delta_{(z_k)}^{(N-2)} dz_k \\ &= J_\lambda^{(1/2)}(x) \Big|_{t_i = \frac{1}{i} \sum_k x_k^i} \text{ for } \lambda = \underbrace{(N-2n, \dots, N-2n)}_n \end{aligned}$$

where the $m_{2n}(t)$'s are the $2n \times 2n$ upper-left hand corners of

$$m_N(t) = ((j-i)\tilde{\mathbf{s}}_{N-i-j-1})_{0 \leq i, j \leq N-1}. \quad (1.0.25)$$

upon setting $\tilde{\mathbf{s}}_n(t) := \mathbf{s}_n(2t)$.

Example 2: Two-row Jack polynomials

Proposition 1.6 *For even N , choosing⁴*

$$\begin{cases} b_0 = \dots = b_{\frac{p}{2}-1} = 0 \\ b_{\frac{p}{2}+k} = \frac{(1-\alpha)_k (p+1)_k}{k! (\alpha+p+1)_k}, \text{ for } k = 0, \dots, \frac{N-2-p}{2}, \end{cases} \quad (1.0.26)$$

$${}^4(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1)$$

one finds the most general two-row Jack polynomial for τ_2 , for arbitrary α ,

$$\begin{aligned}
\tau_2(t) &= pf \, m_2(t) \\
&= J_{\left(\frac{N+p-2}{2}, \frac{N-p-2}{2}\right)}^{(1/\alpha)}(t/\alpha) \\
&= c \oint \frac{dx}{2\pi i} \frac{dy}{2\pi i} \frac{(y-x)^{2\alpha}}{(xy)^{\alpha+\frac{N}{2}}} e^{\sum_1^\infty t_i(x^i+y^i)} \left(\frac{x}{y}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{y}{x}\right).
\end{aligned} \tag{1.0.27}$$

Then $\tau_\ell(t)$ for $\ell \geq 4$ is given by an integral of the same hypergeometric function in the integrand above.

Acknowledgment: We thank Michael Kleber for useful discussions and insights.

2 The vector fields $\partial m / \partial t_k = \Lambda^k m + m \Lambda^\top k$ and the finite Pfaff lattice

The $\ell \times N$ matrix defined in (1.0.12) reads

$$E_{\ell,N}(t) = \left(\begin{array}{ccccc|ccc} 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \dots & \mathbf{s}_{\ell-1}(t) & \mathbf{s}_\ell(t) & \dots & \mathbf{s}_{N-1}(t) \\ 0 & 1 & \mathbf{s}_1(t) & \dots & \mathbf{s}_{\ell-2}(t) & \mathbf{s}_{\ell-1}(t) & \dots & \mathbf{s}_{N-2}(t) \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \dots & \mathbf{s}_{N-\ell+1}(t) \\ 0 & 0 & 0 & \dots & 1 & \mathbf{s}_1(t) & \dots & \mathbf{s}_{N-\ell}(t) \end{array} \right)$$

The main claim of this section can be summarized in the following statement:

Proposition 2.1 *The commuting equations (for definition of Λ , see footnote 2)*

$$\frac{\partial m_N}{\partial t_k} = \Lambda^k m_N + m_N \Lambda^\top k, \tag{2.0.1}$$

with $N \times N$ skew-symmetric initial condition $m(0)$, have the following solution

$$m_N(t) = E_{N,N}(t)m_N(0)E_{N,N}^\top(t). \quad (2.0.2)$$

In particular, each $\ell \times \ell$ upper-left block of $m(t)$ equals

$$m_\ell(t) = E_{\ell,N}(t)m_N(0)E_{\ell,N}^\top(t), \quad (2.0.3)$$

Proof: Define $m_\infty(0)$ as the semi-infinite matrix formed by putting $m_N(0)$ in the upper-left corner and setting all other entries equal to 0 and let Λ_∞ be the semi-infinite shift matrix. Then the solution to the differential equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda_\infty^k m_\infty + m_\infty \Lambda_\infty^{\top k} \quad (2.0.4)$$

is given by

$$m_\infty(t) = e^{\sum_1^\infty t_k \Lambda_\infty^k} m_\infty(0) e^{\sum_1^\infty t_k \Lambda_\infty^{\top k}}. \quad (2.0.5)$$

Result (2.0.1) follows from the Taylor expansion

$$e^{\sum_1^\infty t_k \Lambda_\infty^k} = \sum_{k=0}^{\infty} \mathbf{s}_k(t) \Lambda_\infty^k,$$

which is an upper-triangular semi-infinite matrix, and considering only the upper-left $\ell \times \ell$ block. Each upper-left $\ell \times \ell$ block of $m_\infty(t)$ for $\ell \leq N$, equals

$$\begin{aligned} m_\ell(t) &= E_{\ell,\infty}(t)m_\infty(0)E_{\ell,\infty}^\top(t) \\ &= E_{\ell,N}(t)m_N(0)E_{\ell,N}^\top(t) \end{aligned}$$

from which (2.0.3) follows, and (2.0.2) setting $\ell = N$. ■

Remark: The flow (2.0.4) maintains the finite upper-left hand corner of m_∞ and on that locus it is equivalent to the finite flow (2.0.1). Therefore, the whole semi-infinite theory can be applied to this case. It is possible to give a proof of Theorem 2.1 purely within finite matrices.

Theorem 2.2 Consider the commuting equations on the $N \times N$ matrix in

$$\frac{\partial m_N}{\partial t_i} = \Lambda^i m_N + m_N \Lambda^i \quad (2.0.6)$$

with skew-symmetric initial condition $m_N(s)$ and its “skew-Borel decomposition”

$$m_N = Q^{-1} J Q^{-1\top}, \text{ for } Q \in G_{\mathfrak{k}}. \quad (2.0.7)$$

When N is odd, we further impose the differential equations for the last entry Q_{NN} of Q :

$$\frac{\partial Q_{NN}}{\partial t_i} = -\frac{1}{2} Q_{N,N-i}. \quad (2.0.8)$$

Then for arbitrary $N > 0$ the matrix Q evolves according to the equations

$$\frac{\partial Q}{\partial t_i} Q^{-1} = -\pi_{\mathfrak{k}}(Q \Lambda^i Q^{-1}) \quad (2.0.9)$$

and the matrix $L := Q \Lambda Q^{-1}$ provides a solution to the Lax pair

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathfrak{k}} L^i, L] = [\pi_{\mathfrak{n}} L^i, L]. \quad (2.0.10)$$

Proof: For a matrix A , consider the projections

$$\begin{aligned} A_0 &= \begin{pmatrix} * & * & & O \\ * & * & & \\ & & \ddots & \\ O & & & * & * \\ & & & * & * \end{pmatrix}, \text{ for } N \text{ even} \\ &= \begin{pmatrix} * & * & & O \\ * & * & & \\ & & \ddots & \\ O & & & * & * \\ & & & * & * \\ & & & & * \end{pmatrix}, \text{ for } N \text{ odd,} \end{aligned}$$

and

$$\begin{aligned}
A_{00} &= A_0, \text{ for } N \text{ even} \\
&= \begin{pmatrix} * & * & & & & \\ & * & * & & & \\ & & \ddots & & & \\ & & & O & & \\ & & & & * & * \\ & O & & & * & * \\ & & & & & 0 \end{pmatrix}, \text{ for } N \text{ odd.}
\end{aligned}$$

The main point is to prove that⁵

$$\begin{aligned}
0 &= \frac{\partial Q}{\partial t_i} Q^{-1} + \pi_{\mathfrak{k}} L^i \\
&= \frac{\partial Q}{\partial t_i} Q^{-1} + (L_-^i - J(L_+^i)^\top J) + \frac{1}{2}(L_0^i - J(L_0^i)^\top J) \\
&=: A.
\end{aligned}$$

Also define

$$\left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1} \right) - J \left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1} \right)^\top J =: B.$$

we have, setting $\dot{\cdot} = \frac{\partial}{\partial t_i}$,

$$\begin{aligned}
0 &= Q \left(\Lambda^i m + m \Lambda^{\top i} - \frac{\partial m}{\partial t_i} \right) Q^\top \\
&= (Q \Lambda^i Q^{-1}) J + J Q^{-1\top} \Lambda^{\top i} Q^\top + (\dot{Q} Q^{-1}) J + J Q^{-1\top} \dot{Q}^\top \\
&= (L^i + \dot{Q} Q^{-1}) J + J (L^i + \dot{Q} Q^{-1})^\top.
\end{aligned}$$

⁵ $L_+^i := (L^i)_+$ and $L_0^i := (L^i)_0$.

Hence⁶

$$\begin{aligned}
0 &= \left(Q \left(\Lambda^i m + m \Lambda^{\top i} - \frac{\partial m}{\partial t_i} \right) Q^{\top} \right)_{-,00} \\
&= \left(\left((L^i + \dot{Q} Q^{-1}) - J(L^i + \dot{Q} Q^{-1})^{\top} J \right) J \right)_{-,00} \\
&= \left((L^i + \dot{Q} Q^{-1}) - J(L^i + \dot{Q} Q^{-1})^{\top} J \right)_{-,00} J \\
&= B_{-,00} J.
\end{aligned}$$

Therefore

$$0 = B_{-,00} J^2 = \begin{cases} B_{-,0}, & \text{for } N \text{ even} \\ B_{-,00} \begin{pmatrix} I_{N-1} & O \\ O & 0 \end{pmatrix} & \text{for } N \text{ odd} \end{cases}$$

and so

$$B_- = 0 \quad \text{and} \quad B_{00} = 0. \quad (2.0.11)$$

But

$$\begin{aligned}
B_- &= (L^i + \dot{Q} Q^{-1} - J(L_+^i)^{\top} J)_- \\
&= (\dot{Q} Q^{-1})_- + ((L^i)_- - J(L_+^i)^{\top} J) \\
&= A_-
\end{aligned} \quad (2.0.12)$$

and

$$\begin{aligned}
B_{00} &= 2(\dot{Q} Q^{-1})_{00} + (L^i - J(L^i)^{\top} J)_{00} \\
&= 2A_{00}.
\end{aligned} \quad (2.0.13)$$

Then, by (2.0.12) and (2.0.13),

$$0 = B_- + \frac{1}{2} B_{00} = A_- + A_{00} = A_- + A_{00} + A_+, \quad \text{since } A_+ = 0.$$

Therefore, when N is even, $A = 0$ and the proof is finished. When N is odd, we have

⁶ $A_{-,00} = A_- + A_{00}.$

$A = 0$, except for the (N, N) -entry.

But the (N, N) th entry of L^i is given by, since Q is lower-triangular,

$$(L^i)_{NN} = (Q\Lambda^i Q^{-1})_{NN} = \frac{Q_{N,N-i}}{Q_{NN}},$$

and thus we have, using the fact that the (N, N) th entry of $J(L^i)_0 J$ vanishes,

$$\begin{aligned} A_{NN} &= \frac{\partial}{\partial t_i} \log Q_{NN} + \frac{1}{2}(L^i)_{NN} \\ &= \frac{1}{Q_{NN}} \left(\frac{\partial Q_{NN}}{\partial t_i} + \frac{1}{2}Q_{N,N-i} \right), \\ &= 0, \text{ by the assumption (2.0.8),} \end{aligned}$$

thus ending the proof of Theorem 2.2. ■

3 The solution to the Pfaff lattice with anti-diagonal skew-symmetric initial condition

Consider the equations

$$\frac{\partial m_N}{\partial t_i} = \Lambda^i m_N + m_N \Lambda^{\top i}, \tag{3.0.1}$$

with initial condition,

$$\begin{aligned}
m_N(0) &= \begin{pmatrix} & & O & & b_{\frac{N-2}{2}} \\ & & & \nearrow & \\ & & b_0 & & \\ & & & \nearrow & \\ & -b_0 & & & \\ -b_{\frac{N-2}{2}} & & \nearrow & & O \end{pmatrix}, \text{ for } N \text{ even,} \\
&= \begin{pmatrix} & & O & & b_{\frac{N-3}{2}} \\ & & & \nearrow & \\ & & b_0 & & \\ & & & \nearrow & \\ & 0 & & & \\ & & \nearrow & & \\ -b_0 & & & & \\ -b_{\frac{N-3}{2}} & & \nearrow & & O \end{pmatrix}, \text{ for } N \text{ odd.}
\end{aligned} \tag{3.0.2}$$

Proposition 3.1 *The system of equations (3.0.1), with initial condition (3.0.2) have for solution the matrix $m_N(t)$, with entries, for $0 \leq \ell < k \leq N$,*

$$\begin{aligned}
\mu_{\ell,k}(t) &= - \sum_{j=0}^{[\frac{N-2}{2}]-k} \mathbf{s}_j \mathbf{s}_{N-\ell-k-j-1} (b_{[\frac{N-2}{2}]-k-j} - b_{[\frac{N-2}{2}]-\ell-j}) \\
&\quad - \sum_{j=[\frac{N-2}{2}]-k+1}^{[\frac{N-2}{2}]-\ell} \mathbf{s}_j \mathbf{s}_{N-\ell-k-j-1} (-b_{[\frac{N-2}{2}]-\ell-j}). \tag{3.0.3}
\end{aligned}$$

In particular

$$\begin{aligned}
\mu_{01}(t) &= \sum_{i=0}^{\frac{N-2}{2}} b_i \mathbf{s}_{\frac{N-2}{2}+i, \frac{N-2}{2}-i}(t), \quad \text{for } N \text{ even} \\
&= \sum_{i=0}^{\frac{N-3}{2}} b_i \mathbf{s}_{\frac{N-1}{2}+i, \frac{N-3}{2}-i}(t), \quad \text{for } N \text{ odd.} \tag{3.0.4}
\end{aligned}$$

Proof: Equation (3.0.3) is established by explicit computation of

$$\begin{aligned} m_N(t) &= E_{N,N}(t)m_N(0)E_{N,N}(t)^\top \\ &= \left(\sum_{i,j=0}^{N-\ell-1} \mathbf{s}_i(t)\mu_{i+\ell,j+k}(0)\mathbf{s}_j(t) \right)_{0 \leq \ell, k \leq N-1}. \end{aligned}$$

From (3.0.3), one computes, for N even,

$$\begin{aligned} \mu_{01}(t) &= \mathbf{s}_0\mathbf{s}_{N-2}(b_{\frac{N}{2}-1} - b_{\frac{N}{2}-2}) + \mathbf{s}_1\mathbf{s}_{N-3}(b_{\frac{N}{2}-2} - b_{\frac{N}{2}-3}) + \\ &\quad \dots + \mathbf{s}_{\frac{N}{2}-2}\mathbf{s}_{\frac{N}{2}}(b_1 - b_0) + (\mathbf{s}_{\frac{N}{2}-1})^2 b_0 \\ &= \sum_{i=0}^{\frac{N}{2}-1} b_i (\mathbf{s}_{\frac{N}{2}-1-i}\mathbf{s}_{\frac{N}{2}-1+i} - \mathbf{s}_{\frac{N}{2}-2-i}\mathbf{s}_{\frac{N}{2}+i}) \\ &= \sum_{i=0}^{\frac{N}{2}-1} b_i \mathbf{s}_{\frac{N}{2}-1+i, \frac{N}{2}-1-i}(t), \end{aligned}$$

and for N odd,

$$\begin{aligned} \mu_{01}(t) &= \mathbf{s}_0\mathbf{s}_{N-2}(b_{\frac{N-3}{2}} - b_{\frac{N-5}{2}}) + \mathbf{s}_1\mathbf{s}_{N-3}(b_{\frac{N-5}{2}} - b_{\frac{N-7}{2}}) + \\ &\quad \dots + \mathbf{s}_{\frac{N-5}{2}}\mathbf{s}_{\frac{N+1}{2}}(b_1 - b_0) + \mathbf{s}_{\frac{N-3}{2}}\mathbf{s}_{\frac{N-1}{2}}b_0 \\ &= \sum_{i=0}^{\frac{N-3}{2}} b_i \mathbf{s}_{\frac{N-1}{2}+i, \frac{N-3}{2}-i}(t), \end{aligned}$$

ending the proof of Proposition 3.1. ■

Define⁷

$$\begin{aligned} m_N(0; z) &:= m_N(0) \quad , \quad \text{for } N \text{ even,} \\ m_N(0; z) &:= m_N(0) + z^2 \varepsilon_{\frac{N+1}{2}, \frac{N+1}{2}} \quad , \quad \text{for } N \text{ odd,} \end{aligned}$$

$$= \begin{pmatrix} & & O & & b_{\frac{N-3}{2}} \\ & & & & / \\ & & & b_0 & \\ & & z^2 & & \\ & & -b_0 & & \\ / & & & & \\ -b_{\frac{N-3}{2}} & & & & O \end{pmatrix}$$

⁷ $\varepsilon_{i,j}$ denotes the matrix with all zero entries, except for a 1 at the (i,j) th entry.

(3.0.5)

Proposition 3.2

$$\begin{aligned}
& \det^{1/2} (E_{\ell,N}(t) m_N(0; z) E_{\ell,N}^\top(t)) \\
&= z^{\eta(N,\ell)} \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - \lfloor \frac{N+1}{2} \rfloor} \right) \mathbf{s}_{\lambda_1 \geq \dots \geq \lambda_\ell}(t).
\end{aligned}$$

with

$$\eta(N, \ell) = \begin{cases} 1 & , \text{ for } N \text{ and } \ell \text{ odd.} \\ 0 & , \text{ otherwise.} \end{cases}$$

Lemma 3.3 Consider an arbitrary $N \times N$ matrix $A = (A_{ij})_{1 \leq i, j \leq N}$, with $r = \left\lfloor \frac{N}{2} \right\rfloor$ and $A_\ell := (A_{ij})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq N}}$ and consider the anti-diagonal matrix

$$\begin{aligned}
m_N &= \begin{pmatrix} & & & c_r \\ & O & & / \\ & & c_1 & \\ & -c_1 & & \\ / & & & O \\ -c_r & / & & \end{pmatrix} \text{ for } N \text{ even} \\
&= \begin{pmatrix} & & & c_r \\ & O & & / \\ & & c_1 & \\ & z^2 & & \\ & -c_1 & & \\ / & & & O \\ -c_r & / & & \end{pmatrix} \text{ for } N \text{ odd.}
\end{aligned}$$

Setting⁸

$$m_\ell^A(z) := A_\ell m_N(z) A_\ell^\top$$

and

⁸ $B_{(j_1, \dots, j_n)}$ denotes the matrix formed with the columns $j_1, \dots, j_n$ of B

$$\begin{aligned}
P_{N,\ell} &= \sum_{1 \leq i_1 \leq \dots \leq i_{\lfloor \ell/2 \rfloor} \leq r} c_{i_1} \dots c_{i_{\lfloor \ell/2 \rfloor}} \\
&\times \begin{cases} \det(A_\ell)_{(r-i_{\lfloor \ell/2 \rfloor}+1, \dots, r-i_1+1, r+i_1, \dots, r+i_{\lfloor \ell/2 \rfloor})} & \text{for } N \text{ even, } \ell \text{ even} \\ \det(A_\ell)_{(r-i_{\lfloor \ell/2 \rfloor}+1, \dots, r-i_1+1, r+i_1+1, \dots, r+i_{\lfloor \ell/2 \rfloor}+1)} & \text{for } N \text{ odd, } \ell \text{ even} \\ \det(A_\ell)_{(r-i_{\lfloor \ell/2 \rfloor}+1, \dots, r-i_1+1, r+1, r+i_1+1, \dots, r+i_{\lfloor \ell/2 \rfloor}+1)} & \text{for } N \text{ odd, } \ell \text{ odd} \end{cases} \\
&\hspace{15em} (3.0.6)
\end{aligned}$$

we have

$$\det m_\ell^A = \begin{cases} 0 & \text{for } N \text{ even, } \ell \text{ odd} \\ (pf \ m_\ell^A)^2 = (P_{N,\ell})^2 & \text{for } N \text{ even, } \ell \text{ even} \\ z^2 P_{N,\ell}^2 & \text{for } N \text{ odd, } \ell \text{ odd} \\ (pf \ m_\ell^A(0))^2 = (P_{N,\ell})^2 & \text{for } N \text{ odd, } \ell \text{ even.} \end{cases}$$

Proof: Let $w_i \in \mathbb{C}^\ell$ be the columns of A_ℓ

$$A_\ell = [w_0, w_1, \dots, w_{2r}],$$

and observe

$$\begin{aligned}
m_\ell^A(z) &= A_\ell m_N(z) A_\ell^\top = A_\ell (z^2 \varepsilon_{r+1, r+1} + m_N(0)) A_\ell^\top \\
&= z^2 w_r \otimes w_r + m_\ell^A(0).
\end{aligned}$$

Let U be a $\ell \times \ell$ matrix, rational in the a_{ij} , such that

$$U w_r = \alpha e_1, \quad \det U = 1.$$

Then, using $U(x \otimes y)V = (Ux) \otimes (V^\top y)$ and setting $M := U m_\ell^A(0) U^\top$, which is skew-symmetric, we find

$$\begin{aligned}
\det m_\ell^A(z) &= \det U m_\ell^A(z) U^\top \\
&= \det (z^2 U(w_r \otimes w_r) U^\top + U m_\ell^A(0) U^\top) \\
&= \det (z^2 \alpha^2 e_1 \otimes e_1 + U m_\ell^A(0) U^\top), \\
&= \det \left(\begin{array}{c|cccc} (z\alpha)^2 & M_{12} & M_{13} & \dots & M_{1\ell} \\ \hline -M_{12} & 0 & M_{23} & \dots & M_{2\ell} \\ -M_{13} & -M_{23} & & & \\ \vdots & \vdots & & & \vdots \\ -M_{1\ell} & -M_{2\ell} & \dots & & 0 \end{array} \right) \\
&= (z\alpha)^2 \det(M_{ij})_{2 \leq i,j \leq \ell} + \det(M_{ij})_{1 \leq i,j \leq \ell},
\end{aligned}$$

with $M_{ij} = -M_{ji}$. Therefore

$$\begin{aligned}
\det m_\ell^A(z) &= \det m_\ell^A(0) = (pf m_\ell^A(0))^2, & \text{for } \ell \text{ even} \\
&= (z\alpha)^2 \det(M_{ij})_{2 \leq i,j \leq \ell} = (z\alpha pf(M_{ij})_{2 \leq i,j \leq \ell})^2, & \text{for } \ell \text{ odd,} \\
& & (3.0.7)
\end{aligned}$$

the latter being the square of a polynomial in z , the c_i and the entries of the matrix A .

Using the Cauchy-Bonnet formula twice, one computes, say, for N and ℓ odd,

$$\begin{aligned}
\det m_\ell^A(z) &= \det A_\ell m_N(z) A_\ell^\top \\
&= \sum_{1 \leq \alpha_1 < \dots < \alpha_\ell \leq N} \det ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \det ((A_\ell m^\top)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \\
&= \sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_\ell \leq N \\ 1 \leq \beta_1 < \dots < \beta_\ell \leq N \\ \alpha_i + \beta_{\ell-i+1} = N+1}} \det ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \det ((A_\ell)_{i, \beta_j})_{1 \leq i, j \leq \ell} \det ((m^\top)_{\beta_i, \alpha_j})_{1 \leq i, j \leq \ell} \\
&= \sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_\ell \leq N \\ 1 \leq \beta_1 < \dots < \beta_\ell \leq N \\ \alpha_i + \beta_{\ell-i+1} = N+1 \\ \text{for } 1 \leq i \leq \ell}} \det ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \det ((A_\ell)_{i, \beta_j})_{1 \leq i, j \leq \ell} \det (m_{\alpha_i, \beta_j})_{1 \leq i, j \leq \ell} \\
&= \left(\sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_\ell \leq N \\ 1 \leq \beta_1 < \dots < \beta_\ell \leq N \\ (\alpha_1, \dots, \alpha_\ell) = (\beta_1, \dots, \beta_\ell) \\ \alpha_i + \beta_{\ell-i+1} = N+1 \\ \text{for } 1 \leq i \leq \ell}} + \sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_\ell \leq N \\ 1 \leq \beta_1 < \dots < \beta_\ell \leq N \\ (\alpha_1, \dots, \alpha_\ell) \neq (\beta_1, \dots, \beta_\ell) \\ \alpha_i + \beta_{\ell-i+1} = N+1 \\ \text{for } 1 \leq i \leq \ell}} \right) \det ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \det ((A_\ell)_{i, \beta_j})_{1 \leq i, j \leq \ell} \\
&\quad \times \det (m_{\alpha_i, \beta_j})_{1 \leq i, j \leq \ell} \\
&\stackrel{*}{=} z^2 \sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_{\frac{\ell-1}{2}} < \frac{N+1}{2} \\ \alpha_{\frac{\ell+1}{2}+i} + \alpha_{\frac{\ell+1}{2}-i} = N+1 \\ \text{for } 0 \leq i \leq \frac{\ell-1}{2}}} c_{\frac{N+1}{2}-\alpha_1}^2 \dots c_{\frac{N+1}{2}-\alpha_{\frac{\ell-1}{2}}}^2 \det^2 ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} + \dots \\
&= \left(z \sum_{\substack{1 \leq \alpha_1 < \dots < \alpha_{\frac{\ell-1}{2}} < \frac{N+1}{2} \\ \alpha_{\frac{\ell+1}{2}+i} + \alpha_{\frac{\ell+1}{2}-i} = N+1}} c_{\frac{N+1}{2}-\alpha_1} \dots c_{\frac{N+1}{2}-\alpha_{\frac{\ell-1}{2}}} \det ((A_\ell)_{i, \alpha_j})_{1 \leq i, j \leq \ell} \right)^2 \text{ using (3.0.7)} \\
&= \left(z \sum_{1 \leq i_1 < \dots < i_{\frac{\ell-1}{2}} \leq \frac{N-1}{2}} c_{i_{\frac{\ell-1}{2}}} \dots c_{i_1} \det(A_\ell) \left(\frac{N+1}{2} - i_{\frac{\ell-1}{2}}, \dots, \frac{N+1}{2} - i_1, \frac{N+1}{2}, \frac{N+1}{2} + i_1, \dots, \frac{N+1}{2} + i_{\frac{\ell-1}{2}} \right) \right)^2.
\end{aligned}$$

In $\stackrel{*}{=}$ we have used the fact that

$$(\alpha_1, \dots, \alpha_\ell) = (\beta_1, \dots, \beta_\ell) \left. \vphantom{\begin{matrix} \alpha_1, \dots, \alpha_\ell \\ \beta_1, \dots, \beta_\ell \end{matrix}} \right\} \alpha_i + \beta_{\ell-i+1} = N+1 \Bigg\} \iff \begin{cases} \alpha_{\frac{\ell+1}{2}+i} + \alpha_{\frac{\ell+1}{2}-i} = N+1, \\ \text{for } 0 \leq i \leq \frac{\ell-1}{2} \\ \beta_{\ell-i+1} = N+1 - \alpha_i. \end{cases} \quad (3.0.8)$$

Indeed, for N odd consider sequences α_i symmetric about

$$\alpha_{\frac{\ell+1}{2}} = \frac{N+1}{2} \quad (3.0.9)$$

i.e.

$$\alpha_{\frac{\ell+1}{2}+i} + \alpha_{\frac{\ell+1}{2}-i} = N+1, \quad \text{for } 0 \leq i \leq \frac{\ell-1}{2}. \quad (3.0.10)$$

Then, using (3.0.8) and (3.0.10)

$$\beta_{\frac{\ell+1}{2}-i} = N+1 - \alpha_{\frac{\ell+1}{2}+i} = \alpha_{\frac{\ell+1}{2}-i},$$

thus implying

$$(\alpha_1, \dots, \alpha_\ell) = (\beta_1, \dots, \beta_\ell).$$

Vice versa, the latter implies (3.0.8) and thus (3.0.9). This establishes Lemma 3.3 for the case N and ℓ odd; for the other cases, one proceeds in a similar fashion.

Proof of Proposition 3.2: Apply Lemma 3.3 to $A_\ell = E_{\ell,N}(t) = (\mathbf{s}_{j-i})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq N}}$, with $1 \leq k_1 < k_2 < \dots < k_\ell$:

$$\begin{aligned} \det(A_\ell)_{k_1, \dots, k_\ell} &= \det \begin{pmatrix} \mathbf{s}_{k_1-1} & \dots & \mathbf{s}_{k_{\ell-1}-1} & \mathbf{s}_{k_\ell-1} \\ \vdots & & \vdots & \vdots \\ \mathbf{s}_{k_1-\ell} & \dots & \mathbf{s}_{k_{\ell-1}-\ell} & \mathbf{s}_{k_\ell-\ell} \end{pmatrix} \\ &= \det \begin{pmatrix} \mathbf{s}_{k_\ell-\ell} & \mathbf{s}_{k_\ell-\ell+1} & \dots & \mathbf{s}_{k_\ell-1} \\ \mathbf{s}_{k_{\ell-1}-\ell} & \mathbf{s}_{k_{\ell-1}-\ell+1} & \dots & \mathbf{s}_{k_{\ell-1}-1} \\ \vdots & & & \vdots \\ \mathbf{s}_{k_1-\ell} & & \dots & \mathbf{s}_{k_1-1} \end{pmatrix} \\ &= \mathbf{s}_{k_\ell-\ell, k_{\ell-1}-\ell+1, \dots, k_1-\ell+(\ell-1)} \\ &= \mathbf{s}_{\lambda_1 \geq \dots \geq \lambda_\ell} \\ &= \mathbf{s}_\lambda \end{aligned} \quad (3.0.11)$$

where

$$\lambda_i = k_{\ell-i+1} - \ell + i - 1, \text{ for } 1 \leq i \leq \ell. \quad (3.0.12)$$

In order to apply Lemma 3.3, the k_i inherent in formula (3.0.6) must be as in formula (6.0.4); i.e., setting $r = \lfloor N/2 \rfloor$, the k_j 's must satisfy

$$k_j = \left\lfloor \frac{N}{2} \right\rfloor - i_{\lfloor \ell/2 \rfloor - j + 1} + 1 = N + 1 - k_{\ell-j+1}, \text{ for } 1 \leq j \leq \left\lfloor \frac{\ell+1}{2} \right\rfloor \quad (3.0.13)$$

and thus

$$\begin{aligned} i_{\lfloor \ell/2 \rfloor + 1 - j} - 1 &= k_{\ell+1-j} - \left\lfloor \frac{N+1}{2} \right\rfloor - 1 \\ &= \lambda_j + \ell - j - \left\lfloor \frac{N+1}{2} \right\rfloor. \end{aligned}$$

Therefore, formula (3.0.6) can be applied with

$$c_{i_{\lfloor \ell/2 \rfloor - j + 1}} = b_{\lambda_j + \ell - j - \lfloor (N+1)/2 \rfloor}, \text{ for } 1 \leq j \leq \left\lfloor \frac{\ell}{2} \right\rfloor.$$

From (3.0.12) and (3.0.13), it follows that

$$\lambda_i + \lambda_{\ell+1-i} = k_{\ell+1-i} + k_i - \ell - 1 = N + 1 - \ell - 1 = N - \ell,$$

showing that

$$\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)},$$

establishing Proposition 3.2. ■

4 Proof of Theorem 1.1

Using the standard notation for the partition $1^j = \overbrace{(1, \dots, 1)}^j$, we state

Lemma 4.1 *For*

$$\left. \begin{aligned} &\mathbf{s}_i(-\tilde{\partial})\mathbf{s}_{1^j}(t) \\ &\left(-\frac{\partial}{\partial t_i}\right)\mathbf{s}_{1^j}(t) \end{aligned} \right\} = (-1)^i \mathbf{s}_{1^{j-i}}(t). \quad (4.0.1)$$

Proof: Using the usual inner-product between symmetric functions, we have

$$\begin{aligned}
\mathbf{s}_i(\tilde{\partial})\mathbf{s}_j(t) &= \langle \mathbf{s}_i(t+u) \cdot 1, \mathbf{s}_j(t+u) \rangle \\
&= \langle \mathbf{s}_j(t+u), \mathbf{s}_i(t+u) \cdot 1 \rangle \\
&= \langle \mathbf{s}_{j-i}(t+u), 1 \rangle \\
&= \langle 1, \mathbf{s}_{j-i}(t+u) \rangle \\
&= \mathbf{s}_{j-i}(t+u)|_{u=0} \\
&= \mathbf{s}_{j-i}(t)
\end{aligned}$$

and so, changing $t \mapsto -t$,

$$\mathbf{s}_i(-\tilde{\partial})\mathbf{s}_j(-t) = \mathbf{s}_{j-i}(-t),$$

from which this first relation follows upon noticing that

$$\mathbf{s}_j(-t) = (-1)^j \mathbf{s}_{1^j}(t). \quad (4.0.2)$$

This last relation (4.0.2) also leads to the second identity (4.0.1), using $(\partial/\partial t_i)\mathbf{s}_j(t) = \mathbf{s}_{j-i}(t)$. $\blacksquare$

Proof of Theorem 1.1: By Proposition 2.1, the equations for the $N \times N$ matrix m_N

$$\frac{\partial m_N}{\partial t_k} = \Lambda^k m_N + m_N \Lambda^k,$$

with skew-symmetric initial condition $m_N(0)$ has the following solution

$$m_N(t) = E_{\ell,N} m_N(0) E_{\ell,N}^\top(t),$$

which remains skew-symmetric in time. Define a t -dependent skew-inner product such that $\langle y^i, z^j \rangle_t = m_{ij}(t)$, i.e.⁹,

$$\langle \chi_N(y) \chi(z)^\top \rangle = m_N(t).$$

Performing the skew Borel decomposition

$$m_N(t) = Q^{-1}(t) J Q^{-1\top}, \quad \text{with } Q(t) \in G_k \quad (4.0.3)$$

⁹ $\chi(y) := (1, y, y^2, \dots)^\top$.

is tantamount to the process of finding a finite set of skew-orthonormal polynomials; that is, satisfying

$$\left(\langle q_i(t; z), q_j(t; z) \rangle \right)_{1 \leq i, j \leq N} = J.$$

Indeed, the polynomials $q_i(t; z)$ in z , depending on t ,

$$\begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-1} \end{pmatrix} = Q \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{N-1} \end{pmatrix}$$

satisfy

$$\begin{aligned} \left(\langle q_i(t; y), q_j(t; z) \rangle \right)_{0 \leq i, j \leq N-1} &= \langle Q(t) \chi_N(y), Q(t) \chi_N(z) \rangle \\ &= \langle Q(t) \chi_N(y) \chi_N(z) Q^\top(t) \rangle \\ &= Q(t) \langle \chi_N(y) \chi_N(z) \rangle Q^\top(t) \\ &= Q(t) m_N(t) Q^\top(t) \\ &= J. \end{aligned}$$

According to [2], the skew-orthogonal polynomials are related to the τ -functions ($\tau_0 = 1$, $\tau_N = c$)

$$\tau_\ell(t) = pf \ m_\ell(t)$$

as follows

$$\begin{aligned} q_{2n} &= \frac{z^{2n}}{\sqrt{\tau_{2n} \tau_{2n+2}}} \tau_{2n}(t - [z^{-1}]) \\ q_{2n+1} &= \frac{z^{2n}}{\sqrt{\tau_{2n} \tau_{2n+2}}} \left(z + \frac{\partial}{\partial t_1} \right) \tau_{2n}(t - [z^{-1}]), \quad 0 \leq 2n \leq N-2. \end{aligned}$$

This ends the proof of Theorem 1.1 for N even. However for odd N , we must verify condition (2.0.8) of Theorem 2.2. This requires knowing $q_{N-1}(t; z)$ explicitly. For later purposes we shall also need $q_{N-1}(t; z)$ for even N .

For N even, q_{N-1} takes on the following form

$$q_{N-1}(t; z) = \frac{z^{N-2}}{\sqrt{\tau_{N-2}\tau_N}} \left(z + \frac{\partial}{\partial t_1} \right) \tau_{N-2}(t - [z^{-1}]),$$

with (using Proposition 3.2)

$$\tau_{N-2}(t) = \sum_{\lambda \in \mathbb{Y}_{N-2}^{(N-2)}} \left(\prod_1^{\frac{N-2}{2}} b_{\lambda_i - i + \frac{N}{2} - 2} \right) \mathbf{s}_\lambda(t),$$

where

$$\mathbb{Y}_{N-2}^{(N-2)} = \{1^{N-2}, (2, 1^{N-4}), (2^2, 1^{N-6}), \dots, (2^i, 1^{N-2i-2}), \dots\}.$$

For N odd, q_{N-1} has the form

$$q_{N-1}(t; z) = \frac{z^{N-1}}{\sqrt{\tau_{N-1}}} \tau_{N-1}(t - [z^{-1}])$$

with

$$\tau_{N-1}(t) = b_0 \dots b_{\frac{N-3}{2}} \mathbf{s}_{\left(1^{\frac{N-1}{2}}\right)}(t). \quad (4.0.4)$$

Indeed, observe that the set of partitions

$$\begin{aligned} \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^\ell \Big|_{\ell=N-1} &= \mathbb{Y}_{\frac{N-1}{2}}^{(N-1)} = \left\{ \begin{array}{l} (\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{Y}_{\frac{N-1}{2}} \\ \text{with } \lambda_i + \lambda_{\ell+1-i} = 1 \end{array} \right\} \\ &= \left\{ 1^{\frac{N-1}{2}} \right\} \end{aligned}$$

consists of one element $1^{\frac{N-1}{2}}$. Therefore, setting $\lambda_i = 1$ for $1 \leq i \leq \frac{N-1}{2}$ one finds, again by Proposition 3.2,

$$\tau_{N-1}(t) = b_0 \dots b_{\frac{N-3}{2}} \mathbf{s}_{\left(1^{\frac{N-1}{2}}\right)}(t).$$

The last row of $\tilde{Q}$ is given by:

$$\begin{aligned}
\sum_0^{N-1} \tilde{Q}_{N,j+1} z^j &= z^{N-1} \tau_{N-1}(t - [z^{-1}]) \\
&= \sum_{i=0}^{N-1} \mathbf{s}_i(-\tilde{\partial}) \tau_{N-1}(t) z^{N-1-i} \\
&= b_0 \dots b_{\frac{N-3}{2}} \sum_{i=0}^{N-1} \mathbf{s}_i(-\tilde{\partial}) \mathbf{s}_{\left(1 \frac{N-1}{2}\right)}(t) z^{N-1-i} \\
&= b_0 \dots b_{\frac{N-3}{2}} \sum_{i=0}^{\frac{N-1}{2}} z^{N-1-i} (-1)^i \mathbf{s}_{\left(1 \frac{N-1}{2}-i\right)}(t),
\end{aligned}$$

using Lemma 4.1, and so

$$\tilde{Q}_{N,N-i} = (-1)^i \left(\prod_0^{\frac{N-3}{2}} b_k \right) \mathbf{s}_{\left(1 \frac{N-1}{2}-i\right)}.$$

So, the last row of $\tilde{Q}$ reads

$$\prod_0^{\frac{N-3}{2}} b_i \left(\underbrace{0, \dots, 0}_{\frac{N-1}{2}}, (-1)^{\frac{N-1}{2}}, (-1)^{\frac{N-3}{2}} \mathbf{s}_1(t), (-1)^{\frac{N-5}{2}} \mathbf{s}_{(1^2)}(t), \dots, \mathbf{s}_{\left(1 \frac{N-1}{2}\right)}(t) \right)$$

and and the last row of $Q = D\tilde{Q}$:

$$\begin{aligned}
Q_{N,N-i} = (D\tilde{Q})_{N,N-i} &= (-1)^i \prod_0^{\frac{N-3}{2}} b_k \frac{\mathbf{s}_{\left(1 \frac{N-1}{2}-i\right)}(t)}{\sqrt{\tau_{N-1}}} \\
&= (-1)^i \left(\prod_0^{\frac{N-3}{2}} b_k \right)^{1/2} \frac{\mathbf{s}_{\left(1 \frac{N-1}{2}-i\right)}(t)}{\left(\mathbf{s}_{\left(1 \frac{N-1}{2}\right)}(t) \right)^{1/2}}
\end{aligned}$$

and so, using Lemma 4.1,

$$\frac{\partial Q_{N,N}}{\partial t_i} = -\frac{(-1)^i}{2} \left(\prod_0^{N-3} b_k \right)^{1/2} \frac{\mathbf{s} \left(\begin{smallmatrix} N-1-i \\ 1 \end{smallmatrix} \right) (t)}{\left(\mathbf{s} \left(\begin{smallmatrix} N-1 \\ 1 \end{smallmatrix} \right) (t) \right)^{1/2}} = -\frac{1}{2} Q_{N,N-i}.$$

Having checked (2.0.6), (2.0.7) and (2.0.8) (in the odd case) of Theorem 2.2, we have found a solution of the Pfaff lattice. This finally concludes the proof of Theorem 1.1. $\blacksquare$

Proof of Theorem 1.2: According to [2], Pfaff τ -functions satisfy bilinear relations¹⁰: for all $t, t' \in \mathbb{C}^\infty$ and m, n positive integers

$$\oint_{z=\infty} \tau_{2n}(t - [z^{-1}]) \tau_{2m+2}(t' + [z^{-1}]) e^{\sum_{i=0}^{\infty} (t_i - t'_i) z^i} z^{2n-2m-2} dz \\ + \oint_{z=0} \tau_{2n+2}(t + [z]) \tau_{2m}(t' - [z]) e^{\sum_{i=0}^{\infty} (t'_i - t_i) z^{-i}} z^{2n-2m} dz = 0,$$

Shifting appropriately and taking residues leads to the “differential Fay identity”:

$$\{\tau_{2n}(t - [u]), \tau_{2n}(t - [v])\} \\ + (u^{-1} - v^{-1})(\tau_{2n}(t - [u])\tau_{2n}(t - [v]) - \tau_{2n}(t)\tau_{2n}(t - [u] - [v])) \\ = uv(u - v)\tau_{2n-2}(t - [u] - [v])\tau_{2n+2}(t), \quad (4.0.5)$$

and Hirota bilinear equations, involving nearest neighbors:

$$\left(\mathbf{s}_{k+4}(\tilde{\partial}) - \frac{1}{2} \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{k+3}} \right) \tau_{2n} \cdot \tau_{2n} = \mathbf{s}_k(\tilde{\partial}) \tau_{2n+2} \cdot \tau_{2n-2}. \quad (4.0.6)$$

It only remains to check the “boundary condition”:

$$\begin{cases} \tau_N = \prod_0^{\frac{N-2}{2}} b_i, & \text{for even } N, \\ \tau_{N+1} = 0, & \text{for odd } N. \end{cases} \quad (4.0.7)$$

¹⁰ $\tilde{\partial} = (\partial/\partial t_1, (1/2)\partial/\partial t_2, (1/3)\partial/\partial t_3, \dots)$, $\tilde{D} = (D_1, (1/2)D_2, (1/3)D_3, \dots)$ is the corresponding Hirota symbol: $P(\tilde{D})f \cdot g := P(\partial/\partial y_1, (1/2)\partial/\partial y_2, \dots) f(t+y)g(t-y)|_{y=0}$, and $\mathbf{s}_k$ are the previously defined elementary Schur functions: $\sum_{k=0}^{\infty} \mathbf{s}_k(t) z^k := \exp(\sum_{i=1}^{\infty} t_i z^i)$. For further notations, see Dickey [5].

Indeed, for even N , using $\det E_{NN}(t) = 1$ and the matrix (3.0.2), we have that

$$(pf \ m_N(t))^2 = \det (E_{N,N}(t)m_N(0)E_{N,N}^\top(t)) = \det m_N(0) = \prod_0^{\frac{N-2}{2}} b_i.$$

Moreover, for odd N , according to (4.0.4), τ_{N-1} is a pure Schur polynomial, which is known to satisfy the KP Fay identity; i.e., the equation (4.0.5), without right hand side. This justifies setting $\tau_{N+1} = 0$ for odd N . ■

In the next Proposition, we show that the finite vector of skew-orthogonal polynomials form an eigenvector of the matrix L , with modified boundary condition.

Proposition 4.2 *For even N , the skew-orthonormal polynomials $q = (q_0, \dots, q_{N-1})^\top = Q \ (1, \dots, z^{N-1})^\top$ are eigenfunctions for L , with the boundary condition:*

$$Lq = zq - (0, \dots, 0, z^N) \sqrt{pf \ m_{N-2}} \left(\prod_0^{\frac{N-2}{2}} b_i \right)^{-1/2}.$$

Proof: Indeed

$$\begin{aligned} Lq &= Q \wedge Q^{-1} Q \begin{pmatrix} 1 \\ \vdots \\ z^{N-1} \end{pmatrix} \\ &= Q \wedge \begin{pmatrix} 1 \\ \vdots \\ z^{N-1} \end{pmatrix} \\ &= Q \begin{pmatrix} 1 \\ \vdots \\ z^{N-2} \\ 0 \end{pmatrix} \\ &= z \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-2} \\ \bar{q}_{N-1} \end{pmatrix} = zq + z(0, \dots, 0, \bar{q}_{N-1} - q_{N-1}), \end{aligned}$$

where $\bar{q}_{N-1}$ is the same as q_{N-1} , but without leading term, i.e., $\bar{q}_{N-1} = q_{N-1} - Q_{NN}z^{N-1}$, where by (4.0.7) we have

$$Q_{NN} = \sqrt{\frac{\tau_{N-2}}{\tau_N}} = \sqrt{pf \, m_{N-2}} \left(\prod_0^{\frac{N-2}{2}} b_i \right)^{-1/2},$$

ending the proof of Proposition 4.2. ■

5 Vertex operators

The purpose of this section is to prove Theorem 1.3 and Corollary 1.4. Define as in (1.0.23),

$$\beta := \frac{N}{2} - \ell + 1. \quad (5.0.1)$$

Remembering from (1.0.21) the vertex operator $X(t; z)$, consider now its formal expansion in powers of z

$$X(t; z) = e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}} =: \sum_{i \in \mathbb{Z}} B_i z^i, \quad (5.0.2)$$

with differential operators (see footnote 10)

$$B_i := B_i^{(\alpha)} \Big|_{\alpha=1} \quad \text{and} \quad B_i^{(\alpha)} := \sum_{j \geq 0} \mathbf{s}_{i+j}(\alpha t) \mathbf{s}_j(-\alpha \tilde{\partial}_t). \quad (5.0.3)$$

Also define as in (1.0.22) the vector vertex operator¹¹

$$\mathbb{X}(t; z) = \Lambda^\top e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}} \chi(z). \quad (5.0.4)$$

Also remember the definitions of the *integrated vertex operator*, in terms of the vertex operator (5.0.2) and a function ρ_b , defined in (5.0.8) below,

$$Y_\beta(t) := \frac{1}{(2\pi i)^2} \oint \oint_{\infty} X(t; y) X(t; z) \frac{\rho_b(y/z) dy \, dz}{z^2 (yz)^\beta}$$

¹¹ $\chi(z) := (z^i)_{i \geq 0}$.

and the *integrated vector vertex operator*, in terms of (5.0.4),

$$\mathbb{Y}_N(t) = \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \mathbb{X}(t; y) \mathbb{X}(t; z) \frac{\rho_b(y/z) dy dz}{2(yz)^{N/2} z}. \quad (5.0.5)$$

In both cases, the double integral around two contours about ∞ amounts to computing the coefficient of $1/yz$. The next Theorem is nothing but a rephrasing of Theorem 1.3 and Corollary 1.4.

Theorem 5.1 *For a given set of b_i , the sequence of τ -functions $\tau_0, \tau_2, \tau_4, \dots$, defined in (1.0.15), is generated by the vertex operators Y_β :*

$$Y_\beta \tau_{\ell-2} = \ell \tau_\ell. \quad (5.0.6)$$

The vector $I = (I_0, I_2, I_4, \dots)$, with $I_\ell = \left(\frac{\ell}{2}\right)! \tau_\ell$ is a fixed point for the vector vertex operator $\mathbb{Y}_N$, namely

$$(\mathbb{Y}_N I)_\ell = I_\ell, \text{ for even } \ell. \quad (5.0.7)$$

We shall first need a few propositions.

Proposition 5.2 *Defining*

$$\rho_b(x) := \begin{cases} \rho_b^{(e)}(x) := \sum_{i \geq 0} b_i (x^{-i-1} - x^i), & \text{for } N \text{ even,} \\ \rho_b^{(o)}(x) := x^{-1/2} \sum_{i \geq 0} b_i (x^{-i-1} - x^{i+1}), & \text{for } N \text{ odd,} \end{cases} \quad (5.0.8)$$

we have

$$\begin{aligned} Y_\beta(t) &= \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} X(t; y) X(t; z) \frac{\rho_b(y/z) dy dz}{z^2 (yz)^\beta} \\ &= \begin{cases} \sum_{j \geq 0} b_j (B_{\beta+j} B_{\beta-j} - B_{\beta-j-1} B_{\beta+j+1}), & \text{for } N \text{ even} \\ \sum_{j \geq 0} b_j (B_{\beta+j+1/2} B_{\beta-j-1/2} - B_{\beta-j-3/2} B_{\beta+j+3/2}), & \text{for } N \text{ odd.} \end{cases} \end{aligned} \quad (5.0.9)$$

Proof: Compute for N even,

$$\begin{aligned}
\frac{X(t; y)X(t; z)}{(yz)^\beta} &= \sum_{i \in \mathbb{Z}} B_i y^{i-\beta} \sum_{j \in \mathbb{Z}} B_j z^{j-\beta} \\
&= \sum_{i \in \mathbb{Z}} B_{\beta+i} y^i \cdot \sum_{j \in \mathbb{Z}} B_{\beta-j} z^{-j} \\
&= \sum_{i, j \in \mathbb{Z}} B_{\beta+i} B_{\beta-j} \frac{y^i}{z^j} \\
&= \sum_{j \in \mathbb{Z}} B_{\beta+j} B_{\beta-j} \left(\frac{y}{z}\right)^j + \sum_{i \neq j \in \mathbb{Z}} a_{ij} \frac{y^i}{z^j}
\end{aligned}$$

and so,

$$\begin{aligned}
\rho_b^{(e)} \left(\frac{y}{z}\right) \cdot \frac{X(t; y)X(t; z)}{z^2(yz)^\beta} &= \frac{1}{z^2} \left(\sum_{i \geq 0} b_i \left[\left(\frac{y}{z}\right)^{-(i+1)} - \left(\frac{y}{z}\right)^i \right] \right) \cdot \left(\sum_{j \in \mathbb{Z}} B_{\beta+j} B_{\beta-j} \left(\frac{y}{z}\right)^j + \sum_{i \neq j \in \mathbb{Z}} a_{ij} \frac{y^i}{z^j} \right) \\
&= \frac{1}{yz} \left(\sum_{j \geq 0} b_j (B_{\beta+j} B_{\beta-j} - B_{\beta-j-1} B_{\beta+j+1}) \right) + \sum_{i \text{ or } j \neq 0} c_{ij} y^{i-1} z^{j-1}
\end{aligned}$$

and therefore, upon taking the double residue,

$$\oint \oint_{\infty \infty} \frac{\rho_b^{(e)}(y/z) X(t; y) X(t; z)}{z^2(yz)^\beta} \frac{dy dz}{(2\pi i)^2} = \sum_{j \geq 0} b_j (B_{\beta+j} B_{\beta-j} - B_{\beta-j-1} B_{\beta+j+1}).$$

For N odd,

$$\frac{X(t; y)X(t; z)}{(yz)^\beta (y/z)^{1/2}} = \sum_{j \in \mathbb{Z}} B_{\beta+\frac{1}{2}+j} B_{\beta-\frac{1}{2}-j} \left(\frac{y}{z}\right)^j + \sum_{i \neq j \in \mathbb{Z}} a_{ij} \frac{y^i}{z^j}$$

and so

$$\begin{aligned}
\rho_b^{(o)} \left(\frac{y}{z}\right) \frac{X(t; y)X(t; z)}{z^2(yz)^\beta} &= \frac{1}{yz} \sum_{j \geq 0} b_j \left(B_{\beta+j+\frac{1}{2}} B_{\beta-j-\frac{1}{2}} - B_{\beta-j-\frac{3}{2}} B_{\beta+j+\frac{3}{2}} \right) + \sum_{i \text{ or } j \neq 0} c_{ij} y^{i-1} z^{j-1}
\end{aligned}$$

and therefore

$$\oint_{\infty} \oint_{\infty} \frac{\rho_b^{(o)}(y/z) X(t; y) X(t; z)}{z^2 (yz)^\beta} \frac{dy dz}{(2\pi i)^2} = \sum_{j \geq 0} b_j \left(B_{\beta+j+\frac{1}{2}} B_{\beta-j-\frac{1}{2}} - B_{\beta-j-\frac{3}{2}} B_{\beta+j+\frac{3}{2}} \right),$$

ending the proof of Proposition 5.2. $\blacksquare$

Defining the set

$$\mathbb{S}_N^{(\ell)} := \left\{ \begin{array}{l} \sigma_1 > \sigma_2 > \dots > \sigma_{\ell/2}, \quad \sigma_i \in \mathbb{Z} \\ \frac{\ell}{2} \leq \sigma_i + i \leq \left\lfloor \frac{N}{2} \right\rfloor \end{array} \right\}, \quad (5.0.10)$$

the map

$$\sigma : \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)} \longrightarrow \mathbb{S}_N^{(\ell)} : \lambda \longmapsto \sigma(\lambda) = \left(\lambda_i - i + \ell - \left\lfloor \frac{N+1}{2} \right\rfloor \right)_{1 \leq i \leq n/2}. \quad (5.0.11)$$

is a bijection.

Indeed, $\lambda_1 \geq \lambda_2 \geq \dots$ implies at once the strict inequalities $\sigma_1 > \sigma_2 > \dots$ and also implies, together with the fact that for $\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}$ and $1 \leq i \leq \ell/2$,

$$2\lambda_i \geq \lambda_i + \lambda_{\ell+1-i} = N - \ell \quad \text{and, clearly} \quad \lambda_i \leq N - \ell.$$

Conversely, every $\sigma \in \mathbb{S}_N^{(\ell)}$ comes from a $\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}$.

Lemma 5.3 *For a given partition*

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell-2}) \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)},$$

and $j \geq 0$, the following holds

$$B_{\beta+j} B_{\beta-j} \mathbf{s}_\lambda = -B_{\beta-j-1} B_{\beta+j+1} \mathbf{s}_\lambda = \begin{cases} 0, & \text{if } \beta + j = \text{some } \lambda_\nu - \nu - 1 \\ & \text{for } 1 \leq \nu \leq \ell/2 - 1, \\ & \text{or if } j \geq N/2 \\ \mathbf{s}_{\lambda'}, & \text{if } \beta + j \neq \text{every } \lambda_\nu - \nu - 1 \\ & \text{for } 1 \leq \nu \leq \ell/2 - 1, \end{cases} \quad (5.0.12)$$

where

$$\begin{aligned}
\lambda' = \left(\lambda_1 - 2 \geq \dots \geq \lambda_\nu - 2 \geq \beta + j + \nu \geq \lambda_{\nu+1} - 1 \geq \dots \geq \lambda_{\frac{\ell}{2}-1} - 1 \right. \\
\geq \lambda_{\frac{\ell}{2}} - 1 \geq \dots \geq \lambda_{\ell-2-\nu} - 1 \geq (N - \ell) - (\beta + j + \nu) \\
\left. \geq \lambda_{\ell-1-\nu} \geq \dots \geq \lambda_{\ell-2} \right) \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}. \tag{5.0.13}
\end{aligned}$$

Moreover for j 's such that $\beta + j \neq$ every $\lambda_\nu - \nu - 1$, the maps $B_{\beta+j}B_{\beta-j}$ induce maps

$$B_{\beta+j}B_{\beta-j} : \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)} \longrightarrow \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)} : \lambda \longmapsto \lambda' \tag{5.0.14}$$

having, as a whole, a “surjectivity property”, meaning that to each $\lambda' \in \mathbb{Y}_{(\ell-2)(N-\ell)/2}^{(\ell-2)}$, there are $\ell/2$ choices of $j \geq 0$ and $\lambda \in \mathbb{Y}_{(\ell-2)(N-\ell+2)/2}^{(\ell-2)}$ mapping to λ' , by means of the map $B_{\beta+j}B_{\beta-j}$, as in (5.0.12).

At the level of the $\mathbb{S}$ -spaces, the maps $B_{\beta+j}B_{\beta-j}$ induce maps

$$\mathbb{S}_N^{(\ell-2)} \longrightarrow \mathbb{S}_N^{(\ell)} : \sigma = (\sigma_1, \dots, \sigma_{\frac{\ell-2}{2}}) \longmapsto \sigma' = (\sigma_1, \dots, \sigma_\nu, j, \sigma_{\nu+1}, \dots, \sigma_{\frac{\ell-2}{2}}), \tag{5.0.15}$$

having the same “surjectivity property” as above.

For N odd, all formulae above remain the same, except for the substitution $j \mapsto j + \frac{1}{2}$ in (5.0.12) and (5.0.13).

Proof: Extending a classic identity (see MacDonald [8]) to arbitrary sequences $(\lambda_1, \dots, \lambda_n)$, we have

$$B_{\lambda_1}, \dots, B_{\lambda_n}(1) = (\lambda_1, \dots, \lambda_n) := \det(\mathbf{s}_{\lambda_i+j-i}(t))_{1 \leq i, j \leq n}$$

and, in particular, for a partition $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$, we have, for an arbitrary choice of $j \geq 0$,

$$\begin{aligned}
B_{\beta+j}B_{\beta-j}\mathbf{s}_{(\lambda_1, \dots, \lambda_{\ell-2})} &= \mathbf{s}_{(\beta+j, \beta-j, \lambda_1, \dots, \lambda_{\ell-2})} \\
&= \det \begin{pmatrix} \mathbf{s}_{\beta+j} & \mathbf{s}_{\beta+j+1} & \mathbf{s}_{\beta+j+2} & \dots & \mathbf{s}_{\beta+j+\ell-1} \\ \mathbf{s}_{\beta-j-1} & \mathbf{s}_{\beta-j} & \mathbf{s}_{\beta-j+1} & \dots & \mathbf{s}_{\beta-j+\ell-2} \\ \mathbf{s}_{\lambda_1-2} & \mathbf{s}_{\lambda_1-1} & \mathbf{s}_{\lambda_1} & \dots & \mathbf{s}_{\lambda_1+\ell-3} \\ \vdots & & & \ddots & \vdots \\ \mathbf{s}_{\lambda_{\ell-2}-\ell+1} & \dots & \dots & \dots & \mathbf{s}_{\lambda_{\ell-2}} \end{pmatrix}. \tag{5.0.16}
\end{aligned}$$

Using the value (5.0.1) of β , it is immediately clear, from the matrix (5.0.16), that for $j \geq N/2$, the second row of the matrix (5.0.16) vanishes and therefore the determinant. Therefore we assume $0 \leq j \leq \frac{N}{2} - 1$. We give the proof for even N ; for odd N , it is identical with $j \mapsto j + 1/2$.

The first column of the matrix above involves the indices

$$\frac{N}{2} - \ell + 1 + j, \frac{N}{2} - \ell - j, \lambda_1 - 2, \lambda_2 - 3, \dots, \lambda_{\frac{\ell}{2}} - \frac{\ell}{2} - 1, \dots, \lambda_{\ell-2} - \ell + 1. \quad (5.0.17)$$

Consider now an arbitrary integer $j \geq 0$ and an arbitrary partition

$$\lambda \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)};$$

it has the property that

$$\lambda_i + \lambda_{\ell-1-i} = N - \ell + 2 \text{ for } 1 \leq i \leq \frac{\ell-2}{2}.$$

Hence, for $i = \frac{\ell-2}{2}$

$$2\lambda_{\frac{\ell}{2}} \leq \lambda_{\frac{\ell}{2}-1} + \lambda_{\frac{\ell}{2}} = N - \ell + 2,$$

and so

$$\lambda_{\ell/2} \leq \frac{N - \ell + 2}{2};$$

thus, for the arbitrary $j \geq 0$ chosen above

$$\lambda_{\ell/2} - \ell/2 - 1 \leq \frac{N}{2} - \ell < \frac{N}{2} - \ell + j + 1.$$

The partition $\lambda_1 \geq \lambda_2 \geq \dots$ implies the strict inequalities

$$\lambda_1 - 1 - 1 > \lambda_2 - 2 - 1 > \lambda_3 - 3 - 1 > \dots > \lambda_{\nu+1} - (\nu+1) - 1 > \dots > \lambda_{\ell/2} - \ell/2 - 1$$

and therefore, there exist $0 \leq \nu \leq \frac{\ell}{2} - 1$ such that

$$\lambda_{\nu} - \nu - 1 \geq \frac{N}{2} - \ell + j + 1 \geq \lambda_{\nu+1} - \nu - 2.$$

These inequalities together with the fact that

$$\lambda_{\nu} + \lambda_{\ell-1-\nu} = N - \ell + 2, \quad \lambda_{\nu+1} + \lambda_{\ell-2-\nu} = N - \ell + 2$$

also imply

$$\lambda_{\ell-2-\nu} - (\ell - 1 - \nu) \geq \frac{N}{2} - \ell - j \geq \lambda_{\ell-1-\nu} - (\ell - \nu).$$

Therefore, the indices (5.0.17) of the first column of the matrix (5.0.16) are now rearranged by order, as follows:

$$\begin{aligned} \lambda_1 - 2 &> \lambda_2 - 3 > \dots > \lambda_\nu - \nu - 1 \geq \frac{N}{2} - \ell + 1 + j \geq \lambda_{\nu+1} - \nu - 2 \\ &> \dots > \lambda_{\frac{\ell}{2}-1} - \frac{\ell}{2} > \lambda_{\frac{\ell}{2}} - \frac{\ell}{2} - 1 > \dots > \lambda_{\ell-2-\nu} - (\ell - 1 - \nu) \\ &\geq \frac{N}{2} - \ell - j \geq \lambda_{\ell-1-\nu} - (\ell - \nu) > \dots > \lambda_{\ell-2} - \ell + 1. \end{aligned} \quad (5.0.18)$$

Notice that the determinant (5.0.16) vanishes if any of the equalities hold in (5.0.18) above. Therefore we may assume strict inequalities. Upon rearranging the rows of the matrix (5.0.16) according to the order in (5.0.18), we now list the corresponding partitions by looking at the indices on the diagonal. This amounts to adding $i - 1$ to the i^{th} entry of (5.0.18), thus leading to

$$\begin{aligned} \lambda' = & \left(\begin{array}{ccccccc} \lambda_1 - 2 \geq & \lambda_2 - 2 & \geq \dots \geq & \lambda_\nu - 2 \geq & \frac{N}{2} - \ell + 1 + j + \nu \geq & \lambda_{\nu+1} - 1 & \geq \dots \geq & \lambda_{\frac{\ell}{2}-1} - 1 \\ \uparrow & \uparrow & & \uparrow & \nu \uparrow + 1 & \nu \uparrow + 2 & & \ell/2 \uparrow \end{array} \right. \\ & \geq \left. \begin{array}{ccccccc} \lambda_{\frac{\ell}{2}} - 1 & \geq \dots \geq & \lambda_{\ell-2-\nu} - 1 \geq & \frac{N}{2} - j - \nu - 1 \geq & \lambda_{\ell-1-\nu} & \geq \dots \geq & \lambda_{\ell-2} \\ \ell/2 \uparrow + 1 & & \ell - 2 - \nu \uparrow + 1 & \ell - 2 - \nu \uparrow + 2 & \ell - 2 - \nu \uparrow + 3 & & \ell \uparrow \end{array} \right) \end{aligned} \quad (5.0.19)$$

The rearrangement does not change the sign of the determinant (5.0.16). Knowing that $\lambda \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)}$, we now prove that the new partition λ' (obtained in (5.0.19)),

$$\lambda' \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)};$$

i.e., we prove

$$\begin{aligned}
(i) \quad \sum_i^\ell \lambda'_i &= \sum_1^{\ell-2} \lambda_i - 2\nu - (\ell - 2 - 2\nu) + \left(\frac{N}{2} - \ell + 1 + j + \nu\right) \\
&\quad + \left(\frac{N}{2} - j - \nu - 1\right) \\
&= \frac{\ell(N - \ell)}{2},
\end{aligned}$$

and

$$(ii) \quad \lambda'_i + \lambda'_{\ell+1-i} = N - \ell \quad \text{for all } 1 \leq i \leq \frac{\ell}{2};$$

e.g.,

$$\lambda'_i + \lambda'_{\ell+1-i} = \lambda_i - 2 + \lambda_{\ell-1-i} = N - \ell \quad \text{for } 1 \leq i \leq \nu$$

$$\lambda'_{\nu+1} + \lambda'_{\ell-\nu} = \left(\frac{N}{2} - \ell + 1 + j + \nu\right) + \left(\frac{N}{2} - j - \nu - 1\right) = N - \ell$$

$$\lambda'_i + \lambda'_{\ell+1-i} = \lambda_{i-1} - 1 + \lambda_{\ell-i} - 1 = N - \ell \quad \text{for } \nu + 2 \leq i \leq \ell/2.$$

So far, we have shown that to an arbitrary integer $j \geq 0$ and a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell-2}) \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)},$$

such that the inequalities in (5.0.19) are strict, corresponds a new partition

$$\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_\ell) \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)},$$

with λ' totally determined by (5.0.19). Then $\ell/2$ different choices of $\lambda \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell+2)}{2}}^{(\ell-2)}$ and $j \geq 0$ will lead to the same sequence of numbers (5.0.19), as appears from the next argument.

In view of the σ -map in (5.0.11), it is obvious to see that the $\nu + 1$ -st number in λ' of (5.0.19) gets mapped by σ into j , namely

$$\frac{N}{2} - \ell + 1 + j + \nu \mapsto j,$$

and, in general, (5.0.15) holds. The “surjectivity property” is straightforward in this description, since given a sequence $\sigma' \in \mathbb{S}_N^{(\ell)}$, you may choose j to be any of the $\ell/2$ numbers appearing in σ' ; then σ is the sequence formed by the remaining numbers in order. This establishes Lemma 5.3. $\blacksquare$

Proposition 5.4 *Given positive integers N and ℓ with even ℓ and the operator*

$$Y_\beta = \begin{cases} \sum_{j \geq 0} b_j (B_{\beta+j} B_{\beta-j} - B_{\beta-j-1} B_{\beta+j+1}), & N \text{ even}, \\ \sum_{j \geq 0} b_j \left(B_{\beta+j+\frac{1}{2}} B_{\beta-j-\frac{1}{2}} - B_{\beta-j-\frac{3}{2}} B_{\beta+j+\frac{3}{2}} \right), & N \text{ odd}, \end{cases}$$

we have

$$Y_{\frac{N}{2}-\ell+1} \tau_{\ell-2} = \ell \tau_\ell.$$

Proof: The indices of the b_i in the ℓ th tau-function can now be expressed in terms of the σ -map, as follows

$$\tau_\ell(t) = \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{\ell/2} b_{\lambda_i - i + \ell - \lfloor \frac{N+1}{2} \rfloor} \right) \mathbf{s}_\lambda(t) = \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{\ell/2} b_{\sigma_i(\lambda)} \right) \mathbf{s}_\lambda(t).$$

We give the proof for even N . From (5.0.15), it follows at once that

$$b_j \prod_1^{(\ell-2)/2} b_{\sigma_i(\lambda)} = \prod_1^{\ell/2} b_{\sigma_i(\lambda')}. \quad (5.0.20)$$

Setting $Y_\beta = \sum_{i \geq 0} b_i \Gamma_i$, one computes, using Lemma 5.3, (5.0.20) and in $\stackrel{*}{=}$ the $\ell/2$ -to-1 “surjectivity” of the maps (5.0.14) or (5.0.15):

$$\begin{aligned} Y_{\frac{N}{2}-\ell+1} \tau_{\ell-2}(t) &= \sum_{\lambda \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell-2)}{2}}^{(\ell-2)}} \left(\prod_1^{\frac{\ell-2}{2}} b_{\sigma_i(\lambda)} \right) Y_\beta(s_\lambda(t)) \\ &= \sum_{\lambda \in \mathbb{Y}_{\frac{(\ell-2)(N-\ell-2)}{2}}^{(\ell-2)}} \sum_{j \geq 0} \left(\prod_1^{\frac{\ell-2}{2}} b_{\sigma_i(\lambda)} \right) b_j \Gamma_j(s_\lambda(t)) \\ &\stackrel{*}{=} \frac{\ell}{2} \sum_{\lambda' \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \prod_1^{\ell/2} b_{\sigma_i(\lambda')} 2s_{\lambda'}(t) \\ &= \ell \tau_\ell(t), \end{aligned}$$

ending the proof of Proposition 5.4. ■

Proof of Theorem 5.1: Formula (5.0.6) follows at once from Propositions 5.2 and 5.4. To prove (5.0.7), first notice that, upon setting $I_\ell := \left(\frac{\ell}{2}\right)! \tau_\ell$,

$$(\mathbb{X}(t; y)\mathbb{X}(t; z)I)_\ell = y^{\ell-1}z^{\ell-2}X(t; y)X(t; z)I_{\ell-2}.$$

Then

$$\begin{aligned} (\mathbb{Y}(t)I)_\ell &= \left(\frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \mathbb{X}(t; y)\mathbb{X}(t; z) \frac{\rho_b(y/z) dy dz}{2z(yz)^{N/2}} I \right)_\ell \\ &= \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \frac{dy dz}{2z^2} \frac{\rho_b(y/z)}{(yz)^{N/2-\ell+1}} X(t; y)X(t; z)I_{\ell-2} \\ &= \frac{1}{2} Y_{\frac{N}{2}-\ell+1} I_{\ell-2}, \quad \text{by definition (5.0.5) of } Y_\beta, \\ &= \frac{1}{2} Y_{\frac{N}{2}-\ell+1} \left(\frac{\ell-2}{2} \right)! \tau_{\ell-2} \\ &= \left(\frac{\ell}{2} \right)! \tau_\ell, \quad \text{using (5.0.6)} \\ &= I_\ell, \end{aligned}$$

ending the proof of Theorem 5.1. ■

Example: For $b_i = 2i + 1$ and even N , the function $\rho_b(x)$, defined in (5.0.8), equals¹²

$$\rho_b(x) = \sum_{i \geq 0} b_i (x^{-i-1} - x^i) = -\frac{1+x}{(1-x)^2} + x^{-1} \frac{1+x^{-1}}{(1-x^{-1})^2}. \quad (5.0.21)$$

The corresponding vertex operator (5.0.9) takes on a particularly simple form:

$$Y_{\frac{N}{2}-\ell+1} = 2B_{N-2\ell+2}^{(2)} = 2 \int_{\mathbb{R}} du \delta^{(N-2)}(u) u^{2\ell-4} X^{(2)}(u), \quad (5.0.22)$$

¹² $\rho_b(x)$ is actually a distribution!

where $\delta^{(N-2)}$ is the $(N-2)$ nd derivative of the customary δ -function and where the $B_i^{(2)}$ are the differential operators (5.0.3) in the t_i ,

$$B_i^{(2)} := \sum_{j \geq 0} \mathbf{s}_{i+j}(2t) \mathbf{s}_j(-2\tilde{\partial}_t),$$

given by the coefficients of the expansion in powers of z of the vertex operator

$$X^{(2)}(z) := e^{2 \sum_1^\infty t_i z^i} e^{-2 \sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}} = \sum_{i \in \mathbb{Z}} B_i^{(2)} z^i.$$

Proof: Formula (5.0.21) follows immediately from the series

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \dots$$

Setting, for convenience,

$$X(t; y, z) := e^{\sum_1^\infty t_i (y^i + z^i)} e^{-\sum_1^\infty \left(\frac{y^{-i} + z^{-i}}{i} \right) \frac{\partial}{\partial t_i}}$$

and using $X(t; y)X(t; z) = \left(1 - \frac{z}{y}\right)X(t; y, z)$ and $X(t; z, z) = X^{(2)}(t; z)$, one

computes $(\beta = \frac{N}{2} - \ell + 1)$

$$\begin{aligned}
Y_\beta &= \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \frac{\rho^e(y/z)}{(yz)^\beta z^2} X(t; y) X(t; z) dy \, dz \\
&= \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \left(\frac{z(1+z/y)}{y(1-z/y)^2} - \frac{(1+y/z)}{(1-y/z)^2} \right) \frac{(1-z/y)}{z^2 (zy)^\beta} X(t; y, z) dy \, dz \\
&= \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} \frac{z}{y} \frac{(1+z/y)}{(1-z/y)^2} \frac{(1-z/y)}{z^2 (zy)^\beta} X(t; y, z) dy \, dz \\
&= \oint_{\infty} \left(\oint_{\infty} \frac{(1+z/y)}{(y-z)z(zy)^\beta} X(t; y, z) \frac{dy}{2\pi i} \right) \frac{dz}{2\pi i} \\
&= 2 \oint_{\infty} \frac{X(t; z, z)}{z^{2\beta+1}} \frac{dz}{2\pi i} \\
&= 2 \oint_{\infty} \frac{X^{(2)}(t; z)}{z^{2\beta+1}} \frac{dz}{2\pi i} \\
&= 2B_{2\beta}^{(2)} = 2B_{N-2\ell+2}^{(2)} = 2 \int_{\mathbb{R}} du \, \delta^{(N-2)}(u) u^{2\ell-4} X^{(2)}(t; u),
\end{aligned}$$

establishing (5.0.22). ■

6 Duality

Proposition 6.1 *For odd N and odd ℓ , the following holds:*

$$\begin{aligned}
\tilde{\tau}_\ell(t) &:= z^{-1} \det^{1/2} \left(E_{\ell, N}(t) \left(m_N(0) + z^2 \varepsilon_{\frac{N+1}{2}, \frac{N+1}{2}} \right) E_{\ell, N}^\top(t) \right) \\
&= \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [\frac{N+1}{2}]} \right) \mathbf{s}_{\lambda_1 \geq \dots \geq \lambda_\ell}(t). \tag{6.0.1}
\end{aligned}$$

Then these functions

$$\tilde{\tau}_\ell(t) = (-1)^{\ell(N-\ell)/2} \left(\prod_0^{\frac{N-3}{2}} b_i \right) \left(\tau_{N-\ell}(-t)|_{b_i \rightarrow b_i^{-1}} \right), \quad \text{for } \ell \text{ odd.} \quad (6.0.2)$$

are the τ -functions $\tau_k(t)$ (in reverse order and modulo a multiplicative factor) of the Pfaff lattice for odd N and even k , with $t \mapsto -t$, and with initial condition

$$\begin{pmatrix} & & O & & b_{\frac{N-3}{2}}^{-1} \\ & & & & / \\ & & & b_0^{-1} & \\ & & 0 & & \\ & -b_0^{-1} & & & \\ & / & & & \\ -b_{\frac{N-3}{2}}^{-1} & & & O & \end{pmatrix}. \quad (6.0.3)$$

Proof: Defining k_i and $k_i^\top$ by

$$\lambda_i = k_i - \ell + i, \quad \lambda_i^\top = k_i^\top - (N - \ell) + i, \quad (6.0.4)$$

it is easy to see the one-to-one correspondence between

$$\mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)} \longleftrightarrow \left\{ \begin{array}{l} N-1 \geq k_1 > k_2 > \dots > k_\ell \geq 0 \\ \text{with } k_i + k_{\ell+1-i} = N-1 \text{ for } 1 \leq i \leq \frac{\ell+1}{2} \end{array} \right\}$$

and also between

$$\mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(N-\ell)} \longleftrightarrow \left\{ \begin{array}{l} N-1 \geq k_1^\top > k_2^\top > \dots > k_{N-\ell}^\top \geq 0 \\ \text{with } k_i^\top + k_{N-\ell+1-i}^\top = N-1 \text{ for } 1 \leq i \leq \frac{N-\ell}{2} \end{array} \right\}. \quad (6.0.5)$$

Lemma 6.2 (1) *The following correspondence holds*

$$\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)} \longleftrightarrow \lambda^\top \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(N-\ell)}, \quad (6.0.6)$$

(2) *For λ and $\lambda^\top$, we have the following disjoint union*

$$\{k_1 > \dots > k_\ell\} \cup \{k_1^\top > \dots > k_{N-\ell}^\top\} = \{0, 1, \dots, N-1\}. \quad (6.0.7)$$

Proof: Considering

$$(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell) \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^\ell,$$

we have

$$\begin{aligned} \lambda_1^\top &= \dots = \lambda_{\lambda_\ell}^\top = \ell \\ \lambda_{\lambda_\ell+1}^\top &= \dots = \lambda_{\lambda_{\ell-1}}^\top = \ell - 1 \\ \lambda_{\lambda_{\ell-1}+1}^\top &= \dots = \lambda_{\lambda_{\ell-2}}^\top = \ell - 2 \\ &\vdots \end{aligned} \tag{6.0.8}$$

and so, since

$$\begin{aligned} k_i = N - 1 - k_{\ell+1-i} &= N - 1 - (\lambda_{\ell+1-i} + \ell - (\ell + 1 - i)) \\ &= N - (\lambda_{\ell+1-i} + i) \end{aligned}$$

we have, on the one hand

$$k_1 = N - \lambda_\ell - 1 > k_2 = N - \lambda_{\ell-1} - 2 > k_3 = N - \lambda_{\ell-2} - 3, \tag{6.0.9}$$

and on the other hand, using (6.0.4) and (6.0.8),

$$\begin{aligned} k_1^\top &= N - 1 > k_2^\top = N - 2 > \dots > k_\alpha^\top = N - \alpha > \dots > k_{\lambda_\ell}^\top = N - \lambda_\ell > \\ k_{\lambda_\ell+1}^\top &= N - \lambda_\ell - 2 > \dots > k_\beta^\top = N - \beta - 1 > \dots > k_{\lambda_{\ell-1}}^\top = N - \lambda_{\ell-1} - 1 > \\ k_{\lambda_{\ell-1}+1}^\top &= N - \lambda_{\ell-1} - 3 > \dots > k_\gamma^\top = N - \gamma - 2 > \dots > k_{\lambda_{\ell-2}}^\top = N - \lambda_{\ell-2} - 2 > \dots \end{aligned} \tag{6.0.10}$$

So the gaps in (6.0.10) coincide with the sequence (6.0.9). This ends the proof of Lemma 6.2. ■

Proof of Proposition 6.1: One checks, using Proposition 3.2,

$$\begin{aligned}
\tilde{\tau}_\ell(t) &= \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \prod_1^{\frac{\ell-1}{2}} b_{\lambda_i - i + \ell - \frac{N+1}{2}} \mathbf{s}_\lambda(t) \\
&= (-1)^{|\lambda|} \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \prod_1^{\frac{\ell-1}{2}} b_{k_i - \frac{N+1}{2}} \mathbf{s}_{\lambda^\top}(-t) \\
&= (-1)^{|\lambda|} \prod_0^{\frac{N-3}{2}} b_i \sum_{\lambda \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(\ell)}} \frac{1}{\prod_1^{\frac{N-\ell}{2}} b_{k_i^\top - \frac{N+1}{2}}} \mathbf{s}_{\lambda^\top}(-t), \quad \text{using Lemma 6.2,} \\
&= (-1)^{|\lambda|} \prod_0^{\frac{N-3}{2}} b_i \sum_{\lambda^\top \in \mathbb{Y}_{\frac{\ell(N-\ell)}{2}}^{(N-\ell)}} \prod_1^{\frac{N-\ell}{2}} b_{\lambda_i^\top - i + \ell - \frac{N+1}{2}}^{-1} \mathbf{s}_{\lambda^\top}(-t) \\
&= (-1)^{\frac{\ell(N-\ell)}{2}} \prod_0^{\frac{N-3}{2}} b_i \left(\tau_{N-\ell}(-t) \Big|_{b_i \rightarrow b_i^{-1}} \right),
\end{aligned}$$

which is, using Theorem 1.1, the τ -function (modulo a constant) for the case where N is odd and $N - \ell$ even, concluding the proof of the Proposition. ■

7 Examples

Example 1: Rectangular Jack polynomials

Proposition 7.1 *When*

$$\begin{aligned}
b_i &= 2i + 1 \quad \text{for } N \text{ even} \\
&= 2i + 2 \quad \text{for } N \text{ odd,}
\end{aligned} \tag{7.0.1}$$

then the $\tau_{2n}(t)$'s are Jack polynomials for rectangular partitions, with $n \leq [N/2]$,

$$\begin{aligned}
\tau_{2n}(t) &= pf \ m_{2n}(t) \\
&= \sum_{\lambda \in \mathbb{Y}_{n(N-2n)}^{(2n)}} \prod_1^n (k_i - k_{2n+1-i}) \mathbf{s}_\lambda(t), \text{ where } k_i = \lambda_i - i + 2n \\
&= J_\lambda^{(1/2)}(x) \Big|_{t_i = \frac{1}{i} \sum_k x_k^i} \text{ for the partition } \lambda = (N-2n)^n \\
&= \frac{1}{n!} \int_{\mathbb{R}^n} \Delta(z)^4 \prod_{k=1}^n e^{2 \sum_1^\infty t_i z_k^i} \delta^{(N-2)}(z_k) dz_k. \tag{7.0.2}
\end{aligned}$$

Then

$$m_\ell(t) = E_{\ell,N}(t)m_N(0)E_{\ell,N}^\top(t),$$

with

$$m_N(0) = \begin{pmatrix} & & O & & & \\ & & & N-3 & N-1 & \\ & & & & & \\ & & -1 & 1 & / & \\ & & & & & \\ -N+1 & -N+3 & / & & & O \end{pmatrix}, \text{ for } N \text{ even},$$

$$m_N(0) = \begin{pmatrix} & & O & & & \\ & & & N-3 & N-1 & \\ & & & & & \\ & & & & 2 & / \\ & & -2 & 0 & 2 & \\ -N+1 & -N+3 & / & & & O \end{pmatrix}, \text{ for } N \text{ odd}$$

where (setting $\tilde{\mathbf{s}}_n(t) = \mathbf{s}_n(2t)$)

$$\begin{aligned}
m_N(t) &= ((j-i)\tilde{\mathbf{s}}_{N-i-j-1})_{0 \leq i, j \leq N-1} \\
&= \begin{pmatrix} 0 & \tilde{\mathbf{s}}_{N-2} & 2\tilde{\mathbf{s}}_{N-3} & \dots & (N-2)\tilde{\mathbf{s}}_1 & N-1 \\ -\tilde{\mathbf{s}}_{N-2} & 0 & \tilde{\mathbf{s}}_{N-4} & \dots & N-3 & \\ -2\tilde{\mathbf{s}}_{N-3} & -\tilde{\mathbf{s}}_{N-4} & 0 & \diagup & & \\ & & & 1 & & \\ \vdots & \vdots & & -1 & & \\ & & & & \diagup & \\ -(N-2)\tilde{\mathbf{s}}_1 & -N+3 & & & & \\ & & & & & O \\ -N+1 & & & & & \end{pmatrix} \\
&\quad \text{for } N \text{ even} \\
&= \begin{pmatrix} 0 & \tilde{\mathbf{s}}_{N-2} & 2\tilde{\mathbf{s}}_{N-3} & \dots & (N-2)\tilde{\mathbf{s}}_1 & N-1 \\ -\tilde{\mathbf{s}}_{N-2} & 0 & \tilde{\mathbf{s}}_{N-4} & \dots & N-3 & \\ -2\tilde{\mathbf{s}}_{N-3} & -\tilde{\mathbf{s}}_{N-4} & 0 & \diagup & & \\ & & & 2 & & \\ & & & 0 & & \\ \vdots & \vdots & & -2 & & \\ & & & & \diagup & \\ -(N-2)\tilde{\mathbf{s}}_1 & -N+3 & & & & \\ & & & & & O \\ -N+1 & & & & & \end{pmatrix} \\
&\quad \text{for } N \text{ odd}
\end{aligned} \tag{7.0.3}$$

Proof: Setting

$$t_k = \frac{1}{k} \sum_{i=1}^{\ell} x_i^k$$

we have

$$\begin{aligned} e^{\beta \sum_{k=1}^{\infty} t_k z^k} &= e^{\beta \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \frac{1}{k} (x_i z)^k} \\ &= \prod_{i=1}^{\ell} \left(\sum_{k=1}^{\infty} \frac{1}{k} (x_i z)^k \right)^{\beta} \\ &= \prod_{i=1}^{\ell} (1 - x_i z)^{-\beta} \end{aligned}$$

According to Awata et al. [4], the Jack polynomials for rectangular partitions s^n have the following integral representation: (for connections with random matrix theory, see [10])

$$\begin{aligned} cJ_{s^n}^{1/\beta} &= \oint_{z_1=\dots=z_n=0} |\Delta(z)|^{2\beta} \prod_{j=1}^n z_j^{-(n-1)\beta-s} \prod_{i=1}^{\ell} (1 - x_i z_j)^{-\beta} \frac{dz_j}{2\pi i z_j} \\ &= \oint_{z_1=\dots=z_n=0} |\Delta(z)|^{2\beta} \prod_{j=1}^n z_j^{-(n-1)\beta-s} e^{\beta \sum_{k=1}^{\infty} t_k z_j^k} \frac{dz_j}{2\pi i z_j} \\ &= c_n \int_{\mathbb{R}^n} |\Delta_n(z)|^{2\beta} \prod_{j=1}^n e^{\beta \sum_{k=1}^{\infty} t_k z_j^k} \delta^{s+(n-1)\beta}(z_j) dz_j. \end{aligned}$$

Setting $\beta = 2$, $s = N - 2n$ and $2 \leq 2n \leq N$ in the last integral, we have, using the standard derivation of the “symplectic” matrix integral, (see [2])

$$\begin{aligned}
& \frac{1}{n!} \int_{\mathbb{R}^n} \Delta_n^4(z) \prod_{k=1}^n e^{2 \sum_{j=1}^{\infty} t_k z_j^k} \delta^{N-2}(z_j) dz_j \\
&= pf \left(\int_{\mathbb{R}} \{y^k, y^\ell\} e^{2 \sum_{i=1}^{\infty} t_i y^i} \delta^{(N-2)}(y) dy \right)_{0 \leq k, \ell \leq 2n-1} \\
&= pf \left((k - \ell) \int_{\mathbb{R}} y^{k+\ell-1} e^{2 \sum_{i=1}^{\infty} t_i y^i} \delta^{(N-2)}(y) dy \right)_{0 \leq k, \ell \leq 2n-1} \\
&= pf \left((k - \ell) \sum_{i=0}^{\infty} \tilde{\mathbf{s}}_i(t) \int_{\mathbb{R}} y^{i+k+\ell-1} \delta^{(N-2)}(y) dy \right) \\
&= pf \left((-1)^{N-2} (N-2)! (k - \ell) \tilde{\mathbf{s}}_{N-1-k-\ell}(t) \right)_{0 \leq k, \ell \leq 2n-1} \\
&= c_{N,n} pf \left((\ell - k) \tilde{\mathbf{s}}_{N-1-k-\ell}(t) \right)_{0 \leq k, \ell \leq 2n-1}. \tag{7.0.4}
\end{aligned}$$

In order to find the initial condition $m_N(0)$, one sets $t = 0$ in the last matrix appearing in (7.0.3), to yield

$$\left((\ell - k) \tilde{\mathbf{s}}_{N-1-k-\ell}(0) \right)_{0 \leq k, \ell \leq N-1}.$$

All entries of this matrix vanish, except the antidiagonal, from which one reads off the b_i 's:

For N even, we have $b_i = 2i + 1$ and thus

$$\begin{aligned}
b_{\lambda_i - i + \ell - N/2} &= 2 \left(\lambda_i - i + \ell - \frac{N}{2} \right) + 1 \\
&= \lambda_i - \lambda_{\ell+1-i} - 2i + \ell + 1 \quad \text{using } \lambda_i + \lambda_{\ell+1-i} = N - \ell \\
&= k_i - k_{\ell+1-i} \quad \text{using } k_i = \lambda_i - i + 2n.
\end{aligned}$$

For N odd, we have $b_i = 2i + 2$ and thus

$$\begin{aligned}
b_{\lambda_i - i + \ell - (N+1)/2} &= 2 \left(\lambda_i - i + \ell - \frac{N+1}{2} \right) + 2 \\
&= \lambda_i - \lambda_{\ell+1-i} - 2i + \ell + 1 \quad \text{using } \lambda_i + \lambda_{\ell+1-i} = N - \ell \\
&= k_i - k_{\ell+1-i},
\end{aligned}$$

ending the proof of Proposition 7.1. ■

Example: For $n = 4$ and $b_0 = 1$, $b_1 = 3$, the solution to the system (1.0.8) is given by

$$L = \frac{1}{(t_2 + t_1^2)^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ t_1 & 2(t_2 - t_1^2) & -\sqrt{3}t_1 & 0 \\ \frac{2}{\sqrt{3}}(t_2 - t_1^2) & -\frac{16}{\sqrt{3}}t_1t_2 & -2(t_2 - t_1^2) & 1 \\ -\sqrt{3}t_1 & -2\sqrt{3}(t_2 - t_1^2) & 3t_1 & 0 \end{pmatrix}. \quad (7.0.5)$$

Indeed

$$m_4 = \begin{pmatrix} 0 & -\tilde{s}_2 & -2\tilde{s}_1 & -3 \\ \tilde{s}_2 & 0 & -1 & 0 \\ 2\tilde{s}_1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} = Q^{-1} J Q^{\top -1},$$

with

$$Q = D \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2\tilde{s}_1 & \tilde{s}_2 & 0 \\ 0 & -3 & 0 & \tilde{s}_2 \end{pmatrix}$$

where

$$D = \text{diag} \left(\frac{1}{\sqrt{\tilde{s}_2}}, \frac{1}{\sqrt{\tilde{s}_2}}, \frac{1}{\sqrt{3\tilde{s}_2}}, \frac{1}{\sqrt{3\tilde{s}_2}} \right).$$

Therefore

$$\begin{aligned} L &= Q \Lambda Q^{-1} \\ &= \frac{1}{\tilde{s}_2^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2\tilde{s}_1 & 4(\tilde{s}_2 - \tilde{s}_1^2) & -2\sqrt{3}\tilde{s}_1 & 0 \\ \frac{4}{\sqrt{3}}(\tilde{s}_2 - \tilde{s}_1^2) & -\frac{8\tilde{s}_1}{\sqrt{3}}(2\tilde{s}_2 - \tilde{s}_1^2) & -4(\tilde{s}_2 - \tilde{s}_1^2) & 1 \\ -\frac{6}{\sqrt{3}}\tilde{s}_1 & -\frac{12}{\sqrt{3}}(\tilde{s}_2 - \tilde{s}_1^2) & 6\tilde{s}_1 & 0 \end{pmatrix}, \end{aligned}$$

leads to formula (7.0.5).

Example 2: Two-column Jack polynomials

Proposition 7.2 *For even N , choosing¹³*

$$\begin{cases} b_0 = \dots = b_{\frac{p}{2}-1} = 0 \\ b_{\frac{p}{2}+k} = \frac{(1-\alpha)_k (p+1)_k}{k! (\alpha+p+1)_k}, \text{ for } k = 0, \dots, \frac{N-2-p}{2}, \end{cases} \quad (7.0.6)$$

one finds the most general two-row Jack polynomial for τ_2 , for arbitrary α ,

$$\begin{aligned} \tau_2(t) &= pf \, m_2(t) \\ &= J_{\left(\frac{N+p-2}{2}, \frac{N-p-2}{2}\right)}^{(1/\alpha)}(t/\alpha) \\ &= c \oint \frac{dx}{2\pi i} \frac{dy}{2\pi i} \frac{(y-x)^{2\alpha}}{(xy)^{\alpha+\frac{N}{2}}} e^{\sum_{i=1}^{\infty} t_i (x^i + y^i)} \left(\frac{x}{y}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{y}{x}\right) \end{aligned} \quad (7.0.7)$$

and for general $\ell \geq 2$,

$$\begin{aligned} \tau_\ell(t) &= \frac{2c}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha-1}}{z_2 (z_1 z_2)^{\alpha-1}} \left(\frac{z_1}{z_2}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{z_2}{z_1}\right) \\ &\quad \frac{\prod_{i=2}^{\ell/2} \rho(z_{2i}/z_{2i-1})}{\prod_{i=1}^{\ell/2} \frac{z_{\frac{N}{2}-2i+3}}{z_{2i-1}} \frac{z_{\frac{N}{2}-2i+1}}{z_{2i}}} \prod_{1 \leq i < j \leq \ell} \left(1 - \frac{z_i}{z_j}\right) \prod_{j=1}^{\ell} e^{\sum_{k=1}^{\infty} t_k z_j^k} \frac{dz_j}{2\pi i}, \end{aligned} \quad (7.0.8)$$

where

$$\rho(x) = \sum_{i=0}^{\frac{N-2}{2}} b_i (x^{-i-1} - x^i). \quad (7.0.9)$$

¹³ $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1)$

Proof: According to a formula by Stanley [9], two-column Jack polynomials can be expressed as a linear combination of two-column Schur polynomials. So, setting in the end $2s = N - 2 - p$, we have

$$\begin{aligned}
\tau_2(t) &= \sum_{k=0}^{\frac{N-2}{2}} b_k \mathbf{s}_{\frac{N-2}{2}+k, \frac{N-2}{2}-k}(t), \quad \text{with } b_k \text{ as in (7.0.6),} \\
&= \sum_{k=\frac{p}{2}}^{\frac{N-2}{2}} \frac{(1-\alpha)_{k-p/2}(p+1)_{k-p/2}(-1)^{N-2}}{(k-p/2)!(\alpha+p+1)_{k-p/2}} \mathbf{s}_{\frac{N-2}{2}+k, \frac{N-2}{2}-k}(t) \\
&= \sum_{k=\frac{p}{2}}^{\frac{N-2}{2}} \frac{(1-\alpha)_{k-p/2}(p+1)_{k-p/2}}{(k-p/2)!(\alpha+p+1)_{k-p/2}} \mathbf{s}_{2^{\frac{N-2}{2}-k} 1^{2k}}(-t) \\
&= \sum_{k=0}^{\frac{N-p-2}{2}} \frac{(1-\alpha)_k(p+1)_k}{k!(\alpha+p+1)_k} \mathbf{s}_{2^{\frac{N-2}{2}-k-p/2} 1^{2k+p}}(-t) \\
&= \sum_{k=0}^s \frac{(1-\alpha)_k(p+1)_k}{k!(\alpha+p+1)_k} \mathbf{s}_{2^{s-k} 1^{2k+p}}(-t) \\
&= J_{2^s 1^p}^{(\alpha)}(-t) \quad (\text{Stanley's formula}) \\
&= J_{(p+s, s)}^{(1/\alpha)}(t/\alpha) \quad \text{using duality,}
\end{aligned}$$

showing that any two-row Jack polynomial can serve as Pfaff τ -function τ_2 .

According to [4], Jack polynomials also have an integral representation, and so $\tau_2(t)$ can also be expressed as

$$\begin{aligned}
\tau_2(t) &= J_{(p+s, s)}^{(1/\alpha)}(t/\alpha) \\
&= c' \oint \frac{dx}{2\pi i x} \frac{dy}{2\pi i y} \frac{dz}{2\pi i z} \frac{(x-y)^{2\alpha}(xy)^{-s}z^{-p}}{((x-z)(y-z))^\alpha} e^{\sum_{i=1}^\infty t_i(x^i+y^i)} \\
&= c' \oint \frac{dx}{2\pi i x} \frac{dy}{2\pi i y} (x-y)^{2\alpha}(xy)^{-s} e^{\sum_{i=1}^\infty t_i(x^i+y^i)} D_z^p((x-z)(y-z))^{-\alpha} \Big|_{z=0} \\
&= c'(\alpha)_p \oint \frac{dx}{2\pi i x} \frac{dy}{2\pi i y} \frac{(x-y)^{2\alpha}}{(xy)^{\alpha+s}y^p} e^{\sum_{i=1}^\infty t_i(x^i+y^i)} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{y}{x}\right),
\end{aligned}$$

where we used the identity:

$$\begin{aligned}
& D_z^p((x-z)(y-z))^{-\alpha} \Big|_{z=0} \\
&= (xy)^{-\alpha} D_z^p \left(\left(1 - \frac{z}{x}\right)^{-\alpha} \left(1 - \frac{z}{y}\right)^{-\alpha} \right) \Big|_{z=0} \\
&= (xy)^{-\alpha} D_z^p \left(\sum_{k,\ell=0}^{\infty} \frac{(\alpha)_k (\alpha)_\ell}{k! \ell!} \frac{z^{k+\ell}}{x^k y^\ell} \right) \Big|_{z=0} \\
&= p! (xy)^{-\alpha} \sum_{k+\ell=p} \frac{(\alpha)_k (\alpha)_\ell}{k! \ell!} x^{-k} y^{-\ell} \\
&= p! (xy)^{-\alpha} y^{-p} \sum_{k=0}^p \frac{(\alpha)_k (\alpha)_{p-k}}{k! (p-k)!} \left(\frac{y}{x}\right)^k \\
&= (\alpha)_p (xy)^{-\alpha} y^{-p} \sum_{k=0}^p \frac{(\alpha)_k (-p)_k}{k! (1-\alpha-p)_k} \left(\frac{y}{x}\right)^k \\
&\quad \text{using } \frac{p! (\alpha)_{p-k}}{(p-k)! (\alpha)_p} = \frac{(-p)_k}{(1-\alpha-p)_k} \\
&= (\alpha)_p (xy)^{-\alpha} y^{-p} {}_2F_1 \left(\alpha, -p; 1-\alpha-p; \frac{y}{x} \right).
\end{aligned}$$

This proves identity (7.0.7).

Applying Theorem 1.3, we find the higher τ_ℓ 's, by applying the integrated vertex operator

$$Y_{\frac{N-2}{2}-2j}(t) = \frac{1}{(2\pi i)^2} \oint_{\infty} \oint_{\infty} X(t; z_{2j+2}) X(t; z_{2j+1}) \frac{\rho_b(z_{2j+2}/z_{2j+1}) dz_{2j+2} dz_{2j+1}}{z_{2j+1}^2 (z_{2j+1} z_{2j+2})^{\frac{N-2}{2}-2j}}, \quad (7.0.10)$$

for $j = 1, 2, \dots, (\ell - 2)/2$ to τ_2 (see formula (7.0.7)); so, one finds¹⁴

$$\begin{aligned}
\tau_\ell &= \frac{2}{\ell!!} Y_{\frac{N}{2}-\ell+1} \dots Y_{\frac{N}{2}-5} Y_{\frac{N}{2}-3} \tau_2 \\
&= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha}}{(z_1 z_2)^{\alpha + \frac{N}{2}}} \left(\frac{z_1}{z_2} \right)^{p/2} {}_2F_1 \left(\alpha, -p; 1 - \alpha - p; \frac{z_2}{z_1} \right) \\
&\quad \frac{\rho(z_\ell/z_{\ell-1}) \dots \rho(z_4/z_3)}{(z_3 z_5 \dots z_{\ell-1})^2 (z_3 z_4)^{\frac{N}{2}-3} (z_5 z_6)^{\frac{N}{2}-5} \dots (z_{\ell-1} z_\ell)^{\frac{N}{2}-\ell+1}} \\
&\quad X(t; z_\ell) X(t; z_{\ell-1}) \dots X(t; z_4) X(t; z_3) e^{\sum_{k=1}^\infty t_k (z_1^k + z_2^k)} \prod_{j=1}^\ell \frac{dz_j}{2\pi i} \\
&= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha} z_1^2 (z_1 z_2)^{N/2-1}}{(z_1 z_2)^{\alpha + \frac{N}{2}}} \left(\frac{z_1}{z_2} \right)^{p/2} {}_2F_1 \left(\alpha, -p; 1 - \alpha - p; \frac{z_2}{z_1} \right) \\
&\quad \left(1 - \frac{z_1}{z_2} \right)^{-1} \frac{\rho(z_\ell/z_{\ell-1}) \dots \rho(z_4/z_3)}{\prod_{i=1}^{\ell/2} z_{2i-1}^2 (z_{2i-1} z_{2i})^{\frac{N}{2}-2i+1}} \prod_{1 \leq i < j \leq \ell} \left(1 - \frac{z_i}{z_j} \right) \prod_{j=1}^\ell e^{\sum_{k=1}^\infty t_k z_j^k} \frac{dz_j}{2\pi i} \\
&= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha-1}}{z_2 (z_1 z_2)^{\alpha-1}} \left(\frac{z_1}{z_2} \right)^{p/2} {}_2F_1 \left(\alpha, -p; 1 - \alpha - p; \frac{z_2}{z_1} \right) \\
&\quad \frac{\prod_{i=2}^{\ell/2} \rho(z_{2i}/z_{2i-1})}{\prod_{i=1}^{\ell/2} z_{2i-1}^{\frac{N}{2}-2i+3} z_{2i}^{\frac{N}{2}-2i+1}} \prod_{1 \leq i < j \leq \ell} \left(1 - \frac{z_i}{z_j} \right) \prod_{j=1}^\ell e^{\sum_{k=1}^\infty t_k z_j^k} \frac{dz_j}{2\pi i},
\end{aligned}$$

establishing formula (7.0.8). ■

Alternative formula: The following formula has the advantage to be more

¹⁴replacing x, y in τ_2 with z_1, z_2 .

symmetric, but the disadvantage to have many more integrations:

$$\tau_\ell(t) = \oint \prod_{i=1}^{\ell} \prod_{j=1}^i \frac{dz_j^{(i)}}{z_j^{(i)}} \prod_{i=1}^{\ell} e^{\sum_1^{\infty} t_k (z_i^{(\ell)})^{-k}} \frac{\prod_{k=1}^{\ell} \prod_{\substack{1 \leq i, j \leq k \\ i \neq j}} \left(1 - \frac{z_i^{(k)}}{z_j^{(k)}}\right)}{\prod_{k=1}^{\ell-1} \prod_{\substack{1 \leq i \leq k+1 \\ 1 \leq j \leq k}} \left(1 - \frac{z_i^{(k+1)}}{z_j^{(k)}}\right)} K_{N,p,\ell}(Z)$$

with

$$K_{N,p,\ell} = \frac{\left(\prod_{j=1}^{\ell} z_j^{(\ell)}\right)^{\frac{N-p}{2}-1} \left(\prod_{j=1}^{\ell/2} z_j^{(\ell/2)}\right)^{p+1}}{\prod_{i=1}^{\ell-1} \prod_{j=1}^i z_j^{(i)}} \prod_{i=1}^{\ell/2} {}_2F_1 \left(1 - \alpha, p + 1; 1 + \alpha + p; \frac{\prod_{j=1}^i z_j^{(i)} \prod_{j=1}^{\ell-i} z_j^{(\ell-i)}}{\prod_{j=1}^{i-1} z_j^{(i-1)} \prod_{j=1}^{\ell+i-i} z_j^{(\ell+1-i)}} \right).$$

References

- [1] M. Adler, T. Shiota and P. van Moerbeke: *Pfaff τ -functions*, Math. Annalen, **322**, 423-476 (2002) (arXiv: solv-int/9909010)
- [2] M. Adler and P. van Moerbeke: *Toda versus Pfaff Lattice and related polynomials*, Duke Math. J., **112**, 2002 (arXiv: solv-int/9912008)
- [3] M. Adler, E. Horozov and P. van Moerbeke: *The Pfaff lattice and skew-orthogonal polynomials*, Intern. Math. Res. Notices, **11**, 569–588 (1999)
- [4] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi: *Excited states of the Calogero-Sutherland model and singular vectors of the W_N algebra*, Nuclear Phys. B **449**, 347–374 (1995)
- [5] L. Dickey: *Soliton equations and integrable systems*, World Scientific (1991).

- [6] V.B. Kuznetsov and E.K. Sklyanin: *Separation of variables in A_2 type Jack polynomials*, RIMS Kokyuroku, **919**, 27-34 (1995)
- [7] V.B. Kuznetsov and E.K. Sklyanin: *Factorisation of Macdonald polynomials*, in: *Symmetries and Integrability of Difference Equations*, London Mathematical Society, Lecture Note Series, **255**, 370–384 (1999); (Eds P.A. Clarkson and F.W. Nijhoff, Cambridge University Press).
- [8] I.G. MacDonald: “Symmetric functions and Hall polynomials”, Clarendon Press, 1995.
- [9] R. P. Stanley: *Some combinatorial properties of Jack symmetric functions*, Adv. in Math., **77**, 76–115 (1989).
- [10] P. van Moerbeke: *Integrable lattices: random matrices and random permutations*, ”Random matrices and their applications”, Mathematical Sciences research Institute Publications #40, Cambridge University Press, pp 321-406, (2001).