

K-inflationary Power Spectra at Second Order

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Abstract. Within the class of inflationary models, k-inflation represents the most general single field framework that can be associated with an effective quadratic action for the curvature perturbations and a varying speed of sound. The incoming flow of high-precision cosmological data, such as those from the Planck satellite and small scale Cosmic Microwave Background (CMB) experiments, calls for greater accuracy in the inflationary predictions. In this work, we calculate for the first time the next-to-next-to-leading order scalar and tensor primordial power spectra in k-inflation needed in order to obtain robust constraints on the inflationary theory. The method used is the uniform approximation together with a second order expansion in the Hubble and sound flow functions. Our result is checked in various limits in which it reduces to already known situations.

Keywords: Cosmic Inflation, Slow-Roll, Cosmic Microwave Background

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1 Introduction

Inflation [1, 2] (for reviews, see Refs. [3–7]), which is currently the leading paradigm to describe the physical conditions that prevailed in the very early Universe, is now entering a new phase. With the advent of new high-accuracy cosmological data [8–21], among which are the Planck data [22], one can hope to obtain very tight constraints on the inflationary theory and even to pin-point the correct model of inflation. In order to achieve this ambitious goal, one must be able to compare the inflationary predictions to the data. The problem is that the inflationary landscape is very large [23] and that there is a whole zoo of different models making different predictions. Moreover, for many of these models, predictions can only be worked out by numerical methods. It is therefore not obvious how to extract model-independent constraints on the inflationary scenario.

How then should we proceed? Clearly, one can approach the problem step by step and start with the simplest models. In other words, it seems reasonable to consider more complicated models only if the data force us to do so and tell us that the simplest models are not enough. Then comes the question of identifying these models. One can convincingly argue that slow-roll Single Field with a Minimal Kinetic term (SFMK) scenarios are the simplest inflationary models since they are just characterized by one function, the potential $V(\phi)$. In order to establish their observational consequences, a possible approach is to scan models one by one and calculate the predictions exactly [24–29], most of the time numerically [30, 31]¹. This leads to an exact mapping of the inflationary landscape within this class of scenarios but, given that the number of SFMK models remains large, it would represent a huge effort. Another approach consists in developing a scheme of approximation allowing us to derive analytical, or semi-analytical, predictions. Although this is not always possible, such a method is available for the SFMK models and one can explicitly write a functional form for the primordial power spectrum of the cosmological perturbations [32], and even their higher order correlation functions [33–38].

In fact, one can enlarge the class of what we consider as the simplest models of inflation and assume that these ones are k-inflationary scenarios. K-inflation [39, 40] encompasses standard inflation and is more general since not only the potential but also the kinetic term

¹See for instance <http://theory.physics.unige.ch/~ringeal/fieldinf.html>.

is now a free function. At the perturbation level, the action for the comoving curvature perturbation has a varying speed of sound and this describes all possible quadratic terms within the effective field theory formalism [41, 42]. But, more interestingly, and despite the fact that this class of scenarios is more complicated to analyze, a properly generalized slow-roll approximation can still be used.

1.1 State-of-the-art

At this stage, it is interesting to recall the present status of the techniques that enable us to calculate the two-point correlation function for the primordial cosmological perturbations.

The spectrum of density perturbations during inflation was computed for the first time in Refs. [43, 44] and for the gravity waves in Ref. [45]. Then, in Ref. [46], it was realized that it can be evaluated exactly in the case of power-law inflation. The first calculation at first order in the so-called “horizon flow parameters” and using the slow-roll approximation was performed in Ref. [32]. This calculation was done for the SFMK models. This is a fundamental result since it allows to connect the deviations from scale invariance to the microphysics of inflation. This result was re-derived using the Green function methods in Ref. [47], using the Wentzel-Kramers-Brillouin (WKB) method in Ref. [48] and using the uniform approximation in Refs. [49, 50]. In fact, the Green function method of Ref. [47] made possible the first determination of the scalar power spectrum at second order in the “horizon flow parameters”. Indeed, at second order, the mode equation describing the evolution of the cosmological perturbations can no longer be solved exactly, hence the need for a new method of approximation. Higher order corrections were also obtained in Ref. [51]. The first derivation of the tensor power spectrum at second order using the Green function method was presented in Ref. [52]. In Refs. [53, 54], it was also shown how to improve the WKB method by adding more adiabatic terms. This improved WKB method has allowed a re-derivation of the scalar and tensor power spectra at second order and confirmed the results of the Green function approach.

After the advent of k-inflation, various attempts have been made to derive the corresponding power spectra. The problem is complicated due to the fact that density perturbations now propagate with a time-dependent speed (the speed of sound). In Ref. [55], the Green function method has been used but with some extra-assumptions on the behavior of the sound speed. The question was also considered in Refs. [56, 57] but the results obtained in those articles were not totally correct since the sound speed was (implicitly) assumed to be constant which is not the case in most of the k-inflationary scenarios (this result was also used afterward in Ref. [58]). These works also missed the influence of the sound speed in the tensor power spectrum due to the shift between the scalar and tensor pivot scales [59, 60]. The first fully consistent result for the k-inflationary scalar power spectrum was presented in Ref. [61]. The latter has been re-derived using the uniform approximation in Ref. [59] together with the first fully consistent calculation of the tensor power spectrum at the same pivot scale. These spectra were compared to Cosmic Microwave Background Anisotropy (CMB) data first in Ref. [62]. However, all of these calculations have been derived at first order only and no complete result at second order exists in the literature.

The main purpose of this article is to close this gap and to derive the slow-roll power spectra for the density and tensor perturbations in k-inflation, at second order in the Hubble and sound flow functions². This calculation is interesting for two reasons. Firstly, the second order result is available for SFMK models and, for completeness, it should also be done for the

²Conforming to the modern usage, we will prefer the denomination of “Hubble flow functions” and “sound

k-inflationary models. Secondly, according to the "blue book" [63], Planck will measure the spectral index with accuracy $\Delta n_s \simeq 0.005$. Even if one expects the Hubble flow parameters to be less than 10^{-2} , second order corrections will be of order 10^{-4} , that is to say relevant for high-accuracy measurements of n_s and/or estimation of the corresponding error bars. Moreover, having at hand the second order terms allows to marginalize over, a procedure that should always be carried on to get robust Bayesian constraints on the first order terms.

Before moving to the calculation, let us briefly recall some well-known results about k-inflation at the background and perturbation levels.

1.2 K-inflation in brief

K-inflation corresponds to a class of models where gravity is described by General Relativity and where the action for the inflaton field is an arbitrary function, $P(\phi, X)$, the quantity X being defined by $X \equiv -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. This action can be written as

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left[R + \frac{2}{M_{\text{Pl}}^2} P(X, \phi) \right], \quad (1.1)$$

where M_{Pl} is the reduced Planck mass. In fact, in order to satisfy the requirements that the Hamiltonian is bounded from below and that the equations of motion remain hyperbolic, the function $P(X, \phi)$ must satisfy the following two conditions [64]

$$\frac{\partial P}{\partial X} > 0, \quad 2X \frac{\partial^2 P}{\partial X^2} + \frac{\partial P}{\partial X} > 0. \quad (1.2)$$

The general action (1.1) includes standard inflation for which $P = X - V(\phi)$, where $V(\phi)$ is the inflaton potential. This class of model is in fact characterized by an arbitrary function of ϕ only. K-inflation also includes the Dirac-Born-Infeld (DBI) class of inflationary models [65]. For those, one has $P = -T(\phi)\sqrt{1 - 2X/T(\phi)} + T(\phi) - V(\phi)$. This kind of action typically appears in brane inflation and $T(\phi)$ is interpreted as a warping function representing the bulk geometry in which various branes can move. It is of course possible to find even more complicated examples but, in the following, we will not need to specify explicitly the function $P(X, \phi)$.

As in standard inflation, the dynamics of the background space-time can be described by the Hubble flow functions ϵ_n defined by

$$\epsilon_{n+1} = \frac{d \ln \epsilon_n}{dN}, \quad \epsilon_0 \equiv \frac{H_{\text{ini}}}{H}, \quad (1.3)$$

where $N \equiv \ln(a/a_{\text{ini}})$ is the number of e-folds. Inflation occurs if $\epsilon_1 < 1$ and the slow-roll approximation assumes that all these parameters are small during inflation $\epsilon_n \ll 1$. Let us notice that it is difficult to have an inflationary model without such a condition because otherwise one would obtain a deviation from scale invariance which would be too strong to be compatible with the cosmological data (see however Ref. [42]).

At the perturbed level, we have density perturbations and gravity waves. As usual, rotational perturbations are unimportant since they quickly decay. Obviously, the tensorial sector of the theory is standard since the gravitational part of (1.1) is the ordinary Einstein-Hilbert

flow functions" to refer to the original, but confusing, appellation "horizon flow parameters". See Sec. 1.2 for the definition of the Hubble and sound flow functions.

action. As a consequence, the equation of motion for the amplitude $\mu_{\mathbf{k}}$ of gravity waves (rescaled by a factor $1/a$ for convenience, where a is the Friedman-Lemaître-Robertson-Walker scale factor) takes the usual form, namely

$$\mu_{\mathbf{k}}'' + [k^2 - U_{\text{T}}(\eta)] \mu_{\mathbf{k}} = 0, \quad (1.4)$$

where η is the conformal time and a prime denotes a derivative with respect to η . The effective potential for the tensorial modes can be written as $U_{\text{T}} = a^2 H^2 (2 - \epsilon_1)$, i.e. only depends on the first Hubble flow function ($H = a'/a^2$ is the Hubble parameter).

For the density perturbations, the situation is slightly more complicated. One can show that the comoving curvature perturbation in Fourier space, $\zeta_{\mathbf{k}}$, can be written in terms of a modified Mukhanov-Sasaki variable $v_{\mathbf{k}}$ by means of the following expression, $v_{\mathbf{k}} = (a\sqrt{\epsilon_1})\zeta_{\mathbf{k}}/c_{\text{s}}$ (in Planck units) where the quantity c_{s} is defined by the following equation

$$c_{\text{s}}^2 \equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}, \quad (1.5)$$

a subscript “ X ” denoting differentiation with respect to X . This quantity can be interpreted as the “sound speed” of density fluctuations. Notice that, because of the two consistency relations (1.2), we have $c_{\text{s}}^2 > 0$. The fact that c_{s} is the sound speed can be most easily seen if one writes down the equation of motion of the Mukhanov-Sasaki variable. It reads

$$v_{\mathbf{k}}'' + [c_{\text{s}}^2(\eta)k^2 - U_{\text{s}}(\eta)] v_{\mathbf{k}} = 0. \quad (1.6)$$

This is similar to the equation of motion of a parametric oscillator. The quantity U_{s} is the effective potential for the density perturbations and is a function of time only. As expected, c_{s}^2 appears in front of the k^2 term, which is nothing but a gradient term in Fourier space and this confirms its interpretation as a time dependent sound speed. Since $c_{\text{s}}(\eta)$ is not known a priori, one can introduce a second hierarchy of flow functions in order to describe its behavior. Therefore, we define the sound flow functions δ_n ’s by

$$\delta_{n+1} \equiv \frac{d \ln \delta_n}{dN}, \quad \delta_0 \equiv \frac{c_{\text{sini}}}{c_{\text{s}}}. \quad (1.7)$$

Consistent models of inflation are obtained if $\delta_n \ll 1$, that is to say if the sound speed does not change too abruptly [59, 61]. A remark about terminology is in order at this point. In terms of the Hubble and sound flow functions, the effective potential for the density perturbations can be expressed as

$$U_{\text{s}} = a^2 H^2 \left[2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3 + (3 - \epsilon_1 + \epsilon_2)\delta_1 + \delta_1^2 + \delta_1\delta_2 \right]. \quad (1.8)$$

The quantity U_{s} depends on the ϵ_n ’s up to ϵ_3 only and on the δ_n ’s up to δ_2 only. Despite this last property, it is important to remember that the above expression of U_{s} is *exact* and that no approximation has been made at this stage.

The cosmological observables we are interested in are the two point correlation functions of the fluctuations, i.e. in Fourier space, the power spectra of both gravity waves and density perturbations:

$$\mathcal{P}_h = \frac{2k^3}{\pi^2} \left| \frac{\mu_{\mathbf{k}}}{a} \right|^2, \quad \mathcal{P}_{\zeta} = \frac{k^3}{2\pi^2} |\zeta_{\mathbf{k}}|^2 = \frac{k^3}{4\pi^2} \frac{c_{\text{s}}^2 |v_{\mathbf{k}}|^2}{M_{\text{Pl}}^2 a^2 \epsilon_1}. \quad (1.9)$$

They have to be evaluated at the end of inflation and on large scales. After the inflationary era, and for single field models, these power spectra remain constant and can be directly used to compute various observable quantities such as the CMB anisotropies or the matter power spectrum. Our goal is now to integrate the equations of motion of $\mu_{\mathbf{k}}$ and $v_{\mathbf{k}}$ in order to explicitly evaluate the above power spectra.

This article is organized as follows. In the next section, after having very quickly reviewed how the uniform approximation can be used in the cosmological context, we apply it to the calculation of the scalar and tensor primordial power spectra. Our results are discussed Sec. 3, in which we compare them, in the appropriate limits, with the existing literature and we present our conclusions.

2 K-inflationary Power Spectra

2.1 The Uniform Approximation

In this section, we use the uniform approximation to calculate the power spectrum of the density fluctuations in k-inflation, at second order in the Hubble and sound flow functions. We have seen in the previous section that the density perturbations in k-inflation propagate with a time-dependent velocity $c_s(\eta)$. As the mode equation can no longer be solved exactly in terms of Bessel functions (even at first order for the sound flow functions), this prompts for the use of new techniques. Here, we choose to work with the well-suited uniform approximation [50]. The idea is to rewrite the effective potential according to $U_s = (\nu^2 - 1/4)/\eta^2$, an equation which has to be understood as the definition of the function $\nu(\eta)$. Then, we introduce two new functions

$$g(\eta) \equiv \frac{\nu^2}{\eta^2} - c_s^2 k^2, \quad f(\eta) \equiv \frac{|\eta - \eta_*|}{\eta - \eta_*} \left| \frac{3}{2} \int_{\eta_*}^{\eta} d\tau \sqrt{g(\tau)} \right|^{2/3}, \quad (2.1)$$

where the turning point time $\eta_*(k)$ is defined by the condition $g(\eta_*) = 0$, that is to say $\eta_* \equiv -\nu(\eta_*)/[kc_s(\eta_*)]$. According to the uniform approximation, the Mukhanov-Sasaki variable can then be expressed as

$$v_{\mathbf{k}}(\eta) = A_{\mathbf{k}} \left(\frac{f}{g} \right)^{1/4} \text{Ai}(f) + B_{\mathbf{k}} \left(\frac{f}{g} \right)^{1/4} \text{Bi}(f), \quad (2.2)$$

where the two constants $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are fixed by the choice of the initial conditions and where Ai and Bi denotes the Airy function of the first and second kind respectively. Since one needs to compute $v_{\mathbf{k}}$ on large scales, only the asymptotic behavior of the Airy functions is needed and one arrives at a simpler formula, namely

$$\lim_{c_s k \eta \rightarrow 0} v_{\mathbf{k}}(\eta) = \frac{B_{\mathbf{k}}}{g^{1/4} \pi^{1/2}} \exp \left(\frac{2}{3} f^{3/2} \right). \quad (2.3)$$

Here, the function $g(\eta)$ should be taken in its asymptotic limit, i.e. $g^{1/2} \simeq -\nu(\eta)/\eta$. Inserting the last equation for $v_{\mathbf{k}}(\eta)$ into the formula (1.9), one obtains the following expression for $\mathcal{P}_{\zeta}$

$$\mathcal{P}_{\zeta} = -\frac{k^3 |B_{\mathbf{k}}|^2}{4\pi^3 M_{\text{Pl}}^2} \frac{\eta c_s^2}{a^2 \nu \epsilon_1} e^{2\Psi}, \quad (2.4)$$

where we have defined $\Psi \equiv 2f^{3/2}/3$. One verifies that $\mathcal{P}_{\zeta}$ is positive definite since the conformal time is negative during inflation. Therefore, the only thing which remains to be done is to express the combination $c_s^2/(a^2 \nu \epsilon_1)$ and the quantity Ψ at second order in the Hubble and sound flow functions.

2.2 Hubble and sound flow expansion

The first step of the calculation consists in determining the functions $a(\eta)$, $c_s(\eta)$, $\nu(\eta)$ and $\epsilon_1(\eta)$ at second order in the Hubble and sound flow functions. Here, we first briefly explain the method in the case of the scale factor. By definition, the conformal time is given by $\eta = -\int dt/a(t)$, where t is the cosmic time. By successive integrations by parts, one can re-write η as

$$\eta = \frac{1}{\mathcal{H}} \left\{ 1 + \epsilon_1 + \epsilon_1^2 + \epsilon_1 \epsilon_2 - aH \int \frac{1}{a} \frac{d}{da} \left[\frac{1}{H} (\epsilon_1^2 + \epsilon_1 \epsilon_2) \right] da \right\}, \quad (2.5)$$

where $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter. It is important to stress that this equation is exact. In the last term, the integrand is third order in the ϵ_i . Indeed, differentiating the term $1/H$ produces a ϵ_1 which, multiplied with $(\epsilon_1^2 + \epsilon_1 \epsilon_2)$, is third order. We also have to differentiate expressions quadratic in the Hubble flow functions but, since $d\epsilon_n/dN = \epsilon_n \epsilon_{n+1}$, this also gives third order quantities. Therefore, the last term is $\mathcal{O}(\epsilon^3)$ and can be dropped for a second order calculation. In other words

$$\mathcal{H} = -\frac{1}{\eta} (1 + \epsilon_1 + \epsilon_1^2 + \epsilon_1 \epsilon_2) + \mathcal{O}(\epsilon^3). \quad (2.6)$$

In fact, this equation is not exactly what we want yet because, although the second order terms ϵ_1^2 and $\epsilon_1 \epsilon_2$ can be considered as constant in time³, this is not the case for the term ϵ_1 which is first order. In order to render explicit the time-dependence, let us notice that the equations defining the Hubble-flow functions can also be written as $d\epsilon_n/d\eta = \mathcal{H} \epsilon_n \epsilon_{n+1}$. Given that $\epsilon_n \epsilon_{n+1}$ is already a second-order term, we can just replace $\mathcal{H}$ with $-1/\eta$ in this expression and one gets $\epsilon_n = \epsilon_{n*} - \epsilon_{n*} \epsilon_{n+1*} \ln(\eta/\eta_*) + \mathcal{O}(\epsilon^3)$, where we have chosen the integration constant such that this approximation is accurate around the time η_* of the turning point. Inserting this expression into Eq. (2.6) gives

$$\mathcal{H} = -\frac{1}{\eta} (1 + \epsilon_{1*} + \epsilon_{1*}^2 + \epsilon_{1*} \epsilon_{2*}) + \epsilon_{1*} \epsilon_{2*} \frac{1}{\eta} \ln \left(\frac{\eta}{\eta_*} \right) + \mathcal{O}(\epsilon^3), \quad (2.7)$$

and, this time, the η -dependence of $\mathcal{H}$ is explicit. This equation can be further integrated leading to an expression for the e-folds number N , namely

$$N - N_* = \ln \left(\frac{a}{a_*} \right) \simeq - (1 + \epsilon_{1*} + \epsilon_{1*}^2 + \epsilon_{1*} \epsilon_{2*}) \ln \left(\frac{\eta}{\eta_*} \right) + \frac{1}{2} \epsilon_{1*} \epsilon_{2*} \ln^2 \left(\frac{\eta}{\eta_*} \right). \quad (2.8)$$

Finally, by exponentiation the above formula and by expressing the constant $a_* \eta_*$ in terms of $1/H_*$, one obtains the following equation for the scale factor itself

$$a(\eta) \simeq -\frac{1}{H_* \eta} \left[1 + \epsilon_{1*} + \epsilon_{1*}^2 + \epsilon_{1*} \epsilon_{2*} - (\epsilon_{1*} + 2\epsilon_{1*}^2 + \epsilon_{1*} \epsilon_{2*}) \ln \left(\frac{\eta}{\eta_*} \right) + \frac{1}{2} (\epsilon_{1*}^2 + \epsilon_{1*} \epsilon_{2*}) \ln^2 \left(\frac{\eta}{\eta_*} \right) \right]. \quad (2.9)$$

We have reached our first goal, namely find an expression of $a(\eta)$ at second order in the Hubble flow parameters.

³Their derivative is indeed third order, i.e. zero at the order at which we work.

Let us now discuss how the expression of $\epsilon_1(\eta)$ can be obtained. Let us notice that since ϵ_1 is appearing in Eq. (2.4), we need to go to third order since a term $1/\epsilon_{1*}$ will remain in front of the final expression of $\mathcal{P}_\zeta$. This can however be obtained using the above formulas. Taylor expanding ϵ_1 around N_* one has

$$\epsilon_1 = \epsilon_{1*} + \left. \frac{d\epsilon_1}{dN} \right|_* (N - N_*) + \left. \frac{1}{2} \frac{d^2\epsilon_1}{dN^2} \right|_* (N - N_*)^2 + \dots \quad (2.10)$$

Using the fact that $d\epsilon_1/dN = \epsilon_1\epsilon_2$, $d^2\epsilon_1/dN^2 = \epsilon_1\epsilon_2^2 + \epsilon_1\epsilon_2\epsilon_3$ and the expression of the number of e-folds at first order [see Eq. (2.8) above], one arrives at

$$\epsilon_1 = \epsilon_{1*} \left[1 - \epsilon_{2*} (1 + \epsilon_{1*}) \ln \left(\frac{\eta}{\eta_*} \right) + \frac{1}{2} (\epsilon_{2*}^2 + \epsilon_{2*}\epsilon_{3*}) \ln^2 \left(\frac{\eta}{\eta_*} \right) \right] + \mathcal{O}(\epsilon^4). \quad (2.11)$$

The very same method can be used to determine the second order expression of the sound speed. Taylor expanding c_s in e-fold gives

$$c_s^2 = c_{s*}^2 + \left. \frac{dc_s^2}{dN} \right|_* (N - N_*) + \left. \frac{1}{2} \frac{d^2c_s^2}{dN^2} \right|_* (N - N_*)^2 + \dots, \quad (2.12)$$

and from the sound flow hierarchy one has $dc_s^2/dN = -2c_s^2\delta_1$, $d^2c_s^2/dN^2 = -2c_s^2\delta_1\delta_2 + 4c_s^2\delta_1^2$. Together with the expression of $N - N_*$, it follows that

$$c_s^2(\eta) = c_{s*}^2 + 2c_{s*}^2(\delta_{1*} + \delta_{1*}\epsilon_{1*}) \ln \left(\frac{\eta}{\eta_*} \right) - c_{s*}^2(\delta_{1*}\delta_{2*} - 2\delta_{1*}^2) \ln^2 \left(\frac{\eta}{\eta_*} \right) + \mathcal{O}(\epsilon^3, \delta^3). \quad (2.13)$$

As expected the coefficients of the logarithms are expressed in terms of the parameters δ_1 and δ_2 .

Finally, only the expression for $\nu(\eta)$ remains to be found. By definition, one has $\nu^2 = 1/4 + \eta^2 U_s(\eta)$, i.e. Taylor expanding everything from the previous formulas, one gets

$$\nu(\eta) = \nu_* - \left(\epsilon_{1*}\epsilon_{2*} + \frac{1}{2}\epsilon_{2*}\epsilon_{3*} + \delta_{1*}\delta_{2*} \right) \ln \left(\frac{\eta}{\eta_*} \right) + \mathcal{O}(\epsilon^3, \delta^3), \quad (2.14)$$

with

$$\nu_* \equiv \frac{3}{2} + \epsilon_{1*} + \frac{1}{2}\epsilon_{2*} + \delta_{1*} + \epsilon_{1*}^2 + \frac{11}{6}\epsilon_{1*}\epsilon_{2*} + \frac{1}{6}\epsilon_{2*}\epsilon_{3*} + \epsilon_{1*}\delta_{1*} + \frac{1}{3}\delta_{1*}\delta_{2*}. \quad (2.15)$$

2.3 Comoving Curvature Power Spectrum

We have now determined explicitly the four functions appearing in the expression of the power spectrum $\mathcal{P}_\zeta$, see Eq. (2.4). It is straightforward, although lengthy, to calculate, at second order, the relevant combination $c_s^2/(a^2\nu\epsilon_1)$ appearing in that expression. Moreover, we must also find Ψ . Upon using the expression of the function $g(\eta)$, one gets

$$\Psi = \int_{\eta_*}^{\eta} d\tau \sqrt{\frac{\nu^2(\tau)}{\tau^2} - c_s^2(\tau)k^2}. \quad (2.16)$$

Inserting Eqs. (2.13) and (2.14) into the previous formula and expanding everything to second order, the integrand in Eq. (2.16) reads

$$\begin{aligned}
\sqrt{\frac{\nu^2(\tau)}{\tau^2} - c_s^2(\tau)k^2} &= -\frac{\nu_*}{\tau} \left(1 - \frac{c_{s*}^2 k^2 \tau^2}{\nu_*^2}\right)^{1/2} \\
&+ \frac{3}{2\nu_*} \left(\epsilon_{1*}\epsilon_{2*} + \frac{1}{2}\epsilon_{2*}\epsilon_{3*} + \delta_{1*}\delta_{2*} \right) \frac{1}{\tau} \left(1 - \frac{c_{s*}^2 k^2 \tau^2}{\nu_*^2}\right)^{-1/2} \ln\left(\frac{\tau}{\eta_*}\right) \\
&+ \frac{c_{s*}^2}{\nu_*} (\delta_{1*} + \epsilon_{1*}\delta_{1*}) \frac{1}{\tau} \left(1 - \frac{c_{s*}^2 k^2 \tau^2}{\nu_*^2}\right)^{-1/2} k^2 \tau^2 \ln\left(\frac{\tau}{\eta_*}\right) \\
&- \frac{c_{s*}^2}{2\nu_*} (\delta_{1*}\delta_{2*} - 2\delta_{1*}^2) \frac{1}{\tau} \left(1 - \frac{c_{s*}^2 k^2 \tau^2}{\nu_*^2}\right)^{-1/2} k^2 \tau^2 \ln^2\left(\frac{\tau}{\eta_*}\right) \\
&+ \frac{c_{s*}^4}{2\nu_*^3} (\delta_{1*} + \epsilon_{1*}\delta_{1*})^2 \frac{1}{\tau} \left(1 - \frac{c_{s*}^2 k^2 \tau^2}{\nu_*^2}\right)^{-3/2} k^4 \tau^4 \ln^2\left(\frac{\tau}{\eta_*}\right). \quad (2.17)
\end{aligned}$$

Therefore, we have five different integrals to calculate in order to evaluate the term Ψ . In the following, we write

$$\Psi = \sum_{i=1}^{i=5} I_i, \quad (2.18)$$

and calculate each of the I_i separately. Let us also notice that the way Eq. (2.17) has been written is not yet fully consistent since all the terms have to be expanded to second-order. For instance, terms like $(\delta_{1*} + \epsilon_{1*}\delta_{1*})/\nu_*$ (in front of the second integral I_2) should clearly be expanded further on in order to keep only second order expressions. For the moment, however, we will be keeping them this way in order to maintain clarity. Only at the end of the calculation these terms will be expanded.

Let us now calculate the five integrals. Defining $w \equiv c_{s*}k\eta/\nu_*$, which implies that $w_* \equiv c_{s*}k\eta_*/\nu_* = -1$, the first integral, I_1 , can be calculated exactly and reads

$$I_1 = -\nu_* \left[(1-u^2)^{1/2} + \ln|u| - \ln\left|1 + (1-u^2)^{1/2}\right| \right] \Big|_{u=w_*}^{u=w}. \quad (2.19)$$

On large scales, w approaches zero and one obtains

$$\lim_{w \rightarrow 0} I_1 = -\nu_* (1 + \ln|w| - \ln 2). \quad (2.20)$$

The second integral is slightly more complicated but can also be carried out exactly. The result can be expressed as

$$\begin{aligned}
I_2 &= \frac{3}{16\nu_*} \left(\epsilon_{1*}\epsilon_{2*} + \frac{1}{2}\epsilon_{2*}\epsilon_{3*} + \delta_{1*}\delta_{2*} \right) \left[4\ln^2|u| - 8\ln|u| \ln\left|\frac{1}{2}(1 + \sqrt{1-u^2})\right| \right. \\
&\quad \left. + 2\ln^2\left|\frac{1}{2}(1 + \sqrt{1-u^2})\right| - 4\text{Li}_2\left(\frac{1}{2} - \frac{\sqrt{1-u^2}}{2}\right) \right] \Big|_{u=w_*}^{u=w}, \quad (2.21)
\end{aligned}$$

where Li_2 denotes the Polygamma function of order two, or dilogarithm function [66]. On large scales the previous expression takes the form

$$\lim_{w \rightarrow 0} I_2 = \frac{3}{16\nu_*} \left(\epsilon_{1*}\epsilon_{2*} + \frac{1}{2}\epsilon_{2*}\epsilon_{3*} + \delta_{1*}\delta_{2*} \right) \left(4\ln^2|w| - 4\ln^2 2 + \frac{\pi^2}{3} \right), \quad (2.22)$$

where we have used $\text{Li}_2(0) = 0$ and $\text{Li}_2(1/2) = \pi^2/12 - (\ln^2 2)/2$. We notice that I_1 and I_2 are logarithmically divergent in the limit $w \rightarrow 0$. We will see that this is not a problem and that those terms cancel out in the final expression of $\mathcal{P}_\zeta$. This is expected since we know that the power spectrum remains constant on large scales, and as such an exact cancellation of those terms constitutes a consistency check of the method. On the contrary, the integrals I_3 , I_4 and I_5 are convergent and can be directly computed. They read

$$\begin{aligned} I_3 &= \nu_* (\delta_{1*} + \epsilon_{1*} \delta_{1*}) (1 - \ln 2), \quad I_4 = -\frac{\nu_*}{2} (\delta_{1*} \delta_{2*} - 2\delta_{1*}^2) \left(\frac{\pi^2}{12} - 2 + 2 \ln 2 - \ln^2 2 \right), \\ I_5 &= \frac{\nu_*}{2} (\delta_{1*} + \epsilon_{1*} \delta_{1*})^2 \left(2 - \frac{\pi^2}{6} - 2 \ln 2 + 2 \ln^2 2 \right). \end{aligned} \quad (2.23)$$

This completes our calculation of the quantity Ψ and we can now evaluate the expression (2.4). Collecting the expressions of a , ϵ_1 , c_s and ν established previously, one gets $c_s^2/(a^2 \nu \epsilon_1)$ that has to be combined with $e^{2\Psi}$ using the above integrals. After some lengthy but straightforward manipulations, one obtains

$$\begin{aligned} \mathcal{P}_\zeta &= \frac{H_*^2 (18e^{-3})}{8\pi^2 M_{\text{Pl}}^2 \epsilon_{1*} c_{s*}} \left[1 + \left(-\frac{8}{3} + 2 \ln 2 \right) \epsilon_{1*} + \left(-\frac{1}{3} + \ln 2 \right) \epsilon_{2*} + \left(\frac{7}{3} - \ln 2 \right) \delta_{1*} \right. \\ &\quad + \left(\frac{23}{18} - \frac{4}{3} \ln 2 + \frac{1}{2} \ln^2 2 \right) \delta_{1*}^2 + \left(\frac{25}{9} - \frac{\pi^2}{24} - \frac{7}{3} \ln 2 + \frac{1}{2} \ln^2 2 \right) \delta_{1*} \delta_{2*} \\ &\quad + \left(-\frac{25}{9} + \frac{13}{3} \ln 2 - 2 \ln^2 2 \right) \epsilon_{1*} \delta_{1*} + \left(\frac{13}{9} - \frac{10}{3} \ln 2 + 2 \ln^2 2 \right) \epsilon_{1*}^2 \\ &\quad + \left(-\frac{2}{9} + \frac{5}{3} \ln 2 - \ln^2 2 \right) \epsilon_{2*} \delta_{1*} + \left(-\frac{25}{9} + \frac{\pi^2}{12} + \frac{1}{3} \ln 2 + \ln^2 2 \right) \epsilon_{1*} \epsilon_{2*} \\ &\quad \left. + \left(-\frac{1}{18} - \frac{1}{3} \ln 2 + \frac{1}{2} \ln^2 2 \right) \epsilon_{2*}^2 + \left(-\frac{1}{9} + \frac{\pi^2}{24} + \frac{1}{3} \ln 2 - \frac{1}{2} \ln^2 2 \right) \epsilon_{2*} \epsilon_{3*} \right]. \end{aligned} \quad (2.24)$$

Several remarks are in order at this stage. Firstly, in the above calculation, we have assumed that the initial state of the perturbations is the Bunch-Davies vacuum. This implies that $|B_k|^2 = \pi/2$. Notice that, in the context of k-inflation, this is a non-trivial choice since, as discussed in Ref. [59], the time dependence of the sound speed could be such that the adiabatic regime is not available anymore⁴. In this paper, we assume that this does not occur and that the function $c_s(\eta)$ is initially smooth enough. Secondly, as announced above, all the time-dependent terms $\ln|w|$ have canceled out and the expression of $\mathcal{P}_\zeta$ is time-independent. Thirdly, Eq. (2.24) should be compared with Eq. (51) of Ref. [59]. These two expressions coincide at first order, which is another indication that the above formula for $\mathcal{P}_\zeta$ is correct. Fourthly, in the overall amplitude, we notice the presence of the factor $18e^{-3}$. As explained in Refs. [48] and [59], this is typical in an approximation scheme based on the WKB method or its extension (such as the uniform approximation). This leads to a $\simeq 10\%$ error in the estimation of the amplitude. In Refs. [53, 54], it was shown that, by taking into account higher order terms in the adiabatic expansion, this shortcomings can easily be fixed. In that case, one obtains a new overall coefficient which dramatically reduces the error in the

⁴Let us notice however that one can still re-define a new time variable to absorb the c_s -dependence in the mode equation [42]. In terms of that new time variable, one could always set Bunch-Davies initial conditions for the scalar, but this would not be compatible with those of the tensor modes.

amplitude. As a consequence, we do not really need to worry about the term $18e^{-3}$ and, for practical applications, one can simply renormalize it to one.

Finally, the above expression of $\mathcal{P}_\zeta$ depends on η_* which depends on k . Our goal is now to make this hidden scale dependence explicit and to re-express the power spectrum at a unique pivot scale defined by

$$k_\diamond \eta_\diamond \equiv -\frac{1}{c_{s\diamond}}. \quad (2.25)$$

This is achieved by re-writing all the quantities appearing in the power spectrum at a single time, $\eta = \eta_\diamond$. Technically, this means that, say, ϵ_{1*} should be written as $\epsilon_{1\diamond} = \epsilon_{1\diamond} - \epsilon_{1\diamond}\epsilon_{2\diamond} \ln(\eta_\diamond/\eta_*) + \mathcal{O}(\epsilon^3)$ and that the dependence in $\eta_\diamond/\eta_*$ should be replaced with a dependence in $k_\diamond/k$. This is performed by making use of the relation between the time $\eta_\diamond$ and η_* :

$$\frac{\eta_*}{\eta_\diamond} = \frac{k_\diamond}{k} \nu_* \frac{c_{s\diamond}}{c_{s*}}. \quad (2.26)$$

Working out the previous equation at second order, one obtains that

$$\begin{aligned} \ln\left(\frac{\eta_*}{\eta_\diamond}\right) &= \left(\ln\frac{3}{2} + \ln\frac{k_\diamond}{k}\right) \left(1 - \delta_{1\diamond} - \epsilon_{1\diamond}\delta_{1\diamond} - \frac{2}{3}\epsilon_{1\diamond}\epsilon_{2\diamond} - \frac{1}{3}\epsilon_{2\diamond}\epsilon_{3\diamond} - \frac{2}{3}\delta_{1\diamond}\delta_{2\diamond} + \delta_{1\diamond}^2\right) \\ &\quad + \frac{2}{3}\epsilon_{1\diamond} + \frac{1}{3}\epsilon_{2\diamond} + \frac{2}{3}\delta_{1\diamond} + \epsilon_{1\diamond}\epsilon_{2\diamond} - \frac{4}{9}\epsilon_{1\diamond}\delta_{1\diamond} + \frac{1}{9}\epsilon_{2\diamond}\epsilon_{3\diamond} + \frac{2}{9}\delta_{1\diamond}\delta_{2\diamond} + \frac{4}{9}\epsilon_{1\diamond}^2 - \frac{1}{18}\epsilon_{2\diamond}^2 \\ &\quad - \frac{8}{9}\delta_{1\diamond}^2 - \frac{5}{9}\epsilon_{2\diamond}\delta_{1\diamond} + \frac{1}{2}\delta_{1\diamond}\delta_{2\diamond} \ln^2\frac{3}{2} + \delta_{1\diamond}\delta_{2\diamond} \ln\frac{3}{2} \ln\frac{k_\diamond}{k} + \frac{1}{2}\delta_{1\diamond}\delta_{2\diamond} \ln^2\frac{k_\diamond}{k}. \end{aligned} \quad (2.27)$$

This finally leads to one of the two main new results of this paper, namely the expression of the scalar power spectrum in k-inflation at second order in the Hubble and sound flow functions

$$\begin{aligned} \mathcal{P}_\zeta &= \frac{H_\diamond^2 (18e^{-3})}{8\pi^2 M_{\text{Pl}}^2 \epsilon_{1\diamond} c_{s\diamond}} \left\{ 1 - 2(1+D)\epsilon_{1\diamond} - D\epsilon_{2\diamond} + (2+D)\delta_{1\diamond} + \left(\frac{2}{9} + D + \frac{D^2}{2}\right)\delta_{1\diamond}^2 \right. \\ &\quad + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\diamond}\delta_{2\diamond} + \left(-\frac{8}{9} - 3D - 2D^2\right)\epsilon_{1\diamond}\delta_{1\diamond} + \left(\frac{17}{9} + 2D + 2D^2\right)\epsilon_{1\diamond}^2 \\ &\quad + \left(\frac{5}{9} - D - D^2\right)\epsilon_{2\diamond}\delta_{1\diamond} + \left(-\frac{11}{9} - D + D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\diamond}\epsilon_{2\diamond} + \left(\frac{2}{9} + \frac{D^2}{2}\right)\epsilon_{2\diamond}^2 \\ &\quad + \left(\frac{\pi^2}{24} - \frac{1}{18} - \frac{D^2}{2}\right)\epsilon_{2\diamond}\epsilon_{3\diamond} + [-2\epsilon_{1\diamond} - \epsilon_{2\diamond} + \delta_{1\diamond} + (1+D)\delta_{1\diamond}^2 \\ &\quad + (2+D)\delta_{1\diamond}\delta_{2\diamond} - (3+4D)\epsilon_{1\diamond}\delta_{1\diamond} + 2(1+2D)\epsilon_{1\diamond}^2 - (1+2D)\epsilon_{2\diamond}\delta_{1\diamond} - (1-2D)\epsilon_{1\diamond}\epsilon_{2\diamond} \\ &\quad + D\epsilon_{2\diamond}^2 - D\epsilon_{2\diamond}\epsilon_{3\diamond}] \ln\frac{k}{k_\diamond} + \left(2\epsilon_{1\diamond}^2 + \epsilon_{1\diamond}\epsilon_{2\diamond} + \frac{1}{2}\epsilon_{2\diamond}^2 - \frac{1}{2}\epsilon_{2\diamond}\epsilon_{3\diamond} + \frac{1}{2}\delta_{1\diamond}^2 + \frac{1}{2}\delta_{1\diamond}\delta_{2\diamond} \right. \\ &\quad \left. \left. - 2\epsilon_{1\diamond}\delta_{1\diamond} - \epsilon_{2\diamond}\delta_{1\diamond}\right) \ln^2\frac{k}{k_\diamond} \right\}, \end{aligned} \quad (2.28)$$

where we have introduced the quantity D defined by $D \equiv 1/3 - \ln 3$. One easily checks that, at first order, this expression exactly coincides with Eq. (53) of Ref. [59]. More details in the comparison of the above formula with the existing literature can be found in Sec. 3.

Using the method of Ref. [67], one can also deduce the expression of the scalar spectral index which reads

$$\begin{aligned}
n_s - 1 = & -2\epsilon_{1\circ} - \epsilon_{2\circ} + \delta_{1\circ} - 2\epsilon_{1\circ}^2 - (2D + 3)\epsilon_{1\circ}\epsilon_{2\circ} + 3\epsilon_{1\circ}\delta_{1\circ} + \epsilon_{2\circ}\delta_{1\circ} - D\epsilon_{2\circ}\epsilon_{3\circ} \\
& - \delta_{1\circ}^2 + (D + 2)\delta_{1\circ}\delta_{2\circ} - 2\epsilon_{1\circ}^3 - \left(\frac{47}{9} + 6D\right)\epsilon_{1\circ}^2\epsilon_{2\circ} + 5\epsilon_{1\circ}^2\delta_{1\circ} \\
& + \left(-\frac{20}{9} - 3D - D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\circ}\epsilon_{2\circ}^2 + \left(-\frac{11}{9} - 4D - D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} \\
& + \left(\frac{73}{9} + 5D\right)\delta_{1\circ}\epsilon_{1\circ}\epsilon_{2\circ} - 4\epsilon_{1\circ}\delta_{1\circ}^2 + \left(\frac{46}{9} + 4D\right)\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} + \frac{4}{9}\epsilon_{2\circ}^2\epsilon_{3\circ} \\
& + \left(-\frac{1}{18} - \frac{D^2}{2} + \frac{\pi^2}{24}\right)\epsilon_{2\circ}\epsilon_{3\circ}^2 + \left(\frac{5}{9} + 2D\right)\delta_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} \\
& + \left(-\frac{1}{18} - \frac{D^2}{2} + \frac{\pi^2}{24}\right)\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} - \delta_{1\circ}^2\epsilon_{2\circ} + \left(\frac{5}{9} + D\right)\delta_{1\circ}\delta_{2\circ}\epsilon_{2\circ} + \delta_{1\circ}^3 \\
& - \left(\frac{50}{9} + 3D\right)\delta_{1\circ}^2\delta_{2\circ} + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\circ}\delta_{2\circ}^2 \\
& + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24}\right)\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}
\end{aligned} \tag{2.29}$$

At first order in the flow parameters, one recovers the standard expression, i.e. $n_s - 1 = -2\epsilon_{1\circ} - \epsilon_{2\circ} + \delta_{1\circ}$. One can also check that the second order corrections are similar to those found in Ref. [59]. Here, for the first time, we have given the formula of the spectral index at third order. This is of course possible only because we have determined the overall amplitude at second order. This also allows us to determine the higher order corrections to the running and to the running of the running. For instance, one can calculate α_s at the fourth order and the running of the running at the fifth order. Here, in order to illustrate the efficiency of the method, we just present the expression of α_s . It reads

$$\begin{aligned}
\alpha_s = & -2\epsilon_{1\circ}\epsilon_{2\circ} - \epsilon_{2\circ}\epsilon_{3\circ} + \delta_{1\circ}\delta_{2\circ} - 6\epsilon_{1\circ}^2\epsilon_{2\circ} - (3 + 2D)\epsilon_{1\circ}\epsilon_{2\circ}^2 - (4 + 2D)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} + 5\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} \\
& + 4\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} - D\epsilon_{2\circ}\epsilon_{3\circ}^2 - D\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} + 2\delta_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} + \delta_{1\circ}\delta_{2\circ}\epsilon_{2\circ} - 3\delta_{1\circ}^2\delta_{2\circ} + (2 + D)\delta_{1\circ}\delta_{2\circ}^2 \\
& + (2 + D)\delta_{1\circ}\delta_{2\circ}\delta_{3\circ} - 12\epsilon_{1\circ}^3\epsilon_{2\circ} - \left(\frac{139}{9} + 14D\right)\epsilon_{1\circ}^2\epsilon_{2\circ}^2 - \left(\frac{83}{9} + 8D\right)\epsilon_{1\circ}^2\epsilon_{2\circ}\epsilon_{3\circ} \\
& + 21\delta_{1\circ}\epsilon_{1\circ}^2\epsilon_{2\circ} + 9\delta_{1\circ}\delta_{2\circ}\epsilon_{1\circ}^2 + \left(-\frac{20}{9} - 3D - D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\circ}\epsilon_{2\circ}^3 \\
& + \left(-\frac{20}{3} - 10D - 3D^2 + \frac{\pi^2}{4}\right)\epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} + \left(\frac{100}{9} + 7D\right)\delta_{1\circ}\epsilon_{1\circ}\epsilon_{2\circ}^2 \\
& + \left(-\frac{11}{9} - 5D - D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 + \left(-\frac{11}{9} - 5D - D^2 + \frac{\pi^2}{12}\right)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} \\
& + \left(\frac{127}{9} + 7D\right)\delta_{1\circ}\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} - 9\delta_{1\circ}^2\epsilon_{1\circ}\epsilon_{2\circ} + \left(\frac{137}{9} + 9D\right)\delta_{1\circ}\delta_{2\circ}\epsilon_{1\circ}\epsilon_{2\circ} - 15\delta_{1\circ}^2\delta_{2\circ}\epsilon_{1\circ} \\
& + \left(\frac{64}{9} + 5D\right)\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}^2 + \left(\frac{64}{9} + 5D\right)\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ} + \frac{8}{9}\epsilon_{2\circ}^2\epsilon_{3\circ}^2 + \frac{4}{9}\epsilon_{2\circ}^2\epsilon_{3\circ}\epsilon_{4\circ} \\
& + \left(-\frac{1}{18} - \frac{D^2}{2} + \frac{\pi^2}{24}\right)\epsilon_{2\circ}\epsilon_{3\circ}^3 + \left(-\frac{1}{6} - \frac{3D^2}{2} + \frac{\pi^2}{8}\right)\epsilon_{2\circ}\epsilon_{3\circ}^2\epsilon_{4\circ} + \left(\frac{5}{9} + 3D\right)\delta_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{1}{18} - \frac{D^2}{2} + \frac{\pi^2}{24} \right) \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ}^2 + \left(-\frac{1}{18} - \frac{D^2}{2} + \frac{\pi^2}{24} \right) \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ} \epsilon_{5\circ} \\
& + \left(\frac{5}{9} + 3D \right) \delta_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ} - 3\delta_{1\circ}^2 \epsilon_{2\circ} \epsilon_{3\circ} + \left(\frac{10}{9} + 3D \right) \delta_{1\circ} \delta_{2\circ} \epsilon_{2\circ} \epsilon_{3\circ} - 3\delta_{1\circ}^2 \delta_{2\circ} \epsilon_{2\circ} \\
& + \left(\frac{5}{9} + D \right) \delta_{1\circ} \delta_{2\circ}^2 \epsilon_{2\circ} + \left(\frac{5}{9} + D \right) \delta_{1\circ} \delta_{2\circ} \delta_{3\circ} \epsilon_{2\circ} + 6\delta_{1\circ}^3 \delta_{2\circ} - \left(\frac{118}{9} + 7D \right) \delta_{1\circ}^2 \delta_{2\circ}^2 \\
& - \left(\frac{68}{9} + 4D \right) \delta_{1\circ}^2 \delta_{2\circ} \delta_{3\circ} + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24} \right) \delta_{1\circ} \delta_{2\circ}^3 \\
& + \left(\frac{37}{6} + 6D + \frac{3D^2}{2} - \frac{\pi^2}{8} \right) \delta_{1\circ} \delta_{2\circ}^2 \delta_{3\circ} + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24} \right) \delta_{1\circ} \delta_{2\circ} \delta_{3\circ}^2 \\
& + \left(\frac{37}{18} + 2D + \frac{D^2}{2} - \frac{\pi^2}{24} \right) \delta_{1\circ} \delta_{2\circ} \delta_{3\circ} \delta_{4\circ}
\end{aligned} \tag{2.30}$$

One can check that the second and third order corrections match the expression already found in Ref. [59]. The fourth order corrections represent a new result.

2.4 Tensor Power Spectrum

In this section, we repeat the previous analysis but for tensor perturbations. Since the method is the same and, fortunately, the calculations are easier, the details will be skipped. The main difference between gravity waves and density perturbations is that their effective potential is not the same, see Eqs. (1.4) and (1.6). This implies that the function $\nu(\eta)$ for tensors is different from the one of the scalars. One gets for the tensor

$$\nu^2(\eta) = \frac{9}{4} + 3\epsilon_{1*} + 4\epsilon_{1*}^2 + 4\epsilon_{1*}\epsilon_{2*} - 3\epsilon_{1*}\epsilon_{2*} \ln\left(\frac{\eta}{\eta_*}\right) + \mathcal{O}(\epsilon^3). \tag{2.31}$$

As a consequence, the functions $g(\eta)$, $f(\eta)$, and hence Ψ , are also different. Using the uniform approximation to evaluate $\mu_{\mathbf{k}}$ and inserting the corresponding formula into the expression of $\mathcal{P}_h$ given by Eq. (1.9), one obtains

$$\begin{aligned}
\mathcal{P}_h = & \frac{2(18e^{-3})H_*^2}{\pi^2 M_{\text{Pl}}^2} \left[1 + \left(-\frac{8}{3} + 2\ln 2 \right) \epsilon_{1*} + \left(\frac{\pi^2}{12} - \frac{26}{9} + \frac{8}{3} \ln 2 - \ln^2 2 \right) \epsilon_{1*}\epsilon_{2*} \right. \\
& \left. + \left(\frac{13}{9} - \frac{10}{3} \ln 2 + 2\ln^2 2 \right) \epsilon_{1*}^2 \right].
\end{aligned} \tag{2.32}$$

This equation is for the tensors what Eq. (2.24) is for the scalars. As explained before, one has still to make explicit the scale dependence hidden in η_* . In the case of tensors, the pivot point is usually defined by $k_*\eta_* = -1$ since gravity waves propagate at the speed of light. This leads to the following expression for the power spectrum

$$\begin{aligned}
\mathcal{P}_h = & \frac{2(18e^{-3})H_*^2}{\pi^2 M_{\text{Pl}}^2} \left\{ 1 - 2(1+D)\epsilon_{1*} + \left(\frac{17}{9} + 2D + 2D^2 \right) \epsilon_{1*}^2 + \left(-\frac{19}{9} + \frac{\pi^2}{12} \right. \right. \\
& \left. \left. - 2D - D^2 \right) \epsilon_{1*}\epsilon_{2*} + [-2\epsilon_{1*} + 2(1+2D)\epsilon_{1*}^2 - 2(1+D)\epsilon_{1*}\epsilon_{2*}] \ln \frac{k_*}{k} \right. \\
& \left. + (2\epsilon_{1*}^2 - \epsilon_{1*}\epsilon_{2*}) \ln^2 \frac{k_*}{k} \right\}.
\end{aligned} \tag{2.33}$$

Let us notice that, in order to obtain this relationship, we have used the initial conditions for gravity waves $|B_{\mathbf{k}}| = 1/M_{\text{Pl}}^2$. Otherwise, one notices the presence of the WKB factor $18e^3$ and one can check that, at first order, it coincides with the known expression for the tensor power spectrum. The above formula, being expressed at the time η_* , is convenient for SFMK models only, but not for k-inflation. Indeed, all parameters here are functions evaluated at the time η_* which is different that the one at which the scalar power spectrum is calculated, namely $\eta_\diamond$. It has become a common mistake to try fitting data with both Eq. (2.28) and Eq.(2.33) while implicitly assuming that all Hubble and sound flow “parameters” are the same. As we have explicitly shown before, they do differ and such a fit would absolutely make no sense.

However, within slow-roll, one can re-express the tensor power spectrum at the same pivot point as for the scalar power spectrum. As before, each quantity in the tensor power spectrum should be re-expressed at the scalar pivot point, as for instance $\epsilon_{1\star} = \epsilon_{1\diamond} - \epsilon_{1\diamond}\epsilon_{2\diamond}\ln c_{\text{s}\diamond} + \mathcal{O}(\epsilon^3, \delta^3)$. The quantity $c_{\text{s}\diamond}$ appears because it is present in the ratio of the tensor to scalar pivot points. It follows that the final expression for the tensor power spectrum for k-inflation is

$$\begin{aligned} \mathcal{P}_h = & \frac{2(18e^{-3})H_\diamond^2}{\pi^2 M_{\text{Pl}}^2} \left\{ 1 - 2(1 + D - \ln c_{\text{s}\diamond})\epsilon_{1\diamond} + \left[\frac{17}{9} + 2D + 2D^2 + 2\ln^2 c_{\text{s}\diamond} \right. \right. \\ & - 2(1 + 2D)\ln c_{\text{s}\diamond} \left. \right] \epsilon_{1\diamond}^2 + \left[-\frac{19}{9} + \frac{\pi^2}{12} - 2D - D^2 + 2(1 + D)\ln c_{\text{s}\diamond} - \ln^2 c_{\text{s}\diamond} \right] \epsilon_{1\diamond}\epsilon_{2\diamond} \\ & + \left[-2\epsilon_{1\diamond} + (2 + 4D - 4\ln c_{\text{s}\diamond})\epsilon_{1\diamond}^2 + (-2 - 2D + 2\ln c_{\text{s}\diamond})\epsilon_{1\diamond}\epsilon_{2\diamond} \right] \ln \frac{k_\diamond}{k} \\ & \left. + (2\epsilon_{1\diamond}^2 - \epsilon_{1\diamond}\epsilon_{2\diamond}) \ln^2 \frac{k_\diamond}{k} \right\}, \end{aligned} \quad (2.34)$$

where now “diamonded” terms are evaluated at the scalar pivot point. This new formula is the second main result of the present paper. It extends to second order the results of Ref. [59]. As for the scalar modes, this expression also allows us to calculate the tensor spectral index at third order. One obtains

$$\begin{aligned} n_{\text{T}} = & -2\epsilon_{1\diamond} - 2\epsilon_{1\diamond}^2 + (-2 - 2D + 2\ln c_{\text{s}\diamond})\epsilon_{1\diamond}\epsilon_{2\diamond} - 2\epsilon_{1\diamond}^3 + \left(-\frac{38}{9} - 6D + 6\ln c_{\text{s}\diamond} \right) \epsilon_{1\diamond}^2\epsilon_{2\diamond} \\ & + \left(-\frac{19}{9} - 2D - D^2 + \frac{\pi^2}{12} + 2\ln c_{\text{s}\diamond} + 2D\ln c_{\text{s}\diamond} - \ln^2 c_{\text{s}\diamond} \right) \epsilon_{1\diamond}\epsilon_{2\diamond}^2 \\ & + \left(-\frac{19}{9} - 2D - D^2 + \frac{\pi^2}{12} + 2\ln c_{\text{s}\diamond} + 2D\ln c_{\text{s}\diamond} - \ln^2 c_{\text{s}\diamond} \right) \epsilon_{1\diamond}\epsilon_{2\diamond}\epsilon_{3\diamond} \end{aligned} \quad (2.35)$$

Of course, at first order, one recovers the standard formula, $n_{\text{T}} = -2\epsilon_{1\diamond}$. We have already discussed before the relevance of higher order corrections for Bayesian parameter estimation. Notice that, in the case of primordial gravitational waves and as discussed in Ref. [68], another motivation is the possibility of detecting them directly. Indeed, in that case, one needs to estimate their power spectrum today and, due to the very large lever arm between the cosmological scales and the smaller scales where a direct detection can be performed, it is necessary to calculate the power spectrum at the end of inflation very precisely. In this context, higher order corrections become mandatory. Similarly, the running of the tensors is

obtained at fourth order and reads

$$\begin{aligned}
\alpha_T = & -2\epsilon_{1\circ}\epsilon_{2\circ} - 6\epsilon_{1\circ}^2\epsilon_{2\circ} + (-2 - 2D + 2\ln c_{s\circ})\epsilon_{1\circ}\epsilon_{2\circ}^2 + (-2 - 2D + 2\ln c_{s\circ})\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} \\
& - 12\epsilon_{1\circ}^3\epsilon_{2\circ} + \left(-\frac{112}{9} - 14D + 14\ln c_{s\circ}\right)\epsilon_{1\circ}^2\epsilon_{2\circ}^2 + \left(-\frac{56}{9} - 8D + 8\ln c_{s\circ}\right)\epsilon_{1\circ}^2\epsilon_{2\circ}\epsilon_{3\circ} \\
& + \left[-\frac{19}{9} - 2D - D^2 + \frac{\pi^2}{12} + 2(1+D)\ln c_{s\circ} - \ln^2 c_{s\circ}\right](\epsilon_{1\circ}\epsilon_{2\circ}^3 + 3\epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} \\
& + \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}).
\end{aligned} \tag{2.36}$$

Finally, one can also deduce the tensor to scalar ratio at the third order. It reads

$$\begin{aligned}
r = & 16\epsilon_{1\circ}c_{s\circ}\left[1 + 2\epsilon_{1\circ}\ln c_{s\circ} + D\epsilon_{2\circ} - (2+D)\delta_{1\circ} + \left(\frac{34}{9} + 3D + \frac{D^2}{2}\right)\delta_{1\circ}^2\right. \\
& + \left(-\frac{37}{18} - 2D - \frac{D^2}{2} + \frac{\pi^2}{24}\right)\delta_{1\circ}\delta_{2\circ} - \left(\frac{5}{9} + 3D + D^2\right)\delta_{1\circ}\epsilon_{2\circ} + \left(-\frac{2}{9} + \frac{D^2}{2}\right)\epsilon_{2\circ}^2 \\
& + \frac{1}{72}(4 + 36D^2 - 3\pi^2)\epsilon_{2\circ}\epsilon_{3\circ} + 2\epsilon_{1\circ}^2(1 + \ln c_{s\circ})\ln c_{s\circ} \\
& + \left(-\frac{28}{9} - 3D - 4\ln c_{s\circ} - 2D\ln c_{s\circ}\right)\delta_{1\circ}\epsilon_{1\circ} \\
& \left. + \left(-\frac{8}{9} + D + 2\ln c_{s\circ} + 4D\ln c_{s\circ} - \ln^2 c_{s\circ}\right)\epsilon_{1\circ}\epsilon_{2\circ}\right].
\end{aligned} \tag{2.37}$$

As usual the leading term is proportional to $\epsilon_{1\circ}c_{s\circ}$ and the above formula shows that the corresponding corrections depend on the flow parameters but also on the sound speed.

3 Discussion and Conclusions

The power spectra of Eqs. (2.28) and (2.34) represent the main result of this article. There are the first calculation, at second order in the Hubble and sound flow functions, of the scalar and tensor power spectra in k-inflation within the uniform approximation. In this section, we discuss our results and check their consistency. In particular, in some limits, our calculation should reproduce known results already derived in the literature. As we show below, this is indeed the case.

We have seen before that the power spectrum is obtained as an expansion around the pivot scale and that the most general expression of $\mathcal{P}_\zeta$ can be written as

$$\mathcal{P}_\zeta(k) = \tilde{\mathcal{P}}_\zeta(k_\circ) \sum_{n \geq 0} \frac{a_n}{n!} \ln^n \frac{k}{k_\circ}, \tag{3.1}$$

where $\tilde{\mathcal{P}}_\zeta$ is the overall amplitude and the coefficients a_n are functions of the horizon flow parameters. The expression of a_n always starts at order n , i.e. a_0 starts with one, a_1 starts with a term of order $\mathcal{O}(\epsilon, \delta)$, a_2 with a term of order $\mathcal{O}(\epsilon^2, \delta^2, \epsilon\delta)$ and so on. As already mentioned before, k-inflationary power spectra were determined at first order in Ref. [59]. This means that the expression found in that paper included only the first two terms, proportional to a_0 and a_1 . There is however a trick derived in Ref. [67] which allows us to determine some higher order terms. Indeed, the power spectrum should not depend

on the choice of the pivot scale, which is arbitrary. As a consequence, one can establish the following recursion relation

$$a_{n+1} = \frac{d \ln \tilde{\mathcal{P}}_\zeta}{d \ln k_\diamond} a_n + \frac{da_n}{d \ln k_\diamond}. \quad (3.2)$$

Given that a_0 was given at first order in Ref. [59], it was then possible to calculate a_1 up to second order and a_2 to third order [see Eqs. (64) and (65) in that reference]. Therefore, one can compare those formulas to the expression obtained in this article. One finds that they are the same, indicating the consistency of our results.

Another way to verify the validity of our expressions is to take the limit $c_s = 1$ and to compare the resulting formulas to the results already obtained in the literature for SFMK models. As mentioned in the introduction, second order results were first obtained using the Green function method in Ref. [47]. The corresponding expression for the scalar power spectrum reads

$$\begin{aligned} \mathcal{P}_\zeta = \frac{H^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_1} & \left\{ 1 - 2(C+1)\epsilon_1 - C\epsilon_2 + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right) \epsilon_1^2 \right. \\ & + \left(C^2 - C + \frac{7\pi^2}{12} - 7\right) \epsilon_1 \epsilon_2 + \left(\frac{C^2}{2} + \frac{\pi^2}{8} - 1\right) \epsilon_2^2 + \left(-\frac{C^2}{2} + \frac{\pi^2}{24}\right) \epsilon_2 \epsilon_3 \\ & + [-2\epsilon_1 - \epsilon_2 + 2(2C+1)\epsilon_1^2 + (2C-1)\epsilon_1 \epsilon_2 + C\epsilon_2^2 - C\epsilon_2 \epsilon_3] \ln \frac{k}{k_\otimes} \\ & \left. + \left(2\epsilon_1^2 + \epsilon_2 \epsilon_2 + \frac{1}{2}\epsilon_2^2 - \frac{1}{2}\epsilon_2 \epsilon_3\right) \ln^2 \frac{k}{k_\otimes} \right\}, \end{aligned} \quad (3.3)$$

where the constant C is defined by $C \equiv \gamma + \ln 2 - 2 \simeq -0.7296$, γ being the Euler constant, while the expression of the gravity wave power spectrum can be written as

$$\begin{aligned} \mathcal{P}_h = \frac{2H^2}{\pi^2 M_{\text{Pl}}^2} & \left\{ 1 - 2(C+1)\epsilon_1 + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right) \epsilon_1^2 + \left(-C^2 - 2C + \frac{\pi^2}{12} - 2\right) \epsilon_1 \epsilon_2 \right. \\ & \left. + [-2\epsilon_1 + 2(2C+1)\epsilon_1^2 - 2(C+1)\epsilon_1 \epsilon_2] \ln \frac{k}{k_\otimes} + \left(2\epsilon_1^2 - \epsilon_1 \epsilon_2\right) \ln^2 \frac{k}{k_\otimes} \right\}. \end{aligned} \quad (3.4)$$

In the two previous formulas (3.3) and (3.4), the Hubble flow functions are evaluated at time $\eta_\otimes$ such that $a(\eta_\otimes)H(\eta_\otimes) = k_\otimes$ which slightly differs from the time $k_\diamond \eta_\diamond = -1$ (for $c_{s\diamond} = 1$) used in the present paper. Therefore, if we want to compare Eqs. (2.28) and (2.34) with $c_{s\diamond} = 1$ to Eqs. (3.3) and (3.4), one should first re-express the latter in terms of the Hubble flow parameters evaluated at time $k_\diamond \eta_\diamond = -1$. In the following, in order to simplify the discussion, we focus only on the scalar case but the tensor case could be treated in the same manner. From the definition of $\eta_\otimes$ one has $\eta_\otimes/\eta_\diamond = 1 + \epsilon_{1\diamond} + \epsilon_{1\diamond}^2 + \epsilon_{1\diamond}\epsilon_{2\diamond} + \mathcal{O}(\epsilon^3)$. As consequence, in Eqs. (3.3) and (3.4), one should just replace ϵ_1, ϵ_2 with $\epsilon_{1\diamond}, \epsilon_{2\diamond}$ and H^2/ϵ_1

with $H_\diamond^2/\epsilon_{1\diamond}(1 + 2\epsilon_{1\diamond}^2 + \epsilon_{1\diamond}\epsilon_{2\diamond})$. This yields the following expression

$$\begin{aligned} \mathcal{P}_\zeta = & \frac{H_\diamond^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_{1\diamond}} \left\{ 1 - 2(C+1)\epsilon_{1\diamond} - C\epsilon_{2\diamond} + \left(2C^2 + 2C + \frac{\pi^2}{2} - 3\right) \epsilon_{1\diamond}^2 \right. \\ & + \left(C^2 - C + \frac{7\pi^2}{12} - 6\right) \epsilon_{1\diamond}\epsilon_{2\diamond} + \left(\frac{C^2}{2} + \frac{\pi^2}{8} - 1\right) \epsilon_{2\diamond}^2 + \left(-\frac{C^2}{2} + \frac{\pi^2}{24}\right) \epsilon_{2\diamond}\epsilon_{3\diamond} \\ & + [-2\epsilon_{1\diamond} - \epsilon_{2\diamond} + 2(2C+1)\epsilon_{1\diamond}^2 + (2C-1)\epsilon_{1\diamond}\epsilon_{2\diamond} + C\epsilon_{2\diamond}^2 - C\epsilon_{2\diamond}\epsilon_{3\diamond}] \ln \frac{k}{k_\diamond} \\ & \left. + \left(2\epsilon_{1\diamond}^2 + \epsilon_{1\diamond}\epsilon_{2\diamond} + \frac{1}{2}\epsilon_{2\diamond}^2 - \frac{1}{2}\epsilon_{2\diamond}\epsilon_{3\diamond}\right) \ln^2 \frac{k}{k_\diamond} \right\}, \end{aligned} \quad (3.5)$$

that can be now compared to Eq. (2.28). As already discussed, the overall amplitude differs by the WKB factor $18e^{-3}$. We also notice that the terms in D in Eq. (2.28) exactly corresponds to the term in C in Eq. (3.5). For instance, the coefficient of $\epsilon_{1\diamond}^2$ in Eq. (2.28) contains a term $2D^2 + 2D$ while the coefficient of $\epsilon_{1\diamond}^2$ in Eq. (3.5) contains a $2C^2 + 2C$. One easily checks that this is the rule for all first and second order terms. Provided one substitutes D with C , the first order term in the amplitude, the coefficient of $\ln k/k_\diamond$ and the coefficient of $\ln^2(k/k_\diamond)$ are identical. The only difference appears in the second order terms in the amplitude. For instance, the coefficients of $\epsilon_{1\diamond}^2$ in Eq. (2.28) is $2D^2 + 2D + 17/9$ while it is $2C^2 + 2C + \pi^2/2 - 3$ in Eq. (3.5). But $17/9 \simeq 1.88$ and $\pi^2/2 - 3 \simeq 1.93$ and, therefore, the two terms are in fact numerically very close. The same is true for all the other terms in the amplitude. Therefore, we conclude that our result (2.28), specialized to SFMK models, is fully consistent with Eq. (3.5) that comes from another approximation scheme. This confirms its validity.

Let us now compare our result to the one of Refs. [53, 54] calculated with the help of the WKB approximation. The scalar power spectrum obtained in those articles reads

$$\begin{aligned} \mathcal{P}_\zeta = & \frac{H^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_1} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1)\epsilon_1 - D_{\text{WKB}}\epsilon_2 + \left(2D_{\text{WKB}}^2 + 2D_{\text{WKB}} - \frac{1}{9}\right) \epsilon_1^2 \right. \\ & + \left(D_{\text{WKB}}^2 - D_{\text{WKB}} + \frac{\pi^2}{12} - \frac{20}{9}\right) \epsilon_1\epsilon_2 + \left(\frac{D_{\text{WKB}}^2}{2} + \frac{2}{9}\right) \epsilon_2^2 + \left(-\frac{D_{\text{WKB}}^2}{2} + \frac{\pi^2}{24} - \frac{1}{18}\right) \epsilon_2\epsilon_3 \\ & + [-2\epsilon_1 - \epsilon_2 + 2(2D_{\text{WKB}} + 1)\epsilon_1^2 + (2D_{\text{WKB}} - 1)\epsilon_1\epsilon_2 + D_{\text{WKB}}\epsilon_2^2 - D_{\text{WKB}}\epsilon_2\epsilon_3] \ln \frac{k}{k_\otimes} \\ & \left. + \left(2\epsilon_1^2 + \epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right) \ln^2 \frac{k}{k_\otimes} \right\}, \end{aligned} \quad (3.6)$$

while the tensor power spectrum is given by the following formula

$$\begin{aligned} \mathcal{P}_h = & \frac{2H^2}{\pi^2 M_{\text{Pl}}^2} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1)\epsilon_1 + \left(2D_{\text{WKB}}^2 + 2D_{\text{WKB}} - \frac{1}{9}\right) \epsilon_1^2 + \left(-D_{\text{WKB}}^2 - 2D_{\text{WKB}} \right. \right. \\ & + \left.\frac{\pi^2}{12} - \frac{19}{9}\right) \epsilon_1\epsilon_2 + [-2\epsilon_1 + 2(2D_{\text{WKB}} + 1)\epsilon_1^2 - 2(D_{\text{WKB}} + 1)\epsilon_1\epsilon_2] \ln \frac{k}{k_\otimes} \\ & \left. + (2\epsilon_1^2 - \epsilon_1\epsilon_2) \ln^2 \frac{k}{k_\otimes} \right\}. \end{aligned} \quad (3.7)$$

In these equations, $A_{\text{WKB}} = 18e^{-3}$ and $D_{\text{WKB}} = 1/3 - \ln 3$, that is to say exactly what was found by means of the uniform approximation as $D_{\text{WKB}} = D$. As already mentioned, Refs. [53, 54] have shown that, by taking the next order in the adiabatic approximation into account, one obtains a new value for these two constants (in some sense, they are renormalized), namely A_{WKB} becomes $361/(18e^3) \simeq 0.99$ and $D_{\text{WKB}} = 7/19 - \ln 3 \simeq -0.7302$. In particular, the new value of D_{WKB} is closer to the constant C than the non-renormalized one. Both Eqs. (3.6) and (3.7) are evaluated at the pivot time η_* and have to be time-shifted to η_0 to be compared with our results. Proceeding as previously, it is easy to show that this modifies the coefficients of ϵ_1^2 which now becomes $2D_{\text{WKB}}^2 + 2D_{\text{WKB}} + 17/9$, and the coefficient of $\epsilon_1\epsilon_2$ which becomes $2D_{\text{WKB}}^2 - D_{\text{WKB}} + \pi^2/12 - 11/9$. In other words Eqs. (2.28) and Eq. (3.6) expressed at η_0 are exactly the same for $c_{s_0} = 1$. This is maybe not so surprising considering the fact that the WKB and uniform approximations are closely related methods.

A few words are in order about Ref. [55]. Historically, this is probably the first paper that attempted to evaluate the k-inflationary power spectrum at second order in some equivalent of the Hubble and sound flow functions used here. The method chosen is the Green function expansion discussed before. However, a specific form for the sound speed, which in the language of the present paper would be a first order approximation of c_s^2 , was also postulated. Together with a k -dependence kept implicit, this makes the comparison with the present work difficult. For this reason, we do not investigate further this issue.

To conclude, let us briefly recap our main result and discuss directions for future works. In this paper, using the uniform approximation, we have calculated the scalar and tensor power spectra in k-inflation, at second order in the Hubble and sound flow parameters, see Eqs. (2.28) and (2.34). We have carefully checked that, in the various limits where our calculation reduces to known cases, consistent results are obtained. The next step is clearly to use these power spectra in order to constrain the values of the Hubble and sound flow parameters using CMB observations. This was done in Ref. [62] but only for the first order power spectra (since only this result was available at that time). Given the on-going flux of high precision data, such as those from the Planck satellite, the results obtained in this article should be important to keep theoretical uncertainties at a minimal level. In this way, as discussed in the introduction, one may hope to obtain unprecedented information on the inflationary scenario.

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