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REPRINT | 3286

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Minimal farsighted instability

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Accepted: 4 February 2024

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Abstract

We propose the notion of minimal farsighted instability to determine the states that are more likely to emerge in the long run when agents are farsighted. A state is minimally farsighted unstable if there is no other state which is more farsightedly stable. To formulate what it means to be more farsightedly stable, we compare states by comparing (in the set inclusion or cardinal sense) their sets of farsighted defeating states. We next compare states in terms of their absorbtiveness by comparing both their sets of farsighted defeating states (i.e. in terms of their stability) and their sets of farsighted defeated states (i.e. in terms of their reachability). A state is maximally farsighted absorbing if there is no other state which is more farsightedly absorbing. We provide general results for characterizing minimally farsighted unstable states and maximally farsighted absorbing states, and we study their relationships with alternative notions of farsightedness. Finally, we use experimental data to show the relevance of the new solution concepts.

Keywords Social environments · States · Stability comparisons · Farsighted players

JEL classification: A14 · C70 · D20

1 Introduction

The organization of individual agents into networks and groups has an important role in the determination of the outcome of many social and economic interactions. For instance, networks of relationships help to determine the careers that people choose, the jobs they obtain, the products they buy, and how they vote. Partitioning of societies into groups is also important in many contexts, such as the provision of public goods and formation of alliances, cartels and federations. Social

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environments constitute a framework in which it is possible to study how groups of agents interact in a society (see Chwe 1994). The framework is general enough to analyze both non-cooperative and cooperative games as well as models of coalition and network formation, and matching markets. In the theoretical literature on coalition and network formation several approaches and related solution concepts have been developed, depending on whether it is assumed that agents are myopic or farsighted. Myopic agents do not forecast how others might react to their actions. However, farsighted agents might not add a link (might add a link) that appears valuable to them (that is not valuable to them) given the current network, as that might in turn lead to the formation of other links and ultimately lower (ultimately increase) the payoffs of the original deviating agents.

Stability is considered to be a crucial property in social environments. When agents are myopic, a state is stable if it is not dominated by any other state, with one state y dominating state x if some coalition has the power or ability to change the state from x to y and all members of the coalition prefer y to x . When agents are farsighted, a state is farsightedly stable if it is not farsightedly dominated by any other state. A state farsightedly defeats (or dominates) another state if there is a farsighted improving path from the latter to the former state. A farsighted improving path is a sequence of states that can emerge when agents deviate based on the improvement the end state offers relative to the current state.¹ But, there are many social environments where a farsightedly stable state fails to exist. In fact, requiring a state to be immune to any profitable farsighted deviations is very demanding. Indeed, farsighted deviations are not limited to a specific horizon and so they even include myopic deviations. Hence, we propose the notion of minimal farsighted instability to predict the states that are most likely to emerge in the long run when agents are farsighted.

A state is said to be minimally farsighted unstable if there is no other state which is more farsightedly stable. To formulate what it means to be more farsightedly stable, we consider comparing states by comparing their sets of farsighted defeating states.² A state is more farsightedly stable than another state if the set of farsighted defeating states in the former is a proper subset of the set of farsighted defeating states in the latter state. We also consider a corresponding cardinal version such that a state is cardinally more farsightedly stable than another state if the number of farsighted defeating states in the former is less than the number of farsighted defeating states in the latter state. A state is said to be cardinally minimally farsighted unstable if there is no other state which is cardinally more farsightedly stable.³ Since

¹ Jackson (2008) defines the notion of farsightedly pairwise stable network: a network is a farsightedly pairwise stable network if and only if it is not farsightedly defeated.

² The set of farsighted defeating states of a given state is the set of all states that farsightedly defeat this state.

³ The existence of a farsighted improving path means that some agents could leave the current state anticipating further deviations from other agents leading to a final state, where all the agents deviating along the path are better off than at the current state when they deviate. Thus, it is a potential source of instability. And the higher the number of farsighted improving paths emanating from a state, the higher the potential instability of such state.

both binary relations are transitive, (cardinally) minimally farsighted unstable states always exist.

Clearly, if a state is more farsightedly stable than another state, then it is also cardinally more farsightedly stable. Hence, the set of cardinally minimally farsighted unstable states refines the set of minimally farsighted unstable states. We also provide results that are helpful for characterizing the minimally farsighted unstable states. If there is a farsighted improving path from one state to another one, then the former cannot be more farsightedly stable than the latter state. Hence, if there is a farsighted improving path from all other states to some state, then this state is minimally farsighted unstable. Moreover, if the set of states that defeat a state is a proper subset of the set of states that defeat any other state, then the former state is the unique minimally farsighted unstable state.

Our method to compare states by their farsighted stability is inspired by methods used in recent matching studies. Doğan and Ehlers (2021) compare assignments in school choice in terms of (myopic) stability by comparing (in the set inclusion or cardinal sense) their sets of blocking pairs or comparing (in the set inclusion or cardinal sense) their sets of blocking students involved in at least one blocking pair. Recently, Doğan and Ehlers (2022) use similar methods to compare assignments in the context of priority-based allocation of objects.⁴ Here, we adapt their methods for matching problems with myopic agents to social environments with farsighted agents.

In addition of comparing states in terms of their stability, one may also require to compare them in terms of their reachability. Indeed, selecting a state that is more farsightedly stable than another one might be a more robust prediction if at the same time the former is more likely to be reached than the latter one. A state is said to be maximally farsighted absorbing if there is no other state which is more farsightedly absorbing. To formulate what it means to be more farsightedly absorbing, we consider comparing states by comparing both their sets of farsighted defeating states (i.e. in terms of their stability) and their sets of farsighted defeated states (i.e. in terms of their reachability).⁵ A state is more farsightedly absorbing than another state if (i) the set of farsighted defeating states in the former is a proper subset of the set of farsighted defeating states in the latter state and (ii) the set of farsighted defeated states in the latter is a proper subset of the set of farsighted defeated states in the former state. We also consider a corresponding cardinal version such that a state is cardinally more farsightedly absorbing than another state if (i) the number of farsighted defeating states in the former is less than the number of farsighted defeating states in the latter state and (ii) the number of farsighted defeated states in the latter is less than the number of farsighted defeated states in the former state. A state is said to be cardinally maximally farsighted absorbing if there is no other state

⁴ Minimally unstable assignments (in the set inclusion sense) are considered by Abdulkadiroğlu et al. (2020) and Tang and Zhang (2021) for school choice problems and by Combe et al. (2022) for teacher assignment problems.

⁵ The set of farsighted defeated states of a given state is the set of all states that this state farsightedly defeats.

which is cardinally more farsightedly absorbing. Again, since both binary relations are transitive, (cardinally) maximally farsighted absorbing states always exist.

Obviously, if a state is more farsightedly absorbing than another state, then it is also cardinally more farsightedly absorbing. Hence, the set of cardinally maximally farsighted absorbing states refines the set of maximally farsighted absorbing states. Moreover, we show that, if there is a farsighted improving path from all other states to some state, then this state is minimally farsighted unstable, maximally farsighted absorbing and Pareto efficient. Finally, every closed cycle contains a maximally farsighted absorbing state.

Another common approach for analysing states that emerge in the long run when agents are farsighted are set-valued concepts like the vNM farsighted stable set (Herings et al. 2009; Mauleon et al. 2011; Ray and Vohra 2015), the largest consistent set (Chwe 1994; Page et al. 2005), the DEM farsighted stable set (Herings et al. 2009) or the horizon- K farsighted set (Herings et al. 2019).⁶ While vNM farsighted stable sets do not always exist, DEM farsighted stable sets and horizon- K farsighted sets always exist. However, it is often complex to compute all DEM farsighted stable sets or horizon- K farsighted sets. Moreover, as set-valued concepts, one cannot discriminate among the different states contained in one of these sets. Hence, minimal farsighted instability or maximal farsighted absorbtiveness could be useful for selecting or comparing states belonging to set-valued concepts like vNM farsighted stable sets. For instance, by simply comparing the set of states that farsightedly dominate or are farsightedly dominated by two given states, we are able to provide two different, though not complete, orderings of the states based on their farsighted stability and their farsighted absorbtiveness. Thus, our newly defined concepts are not only an alternative to set-valued concepts but can also complement them for providing more precise or accurate predictions.

With respect to the relationships between set-valued concepts and our newly defined concepts, we show that if a singleton set is a vNM farsighted stable set (a DEM farsighted stable set) then this single state in the set is both minimally farsighted unstable and maximally farsighted absorbing. Also, any vNM farsighted stable set (any DEM farsighted stable set) contains a minimally farsighted unstable state. Moreover, if there is a unique DEM farsighted stable set, then this set coincides with the set of (cardinally) minimally farsighted unstable states.

We use experimental data from Kirchsteiger et al. (2016) on network formation to test the relevance of the new solution concepts. We find that the set of minimally farsighted unstable networks is not significantly different than the set of maximally farsighted absorbing networks. Moreover, the set of minimally farsighted unstable networks predicts most networks that occur with positive probability while the set of cardinally minimally farsighted unstable networks is good at selecting the network with the highest frequency. Finally, both the set of minimally farsighted unstable

⁶ Alternative notions of farsightedness are suggested by Bloch and van den Nouweland (2020); Diamantoudi and Xue (2003); Dutta et al. (2005); Dutta and Vohra (2017); Ray and Vohra (2019); Herings et al. (2004); Kimya (2020); Mauleon and Vannetelbosch (2004); Page and Wooders (2009); Xue (1998) among others.

networks and the set of cardinally minimally farsighted unstable networks outperform on average the vNM farsighted stable set.

The paper is organized as follows. In Sect. 2 we introduce some notation. In Sect. 3 we define the notion of (cardinally) minimally unstable state. In Sect. 4 we define the notion of (cardinally) maximally absorbing state. In Sect. 5 we study the relationships with set-valued concepts. In Sect. 6 we test the relevance of the new solution concepts using experimental data on network formation. Finally, in Sect. 7 we conclude.

2 Social environments

We consider a general setting described by a social environment, $\Gamma = (N, X, E, u_i(\cdot))$, where $N = \{1, 2, \dots, n\}$ is the set of players and X is the set of states. Let $\mathcal{N}$ denote the set of all subsets of N . The effectivity correspondence, $E : X \times X \rightarrow \mathcal{N}$, specifies the coalitions that have the ability to enforce a state over another state: for $x, y \in X$, $E(x, y)$ is the (possibly empty) set of coalitions that can enforce y over x . Finally, $u_i : X \rightarrow \mathbb{R}$ specifies the utility function of player $i \in N$.

The set of states as well as the effectivity correspondence depend on the structure of the model being studied. For instance, in a model of network (coalition) formation, a state refers to a network structure (coalition structure) and the effectivity correspondence indicates the coalitions that can enforce a network g' (partition p') over a network g (partition p). In a non-cooperative normal form game, a state refers to a strategy profile and the effectivity correspondence only allows a single player to change her strategy.

For a given social environment $\Gamma = (N, X, E, u_i(\cdot))$, we say that a state $y \in X$ dominates another state $x \in X$ if there is a coalition such that (i) it can enforce y over x and (ii) at least one of its members is better off at y while all other members are at least as well off. Formally, a state $y \in X$ dominates a state $x \in X$ if there exists a coalition $S \in E(x, y)$ such that for all $i \in S$, $u_i(y) \geq u_i(x)$ and there is at least one $j \in S$ such that $u_j(y) > u_j(x)$. Let some state $x \in X$ be given. The subset of X consisting of all states that dominate x is denoted by $\rho(x)$. We say that a state $x \in X$ is stable if it is not dominated by any other state. Formally, a state $x \in X$ is stable if $\rho(x) = \emptyset$. The set of stable states $\mathcal{Z} \subseteq X$ is defined as follows: $\mathcal{Z} = \{x \in X \mid \rho(x) = \emptyset\}$. This stability notion assumes that players are myopic since they do not anticipate that other players may react to their changes. In general, there is no guarantee that the set $\mathcal{Z}$ is non-empty.⁷

In evaluating societal welfare, we may take various perspectives. A state $x \in X$ is strongly efficient relative to u if it maximizes $\sum_{i \in N} u_i(x)$; i.e. if $\sum_{i \in N} u_i(x) \geq \sum_{i \in N} u_i(x')$ for all $x' \in X$. A state $x \in X$ Pareto dominates a state $x' \in X$ relative to u if $u_i(x) \geq u_i(x')$ for all $i \in N$, with strict inequality for at least

⁷ Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

one $i \in N$. A state $x \in X$ is Pareto efficient relative to u if it is not Pareto dominated, and a state $x \in X$ is Pareto dominant if it Pareto dominates any other state.

We next illustrate how social environments are useful to study specific models that have been studied extensively in the literature: network formation, normal-form games and coalition formation.

2.1 Network formation

A network g is a list of which pairs of players are linked to each other and $ij \in g$ indicates that i and j are linked under g . The set of all possible networks on N is denoted by $\mathcal{G}$. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. A network g' is adjacent to a network g if $g' = g + ij$ or $g' = g - ij$ for some ij . A network utility function (or payoff function) is a mapping $u : \mathcal{G} \rightarrow \mathbb{R}^N$ that assigns to each network g a utility $u_i(g)$ for each player $i \in N$. We identify X with the set of all possible networks $\mathcal{G}$. We follow Jackson and Wolinsky (1996) and consider only pairwise deviations involving a single link at a time. Moreover, link addition is bilateral (two players that would be involved in the link must agree to add the link), link deletion is unilateral (at least one player involved in the link must agree to delete the link), and network changes take place one link at a time. Thus, the effectivity correspondence is defined as follows: $E(g, g + ij) = \{i, j\}$, $E(g, g - ij) \cap \{i, j\} \neq \emptyset$ and $E(g, g') = \emptyset$ for all g, g' that are not adjacent networks. This completes the description of our social environment for the network formation model. It is straightforward to adjust the effectivity correspondence to incorporate models of network formation where coalitions of any size can change more than one link at a time. See (Page and Wooders 2009) for a more extensive discussion of alternative rules of network formation.⁸

To determine which networks can be formed in the long run, Jackson and Wolinsky (1996) propose a myopic notion of stability: a network g is pairwise stable with respect to u if and only if (i) for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and (ii) for all $ij \notin g$, if $u_i(g) < u_i(g + ij)$ then $u_j(g) > u_j(g + ij)$. The set of stable states $\mathcal{Z}$ is then equal to the set of pairwise stable networks $\mathcal{P}$.

2.2 Normal-form games

A normal-form game $G = (N, \{\Sigma_i\}_{i \in N}, \{u_i\}_{i \in N})$ consists of a finite set of players N and for each player $i \in N$ a nonempty and finite set of pure strategies Σ_i and a utility function $u_i : \Sigma \rightarrow \mathbb{R}$ over the set of strategy profiles $\Sigma = \prod_{i \in N} \Sigma_i$. A typical element of Σ is denoted by s . We write (s_i, s_{-i}) for the strategy profile where s_i is the strategy of player i and $s_{-i} = (s_j)_{j \in N \setminus \{i\}}$ is the list of strategies of all other players.

⁸ Mauleon and Vannetelbosch (2016) provide a comprehensive overview of the solution concepts for solving network formation games.

We identify X with the set of strategy profiles Σ . We only allow for individual deviations and then the effectivity correspondence is as follows: $E(s, s') = \{i\}$ if $s_i \neq s'_i$ and $s_{-i} = s'_{-i}$, and $E(s, s') = \emptyset$ otherwise.

A strategy profile $s \in \Sigma$ is said to be a pure strategy Nash equilibrium of the game G if for every $i \in N$ and for every $s'_i \in \Sigma_i$ it holds that $u_i(s) \geq u_i(s'_i, s_{-i})$. The set of pure strategy Nash equilibria $\mathcal{N}$ are then the set of stable states $\mathcal{Z}$ of the corresponding social environment.

2.3 Coalition formation

A coalition structure $p = \{S_1, S_2, \dots, S_m\}$ is a partition of the player set $N = \{1, 2, \dots, n\}$ such that $S_i \neq \emptyset$ for $i = 1, \dots, m$, $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^m S_i = N$. The set of all possible coalition structures on N is denoted by $\mathcal{C}$. Gains from cooperation are described by a valuation $u : \mathcal{C} \rightarrow \mathbb{R}^N$ that assigns to each coalition structure p a utility $u_i(p)$ for each player $i \in N$.

We identify X with the set of all possible coalition structures $\mathcal{C}$. Starting at p , a coalition $T \subseteq N$ can enforce a coalition structure p' if (i) the non-deviating players do not move when players in T leave their respective coalition(s) in p , and (ii) the deviating players in T form one or several coalitions in p' among themselves. Formally, the effectivity correspondence is as follows: $E(p, p') = \{T\}$ if (i) $\{S' \in p' \mid S' \subseteq N \setminus T\} = \{S \setminus T \mid S \in p, S \setminus T \neq \emptyset\}$ and (ii) $\exists \{S'_1, S'_2, \dots, S'_m\} \subseteq p'$ such that $\bigcup_{i=1}^m S'_i = T$, and $E(p, p') = \emptyset$ otherwise.

A coalition structure p is stable if for any $T \subseteq N$, p' that can be enforced from p by T and $i \in T$ such that $u_i(p') > u_i(p)$, there exists $j \in T$ such that $u_j(p') < u_j(p)$. The set of stable states $\mathcal{Z}$ is then equal to the set of stable coalition structures $\mathcal{V}$.

3 Minimally farsighted unstable states

The notion of stable states presumes that players are myopic and assume that deviations cannot be countered by subsequent deviations. However, farsighted players consider the possibility that, once they act, another coalition might react, a third coalition might in turn react, and so on without limit. Following the definition of a stable state, we say that a state $x \in X$ is farsightedly stable if it is not farsightedly dominated by any other state. We use the notion of farsighted improving path in order to describe the notion of farsighted dominance. The notion of farsighted improving path captures the fact that farsighted players consider the end outcome that their move(s) may lead to.

Definition 1 A farsighted improving path from a state x to a state $y \neq x$ is a finite sequence of states $y_1, \dots, y_K$ with $y_1 = x$ and $y_K = y$ such that for any $k \in \{1, \dots, K-1\}$ y_{k+1} is enforced from y_k via some coalition $S_k \in E(y_k, y_{k+1})$, and $u_i(y) \geq u_i(y_k)$ for all $i \in S_k$ and $u_i(y) > u_i(y_k)$ for some $i \in S_k$.

That is, a farsighted improving path is a sequence of states that can emerge when farsighted players act based on the improvement the end state offers relative to the current state.⁹ In a network formation problem where the effectivity correspondence only allows for pairwise deviations, each network in the sequence differs from the previous one in that either a new link is formed between two players or an existing link is deleted. If a link ij is deleted, then it must be that either player i or player j prefers the end network to the current network. If a link is added between player i and player j , then both player i and player j must prefer the end network to the current network.

If there exists a farsighted improving path from a state x to a state x' , then we write $x \rightarrow x'$. For a given state $x \in X$, let $\phi(x)$ be the set of all states that can be reached from x by a farsighted improving path. That is,

$$\phi(x) = \{x' \in X \mid x \rightarrow x'\}.$$

Definition 2 A state $x' \in X$ farsightedly defeats $x \in X$ ($x \neq x'$) if $x' \in \phi(x)$.

Hence, $\phi(x)$ gives us the set of states that farsightedly defeat the state $x \in X$.

Definition 3 A state $x \in X$ is farsightedly stable if $\phi(x) = \emptyset$.

Let $\mathcal{Z}_\infty = \{x \in X \mid \phi(x) = \emptyset\}$ be the farsighted core of $\Gamma = (N, X, E, u_i(\cdot))$,¹⁰ i.e. the set of farsightedly stable states. For network formation problems, Jackson (2008) defines the set of farsightedly pairwise stable networks P_∞ as those that are immune to farsighted pairwise deviations. Thus, $\mathcal{Z}_\infty = P_\infty$. Let $P_1 = \mathcal{P}$ be the set of pairwise stable networks. There is no guarantee that the set P_1 is non-empty. Since $P_1 \supseteq P_\infty$, emptiness or instability is more likely to become a problem when players are farsighted.¹¹

Example 1 (Network formation) In Bloch and Jackson (2007) or Jackson (2008) distance-based network model, if player i is connected to player j by a path of t links, then player i receives a benefit of $b(t)$ from her indirect connection with player j . It is assumed that $b(t) \geq b(t+1) > 0$ for any t . Each direct link $ij \in g$ results in a benefit $b(1)$ and a cost c to both i and j . This cost can be interpreted as the time a player

⁹ The notion of farsighted improving path is introduced by Jackson (2008) and Herings et al. (2009) in network formation, Herings et al. (2010) in coalition formation, and Mauleon et al. (2022) in normal-form games. Jackson and Watts (2002) define the notion of improving path in network formation when all players are myopic. Herings et al. (2020); Luo et al. (2021) and Mauleon et al. (2022) extend this notion to a mixed population composed of both myopic and farsighted players in matching markets, network formation problems and coordination games, respectively.

¹⁰ Diamantoudi and Xue (2003) study the farsighted core in hedonic games.

¹¹ Mauleon et al. (2022) show that there always exists a farsightedly stable strategy profile in coordination games but, in general, there is no guarantee that there exist farsightedly stable strategy profiles.

must spend with another player in order to maintain a direct link. Player i 's distance-based payoff from a network g is given by

$$u_i(g) = \sum_{j \neq i} b(t(ij)) - \#\{j \in N \mid ij \in g\} \cdot c, \quad (1)$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j), $c \geq 0$ is a cost per link, and b is a non-increasing function. The symmetric connections model ($b(t) = \delta^t$) and the truncated connections model of Jackson and Wolinsky (1996) are special cases of distance-based payoffs. In Fig. 1 we have depicted the 3-player case for $b(1) = 3$, $b(2) = 1.5$, and $c = 2$. We have $\phi(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi(g_1) = \{g_4, g_5, g_6\}$, $\phi(g_2) = \{g_4, g_5, g_6\}$, $\phi(g_3) = \{g_4, g_5, g_6\}$, $\phi(g_4) = \{g_5, g_6\}$, $\phi(g_5) = \{g_4, g_6\}$, $\phi(g_6) = \{g_4, g_5\}$ and $\phi(g_7) = \{g_4, g_5, g_6\}$. Hence, there is no farsightedly pairwise stable network, $P_\infty = \emptyset$. But, are some networks less farsightedly unstable than others?

A farsighted stability comparison is a function σ associating with each social environment a binary relation $\succsim^\sigma$ over states, where $x \succsim^\sigma x'$ means that x is σ -more farsightedly stable than x' . We write $x \succsim^\sigma x'$ instead of $[x \succsim^\sigma x' \text{ and not } x' \succsim^\sigma x]$. We consider two primitive stability comparisons based on farsighted defeating states.

- The farsighted defeating states inclusion comparison (denoted $\sigma(\subseteq)$) is defined as follows. For each $x, x' \in X$,

$$x \succsim^{\sigma(\subseteq)} x' \Leftrightarrow \phi(x) \subseteq \phi(x').$$

- The farsighted defeating states cardinality comparison (denoted $\sigma(\#)$) is defined as follows. For each $x, x' \in X$,

$$x \succsim^{\sigma(\#)} x' \Leftrightarrow \#\phi(x) \leq \#\phi(x').$$

Any two states can be compared with respect to farsighted defeating states cardinality but not necessarily with respect to farsighted defeating states inclusion. If $x \succsim^{\sigma(\subseteq)} x'$, we say that x is more farsightedly stable than x' , while if $x \succsim^{\sigma(\#)} x'$, we say that x is $\#$ -more farsightedly stable than x' (or x is cardinally more farsightedly stable than x'). Notice that $\succsim^{\sigma(\#)}$ is complete and transitive while $\succsim^{\sigma(\subseteq)}$ is transitive but not complete.

We now provide the definition of a minimally farsighted unstable state as well as the definition of a cardinally minimally farsighted unstable state.

Definition 4 A state $x \in X$ is said to be

- *Minimally farsighted unstable* if there is no $x' \neq x$ such that $x' \succsim^{\sigma(\subseteq)} x$.
- *$\#$ -Minimally farsighted unstable* if there is no $x' \neq x$ such that $x' \succsim^{\sigma(\#)} x$.

Let F be the set of minimally farsighted unstable states and let $F_{\#}$ be the set of $\#$ -minimally farsighted unstable states (or cardinally minimally farsighted unstable states). Since $\succsim^{\sigma(\subseteq)}$ and $\succsim^{\sigma(\#)}$ are both transitive, it holds that F and $F_{\#}$ are non-empty sets in any social environment with a finite set of states. Notice that if x is more farsightedly stable than x' then x is also $\#$ -more farsightedly stable than x' . Hence, we do have that $F_{\#}$ is a refinement of F , i.e. $F_{\#} \subseteq F$. Also, for any farsightedly stable state $x \in Z_{\infty}$, there is no other state $x' \neq x$ which is more farsightedly stable ($\#$ -more farsightedly stable) than x .

Proposition 1 (i) $F_{\#} \subseteq F$;
(ii) If $Z_{\infty} \neq \emptyset$ then $F_{\#} = F = Z_{\infty}$.

For the distance-based network formation model of Fig. 1 we have that the set of minimally farsighted unstable networks is given by $F = \{g_4, g_5, g_6\}$. Indeed, g_4 , g_5 and g_6 are more farsightedly stable than any $g' \in \{g_0, g_1, g_2, g_3, g_7\}$ and no additional comparison among g_4 , g_5 and g_6 can be made with respect to farsighted defeating networks inclusion comparison. With respect to the farsighted defeating networks cardinality comparison, we obtain the following ranking: $\#\phi(g_4) = \#\phi(g_5) = \#\phi(g_6) = 2$, $\#\phi(g_1) = \#\phi(g_2) = \#\phi(g_3) = \#\phi(g_7) = 3$, and $\#\phi(g_0) = 6$. Hence, the set of $\#$ -minimally farsighted unstable networks leads to the same prediction, $F_{\#} = \{g_4, g_5, g_6\}$.

We now provide an example where the inclusion comparison and the cardinality comparison lead to two different sets.

Example 2 (Network formation) Consider the situation where three players can form links and where the payoffs are given in Fig. 2. We have $\phi(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi(g_1) = \{g_2, g_4, g_5, g_6\}$, $\phi(g_2) = \{g_3, g_4, g_5, g_6\}$, $\phi(g_3) = \{g_1, g_4, g_5, g_6\}$, $\phi(g_4) = \{g_2, g_5, g_6\}$, $\phi(g_5) = \{g_1, g_4, g_6\}$, $\phi(g_6) = \{g_3, g_4, g_5\}$ and $\phi(g_7) = \{g_5, g_6\}$. Hence, we obtain that $F = \{g_5, g_6, g_7\}$ while $F_{\#} = \{g_7\}$.

We now provide some immediate results that are helpful for characterizing the minimally farsighted unstable states.

Lemma 1 If $x' \in \phi(x)$ then x cannot be more farsightedly stable than x' .

Proof Take x' such that $x' \in \phi(x)$. Since $x' \notin \phi(x')$, we cannot have that $\phi(x) \subseteq \phi(x')$. $\square$

Lemma 1 tells us that, if there is a farsighted improving path from one state to another one, then the former cannot be more farsightedly stable than the latter state. Thus, if there is a farsighted improving path from all other states to some state, then this state is minimally farsighted unstable:

(i) If $x \in \phi(x')$ for every $x' \in X \setminus \{x\}$, then $x \in F$.

From Definition 4, it immediately follows that:

- #### 4 Maximally farsighted absorbing states

$$\phi^{-1}(x) = \{x' \in X \mid x' \rightarrow x\}.$$

Hence, $\phi^{-1}(x)$ gives us the set of states that are farsightedly defeated by the state $x \in X$.

- The farsighted defeating/defeated states inclusion comparison (denoted $\alpha(\subseteq)$) is defined as follows. For each $x, x' \in X$,

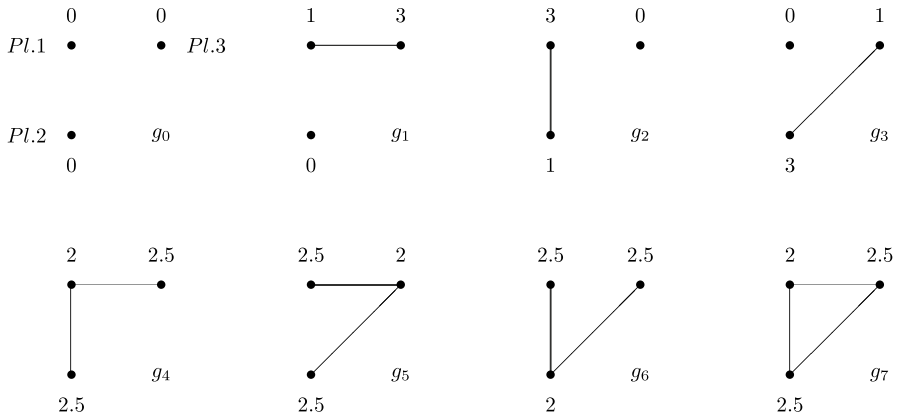


Fig. 2 An example with three players where $F_{\#} \subsetneq F$

$$x \succsim^{\alpha(\subseteq)} x' \Leftrightarrow \phi(x) \subseteq \phi(x') \text{ and } \phi^{-1}(x') \subseteq \phi^{-1}(x).$$

That is, x is more farsightedly absorbing than x' , or $x \succsim^{\alpha(\subseteq)} x'$, if and only if $\phi(x) \subseteq \phi(x')$ and $\phi^{-1}(x') \subseteq \phi^{-1}(x)$, with one inclusion holding strictly.

- The farsighted defeating/defeated states cardinality comparison (denoted $\alpha(\#)$) is defined as follows. For each $x, x' \in X$,

$$x \succsim^{\alpha(\#)} x' \Leftrightarrow \#\phi(x) \leq \#\phi(x') \text{ and } \#\phi^{-1}(x') \leq \#\phi^{-1}(x).$$

That is, x is $\#$ -more farsightedly absorbing than x' , or $x \succsim^{\alpha(\#)} x'$, if and only if $\#\phi(x) \leq \#\phi(x')$ and $\#\phi^{-1}(x') \leq \#\phi^{-1}(x)$, with one inequality holding strictly.

Any two states cannot be compared with respect to farsighted defeating/defeated states inclusion (or cardinality). Notice that $\succsim^{\alpha(\subseteq)}$ and $\succsim^{\alpha(\#)}$ are transitive but not complete. A state is said to be maximally farsighted absorbing if there is no other state which is more farsightedly absorbing. Similarly, a state is said to be cardinally maximally farsighted absorbing if there is no other state which is cardinally more farsightedly absorbing.

Definition 6 A state $x \in X$ is said to be

- *Maximally farsighted absorbing* if there is no $x' \neq x$ such that $x' \succsim^{\alpha(\subseteq)} x$;
- *$\#$ -Maximally farsighted absorbing* if there is no $x' \neq x$ such that $x' \succsim^{\alpha(\#)} x$.

Let A be the set of maximally farsighted absorbing states and let $A_{\#}$ be the set of $\#$ -maximally farsighted absorbing states (or cardinally maximally absorbing states). Since $\succsim^{\alpha(\subseteq)}$ and $\succsim^{\alpha(\#)}$ are transitive, it holds that A and $A_{\#}$ are non-empty sets in any social environment with a finite set of states. Obviously, we have that $A_{\#}$ is a refinement of A (i.e. $A_{\#} \subseteq A$).

Proposition 2 $A_{\#} \subseteq A$.

Example 1 (Continued) Remember that we have $\phi(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi(g_1) = \{g_4, g_5, g_6\}$, $\phi(g_2) = \{g_4, g_5, g_6\}$, $\phi(g_3) = \{g_4, g_5, g_6\}$, $\phi(g_4) = \{g_5, g_6\}$, $\phi(g_5) = \{g_4, g_6\}$, $\phi(g_6) = \{g_4, g_5\}$ and $\phi(g_7) = \{g_4, g_5, g_6\}$. In addition, we have $\phi^{-1}(g_0) = \emptyset$, $\phi^{-1}(g_1) = \{g_0\}$, $\phi^{-1}(g_2) = \{g_0\}$, $\phi^{-1}(g_3) = \{g_0\}$, $\phi^{-1}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\}$, $\phi^{-1}(g_5) = \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\}$, $\phi^{-1}(g_6) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\}$ and $\phi^{-1}(g_7) = \emptyset$. Hence, we obtain that $A = A_{\#} = \{g_4, g_5, g_6\}$, and so the minimally farsighted unstable networks are also the maximally farsighted absorbing ones.

Example 2 (Continued) Remember that we have $\phi(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi(g_1) = \{g_2, g_4, g_5, g_6\}$, $\phi(g_2) = \{g_3, g_4, g_5, g_6\}$, $\phi(g_3) = \{g_1, g_4, g_5, g_6\}$, $\phi(g_4) = \{g_2, g_5, g_6\}$, $\phi(g_5) = \{g_1, g_4, g_6\}$, $\phi(g_6) = \{g_3, g_4, g_5\}$ and $\phi(g_7) = \{g_5, g_6\}$. We have $\phi^{-1}(g_0) = \emptyset$, $\phi^{-1}(g_1) = \{g_0, g_3, g_5\}$, $\phi^{-1}(g_2) = \{g_0, g_1, g_4\}$, $\phi^{-1}(g_3) = \{g_0, g_2, g_6\}$, $\phi^{-1}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6\}$, $\phi^{-1}(g_5) = \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\}$, $\phi^{-1}(g_6) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\}$ and $\phi^{-1}(g_7) = \emptyset$. Hence, we get that $A = \{g_4, g_5, g_6, g_7\}$ and $A_{\#} = \{g_5, g_6, g_7\}$, while $F = \{g_5, g_6, g_7\}$ and $F_{\#} = \{g_7\}$.

Example 3 (Normal form game) Consider the normal form game where player 1 has two strategies (s_1 and s'_1) and player 2 has two strategies (s_2 and s'_2).

	s_2	s'_2
s_1	5, 3	3, 5
s'_1	2, 2	5, 1

It follows that $\phi((s_1, s_2)) = \{(s_1, s'_2)\}$, $\phi((s_1, s'_2)) = \{(s_1, s_2), (s'_1, s'_2)\}$, $\phi((s'_1, s_2)) = \{(s_1, s_2), (s_1, s'_2)\}$, and $\phi((s'_1, s'_2)) = \{(s_1, s_2), (s_1, s'_2), (s'_1, s_2)\}$. Hence, $\phi^{-1}((s_1, s_2)) = \{(s_1, s'_2), (s'_1, s'_2), (s'_1, s_2)\}$, $\phi^{-1}((s_1, s'_2)) = \{(s_1, s_2), (s'_1, s'_2), (s'_1, s_2)\}$, while $\phi^{-1}((s'_1, s_2)) = \{(s'_1, s'_2)\}$, and $\phi^{-1}((s'_1, s'_2)) = \{(s_1, s'_2)\}$. Thus, $F = A = \{(s_1, s_2), (s_1, s'_2)\}$ and $F_{\#} = A_{\#} = \{(s_1, s_2)\}$.

The next proposition shows that, if there is a farsighted improving path from all other states to some state, then this state is minimally farsighted unstable, maximally farsighted absorbing and Pareto efficient.

Proposition 3 If $x \in \phi(x')$ for all $x' \in X$, $x' \neq x$, then x is minimally farsighted unstable, maximally farsighted absorbing and Pareto efficient.

Proof If $x \in \phi(x')$ for all $x' \in X$, $x' \neq x$, we know from Lemma 1 that x is minimally farsighted unstable, and so there is no $x' \neq x$ that is more farsightedly stable than x (i.e. no x' such that $\phi(x') \subsetneq \phi(x)$). In addition, $\phi^{-1}(x) = X \setminus \{x\}$ and so there is no x'

such that $\phi^{-1}(x) \not\subseteq \phi^{-1}(x')$. Thus, x is maximally farsighted absorbing. Given the definition of a farsighted improving path, if $x \in \phi(x')$ then there is some player i such that $u_i(x) > u_i(x')$. Hence, x' does not Pareto dominates x . Since $x \in \phi(x')$ for all $x' \in X$, $x' \neq x$, x is Pareto efficient. $\square$

Consider again the distance-based network formation model (Example 1) where the utility function is given by (1). A star network is a network such that there exists some player i (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links: g is such that $\{l \in N \mid il \in g\} = N \setminus \{i\}$ and $\{l \in N \mid lj \in g\} = \{i\}$ for all $j \in N \setminus \{i\}$.

Proposition 4 *Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1) - b(3)$. We have $F \supseteq \{g \in \mathcal{G} \mid g \text{ is a star network}\}$ and $A \supseteq \{g \in \mathcal{G} \mid g \text{ is a star network}\}$.*

Proof Suppose that $b(1) - b(2) < c < b(1) - b(3)$. From Grandjean et al. (2011) and Luo et al. (2021) we have that, for each star network g , $g \in \phi(g')$ for all $g' \in \mathcal{G} \setminus \{g\}$. Hence, by Proposition 3, it follows that each star network is both minimally farsighted unstable and maximally farsighted absorbing. That is, $F \supseteq \{g \in \mathcal{G} \mid g \text{ is a star network}\}$ and $A \supseteq \{g \in \mathcal{G} \mid g \text{ is a star network}\}$. $\square$

Other networks could be minimally farsighted unstable or maximally farsighted absorbing. A d -regular network is a network where all players have d links (i.e. $\#\{j \in N \mid ij \in g\} = d$ for all $i \in N$). In the case of four players, Luo et al. (2021) show that, for each 2-regular network g (i.e. each circle network), $g \in \phi(g')$ for all $g' \in \mathcal{G} \setminus \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$ and $g \notin \phi(g')$ for all $g' \in \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$, $g' \neq g$. Hence, $F \supseteq \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$ and $A \supseteq \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$.¹²

Alternatively to maximally farsighted absorbing states, one may look for states belonging to closed cycles. Let $\phi^2(x) = \phi(\phi(x)) = \{x'' \in X \mid \exists g' \in \phi(x) \text{ such that } x'' \in \phi(g')\}$ be the set of states that can be reached by a composition of two farsighted improving paths from x . Note now $\phi^r(x)$ the set of states that can be reached through a combination of r farsighted improving paths. The transitive closure of ϕ is denoted by ϕ^∞ and defined as $\phi^\infty(x) = \bigcup_{r \in \mathbb{N}} \phi^r(x)$. The notion of closed cycles from Jackson and Watts (2002) has been extended in the literature to populations of farsighted agents.¹³ A set of states Q forms a cycle if for any $x \in Q$ and $x' \in Q$ there exists a sequence of farsighted improving paths connecting x to x' , i.e., $x' \in \phi^\infty(x)$. A cycle Q is a closed cycle if no state in Q lies on a farsighted improving path leading to a state that is not in Q , i.e., $\bigcup_{x \in Q} \phi^\infty(x) = Q$. Next proposition relates closed cycles to maximally farsighted absorbing states.

¹² Take $b(1) = 3$, $b(2) = 1.5$, $b(3) = 0.5$, $c = 2$ and $n = 4$. Computing the number of defeating networks ($\#\phi(g)$) and number of defeated networks ($\#\phi^{-1}(g)$) for each network, we obtain that $F_\# = \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$ while $A_\# = \{g \in \mathcal{G} \mid g \text{ is a star network}\} \cup \{g \in \mathcal{G} \mid g \text{ is a 2-regular network}\}$.

¹³ See Luo et al. (2021).

Proposition 5 *If Q is a closed cycle then $A \cap Q \neq \emptyset$.*

Proof Suppose $x \in Q$. By definition of closed cycle, there exists some state $x' \in Q$ such that $x \in \phi(x')$. Moreover, $\forall x'' \notin Q$ we have $x'' \notin \phi(x)$ for any $x \in Q$. Thus no state $x'' \notin Q$ can be more farsighted absorbing than any state $x \in Q$. It then holds that only some state $x' \in Q$ can be more farsighted absorbing than another state $x \in Q$. Because Q has a finite cardinality, there must be at least one state in Q which is maximally farsighted absorbing. $\square$

Notice that Proposition 5 does not hold for minimally farsighted unstable states since $F \cap Q$ may be empty.

5 Relationship with set concepts

A set of states Y is said to be a vNM farsighted stable set (FSS) if the following two conditions hold. Internal stability (**IS**): For any two states x and x' in the set Y there is no farsighted improving path from x to x' (and vice versa). External stability (**ES**): From any state outside the set Y there is a farsighted improving path to some state within the set Y .¹⁴

Definition 7 A set of states $Y \subseteq X$ is a vNM farsighted stable set if: (**IS**) for every $x, x' \in Y$, it holds that $x' \notin \phi(x)$; and (**ES**) for every $x \in X \setminus Y$, it holds that $\phi(x) \cap Y \neq \emptyset$.

Proposition 6 *If $\{x\}$ is a vNM farsighted stable set then x is minimally farsighted unstable ($x \in F$) and maximally farsighted absorbing ($x \in A$).*

Proof If $\{x\}$ is a vNM farsighted stable set then $\{x\}$ satisfies **ES** and so, for every $x' \in X$, $x' \neq x$, $\phi(x') \cap \{x\} \neq \emptyset$. Hence, from Proposition 3 we have that $x \in F$ and $x \in A$. $\square$

Remark that if $\{x\}$ is a vNM farsighted stable set and there is no $x' \neq x$ such that $\{x'\}$ is a vNM farsighted stable set, then x is also a #-maximally farsighted absorbing state ($x \in A_{\#}$).

Next proposition shows that any non-singleton vNM farsighted stable set Y , contains a minimally unstable state.

¹⁴ Herings et al. (2020) and Luo et al. (2021) define the notion of myopic-farsighted stable set that extends the notion of vNM farsighted stable set to a mixed population of myopic and farsighted players.

Proposition 7 *If $Y \subseteq X$ is a vNM farsighted stable set then $Y \cap F \neq \emptyset$.*

Proof By **ES** of Y , from any state outside the set Y there is a farsighted improving path to some state within the set Y . By Lemma 1, no state outside the set Y can be more farsightedly stable than all the states in the set Y . Thus, there exist some state $x \in Y$ that is minimally farsightedly unstable, $x \in F$. $\square$

Notice that vNM farsighted stable sets do not always exist.

Example 4 (Network formation) Consider the situation where three players can form links and where the payoffs are given in Fig. 3. With respect to defeating networks, we have $\phi(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi(g_1) = \{g_2, g_3\}$, $\phi(g_2) = \{g_3, g_4, g_5\}$, $\phi(g_3) = \{g_4, g_5\}$, $\phi(g_4) = \{g_1, g_5, g_6\}$, $\phi(g_5) = \{g_1, g_6\}$, $\phi(g_6) = \{g_1, g_2, g_3\}$ and $\phi(g_7) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$. With respect to defeated networks, we have $\phi^{-1}(g_0) = \emptyset$, $\phi^{-1}(g_1) = \{g_0, g_4, g_5, g_6, g_7\}$, $\phi^{-1}(g_2) = \{g_0, g_1, g_6, g_7\}$, $\phi^{-1}(g_3) = \{g_0, g_1, g_2, g_6, g_7\}$, $\phi^{-1}(g_4) = \{g_0, g_2, g_3, g_7\}$, $\phi^{-1}(g_5) = \{g_0, g_2, g_3, g_4, g_7\}$, $\phi^{-1}(g_6) = \{g_0, g_4, g_5, g_7\}$ and $\phi^{-1}(g_7) = \emptyset$. Herings et al. (2009) show that there is no vNM farsighted stable set. However, $F = F_{\#} = \{g_1, g_3, g_5\}$ and $A = A_{\#} = \{g_1, g_3, g_5\}$.

Replacing the internal stability condition in the definition of the vNM farsighted stable set by deterrence of external deviations and minimality, leads to a stability concept, the DEM farsighted stable set (DEM-FSS), that is always non-empty.¹⁵ A set of states Y is said to be a DEM farsighted stable if the following three conditions hold. Deterrence of external deviations (**DED**): All deviations from any state $x \in Y$ to a state outside Y are deterred by a credible threat of ending worse off or equally well off. External stability (**ES**): There exists a farsighted improving path from any state outside the set leading to some state in the set. Minimality (**MIN**): There is no proper subset of Y satisfying (**DED**) and (**ES**).

Definition 8 A set of states $Y \subseteq X$ is a DEM farsighted stable set if:

- (**DED**) $\forall x \in Y, \forall x' \notin Y$, such that x' can be enforced from x by some coalition $S \in E(x, x')$, $\exists y \in \phi(x') \cap Y$ such that we do not have $u_i(x) \leq u_i(y) \forall i \in S$ and $u_i(x) < u_i(y)$ for some $i \in S$;
- (**ES**) $\forall x' \in X \setminus Y, \phi(x') \cap Y \neq \emptyset$;
- (**MIN**) $\nexists Y' \subsetneq Y$ such that Y' satisfies (**DED**) and (**ES**).

Condition (**DED**) captures that any deviation to a state outside Y , is deterred by the threat of ending in some state y . Here y is such that there is a farsighted improving path from x' to y . Moreover, y belongs to Y , which makes y a credible threat.

¹⁵ Herings et al. (2009, 2010) introduce the DEM farsighted stable set for network formation and coalition formation problems.

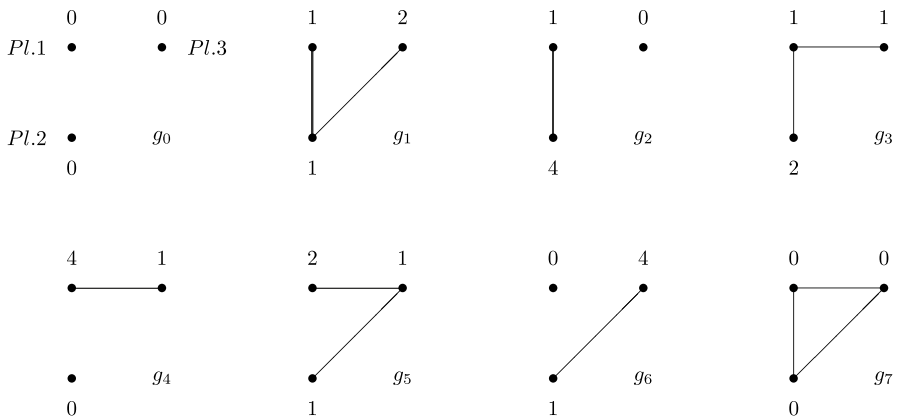


Fig. 3 Non-existence of a vNM farsightedly stable set

Since the finite set X (trivially) satisfies **(DED)** and **(ES)**, a minimality condition **(MIN)** is required.

Since a vNM farsighted stable set is a DEM farsighted stable set (Corollary 3 in Herings et al. (2009)), Proposition 6 and Proposition 7 extend to DEM farsighted stable sets. That is, if $\{x\}$ is a DEM farsighted stable set then x is minimally farsighted unstable ($x \in F$) and maximally farsighted absorbing ($x \in A$). Moreover, if there is no $x' \neq x$ such that $\{x'\}$ is a DEM farsighted stable set, then x is also a #-maximally farsighted absorbing state ($x \in A_{\#}$). Also, if $Y \subseteq X$ is a DEM farsighted stable set then $Y \cap F \neq \emptyset$.

Theorem 5 in Herings et al. (2009) shows that a set $Y \subseteq X$ is the unique DEM farsighted stable set if and only if $Y = \{x \in X \mid \phi(x) = \emptyset\}$ and for any $x' \in X \setminus Y$, $\phi(x') \cap Y \neq \emptyset$.

Proposition 8 *If $Y \subseteq X$ is the unique DEM farsighted stable set then $F = F_{\#} = Y$.*

Proof Take Y such that $Y = \{x \in X \mid \phi(x) = \emptyset\}$ and for any $x' \in X \setminus Y$, $\phi(x') \cap Y \neq \emptyset$. It follows that $\phi(x) \subsetneq \phi(x')$ for all x, x' such that $x \in Y, x' \notin Y$. Hence, $F = F_{\#} = Y$. $\square$

Replacing the internal and external stability conditions of the vNM farsighted stable set by the conditions that internal and external deviations should be deterred, Chwe (1994) introduce the notion of largest consistent set (LCS) for social environments. The largest consistent set contains vNM farsighted stable sets.¹⁶

¹⁶ Mauleon and Vannetelbosch (2004) define the largest (cautious) consistent set for coalition formation problems while Herings et al. (2009) define the largest pairwise consistent set for network formation problems (see also Page et al. 2005).

Definition 9 A set of states $Y \subseteq X$ is a *consistent set* if $\forall x \in Y: \forall x' \in X, x' \neq x$, such that x' can be enforced from x by some coalition $S \in E(x, x')$, there exists $x'' \in Y$, where $x'' = x'$ or $x'' \in \phi(x')$ such that we do not have $u_i(x) \leq u_i(y) \forall i \in S$ and $u_i(x) < u_i(y)$ for some $i \in S$. The *largest consistent set* is the consistent set that contains any consistent set.

It follows from Chwe (1994) that the largest consistent set exists, is non-empty, and satisfies (ES).

In Table 1 we give the minimally farsighted unstable states and the maximally farsighted absorbing states as well as the vNM farsightedly stable sets (FSS), the DEM farsightedly stable sets (DEM-FSS) and the largest consistent set (LCS) found in Examples 1–4. We observe that FSS, DEM-FSS and LCS often support more networks or strategy profiles than F , $F_{\#}$, A or $A_{\#}$.

6 Experimental data

Consider a network formation problem where a group of players forms the network g . The existence of a farsighted improving path emanating from g to g' means that at least one player can alter the network g to increase her utility in the end, anticipating further changes made by other players leading to g' . A farsighted improving path emanating from g is thus a source of instability for g . On the opposite, being the end network of a farsighted improving path is a source of attractiveness for g' .

Of course, the existence of a farsighted improving path from g to g' does not imply that players will actually follow this path. There may be several farsighted improving paths emanating from the current network g and each farsighted improving path requires some coordination on behalf of the players involved in the path. Hence, a player could restrain herself from starting a path by fear of unwanted deviations along the path leading to another network than g' or even by fear of potential subsequent deviations after having reached g' .

Nevertheless, the existence of a farsighted improving path from g means that players could be willing to leave the current network g . It is thus a potential source of instability. And the smaller the number of farsighted improving paths emanating from g , the greater its stability. Similarly, the greater the number of networks farsightedly dominated by g , the greater its attractiveness. This is the reason why we adopt the notions of (cardinally) minimally farsighted unstable networks and (cardinally) maximally farsighted absorbing networks. Those concepts are descriptive: their aim is to predict the networks that will emerge and persist in the long run. However, despite the concepts are descriptive, their empirical performance may be linked to their correlation with normative criteria: everything else being equal, a network g is more likely to be immune to farsighted deviations and is more likely to farsightedly dominate others networks if g is efficient and/or fair. We now test the empirical performance of those notions using data from an experiment on network formation.

Kirchsteiger et al. (2016) test whether players are either myopic or farsighted when forming links between them. In the experiment, participants are grouped by

Table 1 Solutions for the four examples

Concept	Example 1	Example 2	Example 3	Example 4
F	g_4, g_5, g_6	g_5, g_6, g_7	$(s_1, s_2), (s_1, s'_2)$	g_1, g_3, g_5
$F_{\#}$	g_4, g_5, g_6	g_7	(s_1, s_2)	g_1, g_3, g_5
A	g_4, g_5, g_6	g_4, g_5, g_6, g_7	$(s_1, s_2), (s_1, s'_2)$	g_1, g_3, g_5
$A_{\#}$	g_4, g_5, g_6	g_5, g_6, g_7	(s_1, s_2)	g_1, g_3, g_5
FSS	$\{g_4\}, \{g_5\}, \{g_6\}$	$\{g_5\}, \{g_6\}, \{g_4, g_7\}$	$\{(s_1, s_2)\}, \{(s_1, s'_2)\}$	None
DEM-FSS	$\{g_4\}, \{g_5\}, \{g_6\}$	$\{g_5\}, \{g_6\},$ $\{g_4, g_7\},$ $\{g_1, g_2, g_3, g_7\}$	$\{(s_1, s_2)\}, \{(s_1, s'_2)\}$	$\{g_1, g_2, g_3\},$ $\{g_3, g_4, g_5\},$ $\{g_1, g_5, g_6\}$
LCS	$\{g_4, g_5, g_6\}$	$\{g_1, g_2, g_3, g_4,$ $g_5, g_6, g_7\}$	$\{(s_1, s_2), (s_1, s'_2)\}$	$\{g_1, g_2, g_3,$ $g_4, g_5, g_6\}$

four to play a sequential network formation game. Players are initially unconnected (i.e. they start from the empty network). At each stage, a potential link between two players is randomly selected. If the link exists in the current network, then both players choose simultaneously whether to keep the link or to delete the link. If one of the players decides to delete the link, then the link is cut. If the link is not yet formed in the current network, then both players choose simultaneously whether or not to add this link. Only if both players decide to add the link, then the link is formed. The four players are informed about the linking choices of the players involved in the selected link. All players know perfectly the resulting network and its payoffs. Before moving to the next stage where another potential link is randomly selected, players declare if they want to do further changes to the network (satisfaction choice). The game ends either when all players do not want to modify the current network or when it is randomly stopped after stage 26.¹⁷ The game is repeated three times and groups are kept the same through repetitions. A vector of payoffs that allocates a number of points to each player is associated to every network. Players receive points depending only on the final network of each repetition. Participants are informed about the payoffs associated to every possible network.

Kirchsteiger et al. (2016) run three treatments (T1, T2, T3) where the payoffs are modified in some networks to get vNM farsighted stable sets with different properties. Figures 4, 5 and 6 display, respectively, the payoffs that were used in the three treatments for each class of networks C_k ($k = 1, \dots, 11$). Each class C_k regroups networks that share the same architecture but differ in the position occupied by each player. To identify F , $F_{\#}$, A and $A_{\#}$ we need to compute $\phi(g)$ and $\phi^{-1}(g)$ for every g .

We now provide the predictions of the stability notions in the experiment of Kirchsteiger et al. (2016):

¹⁷ The design of the termination rule allows each individual to decide unilaterally to continue playing (at least for the first 26 stages).

$$\mathbf{T1:} \quad F = F_{\#} = A = A_{\#} = \{g^N\}.$$

$$\mathbf{T2:} \quad F_{\#} = \{g^{\emptyset}\}, F = A = A_{\#} = \{g^{\emptyset}\} \cup \{g \mid g \in C_5\} \cup \{g \mid g \in C_9\}.$$

$$\mathbf{T3:} \quad F_{\#} = \{g^{\emptyset}\}, \quad F = A_{\#} = \{g^{\emptyset}\} \cup \{g \mid g \in C_7\}, \\ A = \{g^{\emptyset}\} \cup \{g \mid g \in C_7\} \cup \{g \mid g \in C_{10}\}.$$

In T1, since $\phi(g^N) = \emptyset$ and $g^N \in \phi(g)$ for all $g \neq g^N$ (so $\phi^{-1}(g^N) = \{g \mid g \neq g^N\}$) it follows directly that all concepts single out the complete network g^N . Hence, data from T1 cannot be used to discriminate between these concepts. In T2 and T3, the empty network is the unique $\#$ -minimally farsighted unstable network. The set of minimally farsighted unstable networks and the set of $\#$ -maximally farsighted absorbing networks coincide in both T2 and T3 (i.e. $F = A_{\#}$ in T2, T3). Hence, data from the experiment cannot be used to discriminate between F and $A_{\#}$. In both T2 and T3 we get $F_{\#} \subsetneq F$. Finally, we have $A_{\#} = A$ in T2 but $A_{\#} \subsetneq A$ in T3. Predictions are summarized in Table 2 included the vNM farsighted stable sets.¹⁸ In T2 and T3 of the experiment, we have $F_{\#} \subsetneq F = A_{\#} \subseteq A$.

We know from Proposition 1 and Proposition 2 that some concepts always include others: $F_{\#} \subseteq F$ and $A_{\#} \subseteq A$. Notice that the relations are weak inclusion since in some social environments the notions may coincide. In the first step of the empirical analysis, we narrow the group of notions under scrutiny by testing if a narrower "sub-notion" is significantly different from its directly larger "upper-notion". In practice we test if the proportion of final networks that were accurately predicted by the sub-notion is significantly lower than the proportion of networks accurately predicted by its upper-notion (at a simple proportion test). We then exclude the upper-notions that fail to have a significantly larger proportion than their sub-notions as their enlargement do not come along with a (significant) higher proportion of accurately predicted networks and are thus judged useless for this analysis from a descriptive point of view. Hence, we will test the following two hypotheses:

$$\mathbf{H1:} \quad q_{F_{\#}} < q_F \text{ and } \mathbf{H2:} \quad q_{A_{\#}} < q_A.$$

where q_F , $q_{F_{\#}}$, q_A and $q_{A_{\#}}$ are, respectively, the proportion of networks accurately predicted by the stability notions F , $F_{\#}$, A and $A_{\#}$.

The experiment took place at the University of Milan-Bicocca in 2010 for T1 and 2012 for T2 and T3. Kirchsteiger et al. (2016) run 16 sessions for a total of 288 participants and 72 groups (36 for T1, 18 for T2, 18 for T3). Each group played three consecutive rounds. Table 3 gives us the number of groups ending on average in each network of each class of networks in the experiment. The column $\#C$ gives us the number of different networks within each class.¹⁹ Table 3 can be interpreted

¹⁸ Since the empty network is pairwise stable in all treatments, myopia predicts the empty network in T1, T2 and T3.

¹⁹ From the experimental data, we only have the number of groups ending in each class in the experiment. The average number of groups ending in any given network is then equal to the number of groups ending in each class divided by the number of different networks in the class.

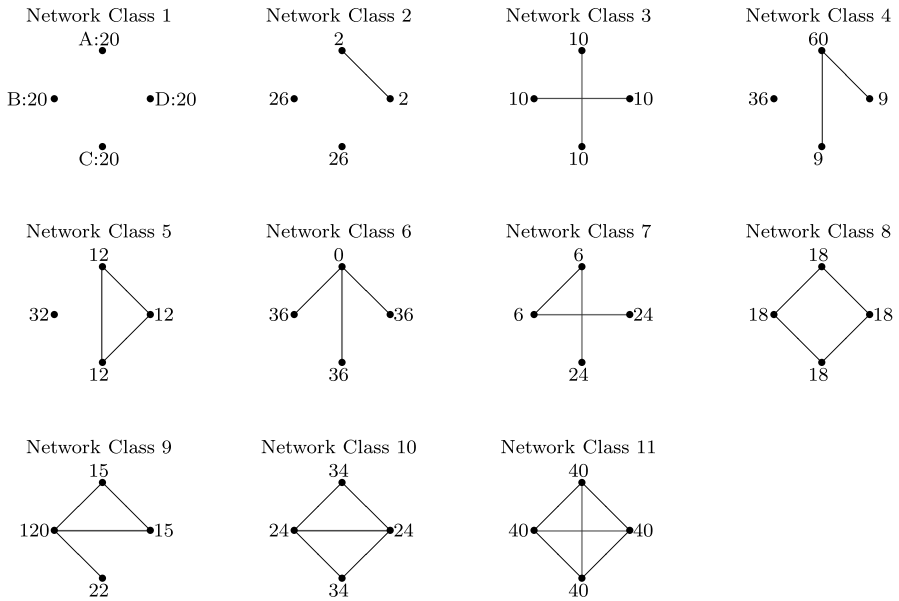


Fig. 4 Payoffs in treatment T1

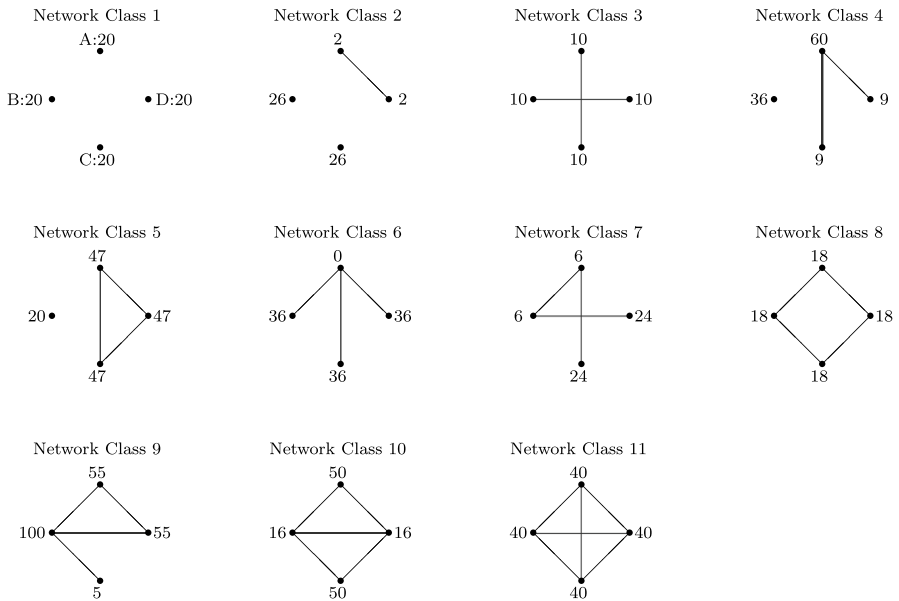


Fig. 5 Payoffs in treatment T2

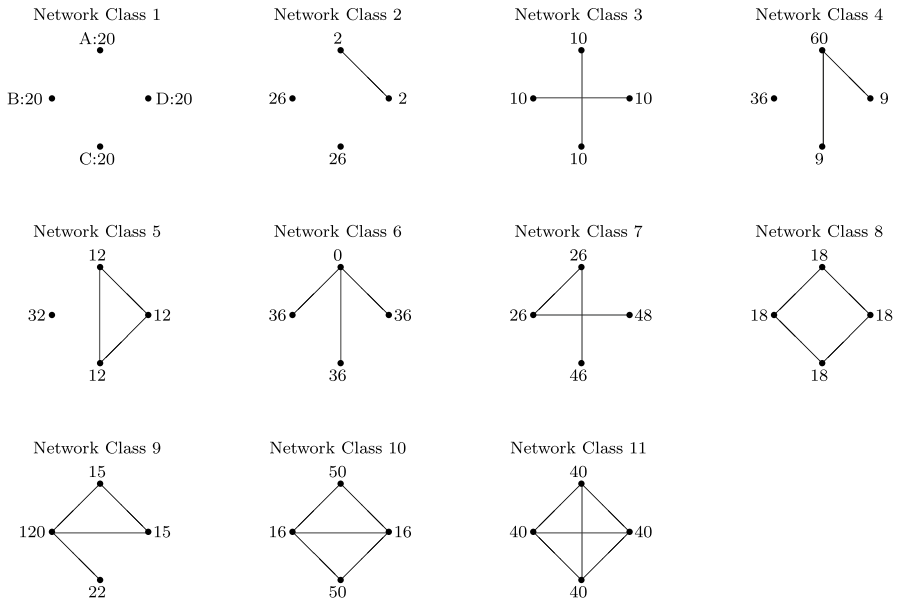


Fig. 6 Payoffs in treatment T3

Table 2 Solutions in the three treatments of the experiment

Concept	T1	T2	T3
F	g^N	$g^\emptyset, g \in C_5, C_9$	$g^\emptyset, g \in C_7$
$F_\#$	g^N	$g^\emptyset$	$g^\emptyset$
A	g^N	$g^\emptyset, g \in C_5, C_9$	$g^\emptyset, g \in C_7, C_{10}$
$A_\#$	g^N	$g^\emptyset, g \in C_5, C_9$	$g^\emptyset, g \in C_7$
FSS	$\{g^N\}$	$\{g \in \mathcal{G} \mid g \in C_5\}$	$\{g, g' \in \mathcal{G} \mid g, g' \in C_7, \#N_i(g) = \#N_i(g') \forall i\}$

as follows. In T1, 19.4% of the groups end up in the empty network while 60.2% end up in the complete network. In T2, 22.2% of the groups end up in the empty network while 13.4% end up in a network where player i is isolated and players j, k, l form a complete network between them.²⁰ In T3, half of the groups end up in the empty network.

Using a one-sided proportion test, it turns out that H1 is consistent with the data: $F_\#$ is significantly lower than F (P -value < 0.001). But H2 is rejected (P -value of 0.42). It is not surprising since in T2 both $A_\#$ and A lead to the same predictions. In T3, A predicts more networks but networks in which nearly no groups were ending up. Hence, $A_\#$ is even not significantly lower from A in T3 (P -value of 0.34).

²⁰ There are four different networks in class C_5 . In total 53.6% of the groups end up in class C_5 .

Table 3 Number of groups ending on average in each network of each class of networks in the experiment

Network	#C	T1	T2	T3	T1	T2	T3
$g \in C_1$	1 (1)	21	12	26	19.4	22.2	48.1
$g \in C_2$	6 (6)	11/6	2/6	7/6	1.7	0.6	2.2
$g \in C_3$	3 (1)	0	0	0	0.0	0.0	0.0
$g \in C_4$	12 (12)	4/12	0	6/12	0.3	0.0	0.9
$g \in C_5$	4 (4)	1/4	29/4	1/4	0.2	13.4	0.5
$g \in C_6$	4 (4)	0	0	0	0.0	0.0	0.0
$g \in C_7$	12 (6)	1/12	2/12	10/12	0.1	0.3	1.5
$g \in C_8$	3 (1)	0	0	0	0.0	0.0	0.0
$g \in C_9$	12 (12)	0	8/12	0	0.0	1.2	0.0
$g \in C_{10}$	6 (6)	5/6	0	2/6	0.7	0.0	0.6
$g \in C_{11}$	1 (1)	65	1	2	60.2	1.9	3.7
Average number					Average percentage		

Column #C: in parenthesis the number of payoff relevant networks in each class

Table 4 Percentage of networks predicted by each concept among the networks in which at least $x\%$ of the groups were ending up

Concept	T1	T2	T3	T1	T2	T3	T3*
F	60.2	90.7	66.7	75.6	100	92.8	94.7
$F_{\#}$	60.2	22.2	48.1	75.6	29.3	92.8	68.9
A	60.2	90.7	70.4	75.6	100	92.8	94.7
$A_{\#}$	60.2	90.7	66.7	75.6	100	92.8	94.7
FSS	60.2	53.7	18.5	75.6	70.7	0	25.8
x	0			2.5			2.5

In the column T3* the percentage is computed considering only the number of payoff relevant networks in each class

Result 1 In the experiment:

- (i) The proportion of final networks accurately predicted by F , q_F , is significantly larger than the proportion accurately predicted by $F_{\#}$, $q_{F_{\#}}$.
- (ii) The proportion of final networks accurately predicted by A , q_A , is not significantly larger than the proportion accurately predicted by $A_{\#}$, $q_{A_{\#}}$.

Since the proportion accurately predicted by A , i.e. q_A , is not significantly larger than the proportion accurately predicted by $A_{\#}$ (and F), i.e. $q_{A_{\#}}$ (and q_F), we only focus on F and $F_{\#}$ and we compare those concepts with respect to the following two

Table 5 Percentage of networks predicted by each concept among the networks in which less than 2.5% of the groups were ending

Concept	T1	T2	T3	T3*
F	0	59.5	37.3	0
$F_{\#}$	0	0	0	0
A	0	59.5	44.8	11.9
$A_{\#}$	0	59.5	37.3	0
FSS	0	0	37.3	0

criteria. A first criteria is the percentage of networks predicted by each concept. A higher percentage is better. Table 4 gives us the percentage of networks predicted by each concept among the networks in which at least $x\%$ of the groups were ending up.²¹ A second criteria is the percentage of networks predicted by each concept among the networks in which less than 2.5% of the groups were ending up. A lower percentage is better. Table 5 gives us the percentage of networks predicted by each concept among the networks in which less than 2.5% of the groups were ending up. We observe that, with respect to the first criteria, $F_{\#}$ is (weakly) dominated by F , but with respect to the second criteria, $F_{\#}$ is not dominated by one of the other concepts. Conversely, with respect to the first criteria, F is not dominated by one of the other concepts, while with respect to the second criteria, F is (weakly) dominated by $F_{\#}$.²² Based on these two criteria, we cannot discriminate between F (and $A_{\#}$) and $F_{\#}$. However, both perform better than FSS for fitting the data. F (and $A_{\#}$) predicts most networks that occur with positive probability while $F_{\#}$ is good at selecting the network with the highest percentage among those predicted by F . This observation is confirmed by Table 6 that gives us the mean percentage of a network predicted by a given concept. Thus, F (and $A_{\#}$) predicts the most relevant networks²³ while $F_{\#}$ provides narrower but more robust predictions.

The analysis suggests that the frequency of a network g depends negatively on $\#\phi(g)$ and positively on $\#\phi^{-1}(g)$. Table 7 shows the naive regression of the percentage per representative network on $\#\phi(g)$, $\#\phi^{-1}(g)$ and $\#\phi^{-1}(g) - \#\phi(g)$, and it confirms this intuition. Moreover, $\#\phi(g)$ impacts more the frequency of a network than $\#\phi^{-1}(g)$. Thus, both the stability requirement and the reachability requirement seem to matter for predicting the most frequent networks with a higher importance of the former.

Notice that in the three treatments of the experiment, the set of (cardinally) minimally farsighted unstable states, F (or $F_{\#}$), refines and performs better than the set

²¹ In the column T3* the percentage is computed considering the number of payoff relevant networks in each class. For instance, the number of groups ending on average in a representative network of class C_7 becomes 3% (instead of 1.5% in Table 3) since there are only 6 different payoff relevant networks in this class.

²² With respect to the first criteria, FSS is (weakly) dominated by F , and with respect to the second criteria, FSS is (weakly) dominated by $F_{\#}$.

²³ When we separate the classes of networks between those with a frequency higher than 1% and the others, F (and $A_{\#}$) has the highest correlation with this binary criteria.

Table 6 Mean percentage of a network predicted by a given concept

Treatment	F	$F_{\#}$	A	$A_{\#}$	FSS
T2	5.31	22.20	5.31	5.31	13.40
T3	5.09	48.10	3.67	5.09	1.50
T2 + T3	5.20	35.15	4.49	5.20	7.45

of (cardinally) maximally farsighted absorbing states, A ($A_{\#}$). Both F and $F_{\#}$ outperform the vNM farsighted stable set that selects or predicts too many networks. Since every vNM farsighted stable set is a DEM farsighted stable set and is included in the largest consistent set, these alternative set-valued concepts would even perform worse than the vNM farsighted stable set.

However, one cannot conclude that the set of (cardinally) minimally farsighted unstable states will always outperform the set of (cardinally) maximally farsighted absorbing states and set-valued concepts. In the three treatments of the experiment, the set of (cardinally) minimally farsighted unstable states has a non-empty intersection with the set of networks belonging to closed cycles. But, in general, minimally farsighted unstable states may have an empty intersection with closed cycles, while maximally farsighted absorbing states always intersect closed cycles (see Proposition 5). We conjecture that minimal farsighted instability will perform better when it refines maximal farsighted absorbtiveness and has a non-empty intersection with closed cycles.

7 Conclusion

We have proposed the notion of minimal farsighted instability to predict the states that are more likely to be observed in the long run when agents are farsighted.²⁴ A state is said to be minimally farsighted unstable if there is no other state which is more farsightedly stable, where the comparison between two networks is made by comparing (in the set inclusion or cardinal sense) their sets of farsighted defeating states. But, selecting a state that is more farsightedly stable than another one might be a more robust prediction if at the same time the former is more likely to be reached than the latter one. Hence, we have also proposed to compare states in terms of their reachability. A state is said to be maximally farsighted absorbing if there is no other state which is more farsightedly absorbing, where the comparison between two states is now made by comparing both their sets of farsighted defeating states (i.e. in terms of their stability) and their sets of farsighted defeated states (i.e. in terms of their reachability).

We have provided some general results that are helpful for characterizing minimally farsighted unstable states and maximally farsighted absorbing states, and we

²⁴ An alternative approach for solving the lack of farsighted stability is to require the consent of partners or neighbours for adding or deleting links. See (Caulier et al. 2013) and (Caulier et al. 2013).

Table 7 Regression of the percentage per representative network on $\#\phi(g)$, $\#\phi^{-1}(g)$ and $\#\phi^{-1}(g) - \#\phi(g)$

	T1	T2	T3	All	All	All	All
$\#\phi(g)$	-1.602 (0.211)	-0.383 (0.294)	-1.119 (0.166)	-0.640+ (0.063)	-1.059** (0.001)		
$\#\phi^{-1}(g)$	0.143 (0.655)	0.124 (0.355)	0.191 (0.315)	0.219+ (0.050)		0.347** (0.000)	
$\#\phi^{-1}(g) - \#\phi(g)$							0.304** (0.000)
Constant	24.45 (0.290)	8.309 (0.329)	24.63 (0.182)	12.02 (0.113)	23.42** (0.000)	-1.358 (0.609)	4.650* (0.021)
Num. obs	11	11	11	33	33	33	33
Adj. R-sq	0.433	0.449	0.227	0.369	0.305	0.314	0.368

P-values in parentheses, + $P < 0.10$, * $P < 0.05$, ** $P < 0.001$

have investigated their relationships with set-valued notions of farsightedness.²⁵ In terms of computational complexity, minimal farsighted instability turns to be much less demanding than set-valued concepts of farsightedness. Indeed, in addition to the computation of the set of farsighted defeating states for each state, most set-valued concepts require to consider all possible combinations of states into a set. We have used experimental data on network formation to show the relevance of the new solution concepts. It turns out that the set of minimally farsighted unstable networks (F) is not significantly different than the set of maximally farsighted absorbing networks (A). Finally, the set of minimally farsighted unstable networks (F) predicts most networks that occur with positive probability while the set of cardinally minimally farsighted unstable networks ($F_{\#}$) is good at selecting the network with the highest percentage among those predicted by F . Both F and $F_{\#}$ outperform on average FSS.

Recently, the literature (Dutta and Vohra 2017; Ray and Vohra 2019) has introduced set-valued concepts that incorporate maximality regarding the continuation path. However, our definitions of minimally farsighted unstable states and maximally farsighted absorbing states do not restrict a priori the correspondence $\phi(x)$. Incorporating maximality arguments into the correspondence would lead to minimal farsighted instability or maximal farsighted absorbtiveness satisfying the maximality constraints.

²⁵ Notice that the vNM farsighted stable set is also based on the notion of farsighted improving path, given by the correspondence $\phi(x)$, that ignores the possibility that one might avoid enforcing a new state that is valuable to her anticipating further changes making her worse off. This drawback was already pointed out by Chwe (1994) and partially solved by the largest consistent set and the DEM farsighted stable set.

Appendix

Experimental data

Let g^i be a generic network in class C_i and $c_i \subseteq C_i$ a generic proper subset of the corresponding class. We will write $g^i \rightarrow g$ with $g \in C_j$, and $g^i \rightarrow g$ with $g \in c_j$, when the generic network g^i in class C_i reaches with a farsighted improving path all the networks in class C_j or only a proper subset c_j of C_j , respectively.

Minimally unstable and maximally absorbing networks in T1

In treatment 1 (T1) the set of networks that can be reached from g^i by some farsighted improving path are:

$$\begin{aligned}\phi(g^0) &= \{g \mid g \in C_{10} \cup C_{11}\} \\ \phi(g^2) &= \{g \mid g \in C_1 \cup C_{10} \cup C_{11}\} \\ \phi(g^3) &= \{g \mid g \in C_1 \cup c_2 \cup c_5 \cup C_{10} \cup C_{11}\} \\ \phi(g^4) &= \{g \mid g \in C_1 \cup c_2 \cup c_5 \cup C_{10} \cup C_{11}\} \\ \phi(g^5) &= \{g \mid g \in C_1 \cup c_2 \cup C_{10} \cup C_{11}\} \\ \phi(g^6) &= \{g \mid g \in C_1 \cup c_2 \cup c_4 \cup c_5 \cup C_{10} \cup C_{11}\} \\ \phi(g^7) &= \{g \mid g \in C_1 \cup c_2 \cup c_3 \cup c_4 \cup c_5 \cup C_{10} \cup C_{11}\} \\ \phi(g^8) &= \{g \mid g \in C_1 \cup c_2 \cup c_4 \cup c_5 \cup c_7 \cup C_{10} \cup C_{11}\} \\ \phi(g^9) &= \{g \mid g \in C_1 \cup c_2 \cup c_4 \cup c_5 \cup c_6 \cup c_7 \cup C_{10} \cup C_{11}\} \\ \phi(g^{10}) &= \{g \mid g \in c_2 \cup c_4 \cup c_5 \cup c_6 \cup C_{11}\} \\ \phi(g^N) &= \emptyset.\end{aligned}$$

Remember that a network $g \in \mathcal{G}$ is minimally farsighted unstable if there is no $g' \neq g$ such that $g' \succ_{\mathcal{F}}^{\sigma(\subseteq)} g$; i.e., $\phi(g') \subsetneq \phi(g)$. It follows that g^N is the unique minimally farsighted unstable network. Notice that g^0 is more farsightedly stable than all networks $g \neq g^{10}, g^N$.

A network $g \in \mathcal{G}$ is maximally farsighted absorbing if there is no $g' \neq g$ such that $g' \succ_{\mathcal{F}}^{\sigma(\subseteq)} g$; i.e., $\phi(g') \subseteq \phi(g)$ and $\phi^{-1}(g) \subseteq \phi^{-1}(g')$ with one inclusion holding strictly. Since $\phi(g^N) = \emptyset$ and $g^N \in \phi(g^i)$ for all $g^i \neq g^N$, it also holds that g^N is the unique maximally farsighted absorbing network.

A network $g \in \mathcal{G}$ is $\#$ -minimally farsighted unstable if there is no $g' \neq g$ such that $g' \succ_{\mathcal{F}}^{\sigma(\#)} g$; i.e., $\#\phi(g') < \#\phi(g)$. Thus, g^N is the $\#$ -minimally farsighted unstable network.

A network $g \in \mathcal{G}$ is $\#$ -maximally farsighted absorbing if there is no $g' \neq g$ such that $g' \succ_{\mathcal{F}}^{\sigma(\#)} g$; i.e., $\#\phi(g') \leq \#\phi(g)$ and $\#\phi^{-1}(g) \leq \#\phi^{-1}(g')$ with one inequality holding strictly. Then, g^N is the $\#$ -maximally farsighted absorbing network.

Minimally unstable and maximally absorbing networks in T2

Let us denote B_g the networks that are adjacent to g ,

$$B_g = \{g' \mid g' = g + ij \vee g - ij, \text{ for some } ij\},$$

and let $\bar{B}_g$ be its complement. In treatment 2 (T2) the set of networks that can be reached from g^i by some farsighted improving path are:

$$\begin{aligned}\phi(g^\emptyset) &= \{g \mid g \in C_5\} \\ \phi(g^2) &= \{g \mid g \in C_1 \cup C_5 \cup C_9\} \\ \phi(g^3) &= \{g \mid g \in C_1 \cup C_2 \cup C_5 \cup C_9\} \\ \phi(g^4) &= \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_9\} \\ \phi(g^5) &= \{g \mid g \in C_9 \cap \bar{B}_{g^5}\} \\ \phi(g^6) &= \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_9\} \\ \phi(g^7) &= \{g \mid g \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_9\} \\ \phi(g^8) &= \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_7 \cup C_9\} \\ \phi(g^9) &= \{g \mid g \in C_4 \cup (C_5 \cap B_{g^9}) \cup (C_9 \setminus g^9)\} \\ \phi(g^{10}) &= \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_7 \cup C_8 \cup C_9\} \\ \phi(g^N) &= \{g \mid g \in C_5 \cup C_9 \cup C_{10}\}.\end{aligned}$$

Note that $g^\emptyset$ is more farsightedly stable than all networks $g \neq g^5, g^9$. Moreover, neither g^5 nor g^9 are more farsightedly stable than $g^\emptyset$. Thus, $g^\emptyset$ is minimally farsighted unstable. Note also that it is not always true that $(C_9 \setminus g^9) \supseteq (C_9 \cap \bar{B}_{g^5})$ since g^9 could belong to $(C_9 \cap \bar{B}_{g^5})$. Then, also g^5, g^9 are minimally farsighted unstable.

The maximally farsighted absorbing networks are $g^\emptyset, g^5$ and g^9 . Indeed:

$$\begin{aligned}\phi^{-1}(g^\emptyset) &= \{g \mid g \in C_2 \cup C_3 \cup C_4 \cup C_6 \cup C_7 \cup C_8 \cup C_{10}\}; \\ \phi^{-1}(g^5) &= \{g \mid g \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11}\}\end{aligned}$$

and

$$\phi^{-1}(g^9) = \{g \mid g \in C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11}\}.$$

From Table 8, we have that $g^\emptyset$ is the #-minimally farsighted unstable network. Notice that the network $g^\emptyset$ is #-more farsightedly absorbing than the network g^2 , but not more than g^5 and g^9 .

Minimally unstable and maximally absorbing networks in T3

In treatment 3 the set of networks that can be reached from g^i by some farsighted improving path are:

Table 8 Number of defeating networks ($\#\phi(g)$) and number of defeated networks ($\#\phi^{-1}(g)$) for each network in each class of networks of the experiment

Network	T1		T2		T3	
	$\#\phi(g)$	$\#\phi^{-1}(g)$	$\#\phi(g)$	$\#\phi^{-1}(g)$	$\#\phi(g)$	$\#\phi^{-1}(g)$
$g \in C_1$	7	56	4	46	12	50
$g \in C_2$	8	29	11	19	14	21
$g \in C_3$	14	4	19	4	21	0
$g \in C_4$	11	7	13	7	19	5
$g \in C_5$	11	30	9	51	19	12
$g \in C_6$	17	6	23	0	28	6
$g \in C_7$	18	3	23	3	16	62
$g \in C_8$	24	0	29	2	23	2
$g \in C_9$	18	0	14	52	27	0
$g \in C_{10}$	11	57	28	1	29	49
$g \in C_{11}$	0	63	22	0	18	0

$$\phi(g^\emptyset) = \{g \mid g \in C_7\}$$

$$\phi(g^2) = \{g \mid g \in C_1 \cup C_7 \cup C_{10}\}$$

$$\phi(g^3) = \{g \mid g \in C_1 \cup C_2 \cup C_7 \cup C_{10}\}$$

$$\phi(g^4) = \{g \mid g \in C_1 \cup C_2 \cup C_5 \cup C_7 \cup C_{10}\}$$

$$\phi(g^5) = \{g \mid g \in C_1 \cup C_2 \cup C_7 \cup C_{10}\}$$

$$\phi(g^6) = \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_7 \cup C_{10}\}$$

$$\phi(g^7) = \{g \mid g \in (C_7 \setminus g \text{ s.t. } d_i(g) = d_i(g^7) \text{ for all } i \in N) \cup C_{10}\}$$

$$\phi(g^8) = \{g \mid g \in C_1 \cup C_4 \cup C_7 \cup C_{10}\}$$

$$\phi(g^9) = \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_{10}\}$$

$$\phi(g^{10}) = \{g \mid g \in C_1 \cup C_2 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_{10} \setminus g^{10}\}$$

$$\phi(g^{11}) = \{g \mid g \in C_7 \cup C_{10}\}.$$

Note that $g^\emptyset$ is more farsightedly stable than all networks $g \neq g^7$. Moreover, g^7 is not more farsightedly stable than $g^\emptyset$. Thus, $g^\emptyset$ and each g^7 are minimally farsighted unstable networks. The maximally farsighted absorbing networks are also $g^\emptyset$, g^7 and g^{10} . Indeed:

$$\phi^{-1}(g^\emptyset) = \{g \mid g \in C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_8 \cup C_9 \cup C_{10}\};$$

$$\phi^{-1}(g^7) = \{g \mid g \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11}\}$$

and

$$\phi^{-1}(g^{10}) = \{g \mid g \in C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \setminus g^{10} \cup C_{11}\}.$$

From Table 8, we have that g^θ is the #-minimally farsighted unstable network. Remark that the network g^7 is #-more farsightedly absorbing than all other networks except g^θ .

Acknowledgements Ana Mauleon and Vincent Vannetelbosch are, respectively, Research Director and Senior Research Associate of the National Fund for Scientific Research (FNRS). Financial support from the Fonds de la Recherche Scientifique - FNRS research grant T.0143.18 is gratefully acknowledged.

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