Rational homotopy type of subspace arrangements

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Contents

Introduction **vii**

Chapter 1 Homotopy and rational homotopy theory **1**

1.1. Rational homotopy type 1
1.2. Sullivan models 3
1.2.1. Definitions 3
1.2.2. Bigrading of minimal Sullivan models 4
1.3. Formal spaces 5
1.4. Dichotomy 6
1.5. Homotopy Lie algebras 7
1.6. Holonomy of a fibration 8
1.7. Fibrations and wedges of spheres 9

Chapter 2 Topology of subspace arrangements **13**

2.1. Definitions 13
2.2. The lattice $L(\mathcal{A})$ 15
2.3. Independant subsets 18
2.4. Fundamental group of the complement space 19

Chapter 3 Rational models **23**

3.1. A simple model for central arrangements 25
3.1.1. Description of $D(\mathcal{A})$ 25
3.1.2. Arrangements with a geometric lattice 28
3.1.3. Restricted arrangements 29
3.2. Rational model of coordinate subspace arrangements . 31

Chapter 4 Deleted and restricted arrangements **35**

4.1. Introduction 35
4.2. A long exact sequence in cohomology 37

4.2.1. Proof of existence 37
4.2.2. Euler characteristic 44
4.3. Deletion-restriction theorem 45
4.3.1. Arrangements with a separator 45
4.3.2. Arrangements with a geometric lattice 46
4.4. A $\mathbb{Z}_2$-isomorphism in cohomology 48
4.5. Examples 51

Chapter 5 Rational wedges of spheres 55
5.1. Arrangements with a geometric lattice 56
5.2. Coordinate subspace arrangements 59

Chapter 6 Ellipticity of $M(\mathcal{A})$ 63
6.1. Introduction 63
6.2. Poincaré duality 64
6.3. Arrangements with a geometric lattice 69
6.4. Coordinate subspace arrangements 71

Chapter 7 Hyperbolicity of $M(\mathcal{A})$ 77
7.1. Introduction 77
7.2. Technical lemmas 78
7.3. Non-ellipticity of the complement space 85
7.4. Arrangements with a geometric lattice 87
7.5. Coordinate subspace arrangements 91

Bibliography 95

Introduction

Arrangements of hyperplanes

An *arrangement of hyperplanes* is a finite set of hyperplanes (affine subspace of codimension one) in some finite dimensional vector space V (in practice, it will often be $\mathbb{R}^m$ or $\mathbb{C}^m$). The study of this area of mathematics started with the cheese cutting problem :

> *What is the largest number of pieces that we can obtain by cutting a cheese with n cuts? (see figure 1)*

Mathematically, this question can be reformulated in the following way : among the finite sets of n planes in $\mathbb{R}^3$, what is the largest number of connected components? This simple problem is one of the earliest example of a question in the subject of arrangements of hyperplanes.

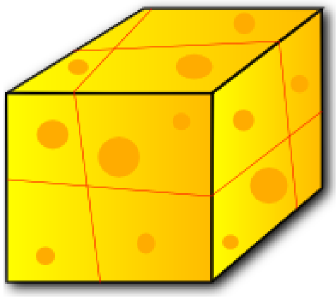


Figure 1: Cheese cut with three planes

The cheese cutting problem is a typical example : with some combinatorial data (number of hyperplanes, dimension of V), we want to know a topological property of the complement space : its number of connected components. A complete solution was given by

L. Schläfli in 1901 (see [29]). Also, this problem is discussed in chapter 4.

Since then, and for various reasons, many arrangements of hyperplanes appeared in different areas of science and were studied by mathematicians. As illustrated by the cheese cutting problem, the earlier apparitions of arrangements involved partition problems in Euclidean and projective spaces. But it was discovered that the notion of arrangement of hyperplanes connects many aspects of mathematics which seemed to be quite unrelated, in particular, topology, geometry and combinatorics.

In 1971, B. Grünbaum published a survey of the new field (see [20]), and was one of the first to use the term *arrangement of hyperplanes*. Since then, arrangements have been studied with various techniques. An important milestone was reached when Orlik & Terao published their book *Arrangements of hyperplanes* in 1992 (see [26]). This book contains most foundational results of the theory.

Subspace arrangements

Clearly, the definition of arrangements of hyperplanes can be generalized : since we only consider subspaces of codimension one, it is natural to consider the more general notion of *subspace arrangements* which are finite sets of affine subspaces of arbitrary dimension.

Obviously, subspace arrangements are much more general than arrangements of hyperplanes, and therefore much more complicated. It took a long time before some significant results were proved. One of the most important early result is the computation of the cohomology groups (not algebras) of the complement space, by Goresky & MacPherson, in 1988. But even if it is not easy, the study of subspace arrangements gives us a new perspective on many properties of arrangements of hyperplanes.

Note that subspace arrangements are not just an abstract concept born in the imagination of a mathematician. They actually play a role in recent engineering works related to processing, representing, storing, retrieving and interpreting multidimensional data[1]. Typically, data extracted from images and videos is represented as

[1] See the slides of the talk given by R. Fossum, titled *Subspace Arrangements in Theory and Practice*, available at `http://www.ima.umn.edu/2006-2007/`

points in spaces of large dimension. While the data is not often linear, linear models are easier to use. And being able to compute some topological/geometrical data out of these models can be useful. Some examples of such applications are : finger print identification, clustering of DNA sequences, segmenting images in regions of homogeneous textures, face recognition, ...

In recent works about arrangements, various rational models of complement of subspace arrangements have been obtained in various papers (see [8], [31], [32], [7]). These models are algebras which contain interesting information about various topological properties of the space that they represent : the rational homotopy type.

Rational homotopy theory and models

Homotopy theory is the study of the invariants and properties of topological spaces X and continuous maps f that depend only on the homotopy type of the space and the homotopy class of the map. Classical examples include the homology groups $H_*(X;\mathbb{Z})$, the cohomology algebra $H^*(X;\mathbb{Z})$, the homotopy groups $\pi_*(X)$.

But in practice, those invariants are sometimes difficult to compute, even for simple spaces. A solution is to discard some information until we keep something that we can manipulate. The problem is that if we discard too much information, we don't have anything interesting left. A good compromise has been discovered with the rational homotopy theory : we keep the rational information by localizing with respect to $\mathbb{Q}$. The result is a theory that is both interesting and usable in practice. Invariant of rational homotopy theory include the rational homology groups $H_*(X;\mathbb{Q})$, the rational cohomology algebra $H^*(X;\mathbb{Q})$, the rational homotopy groups $\pi_*(X)\otimes\mathbb{Q}$ (see the book [16] for a very complete reference on that topic).

An important aspect of that theory is the concept of rational model : each space is replaced with algebraic models that mimic its properties. In this work, we will study a few models of complement of subspace arrangements. Understanding how to extract the information out of those models will be the key to most results.

In this work, we will use a lot a rational model obtained by Yuzvinsky and Feichtner. This model is only valid in the case of subspace arrangements with only vector subspaces of $\mathbb{C}^m$. So, from chapter 3 until the end, we will only consider such arrangements.

Main results

The first results presented in this thesis concern recurrence formulas. The idea is that various topological invariants (in particular, the cohomology groups) can be computed recursively by adding one subspace at a time. For example, we show that the Euler characteristic of the complement space of a complex arrangement is always zero (if the arrangement is not empty).

But the two main results of this thesis are contained in the last two chapters. The first theorem is the following :

Theorem. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a central complex coordinate subspace arrangement or a complex subspace arrangement with a geometric lattice. Then the following conditions are equivalent :*

1. *The complement space $\mathcal{A} \setminus \bigcup_{x \in \mathcal{A}} x$ is a nilpotent rationally elliptic space,*
2. $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$*,*
3. *The complement space has the homotopy type of a finite product of odd dimensional spheres,*

This result is interesting for several reasons. First, it connects a rational homotopy property (condition 1), a geometrical property (condition 2. It means that the subspaces are in the most *disjoint* situation) and a topological property (condition 3). Then, it is proved for two types of subspace arrangements (coordinate subspace arrangements and subspace arrangements with a geometric lattice) with totally different proofs. With this theorem in mind, an interesting question is to determine if these three conditions are equivalent in general, for every subspace arrangement (it is the question 6.1).

The theorem mentioned above shows that most subspace arrangements are not rationally elliptic. Therefore, it is interesting to consider rationally hyperbolic subspace arrangements. This brings us to the next theorem…

Theorem. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex coordinate subspace arrangement or a complex subspace arrangement with a geometric lattice. If every $x \in \mathcal{A}$ has $\operatorname{codim}(x) \geq 2$, then the complement space $M(\mathcal{A})$ is rationally hyperbolic if and only if there is an injective map $\mathbb{L}(u, v) \to \pi_*(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$.*

Here again, this theorem is obtained for two types of subspace arrangements, with totally different proofs. Note that it is a particular case of a conjecture by Avramov & Félix.

Conjecture (Avramov-Félix)**.** *If X is a nilpotent finite dimensional, rationally hyperbolic space, then the homotopy Lie algebra $L_X = \pi_*(\Omega X)) \otimes \mathbb{Q}$ contains a free Lie subalgebra on two generators.*

Overview of this thesis

Chapter 1 contains most of the preliminaries from rational homotopy theory that we will need. It is not self-contained (but it gives a few references to fill the gaps!), its purpose is to give a proper definition of the various concepts that we need and to recall a few important results.

In chapter 2, we present the various definitions and results in the field of subspace arrangements. In particular, we will explain what a coordinate subspace arrangement and a geometric lattice are.

Chapter 3 is a presentation of the two main models that we will use in this work : a rational model for central complex subspace arrangements $D(\mathcal{A})$, and a rational model $BP(\mathcal{A})$, only valid for complex coordinate subspace arrangements.

The next chapter, chapter 4, is a generalization to arbitrary subspace arrangements of various results about arrangements of hyperplanes obtained by studying the *deleted* and *restricted* arrangements. For a given arrangement $\mathcal{A}$ with $x \in \mathcal{A}$, the idea is to consider the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{x\}$ and the restricted arrangement $\mathcal{A}'' = \{x \cap y : y \in \mathcal{A}'\}$. The triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is the basis for some recurrence formulas.

Chapter 5 is a short chapter which shows the power of the model $D(\mathcal{A})$. By considering the situation where all the products are zero in (rational) cohomology, we can prove that if $\mathcal{A}$ is an arrangement with a geometric lattice, then the complement space is a rational wedge of spheres if and only if for all $x, y \in \mathcal{A}$, $x + y \neq V$.

Finally, the last two chapters (6 and 7) study the subspace arrangements with the rational dichotomy in mind. They contain the proofs of the two theorems mentioned in the previous section.

Notations

Finally, let us end this introduction with some explanations about the notations.

Subspace arrangements (and arrangements of hyperplanes) are denoted by calligraphic letters : $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}, \ldots$ Subspaces (elements of arrangements) are denoted by lowercase letters : $x, y, z, \ldots$

The letter m will be reserved for the dimension of the ambient space, so, for example, we will often consider arrangements in $\mathbb{C}^m$. The letter n will mostly be used for the numbers of subspaces of a given arrangement. For example, $\mathcal{A} = \{x_1, \ldots, x_n\}$.

When we will consider the lattice $L(\mathcal{A})$ of a given arrangement $\mathcal{A}$, we will use uppercase letters for elements of $L(\mathcal{A})$: $M, N, P, \ldots$

The notation $[n]$ represents the set $\{1, \ldots, n\}$. And the last notation might seem unusual, but will be useful : $\{i_1, \ldots, i_n\}_<$ represents the set $\{i_1, \ldots, i_n\}$, but with the additional assumption that $i_1 < i_2 < \cdots < i_n$.

CHAPTER 1

Homotopy and rational homotopy theory

This chapter contains definitions and results from (rational) homotopy theory. Since this is such a rich and broad topic, this work can only scratch the surface.

As we will see later in this chapter, rational homotopy theory is the study of the rational homotopy type of a topological space and of properties invariant under rational homotopy equivalences. While the rational homotopy type of a space X contains less information than its homotopy type (it does not know about torsion), it can be computed much more easily.

The key is the fact that there is a correspondence between rational homotopy types of simply connected spaces and some commutative differential graded algebras. These algebras (that we will call *minimal models*) encode all the rational homotopy information of the corresponding space.

The interested reader will find more information about rational homotopy theory in the classical reference [16].

1.1 Rational homotopy type

In this section, we present the main results and definitions needed to be able to properly establish the foundations of rational homotopy theory. This section is inspired by [23].

Definition 1.1. A simply connected space X is *rational* if $\pi_* X$ is a $\mathbb{Q}$-vector space. A map $f\colon X \to Y$ is a *rationalization* of X if Y is simply connected, rational and if

$$\pi_* f \otimes \mathbb{Q}\colon \pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q} \cong \pi_* Y$$

is an isomorphism.

Clearly, a rational space can not have torsion in its homotopy. Being a rational space is a big constraint, it is not obvious that rational spaces/rationalization exists. Luckily, the next theorem proves their existence and unicity.

Theorem 1.2. *Let X be a simply connected space. There exists a rational space $X_{\mathbb{Q}}$ and a rationalization $j\colon X \to X_{\mathbb{Q}}$. Also, if $Y_{\mathbb{Q}}$ is a simply connected rational space, then any continuous map $f\colon X \to Y_{\mathbb{Q}}$ can be extended over $X_{\mathbb{Q}}$, i.e., there is a continuous map $g\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ which is unique up to homotopy, such that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y_{\mathbb{Q}} \\ & X_{\mathbb{Q}} \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

This theorem shows that if we have a continuous map $f\colon X \to Y$, then there exists a unique (up to homotopy) map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. Now, we can state the two main definitions of this section.

Definition 1.3. The *rational homotopy type* of a simply connected space X is the weak homotopy type of its rationalization $X_{\mathbb{Q}}$.

Definition 1.4. A continuous map $f\colon X \to Y$ between simply connected spaces is a *rational homotopy equivalence* if the following equivalent conditions are satisfied :

1. $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism,

2. $H_*(f;\mathbb{Q})$ is an isomorphism,

3. $H^*(f;\mathbb{Q})$ is an isomorphism,

4. $f_{\mathbb{Q}}\colon X_Q \to Y_{\mathbb{Q}}$ is a weak homotopy equivalence.

These two concepts are what rational homotopy theory is about, namely the study of rational homotopy types of spaces and of the properties that are invariant under rational homotopy equivalences.

1.2 Sullivan models

1.2.1 Definitions

Since rational homotopy theory deals with the interactions between topology and algebraic objects, it makes sense to define a few basic concepts from algebra.

Definition 1.5. A *commutative differential graded algebra* (cdga) (A, d) is a graded algebra $A = \oplus_{n\in\mathbb{N}} A^n$ over $\mathbb{Q}$ with a map $d\colon A^n \to A^{n+1}$ such that

$$a \cdot b = (-1)^{|a||b|} b \cdot a \quad \text{ and } \quad d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

for all $a, b \in A$ ($|a|$ denotes the degree of a : $a \in A^n$ iff $|a| = n$).

The morphisms of cdga's are the obvious ones : maps of graded algebras that preserve the differential. When a map $\varphi\colon (A, d) \to (B, d')$ induces an isomorphism in cohomology, it is said to be a *quasi-isomorphism*.

An important case occurs when $d = 0$. For example, the cohomology algebra $(H^*(X; \mathbb{Q}), 0)$ is a cdga.

Definition 1.6. Let V be a graded vector space. The *free commutative algebra on* V, ΛV, is the tensor product of the symmetric algebra on V^{even} by the exterior algebra on V^{odd}.

Definition 1.7. A *Sullivan algebra* is a cdga of the form $(\Lambda V, d)$ such that there is a basis of V, $(x_a)_{a\in A}$, indexed by a well-ordered set with the property that $d(x_a) \in \Lambda(x_b)_{b<a}$. A Sullivan algebra is *minimal* if $d(V) \subseteq \Lambda^{\geq 2} V$.

Minimal Sullivan algebras are very nice : they do not contain redundant information and the order property on a basis of V is the key to many proofs by induction. As we will see, they possess some nice properties.

Definition 1.8. Let (A, d) be a cdga. A *Sullivan model* of (A, d) is given by a Sullivan algebra $(\Lambda V, d)$ and a quasi-isomorphism of cdga $\varphi\colon (\Lambda V, d) \to (A, d)$. If $(\Lambda V, d)$ is minimal, then it is a *minimal model*.

The link between these algebraic definitions and topology is given by the functor of piecewise linear forms $A_{PL}\colon \mathrm{Top} \to \mathrm{CDGA}$ (again, see [16] for details). A nice property of this functor is that for all topological spaces X, we have $H^*(A_{PL}(X)) \cong H^*(X;\mathbb{Q})$.

Definition 1.9. A *Sullivan model* for a topological space X is a Sullivan algebra $(\Lambda V, d)$ with a quasi-isomorphism $\varphi\colon (\Lambda V, d) \to A_{PL}(X)$. A Sullivan model is *minimal* if $(\Lambda V, d)$ is a minimal algebra.

Now, for some spaces, a minimal model exists and possesses some nice properties.

Proposition 1.10. *Let X be a simply connected space with finite Betti numbers. Then the space X has a minimal model $\varphi\colon (\Lambda V, d) \to A_{PL}(X)$ (unique up to isomorphism). Also,* $\dim V^n = \dim \pi_n(X) \otimes \mathbb{Q}$*.*

Finally, we'll end this subsection with a technical definition.

Definition 1.11. The *linear part* of a map $f\colon (\Lambda V, d) \to (\Lambda W, d)$ between Sullivan algebras is the linear map $Qf\colon V \to W$ defined by $f(v) - Qf(v) \in \Lambda^{\geq 2} W$ for all $v \in V$.

1.2.2 Bigrading of minimal Sullivan models

Let $(A, 0)$ be a 1-connected cdga (so $A^0 = \mathbb{Q}$ and $A^1 = 0$) where $A = \oplus_{q\geq 0} A^q$. It is well known that there exists a minimal Sullivan model $(\Lambda V, d) \xrightarrow{\simeq} (A, 0)$. The algebra ΛV is graded by $V = \oplus_{n\in\mathbb{N}} V^n$.

However, by taking a closer look to the proof of the existence of the minimal model $(\Lambda V, d) \xrightarrow{\simeq} (A, 0)$, one can see that V has an additional grading $V = \oplus_{q\geq 0} V_q$ such that $d(V_q) \subset (\Lambda V)_{q-1}$, given by the induction step.

So, let's describe how V is constructed. First, let $V_0 = A^+/(A^+ \cdot A^+)$ and define the differential on V_0 by $d(V_0) = 0$. There is a natural map $\varphi_0\colon \Lambda V_0 \to A$ such that $H(\varphi_0) = \varphi_0$ is surjective. Suppose that φ_0 has been extended to $\varphi_k\colon (\Lambda(\oplus_{i=0}^k V_i), d) \to (A, 0)$. Let $(z_\alpha)_{\alpha\in I}$ be cocycles in $\Lambda(\oplus_{i=0}^k V_i)$ such that $(z_\alpha)_{\alpha\in I}$ is a basis of $\ker H(\varphi_k)$. Then denote by V_{k+1} a graded vector space with basis $(v_\alpha)_{\alpha\in I}$ and $\deg v_\alpha = \deg z_\alpha - 1$. Extend d to $\Lambda(\oplus_{i=0}^{k+1} V_i)$ by setting $dv_\alpha = z_\alpha$. We define a morphism of graded differential algebras $\varphi_{k+1}\colon (\Lambda(\oplus_{i=0}^{k+1} V_i), d) \to (A, 0)$ by sending $\varphi_{k+1}(v_\alpha) = 0$.

This construction can be repeated as many times as necessary. It essentially gives the proof of the following proposition.

Proposition 1.12. *Let $(A, 0)$ be a 1-connected cdga. There exists a minimal model $\varphi\colon (\Lambda V, d) \xrightarrow{\simeq} (A, 0)$ with the following properties :*

1. *$V = \oplus_{p\geq 0, q\geq 1} V_p^q$ is a bigraded vector space,*
2. *$d(V_p^q) \subseteq (\Lambda^{\geq 2} V)_{p-1}^{q+1}$,*
3. *$H_q(\Lambda V, d) = A$ if $q = 0$ and $H_q(\Lambda V, d) = 0$ otherwise.*

The bigrading on V is sometimes useful. For example, given $1 \leq s \leq k$, let's consider the two following differential cochain algebras :

$$(A, 0) = (\mathbb{Q}[x_1, x_2, \ldots, x_k]/(x_1 \cdots x_s), 0),$$
$$(B, d) = (\Lambda(x_1, \ldots, x_k, u), d).$$

with $dx_i = 0, du = x_1 \cdots x_s, |x_i| = 2$, and $|u| = 2s - 1$.

Let's define a second grading on A and B : we say that x_i has a lower degree 0 and u has a lower degree 1. This new grading gives a quick proof of the following proposition.

Proposition 1.13. *The map $\varphi\colon (B, d) \to (A, 0)$ defined by $\varphi(x_i) = x_i$ and $\varphi(u) = 0$ is a quasi-isomorphism.*

Proof. In lower degree 0, we have : $H_0(B, d) = A$. Since there is no element of lower degree greater than 2, we have $H_p(B, d) = 0$ if $p \geq 2$. Now, suppose that ω is a cycle of lower degree 1. Then $\omega = au$ with $a \in \Lambda(x_1, \ldots, x_k) = \mathbb{Q}[x_1, \ldots, x_k]$ and $0 = d\omega = ax_1 \ldots x_k$. Hence $a = 0$ and $H_*(B, d) = H_0(B, d) = A$. □

1.3 Formal spaces

Some spaces have a nice property : their rational homotopy type is completely determined by their cohomology.

Definition 1.14. Let (A, d) be a cdga and $(\Lambda V, d) \xrightarrow{\simeq} (A, d)$ be its minimal model. We say that (A, d) is *formal* if there exists a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} (H^*(A, d), 0)$. A topological space X is *formal* if $A_{PL}(X)$ is formal.

Clearly, an equivalent definition of formality is given by the next proposition.

Proposition 1.15. *A rational cdga (A, d) is formal if and only if there exists a chain of quasi-isomorphisms connecting (A, d) and $(H^*(A, d), 0)$.*

If X and Y are two formal spaces, then the wedge $X \vee Y$ and the product $X \times Y$ are both formal.

1.4 Dichotomy

The dichotomy theorem in rational homotopy theory is a beautiful result. It states that finite simply connected CW-complexes can behave in two radically different ways : either the sum of the dimensions of the rational homotopy groups is finite (elliptic case), or the sequence of dimensions of the rational homotopy groups grow exponentially (hyperbolic case).

But this result is still true if we consider a slightly wider class of spaces : the nilpotent spaces.

Definition 1.16. A space X is *nilpotent* if $\pi_1(X)$ is nilpotent (as a group) and acts in a nilpotent way on each $\pi_r(X)$, $r \geq 2$.

Obviously, simply connected spaces are nilpotent.

Definition 1.17. A nilpotent space X is *rationally elliptic* if

$$\dim H^*(X; \mathbb{Q}) < \infty \text{ and } \dim \pi_*(X) \otimes \mathbb{Q} < \infty.$$

For example, spheres are elliptic spaces, products and retracts of elliptic spaces are elliptic.

Now, the precise definition of hyperbolic spaces in [16] is slightly more technical than what we will need in this thesis, so, we will use a simpler definition, which is a little more restrictive.

Definition 1.18. A nilpotent space X with the homotopy type of a finite CW-complexe is *rationally hyperbolic* if $\dim \pi_*(X) \otimes \mathbb{Q} = \infty$.

If X and Y are finite dimensional and are not rationally contractible, then $X \vee Y$ is an hyperbolic space.

Note that this defines a dichotomy for spaces with the homotopy type of a nilpotent finite CW-complexes X : they are either rationally elliptic (with $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$) or they are rationally hyperbolic (with $\dim \pi_*(X) \otimes \mathbb{Q} = \infty$).

The dichotomy elliptic/hyperbolic is an useful result. In fact, it has been studied with the help of Sullivan models and it gave many results about the dimensions of V, V^{odd}, V^{even}. In particular, elliptic spaces have very precise constraints on the form of their minimal models, and these constraints are the key to results about the category, the Euler characteristic, the Poincaré polynomial and in general, about pretty much any dimensional/combinatorial information contained in a space.

The terminology *hyperbolic* comes from the following theorem.

Theorem 1.19. *If X is a rationally hyperbolic space, then there exist $C > 1$ and $N \in \mathbb{N}$ such that $\sum_{i=0}^{n} \dim \pi_i(X) \otimes \mathbb{Q} \geq C^n$ for all $n \geq N$.*

Proof. See theorem 33.2 in [16]. □

When a space X is elliptic and formal, Félix and Halperin proved that the explicit construction of the minimal model $\varphi\colon (\Lambda V, d) \to (H^*(X), 0)$ given in section 1.2 stops at the second step. More precisely, they obtain the following theorem.

Theorem 1.20 (Félix-Halperin)**.** *If a space X is formal and if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$, then its minimal model has the form $(\Lambda V, d) = (\Lambda(V_0 \oplus V_1), d)$ with $V_1 = V_1^{odd}$, $dV_0 = 0$ and $dV_1 \subset \Lambda V_0$. Moreover, the injection $(\Lambda V_0^{odd}, 0) \to (\Lambda V, d)$ induces an injective map in cohomology.*

Proof. See [15], theorem II. □

1.5 Homotopy Lie algebras

The homotopy Lie algebra L_X of a simply connected space X is obtained by giving to $\pi_*(\Omega X) \otimes \mathbb{Q}$ a Lie algebra structure whose bracket is defined by the Whitehead bracket.

If $(\Lambda V, d)$ is the minimal model of X, denote by d_1 the quadratic part of d. Then, as shown in [16], L_X can be deduced from $(\Lambda V, d_1)$. More precisely, $(\Lambda V, d_1)$ is the cochain algebra on L_X.

In general, the homotopy Lie algebra can be defined for any cdga :

Definition 1.21. Let A be a commutative differential graded algebra. The algebra $\mathrm{Ext}_A(\mathbb{Q},\mathbb{Q})$ is the universal enveloping algebra $U(L_A)$ of a Lie algebra L_A. This Lie algebra is called the *homotopy Lie algebra* of A.

If $\varphi\colon (A,d) \to (B,d)$ is a quasi-isomorphism of cdga's, then their associated homotopy Lie algebras are the same : $L_A \cong L_B$. So, for any cdga (A,d), $L_{(A,d)}$ is the homotopy Lie algebra of its Sullivan model.

The homotopy Lie algebra of a space X and the homotopy Lie algebra of its minimal model are isomorphic, which is a good sign that the minimal model of a space reflects its properties :

Proposition 1.22. *Let $(\Lambda V,d)$ be the minimal model of a nilpotent space X, then the homotopy Lie algebra L_X and $L_{(\Lambda V,d)}$ are isomorphic.*

1.6 Holonomy of a fibration

Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. In this section, we'll show that we can define a map $h\colon \Omega B \times F \to F$ such that the map h is the *holonomy of the fibration* and can be easily understood : we fix a point in the fiber and a loop in B, then we follow the loop B and look at the corresponding path in E. The end of this path is necessarily in F, so we choose this point as the image of h.

Let's consider the commutative diagram

$$\begin{tikzcd} \Omega B \times F \times \{0\} \cup \{*\} \times F \times I \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Omega B \times F \times I \arrow[r, "l"'] & B \end{tikzcd}$$

defined by $j(w,x,0) = x$, $j(*,x,t) = x$ and $l(w,x,t) = w(t)$. By the homotopy lifting property, there exists a lifting map $H : \Omega B \times F \times I \to E$ such that $p \circ H = l$ and $H \circ i = j$.

Definition 1.23. The *holonomy of the fibration* $p : E \to B$ is the map $h\colon \Omega B \times F \to F$ which maps (w,x) to $H(w,x,1)$.

It is well known that there is a long exact sequence in homotopy

$$\cdots \to \pi_{n+1}(B) \xrightarrow{\partial_n} \pi_n(F) \to \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(B) \to \cdots$$

where the map $\partial_n \colon \pi_{n+1}(B) \to \pi_n(F)$ is the connecting map of the fibration. Let us show that we can derive the map ∂_n from the holonomy of the fibration : the map h can be restricted to $\delta\colon \Omega B \to F; w \mapsto h(w,*)$. Then

$$\pi_n(\delta) = \partial_n \circ \Sigma_n \colon \pi_n(\Omega B) \to \pi_n(F)$$

where $\Sigma_n \colon \pi_n(\Omega B) \to \pi_{n+1}(B)$ is given by the adjunction

$$[S^n, \Omega B] \simeq [S^{n+1}, B].$$

1.7 Fibrations and wedges of spheres

The holonomy map defined above allows us to study the image of the connecting map of a fibration.

Proposition 1.24. *The image of the map $\pi_*(\Omega(\delta))\otimes\mathbb{Q}$ is contained in the center of the homotopy Lie algebra $\pi_*(\Omega F)\otimes\mathbb{Q}$.*

Proof. Let $x \in \pi_n(\Omega^2 B)$ and $y \in \pi_k(\Omega F)$. The holonomy map $h\colon \Omega B \times F \to F$ induces a Lie algebra morphism

$$\pi_*(\Omega h)\colon (\pi_*(\Omega^2 B)\otimes\mathbb{Q}) \oplus (\pi_*(\Omega F)\otimes\mathbb{Q}) \to \pi_*(\Omega F)\otimes\mathbb{Q}$$

such that $\pi_*(\Omega h)(0,y) = y$ and $\pi_*(\Omega h)(x,0) = \pi_*(\Omega\delta)(x)$. We have :

$$\begin{aligned}[\pi_*(\Omega\delta)(x), y] &= [\pi_*(\Omega h)(x,0), \pi_*(\Omega h)(0,y)] \\ &= \pi_*(\Omega h)([(x,0),(0,y)]) = 0.\end{aligned}$$

$\square$

Lemma 1.25. *If W is a finite dimensional graded vector space such that $\dim W \geq 2$, then the center of the free graded Lie algebra $\mathbb{L}(W)$ is zero.*

Proof. Let $n \geq 2$ and $x_1, \ldots, x_n$ be a basis of W. If a is in the center of the free graded Lie algebra $\mathbb{L}(W) = \mathbb{L}(x_1, \ldots, x_n)$, then a is in the center of the universal enveloping algebra $U\mathbb{L}(x_1, \ldots, x_n)$, which is the tensor algebra $T(x_1, \ldots, x_n)$.

Let's use the lexicographical order on $T(x_1, \ldots, x_n)$ and write a as a sum of monomials $a = \sum_{i=1}^{k} a_i$ with $a_1 > a_2 > \cdots > a_k$. Since $\mathbb{L}(W)$ is a graded Lie algebra, we can assume that $\deg(a_1) = \deg(a_2) = \cdots = \deg(a_k)$.

Suppose that $a_1 = x_1^r uv$ with $u \in \{x_2, \ldots, x_k\}$ and that $v \in T(x_1, \ldots, x_n)$. Notice that $x_1 a = \sum_{i=1}^{k} x_1 a_i$ with $x_1 a_1 > x_1 a_2 > \ldots > x_1 a_k$ and $a x_1 = \sum_{i=1}^{k} a_i x_1$ with $a_1 x_1 > a_2 x_1 > \ldots > a_k x_1$. From $x_1 a = a x_1$, we deduce that $x_1 a_i = a_i x_1$ for $i = 1, \ldots, k$. But this is a contradiction, since $x_1^{r+1} uv$ cannot be equal to $x_1^r uv x_1$. So, $a_1 = \alpha x_1^r$ where α is a scalar.

If $a_2 \neq 0$, then the very same reasoning shows that $a_2 = \beta x_1^s$ for some scalar β and $s \in \mathbb{N}$. But $a_1 > a_2$, so $r \neq s$ and $\deg(a_1) = \deg(a_2)$, which is a contradiction, unless $a_2 = 0$ and $a = a_1 = \alpha x_1^r$. But the monomial x_1^r does not commute with x_2, unless $r = 0$, which means that $a = \alpha$. Since $a \in \mathbb{L}(x_1, \ldots, x_k)$, it can only be zero. □

Proposition 1.26. *Let $k \geq 2$ and $F = \vee_{i=1}^{k} S^{d_i}$ be a wedge of spheres. If F is the fiber of a fibration $F \hookrightarrow E \xrightarrow{p} B$, then for every $n \in \mathbb{N}$, the connecting map*

$$\partial_n \otimes \mathbb{Q} \colon \pi_{n+1}(B) \otimes \mathbb{Q} \to \pi_n(\vee_{i=1}^{k} S^{d_i}) \otimes \mathbb{Q}$$

in the long exact sequence in (rational) homotopy is zero.

Proof. For $n = 0$, the map $\pi_1(B) \otimes \mathbb{Q} \to \pi_0(\vee_i S^{d_i}) \otimes \mathbb{Q}$ is zero because $\vee_i S^{d_i}$ is a connected space.

Suppose that $n \geq 1$. Since $\pi_n(\delta) = \partial_n \circ \Sigma_n$ and Σ_n is an isomorphism, it is sufficient to show that $\pi_n(\delta)$ is the zero map. We have a commutative diagram

$$\begin{tikzcd}
\pi_{n-1}(\Omega^2 B) \otimes \mathbb{Q} \arrow[r, "\pi_{n-1}(\Omega\delta)\otimes\mathbb{Q}"] \arrow[d] & \pi_{n-1}(\Omega \vee_i S^{d_i}) \otimes \mathbb{Q} \arrow[d] \\
\pi_n(\Omega B) \otimes \mathbb{Q} \arrow[r, "\pi_n(\delta)\otimes\mathbb{Q}"'] & \pi_n(\vee_i S^{d_i}) \otimes \mathbb{Q}
\end{tikzcd}$$

where the two vertical maps are the isomorphisms obtained by the adjunctions $[S^n, \Omega B] \simeq [S^{n+1}, B]$. The proposition 1.24 and the lemma 1.25 imply that the map $\pi_{n-1}(\Omega\delta)\otimes\mathbb{Q}$ is the zero map. Therefore, the map $\pi_n(\delta) \otimes \mathbb{Q}$ is also the zero map. □

Corollary 1.27. *If $F = \vee_{i=1}^{k} S^{d_i}$ is a wedge of spheres ($k \geq 2$) and $F \hookrightarrow E \xrightarrow{p} B$ is a fibration, then there exists a short exact sequence of homotopy Lie algebras*

$$0 \to \pi_*(\Omega F) \otimes \mathbb{Q} \to \pi_*(\Omega E) \otimes \mathbb{Q} \to \pi_*(\Omega B) \otimes \mathbb{Q} \to 0.$$

Proof. Let's consider the fibration $\Omega F \to \Omega E \to \Omega B$. The proof of proposition 1.26 shows that the connecting maps $\pi_{n+1}(\Omega B) \otimes \mathbb{Q} \to \pi_n(\Omega F) \otimes \mathbb{Q}$ are trivial. Therefore, the long exact sequence in homotopy split in short exact sequences

$$0 \to \pi_n(\Omega F) \otimes \mathbb{Q} \to \pi_n(\Omega E) \otimes \mathbb{Q} \to \pi_n(\Omega B) \otimes \mathbb{Q} \to 0.$$

This completes the proof. □

CHAPTER 2

Topology of subspace arrangements

Let us start this chapter with the two most important definitions :

Definition 2.1. A *subspace arrangement* $\mathcal{A}$ is a finite set of affine subspaces in a finite dimensional vector space V.

In this work, the vector space V will always be $\mathbb{C}^m$ unless explicitly stated otherwise. But most definitions in this chapter will also work for $V = \mathbb{R}^m$.

Definition 2.2. The *complement space* of a subspace arrangement $\mathcal{A}$ is the topological space $M(\mathcal{A}) = V \setminus \bigcup_{x \in \mathcal{A}} x$.

The topology of complements of subspace arrangement is a well studied area of mathematics. A typical example is the cheese cutting problem mentioned in the introduction. This is a typical example of a problem in the field of subspace arrangements : we start with some combinatorial data (dimension of the space V and number of hyperplanes) and we ask a question about the topology of the complement space (number of connected components).

It illustrates a common situation in this field : the interplay between combinatorics and topology. As we'll see, algebra also plays often an important role.

2.1 Definitions

This complement space is in general complicated. Because of that, its topology is not easy to study. We will need some definitions to distinguish different kind of subspace arrangements

Definition 2.3. Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a subspace arrangement in V. If every subspace x_i is an hyperplane, then $\mathcal{A}$ is an *arrangement of hyperplanes*. If V is the vector space $\mathbb{R}^m$ for some $m \geq 1$, then $\mathcal{A}$ is a *real arrangement*. If V is the vector space $\mathbb{C}^m$ for some $m \geq 1$, then $\mathcal{A}$ is a *complex arrangement*. If all the affine subspaces x_i are vector subspaces, then $\mathcal{A}$ is *central*.

From now on, unless explicitly stated otherwise, we will only work with central subspace arrangements in $\mathbb{C}^m$.

In this work, we will often consider a particular class of central complex arrangements : the coordinate subspace arrangements.

Definition 2.4. A *coordinate subspace* x_σ in $\mathbb{C}^m$ is a subspace defined by $z_{i_1} = z_{i_2} = \cdots = z_{i_r} = 0$ for some $\sigma = \{i_1, \dots, i_r\} \subseteq [m]$. A *coordinate subspace arrangement* $\mathcal{A}$ is a subspace arrangement such that every subspace of $\mathcal{A}$ is a coordinate subspace.

Since our goal is to study the topology of $M(\mathcal{A})$, we will suppose that $\mathcal{A}$ does not contain two distinct subspaces x and y such that $x \subset y$. In the case of coordinate arrangements, it can be rephrased as : if $\sigma_1 \subseteq \sigma_2 \subseteq [m]$, $x_{\sigma_1} \in \mathcal{A}$ and $x_{\sigma_2} \in \mathcal{A}$, then $\sigma_1 = \sigma_2$.

The study of subspace arrangements started with the particular case of arrangement of hyperplanes. In that context, an interesting method was introduced by T. Zaslavsky in [33] : the method of deletion and restriction. If $\mathcal{A}$ is an arrangement of hyperplanes and $x \in \mathcal{A}$, then we can consider the *deleted arrangement* $\mathcal{A}' = \mathcal{A} \setminus \{x\}$ and the *restricted arrangement* $\mathcal{A}'' = \{x \cap y : y \in \mathcal{A}'\}$. The triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ may be used to obtain recursion formulas (see [26] for some examples).

Generalizing the notion of deleted arrangements in the case of arbitrary subspace arrangements is straightforward.

Definition 2.5. Let $\mathcal{A}$ be a subspace arrangement and $x \in \mathcal{A}$. The *deleted arrangement with respect to x* is $\mathcal{A}' = \mathcal{A} \setminus \{x\}$.

However, there is a slight difficulty when we want to extend the idea of restricted arrangements. The obvious way is to define the restricted arrangement by $\{x \cap y : y \in \mathcal{A}'\}$. But there is a problem : this set might contain two distinct subspaces y_1 and y_2 such that $x \cap y_1 \subsetneq x \cap y_2$. Since we want to avoid that situation, it makes sense to adjust slightly our definition.

Definition 2.6. Let $\mathcal{A}$ be a subspace arrangement and $x \in \mathcal{A}$. The *restricted arrangement* with respect to x is the arrangement $\mathcal{A}'' = \{x \cap y : y \in \mathcal{A}'\} \setminus S$, where S is the set

$$S = \{x \cap y : y \in \mathcal{A}' \text{ and } \exists y' \in \mathcal{A}' \text{ with } y \neq y' \text{ and } x \cap y \subseteq x \cap y'\}.$$

Finally, this last proposition shows that complement spaces are well-behaved topological spaces.

Proposition 2.7. *Let $\mathcal{A}$ be a subspace arrangement. Then $M(\mathcal{A})$ has the homotopy type of a finite CW-complex.*

Proof. See [4] for an explicit construction. □

2.2 The lattice $L(\mathcal{A})$

In this section, we will introduce an important combinatorial tool : the lattice associated to a subspace arrangement. It is a very good (and simple) invariant.

Definition 2.8. Let $L(\mathcal{A})$ be the set of all nonempty intersections of elements of $\mathcal{A}$. We agree that $L(\mathcal{A})$ includes V as the intersection of the empty collection of subspaces.

The reverse inclusion defines a partial order on $L(\mathcal{A})$: $x \leq y$ if and only if $x \supseteq y$. This is the reverse inclusion, so the vector space V is the minimal element of the poset $L(\mathcal{A})$.

When the arrangement $\mathcal{A}$ is central, the poset $L(\mathcal{A})$ possess an additional algebraic property : it is a lattice, for an appropriate choice of operations.

Definition 2.9. Let $\mathcal{A}$ be a central arrangement. For every $x, y \in L(\mathcal{A})$, we define two operations $L(\mathcal{A}) \times L(\mathcal{A}) \to L(\mathcal{A})$: the *join* is $x \vee y = x \cap y$ and the *meet* is defined by $x \wedge y = \cap\{z \in L(\mathcal{A}) : x \cup y \subset z\} \in L(\mathcal{A})$.

For $\sigma = \{i_1, \ldots, i_r\} \subseteq [n]$, we will often denote $x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_r}$ by $\vee\{x_{i_1}, \ldots, x_{i_r}\} = \vee\sigma$.

With these two operations, $L(\mathcal{A})$ is a lattice (if $\mathcal{A}$ is central). Notice that the *join* operation is not defined if $\mathcal{A}$ is not central. As we said earlier, the lattice is a good invariant. Richard Randell proved in 1989 the following proposition (see [27] for proper definitions).

Proposition 2.10. *If $\mathcal{A}$ and $\mathcal{B}$ are two subspace arrangements in $\mathbb{R}^m$ or in $\mathbb{C}^m$, connected by a one-parameter lattice isotopy, then their complement spaces $M(\mathcal{A})$ and $M(\mathcal{B})$ are diffeomorphic.*

When we have a lattice, it is useful to consider the rank function.

Definition 2.11. The *rank function* $\mathrm{rk}\colon L(\mathcal{A}) \to \mathbb{N}$ is defined by sending $x \in L(\mathcal{A})$ to the length r of a maximal chain $\mathbb{C}^m < x_1 < \cdots < x_r = x$ of maximal length. The rank of $\mathbb{C}^m$ is $\mathrm{rk}(\mathbb{C}^m) = 0$.

Definition 2.12. The *rank of a subspace arrangement* $\mathcal{A}$ is defined by the formula $\mathrm{rk}\,\mathcal{A} = \max\{\mathrm{rk}\,x : x \in L(\mathcal{A})\}$.

When $\mathcal{A}$ is an arrangement of hyperplanes, the rank of any element $x \in L(\mathcal{A})$ is simply the codimension $\mathrm{codim}(x)$.

Definition 2.13. The lattice $L(\mathcal{A})$ is *geometric* if, for every $x, y \in L(\mathcal{A})$, we have :

$$\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y).$$

The notion of geometric lattice has been studied in the general case of lattice theory (see [18] for more details). In that context, a geometric lattice is a lattice that verifies some additional technical conditions. The definition given above is equivalent when we deal with lattices induced by subspace arrangements.

For example, let's consider the arrangements $\mathcal{A} = \{x_1, x_2, x_3\}$ and $\mathcal{B} = \{y_1, y_2, y_3\}$ in $\mathbb{C}^3$ given by

$$\begin{array}{r|l} x_1 & \{z_2 = z_3 = 0\} \\ x_2 & \{z_1 = z_3 = 0\} \\ x_3 & \{z_1 = z_2 = 0\} \end{array} \qquad \begin{array}{r|c} y_1 & \{z_1 = 0\} \\ y_2 & \{z_2 = 0\} \\ y_3 & \{z_3 = 0, z_1 = z_2\}. \end{array}$$

As a set, $L(\mathcal{A}) = \{\mathbb{C}^3, x_1, x_2, x_3, 0\}$, where 0 is the zero-dimensional vector space in $\mathbb{C}^3$, and $L(\mathcal{B}) = \{\mathbb{C}^3, y_1, y_2, y_3, y_1 \cap y_2, 0\}$. Figure 2.1 shows the Hasse diagram of these two lattices. In $L(\mathcal{B})$, $\mathrm{rk}(0) = 3, \mathrm{rk}(y_i) = 1, \mathrm{rk}(y_1 \cap y_2) = 2$. The lattice $L(\mathcal{A})$ is geometric, and $L(\mathcal{B})$ is not geometric.

From now on, if x and y are in a given lattice, we will use the notation $x \succ y$ to say that $x > y$ and that there is no z such that $x > z > y$.

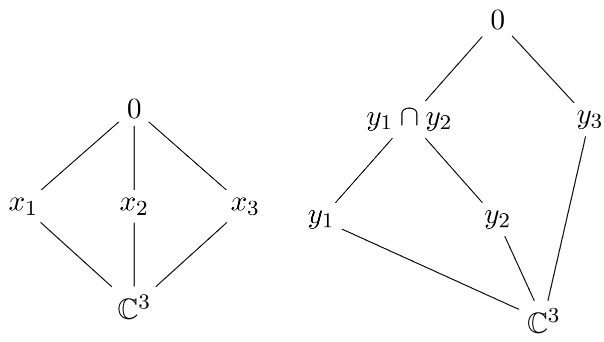


Figure 2.1: Hasse diagram of $L(\mathcal{A})$ and $L(\mathcal{B})$

Proposition 2.14. *Let $\mathcal{A}$ be a subspace arrangement. The following conditions are equivalent :*

1. *$L(\mathcal{A})$ is geometric,*
2. *for all $x, y, z \in L(\mathcal{A})$, $x \prec y$ implies that $x \vee z \prec y \vee z$ or $x \vee z = y \vee z$,*
3. *for all $x, y, z \in L(\mathcal{A})$, if $x \leq y$ and C is a maximal chain in $[x, y]$, then $\{x \vee z : x \in C\}$ is a maximal chain in $[x \vee z, y \vee z]$.*

Proof. See [18], p. 173. □

An interesting property of geometric lattice is that each maximal chain has the same length.

Proposition 2.15. *If L is a geometric lattice and $x, y \in L$ with $x < y$, then every maximal chain $x = x_0 \prec x_1 \prec \cdots \prec x_r = y$ has the same length.*

Proof. See [18] □

However, the converse statement is false, see the example of figure 4.1 for a counterexample.

The next proposition shows that quite a lot of subspace arrangements with a geometric lattice exist.

Proposition 2.16. *If $\mathcal{A}$ is an arrangement of hyperplanes, then $L(\mathcal{A})$ is a geometric lattice.*

Proof. Let $x, y \in L(\mathcal{A})$. Since $\mathcal{A}$ is an arrangement of hyperplanes, the rank of x and y is respectively $\operatorname{codim}(x)$ and $\operatorname{codim}(y)$. Notice that $x + y \subseteq x \wedge y$. So, $\dim(x+y) \leq \dim(x \wedge y)$ and $\operatorname{rk}(x \wedge y) \leq \operatorname{codim}(x+y)$. We have :

$$\begin{aligned}\operatorname{rk}(x \wedge y) + \operatorname{rk}(x \vee y) &\leq \operatorname{codim}(x+y) + \operatorname{codim}(x \cap y)\\ &= \operatorname{codim}(x) + \operatorname{codim}(y)\\ &= \operatorname{rk}(x) + \operatorname{rk}(y),\end{aligned}$$

and the proof is complete. □

2.3 Independant subsets

For a given subspace arrangement $\mathcal{A}$, some subsets $\sigma \subseteq \mathcal{A}$ are independant in the sense of the following definition.

Definition 2.17. A subset $\sigma \subseteq \mathcal{A}$ is *independant* if for all $x \in \sigma$, we have $\vee(\sigma \setminus \{x\}) < \vee\sigma$.

Note that subsets of independant subsets are also independant. Those independant subsets will play a special role, especially when the lattice is geometric.

Proposition 2.18. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice $L(\mathcal{A})$ and $\sigma \subseteq \mathcal{A}$. Then we have*

1. $\operatorname{rk}(\vee\sigma) \leq |\sigma|$ *($|\sigma|$ design the number of elements of σ),*
2. $\operatorname{rk}(\vee\sigma) = |\sigma|$ *if and only if σ is independant.*

Proof. Let's start by proving the second assertion. Suppose $\sigma = \{x_1, \dots, x_k\}$ is independant. Then each subsequence of σ is independant and we have a sequence of inequalities

$$x_1 < x_1 \vee x_2 < x_1 \vee x_2 \vee x_3 < \cdots < \vee\sigma.$$

Therefore, $\operatorname{rk} \vee\sigma \geq k$. If $\operatorname{rk} \vee\sigma > k$, then this sequence is not maximal and there is another element $z \in \mathcal{A}$ such that for some p, we have

$$(x_1 \vee \cdots \vee x_p) < (x_1 \vee \cdots \vee x_p) \vee z < (x_1 \vee \cdots \vee x_p) \vee x_{p+1}.$$

In that case, we have $\mathrm{rk}(x_1 \vee \cdots \vee x_p \vee x_{p+1}) > \mathrm{rk}(x_1 \vee \cdots \vee x_p) + \mathrm{rk}(x_{p+1})$, which is a contradiction with the definition of a geometric lattice.

Suppose now that $\sigma = \{x_1, \ldots, x_k\}$ is such that $\mathrm{rk} \vee\sigma = |\sigma|$ and consider the sequence

$$x_{\tau_1} \leq x_{\tau_1} \vee x_{\tau_2} \leq \cdots \leq \vee\sigma,$$

obtained from σ by some permutation τ of the set $\{1, 2, \ldots, k\}$. If all inequalities are strict for any τ, then σ is independant. Otherwise, some inequality is in fact an equality and, since the rank of $\vee\sigma$ is k, another part of the sequence extends to the form

$$(x_{\tau_1} \vee \cdots \vee x_{\tau_r}) < (x_{\tau_1} \vee \cdots \vee x_{\tau_r}) \vee z < (x_{\tau_1} \vee \cdots \vee x_{\tau_r}) \vee x_{\tau_{r+1}}.$$

As above, the existence of such a sequence is impossible for a geometric lattice.

Now, let's prove the first assertion by induction on $|\sigma|$. If $|\sigma| = 1$, then $\mathrm{rk}(\vee\sigma) = 1 \leq |\sigma|$. Now suppose that it is true until $k - 1$ and let $\sigma = \{x_1, \ldots, x_k\} \subset \mathcal{A}$. If σ is independant, the first part of the proof shows that $\mathrm{rk}(\vee\sigma) = |\sigma|$. If σ is not independant, there is a $x_i \in \sigma$ such that $\vee(\sigma \setminus \{x_i\}) = \vee\sigma$. So, by induction, we have : $\mathrm{rk}(\vee\sigma) = \mathrm{rk}(\vee(\sigma \setminus \{x_i\})) \leq |\sigma \setminus \{x_i\}| \leq |\sigma|$ and the proof is complete. □

2.4 Fundamental group of the complement space

Some results of rational homotopy theory only apply to simply connected spaces. Therefore, it is convenient to have some insights about the fundamental group of the complement space of a subspace arrangement in general, and about the 1-connectedness in particular.

A good starting point in the study of the fundamental group of an arrangement of hyperplanes is the paper [22] by Hamm, H. and Lê Dũng T. Basically, they proved that the fundamental group of a complex arrangement of hyperplanes can be computed by looking at the intersection of $M(\mathcal{A})$ with a *generic* plane. So, the problem of computation of $\pi_1(M(\mathcal{A}))$ is reduced to the computation of the fundamental group of an affine arrangement of lines in $\mathbb{C}^2$.

A general method to obtain a presentation of the fundamental group in that case was found by Arvola. See the book [26] for a detailed explanation. We will use this presentation to prove a few results about the fundamental group. Note that it does not work if we consider real arrangements.

Theorem 2.19. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^m$, with $m \geq 3$. Let x be an hyperplane such that for every $x_i, x_j \in \mathcal{A}$ with $i \neq j$,* $\operatorname{codim}(x_i \cap x_j \cap x) = 3$. *Then $\pi_1(M(\mathcal{A} \cup \{x\}) = \pi_1(M(\mathcal{A})) \oplus \mathbb{Z}\alpha$ where α is a loop around x.*

Proof. Using the results of [22], we can choose a generic plane α and consider its intersection with $M(\mathcal{A} \cup \{x\})$ and $M(\mathcal{A})$. This reduces the problem to the following : $\widetilde{\mathcal{A}}$ is an arrangement of lines in $\mathbb{C}^2$, $\widetilde{x}$ is a line such that $\widetilde{x} \cap \widetilde{x_i} = \widetilde{x} \cap \widetilde{x_j} \neq \emptyset$ for all $\widetilde{x_i}, \widetilde{x_j} \in \widetilde{\mathcal{A}}$, $\widetilde{x_i} \neq \widetilde{x_j}$. We want to prove that $\pi_1(M(\widetilde{\mathcal{A}} \cup \widetilde{x})) = \pi_1(M(\widetilde{\mathcal{A}})) \oplus \mathbb{Z}\alpha$ where α is a loop around $\widetilde{x}$.

Let P be the set of non-empty intersections of the lines of $\widetilde{\mathcal{A}}$ (so, P is the set of multiple points). Let S be the set of non-empty intersections of $\widetilde{x}$ with lines of $\widetilde{\mathcal{A}}$. Clearly, S is a set of exactly n elements. Arvola's presentation of the fundamental group $\pi_1(M(\widetilde{\mathcal{A}}))$ is given by

$$\langle g_1, g_2, \ldots, g_n \mid \cup_{p \in P} R_p \rangle,$$

for some sets of relations R_p. Arvola's presentation of the fundamental group $\pi_1(M(\widetilde{\mathcal{A}} \cup \{\widetilde{x}\})$ is given by

$$\langle g, g_1, \ldots, g_n \mid \cup_{p \in P} \widetilde{R_p}, \cup_{s \in S} \widetilde{R_s} \rangle$$

for some other sets of relations $\widetilde{R_p}$ and $\widetilde{R_s}$. But the relations $\widetilde{R_s}$ are very simple : since only two lines intersect at a point $s \in S$, namely $\widetilde{x}$ and some $\widetilde{x_i}$, the corresponding relation R_s is just $gg_i = g_i g$. In other words, g commutes with every g_i. Also, since no point of P belong to the line $\widetilde{x}$, the relations $\widetilde{R_p}$ are the same as the relations R_p. Therefore,

$$\langle g, g_1, \ldots, g_n \mid \cup_{p \in P} \widetilde{R_p}, \cup_{s \in S} \widetilde{R_s} \rangle = \langle g_1, g_2, \ldots, g_n \mid \cup_{p \in P} R_p \rangle \oplus \mathbb{Z}g.$$

□

Now, using the previous theorem, we have an interesting result : the fundamental group is not influenced by subspaces of codimension

$$\begin{array}{ccc} M(\mathcal{A} \cup \{y_1, y_2\}) & \longrightarrow & M(\mathcal{A} \cup \{y_1\}) \\ \downarrow & & \downarrow \\ M(\mathcal{A} \cup \{y_2\}) & \longrightarrow & M(\mathcal{A} \cup \{y_1 \cap y_2\}) \end{array}$$

Figure 2.2: Inclusion maps when $k = 2$

greater than 2. It is not really surprising, since the real codimension is then at least 4. It might have been faster to prove that result using transversality arguments : when M is a manifold and $K \subset M$ is a polyhedron, then the injection $M \setminus K \to M$ induces an isomorphism of fundamental groups if the codimension of K is large enough.

Theorem 2.20. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^m$ and $\mathcal{B}$ a subspace arrangement in $\mathbb{C}^m$ such that for all $x \in \mathcal{B}$,* $\operatorname{codim} x \geq 2$. *Then*

$$\pi_1(M(\mathcal{A} \cup \mathcal{B})) = \pi_1(M(\mathcal{A})).$$

Proof. The proof is done by induction on the size of $\mathcal{B}$. Suppose that $\mathcal{B} = \{x_1\}$. We can assume that x_1 doesn't belong to $L(\mathcal{A})$ (otherwise, clearly $M(\mathcal{A} \cup \mathcal{B}) = M(\mathcal{A})$ and the theorem is true). Let $k := \operatorname{codim} x_1 \geq 2$ and let's choose k hyperplanes $y_1, \ldots, y_k$ such that $y_1 \cap \cdots \cap y_k = x_1$ and the y_i are generic in the sense of theorem 2.19. More precisely, we want the following condition : for all $u, v \in \mathcal{A} \cup \{y_1, \ldots, y_k\}$ and for all y_i, if $u \neq v, u \neq y_i$ and $v \neq y_i$, then $\operatorname{codim}(y_i \cap u \cap v) = 3$ (or $y_i \cap u \cap v = \emptyset$ if $\mathcal{A}$ is an arrangement in $\mathbb{C}^2$).

Such a choice is always possible : to construct y_1, choose k linearly independant vectors $a_1, \ldots, a_k$ in $M(\mathcal{A})$ and let $y_1 = x_1 \oplus \operatorname{span}\{a_1, \ldots, a_k\}$. The hyperplane y_2 is constructed in the same way by choosing k linearly independant vectors in $M(\mathcal{A} \cup \{y_1\})$, then y_3, and so on...

Clearly, we have $M(\mathcal{A} \cup \{x_1\}) = \cup_{i=1}^k M(\mathcal{A} \cup \{y_i\})$ and the intersections $M(\mathcal{A} \cup \{y_i\}) \cap M(\mathcal{A} \cup \{y_j\})$ and $M(\mathcal{A} \cup \{y_i\}) \cap M(\mathcal{A} \cup \{y_j\}) \cap M(\mathcal{A} \cup \{y_k\})$ are all path-connected, so we can apply Van Kampen's theorem. By Van Kampen's theorem (and theorem 2.19), we have

$$\begin{aligned}\pi_1(M(\mathcal{A}\cup\{x_1\})) &= *_{i=1}^k \pi_1(M(\mathcal{A}\cup\{y_i\}))/N \\ &= *_{i=1}^k (\pi_1^i(M(\mathcal{A}))\oplus\mathbb{Z}_i)/N\end{aligned}$$

where N identifies each copy $\pi_1^i(M(\mathcal{A}))$ with each $\pi_1^j(M(\mathcal{A}))$ and identifies $z_i \in \mathbb{Z}_i$ with the identity. Therefore, $\pi_1(M(\mathcal{A}\cup\{x_1\}) = \pi_1(M(\mathcal{A}))$. This complete the proof when $\mathcal{B}$ has exactly one subspace.

Suppose now that the result is true when $\mathcal{B} = \{x_1, x_2, \ldots, x_s\}$ and consider an additional subspace x_{s+1} such that $\operatorname{codim} x_{s+1} = k \geq 2$. The only thing that we need to prove is that $\pi_1(M(\mathcal{A}\cup\mathcal{B}\cup\{x_{s+1}\}) = \pi_1(M(\mathcal{A}))$. Like before, we can choose k hyperplanes $y_1, \ldots, y_k$ such that $\cap_{i=1}^k y_i = x_{s+1}$ and for all $u, v \in \mathcal{A}\cup\mathcal{B}\cup\{y_1, \ldots, y_k\}$ and for all y_i, if $u \neq v, u \neq y_i$ and $v \neq y_i$, then $\operatorname{codim}(y_i \cap u \cap v) = 3$. We have : $M(\mathcal{A}\cup\mathcal{B}\cup\{x_{s+1}\}) = \cup_{i=1}^k M(\mathcal{A}\cup\mathcal{B}\cup\{y_i\})$, and the intersections are all path-connected. By Van Kampen's theorem and the induction hypothesis,

$$\begin{aligned}\pi_1(M(\mathcal{A}\cup\mathcal{B}\cup\{x_{s+1}\})) &= *_{i=1}^k \pi_1(M(\mathcal{A}\cup\mathcal{B}\cup\{y_i\}))/N \\ &= *_{i=1}^k (\pi_1^i(M(\mathcal{A}\cup\{y_i\}))\oplus\mathbb{Z}_i)/N \\ &= *_{i=1}^k (\pi_1^i(M(\mathcal{A}))\oplus\mathbb{Z}_i)/N,\end{aligned}$$

where the relations given by N are exactly the same as above. Therefore, $\pi_1(M(\mathcal{A}\cup\mathcal{B}\cup\{x_{s+1}\})) = \pi_1(M(\mathcal{A}))$ and the proof is complete. □

The previous two theorems imply the next corollary, which is a simple yet interesting result.

Corollary 2.21. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$.*

- *If $\mathcal{A}$ is an arrangement of hyperplanes in general position, then $\pi_1(M(\mathcal{A})) = \mathbb{Z}^n$.*
- *If $\operatorname{codim} x_i \geq 2$ for all $i = 1, \ldots, n$, then $\pi_1(M(\mathcal{A})) = 0$.*

Proof. The first statement is a consequence of theorem 2.19. It is proven inductively by starting with an empty arrangement, then adding hyperplanes in general position. The second statement is a consequence of theorem 2.20, when the arrangement $\mathcal{A}$ is empty. □

CHAPTER 3

Rational models

The complement space $M(\mathcal{A})$ of subspace arrangements has been studied extensively over the past twenty years. Many topological invariants have been computed, using various techniques. In this chapter, we'll be mainly interested by the cohomology of $M(\mathcal{A})$.

An early result was obtained by ARNOL'D in 1969. He investigated the cohomology of the complex braid arrangement $\mathcal{A}_{n-1}^{\mathbb{C}}$ (it is the subspace arrangement in $\mathbb{C}^n$ composed of all hyperplanes of the form $z_i - z_j = 0$ for $1 \leq i \leq j \leq n$). In [1], he gave a presentation of the cohomology algebra of $M(\mathcal{A}_{n-1}^{\mathbb{C}})$ as a quotient of a free algebra by an ideal :

Theorem 3.1 (Arnol'd)**.** *Let $\mathcal{A}_{n-1}^{\mathbb{C}}$ be the complex braid arrangement. Then its cohomology algebra is*

$$H^*(M(\mathcal{A}_{n-1}^{\mathbb{C}}); \mathbb{Z}) \cong \Lambda V/J,$$

where $V = \oplus_{1 \leq k < l \leq n} \mathbb{Z} e_{k,l}$ is a vector space of dimension $n(n-1)/2$ with each $e_{k,l}$ in degree 1, and J is the ideal of relations generated by the elements $e_{k,m}e_{l,m} - e_{k,l}e_{l,m} + e_{k,l}e_{k,m}$ for $1 \leq k < l < m \leq n$.

Later, in 1980, hyperplane arrangements became more important with the work of ORLIK & SOLOMON. They used the intersection lattice of complex hyperplane arrangements to derive a presentation of the cohomology algebra, again as a quotient of a free algebra by an ideal (see [25]) :

Theorem 3.2 (Orlik, Solomon)**.** *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex arrangement of hyperplanes. Then its cohomology algebra is $H^*(M(\mathcal{A}); \mathbb{Z}) \cong \Lambda V/J$ where $V = \oplus_{1 \leq i \leq n} \mathbb{Z} e_i$ is a vector space of* *dimension n with each e_i in degree 1 and J is the ideal generated by*

$$\sum_{j=1}^{k}(-1)^j e_{i_1}\cdots\widehat{e_{i_j}}\cdots e_{i_k}$$

for each $\{i_1,\ldots,i_k\}_< \subseteq [n]$ that verify a technical condition, which depends on the lattice $L(\mathcal{A})$.

Recall that $\{i_1,\ldots,i_k\}_<$ is a notation already explained : it is the same as $\{i_1,\ldots,i_k\}$, but with the additional hypothesis that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

The results on cohomology algebras of complex subspace arrangements in general are far less complete. One of the first major results in that area was discovered by Goresky and MacPherson : they obtained a full description of the linear cohomology structure of $M(\mathcal{A})$, using stratified Morse theory. It turns out that the cohomology groups of $M(\mathcal{A})$ are completely determined by the combinatorial data of the arrangement.

Theorem 3.3 (Goresky-MacPherson)**.** *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$. Then*

$$\widetilde{H}^i(M(\mathcal{A})) \simeq \bigoplus_{x\in L(\mathcal{A})\setminus\{\mathbb{C}^m\}} \widetilde{H}_{2\operatorname{codim}(x)-2-i}(\Delta(0,x)),$$

where $\Delta(0,x)$ denotes the order complex[1] *of the interval $[\mathbb{C}^m, x]$ in $L(\mathcal{A})$.*

But this description doesn't give any information about the multiplication structure in $H^*(M(\mathcal{A}))$. In 1995, De Concini and Procesi gave a rational model for the cohomology algebra. However, the result is far from explicit. The De Concini-Procesi rational model has been considerably simplified by Yuzvinsky and Feichtner. This model, which we will note $D(\mathcal{A})$, is explained in the first section of this chapter. Note that it is only valid for central subspace arrangements. There is also a natural integral version of this model : $D_{\mathbb{Z}}(\mathcal{A})$, which will also be discussed in the first section. As we will see later, this model is particularly interesting in the case of subspace arrangements with a geometric lattice, because then, the space $M(\mathcal{A})$ is formal.

[1] see [5] for definitions of order complexes and their homology

The next and last section of this chapter is devoted to another rational model, $BP(\mathcal{A})$, specific to the coordinate subspace arrangements. This model, by Buchstaber and Panov, is a little more complicated to manipulate. It was obtained by studying the subject of toric topology. It was discovered that if $\mathcal{A}$ is a complex coordinate arrangement in $\mathbb{C}^m$, then $M(\mathcal{A})$ has the same homotopy type as the moment-angle complex $\mathcal{Z}_K$ of a simplicial complex K (see section 3.2 or [7] and [13] for more details). But there exists a space $DJ(K)$ such that $\mathcal{Z}_K$ is the homotopy fiber of a fibration

$$\mathcal{Z}_K \to DJ(K) \to (\mathbb{C}P^\infty)^m.$$

From this fibration, Buchstaber and Panov derived a rational model $BP(\mathcal{A})$. The interesting part is that $DJ(K)$ is always a formal space. This will be the key to obtain results for coordinate subspace arrangements which don't have a geometric lattice.

3.1 A simple model for central arrangements

Let $\mathcal{A}$ be a central arrangement of subspaces in $\mathbb{C}^m$. S. Yuzvinsky and E. Feichtner described a simple rational model for the topological space $M(\mathcal{A})$. This model is a graded differential algebra denoted by $D(\mathcal{A})$. See their paper [32] for additional details.

3.1.1 Description of $D(\mathcal{A})$

Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. A rational model for the space $M(\mathcal{A})$ is given as follows : choose a linear order on $\mathcal{A}$ (in this work, we will use $x_1 < x_2 < \cdots < x_n$).

- *Graded vector space $D(\mathcal{A})$:* it is the vector space over $\mathbb{Q}$ which has a basis given by all subsets $\sigma \subseteq \mathcal{A}$, i.e. $D(\mathcal{A}) = \oplus_{\sigma \subseteq \mathcal{A}} \mathbb{Q}[\sigma]$. A grading is given by $\deg(\sigma) = 2\,\mathrm{codim}_{\mathbb{C}} \vee\sigma - |\sigma|$

- *Multiplication structure :* the multiplication is defined, for $\sigma, \tau \subseteq \mathcal{A}$, by

$$\sigma \cdot \tau = \begin{cases} (-1)^{\operatorname{sgn}\epsilon(\sigma,\tau)} \sigma \cup \tau & \text{if } \vee\sigma + \vee\tau = \mathbb{C}^m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\epsilon(\sigma, \tau)$ is the permutation that, applied to $\sigma \cup \tau$ with the induced linear order, places elements of τ after elements of σ, both in the induced linear order. This multiplication turns $D(\mathcal{A})$ into an algebra. Note that the condition $\vee\sigma + \vee\tau = \mathbb{C}^m$ is equivalent to $\operatorname{codim} \vee\sigma + \operatorname{codim} \vee\tau = \operatorname{codim} \vee(\sigma \cup \tau)$.

- *Differential* $d\colon D(\mathcal{A})^* \to D(\mathcal{A})^{*+1}$. For $\sigma = \{x_{i_1}, \ldots, x_{i_r}\}_<$, let $J_\sigma = \{j \in [r] : x_{i_j} \in \sigma \text{ and } \vee(\sigma \setminus \{x_{i_j}\}) = \vee\sigma\}$. Then the differential is defined by the formula

$$d\sigma = \sum_{j \in J_\sigma} (-1)^j (\sigma \setminus \{x_{i_j}\})$$

where the indexing of the elements in σ follows the linear order imposed on $\mathcal{A}$. This differential turns $D(\mathcal{A})$ into a cochain complex.

Theorem 3.4. *If $\mathcal{A}$ is a central subspace arrangement in $\mathbb{C}^m$, then the cdga $D(\mathcal{A})$ is a rational model of $M(\mathcal{A})$ and $H^*(M(\mathcal{A}); \mathbb{Q}) \cong H^*(D(\mathcal{A}))$.*

Proof. A chain of quasi-isomorphisms between the model $D(\mathcal{A})$ and $A_{PL}(M(\mathcal{A})$ is described in [31] and [32]. $\square$

Remark 3.5. There is a natural integral version of $D(\mathcal{A})$. Let $D_\mathbb{Z}(\mathcal{A})$ be the $\mathbb{Z}$-module $D_\mathbb{Z}(\mathcal{A}) = \oplus_{\sigma \subseteq \mathcal{A}} \mathbb{Z}[\sigma]$. By using the exact same definitions, we can give to $D_\mathbb{Z}(\mathcal{A})$ a ring structure and a differential. In general, there is no proof that $H^*(D_\mathbb{Z}(\mathcal{A}))$ is equal to $H^*(M(\mathcal{A}); \mathbb{Z})$, however, it has been proven in the case of arrangements with a geometric lattice.

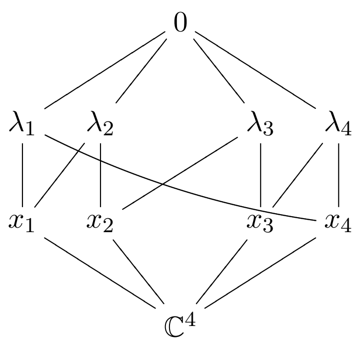
Remark 3.6. For each $X \in L(\mathcal{A})$, let $D_X(\mathcal{A})$ be the vector subspace generated by all the $\sigma \subseteq \mathcal{A}$ such that $\vee\sigma = X$. Note that for every $\sigma \in D_X(\mathcal{A})$, $d\sigma \in D_X(\mathcal{A})$ or $d\sigma = 0$. Therefore, the cochain complex (not algebra!) $D(\mathcal{A})$ decomposes as a direct sum $D(\mathcal{A}) = \oplus_{X \in L(\mathcal{A})} D_X(\mathcal{A})$. In (reduced) cohomology, it implies that

$$\widetilde{H}^*(M(\mathcal{A}); \mathbb{Q}) = \widetilde{H}^*(D(\mathcal{A})) = \bigoplus_{X \in L(\mathcal{A}) \setminus \{\mathbb{C}^m\}} H^*(D_X(\mathcal{A})).$$

This is an equality of groups, not of algebras. Remark that it is exactly a rational version of Goresky-MacPherson's theorem (see theorem 3.3).

Remark 3.7. If $\mathcal{A}$ is an arrangement with n hyperplanes, then a basis of $D(\mathcal{A})$ has $\#\mathcal{PA} = 2^n$ elements, which is the same as the dimension of ΛV if V is a n-dimensional vector space. Orlik and Solomon described the cohomology of $M(\mathcal{A})$ as a quotient $\Lambda V/J$, where the information is contained in the ideal J. In the model $D(\mathcal{A})$, the information is contained both in the multiplication and in the differential.

Example 3.8. Let's consider the arrangement $\mathcal{A} = \{x_1, x_2, x_3, x_4\}$ in $\mathbb{C}^4$ where x_1, x_2, x_3, x_4 are defined, respectively, by the equations $z_1 = z_2 = 0, z_2 = z_3 = 0, z_3 = z_4 = 0$ and $z_4 = z_1 = 0$.



Degree	Basis
0	$\{\}$
3	$\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$
4	$\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}, \{x_1, x_2, x_3, x_4\}$
5	$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}$
6	$\{x_1, x_3\}, \{x_2, x_4\}$

Figure 3.1: Lattice $L(\mathcal{A})$ and basis of $D(\mathcal{A})$

The arrangement $\mathcal{A} = \{x_1, x_2, x_3, x_4\}$ in $\mathbb{C}^4$ is a coordinate subspace arrangement with a non-geometric lattice. See figure 3.1 for a representation of the lattice $L(\mathcal{A})$ and a basis of $D(\mathcal{A})$.

To simplify notations, let's denote $\{x_i\}$ by X_i, $\{x_i, x_j\}$ by X_{ij}, $\{x_i, x_j, x_k\}$ by X_{ijk} and $\{x_1, x_2, x_3, x_4\}$ by X_{1234}. A simple calculation shows that the differential is always zero except for the following elements :

$$\begin{aligned} dX_{1234} &= -X_{234} + X_{134} - X_{124} + X_{123} \\ dX_{123} &= -dX_{134} = X_{13} \\ dX_{234} &= -dX_{124} = X_{24}. \end{aligned}$$

From this, it is easy to see that $H^*(D(\mathcal{A}))$ is concentrated in degree $0, 3, 4, 5$ with

$$\begin{aligned} \dim H^0(D(\mathcal{A})) &= 1, & \dim H^3(D(\mathcal{A})) &= 4, \\ \dim H^4(D(\mathcal{A})) &= 4, & \dim H^5(D(\mathcal{A})) &= 1. \end{aligned}$$

In $D(\mathcal{A})$, the only non trivial products are $X_1 \cdot X_3 = X_{13}$ and $X_2 \cdot X_4 = X_{24}$, however these products vanish in cohomology.

Finally, the next result is a simple consequence of the definition, and will be useful later in this text.

Proposition 3.9. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$ and $\sigma, \tau \subseteq \mathcal{A}$. If $\sigma \cap \tau \neq \emptyset$, then their product vanishes in $D(\mathcal{A})$: $\sigma \cdot \tau = 0$.*

Proof. The condition $\operatorname{codim} \vee\sigma + \operatorname{codim} \vee\tau = \operatorname{codim} \vee(\sigma \cup \tau)$ can be rewritten as $\dim(\vee\sigma + \vee\tau) = m$, or in other words, $\vee\sigma + \vee\tau = \mathbb{C}^m$. But if $x \in \sigma \cap \tau$, then $\vee\sigma + \vee\tau \subseteq x \neq \mathbb{C}^m$. Therefore the product $\sigma \cdot \tau$ is zero. □

3.1.2 Arrangements with a geometric lattice

The rational model $D(\mathcal{A})$ is very nice, because it contains all the rational homotopy type of $M(\mathcal{A})$, and at the same time, is simple to manipulate.

But it's even nicer when the space $M(\mathcal{A})$ is formal, because in that case, the model $D(\mathcal{A})$ can be deduced from the rational cohomology of $M(\mathcal{A})$. Luckily, we are aware of a large class of arrangements such that their complement space is formal : the ones with a geometric lattice. This is not a trivial result and the rational model $D(\mathcal{A})$ was actually introduced by E. Feichtner and S. Yuzvinsky in [32] in order to prove it.

Proposition 3.10. *If $\mathcal{A}$ is a central subspace arrangement such that the lattice $L(\mathcal{A})$ is geometric, then the space $M(\mathcal{A})$ is formal.*

Proof. See [32]. □

In other words, the algebra $H^*(D(\mathcal{A}))$ contains all the rational homotopy informations about $M(\mathcal{A})$. Also noteworthy is the fact is that a geometric lattice give us some control on how the lattice behave, which in practice means that many proofs use this control. This is the reason why many results have been shown only for arrangements with a geometric lattice. For example, let's state a simple, yet useful result about the cohomology $H^*(M(\mathcal{A});\mathbb{Q})$.

Proposition 3.11. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^m$ with a geometric lattice. Then $H^*(D(\mathcal{A}))$ and $H^*(D_{\mathbb{Z}}(\mathcal{A}))$ are generated by the classes $[\sigma]$, with $\sigma \subseteq \mathcal{A}$ independant.*

Proof. See [32]. □

Also, this is not true for arrangements with a non-geometric lattice. The arrangement of example 3.8 is a counter-example. The cohomology of $D(\mathcal{A})$ is not generated by independant elements : there are no independant elements in degree 5, but we have one cohomology class : $[X_{123} + X_{134}] = [X_{234} + X_{124}]$.

As mentioned in remark 3.5, another interesting property of arrangements with a geometric lattice is that their cohomology is described by $D_{\mathbb{Z}}(\mathcal{A})$.

Proposition 3.12. *Let $\mathcal{A}$ be a central subspace arrangement in $\mathbb{C}^m$ with a geometric lattice. Then $H^*(M(\mathcal{A});\mathbb{Z}) = H^*(D_{\mathbb{Z}}(\mathcal{A}))$.*

Proof. See [14]. □

3.1.3 Restricted arrangements

For a given subspace arrangement $\mathcal{A}$, we introduced in the definitions 2.5 and 2.6 the notions of deleted and restricted arrangements ($\mathcal{A}'$ and $\mathcal{A}''$ respectively). The definition of restricted arrangement was not straightforward because we wanted to avoid having two subspaces y_1, y_2 in $\mathcal{A}''$ such that $y_1 \subseteq y_2$.

More precisely, let $x \in \mathcal{A}$ and $\widetilde{\mathcal{A}}'' = \{x \cap y : y \in \mathcal{A}'\}$. In chapter 2, we defined the restricted arrangement $\mathcal{A}'' = \widetilde{\mathcal{A}}'' \setminus S$ for some set S, such that $M(\mathcal{A}'') = M(\widetilde{\mathcal{A}}'')$. In this section, we will show that even if there is y_1, y_2 in $\widetilde{\mathcal{A}}''$ with $y_1 \subset y_2$, then the cdga $D(\widetilde{\mathcal{A}}'')$ is quasi-isomorphic to $D(\mathcal{A}'')$.

Lemma 3.13. *Suppose $\mathcal{B}$ is a subspace arrangement with two subspaces u and v such that $u \subset v$. Let $\mathcal{B}' = \mathcal{B} \setminus \{u\}$. Then the inclusion $D(\mathcal{B}') \hookrightarrow D(\mathcal{B})$ is a quasi-isomorphism.*

Proof. The injective map $D(\mathcal{B}') \to D(\mathcal{B})$ gives us the following short exact sequence of cochain complexes :

$$0 \to D(\mathcal{B}') \to D(\mathcal{B}) \to \frac{D(\mathcal{B})}{D(\mathcal{B}')} \to 0.$$

Since there is a long exact sequence in cohomology, it is sufficient to prove the following statement : $H^*(D(\mathcal{B})/D(\mathcal{B}')) = 0$. The cochain complex $D(\mathcal{B})/D(\mathcal{B}')$ is a vector space generated by all the elements of the form

- $\sigma = \{u, x_{i_1}, \ldots, x_{i_r}\}$ with $x_{i_j} \notin \{u, v\}$. The differential $d\sigma$ is a sum of some $\pm\{u, x_{i_1}, \ldots, \widehat{x_{i_j}}, \ldots, x_{i_r}\}$.
- $\sigma = \{u, v, x_{i_1}, \ldots, x_{i_r}\}$ with $x_{i_j} \notin \{u, v\}$. The differential $d\sigma$ is a sum of some $\{u, x_{i_1}, \ldots, x_{i_r}\}$ and $\pm\{u, v, \ldots, \widehat{x_{i_j}}, \ldots, x_{i_r}\}$.

Let us introduce a new grading on $D(\mathcal{B})/D(\mathcal{B}')$. We say that

$$\deg\{u, x_{i_1}, \ldots, x_{i_r}\} = r \text{ and } \deg\{u, v, x_{i_1}, \ldots, x_{i_r}\} = r + 1.$$

The differential decreases this degree by one, and we have a chain complex.

Let us introduce a filtration on $D(\mathcal{B})/D(\mathcal{B}')$: F_0 is the subcomplex generated by $\{u\}$ and $\{u, v\}$, and in general, F_r is generated by all the $\{u, x_{i_1}, \ldots, x_{i_s}\}$ and $\{u, v, x_{i_1}, \ldots, x_{i_s}\}$ with $s \leq r$. So, we have an increasing filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r \subseteq \cdots$ Let's define the filtration degree d_0 of σ by the number of x_{i_j} in σ, and the complementary degree is 0 if $v \notin \sigma$ and 1 if $v \in \sigma$. This gives us a spectral sequence. The page E^0 is :

- $E^0_{p,0}$ is a vector subspace generated by all the $\{u, x_{i_1}, \ldots, x_{i_p}\}$.
- $E^0_{p,1}$ is generated by all the $\{u, v, x_{i_1}, \ldots, x_{i_p}\}$.
- $E^0_{p,q}$ is zero if $q \geq 2$.

The differential $d_0 \colon E^0_{p,q} \to E^0_{p,q-1}$ is zero for every $q \neq 1$. For $q = 1$, $d_0 \colon E^0_{p,1} \to E^0_{p,0}$ maps $\{u, v, x_{i_1}, \ldots, x_{i_p}\}$ to $\{u, x_{i_1}, \ldots, x_{i_p}\}$. So, it is obvious that every spaces $E^1_{p,q}$ of the next page is simply zero. Therefore, there is no homology in $D(\mathcal{B})/D(\mathcal{B}')$, which completes the proof. $\square$

From this lemma, it is clear that $D(\mathcal{A}'')$ and $D(\widetilde{\mathcal{A}}'')$ are connected by a quasi-isomorphism.

Corollary 3.14. *Let $\mathcal{A}$ be a subspace arrangement. Then the inclusion $D(\mathcal{A}'') \hookrightarrow D(\widetilde{\mathcal{A}}'')$ is a quasi-isomorphism.*

Proof. It is a consequence of definition 2.6 and lemma 3.13. □

3.2 Rational model of coordinate subspace arrangements

In this section, we will describe another rational model $BP(\mathcal{A})$, which only works in the case of coordinate subspace arrangements. The results stated in this section come mainly from the book [7] by Buchstaber and Panov.

Let S_m be the set of simplicial complexes on $\{1, \ldots, m\}$ and C_m be the set of complements of complex coordinate subspace arrangement in $\mathbb{C}^m$. Let us prove that there is a bijection between these two sets. A map $S_m \to C_m$ is defined by sending a simplicial complex K to $M(K) = \mathbb{C}^m \setminus \cup_{\sigma \notin K} x_\sigma$ (recall that x_σ denotes the coordinate subspace associated to σ, see definition 2.4).

Proposition 3.15. *The map $S_m \to C_m; K \mapsto M(K)$ is an order-preserving bijection (for the order defined by the inclusion).*

Proof. Obviously, if $K \subset K'$, then $M(K) \subset M(K')$. A simple computation shows that the inverse map $C_m \to S_m$ is defined by sending $M(\mathcal{A}) \in C_m$ to the simplicial complex $K(\mathcal{A})$, where $K(\mathcal{A})$ is defined by

$$K(\mathcal{A}) = \{\sigma \subset [m] : x_\sigma \not\subset |\mathcal{A}|\}.$$

where $|\mathcal{A}| = \cup_{x \in \mathcal{A}} x$ is the support of $\mathcal{A}$, and $[m] = \{1, \ldots, m\}$. □

Notice that if a coordinate subspace arrangement $\mathcal{A}$ contains an hyperplane, then the complement space $M(\mathcal{A})$ decomposes as $M(\mathcal{A}') \times \mathbb{C}^*$. Also, $\mathcal{A}$ contain the hyperplane $z_i = 0$ if and only if $\{i\}$ is not a vertex of $K(\mathcal{A})$. So, for the rest of this section, we will consider only coordinate subspace arrangements without hyperplanes and simplicial complexes on the vertex set $\{1, \ldots, m\}$.

For each simplicial complex K on $[m]$, Davis and Januszkiewicz introduced a T^m-space $\mathcal{Z}_K$, where T^m is the group $(S^1)^m$. This construction is in the center of attention of toric topologists.

Definition 3.16. For each $\sigma \subseteq [m]$, let $D^\sigma = \{(z_1, \dots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin \sigma\}$ where D^2 is the unit disk in $\mathbb{C}$. The *moment-angle complex* $\mathcal{Z}_K$ associated to an simplicial complex K is

$$\mathcal{Z}_K = \cup_{\sigma \in K} D^\sigma.$$

This space is particularly interesting for us because, for a given coordinate arrangement $\mathcal{A}$, the moment-angle complex $\mathcal{Z}_K$ of $K = K(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$.

Proposition 3.17. *There exists a deformation retraction $M(\mathcal{A}) \to \mathcal{Z}_K$.*

Proof. This is the theorem 8.9 in [7]. □

The group T^m acts on each D^σ and on $(D^2)^m$. So, we can consider the associated Borel fibration

$$\mathcal{Z}_K \to \mathcal{Z}_K \times_{(S^1)^m} (ES^1)^m \to (BS^1)^m = (\mathbb{C}P^\infty)^m.$$

Now, let us define an important space : the Davis-Januszkiewicz space associated to a simplicial complex K. To each simplex $\sigma = \{i_1, \dots, i_r\}$ in K, we associate the product $B_\sigma = (\mathbb{C}P^\infty)^r$ viewed as a subspace of $(\mathbb{C}P^\infty)^m$ by the injection defined by $(z_1, \dots, z_r) \mapsto (u_1, \dots, u_m)$ with $u_{i_j} = z_j$ and $u_s = *$ if $s \notin \sigma$.

Definition 3.18. The *Davis-Januszkiewicz space* associated to a simplicial complex K is $DJ(K) = \cup_{\sigma \in K} B_\sigma \subseteq (\mathbb{C}P^\infty)^m$.

Buchstaber and Panov proved that space $DJ(K)$ possesses the following useful properties.

Proposition 3.19. *Let $\mathcal{A}$ be a coordinate subspace arrangement without hyperplanes and $K = K(\mathcal{A})$ be its associated simplicial complex. Then the Davis-Januszkiewicz space of K is such that*

1. *$DJ(K) \simeq \mathcal{Z}_K \times_{(S^1)^m} (ES^1)^m$,*
2. *$DJ(K)$ is a formal space,*
3. *$H^*(DJ(K); \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_m]/I_K$ where $|x_i| = 2$ and I_K is the ideal generated by all the square-free monomials $x_{i_1} \cdots x_{i_s}$ where $\sigma = \{i_1, \dots, i_s\}$ is such that $x_\sigma \in \mathcal{A}$. (this is the Stanley-Reisner ring associated to K).*

Proof. See [7] and [24]. □

So, $M(\mathcal{A})$ has the same homotopy type as $\mathcal{Z}_K$, we have an homotopy fibration $\mathcal{Z}_K \to DJ(K) \to (\mathbb{C}P^\infty)^m$ and we know rational models for $DJ(K)$ and $(\mathbb{C}P^\infty)^m$. From all these results, we obtain a rational model of $M(\mathcal{A})$.

Theorem 3.20. *If $\mathcal{A}$ is a coordinate subspace arrangement without hyperplanes, then a rational model of $M(\mathcal{A})$ is given by*

$$BP(\mathcal{A}) = (\mathbb{Q}[x_1, \dots, x_m]/I_K \otimes \Lambda(u_1, \dots, u_m), d)$$

with $dx_i = 0, du_i = x_i$, $|x_i| = 2$ and $|u_i| = 1$.

Proof. See [7]. □

Remark 3.21. Actually, Buchstaber and Panov proved a slightly different result : $H^*(M(\mathcal{A}); \mathbb{Z}) = H^*(BP(\mathcal{A}), d)$.

Remark 3.22. In this model, it is easy to see that the differential map preserves the indexes : $d\left(\prod_{i\in I} x_i \prod_{j\in J} u_j\right)$ is a sum of monomials $\prod_{i\in I'} x_i \prod_{j\in J'} u_j$ with $I \cup J = I' \cup J'$. Therefore, $BP(\mathcal{A})$ naturally splits as a direct sum

$$BP(\mathcal{A}) = \bigoplus_{\sigma \subseteq [m]} BP_\sigma(\mathcal{A})$$

where $BP_\sigma(\mathcal{A})$ is the subcomplex generated by the equivalence classes of the monomials $\left(\prod_{i\in I} x_i \prod_{j\in J} u_j\right)$ such that $I \cup J = \sigma$. This decomposition is very similar to the Goresky-MacPherson theorem (theorem 3.3).

Example 3.23. From the definitions, the model $BP(\mathcal{A})$ might not look easy to manipulate, but it's not difficult either. For example, let's consider the arrangement $\mathcal{B} = \{x_1, x_2, x_3, x_4, x_5\}$ in $\mathbb{C}^6$ where $x_i = \{z \in \mathbb{C}^6 : z_i = z_{i+1} = 0\}$. The simplicial complex K associated to $\mathcal{A}$ is

$$K = \{\sigma \subseteq [6] : \{i, i+1\} \not\subseteq \sigma \text{ for } i = 1, \dots, 5\}.$$

So, K has 6 vertexes, 10 edges and 4 triangles, as depicted in figure 3.2.

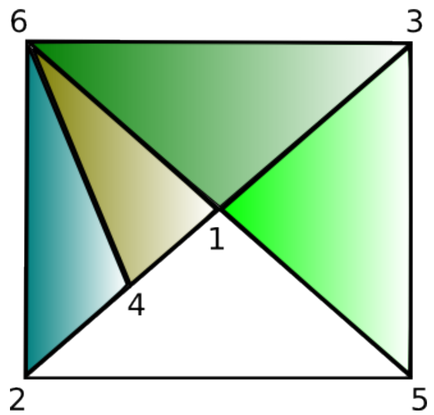


Figure 3.2: Geometric realization of K

The ideal I_K is generated by the monomials x_1x_2, x_2x_3, x_3x_4, x_4x_5 and x_5x_6. So, a rational model of $M(\mathcal{B})$ is $(\mathbb{Q}[x_1, \dots, x_6]/I_K \otimes \Lambda(u_1, \dots, u_6), d)$ with $dx_i = 0$ and $du_i = x_i$. This arrangement is the simplest arrangement such that the space $M(\mathcal{A})$ is not formal (see [13] for more details).

Notice that if $\mathcal{A}$ is a coordinate subspace arrangement in $\mathbb{C}^m$ without hyperplanes, then the space $M(\mathcal{A})$ is simply connected (see corollary 2.21). So it makes sense to ask if $M(\mathcal{A})$ is elliptic or hyperbolic. The long exact sequence of the homotopy fibration

$$\mathcal{Z}_K \to DJ(K) \to (\mathbb{C}P^\infty)^m.$$

defined above gives a short proof of the following lemma.

Lemma 3.24. *The space $\mathcal{Z}_K$ is rationally hyperbolic if and only if $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} = \infty$, and it is rationally elliptic if and only if $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} < \infty$.*

CHAPTER 4

Deleted and restricted arrangements

4.1 Introduction

As mentioned in chapter 2, the notions of deleted and restricted arrangements (definitions 2.5 and 2.6) have been proven useful in the study of arrangements of hyperplanes. Let's illustrate this with a simple problem, that we already mentioned in the introduction : the cheese cutting problem.

To solve this problem, let $f\colon \mathbb{N} \times \mathbb{N}_0 \to \mathbb{N}$ be the function that maps (n, m) to the largest number of connected components of a real arrangement of n hyperplanes in $\mathbb{R}^m$. Clearly, $f(0, m) = 1, f(1, m) = 2$. It is not hard to prove that the function f satisfies the following recurrence formula :

$$f(n+1, m) = f(n, m) + f(n, m-1).$$

This formula can be used to give a complete solution to the problem[1]. What is interesting is what each term represents. Suppose that $\mathcal{A}$ is an arrangement with $n+1$ hyperplanes in $\mathbb{R}^m$, then $f(n+1, m)$ is the answer to the problem for $\mathcal{A}$. The numbers $f(n, m)$ and $f(n, m-1)$ are respectively the solution for the same problem, but for the arrangements $\mathcal{A}'$ and $\mathcal{A}''$.

This is a good example of problem where thinking in terms of deleted and restricted arrangement gives a nice way to obtain a solution.

In this chapter, we will study the relationships between the complement spaces $M(\mathcal{A}), M(\mathcal{A}')$ and $M(\mathcal{A}'')$ from the point of view of

[1] The general solution to the problem is : $f(n, m) = 1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{m}$.

their rational models. Since many maps in this chapter are only morphisms of vector spaces (or of $\mathbb{Z}$-modules if we work with $D_{\mathbb{Z}}(\mathcal{A})$), and not of algebras, most of the results are about the additive structure of $H^*(M(\mathcal{A});\mathbb{Q})$.

The additive structure in rational cohomology is described by the ranks of the cohomology groups, or, in a condensed way, by the generating function of the Betti numbers.

Definition 4.1. Let X be a topological space. The *Poincaré polynomial* of X is the series

$$\mathrm{Poin}(X,t) = \sum_{i=0}^{\infty} b_i(X)t^i.$$

In this chapter, we will present three different results. The next section (section 4.2) is essentially a proof that there is a long exact sequence in cohomology :

$$\to H^q(M(\mathcal{A}');\mathbb{Q}) \to H^q(M(\mathcal{A});\mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}'');\mathbb{Q}) \to \cdots$$

An easy consequence of this sequence is the following : if $\mathcal{A}$ is non-empty, then the Euler characteristic of $M(\mathcal{A})$ is zero (see corollary 4.8). Sometimes, this long exact sequence splits into short exact sequences and we obtain the following formula, which connects the Poincaré polynomials of $M(\mathcal{A})$, $M(\mathcal{A}')$ and $M(\mathcal{A}'')$:

$$\mathrm{Poin}(M(\mathcal{A}),t) = \mathrm{Poin}(M(\mathcal{A}'),t) + t^{\deg(x_0)}\,\mathrm{Poin}(M(\mathcal{A}''),t).$$

This formula is studied in section 4.3, where two conditions for the splitting to occur are proved. One of them is the following : if $L(\mathcal{A})$ is geometric, then the recurrence formula above holds. Notice that this is always true for arrangements of hyperplanes, but this particular case was already well known (see theorem 2.56 of [26]).

Finally, in section 4.4, we examine what happens when both conditions mentioned above happen simultaneously. In that case, a stronger result can be proved : there exists an isomorphism

$$H^*(M(\mathcal{A}');\mathbb{Q}) \to H^*(M(\mathcal{A}'');\mathbb{Q})$$

of $\mathbb{Z}_2$-graded vector spaces. This gives us some information about the way Betti numbers are distributed.

4.2 A long exact sequence in cohomology

4.2.1 Proof of existence

Let's explain briefly the proof of the existence of the long exact sequence in cohomology. Consider the inclusion map $j\colon D(\mathcal{A}') \to D(\mathcal{A})$. This map is a morphism of cochain complexes and induces a short exact sequence of cochain complexes

$$0 \to D(\mathcal{A}') \xrightarrow{j} D(\mathcal{A}) \to \frac{D(\mathcal{A})}{D(\mathcal{A}')} \to 0.$$

This short exact sequence of cochain complexes gives us a long exact sequence in cohomology connecting $H^*(D(\mathcal{A}'))$, $H^*(D(\mathcal{A}))$ and $H^*(D(\mathcal{A})/D(\mathcal{A}'))$. The trick is to prove that $D(\mathcal{A})/D(\mathcal{A}')$ and $D(\mathcal{A}'')$ are connected by a sequence of quasi-isomorphisms. We already know (by corollary 3.14) that the inclusion $D(\mathcal{A}'') \hookrightarrow D(\widetilde{\mathcal{A}}'')$ is a quasi-isomorphism. In this section, we prove the proposition 4.6, which asserts that there exists a quasi-isomorphism

$$\frac{D(\mathcal{A})}{D(\mathcal{A}')} \xrightarrow{\simeq} D(\widetilde{\mathcal{A}}'').$$

Together, these results give us the desired long exact sequence.

Let $\mathcal{A} = \{x_0, x_1, \dots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. Consider the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$ and the arrangement $\widetilde{\mathcal{A}}'' = \{x_0 \cap y : y \in \mathcal{A}'\}$. We will show in this subsection that there exists a quasi-isomorphism $D(\mathcal{A})/D(\mathcal{A}') \to D(\widetilde{\mathcal{A}}'')$.

Let us define an equivalence relation in $\mathcal{A}'$: $x_i \sim x_j$ if and only if $x_0 \cap x_i = x_0 \cap x_j$. This equivalence relation partitions the set $\mathcal{A}'$ into subsets $\mathcal{A}' = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_q$. Let $\mathcal{A}_0 = \{x_0\}$. As mentioned in section 3.1, we can choose any linear order on $\mathcal{A}$. In order to get the signs right, we choose in this section a linear order such that if $i < j$, $y \in \mathcal{A}_i$ and $z \in \mathcal{A}_j$, then $y < z$. Also, the following set will be useful :

$$E = \{(y, z) \in \mathcal{A}' \times \mathcal{A}' : y \sim z \text{ and } y \neq z\}.$$

Before constructing a map $D(\mathcal{A})/D(\mathcal{A}') \to D(\widetilde{\mathcal{A}}'')$, we need to construct a map $\varphi\colon D(\mathcal{A}) \to D(\widetilde{\mathcal{A}}'')$ with good properties. Let $\sigma \subseteq \mathcal{A}$.

- If $x_0 \notin \sigma$ or if $x_0 \in \sigma$ and there exists $(y, z) \in E$ such that $\{y, z\} \in \sigma$, then we let $\varphi(\sigma) = 0$.

- Otherwise, $\sigma = \{x_0, x_{i_1}, \dots, x_{i_r}\} \subseteq \mathcal{A}$ is sent to $(-1)^r\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \subseteq \widetilde{\mathcal{A}}''$ (in particular, $\varphi(\{x_0\}) = 0$).

This map is not multiplicative but it commutes with the differential.

Lemma 4.2. *The map $\varphi\colon D(\mathcal{A}) \to D(\widetilde{\mathcal{A}}'')$ defined above is such that $\varphi d = d\varphi$.*

Proof. Let $\sigma \subseteq \mathcal{A}$. If $x_0 \notin \sigma$, then we have $d\varphi(\sigma) = 0 = \varphi(d\sigma)$. If $x_0 \in \sigma$ and there exists $(y, z) \in E$ with $\{y, z\} \in \sigma$, then $d\varphi(\sigma) = 0$ and

$$\begin{aligned} d\sigma = d\{x_0, y, z, x_{i_1}, \dots, x_{i_r}\} &= \alpha\left(\sigma \setminus \{x_0\}\right) + \{x_0, z, x_{i_1}, \dots, x_{i_r}\} \\ &\quad - \{x_0, y, x_{i_1}, \dots, x_{i_r}\} + \sum_j \alpha_j \left(\sigma \setminus \{x_{i_j}\}\right), \end{aligned}$$

where α and the α_j can be 0 or ± 1. So, by definition of φ, we have : $\varphi(d\sigma) = 0$.

Finally, the last case we need to check is if $x_0 \in \sigma$ and no $(y, z) \in E$ is such that $\{y, z\} \subset \sigma$. In that case, let $\sigma = \{x_0, x_{i_1}, \dots, x_{i_r}\}$ and denote by J_σ the set $J = \{j : \vee(\sigma \setminus \{x_{i_j}\}) = \vee\sigma\}$. We have

$$\begin{aligned} \varphi(d\sigma) &= \varphi\left(\alpha\left(\sigma \setminus \{x_0\}\right) + \sum_{j \in J_\sigma} (-1)^{j+1} \sigma \setminus \{x_{i_j}\}\right) \\ &= (-1)^{r-1} \sum_{j \in J_\sigma} (-1)^{j+1} \{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}, \\ d\varphi(\sigma) &= (-1)^r d\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \\ &= (-1)^r \sum_{j \in J_\sigma} (-1)^j \{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}. \end{aligned}$$

So, the map φ commutes with the differential d. □

The map φ is clearly surjective and is such that $\varphi j = 0$, where j is the injection $D(\mathcal{A}') \hookrightarrow D(\mathcal{A})$. So it induces a surjective map

$$\overline{\varphi}\colon \frac{D(\mathcal{A})}{D(\mathcal{A}')} \to D(\widetilde{\mathcal{A}}'')$$

We will show that this map induces an isomorphism in cohomology. If the set E is empty, then $\overline{\varphi}$ is injective and is an isomorphism. Otherwise, for every $(u, v) \in E$, let $I_{u,v}$ be the vector subspace of $D(\mathcal{A})/D(\mathcal{A}')$ generated by each elements of the form $\{x_0, u, y_{j_1}, \dots y_{j_r}\} - \{x_0, v, y_{j_1}, \dots, y_{j_r}\}$ and $\{x_0, u, v, y_{j_1}, \dots y_{j_r}\}$ such that $y_i \in \mathcal{A} \setminus \{x_0, u, v\}$. Let $V = \sum_{(u,v)\in E} I_{u,v}$. The following three lemmas prove that V is acyclic.

Lemma 4.3. *In $D(A)/D(A')$, we have $V = \ker \overline{\varphi}$.*

Proof. It is clear that $V \subseteq \ker \overline{\varphi}$. Let $u = \sum_{s\in I} \alpha_s \sigma_s \in \ker \overline{\varphi}$, with $\alpha_s \neq 0 \; \forall s \in I$ and $\sigma_s \neq \sigma_t$ if $s \neq t$. To prove that $u \in V$, we show that we can substract elements of V from u until we obtain zero.

- If there exists a $s \in I$ such that $\sigma_s = \{x_0, u, v, y_{j_1}, \dots, y_{j_r}\}$ with $x_0 \cap u = x_0 \cap v$ and $u \neq v$, then $\alpha_s \sigma_s \in V$ and we can substract it from u. Let's repeat this operation until we obtain a sum $\sum_{s\in I'} \alpha_s \sigma_s$ without any such σ_s.

- Let $t \in I'$. Then the element σ_t is necessarily of the form $\{x_0, x_{i_1}, \dots, x_{i_r}\}_<$ with $x_0 \cap x_{i_j} \neq x_0 \cap x_{i_k}$ for $j \neq k$. We have

$$0 = \overline{\varphi}\left(\sum_{s\in I'} \alpha_s \sigma_s\right) = (-1)^r \alpha_t \{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} + \sum_{s\in I'\setminus\{t\}} \alpha_s \overline{\varphi}(\sigma_s).$$

So, there exists a $s \in I' \setminus \{t\}$ such that $\overline{\varphi}(\sigma_s) = \overline{\varphi}(\sigma_t)$. This is possible only if $\sigma_s = \{x_0, y_1, \dots, y_r\}$ with $x_0 \cap x_{i_j} = x_0 \cap y_j$. In that case, the difference $\sigma_s - \sigma_t$ can be rewritten as

$$\sum_{j=1}^{r} \left(\{x_0, y_1, \dots, y_{j-1}, y_j, x_{i_{j+1}}, \dots, x_{i_r}\} - \{x_0, y_1, \dots, y_{j-1}, x_{i_j}, x_{i_{j+1}}, \dots, x_{i_r}\}\right),$$

which is clearly in V. We can substract $\alpha_t(\sigma_t - \sigma_s)$ from the sum and we get another sum $\sum_{s\in I''} \alpha_s \sigma_s$ such that $|I''| < |I'|$. This process can be repeated until we obtain an empty sum.

Therefore, $u \in V$ and $V = \ker \overline{\varphi}$. □

Lemma 4.4. *For every $(u, v) \in E$, $d(I_{u,v}) \subseteq I_{u,v}$, so $I_{u,v}$ is a subcomplex of $D(\mathcal{A})/D(\mathcal{A}')$. Also, the complex $I_{u,v}$ is acyclic.*

Proof. Let's consider a cocycle

$$\begin{aligned}\alpha = \sum_i \alpha_i \left(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} - \{x_0, v, y_{i_1}, \ldots, y_{i_r}\}\right)\\ + \sum_j \beta_j \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\},\end{aligned}$$

with the $y_{i_k}, z_{j_k} \notin \{x_0, u, v\}$. We want to show that α is a coboundary. Let

$$\begin{aligned}K_i &= \{k : \vee\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} = \vee(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\})\},\\ L_j &= \{k : \vee\{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} =\\ &\qquad \vee\left(\{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} \setminus \{z_{j_k}\}\right)\}.\end{aligned}$$

Since α is a cocycle, its differential is zero and we have

$$\begin{aligned}d\alpha = &\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \big(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\}\\ &\qquad\qquad - \{x_0, v, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\}\big)\\ &+ \sum_j \beta_j \big(\{x_0, u, z_{j_1}, \ldots, z_{j_{r-1}}\} - \{x_0, v, z_{j_1}, \ldots, z_{j_{r-1}}\}\big)\\ &+ \sum_j \beta_j \sum_{k \in L_j} (-1)^{k+1} \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} \setminus \{z_{j_k}\} = 0.\end{aligned}$$

Looking at the terms without a v and using the relation $d\alpha = 0$, we have

$$\begin{aligned}\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\}\\ + \sum_j \beta_j \{x_0, u, z_{j_1}, \ldots, z_{j_{r-1}}\} = 0.\end{aligned}$$

But we can add a v into each terms of the previous relation, without making it false. This gives

$$\begin{aligned}-\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \{x_0, u, v, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\}\\ = \sum_j \beta_j \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\}.\end{aligned}$$

Finally, let $\omega = -\sum_i \alpha_i\{x_0, u, v, y_{i_1}, \dots, y_{i_r}\}$. Its differential is

$$d\omega = \sum_i \alpha_i(\{x_0, u, y_{i_1}, \dots, y_{i_r}\} - \{x_0, v, y_{i_1}, \dots, y_{i_r}\}) - \sum_i \alpha_i \sum_{k\in K_i} (-1)^k \{x_0, u, v, y_{i_1}, \dots, y_{i_r}\} \setminus \{y_{i_k}\} = \alpha.$$

Hence, α is a coboundary and $I_{u,v}$ is acyclic. ☐

We shouldn't deduce too quickly from the previous lemma that V is acyclic. It is false that a cocycle α decomposes as a sum of cocycles $\alpha_{u,v} \in I_{u,v}$. The problem comes from the fact that $I_{u,v} \cap I_{v,w}$ is not empty. But with a little more work, it is possible to prove that V is acyclic.

Lemma 4.5. *The vector space V is such that $d(V) \subseteq V$ and V is acyclic.*

Proof. Clearly, $d(V) \subseteq V$. Showing that V is acyclic is technical. We'll do it in three steps. First, we choose a finite number of couple $(u_i, v_i) \in E$ that have some nice properties. Then, using these elements, we construct a sequence of vector subspaces $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = V$. Finally, we prove by induction that all these subspaces are acyclic.

The linear order was chosen in such a way that it respects the equivalence relation $\sim$ on $\mathcal{A}'$ in the following sense : the equivalence classes are $\mathcal{A}' = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_q$ and if $i < j$, $y \in \mathcal{A}_i$ and $z \in \mathcal{A}_j$, then $y < z$. Let $\mathcal{A}_1 = \{x_{j_1}, \dots, x_{j_2-1}\}$, $\mathcal{A}_2 = \{x_{j_2}, \dots, x_{j_3-1}\}, \dots, \mathcal{A}_r = \{x_{j_q}, \dots, x_{j_{q+1}-1}\}$. Let $\mathcal{A}_i$ be the first class with more than one element and $(u_0, v_0) = (x_{j_i}, x_{j_i+1}) \in E$, $(u_1, v_1) = (u_0, x_{j_i+2}), \dots, (u_\ell, v_\ell) = (u_0, x_{j_{i+1}-1})$. Let $\mathcal{A}_k$ be the next equivalence class with more than 1 element and $(u_{\ell+1}, v_{\ell+1}) = (x_{j_k}, x_{j_k+1}), (u_{\ell+2}, v_{\ell+2}) = (x_{j_k}, x_{j_k+2}), \dots$ We can do this until we have a sequence $(u_i, v_i)_{0\le i\le r} \subseteq E$ with the following properties : $V = \sum_{i=0}^r I_{u_i,v_i}$ and $v_\ell \notin \{u_0, v_0, u_1, v_1, \dots, u_{\ell-1}, v_{\ell-1}\}$.

Let $I_0 = I_{u_0,v_0}$ and $I_j = \sum_{i=0}^j I_{u_i,v_i}$. The properties of the (u_i, v_i) implies that we have an increasing sequence

$$I_{u_0,v_0} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = V.$$

Lemma 4.4 shows that I_0 is acyclic. Suppose $I_{\ell-1}$ acyclic and let's show that $I_\ell = I_{\ell-1} + I_{u_\ell,v_\ell}$ is acyclic as well.

Let's start by proving that every element of I_ℓ can be written as a sum $\alpha_1+\alpha_2+d\omega$ where $\alpha_1 \in I_{\ell-1}, \alpha_2 \in I_{u_\ell,v_\ell}$, $\omega \in I_{\ell-1}$ and there is no v_ℓ in any of the terms of α_1. It is sufficient to see that it is true for every element of a generating sequence of $I_{\ell-1}$:

- Let $\sigma = \{x_0, u_i, v_i, \ldots\} \in I_{\ell-1}$. If $v_\ell \in \sigma$ and $u_\ell \in \sigma$, then $\sigma = 0+\sigma+d(0)$. If $v_\ell \in \sigma$ and $u_\ell \notin \sigma$, then $d(\sigma \cup \{u_\ell\}) = \pm\sigma \pm (\sigma\cup\{u_\ell\})\setminus\{v_\ell\}+S$ where S is a sum with $\{u_\ell, v_\ell\}$ in each term, which means that S is in I_{u_ℓ,v_ℓ}. It shows that we have the required decomposition $\sigma = \pm(\sigma\cup\{u_\ell\})\setminus\{v_\ell\}\pm S\pm d(\sigma\cup\{u_\ell\})$, for an appropriate choice of signs.

- Let $\alpha = \{x_0, u_i, y_{j_1}, \ldots, y_{j_s}\} - \{x_0, v_i, y_{j_1}, \ldots, y_{j_s}\} \in I_{\ell-1}$. If we have $v_\ell \in \{y_{j_1}, \ldots, y_{j_s}\}$ and $u_\ell \notin \{y_{j_1}, \ldots, y_{j_s}\}$, then consider $\omega = \{x_0, u_i, y_{j_1}, \ldots, y_{j_s}\} \cup \{u_\ell\} - \{x_0, v_i, y_{j_1}, \ldots, y_{j_s}\} \cup \{u_\ell\} \in I_{\ell-1}$. As in the first case, we obtain a proper decomposition from $d\omega$.

Now, we can show that every cocycle in I_ℓ is a coboundary. Let $\alpha \in I_\ell$ such that $d\alpha = 0$. Since $I_\ell = I_{\ell-1} + I_{u_\ell,v_\ell}$, we can write $\alpha = \alpha_1+\alpha_2+d\omega$ with $\alpha_1 \in I_{\ell-1}, \alpha_2 \in I_{u_\ell,v_\ell}$, $\omega \in I_\ell$ and no v_ℓ in any term of α_1. The vector α_2 is a sum

$$\alpha_2 = \sum_i \gamma_i \left(\{x_0, u_\ell, y_{i_1}, \ldots, y_{i_s}\} - \{x_0, v_\ell, y_{i_1}, \ldots, y_{i_s}\}\right) + \sum_j \beta_j \{x_0, u_\ell, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\}.$$

For appropriate sets K_i and L_j (as in lemma 4.4), its differential is

$$\begin{aligned} d\alpha_2 = &\sum_i \gamma_i \sum_{k\in K_i} (-1)^k \big(\{x_0, u_\ell, y_{i_1}, \ldots, y_{i_s}\}\setminus\{y_{i_k}\} \\ &\qquad - \{x_0, v_\ell, y_{i_1}, \ldots, y_{i_s}\}\setminus\{y_{i_k}\}\big) \\ &+ \sum_j \beta_j \big(\{x_0, u_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\} - \{x_0, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\}\big) \\ &+ \sum_j \beta_j \sum_{k\in L_j} (-1)^{k+1} \{x_0, u_\ell, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\}\setminus\{z_{j_k}\}. \end{aligned}$$

But there is no term in α_1 containing v_ℓ and $d\alpha_1 + d\alpha_2 = 0$. So, looking at the terms with u_ℓ and v_ℓ in $d\alpha_2$, we deduce that

$$\sum_j \beta_j \sum_{k\in L_j} (-1)^{k+1} \{x_0, u_\ell, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\}\setminus\{z_{j_k}\} = 0.$$

Hence the terms in $d\alpha_2$ with only v_ℓ are such that

$$\sum_i \gamma_i \sum_{k\in K_i} (-1)^k \{x_0, v_\ell, y_{i_1}, \dots, y_{i_s}\} \setminus \{y_{i_k}\} + \sum_j \beta_j \{x_0, v_\ell, z_{j_1}, \dots, z_{j_{s-1}}\} = 0.$$

And this relation stays true if we replace v_ℓ by u_ℓ. So, we just proved that $d\alpha_2 = 0$, which implies that $d\alpha_1 = 0$ as well. By the induction hypothesis, there exists a $\omega_1 \in I_{\ell-1}$ such that $d\omega_1 = \alpha_1$ and by lemma 4.4, there exists a $\omega_2 \in I_{u_\ell,v_\ell}$ such that $d\omega_2 = \alpha_2$. So, $\alpha = d(\omega_1 + \omega_2 + \omega)$. The vector subspace I_ℓ is acyclic and the proof by induction is complete. □

Together, these lemmas give us the following proposition.

Proposition 4.6. *The map $\overline{\varphi}\colon D(\mathcal{A})/D(\mathcal{A}') \to D(\widetilde{\mathcal{A}}'')$ is a quasi-isomorphism.*

Proof. By lemma 4.3, we can factorize $\overline{\varphi}$ as shown in the following diagram.

$$\begin{tikzcd}
\frac{D(\mathcal{A})}{D(\mathcal{A}')} \arrow[r, "\overline{\varphi}"] \arrow[d, "p"'] & D(\widetilde{\mathcal{A}}'') \\
\frac{D(\mathcal{A})}{D(\mathcal{A}')\oplus V} \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

where $\tilde{\varphi}$ is the quotient map. Since $V = \ker \overline{\varphi}$ and $\overline{\varphi}$ is surjective, $\tilde{\varphi}$ is an isomorphism. By lemma 4.5, the map p is a quasi-isomorphism, which implies that $\overline{\varphi}$ is a quasi-isomorphism as well. □

Remember that the cdga $D(\widetilde{\mathcal{A}}'')$ and $D(\mathcal{A}'')$ are quasi-isomorphic (see corollary 3.14). So, the quasi-isomorphism $\overline{\varphi}\colon D(\mathcal{A})/D(\mathcal{A}') \to D(\widetilde{\mathcal{A}}'')$ described in the previous section completes the link between $D(\mathcal{A})/D(\mathcal{A}')$ and $D(\mathcal{A}'')$. Now, we state and prove the main result of this section.

Theorem 4.7. *Let $\mathcal{A}$ be a subspace arrangement, $x_0 \in \mathcal{A}$ with $\mathcal{A}'$ and $\mathcal{A}''$ the corresponding deleted and restricted arrangements. Then, there exists a long exact sequence in cohomology :*

$$\cdots \to H^q(M(\mathcal{A}'), \mathbb{Q}) \to H^q(M(\mathcal{A}), \mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}''), \mathbb{Q}) \to H^{q+1}(M(\mathcal{A}'), \mathbb{Q}) \to H^{q+1}(M(\mathcal{A}), \mathbb{Q}) \to \cdots$$

Proof. Notice that the quasi-isomorphism of corollary 3.14 preserves the degree and that $\overline{\varphi}$ decreases the degree by $2\operatorname{codim} x_0 - 1 = \deg(x_0)$. So, the long exact sequence is a direct consequence from the short exact sequence

$$0 \to D(\mathcal{A}') \to D(\mathcal{A}) \to D(\mathcal{A})/D(\mathcal{A}') \to 0,$$

corollary 3.14 and proposition 4.6. □

4.2.2 Euler characteristic

The long exact sequence in cohomology of theorem 4.7 is quite powerful. In this subsection, we will prove a first application : the fact that the Euler characteristic of $M(\mathcal{A})$ is always zero for any non-empty arrangements.

Corollary 4.8. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$. Then, the Euler characteristic of $M(\mathcal{A})$ satisfies : $\chi(M(\mathcal{A})) = 0$ if $\mathcal{A}$ is non-empty, and $\chi(M(\mathcal{A})) = 1$ if $\mathcal{A}$ is empty.*

Proof. If $\mathcal{A}$ is empty, then $M(\mathcal{A}) = \mathbb{C}^m$ and $\chi(M(\mathcal{A})) = 1$. If $|\mathcal{A}| = 1$, then $M(\mathcal{A}) = \mathbb{C}^m \setminus \{x_0\}$ for some vector subspace x_0. So, $M(\mathcal{A})$ has the homotopy type of an odd-dimensional sphere and $\chi(M(\mathcal{A})) = 0$.

By the universal coefficient theorems (in homology and cohomology), we have : $b_i(M(\mathcal{A})) = \dim H^i(M(\mathcal{A}), \mathbb{Q})$, where $b_i(M(\mathcal{A}))$ is the ith Betti number of $M(\mathcal{A})$. The long exact sequence in cohomology from theorem 4.7 gives us the following formula : $\chi(M(\mathcal{A})) = \chi(M(\mathcal{A}')) + (-1)^{\deg(x_0)}\chi(M(\mathcal{A}'')) = \chi(M(\mathcal{A}')) - \chi(M(\mathcal{A}''))$. Now, let's prove by induction on $|\mathcal{A}|$ that $\chi(M(\mathcal{A})) = 0$. Suppose that this is true for arrangements with n or less subspaces. Let $\mathcal{A}$ be an arrangement with $n+1$ subspaces and $x_0 \in \mathcal{A}$. The deleted and restricted arrangements are such that $|\mathcal{A}'| = |\mathcal{A}| - 1$ and $1 \leq |\mathcal{A}''| < |\mathcal{A}|$. So, by induction, $\chi(M(\mathcal{A})) = \chi(M(\mathcal{A}')) - \chi(M(\mathcal{A}'')) = 0 - 0 = 0$. □

Note that this is only true for complex subspace arrangements. Clearly, if $\mathcal{B}$ is the real arrangement with one point in $\mathbb{R}^3$, then $M(\mathcal{B}) \cong S^2$ and its Euler characteristic is 2. The reason is that we used the model $D(\mathcal{A})$, which is not valid in the case of real arrangements.

4.3 Deletion-restriction theorem

The long exact sequence of theorem 4.7 is particularly interesting when it splits in short exact sequences

$$0 \to H^q(M(\mathcal{A}'), \mathbb{Q}) \to H^q(M(\mathcal{A}), \mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}''), \mathbb{Q}) \to 0.$$

In that case, the following formula is true :

$$\operatorname{Poin}(M(\mathcal{A}), t) = \operatorname{Poin}(M(\mathcal{A}'), t) + t^{\deg(x_0)} \operatorname{Poin}(M(\mathcal{A}''), t).$$

While there is no reason for such a splitting to occur in general (there exists many counterexamples), it is possible to find such arrangements. In this section, we will focus on two different kinds of arrangements : the ones with a separator (see definition 4.9) and the ones with a geometric lattice.

4.3.1 Arrangements with a separator

The following definition is a generalization of the concept of separator found in [26]. This notion seems to appear naturally when working with the long exact sequence.

Definition 4.9. Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement. We say that x_i is a *separator* if $\vee\{x_0, \ldots, x_n\} > \vee(\{x_0, \ldots, x_n\} \setminus \{x_i\})$, in the lattice $L(\mathcal{A})$.

Remember that independant subsets are subsets σ such that $\vee\sigma > \vee(\sigma \setminus \{x\})$ for all $x \in \sigma$. So, if $\sigma = \mathcal{A}$ is independant, then each $x \in \mathcal{A}$ is a separator.

As shown by the next proposition, deleted and restricted arrangements with respect to a separator have the desired property.

Proposition 4.10. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$ and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to x_0. If x_0 is a separator, then the long exact sequence of theorem 4.7 splits and we have*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \operatorname{Poin}(M(\mathcal{A}'), t) + t^{\deg(x_0)} \operatorname{Poin}(M(\mathcal{A}''), t).$$

Proof. It is sufficient to show that the injective map $j\colon D(\mathcal{A}') \to D(\mathcal{A})$ induces an injective map $j^*\colon H^*(D(\mathcal{A}')) \to H^*(D(\mathcal{A}))$. Let's define a map $k\colon D(\mathcal{A}) \to D(\mathcal{A}')$ of cochain complexes by $k(\sigma) = 0$ if $x_0 \in \sigma$ and $k(\sigma) = \sigma$ if $x_0 \notin \sigma$. This map commutes with the differential :

- If $x_0 \notin \sigma$, then $dk(\sigma) = kd(\sigma)$,
- if $x_0 \in \sigma$, then $dk(\sigma) = 0$ and $kd(\sigma) = \alpha(\sigma \setminus \{x_0\})$, with $\alpha = 0$ if and only if $\vee(\sigma \setminus \{x_0\}) \neq \vee\sigma$, which is the case because x_0 is a separator.

So, the map k is a map of chain complexes and induces a map $k^*\colon H^*(D(\mathcal{A})) \to H^*(D(\mathcal{A}'))$. Obviously, we have $k^*j^* = \mathrm{id}$, which implies that j^* is injective. Since the sequence of theorem 4.7 is exact, it splits as required. □

4.3.2 Arrangements with a geometric lattice

In this section, inspired by [26], we will study the Poincaré polynomial for arrangements with a geometric lattice. When we deal with such subspaces, an explicit formula for $\mathrm{Poin}(M(\mathcal{A}), t)$ can be derived from the literature. Goresky-MacPherson proved with the theorem 3.3 that we only need to compute the homology of the order complexes $\Delta(0, x)$, for every $x \in L(\mathcal{A})$.

The homology of order complexes is well known in the geometric case. A theorem of Folkman (theorem 4.1 in [17]) states that if L is a geometric lattice with 0 and 1 as the smallest and largest element of L, then the homology of its order complex $\Delta(L)$ is

$$H_i(\Delta(L)) = \begin{cases} 0 & \text{if } i \neq \operatorname{rk} L - 2, \\ \mathbb{Z}^{|\mu(0,1)|} & \text{if } i = \operatorname{rk} L - 2, \end{cases}$$

where $\mu\colon L \times L \to \mathbb{Z}$ is the Mœbius function, defined recursively by the relations :

$$\begin{cases} \mu(x,x) = 1 & \text{if } x \in L, \\ \sum_{z \in [x,y]} \mu(x,z) = 0 & \text{if } x, y, z \in L \text{ and } x < y, \\ \mu(x,y) = 0 & \text{otherwise.} \end{cases}$$

As a consequence, we have the following proposition :

Proposition 4.11. *If $\mathcal{A}$ is a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice $L(\mathcal{A})$, then*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \sum_{x \in L(\mathcal{A})} \mu(0, x)(-t)^{2 \operatorname{codim} x - \operatorname{rk}(x)}.$$

Proof. It is almost a direct consequence of theorem 3.3 and the results of Folkman. The only thing we need to check is the sign of $\mu(0, x)$. But a simple property of the Mœbius function is that $(-1)^{\operatorname{rk}(x)}\mu(0, x) \geq 0$ (see for example theorem 2.47 in [26]). □

The previous proposition is the key to prove a deletion-restriction formula for arrangements with a geometric lattice. But we need first an interpretation of the Mœbius function in the case of subspace arrangements.

Lemma 4.12. *Let $\mathcal{A}$ be a subspace arrangement. For $u, v \in L(\mathcal{A})$ with $u \leq v$, let $\mathcal{A}_u = \{x \in \mathcal{A} : u \subseteq x\}$ and $S(u, v) = \{\mathcal{B} \subseteq \mathcal{A} : \mathcal{A}_u \subseteq \mathcal{B}$ and $T(\mathcal{B}) = v\}$ (where $T(\mathcal{B})$ denotes the maximal element of $L(\mathcal{B})$). Then*

$$\mu(u, v) = \sum_{\mathcal{B} \in S(u,v)} (-1)^{|\mathcal{B} \setminus \mathcal{A}_u|}.$$

Proof. This lemma is the same as lemma 2.35 in [26] except that $\mathcal{A}$ is not necessarily an arrangement of hyperplanes. The proof of [26] is still valid in this case. □

Now, we can adapt the proofs in [26] to obtain a more general result.

Proposition 4.13. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$ and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to x_0. If $L(\mathcal{A})$ is geometric, then we have*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \operatorname{Poin}(M(\mathcal{A}'), t) + t^{\deg(x_0)} \operatorname{Poin}(M(\mathcal{A}''), t)$$

and the long exact sequence of theorem 4.7 splits (or equivalently, the inclusion map $D(\mathcal{A}') \to D(\mathcal{A})$ induces an injection in cohomology).

Proof. First, let's notice that for any arrangement $\mathcal{A}$, we have

$$\begin{aligned}
\mathrm{Poin}(M(\mathcal{A}),t) &= \textstyle\sum_{x\in L(\mathcal{A})} \mu(0,x)(-t)^{2\operatorname{codim} x-\mathrm{rk}(x)} \\
&= \textstyle\sum_{x\in L(\mathcal{A})} \left(\sum_{\mathcal{B}\in S(0,x)}(-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} x-\mathrm{rk}(x)}\right) \\
&= \textstyle\sum_{\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} T(\mathcal{B})-\mathrm{rk}\, T(\mathcal{B})}.
\end{aligned}$$

The last equality come from the fact that every subarrangement $\mathcal{B}\subseteq\mathcal{A}$ occurs in a unique $S(0,x)$. Now, separate the sum over $\mathcal{B}\subseteq\mathcal{A}$ into two sums R' and R'' with R' the sum over those $\mathcal{B}$ which do not contain x_0 and R'' the sum over those $\mathcal{B}$ which contain x_0. Clearly, by using the previous equality, we have $R' = \mathrm{Poin}(M(\mathcal{A}'),t)$. So we proved that $\mathrm{Poin}(M(\mathcal{A}),t)$ is equal to the sum $\mathrm{Poin}(M(\mathcal{A}'),t)+R''$. Now, let us compute R''.

$$\begin{aligned}
R'' &= \textstyle\sum_{x_0\in\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} T(\mathcal{B})-\mathrm{rk}(\mathcal{B})} \\
&= \textstyle\sum_{v\in L(\mathcal{A}'')} \displaystyle\sum_{\mathcal{B}\in S(x_0,v)}(-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} v-\mathrm{rk}(v)} \\
&= -\textstyle\sum_{v\in L(\mathcal{A}'')} \displaystyle\sum_{\mathcal{B}\in S(x_0,v)}(-1)^{|\mathcal{B}\setminus\mathcal{A}_{x_0}|}(-t)^{2\operatorname{codim} v-\mathrm{rk}(v)} \\
&= -\textstyle\sum_{v\in L(\mathcal{A}'')} \mu(x_0,v)(-t)^{2(\operatorname{codim} v-\operatorname{codim} x_0)-(\mathrm{rk}(v)-1)+2\operatorname{codim}(x_0)-1} \\
&= t^{2\operatorname{codim} x_0-1}\,\mathrm{Poin}(M(\mathcal{A}''),t) = t^{\deg(x_0)}\,\mathrm{Poin}(M(\mathcal{A}''),t).
\end{aligned}$$

The third line comes from lemma 4.12 and the last line comes from the fact that $(\operatorname{codim} v - \operatorname{codim} x_0)$ is equal to the codimension of v, seen in the ambient space of $\mathcal{A}''$, and that $\mathrm{rk}(v)-1$ is the rank of v in $L(\mathcal{A}'')$. This complete the proof of the fact that $\mathrm{Poin}(M(\mathcal{A}),t) = \mathrm{Poin}(M(\mathcal{A}'),t) + t^{\deg(x_0)}\,\mathrm{Poin}(M(\mathcal{A}''),t)$, and shows that the long exact sequence of theorem 4.7 splits as requested. □

Note that in theory, this theorem could be proved directly by showing that $D(\mathcal{A}')\to D(\mathcal{A})$ induces an injection in cohomology.

4.4 A $\mathbb{Z}_2$-isomorphism in cohomology

When we suppose that the arrangement $\mathcal{A}$ has a geometric lattice *and* that there is a subspace x_0 which is a separator, we have some stronger results. This is due to the following lemma.

Lemma 4.14. *Let $\mathcal{A} = \{x_0, \dots, x_n\}$ be a subspace arrangement with a geometric lattice such that x_0 is a separator and $\{i_1, \dots, i_{r+1}\} \subseteq [n]$. Then the following conditions are equivalent :*

1. $\vee\{x_{i_1}, \dots, x_{i_r}\} = \vee\{x_{i_1}, \dots, x_{i_r}, x_{i_{r+1}}\}$,
2. $\vee\{x_0, x_{i_1}, \dots, x_{i_r}\} = \vee\{x_0, x_{i_1}, \dots, x_{i_r}, x_{i_{r+1}}\}$.

Proof. Clearly, the condition (1) implies the condition (2). Suppose that we have

$$\vee\{x_0, x_{i_1}, \dots, x_{i_r}\} = \vee\{x_0, x_{i_1}, \dots, x_{i_r}, x_{i_{r+1}}\}.$$

Since x_0 is a separator, $\vee\{x_0, \dots, x_n\} \neq \vee(\{x_0, \dots, x_n\} \setminus \{x_0\})$. An easy consequence is that $\vee\{x_0, x_{i_1}, \dots, x_{i_r}\} \neq \vee\{x_{i_1}, \dots, x_{i_r}\}$. Since the lattice is geometric, proposition 2.14 can be used and we obtain the maximal chain $\vee\{x_{i_1}, \dots, x_{i_r}\} < \vee\{x_0, x_{i_1}, \dots, x_{i_r}\}$ from the maximal chain $\mathbb{C}^m < x_0$. Therefore, we have the two following chains of inequalities :

$$\begin{aligned} \vee\{x_{i_1}, \dots, x_{i_r}\} < \vee\{x_0, x_{i_1}, \dots, x_{i_r}\} = \vee\{x_0, x_{i_1}, \dots, x_{i_{r+1}}\}, \\ \vee\{x_{i_1}, \dots, x_{i_r}\} \leq \vee\{x_{i_1}, \dots, x_{i_{r+1}}\} < \vee\{x_0, x_{i_1}, \dots, x_{i_{r+1}}\}. \end{aligned}$$

Since the lattice is geometric, by proposition 2.15, all maximal chains between two elements have the same length. The first line is a maximal chain of length 2, so the second line has the same length and we have the equality $\vee\{x_{i_1}, \dots, x_{i_r}\} = \vee\{x_{i_1}, \dots, x_{i_{r+1}}\}$. □

Lemma 4.15. *Let $\mathcal{A}$ be a subspace arrangement with a geometric lattice and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to $x_0 \in \mathcal{A}$. If x_0 is a separator, then there exists an isomorphism $H^*(M(\mathcal{A}'); \mathbb{Q}) \to H^*(M(\mathcal{A}''); \mathbb{Q})$ of $\mathbb{Z}_2$-graded vector spaces.*

Proof. Let's define a map $\theta \colon D(\mathcal{A}') \to D(\mathcal{A}'')$ by sending the element $\{x_{i_1}, \dots, x_{i_r}\}$ to $\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_r\}$. It is obviously a surjection. If $x_0 \cap x_i = x_0 \cap x_j$, then $x_0 \cap x_i = x_0 \cap x_i \cap x_j$, and by lemma 4.14, $x_i = x_i \cap x_j$, which is impossible (we always suppose that $\mathcal{A}$ does not have two subspaces x and y such that $x \subset y$). So, the map θ is a bijection.

To show that θ commutes with the differential, consider $\sigma = \{x_{i_1}, \dots, x_{i_r}\}$ and let

$$\begin{aligned}
J_1 &= \{j : 1 \leq j \leq r \text{ and } \vee\sigma = \vee(\sigma \setminus \{x_{i_j}\})\}, \\
J_2 &= \{j : 1 \leq j \leq r \text{ and } \vee(\sigma \cup \{x_0\}) = \vee(\sigma \cup \{x_0\} \setminus \{x_{i_j}\})\}.
\end{aligned}$$

We have :

$$\begin{aligned}
\theta(d\sigma) &= \sum_{j \in J_1} (-1)^j \{x_0 \cap x_{i_1}, \dots, x_{i_r} \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}, \\
d(\theta(\sigma)) &= \sum_{j \in J_2} (-1)^j \{x_0 \cap x_{i_1}, \dots, x_{i_r} \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}.
\end{aligned}$$

Lemma 4.14 shows that $J_1 = J_2$, so θ commutes with the differential. Therefore, the map in cohomology $H^*\theta \colon H^*(M(\mathcal{A}'); \mathbb{Q}) \to H^*(M(\mathcal{A}''); \mathbb{Q})$ is an isomorphism of vector spaces.

Finally, let $\sigma = \{x_{i_1}, \dots, x_{i_r}\} \in D(\mathcal{A}')$ of degree $\deg(\sigma)$. It is mapped to $\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \in D(\mathcal{A}'')$. We have :

$$\begin{aligned}
\deg\sigma - \deg\theta(\sigma) &= (2m - 2\dim \vee\sigma - r) \\
&\qquad - (2\dim x_0 - 2\dim(x_0 \cap \vee\sigma) - r) \\
&= 2m - 2\dim(x_0 + \vee\sigma) = 2\,\mathrm{codim}(x_0 + \vee\sigma)
\end{aligned}$$

So, θ decreases the degree by an even number. □

Notice that when the subspace x_0 is also an hyperplane, the map θ does not change the degree. So, we have a stronger result.

Proposition 4.16. *Let $\mathcal{A}$ be a subspace arrangement with a geometric lattice and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to $x_0 \in \mathcal{A}$. If x_0 is an hyperplane and a separator, then there exists a map $\theta \colon D(\mathcal{A}') \to D(\mathcal{A}'')$ which induces an isomorphism $H^*(\theta) \colon H^*(M(\mathcal{A}'); \mathbb{Q}) \to H^*(M(\mathcal{A}''); \mathbb{Q})$ of $\mathbb{Z}_2$-graded vector spaces, and*

$$\begin{aligned}
\mathrm{Poin}(M(\mathcal{A}'), t) &= \mathrm{Poin}(M(\mathcal{A}''), t), \\
\mathrm{Poin}(M(\mathcal{A}), t) &= (1 + t)\,\mathrm{Poin}(M(\mathcal{A}''), t).
\end{aligned}$$

Proof. Let $\sigma \in \mathcal{A}'$. Since $x_0 \cap \vee\sigma \neq \vee\sigma$, we have $\vee\sigma \not\subset x_0$. But x_0 is an hyperplane, so $\mathrm{codim}(x_0 + \vee\sigma) = 0$. Therefore, the map $H^*\theta$ of lemma 4.15 preserves the degree and is an isomorphism of graded vector spaces. This fact and proposition 4.10 implies directly the two equalities. □

Finally, since the map $H^*(\theta)$ is a $\mathbb{Z}_2$-isomorphism, we obtain some information about the Betti numbers.

Proposition 4.17. *Let $\mathcal{A}$ be a subspace arrangement with a geometric lattice and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to $x_0 \in \mathcal{A}$. Let $b_i = b_i(M(\mathcal{A})), b_i' = b_i(M(\mathcal{A}'))$ and $b_i'' = b_i(M(\mathcal{A}''))$. If x_0 is a separator and $\mathcal{A}'$ is not empty, then we have the following relations :*

$$\sum_{i=0}^{\infty} b''_{2i} = \sum_{i=0}^{\infty} b'_{2i} = \sum_{i=0}^{\infty} b'_{2i+1} = \sum_{i=0}^{\infty} b''_{2i+1},$$
$$\sum_{i=0}^{\infty} b'_i = \sum_{i=0}^{\infty} b''_i, \quad \forall n \in \mathbb{N}, \sum_{i=0}^{n} b'_i \leq \sum_{i=0}^{n} b''_i.$$

Proof. The equality $\sum_{i=0}^{\infty} b_{2i}(M(\mathcal{A}')) = \sum_{i=0}^{\infty} b_{2i+1}(M(\mathcal{A}'))$ is a direct consequence of corollary 4.8, everything else is a consequence of lemma 4.15 $\square$

4.5 Examples

In this last section, we give a few examples of arrangements, in order to get a better picture of what really happens. A few natural questions are stated and (partially) answered.

We proved (in propositions 4.10 and 4.13) that if $\mathcal{A}$ is an arrangement with a geometric lattice or with a separator, then the long exact sequence of section 4.2 splits. So, a question naturally comes to the mind : what happens if x_0 is not a separator or if $L(\mathcal{A})$ is not geometric? Does the result still holds in the general case? That question is answered negatively by the following example.

Example 4.18. In $\mathbb{C}^5$, let $\mathcal{A} = \{h_0, h_1, h_2\}$ where h_0, h_1 and h_2 are defined by the equations $z_1 = z_5 = 0$, $z_1 = z_2 = z_3 = 0$ and $z_3 = z_4 = z_5 = 0$, respectively.

A basis of the cochain complex $D(\mathcal{A})$ is described in table 4.1. From that, it is easy to see that $\text{Poin}(M(\mathcal{A}), t) = 1 + t^3 + 2t^5 + 2t^6$ (notice that $\chi(M(\mathcal{A})) = 0$). Let's consider $\mathcal{A}' = \mathcal{A} \setminus \{h_0\}$ and the corresponding restricted arrangement $\mathcal{A}''$. Some quick computations yield $\text{Poin}(M(\mathcal{A}'), t) = 1 + 2t^5 + t^8$, $\text{Poin}(M(\mathcal{A}''), t) = 1 + 2t^3 + t^4$

and

$$\mathrm{Poin}(M(\mathcal{A}'),t) + t^3\,\mathrm{Poin}(M(\mathcal{A}''),t) = 1 + t^3 + 2t^5 + 2t^6 + t^7 + t^8.$$

$\deg(x)$	x	dx
0	$\{\emptyset\}$	0
3	$\{h_0\}$	0
5	$\{h_1\}$	0
5	$\{h_2\}$	0
6	$\{h_0, h_1\}$	0
6	$\{h_0, h_2\}$	0
7	$\{h_0, h_1, h_2\}$	$-\{h_1, h_2\}$
8	$\{h_1, h_2\}$	0

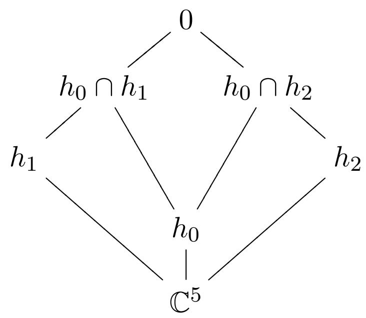


Figure 4.1: Description of $D(\mathcal{A})$ and $L(\mathcal{A})$.

So, the subspace arrangement $\mathcal{A}$ shows that the long exact sequence in cohomology does not always split. Obviously, h_0 is not a separator and $L(\mathcal{A})$ is not geometric (because $\mathbb{C}^5 < h_1$ is a maximal chain while $\mathbb{C}^5 \vee h_2 < h_1 \vee h_2$ is not, see proposition 2.15). However, if we consider the triple $(\mathcal{A}, \widehat{\mathcal{A}}', \widehat{\mathcal{A}}'')$ associated to h_1, then the Poincaré polynomials are :

$$\begin{aligned}
\mathrm{Poin}(M(\widehat{\mathcal{A}'}),t) &= 1 + t^3 + t^5 + t^6,\\
t^5\,\mathrm{Poin}(M(\widehat{\mathcal{A}'}),t) &= t^5(1+t),\\
\mathrm{Poin}(M(\mathcal{A}),t) &= 1 + t^3 + 2t^5 + 2t^6,
\end{aligned}$$

which satisfies the formula of proposition 4.10. But it is easy to see that h_1 is a separator. So, the arrangement $\mathcal{A}$ from example 4.18 is a subspace arrangement such that $L(\mathcal{A})$ is not geometric, with a subspace (h_0) which is not a separator and with a subspace (h_1) which is a separator.

In the previous example, we were able to select a separator. But are there subspace arrangements such that no subspace is a separator? Or is it always possible to choose a separator? That question is very easily answered with the following example.

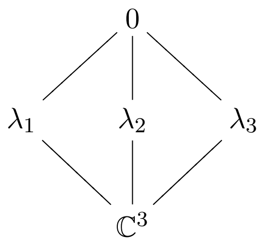


Figure 4.2: Lattice $L(\mathcal{B})$ of example 4.19

Example 4.19. Consider the arrangement $\mathcal{B} = \{\lambda_0, \lambda_1, \lambda_2\}$ in $\mathbb{C}^3$ defined by $z_2 = z_3 = 0$, $z_1 = z_3 = 0$ and $z_1 = z_2 = 0$.

Clearly, it is a subspace arrangement such that no subspace is a separator. Let $\mathcal{B}' = \mathcal{B} \setminus \{\lambda_0\}$ and $\mathcal{B}''$ be the corresponding restricted arrangement. Let us compute the Poincaré polynomials corresponding to $M(\mathcal{B}), M(\mathcal{B}')$ and $M(\mathcal{B}'')$. A basis for $D(\mathcal{B})$ is 1 in degree 0, $\{\lambda_i\}$ in degree 3, $\{\lambda_i, \lambda_j\}$ in degree 4 and $\{\lambda_0, \lambda_1, \lambda_2\}$ in degree 3, with all differentials equal to zero except $d\{\lambda_0, \lambda_1, \lambda_2\} = -\{\lambda_1, \lambda_2\} + \{\lambda_0, \lambda_2\} - \{\lambda_0, \lambda_1\}$. From that, it is easy to compute

$$\begin{aligned}\operatorname{Poin}(M(\mathcal{B}), t) &= 1 + 3t^3 + 2t^4 = 1 + 2t^3 + t^4 + t^3(1+t)\\ &= \operatorname{Poin}(M(\mathcal{B}'), t) + t^{\deg(\lambda_0)} \operatorname{Poin}(M(\mathcal{B}''), t).\end{aligned}$$

But clearly the lattice $L(\mathcal{B})$ of example 4.19 is geometric. It would be interesting to find an example of subspace arrangement $\mathcal{A}$ such that the lattice $L(\mathcal{A})$ is not geometric, and with a subspace x_0 which is not a separator, but such that the long exact sequence in cohomology splits.

CHAPTER 5

Rational wedges of spheres

As we said previously, the rational model $D(\mathcal{A})$ contains all the rational homotopy information on $M(\mathcal{A})$. Two particularly simple situations are when all the products are zero, or when no products are zero. In this chapter, we study what happens in the former situation.

A well-known space with trivial products in cohomology is a wedge of spheres, so an interesting question is if/when $M(\mathcal{A})$ is a wedge of spheres. In general, the question of when the complement of a subspace arrangement is a wedge of spheres seems quite difficult. In a long paper (see [19]), J. Grbic and S. Theriault tackled this problem in the case of coordinate subspace arrangements. They proved that if K is a shifted complex (see [19] for the definition), then $M(\mathcal{A})$ is homotopy equivalent to a wedge of spheres.

Since we work with rational models, $D(\mathcal{A})$ and $BP(\mathcal{A})$, we should not expect to obtain such strong results with them. But we can ask a weaker question : when is $M(\mathcal{A})$ a rational wedge of spheres? In the geometric case, we can answer this question : $M(\mathcal{A})$ is a rational wedge of spheres if and only if for all $x, y \in \mathcal{A}$, $x + y \neq \mathbb{C}^m$. This is corollary 5.4, which is proved in section 5.1. This is probably the best we can get from $D(\mathcal{A})$.

In the case of coordinate subspace arrangements, we do not have the formality nor the nice properties of geometric lattices. As we will see in section 5.2, the theorems of section 5.1 are false in this case, but we still manage to obtain some weaker results by adding some additional restrictions : we prove that if $M(\mathcal{A})$ is a rational wedge of spheres (and a technical condition), then for all $x, y \in \mathcal{A}$, $x + y \neq \mathbb{C}^m$.

5.1 Arrangements with a geometric lattice

The next lemma is the main ingredient of our proof. It shows that in the geometric case, the differential of an element $\{x_i, x_j, x_k\}_<$ is very simple.

Lemma 5.1. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice and $\sigma = \{x_i, x_j, x_k\}_< \in D(\mathcal{A})$. Then, $d\sigma = 0$ or $d\sigma = -\{x_j, x_k\} + \{x_i, x_k\} - \{x_i, x_j\}$.*

Proof. We only need to show that the cases where $d\sigma$ is a sum of one or two terms can not happen if $L(\mathcal{A})$ is geometric. Suppose that this is the case and that $d\sigma = -\{x_j, x_k\}$ or $d\sigma = -\{x_j, x_k\} + \{x_i, x_k\}$ (the exact same proof will apply to any other combination of one or two terms).

By definition of the differential, we have $x_j \vee x_k = x_i \vee x_j \vee x_k$. Since $\{x_j, x_k\}$ is obviously independant, proposition 2.18 tells us that $\mathrm{rk}(x_i \vee x_j \vee x_k) = \mathrm{rk}(x_j \vee x_k) = 2$. Again, by definition of the differential, we have $x_i \vee x_j \vee x_k > x_i \vee x_j$. So, $\mathrm{rk}(x_i \vee x_j \vee x_k) > \mathrm{rk}(x_i \vee x_j) = 2$, which is a contradiction. □

To every subspace arrangement $\mathcal{A}$, we can associate a simple chain complex : for $i \in \mathbb{N}$, let $C_i(\mathcal{A})$ be the free $\mathbb{Q}$-module generated by $\{\sigma \subseteq \mathcal{A} : |\sigma| = i\}$ and $D\colon C_{i+1}(\mathcal{A}) \to C_i(\mathcal{A})$ be the map defined for $\sigma = \{x_{j_1}, \ldots, x_{j_{i+1}}\}_<$ by $D\sigma = \sum_{k=1}^{i+1}(-1)^k(\sigma \setminus \{x_{j_k}\})$. This is almost the same as $D(\mathcal{A})$ except that the grading and the differential are slightly different. As the following lemma shows, $(C_*(\mathcal{A}), D)$ doesn't contain any information.

Lemma 5.2. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. Then the chain complex $(C_*(\mathcal{A}), D)$ is acyclic.*

Proof. Obviously, $D^2 = 0$, so $(C_*(\mathcal{A}), D)$ is a chain complex. Consider the standard $(n-1)$-simplex $\Delta^{n-1} = \langle e_1, \ldots, e_n\rangle \subseteq \mathbb{R}^n$. Let $S_i(\Delta^{n-1}, \mathbb{Q})$ be the free $\mathbb{Q}$-module generated by the i-dimensional faces of Δ^{n-1}. There is an obvious bijection $\varphi_i\colon C_i \to S_i(\Delta^{n-1}, \mathbb{Q})$, which gives us the following commutative diagram, where the map ∂ is the boundary homomorphism as defined in simplicial homology theory.

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{D} & C_{i+1}(\mathcal{A}) & \xrightarrow{\quad D \quad} & C_i(\mathcal{A}) & \xrightarrow{\quad D \quad} & C_{i-1}(\mathcal{A}) & \xrightarrow{\quad D \quad} \cdots \\
& \Big\downarrow{\scriptstyle \varphi_{i+1}} & & \Big\downarrow{\scriptstyle \varphi_i} & & \Big\downarrow{\scriptstyle \varphi_{i-1}} & \\
\cdots \xrightarrow{\partial} & S_{i+1}(\Delta^{n-1}, \mathbb{Q}) & \xrightarrow{\partial} & S_i(\Delta^{n-1}, \mathbb{Q}) & \xrightarrow{\partial} & S_{i-1}(\Delta^{n-1}, \mathbb{Q}) & \xrightarrow{\partial} \cdots
\end{array}$$

So, the homology $H_*(C_*(\mathcal{A}), D)$ actually corresponds to the simplicial homology $H_*(\Delta^{n-1}; \mathbb{Q})$. But Δ^{n-1} is contractible, therefore $H_*(\mathbb{C}_*(\mathcal{A}), D) = 0$. □

Now, we have all the tools required to prove the following theorem.

Theorem 5.3. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice. The following conditions are equivalent.*

1. *The multiplication map*

$$H^+(M(\mathcal{A}); \mathbb{Q}) \otimes H^+(M(\mathcal{A}); \mathbb{Q}) \to H^+(M(\mathcal{A}); \mathbb{Q})$$

   is zero,

2. *for all i, j, the products $\{x_i\} \cdot \{x_j\} = 0$ in $D(\mathcal{A})$ (or in other words : $x_i + x_j \neq \mathbb{C}^m$),*

3. *for all $\sigma, \tau \subseteq \mathcal{A}$, we have $\sigma \cdot \tau = 0$,*

4. *$M(\mathcal{A})$ has the rational homotopy type of a wedge of spheres.*

Proof. We will prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2). Suppose that every products in $H^+(M(\mathcal{A}); \mathbb{Q})$ are zero. Then for all $i < j$, $[\{x_i\} \cdot \{x_j\}] = [\{x_i\}] \cdot [\{x_j\}] = 0$. So, it means that either $\{x_i\} \cdot \{x_j\} = 0$ or $\{x_i\} \cdot \{x_j\} = \{x_i, x_j\}$ is a coboundary. Suppose that $\{x_i, x_j\}$ is a coboundary. Then, by lemma 5.1,

$$\{x_i, x_j\} = \sum_{1 \leq u < v < w \leq n} \alpha_{uvw} \Big(- \{x_v, x_w\} + \{x_u, x_w\} - \{x_u, x_v\} \Big)$$

for some $\alpha_{uvw} \in \mathbb{Q}$. But that would mean that $\{x_i, x_j\}$ is in the image of $D\colon C_3(\mathcal{A}) \to C_2(\mathcal{A})$ (as defined before lemma 5.2). But by lemma 5.2, we know that $H_*(C_*(\mathcal{A}), D) = 0$, so we have $\mathrm{im}(C_3(\mathcal{A}) \xrightarrow{D} C_2(\mathcal{A})) = \ker(C_2(\mathcal{A}) \xrightarrow{D} C_1(\mathcal{A}))$. But this is impossible because $D\{x_i, x_j\} = -\{x_j\} + \{x_i\} \neq 0$. So $\{x_i, x_j\}$ can not be a coboundary and the product $\{x_i\} \cdot \{x_j\}$ is necessarily equal to zero.

(2) $\Rightarrow$ (3). Suppose now that every product $\{x_i\} \cdot \{x_j\}$ is zero (or in other words, the vector space sum $x_i + x_j \neq \mathbb{C}^m$). Let $\sigma, \tau \subseteq \mathcal{A}$ and choose $x_i \in \sigma, x_j \in \tau$. We have : $\vee\sigma + \vee\tau \subseteq x_i + \vee\tau \subseteq x_i + x_j \neq \mathbb{C}^m$. Therefore, $\sigma \cdot \tau = 0$.

(3) $\Rightarrow$ (4). First, if $\mathcal{A}$ contains only one subspace, then $M(\mathcal{A})$ has the homotopy type of a sphere. Now, suppose that $|\mathcal{A}| \geq 2$. For all $x_i, x_j \in \mathcal{A}$, we have $x_i + x_j \neq \mathbb{C}^m$, which proves that x_i and x_j can not be hyperplanes. So $\mathcal{A}$ is an arrangement of subspace of codimension greater or equal to 2. Therefore, by corollary 2.21, the space $M(\mathcal{A})$ is 1-connected. Also, every product in $H^*(M(\mathcal{A}); \mathbb{Q}) = H^*(D(\mathcal{A}))$ is zero, so $H^*(M(\mathcal{A}); \mathbb{Q})$ is a direct sum of copies of $\mathbb{Q}$ in various degrees, with only trivial products. Since $M(\mathcal{A})$ is formal (by proposition 3.10), $M(\mathcal{A})$ has the same minimal model as a wedge of spheres. And since $M(\mathcal{A})$ is 1-connected, it has the same rational homotopy type.

The last step (proving that (4) $\Rightarrow$ (1)) is a well-known property of wedges of spheres. $\square$

This theorem gives us immediately a simple criterion to detect rational wedges of spheres. Also, since this is about arrangements with a geometric lattice, we can use $D_{\mathbb{Z}}(\mathcal{A})$ to get some information about the cohomology with integer coefficients (see proposition 3.12).

Corollary 5.4. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice. Then the three following statements are equivalent :*

1. *$M(\mathcal{A})$ is a rational wedge of spheres,*
2. *for all $x, y \in \mathcal{A}$, $x + y \neq \mathbb{C}^m$,*
3. *all the products in $\widetilde{H}^*(M(\mathcal{A}); \mathbb{Z})$ are trivial.*

Proof. (1) $\iff$ (2) is an immediate consequence of theorem 5.3 and the definition of product in $D(\mathcal{A})$.

(3) $\Rightarrow$ (1). If all the products in $\widetilde{H}^*(M(\mathcal{A}); \mathbb{Z})$ are trivial, then the products in rational cohomology are trivial, which is equivalent to (1) by theorem 5.3.

(2) $\Rightarrow$ (3). The condition (2) means that all the products are trivial in $D(\mathcal{A})$. But then the products are clearly trivial in $D_{\mathbb{Z}}(\mathcal{A})$. So, the products in $\widetilde{H}^*(M(\mathcal{A});\mathbb{Z})$ are trivial. $\square$

5.2 Coordinate subspace arrangements

The results of the previous section do not hold in the general case. The coordinate subspace arrangement studied in example 3.8 is a simple counterexample : $\mathcal{A} = \{x_1, x_2, x_3, x_4\}$ is a coordinate subspace arrangement in $\mathbb{C}^4$ where x_1, x_2, x_3, x_4 are defined, respectively, by the equations $z_1 = z_2 = 0, z_2 = z_3 = 0, z_3 = z_4 = 0$ and $z_4 = z_1 = 0$.

On page 27, we proved that $H^*(D(\mathcal{A}))$ is concentrated in degree $0, 3, 4, 5$ with

$$\dim H^0(D(\mathcal{A})) = 1, \quad \dim H^3(D(\mathcal{A})) = 4,$$
$$\dim H^4(D(\mathcal{A})) = 4, \quad \dim H^5(D(\mathcal{A})) = 1.$$

So, for degree reasons, the products in cohomology are clearly trivial, but $x_1 + x_3 = \mathbb{C}^4$.

In this section, we will show that we can still obtain some results in the case of coordinate subspace arrangements by adding an additional technical condition. However, since those subspace arrangements lack formality, we still won't have results as strong as in the previous section.

Recall that if $\mathcal{A}$ is a coordinate subspace arrangement without hyperplanes, then we have the following rational model of $M(\mathcal{A})$:

$$BP(\mathcal{A}) = (\mathbb{Q}[x_1, \ldots, x_m]/I_K \otimes \Lambda(u_1, \ldots, u_m), d)$$

with $dx_i = 0, du_i = x_i$, $|x_i| = 2, |u_i| = 1$ (this is theorem 3.20).

Lemma 5.5. *Let $\mathcal{A}$ be a coordinate subspace arrangement in $\mathbb{C}^m$ and consider $\sigma_1, \sigma_2 \subseteq [n]$ with $\sigma_1 = \{i_1, \ldots, i_k\}_<, \sigma_2 = \{j_1, \ldots, j_\ell\}_<$ such that $x_{\sigma_1}, x_{\sigma_2} \in \mathcal{A}$. Then*

1. *the cohomology class $[u_{i_1} x_{i_2} \cdots x_{i_k}] = [x_{i_1} u_{i_2} x_{i_3} \cdots x_{i_k}]$ is not trivial (in the model $BP(\mathcal{A})$),*

2. *if $\sigma_1 \cap \sigma_2 = \emptyset$ and if there is no $\sigma_3 \subseteq \sigma_1 \cup \sigma_2$ (with $\sigma_3 \neq \sigma_1$ and $\sigma_3 \neq \sigma_2$) such that $x_{\sigma_3} \in \mathcal{A}$, then the cohomology class $[u_{i_1} x_{i_2} \cdots x_{i_k} u_{j_1} x_{j_2} \cdots x_{j_\ell}]$ is not trivial*

Proof. It is easy to see that

$$du_{i_1}x_{i_2}\cdots x_{i_k} = dx_{i_1}u_{i_2}x_{i_3}\cdots x_{i_k} = du_{i_1}x_{i_2}\cdots x_{i_k}u_{j_1}x_{j_2}\cdots x_{j_\ell} = 0.$$

The equality $[u_{i_1}x_{i_2}\cdots x_{i_k}] = [x_{i_1}u_{i_2}x_{i_3}\cdots x_{i_k}]$ comes from the fact that

$$d(u_{i_1}u_{i_2}x_{i_3}\ldots x_{i_k}) = x_{i_1}u_{i_2}x_{i_3}\cdots x_{i_k} - u_{i_1}x_{i_2}\cdots x_{i_k}.$$

Let's start by proving the first assertion. Recall from remark 3.22 that $BP(\mathcal{A})$ can be decomposed as a direct sum of cochain complexes

$$BP(\mathcal{A}) = \bigoplus_{\sigma\in\{1,\ldots,n\}} BP_\sigma(\mathcal{A}).$$

So, we just need to prove that $[u_{i_1}x_{i_2}\cdots x_{i_k}]$ is not trivial in $BP_\sigma(\mathcal{A})$, where $\sigma = \{i_1,\ldots,i_k\}$. Without loss of generality, we can assume that $\sigma = \{1,\ldots,k\}$. To simplify the proof, we will use the following notations :

- if $1\le i\le k$, then U_i is the monomial $x_1\cdots x_{i-1}x_{i+1}\cdots x_k u_i$,
- if $1\le i<j\le k$, then
$$U_{ij} = x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_k u_i u_j.$$

Note that $dU_{ij} = -U_i + U_j$. With these notations, the problem can be rephrased in the following way : we want to prove that U_1 can not be written as a coboundary $d(\sum\alpha_{ij}U_{ij}) = \sum\alpha_{ij}(U_j - U_i)$. Consider the map θ that maps a sum $\sum\alpha_iU_i$ to $\theta(\sum\alpha_iU_i) = \sum\alpha_i$. We have : $\theta(U_1) = 1$ and $\theta(\sum\alpha_{ij}(U_j-U_i)) = 0$, therefore $U_1 = x_2\cdots x_k u_1$ is not a coboundary and the first assertion is proved.

Now, let us prove that $[u_{i_1}x_{i_2}\cdots x_{i_k}u_{j_1}x_{j_2}\cdots x_{j_\ell}]$ is not trivial. First observation : if every $L_\tau\in\mathcal{A}$ is such that $\tau\not\subseteq\sigma_1\cup\sigma_2$, then $x_{i_2}\cdots x_{i_k}x_{j_2}\cdots x_{j_\ell}$ is not in I_K. So, we only need to prove that $u_{i_1}x_{i_2}\cdots x_{i_k}u_{j_1}x_{j_2}\cdots x_{j_\ell}$ is not a coboundary. We can assume that $\sigma_1 = \{1,\ldots,k\}$ and $\sigma_2 = \{k+1,\ldots,k+\ell\}$. As above, we introduce some new notations to make the proof clearer :

$$V_{ab} := x_1\cdots x_{a-1}x_{a+1}\cdots x_{b-1}x_{b+1}\cdots x_{k+l}u_au_b,$$
$$V_{abc} := x_1\cdots x_{a-1}x_{a+1}\cdots x_{b-1}x_{b+1}\cdots x_{c-1}x_{c+1}\cdots x_{k+l}u_au_bu_c.$$

Notice that $dV_{abc} = -V_{bc} + V_{ac} - V_{ab}$ if $a < b < c$. What we need to prove is that $V_{1(k+1)}$ can not be written as a sum $\sum \alpha_{abc}(dV_{abc}) = \sum \alpha_{abc}(-V_{bc} + V_{ac} - V_{ab})$. This can be proved by using exactly the same argument as in the proof of theorem 5.3.

□

Now, we can state and prove a theorem similar to theorem 5.3, but in the case of coordinate subspace arrangements.

Theorem 5.6. *Let $\mathcal{A}$ be a coordinate subspace arrangement in $\mathbb{C}^m$. If for all $x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3} \in \mathcal{A}$ with $x_{\sigma_3} \neq x_{\sigma_1}$ and $x_{\sigma_3} \neq x_{\sigma_2}$, we have $\sigma_3 \nsubseteq \sigma_1 \cup \sigma_2$, then the following conditions are equivalent.*

1. *The multiplication map*

$$H^+(M(\mathcal{A}); \mathbb{Q}) \otimes H^+(M(\mathcal{A}); \mathbb{Q}) \to H^+(M(\mathcal{A}); \mathbb{Q})$$

   is zero,

2. *for all $x, y \in \mathcal{A}$, the product $\{x\} \cdot \{y\} = 0$ in $D(\mathcal{A})$,*

3. *for all $\sigma, \tau \subseteq \mathcal{A}$, we have $\sigma \cdot \tau = 0$,*

Proof. (1) ⇒ (2). Suppose that every products are trivial in cohomology and choose two subspaces y_1 and y_2 in $\mathcal{A}$ with equations

$$\begin{array}{ll} y_1 & z_{i_1} = \ldots = z_{i_{k_1}} = 0, \\ y_2 & z_{j_1} = \ldots = z_{j_{k_1}} = 0. \end{array}$$

In this case, proving that $\{y_1\} \cdot \{y_2\} = 0$ in $D(\mathcal{A})$ is equivalent to proving that $\{i_1, \ldots, i_{k_1}\} \cap \{j_1, \ldots, j_{k_2}\} \neq \emptyset$.

Suppose that $\{i_1, \ldots, i_{k_1}\} \cap \{j_1, \ldots, j_{k_2}\} = \emptyset$ and consider the cohomology classes $\alpha = [u_{i_1} x_{i_2} \cdots x_{i_{k_1}}]$ and $\beta = [u_{j_1} x_{j_2} \cdots x_{j_{k_1}}]$. By lemma 5.5, α, β and $\alpha \cdot \beta$ are not trivial which contradicts statement (1).

(2) ⇒ (3). Same proof as in theorem 5.3.

(3) ⇒ (1) is obvious. □

The additional technical condition in the previous theorem was needed, the example 3.8 shows that it is not true in general.

We have now the following corollary, similar to corollary 5.4.

Corollary 5.7. *Let $\mathcal{A}$ be a coordinate subspace arrangement in $\mathbb{C}^m$ such that for all $x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3} \in \mathcal{A}$, we have $\sigma_3 \not\subseteq \sigma_1 \cup \sigma_2$. Then $M(\mathcal{A})$ is a rational wedge of spheres if and only if for all $x, y \in \mathcal{A}$, $x + y \neq \mathbb{C}^m$.*

Proof. This is an immediate consequence of theorem 5.6. $\square$

CHAPTER 6

Ellipticity of $M(\mathcal{A})$

6.1 Introduction

Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex subspace arrangement. In this chapter, we explore the consequences of ellipticity of $M(\mathcal{A})$ (see definition 1.17). More precisely, we show in theorems 6.7 and 6.10 that if $L(\mathcal{A})$ is a geometric lattice or if $\mathcal{A}$ is a coordinate subspace arrangement, then the following statements are equivalent :

1. $M(\mathcal{A})$ is a nilpotent rationally elliptic space,
2. $\operatorname{codim} \cap_{i=1}^{n} x_i = \sum_{i=1}^{n} \operatorname{codim} x_i$,
3. $M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres,

These results are curious : they connect a rational homotopy property (ellipticity) with non-rational topological properties. This is explained by examining closely the rational model $D(\mathcal{A})$, and more precisely, the multiplication (see section 3.1). The definition of the multiplication uses a geometric condition, which is the link between rational homotopy and geometry.

From these statements, is is easy to see that most arrangements are not elliptic (for example, arrangements in $\mathbb{C}^m$ with more than m subspaces can not verify the second statement).

The proof of the main theorem, in the case of arrangements with a geometric lattice, is done in the next two sections. In section 6.2, we study what happens when the cohomology of $M(\mathcal{A})$ is a Poincaré duality algebra. The proofs of this section are specific to the geometric case. In section 6.3, we use the previous results and prove the equivalence of the various statements mentioned above.

The proof of the main theorem, in the case of coordinate subspace arrangements, is done in section 6.4. Basically, we use the model $BP(\mathcal{A})$, specific to coordinate subspace arrangements, to prove a similar step than the one proved in section 6.2 with Poincaré duality.

It is interesting to see that the theorem mentioned above is proved in two very different situations (arrangements with a geometric lattice and coordinate subspace arrangement), with different techniques. A question comes immediately to the mind : is it always true, or can we find subspace arrangements which verify some, but not all of the equivalent conditions stated above? I have been unable to find such a counterexample, and my investigations make me believe that it is always true.

Question 6.1. If $\mathcal{A}$ is a subspace arrangement in $\mathbb{C}^m$, then $M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres if and only if $M(\mathcal{A})$ is a nilpotent rationally elliptic space.

Note that a weaker version of some results presented in this chapter has been published in [10].

6.2 Poincaré duality

Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex subspace arrangement with a geometric lattice. In this section, we prove that if $M(\mathcal{A})$ verifies Poincaré duality, then the model $D(\mathcal{A})$ is very simple : it is $(\Lambda(y_1, \ldots, y_n), 0)$ with y_i in odd degrees. By Poincaré duality, we mean that the cohomology is a Poincaré duality algebra :

Definition 6.2. Let H be a connected, finite dimensional, graded algebra over the rationals. H is a *Poincaré duality algebra* if there exists $n \in \mathbb{N}$ with the property that $H^p = 0$ for $p > n$, $\dim H^n = 1$ and for every $0 \leq p \leq n$, multiplication induces a bilinear nondegenerate map $H^p \times H^{n-p} \to H^n$.

Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement with a geometric lattice $L(\mathcal{A})$ in $\mathbb{C}^m$, let M_r be the greatest element in $L(\mathcal{A})$ and $\mathbb{C}^m < M_1 < \cdots < M_{r-1} < M_r$ be a maximal chain in $L(\mathcal{A})$. Notice that $\mathrm{rk}(M_i) = i$ for $1 \leq i \leq r$. Let $X_i = \{x \in \mathcal{A} : x < M_i\}$. For every $1 \leq i \leq r$, we construct a chain complex C^i_* : C^i_p is the module generated by all the linear combinations of the $\sigma \subseteq \mathcal{A}$ such that

$\vee\sigma = M_i$ and $|\sigma| = p$. With the differential defined in the rational model of $M(\mathcal{A})$ (see section 3.1) and $\deg(\sigma) = |\sigma|$, C^i_* is clearly a chain complex.

Lemma 6.3. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex subspace arrangement with geometric lattice. If $H^*(M(\mathcal{A}); \mathbb{Q})$ is a Poincaré duality algebra and $1 \leq k \leq r$, then* $\dim H_k(C^k_*) = 1$.

Proof. To make sure that the signs are correct, we will suppose that the subspaces x_i are numbered in such a way that if $x_i \in X_r$ and $x_j \in X_{r+1} \setminus X_r$, then $i < j$. Such a numbering is always possible.

For $i = 1, \ldots, r$, let E^i_p be the submodule of C^i_p generated by

- all the $\sigma \subseteq \mathcal{A}$ such that $|\sigma| = p$, $\vee\sigma = M_i$ and σ contains at least 2 elements of $X_i \setminus X_{i-1}$,
- each element $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{p-1}}, y_1\} - \{x_{i_1}, \ldots, x_{i_{p-1}}, y_2\}$ with $x_{i_j} \in X_{i-1}$ and $y_1, y_2 \in X_i \setminus X_{i-1}$.

It is easy to check that E^i_* is a subcomplex of C^i_*. We have the following short exact sequences :

$$0 \to E^i_* \to C^i_* \to C^i_*/E^i_* \to 0.$$

Since $L(\mathcal{A})$ is a geometric lattice, proposition 3.11 states that the cohomology is generated by the independant classes $\sigma \subseteq \mathcal{A}$. So, the (reduced) homology of C^i_* is 0 in every degree except possibly the i^{th} degree. In other words, we have : $H_*(C^i_*) = H_i(C^i_*)$.

Notice that the previous paragraph implies that $H_i(C^{i+1}_*) = 0$. Using this observation, let's prove that $H_i(E^{i+1}_*) = 0$. If $\sigma \subseteq \mathcal{A}$ is such that $|\sigma| = i$, $\vee\sigma = M_{i+1}$ and σ contains at least two elements of $X_{i+1} \setminus X_i$, then there exists $\alpha \in C^{i+1}_{i+1}$ such that $d\alpha = \sigma$. But this can only be true if $\alpha \in E^{i+1}_{i+1}$, which implies that $[\sigma] = 0$ in $H_i(E^{i+1}_*)$. If $\sigma_1 - \sigma_2 = \{x_{r_1}, x_{r_2}, \ldots, x_{r_i}, y_1\} - \{x_{r_1}, \ldots, x_{r_i}, y_2\}$ with $x_{r_j} \in X_i$ and $y_1, y_2 \in X_{i+1} \setminus X_i$, then there also exists $\alpha \in C^{i+1}_{i+1}$ such that $d\alpha = \sigma_1 - \sigma_2$. But then $\{x_{r_1}, \ldots, x_{r_i}, y_1, y_2\} \in E^{i+1}_{i+1}$ and its differential is $\sigma_1 - \sigma_2$. This proves that $[\sigma_1 - \sigma_2] = 0$ in $H_i(E^{i+1}_*)$. Therefore, $H_i(E^{i+1}_*) = 0$.

Now, consider the long exact sequence in homology associated with the short exact sequence above. We proved that $H_i(E^{i+1}_*) = 0$, so the map $H_{i+1}(C_*^{i+1}) \to H_{i+1}(C_*^{i+1}/E_*^{i+1})$ is surjective and we have the inequality

$$\dim H_{i+1}(C_*^{i+1}/E_*^{i+1}) \leq \dim H_{i+1}(C_*^{i+1}).$$

The next step is to prove that $H_i(C_*^i) = H_{i+1}(C_*^{i+1}/E_*^{i+1})$. Since $X_{i+1} \setminus X_i$ is non empty (because $\vee X_{i+1} = M_{i+1}$ and $\vee X_i = M_i$), we can choose some $y \in X_{i+1} \setminus X_i$. Let's define a map $\varphi_i \colon C_*^i \to C_*^{i+1}/E_*^{i+1}$ by sending σ to $[\sigma \cup \{y\}]$.

If $\alpha \in C_*^i$ is such that $\varphi_i(\alpha) = 0$, then $\alpha \cup \{y\} \in E_*^{i+1}$. But the only element of $X_{i+1} \setminus X_i$ that appears in $\alpha \cup \{y\}$ is y. So, $\alpha \cup \{y\} \in E_*^{i+1}$ can only happen if $\alpha = 0$, which proves that φ_i is injective. To show that φ_i is surjective, let $[\alpha] \in C_*^{i+1}/E_*^{i+1}$. We can replace α by some $\widetilde{\alpha}$ such that the only element of $X_{i+1} \setminus X_i$ contained in $\widetilde{\alpha}$ is y, and $[\alpha] = [\widetilde{\alpha}]$. Obviously, there exists a $\beta \in C_*^i$ such that $\varphi_i(\beta) = [\widetilde{\alpha}] = [\alpha]$. So, the map φ_i is surjective and is an isomorphism of graded vector spaces.

To prove that φ_i commutes with the differential, we need the geometricity of the lattice $L(\mathcal{A})$. From the definition of φ_i and the differential, we see that we need to prove that for every increasing sequence $1 \leq i_1 < i_2 < \cdots < i_p \leq n$ such that $x_{i_k} \in M_i$ and for every $1 \leq j \leq p$,

$$\vee \{x_{i_1}, \ldots, \widehat{x_{i_j}}, \ldots, x_{i_p}\} = M_i \iff \vee\{x_{i_1}, \ldots, \widehat{x_{i_j}}, \ldots, x_{i_p}, y\} = M_{i+1}.$$

Let $M = \vee\{x_{i_1}, \ldots, \widehat{x_{i_j}}, \ldots, x_{i_p}\}$. We need to prove that $M = M_i \iff M \vee y = M_{i+1}$. Suppose that $M = M_i$. By construction, we have $M = M_i \prec M_{i+1}$. So, proposition 2.14 proves that either $M = M_i \vee y \prec M_{i+1} \vee y$, or $M = M_i \vee y = M_{i+1} \vee y$. The former is impossible (because $M_i \prec M_i \vee y \prec M_{i+1} \vee y = M_{i+1}$), so the latter is true and $M = M_{i+1}$. Now, suppose that $M \vee y = M_{i+1}$. Obviously, $\operatorname{rk} M \leq \operatorname{rk} M_i = i$. Now, the definition of a geometric lattice implies that $\operatorname{rk} M + \operatorname{rk} y \geq \operatorname{rk}(M \vee y) + \operatorname{rk}(M \wedge y)$, or in other words, that $\operatorname{rk} M \geq i$. This prove that $\operatorname{rk} M = i$, so $M = M_i$.

This proves that φ_i is an isomorphism of chain complexes, so we have the equality $\dim H_i(C_*^i) = \dim H_{i+1}(C_*^{i+1}/E_*^{i+1})$. This result, together with the inequality proved above, gives us

$$\dim H_i(C_*^i) \leq \dim H_{i+1}(C_*^{i+1}).$$

By hypothesis, $H^*(M(\mathcal{A});\mathbb{Q})$ is a Poincaré duality algebra, so there is a unique cohomology class in the highest degree in $H^*(D(\mathcal{A}))$. That cohomology class is represented by an independant $\sigma \subseteq \mathcal{A}$ such that $|\sigma| = r$. If $\dim H_r(C_*^r) \geq 2$, then there is another class $[\tau]$ and by Poincaré duality, there is an element $[\rho]$ in $H^*(D(\mathcal{A}))$ such that $[\sigma] = [\rho][\tau]$, but this is impossible by the multiplication law because $\operatorname{codim} \vee\sigma = \operatorname{codim} \vee\tau$. Therefore, $\dim H_r(C_*^r) = 1$. From the following sequence of inequalities

$$1 = \dim H_1(C_*^1) \leq \dim H_2(C_*^2) \leq \cdots \leq \dim H_r(C_*^r) = 1.$$

we deduce that $\dim H_k(C_*^k) = 1$ for all $1 \leq k \leq r$. □

Lemma 6.4. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex subspace arrangement such that $L(\mathcal{A})$ is geometric and $H^*(M(\mathcal{A});\mathbb{Q})$ is a Poincaré duality algebra. If $M \in L(\mathcal{A})$ with* $\operatorname{rk}(M) = k \leq r$, *then $\#\{x \in \mathcal{A} : x \leq M\} = k$.*

Proof. Let $X_M := \{x \in \mathcal{A} : x \leq M\}$. We prove the result by induction on $\operatorname{rk} M$. It is clear for $k = 1$. Now, let's suppose that it is true for all $N \in L(\mathcal{A})$ with $\operatorname{rk} N \leq k-1$ and let $M \in L(\mathcal{A})$ with $\operatorname{rk} M = k$. Denote by $M_1 < M_2 < \cdots < M_k = M \leq M_r$ a maximal sequence in $L(\mathcal{A})$, and write

$$\begin{aligned} \{x \in \mathcal{A} : x \leq M_{k-1}\} &= \{x_1, \ldots, x_{k-1}\}, \\ X_M = \{x \in \mathcal{A} : x \leq M\} &= \{x_1, \ldots, x_{k-1}, x_{k-1+1}, \ldots, x_{k-1+\ell}\}. \end{aligned}$$

We need to prove that $\ell = 1$. Let's consider the chain complex C_*^k defined above. Remark first that if $\sigma = \{x_{n_1}, \ldots, x_{n_{k+1}}\}_< \subset X_M$, then for each s, $\vee(\sigma \setminus \{x_{n_s}\}) = M$, because otherwise $\vee(\sigma \setminus \{x_{n_s}\})$ is an element N in $L(\mathcal{A})$ with $\operatorname{rk} N < k$. So, $\#\{x \in \mathcal{A} : x < N\} \geq k > \operatorname{rk} N$, in contradiction with our induction hypothesis. Therefore, if $\sigma = \{x_{n_1}, \ldots, x_{n_{k+1}}\} \in C_{k+1}^k$, then

$$d\sigma = \sum_{j=1}^{k+1} (-1)^j \sigma \setminus \{x_{n_j}\}.$$

So, if $1 \notin \{j_1, \ldots, j_k\}$, then we have

$$\{x_{j_1}, \ldots, x_{j_k}\} = d\{x_1, x_{j_1}, \ldots, x_{j_k}\} - \sum_{i=2}^{k} (-1)^i \{x_1, \ldots, \hat{x}_{j_i}, \ldots, x_{j_k}\}.$$

This proves that every cycle of degree k in the complex C_*^k is equivalent to a sum $\sum \alpha_j\{x_1, x_{j_2}, \dots, x_{j_k}\}$. The next step of this proof is to show that no cycle of the form $\sum_j \alpha_j\{x_1, x_{j_2}, \dots, x_{j_k}\}$ is a boundary. Suppose that this is the case. Then $\sum_j \alpha_j\{x_1, x_{j_2}, \dots, x_{j_k}\}$ is equal to

$$d\left[\sum\nolimits_m \beta_m\{x_1, x_{m_1}, \dots, x_{m_k}\} + \sum\nolimits_n \gamma_n\{x_{n_1}, \dots, x_{n_{k+1}}\}\right]$$

with $1 \notin \{n_1, \dots, n_{k+1}\}$. By developing the differential and looking at the terms that do not contain x_1, we get

$$\begin{aligned} 0 = -\sum\nolimits_m \beta_m\{x_{m_1}, \dots, x_{m_k}\} \\ + \sum\nolimits_n \gamma_n \left(\sum_{i=1}^{k+1}(-1)^i\{x_{n_1}, \dots, \hat{x}_{n_i}, \dots, x_{n_{k+1}}\}\right). \end{aligned}$$

We deduce that

$$\begin{aligned} d\left(\sum\nolimits_n \gamma_n\{x_1, x_{n_1}, \dots, x_{n_{k+1}}\}\right) &= -\sum_n \gamma_n\{x_{n_1}, \dots, x_{n_{k+1}}\} \\ &\quad - \sum_n \gamma_n \left(\sum_{i=1}^{k+1}(-1)^i\{x_1, x_{n_1}, \dots, \hat{x}_{n_i}, \dots, x_{n_{k+1}}\}\right) \\ &= -\sum\nolimits_n \gamma_n\{x_{n_1}, \dots, x_{n_{k+1}}\} - \sum\nolimits_m \beta_m\{x_1, x_{m_1}, \dots, x_{m_k}\}. \end{aligned}$$

Since $d^2 = 0$, the sum $\sum_j \alpha_j\{x_1, x_{j_2}, \dots, x_{j_i}\}$ is equal to zero. So, every $\sigma \subseteq X_M$ with $|\sigma| = k$ is a generator of $H_k(C_*^k)$.

The above result implies that $\ell = 1$ and $X_M = \{x_1, \dots, x_k\}$, because otherwise $\dim H_k(C_*^k) \geq 2$, in contradiction with lemma 6.3.

□

Proposition 6.5. *Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a complex subspace arrangement. If $L(\mathcal{A})$ is geometric and $H^*(M(\mathcal{A}); \mathbb{Q})$ is a Poincaré duality algebra, then the model $D(\mathcal{A})$ of $M(\mathcal{A})$ is simply the algebra $(\Lambda(y_1, \dots, y_n), 0)$ with $\deg y_i = 2\operatorname{codim} x_i - 1$.*

Proof. Let $\sigma = \{x_{i_1}, \dots, x_{i_r}\} \subseteq \mathcal{A}$. By lemma 6.4, $\operatorname{rk} \vee\sigma = \#\{x \in \mathcal{A} : x < \vee\sigma\} \geq r$. But proposition 2.18 tells us that $\operatorname{rk} \vee\sigma \leq r$. So, $\operatorname{rk} \vee\sigma = r$ and σ is independant. Since any subset of $\mathcal{A}$ is independant, the differential of $D(\mathcal{A})$ is zero and we have $D(\mathcal{A}) = (H^*(M(\mathcal{A}); \mathbb{Q}), 0)$.

Let $\sigma, \tau \subseteq \mathcal{A}$ such that $\sigma \cap \tau = \emptyset$. Let's prove that $\sigma \cdot \tau \neq 0$. By Poincaré duality, there exists a cohomology class $\alpha \subseteq \mathcal{A}$ such that $\sigma \cdot \alpha = \{x_1, \ldots, x_n\}$. But α can only be equal to $\mathcal{A} \setminus \sigma$, so $\tau \subseteq \alpha$. Remember that $\sigma \cdot \alpha \neq 0$ if and only if $\vee\sigma + \vee\alpha = \mathbb{C}^m$. But $\vee\alpha \subseteq \vee\tau$, so we have $\vee\sigma + \vee\tau = \mathbb{C}^m$, which implies that $\sigma \cdot \tau \neq 0$.

Therefore, all the products $\{x_{i_1}\} \cdot \cdots \cdot \{x_{i_l}\} \neq 0$. But by definition, we have $\{x_{i_1}\} \cdot \cdots \cdot \{x_{i_l}\} = \{x_{i_1}, \ldots, x_{i_l}\}$. So, the model $D(\mathcal{A})$ is simply the differential graded algebra $(\Lambda([\{x_1\}], \ldots, [\{x_n\}]), 0)$, with $\deg[\{x_i\}] = 2\operatorname{codim} x_i - 1$. □

6.3 Arrangements with a geometric lattice

Before proving the main theorem, the following lemma gives a geometric interpretation of the statement

$$\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i.$$

If x is a vector subspace, then the notation $(x)^\perp$ denotes its orthogonal subspace.

Lemma 6.6. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement. Then $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$ if and only if the sum $x_1^\perp + \cdots + x_n^\perp$ is a direct sum.*

Proof. First, let's prove by induction on k, $2 \leq k \leq n$, that :

$$\left(\cap_{j=1}^k x_j\right)^\perp = \sum\nolimits_{i=1}^k x_i^\perp.$$

For $k = 2$, it gives $(x_1 \cap x_2)^\perp = x_1^\perp + x_2^\perp$, which is a well-known fact. Now, let's suppose that the formula is true until $k-1$. We have :

$$\left(\cap_{j=1}^k x_j\right)^\perp = (\cap_{j=1}^{k-1} x_j \cap x_k)^\perp = (\cap_{j=1}^{k-1} x_j)^\perp + x_k^\perp.$$

Using the induction hypothesis ends the proof. Now, we can prove the lemma :

$$x_1^\perp + \cdots + x_n^\perp \text{ is a direct sum } \iff \sum \dim x_i^\perp = \dim\left(\sum x_i^\perp\right)$$
$$\iff \sum \operatorname{codim} x_i = \dim (\cap_{i=1}^n x_i)^\perp \iff \sum \operatorname{codim} x_i = \operatorname{codim} \cap_{i=1}^n x_i.$$

□

Now, everything is in place to state and prove the main theorem. Thanks to the previous lemma, this theorem is even better than advertised in the introduction, because we have added a fifth condition.

Theorem 6.7. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a complex subspace arrangement with a geometric lattice. Then the following conditions are equivalent :*

1. *$M(\mathcal{A})$ is a nilpotent rationally elliptic space,*
2. *$M(\mathcal{A})$ is a Poincaré duality space,*
3. $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$,
4. *$M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres,*
5. *the sum $x_1^\perp + \cdots + x_n^\perp$ is a direct sum.*

Proof. We will prove the following implications : (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (3) $\iff$ (5).

- The implication (1) $\Rightarrow$ (2) is proved in [21].
- (2) $\Rightarrow$ (3). We will prove that $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$ by induction on n. Since $H^*(M(\mathcal{A}); \mathbb{Q})$ is a Poincaré duality algebra, proposition 6.5 implies that the model $D(\mathcal{A})$ is $(\Lambda(\{x_1\}, \ldots, \{x_n\}), 0)$. So, we have : $\{x_1\} \cdot \{x_2\} \neq 0$. This means that $\operatorname{codim} x_1 + \operatorname{codim} x_2 = \operatorname{codim}(x_1 \cap x_2)$.

  Now, suppose $\operatorname{codim} \cap_{i=1}^k x_i = \sum_{i=1}^k \operatorname{codim} x_i$ for $k < n$. The product $\{x_1, \ldots, x_k\} \cdot \{x_{k+1}\} \neq 0$, so

  $$\operatorname{codim}(\cap_{i=1}^k x_i) + \operatorname{codim} x_{k+1} = \operatorname{codim}(\cap_{i=1}^{k+1} x_i).$$

  This relation combined with the induction hypothesis proves that (3) holds for any n.
- (3) $\Rightarrow$ (4). If $\cap_{i=1}^n x_i \neq 0$, let's consider the quotient map $p \colon \mathbb{C}^m \to \mathbb{C}^m/(\cap x_i)$. It induces a homotopy equivalence

  $$(\mathbb{C}^m \setminus \cup x_i) \to ((\mathbb{C}^m / \cap x_i) \setminus \cup (x_i / \cap x_i)).$$

Hence we can assume that $\cap x_i = 0$. With that assumption, the condition (3) is simply $\sum \operatorname{codim} x_i = m$. Each subspace $x_i \in \mathcal{A}$ is the kernel of some linear map : $x_i = \ker(\alpha_i \colon \mathbb{C}^m \to \mathbb{C}^{\operatorname{codim} x_i})$, for $i = 1, \dots, n$. Define

$$\alpha \colon \mathbb{C}^m \to \prod \mathbb{C}^{\operatorname{codim} x_i}; p \mapsto (\alpha_1(p), \dots, \alpha_n(p)).$$

The map α is a linear map between vector spaces. Suppose that $p \in \ker \alpha$, then $p \in \ker \alpha_i$ for all $i = 1, \dots, n$. It implies that p is in every subspace x_i, so $p \in \cap_{i=1}^n x_i$. But $\cap_{i=1}^n x_i = 0$, so $p = 0$ and the map is injective.

But we have $\dim \prod \mathbb{C}^{\operatorname{codim} x_i} = \sum \operatorname{codim} x_i = m$, so the map α is surjective. We proved that α is an isomorphism. Therefore,

$$M(\mathcal{A}) \cong \alpha(M(\mathcal{A})) \cong \prod_{i=1}^n \left(\mathbb{C}^{\operatorname{codim} x_i} \setminus \{0\}\right) \cong \prod_{i=1}^n S^{2\operatorname{codim}(x_i)-1}.$$

- (4) $\Rightarrow$ (1). If $M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres, it has the rational homotopy type of a product of odd dimensional spheres, which is an elliptic space. So, $M(\mathcal{A})$ is elliptic.
- (3) $\iff$ (5). Proved in lemma 6.6. □

As a very quick consequence, we get an easy test of hyperbolicity.

Corollary 6.8. *Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a complex subspace arrangement in $\mathbb{C}^m$ with a geometric lattice and such that $\operatorname{codim} x_i \geq 2$ for all i. If $n > m$, then $M(\mathcal{A})$ is rationally hyperbolic.*

Proof. Because of corollary 2.21, the codimension condition implies that $M(\mathcal{A})$ is simply connected. If $n > m$, then $\sum_{i=1}^n \operatorname{codim}(x_i) > \operatorname{codim}(\cap_{i=1}^n x_i)$. So, the condition (3) of theorem 6.7 does not hold and $M(\mathcal{A})$ is not rationally elliptic. □

6.4 Coordinate subspace arrangements

The proof of theorem 6.7 uses the geometric property of the lattice in a few important places. Generalizing these proofs to the general case seems to be a hard problem (it is the conjecture 6.1). However, it is doable in a particular case : the coordinate subspace arrangements, and this is what this section is about.

The differences between arrangements with a geometric lattice and coordinate subspace arrangements explain the need for different methods of proof. The proof in the geometric case is based on the model $D(\mathcal{A})$, and on some nice properties of the lattice. What we will use in the case of coordinate subspace arrangements is the rational model $BP(\mathcal{A})$ from section 3.2.

The main result of this section is theorem 6.10, which is very close to theorem 6.7 but slightly weaker (we don't prove that $M(\mathcal{A})$ having Poincaré duality is equivalent to the other conditions).

A close look to the proof of theorem 6.7 reveals that most of it doesn't depend on the geometricity of the lattice. Actually, the only step specific to geometric lattices is the fact that

$$\text{Poincaré duality} \Rightarrow \operatorname{codim} \cap_{i=1}^{n} x_i = \sum_{i=1}^{n} \operatorname{codim} x_i.$$

It took the entire section 6.2 to prove it in the case of subspace arrangements with a geometric lattice. No proof of this implication is known in the general case, nor in the specific case of coordinate subspaces arrangements. However, using the rational model $BP(\mathcal{A})$, we can prove that

$$M(\mathcal{A}) \text{ is rationally elliptic} \Rightarrow \operatorname{codim} \cap_{i=1}^{n} x_i = \sum_{i=1}^{n} \operatorname{codim} x_i.$$

This will lead us to a weaker version of theorem 6.7. But let's start by looking at the codimension condition.

Remember that there is a bijection between the subsets of $[m] = \{1, \dots, m\}$ and the coordinate subspaces in $\mathbb{C}^m$:

$$c \colon \mathcal{P}[m] \to \mathrm{Coord}_m; \{i_1, \dots, i_r\} \mapsto \{z_{i_1} = \ldots = z_{i_r} = 0\}.$$

Let $I, J \subseteq [m]$. Clearly, we have $\operatorname{codim} c(I) = |I|$ and $c(I) \cap c(J) = c(I \cup J)$. So, the statement $\operatorname{codim} \cap_{x\in\mathcal{A}} x = \sum_{x\in\mathcal{A}} \operatorname{codim} x$ is equivalent to $\sum_{I\in c^{-1}(\mathcal{A})} |I| = |\cup_{I\in c^{-1}(\mathcal{A})} I|$ where $c^{-1}(\mathcal{A})$ is the set

$$c^{-1}(\mathcal{A}) = \{I \subseteq [m] : c(I) \in \mathcal{A}\}.$$

In other words, it is equivalent to the statement that the elements of $c^{-1}(\mathcal{A})$ are pairwise disjoint.

Lemma 6.9. *Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a coordinate subspace arrangement without hyperplane. If $M(\mathcal{A})$ is rationally elliptic, then* $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$.

Proof. Let $I_i = c^{-1}(x_i)$. So the set $c^{-1}(\mathcal{A})$ is $\{I_1, \dots, I_n\}$. If $I = \{i_1, \dots, i_r\}$, let's denote the monomial $x_{i_1} \cdots x_{i_r}$ by x_I. So, by definition, the ideal I_K is generated by the monomials $\{x_I : I \in c^{-1}(\mathcal{A})\}$.

As explained above, we just need to prove that the ellipticity of $M(\mathcal{A})$ implies that the elements of the set $c^{-1}(\mathcal{A})$ are pairwise disjoint.

Suppose that $M(\mathcal{A})$ is elliptic and that there exists two non-disjoint elements in $c^{-1}(\mathcal{A})$. We can assume without loss of generality that $I_1 \cap I_2 \neq \emptyset$. Let $I_1 = \{i_1, \dots, i_r, i_{r+1}, \dots, i_s\}$ and $I_2 = \{i_1, \dots, i_r, i_{s+1}, \dots, i_t\}$ in $c^{-1}(\mathcal{A})$.

Since $M(\mathcal{A})$ is elliptic, $\mathcal{Z}_K$ is also elliptic and lemma 3.24 tells us that $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} < \infty$. By proposition 3.19, the space $DJ(K)$ is formal. Since we know its cohomology (again by proposition 3.19), we can compute a minimal model from it, and the process described in section 1.2 will stop at the second step by theorem 1.20. So, we obtain the following minimal model of $DJ(K)$:

$$(\Lambda(V_0 \oplus V_1), d) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), d)$$

with $dx_i = 0$ and $dy_i = x_{I_i}$. As usual, this model is bigraded. We will only consider the lower grading, induced by the decomposition $V = V_0 \oplus V_1$. Since $d(V_0) = 0$ and $d(V_1) \subseteq \Lambda V_0$, this lower grading is compatible with the differential (which decreases the degree by one). Consider $\alpha = y_1 x_{i_{s+1}} \cdots x_{i_t} - y_2 x_{i_{r+1}} \cdots x_{i_s} \in \Lambda(V)$. A quick computation shows that it is a cycle :

$$d(\alpha) = x_{I_1} x_{i_{s+1}} \cdots x_{i_t} - x_{I_2} x_{i_{r+1}} \cdots x_{i_s} = 0.$$

But $\alpha \in (\Lambda V)_1$ and the cohomology of $(\Lambda V, d)$ is concentrated in (lower) degree 0. Therefore, there exists a $\beta \in (\Lambda V)_2$ such that $d\beta = \alpha$. We can write β as a sum $\sum_{i,j} a_{ij} y_i y_j$ with some coefficients $a_{ij} \in \mathbb{Q}[x_1, \dots, x_m]$. We have :

$$\begin{aligned} \alpha &= y_1 x_{i_{s+1}} \cdots x_{i_t} - y_2 x_{i_{r+1}} \cdots x_{i_s} = d\beta \\ &= \sum_{i,j} a_{ij}(dy_i \cdot y_j - y_i \cdot dy_j) = \sum_{i,j} (a_{ij} x_{I_i}) y_j - (a_{ij} x_{I_j}) y_i. \end{aligned}$$

So, $x_{i_{s+1}} \cdots x_{i_t}$ is a multiple of some x_{I_j}. It implies that $I_j \subseteq \{i_{s+1}, \dots, i_t\} \subsetneq I_2$. But applying the bijection c implies that $c(I_j) \subsetneq c(I_2)$, which is a contradiction, since we supposed that no subspace of $\mathcal{A}$ contain another subspace of $\mathcal{A}$. Therefore the elements of $c^{-1}(\mathcal{A})$ are pairwise disjoints and $\operatorname{codim} \cap_{x\in\mathcal{A}} x = \sum_{x\in\mathcal{A}} \operatorname{codim} x$. □

This lemma is the only missing step to prove the following theorem, which is slightly weaker than theorem 6.7.

Theorem 6.10. *Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a complex coordinate subspace arrangement. Then the following conditions are equivalent :*

1. *$M(\mathcal{A})$ is a nilpotent rationally elliptic space,*
2. $\operatorname{codim} \cap_{i=1}^n x_i = \sum_{i=1}^n \operatorname{codim} x_i$,
3. *$M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres,*
4. *the sum $x_1^\perp + \cdots + x_n^\perp$ is a direct sum.*

Proof. In the proof of theorem 6.7, we showed that for any subspace arrangements, (2) ⇒ (3) ⇒ (1) and (2) ⇔ (4), so we only need to show that (1) ⇒ (2) in the particular case of coordinate arrangements.

If the arrangement $\mathcal{A}$ contains k hyperplanes, then $M(\mathcal{A})$ decomposes as $M(\mathcal{A}') \times (\mathbb{C}^*)^k$ for some arrangement $\mathcal{A}'$ in $\mathbb{C}^{m-k}$. Obviously, $C^* \cong S^1$ and $\mathcal{A}$ is rationally elliptic if and only if $\mathcal{A}'$ is rationally elliptic. So, we can suppose that $\mathcal{A}$ does not contain any hyperplane. In that case, the lemma 6.9 shows that the conditions (1) ⇒ (2). □

As we will see later, it is also true for coordinate subspace arrangements that these conditions are equivalent to the fact that $M(\mathcal{A})$ is a Poincaré duality space. It is a consequence of lemma 7.10 and proposition 7.11.

Here again, we obtain a simple test of hyperbolicity :

Corollary 6.11. *Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a coordinate subspace arrangement in $\mathbb{C}^m$ such that $\operatorname{codim} x_i \geq 2$ for all i. If $n > m$, then $M(\mathcal{A})$ is rationally hyperbolic.*

Proof. Because of corollary 2.21, the codimension condition implies that $M(\mathcal{A})$ is simply connected. If $n > m$, then $\sum_{i=1}^{n} \operatorname{codim}(x_i) > \operatorname{codim}(\cap_{i=1}^{n} x_i)$. So, the condition (2) of theorem 6.10 does not hold and $M(\mathcal{A})$ is not rationally elliptic. $\square$

CHAPTER 7

Hyperbolicity of $M(\mathcal{A})$

7.1 Introduction

In the previous chapter, we discussed the situation where $M(\mathcal{A})$ is rationally elliptic. Some of the equivalent conditions of theorems 6.7 and 6.10 are very strong. Clearly, few arrangements are such that $\mathbb{C}^m$ decomposes as a direct sum $x_1^\perp \oplus \cdots \oplus x_n^\perp$. So, it means that most arrangements are not rationally elliptic.

In the elliptic case (and if the lattice is geometric or if we deal with a coordinate subspace arrangement), we proved that the complement space $M(\mathcal{A})$ is a product of odd-dimensional spheres, so its homotopy Lie algebra is finitely generated and abelian. The hyperbolic case is much more complicated, as suggested by the following conjecture about rationally hyperbolic spaces, by Avramov - Félix.

Conjecture 7.1. *If X is a nilpotent finite dimensional, rationally hyperbolic space, then the homotopy Lie algebra $L_X = \pi_*(\Omega X)) \otimes \mathbb{Q}$ contain a free Lie subalgebra on two generators.*

Some computations have been made. In [28], J. Roos gives some examples of arrangements of hyperplane such that $L_{D(\mathcal{A})}$ is not finitely generated. In some cases, $L_{D(\mathcal{A})}$ can be described more precisely. For example, in [12], Denham and Suciu proved that if $\mathcal{A}$ is an arrangement of hyperplanes which verify some technical conditions, then $L_{D(\mathcal{A})}$ splits as a semi-direct product of a Lie algebra and a free Lie algebra.

Remark that in the previous paragraph, we mentioned some results about $L_{D(\mathcal{A})}$ instead of $L_{M(\mathcal{A})}$. The reason is that $M(\mathcal{A})$ is not necessarily nilpotent if $\mathcal{A}$ is an arrangement of hyperplanes, so we cannot interpret $L_{D(\mathcal{A})}$ as $\pi_*(\Omega M(\mathcal{A})) \otimes \mathbb{Q} = L_{M(\mathcal{A})}$.

The main results of this chapter are theorems 7.14 and 7.18. In the case of arrangements with a geometric lattice or coordinate subspace arrangement, the following conditions are equivalent :

1. $H^*(M(\mathcal{A});\mathbb{Q})$ is not an exterior algebra,

2. there exists an injective map $\mathbb{L}(u,v) \to L_{D(\mathcal{A})}$.

So, if we also take into account the results of the previous chapter, this result means that $M(\mathcal{A})$ is rationally hyperbolic if and only if $L_{M(\mathcal{A})}$ contains a free Lie subalgebra on two generators. This is a particular case of the conjecture 7.1

The proof of theorem 7.14 is quite technical. We will need the next three sections just to properly develop the various concepts needed. Section 7.2 is the most technical : it study various cdgas and their homotopy Lie algebras. Section 7.3 introduces and study the map $\varphi\colon \Lambda(e_1,\ldots,e_n) \to H^*(M(\mathcal{A});\mathbb{Q}); e_i \mapsto [\{x_i\}]$: this map is injective if and only if $M(\mathcal{A})$ is rationally elliptic. Then, section 7.4 contains the proof of theorem 7.14. As usual, we have a different proof in the case of the coordinate subspace arrangements, and this proof is presented in the last section.

Note that the various results presented in this chapter in the geometric case have been published in Algebraic & Geometric Topology, see [11].

Finally, let's introduce an useful notation :

Notation 7.2. For every $\{i_1,i_2,\ldots,i_{r+1}\}_< \subseteq [n]$, we denote by $[[e_{i_1},\ldots,e_{i_{r+1}}]]$ the following element of $\Lambda(e_1,\ldots,e_n)$:

$$[[e_{i_1},\ldots,e_{i_{r+1}}]] := \sum_{j=1}^{r+1}(-1)^j e_{i_1}\cdots\widehat{e_{i_j}}\cdots e_{i_{r+1}}.$$

7.2 Technical lemmas

This section contains the proofs of a few useful lemmas. These lemmas are technical and proving them require some preliminary work on various differential graded algebras. More precisely, we will need the following algebras :

$$(A_1,0)=\left(\frac{\Lambda(e_{i_1},\dots,e_{i_r})}{e_{i_1}e_{i_2}\cdots e_{i_r}},0\right)\qquad (A_2,0)=\left(\frac{\Lambda(e_{j_1},\dots,e_{j_\ell})}{I},0\right)$$
$$(A_3,d)=\left(\frac{\Lambda(e_{i_1},\dots,e_{i_r},t,a)}{te_{i_1},\dots,te_{i_r},t^2},d\right)\quad (A_4,d)=\left(\frac{\Lambda(e_{i_1},\dots,e_{i_r},t)}{te_{i_1},\dots,te_{i_r},t^2},d\right)$$

$$(A_5,0)=\Big(\Lambda(e_{i_2},\dots,e_{i_r})\oplus(\oplus_{s\geq 1}\mathbb{Q}u_s),0\Big)$$

In A_5, the degree of u_s is $|u_s|=\sum_{i=1}^r|e_{i_r}|+(s-1)|e_{i_1}|-s$ and the products $u_se_{i_j}=0$ and $u_su_{s'}=0$ for all s,s' and j. In (A_3,d), the differential is defined by $de_{i_j}=0, dt=e_{i_1}\cdots e_{i_r}$ and $da=e_{i_1}$. In (A_4,d), the differential is $de_{i_j}=0, dt=e_{i_1}\cdots e_{i_r}$. Finally, in A_2, I is the ideal of $\Lambda(e_{j_1},\dots,e_{j_\ell})$ generated by the elements $e_{i_1}\cdots e_{i_{r+1}}$ and $[[e_{i_1},\dots,e_{i_{r+1}}]]$. Notice that the differentials on (A_3,d) and on (A_4,d) increases the degree by one, so the degrees of a and t are well defined.

We will prove six lemmas.

Lemma 7.3. *There exists a quasi-isomorphism* $(A_5,0)\xrightarrow{\simeq}(A_3,d)$. *Also, the projection* $(A_4,d)\xrightarrow{\simeq}(A_1,0)$ *is a quasi-isomorphism.*

Proof. It is clear that the projection $(A_4,d)\to(A_1,0)$ is a quasi-isomorphism, because the kernel is generated (as a vector space) by the elements t and $dt=e_{i_1}\cdots e_{i_r}$. Let's define a quasi-isomorphism $\theta\colon(A_5,0)\to(A_3,d)$. Consider the subalgebra

$$(B,d)=(\Lambda(e_{i_1},\dots,e_{i_r},a),d)$$

of (A_3,d). Since $d(A_3)\subset B$, the differential in A_3/B is zero. Therefore, we have a short exact sequence of complexes :

$$0\to(B,d)\to(A_3,d)\to(A_3/B,0)\to 0,$$

and a long exact sequence in cohomology with $A_3/B=\oplus_{s\geq 0}\mathbb{Q}ta^s$. By the connecting map, an element ta^s of $H^*(A_3/B,0)$ is sent on the cohomology class of $d(ta^s)=e_{i_1}\cdots e_{i_r}a^s$ in B. But

$$d(\frac{1}{s+1}e_{i_2}\cdots e_{i_r}a^{s+1})=e_{i_1}\cdots e_{i_r}a^s.$$

Therefore, the connecting map is zero. It means that we have a short exact sequence in cohomology.

$$0 \to H^*(B,d) \to H^*(A_3,d) \to H^*(A_3/B,0) \to 0.$$

The cohomology of (B,d) is clearly $\Lambda(e_{i_2},\dots,e_{i_r})$ and the cohomology of $(A_3/B,0)$ is A_3/B. Consider the map $\theta\colon (A_5,0) \to (A_3,d)$ defined by $\theta(e_{i_j}) = e_{i_j}$, $j=2,\dots,n$ and $\theta(u_s) = \frac{a^s}{s} e_{i_2}\cdots e_{i_r} - a^{s-1}t$. It is a morphism of differential graded algebras. This gives us the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Lambda(e_{i_2},\dots,e_{i_r}) \arrow[r] \arrow[d, "\simeq"] & \Lambda(e_{i_2},\dots,e_{i_r}) \oplus \oplus_{s\geq 1}\mathbb{Q}u_s \arrow[r] \arrow[d, "H^*\theta"] & \oplus_{s\geq 1}\mathbb{Q}u_s \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^*(B,d) \arrow[r] & H^*(A_3,d) \arrow[r] & H^*(A_3/B,0) \arrow[r] & 0
\end{tikzcd}$$

By the 5-lemma, the map $H^*\theta$ is an isomorphism, or in other words, θ is a quasi-isomorphism. □

Lemma 7.4. *Let $m\colon (\Lambda V,d) \to (A_4,d)$ and $m'\colon (\Lambda W,d) \to (A_3,d)$ be two Sullivan minimal models. Let $f\colon (\Lambda V,d) \to (\Lambda W,d)$ be a minimal model of the injection $(A_4,d) \to (A_3,d)$. Then $Qf\colon V \to W$ is surjective.*

Proof. The map Qf is the linear part of f (see definition 1.11). Let $(v_1,v_2,\dots)$ be a basis of V. Since $de_{i_1}=0$, rational homotopy theory shows that we can construct the map m with the property that $m(v_1)=e_{i_1}$. We form then the relative Sullivan model : $(\Lambda V\otimes\Lambda a,d)$ with $da=v_1$. The map $m\otimes\mathrm{id}\colon (\Lambda V\otimes\Lambda a,d) \to (A_4\otimes\Lambda a,d)$ extends the map m and makes commutative the following diagram.

$$\begin{tikzcd}
(\Lambda V,d) \arrow[r, hook, "i"] \arrow[d, "m"'] & (\Lambda V\otimes\Lambda a,d) \arrow[d, "m\otimes\mathrm{id}"] \\
(A_4,d) \arrow[r, hook, "j"] & (A_4\otimes\Lambda a,d) = (A_3,d)
\end{tikzcd}$$

Since m is a quasi-isomorphism, $m\otimes\mathrm{id}$ is also a quasi- isomorphism (see lemma 14.2 of [16]). So $m\otimes\mathrm{id}$ is a Sullivan model of the map $j\circ m$.

The relative Sullivan algebra $(\Lambda V\otimes\Lambda a,d)$ is a Sullivan algebra, and almost minimal : to make it minimal, we only need to divide by the ideal generated by a and v_1. The projection map $p\colon (\Lambda V \otimes \Lambda a, d) \to (\Lambda(v_2, v_3, \ldots), d)$ is such a quasi-isomorphism. So, $(\Lambda(v_2, v_3, \ldots), d)$ is a minimal model of (A_3, d). We conclude by letting $f = p \circ i$. The map f is such that the linear map Qf is simply the projection $V \to V/v_1$, which is clearly surjective. □

Lemma 7.5. *Let $(\Lambda V, d)$ be a minimal algebra and $f\colon (\Lambda V, d) \to (E, d)$ be a quasi-isomorphism. If there exists $x, y \in V$ such that x and y are linearly independent, $dx = dy = 0$ and $f(xy) = f(x^2) = f(y^2) = 0$, then there exists two morphisms of Lie algebras $\mathbb{L}(u, v) \xrightarrow{i} L_{(\Lambda V, d)} \xrightarrow{p} \mathbb{L}(u, v)$ such that $p \circ i = \mathrm{id}$. In particular, i is injective.*

Proof. Consider the differential graded algebra $(B, 0) = (\mathbb{Q} \oplus \mathbb{Q}x' \oplus \mathbb{Q}y', 0)$ where all products are equal to zero and $|x'| = |x|, |y'| = |y|$. We can define a morphism of cdga $\theta\colon (B, 0) \to (E, d)$ with $\theta(x') = f(x)$ and $\theta(y') = f(y)$.

Notice that $(B, 0)$ is a model of a wedge of two spheres. Its minimal Sullivan model $\rho\colon (\Lambda W, d) \to (B, 0)$ is such that $L_{(\Lambda W, d)} = \mathbb{L}(u, v)$ with $|u| = |x'|-1$ and $|v| = |y'|-1$. Without loss of generality, we can assume that $|x'| \leq |y'|$.

The construction of the Sullivan minimal model is realised by an inductive process. Looking closely at this construction, we can easily (in low degree) construct a basis for W.

- If $|x'|$ is odd or if $|x'| = |y'|$, then $\Lambda W = \Lambda(x', y', t, \ldots)$ with $dt = x'y'$. In degree $\leq |y'|$, W has only two generators : x', y'.
- If $|x'|$ is even and if $|x'| < |y'|$, then $\Lambda W = \Lambda(x', y', t_1, t_2, \ldots)$ with $dt_1 = x'^2$ and $dt_2 = x'y'$.

Let's construct a map $\psi\colon (\Lambda W, d) \to (\Lambda V, d)$. By the lifting lemma, such a map can be obtained by lifting $\theta \circ \rho$ along f. Since the lift ψ is constructed inductively along a basis of W, we can choose $\psi(x') = x$ and $\psi(y') = y$.

$$\begin{tikzcd} & (\Lambda V, d) \arrow[d, "f"] \\ (\Lambda W, d) \arrow[ur, dashed, "\psi"] \arrow[r, "\theta\circ\rho"'] & (E, d) \end{tikzcd}$$

Now, let's look at the induced map $L_{(\Lambda V, d)} \to L_{(\Lambda W, d)} = \mathbb{L}(u, v)$.

- If $|x'|$ is odd or if $|x'| = |y'|$, then the linear map $Q\psi\colon W \to V$ is injective in degree $\leq |y'|$ (it is completely described by $Q\psi(x') = x$ and $Q\psi(y') = y$). So, the dual map is surjective. It implies that $L\psi\colon L_{(\Lambda V,d)} \to \mathbb{L}(u,v)$ is surjective in degree $\leq |v|$, which means that u and v are in the image of $L\psi$.

- If $|x'|$ is even and if $|x'| < |y'|$, then we can do exactly the same construction if $|t_1| > |y'|$. If $|t_1| \leq |y'|$, then there is a slight difference. In that case, $x^2 = \psi(x'^2) = \psi(dt_1) = d\psi(t_1)$. So, x^2 is a boundary. There is a $z \in V$ such that $x^2 = dz$. The map $Q\psi$ in degree $\leq |y'|$ is completely described by $Q\psi(x') = x, Q\psi(y') = y$ and $Q\psi(t_1) = z$. It is injective in degree $\leq |y'|$. So, the dual map is surjective in degree $\leq |v|$, which also means that u and v are in the image of $L\psi$.

In both cases, the map $L\psi\colon L_{(\Lambda V,d)} \to \mathbb{L}(u,v)$ has u and v in its image. Therefore, we can choose $a, b \in L_{(\Lambda V,d)}$ such that $L\psi(a) = u$ and $L\psi(b) = v$. Let $p = L\psi$ and consider the map $i\colon \mathbb{L}(u,v) \to L_{(\Lambda V,d)}$ defined by $i(u) = a$ and $i(v) = b$. These two maps verify $p \circ i = \mathrm{id}$. □

Now, the preliminary work is done. The main lemmas of this section can be proved.

Lemma 7.6. *There exists an injective map $\mathbb{L}(u,v) \to L_{(A_1,0)}$.*

Proof. Let $L_1 = L_{(A_5,0)}$ and $L_2 = L_{(A_1,0)}$. The proof will be done by showing the existence of two injective maps

$$\mathbb{L}(u,v) \xrightarrow{g_1} L_1 \xrightarrow{g_2} L_2.$$

Step 1 : constructing the map g_2. By lemma 7.3, $(A_5,0) \xrightarrow{\simeq} (A_3,d)$ and $(A_1,0) \xrightarrow{\simeq} (A_4,d)$, so $L_1 = L_{(A_3,0)}$ and $L_2 = L_{(A_4,d)}$. The lemma 7.4 gives us a map $f\colon (\Lambda V,d) \to (\Lambda W,d)$ between the Sullivan minimal models of (A_4,d) and (A_3,d). Applying the homotopy Lie algebra functor to the map gives a map $Lf\colon L_1 \to L_2$. The surjectivity of Qf implies that Lf is injective (see [16], chapter 21). Now, $g_2 = Lf$ is the required map.

Step 2 : constructing the map g_1. By lemma 7.5, we only need to show that if $m\colon (\Lambda V,d) \to (A_5,0)$ is a Sullivan minimal model, then there exists $x, y \in V$ such that x, y are linearly independent, $dx = dy = 0$ and $m(xy) = m(x^2) = m(y^2) = 0$.

Since m is a quasi-isomorphism, the map $H^*m\colon H^*(\Lambda V, d) \to (A_5, 0)$ is an isomorphism. So, there exists $[x]$ and $[y]$ in $H^*(\Lambda V, d)$ such that $H^*m([x]) = e_{i_2}$ and $H^*m([y]) = u_1$. It gives us x and y in $(\Lambda V, d)$ such that $dx = dy = 0$, $m(x) = e_{i_2}$ and $m(y) = u_1$. But x and y can not be in $\Lambda^{\geq 2} V$ because, otherwise, $e_{i_2} = m(x)$ would be in $\Lambda^{\geq 2}(e_{i_2}, \ldots, e_{i_r})$ and $u_1 = m(y)$ would be in $\Lambda^{\geq 2}(u_1)/u_1^2$. Therefore, x and y are in V. Finally, the lemma 7.5 gives us the map g_1. $\square$

Lemma 7.7. *There exists an injective map $\mathbb{L}(u, v) \to L_{(A_2,0)}$.*

Proof. Recall that A_2 is the quotient of $\Lambda(e_{j_1}, \ldots, e_{j_\ell})$ by the ideal I, which is generated by the elements $e_{i_1} \cdots e_{i_{r+1}}$ and $[[e_{i_1}, \ldots, e_{i_{r+1}}]]$. So, A_2 is quite simple, except in degree r. The first part of the proof is to find a basis to the vector subspace of $\Lambda(e_{j_1}, \ldots, e_{j_\ell})$ generated by every $[[e_{i_1}, \ldots, e_{i_{r+1}}]]$.

Let's start with some notations. Let $J_1 = \{(i_2, \ldots, i_r) : 1 < i_2 < i_3 < \cdots < i_r \leq n\}$ and $d_1 = \# J_1$. Also, let $J_2 = \{(i_1, \ldots, i_r) : 1 < i_1 < i_2 < \cdots < i_r \leq n\}$ and $d_2 = \# J_2$. If $1 < i_1 < \cdots < i_{r+1} \leq n$, then

$$\sum_{j=1}^{r+1} (-1)^{j+1} [[e_1, e_{i_1}, \ldots, \widehat{e_{i_j}}, \ldots, e_{i_{r+1}}]] =$$

$$\begin{aligned} &\sum_{j=1}^{r+1} (-1)^{j+1} \Big(- e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_{r+1}} + \\ &\sum_{k=1}^{j-1} (-1)^{k+1} e_1 e_{i_1} \cdots \widehat{e_{i_k}} \cdots \widehat{e_{i_j}} \cdots e_{i_{r+1}} \\ &+ \sum_{k=j+1}^{r+1} (-1)^{k} e_1 e_{i_1} \cdots \widehat{e_{i_j}} \cdots \widehat{e_{i_k}} \cdots e_{i_{r+1}} \Big) \\ &= \sum_{j=1}^{r+1} (-1)^{j} e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_{r+1}} = [[e_{i_1}, \ldots, e_{i_{r+1}}]]. \end{aligned}$$

So, the vector space generated by every $[[e_{i_1}, \ldots, e_{i_{r+1}}]]$ and the vector space generated by $[[e_1, e_{i_2}, \ldots, e_{i_{r+1}}]]$ with $(i_2, \ldots, i_{r+1}) \in J_2$ are exactly the same. Also, it implies that $\dim \operatorname{span}\langle [[e_{i_1}, \ldots, e_{i_{r+1}}]] \rangle \leq d_2$.

Let $e_{i_1}\cdots e_{i_r} \in A_2^r$, with $i_1 < \cdots < i_r$. If $i_1 = 1$, then it is trivially a linear combination of elements $e_1 e_{j_2}\cdots e_{j_r}$. If $i_1 > 1$, then from $[[e_1, e_{i_1}, \dots, e_{i_r}]] = 0$ in A_2, we immediately deduce that $e_{i_1}\cdots e_{i_r}$ is a linear combination of some $e_1 e_{j_2}\cdots e_{j_r}$ with $(j_2, \dots, j_r) \in J_1$. So, $\dim A_2^r \leq d_1$.

Notice that d_1 is the dimension of the vector space generated by the elements $e_1 e_{i_2}\cdots e_{i_r}$, with $1 < i_2 < \cdots < i_r$ and d_2 is the dimension of the vector space generated by the elements $e_{i_1} e_{i_2}\cdots e_{i_r}$ with $1 < i_1 < \cdots < i_r$. So we have : $\dim \Lambda^r(e_{j_1}, \dots, e_{j_\ell}) = d_1 + d_2$. Let's consider the following short exact sequence

$$0 \to \mathrm{span}\langle [[e_{i_1}, \dots, e_{i_{r+1}}]]\rangle \to \Lambda^r(e_{j_1}, \dots, e_{j_\ell}) \to A_2^r \to 0.$$

We have

$$\begin{aligned} d_1 + d_2 &= \dim \Lambda^r(e_{j_1}, \dots, e_{j_\ell}) \\ &= \dim A_2^r + \dim \mathrm{span}\langle [[e_{i_1}, \dots, e_{i_{r+1}}]]\rangle. \end{aligned}$$

But we proved that $\dim A_2^r \leq d_1$ and $\dim\langle [[e_{i_1}, \dots, e_{i_{r+1}}]]\rangle \leq d_2$. So, $\dim A_2^r = d_1$ and $\dim \mathrm{span}\langle [[e_{i_1}, \dots, e_{i_{r+1}}]]\rangle = d_2$. Therefore the elements $[[e_1, e_{i_2}, \dots, e_{i_{r+1}}]]$ with $(i_2, \dots, i_{r+1}) \in J_2$ form a basis of $\mathrm{span}\langle [[e_{i_1}, \dots, e_{i_{r+1}}]]\rangle$.

Let (B, d) the differential graded algebra defined by

$$B = \frac{\Lambda(e_{j_1}, \dots, e_{j_\ell})}{\Lambda^{>r}(e_{j_1}, \dots, e_{j_\ell})} \oplus \left(\oplus_{i\in J_2} \mathbb{Q} a_i\right),$$

$d(a_i) = [[e_1, e_{i_2}, \dots, e_{i_{r+1}}]]$ and the product in B is defined by $a_i \cdot a_j = a_i \cdot e_j = 0$. The ideal generated by the a_i and the da_i is acyclic, and the quotient map is a quasi-isomorphism $(B, d) \xrightarrow{\simeq} (A_2, 0)$.

Therefore, the differential graded algebras $(A_2, 0)$ and (B, d) have the same minimal model. Let $\theta\colon (\Lambda W, d) \to (B, d)$ be the minimal model of (B, d). Looking at the construction of $(\Lambda W, d)$, we see that W has a basis that start with $e_1, \dots, e_\ell$ and the $(a_i)_{i\in J_2}$, with $\theta(e_i) = e_i$ and $\theta(a_i) = a_i$. Because $\theta(e_i^2) = \theta(e_i a_j) = \theta(a_j^2) = 0$, the lemma 7.5 shows that $L_{(\Lambda W, d)} = L_{(A_2, 0)}$ contains a Lie subalgebra $\mathbb{L}(u, v)$. $\square$

Lemma 7.8. *Let A and B be two commutative graded algebras such that $A^0 = B^0 = \mathbb{Q}$. If there exists two maps $f\colon A \to B$ and $g\colon B \to$*

A such that $g \circ f = \mathrm{id}_A$, then there exists two morphisms of homotopy Lie algebras $\tilde{f}\colon L_{(B,0)} \to L_{(A,0)}$ and $\tilde{g}\colon L_{(A,0)} \to L_{(B,0)}$ such that $\tilde{f} \circ \tilde{g}$ is an isomorphism. In particular, $\tilde{g}$ is an injective map.

Proof. Let $(\Lambda V, d) \xrightarrow{m} (A, 0)$ and $(\Lambda V', d') \xrightarrow{m'} (B, 0)$ be the minimal Sullivan models of $(A, 0)$ and $(B, 0)$ respectively. Since these maps are quasi-isomorphisms, they are surjective. The lifting lemma shows that there exists maps $\bar{f}$ and $\bar{g}$ such that $m' \circ \bar{f} = f \circ m$ and $m \circ \bar{g} = g \circ m'$.

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\bar{f}"] \arrow[d, "m"'] & (\Lambda V', d') \arrow[r, "\bar{g}"] \arrow[d, "m'"'] & (\Lambda V, d) \arrow[d, "m"] \\
(A, 0) \arrow[r, "f"'] & (B, 0) \arrow[r, "g"'] & (A, 0)
\end{tikzcd}$$

The maps $\bar{f}$ and $\bar{g}$ verify $m \circ (\bar{g} \circ \bar{f}) = (g \circ f) \circ m = m$. Since $g \circ f = \mathrm{id}_A$ and m a quasi-isomorphism, $\bar{g} \circ \bar{f}$ is also a quasi-isomorphism. But quasi-isomorphisms of minimal Sullivan algebras are isomorphisms (see theorem 14.11 of [16] for a proof).

Applying the homotopy Lie algebra functor L to

$$(\Lambda V, d) \xrightarrow{\bar{f}} (\Lambda V', d') \xrightarrow{\bar{g}} (\Lambda V, d)$$

gives us the maps $\tilde{f} = L\bar{f}$ and $\tilde{g} = L\bar{g}$.

$$L_{(A,0)} \xleftarrow{\tilde{f}} L_{(B,0)} \xleftarrow{\tilde{g}} L_{(A,0)}$$

Since L is a functor, $\tilde{f} \circ \tilde{g} = (L\bar{f}) \circ (L\bar{g}) = L(\bar{g} \circ \bar{f})$ is an isomorphism.

□

7.3 Non-ellipticity of the complement space

Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. Let us consider the following morphism of graded algebras :

$$\varphi\colon \Lambda(e_1, \ldots, e_n) \to H^*(M(\mathcal{A}); \mathbb{Q}); e_i \mapsto [\{x_i\}].$$

As the following lemmas will show, the injectivity of this map can give some information on the model $D(\mathcal{A})$ and on the lattice $L(\mathcal{A})$. We will also use it to prove a criterion for ellipticity.

Lemma 7.9. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. If $\varphi|_{\Lambda^{<r}(e_1,\ldots,e_n)}$ is injective, then every $\sigma \subseteq \mathcal{A}$ with $|\sigma| \leq r$ is such that $d\sigma = 0$ (or, in other words, σ is independant).*

Proof. Suppose that $\sigma = \{x_{i_1}, \ldots, x_{i_s}\}_<$ with $s < r$. By definition of the map φ, we have $\varphi(e_{i_1} \cdots e_{i_s}) = \varphi(e_{i_1}) \cdots \varphi(e_{i_s}) = [\{x_{i_1}\}] \cdots [\{x_{i_s}\}]$. But φ is injective on $\Lambda^{<r}(e_1, \ldots, e_n)$, so we know that $[\{x_{i_1}\}] \cdots [\{x_{i_s}\}] \neq 0$. Therefore, $\{x_{i_1}, \ldots, x_{i_s}\}_<$ is a product of cocycles. Also, $d\{x_{i_1}, \ldots, x_{i_s}\}_< = 0$.

Now, we only need to show the result when $|\sigma| = r$. Suppose that $\sigma = \{x_{i_1}, \ldots, x_{i_r}\}_<$. Then $d\sigma$ is a (possibly zero) linear combination $\sum \rho_j \{x_{j_1}, \ldots, x_{j_{r-1}}\}_<$. We have

$$0 = [d\sigma] = \left[\sum \rho_j \{x_{j_1}, \ldots, x_{j_{r-1}}\}_<\right].$$

But each $\{x_{j_1}, \ldots, x_{j_{r-1}}\}_<$ is a cocycle and

$$\varphi(e_{j_1} \cdots e_{j_{r-1}}) = [\{x_{j_1}, \ldots, x_{j_{r-1}}\}].$$

So,

$$0 = [d\sigma] = \varphi\left(\sum \rho_j e_{j_1} \cdots e_{j_{r-1}}\right).$$

Since φ is injective on $\Lambda^{<r}(e_1, \ldots, e_n)$, $\sum \rho_j e_{j_1} \cdots e_{j_{r-1}} = 0$, which implies that each $\rho_j = 0$ and that $d\sigma = 0$. ☐

Lemma 7.10. *Let $\mathcal{A} = \{x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. Then*

1. *φ is injective if and only if φ is an isomorphism,*

2. *if φ is injective, then $L(\mathcal{A})$ is a geometric lattice.*

Proof. Suppose that φ is injective. Then by lemma 7.9, every $\sigma \subseteq \mathcal{A}$ is such that $d\sigma = 0$, so the differential of $D(\mathcal{A})$ is zero and $H^*(M(\mathcal{A}); \mathbb{Q}) = D(\mathcal{A})$. Also, for each $\sigma = \{x_{i_1}, \ldots, x_{i_s}\}_< \subseteq \mathcal{A}$, the injectivity of φ guarantees that $\{x_{i_1}\} \cdots \{x_{i_s}\} \neq 0$. So we have $\sigma = \{x_{i_1}\} \cdots \{x_{i_s}\} = \varphi(e_{i_1} \cdots e_{i_s})$. This proves that the map φ is surjective, so φ is an isomorphism and the statement (1) is proved.

Now, let us prove the second statement. Suppose that φ is an isomorphism, so the differential of $D(\mathcal{A})$ is zero and $D(\mathcal{A}) =$

$H^*(M(\mathcal{A});\mathbb{Q}) \simeq \Lambda(e_1,\dots,e_n)$. We have an obvious surjective morphism of lattice

$$\mu\colon (\mathcal{P}(\mathcal{A}),\subseteq) \to L(\mathcal{A}); \sigma \mapsto \vee\sigma.$$

Let's show that μ is injective. Suppose that $\vee\sigma = \vee\tau$ and choose $y \in \sigma$. If $y \notin \tau$, then $\{y\}\cdot\tau \neq 0$ (because $D(\mathcal{A}) = \Lambda(e_1,\dots,e_n)$). By definition of the multiplication in $D(\mathcal{A})$, we have $\operatorname{codim} y + \operatorname{codim} \vee\tau = \operatorname{codim}(\vee(\tau\cup\{y\}))$. But $y \le \vee\sigma = \vee\tau$, so $\vee(\tau\cup\{y\}) = \vee\tau$. Together, these facts implies that $\operatorname{codim} y + \operatorname{codim}\vee\sigma = \operatorname{codim}\vee\sigma$, or, in other words, $\operatorname{codim} y = 0$ and $y = \mathbb{C}^m$, which is impossible. Therefore we cannot choose such a y, which means that $\sigma \subseteq \tau$. The exact same reasoning shows that $\tau \subseteq \sigma$, so $\sigma = \tau$ and the map μ is injective.

So the lattices $L(\mathcal{A})$ is isomorphic to $(\mathcal{P}(\mathcal{A}),\subseteq)$. But $(\mathcal{P}(\mathcal{A}),\subseteq)$ is clearly lattice-isomorphic to $(\mathcal{P}\{1,\dots,n\},\subseteq)$, which is geometric (the operations $\vee$ and $\wedge$ are respectively $\cup$ and $\cap$). □

In the case where $L(\mathcal{A})$ is a geometric lattice or if $\mathcal{A}$ is a coordinate subspace arrangement, we can use these lemmas and the main results of chapter 6 to obtain the following criterion for ellipticity.

Proposition 7.11. *Let $\mathcal{A} = \{x_1,\dots,x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. If the lattice $L(\mathcal{A})$ is geometric or if $\mathcal{A}$ is a coordinate subspace arrangement, then the following conditions are equivalent :*

1. *$M(\mathcal{A})$ is a nilpotent rationally elliptic space,*
2. *the map $\varphi\colon \Lambda(e_1,\dots,e_n) \to H^*(M(\mathcal{A});\mathbb{Q})$ is injective.*

Proof. Suppose that $M(\mathcal{A})$ is a nilpotent rationally elliptic space. By theorems 6.7 and 6.10, $M(\mathcal{A})$ has the homotopy type of a product of odd dimensional spheres. From that, we can easily see that $D(\mathcal{A}) = \Lambda(e_1,\dots,e_n)$ and that φ is an isomorphism.

Now, suppose that the map φ is injective. By lemma 7.10, φ is an isomorphism and $L(\mathcal{A})$ is a geometric lattice. So $M(\mathcal{A})$ is a Poincaré duality space and by theorem 6.7, $M(\mathcal{A})$ is a nilpotent rationally elliptic space. □

7.4 Arrangements with a geometric lattice

In this section, we consider the geometric case : $\mathcal{A} = \{x_1,\dots,x_n\}$ is a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice $L(\mathcal{A})$. Also, we will suppose that $\ker\varphi \neq 0$ (if $\ker\varphi = 0$, proposition 7.11 shows that we are in the elliptic case, which has already been studied in chapter 6).

Since $\ker\varphi \neq 0$, we can define the following number : $r = \max\{s : \ker\varphi \subseteq \Lambda^{\geq s}e_i\}$. In some sense, this number measure how close (if r is large) or far (if r is small) the space $M(\mathcal{A})$ is from being elliptic.

The map φ can behave in two different ways : either $\ker\varphi$ contains a monomial $e_{i_1}\cdots e_{i_r}$ or $\ker\varphi$ does not contain such a monomial. The first two lemmas of this section shows that there exists an injective map $\mathbb{L}(u,v) \to L_{D(\mathcal{A})}$ in both cases. Then, using these results, we prove the main theorem 7.14. Remember that

$$(A_1,0) = \left(\frac{\Lambda(e_{i_1},\ldots,e_{i_r})}{e_{i_1}e_{i_2}\cdots e_{i_r}},0\right) \text{ and } (A_2,0) = \left(\frac{\Lambda(e_{j_1},\ldots,e_{j_\ell})}{I},0\right).$$

where I is the ideal of $\Lambda(e_{j_1},\ldots,e_{j_\ell})$ generated by the elements $e_{i_1}\cdots e_{i_{r+1}}$ and $[[e_{i_1},\ldots,e_{i_{r+1}}]]$.

Lemma 7.12. *If $\ker\varphi$ contains a monomial $e_{i_1}\cdots e_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq n$, then there exists an injective map $\mathbb{L}(u,v) \to L_{D(\mathcal{A})}$.*

Proof. We construct a map $\psi\colon D(\mathcal{A}) \to (A_1,0)$: if $\{k_1,\ldots,k_t\}_< \subseteq \{i_1,\ldots,i_r\}$, then $\psi(\{k_1,\ldots,k_t\}) = [e_{k_1}\cdots e_{k_t}]$. Otherwise, we say that $\psi(\{k_1,\ldots,k_t\}) = 0$. Since $\ker\varphi \cap \Lambda^{<r}(e_1,\ldots,e_n) = 0$, a simple check shows that ψ is multiplicative. Lemma 7.9 shows that $\psi(d\sigma) = \psi(0) = 0 = d\psi(\sigma)$. Hence, ψ is a morphism of differential graded algebras.

Since $e_{i_1}\cdots e_{i_r} \in \ker\varphi$, we can define another map $\rho\colon (A_1,0) \to H^*(D(\mathcal{A}))$ by letting $\rho([e_{i_s}]) = [\{x_{i_s}\}]$ and extending it multiplicatively. This is clearly a morphism of graded algebras. Now, we have the following maps :

$$A_1 \xrightarrow{\rho} H^*(D(\mathcal{A})) \xrightarrow{H^*\psi} A_1.$$

Those maps verify the following property : $(H^*\psi)\circ\rho = \mathrm{id}$, which means that $H^*\psi$ is a retraction of ρ. By proposition 3.10, the space $M(\mathcal{A})$ is formal. So, $(H^*(D(\mathcal{A})),0) = (H^*(M(\mathcal{A});\mathbb{Q}),0)$ is connected to $D(\mathcal{A})$ by a sequence of quasi-isomorphisms, which shows that $H^*(D(\mathcal{A}))$ and $M(\mathcal{A})$ have the same homotopy Lie algebra. Lemma 7.8 implies that there exists an injective map $L_{(A_1,0)} \to$

$L_{D(\mathcal{A})}$. By lemma 7.6, there is also an injective map $\mathbb{L}(u, v) \to L_{(A_1,0)}$. The composition of these two maps gives us the needed map. □

Lemma 7.13. *If* $\ker \varphi$ *does not contain any monomial* $e_{i_1} \cdots e_{i_r}$*, then there exists an injective map* $\mathbb{L}(u, v) \to L_{D(\mathcal{A})}$.

Proof. Since $\ker \varphi \cap \Lambda^r(e_1, \dots, e_n) \neq \emptyset$, there exists a non zero linear combination $\sum \lambda_{i_1 \cdots i_r} e_{i_1} \cdots e_{i_r}$ such that $\varphi(\sum \lambda_{i_1 \cdots i_r} e_{i_1} \cdots e_{i_r}) = 0$. So, in cohomology, $[\sum \lambda_{i_1 \cdots i_r}\{x_{i_1}, \dots, x_{i_r}\}] = 0$ in $H^*(D(\mathcal{A}))$ and there exists a $\sigma \in D(\mathcal{A})$ such that $d\sigma = \sum \lambda_{i_1 \cdots i_r}\{x_{i_1}, \dots, x_{i_r}\}$. From this, we deduce that there exists a $(r+1)$-uple $1 \leq i_1 < \dots < i_{r+1} \leq n$ such that $d\{x_{i_1}, \dots, x_{i_{r+1}}\} \neq 0$.

Let $X = x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_{r+1}} \in L(\mathcal{A})$. Since $d\{x_{i_1}, \dots, x_{i_{r+1}}\} \neq 0$, X is equal to $\vee(\{x_{i_1}, \dots, x_{i_{r+1}}\} \setminus x_j)$ for some j. By lemma 7.9, $\{x_{i_1}, \dots, x_{i_{r+1}}\} \setminus x_j$ is independant, so $\operatorname{rk} X = r$. Let $B = \{x \in \mathcal{A} : x < X\} = \{x_{j_1}, \dots, x_{j_\ell}\}$. Lemma 7.9 shows that for any subset $\sigma \subseteq B$ with r elements, $\operatorname{rk} \vee\sigma = r = \operatorname{rk} X$, so $\vee\sigma = X$. If $x_k \in B$, we have $\operatorname{codim} \vee\sigma = \operatorname{codim} \vee(\sigma \cup \{x_k\})$. Since $\operatorname{codim} x_k \neq 0$, the product $\sigma \cdot \{x_k\}$ is necessarily zero. It implies that any $r+1$ product $\prod_{i=1}^{r+1}\{x_{k_i}\} = 0$ for x_{k_i} in B. Consequently, the following map can be defined :

$$\rho\colon \frac{\Lambda(e_{j_1}, \dots, e_{j_\ell})}{\Lambda^{\geq r+1}(e_{j_1}, \dots, e_{j_\ell})} \to H^*(D(\mathcal{A})); e_j \mapsto [\{x_j\}].$$

The next step is to prove that $\ker \rho \subseteq \Lambda^r(e_{j_1}, \dots, e_{j_\ell})$ is a vector subspace generated by the $[[e_{i_1}, \dots, e_{i_{r+1}}]]$ with $\{i_1, \dots, i_{r+1}\} \subseteq \{j_1, \dots, j_\ell\}$.

- We showed above that if $\sigma \subset B$ is such that $|\sigma| = r$, then $\vee\sigma = X$. This fact implies that

  $$d\{x_{i_1}, \dots, x_{i_{r+1}}\} = \sum_j (-1)^j \{x_{i_1}, \dots, x_{i_{r+1}}\} \setminus \{x_{i_j}\}.$$

  On the other hand, $\ker \varphi$ does not contain any product of r elements. So ρ is injective on the monomials of $\Lambda^r(e_{j_1}, \dots, e_{j_l})$. Therefore, we have

  $$\begin{aligned}\rho[[e_{i_1}, \dots, e_{i_{r+1}}]] &= \sum_j (-1)^j \{x_{i_1}, \dots, x_{i_{r+1}}\} \setminus \{x_{i_j}\} \\ &= [d\{x_{i_1}, \dots, x_{i_{r+1}}\}] = 0\end{aligned}$$

(the injectivity of ρ on the monomials of $\Lambda^r(e_{j_1}, \ldots, e_{j_l})$ is needed to guarantee that no term of $[[e_{i_1}, \ldots, e_{i_{r+1}}]]$ is sent to zero by ρ).

- Since ρ is injective on the monomials of $\Lambda^r(e_{j_1}, \ldots, e_{j_l})$, if $u \in \ker \rho$ is a sum of s terms, then $\rho(u) = 0 \in H^*(D(\mathcal{A}))$ is also a sum of s terms. So $\rho(u)$ can be equal to 0 only if there exists a linear combination $\sum \alpha_i \sigma_i$ such that $\rho(u) = [d(\sum \alpha_i \sigma_i)]$. But if $\{i_1, \ldots, i_r\} \subseteq \{j_1, \ldots, j_\ell\}$ and $y \in \mathcal{A} \backslash B$, then $d\{x_{i_1}, \ldots, x_{i_r}, y\}$ is a sum with no term equal to $\{x_{i_1}, \ldots, x_{i_r}\}$. Therefore, σ is a linear combination of $\sigma_i \subseteq \{x_{j_1}, \ldots, x_{j_\ell}\}$. Since, $\rho(u) = \sum \alpha_i d\sigma_i$, ρ is injective on monomials and

$$d\sigma_i = \rho[[e_{i_1}, \ldots, e_{i_{r+1}}]],$$

we deduce that u is a linear combination of $[[e_{i_1}, \ldots, e_{i_{r+1}}]]$, as required.

The map ρ induces an injective quotient map $\bar{\rho}\colon A_2 \to H^*(D(\mathcal{A}))$, and we define a map ψ in the opposite direction

$$A_2 \xrightarrow{\bar{\rho}} H^*(D(\mathcal{A})) \xrightarrow{\psi} A_2$$

by sending $\{x_i\}$ to $[e_i]$ if $i \in \{j_1, \ldots, j_\ell\}$ and zero if $i \notin \{j_1, \ldots, j_\ell\}$. These two maps are morphisms of graded algebras and verify the following property : $\psi \circ \bar{\rho} = \mathrm{id}$. Finally, the lemmas 7.7 and 7.8 give us two injective maps $\mathbb{L}(u, v) \to L_{(A_2,0)} \to L_{D(\mathcal{A})}$. $\square$

With the two previous lemmas, the next theorem is almost completely proved. We just need to put everything in place.

Theorem 7.14. *Let $\mathcal{A}$ be a complex subspace arrangement with a geometric lattice. Then the following conditions are equivalent :*

1. *$H^*(M(\mathcal{A}); \mathbb{Q})$ is not an exterior algebra,*
2. *there exists an injective map $\mathbb{L}(u, v) \to L_{D(\mathcal{A})}$.*

Proof. Suppose that $H^*(M(\mathcal{A}); \mathbb{Q})$ is not an exterior algebra. By proposition 6.5 and theorem 6.7, $M(\mathcal{A})$ is not a nilpotent rationally elliptic space. By proposition 7.11, the map $\varphi\colon \Lambda(e_1, \ldots, e_n) \to$

$H^*(M(\mathcal{A});\mathbb{Q})$ is not injective. Therefore $\ker\varphi \neq 0$ and the lemmas 7.12 and 7.13 show that there exists an injective map $\mathbb{L}(u,v) \to L_{D(\mathcal{A})}$.

Now, suppose that $H^*(M(\mathcal{A});\mathbb{Q})$ is an exterior algebra. By proposition 6.5, we have : $D(\mathcal{A}) = (\Lambda(e_1,\ldots,e_n),0)$. So, obviously, $D(\mathcal{A})$ is its own minimal Sullivan algebra and is finitely generated (the degree of each e_i is odd). Therefore, no map $\mathbb{L}(u,v) \to L_{D(\mathcal{A})}$ can be injective. □

When $M(\mathcal{A})$ is simply connected, $L_{D(\mathcal{A})} = \pi_*(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$, which gives the following corollary.

Corollary 7.15. *Let $\mathcal{A}$ be a complex subspace arrangement with a geometric lattice such that every $x \in \mathcal{A}$ has* $\operatorname{codim}(x) \geq 2$. *Then $M(\mathcal{A})$ is rationally hyperbolic if and only if there exists an injective map $\mathbb{L}(u,v) \to \pi_*(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$.*

Proof. It is an immediate consequence of theorem 7.14. □

7.5 Coordinate subspace arrangements

In this section, we will prove a theorem similar to theorem 7.14, but valid in the case of coordinate subspace arrangements.

Let A be an algebra and let's denote by $\langle A\rangle$ a topological space such that $(A,0)$ and $A_{PL}(\langle A\rangle)$ have the same minimal model. The examples we have in mind are $(\mathbb{C}P^\infty)^m = \langle \mathbb{Q}[x_1,\ldots,x_m]\rangle$ and $DJ(K) = \langle \mathbb{Q}[x_1,\ldots,x_m]/I_K\rangle$ (where $DJ(K)$ is defined in section 3.2).

Remember that there is a fibration $\mathcal{Z}_K \hookrightarrow DJ(K) \to (\mathbb{C}P^\infty)^m$. With the above notation, this fibration is :

$$\langle \mathbb{Q}[x_1,\ldots,x_m]/I_K\rangle \to \langle \mathbb{Q}[x_1,\ldots,x_m]\rangle,$$

where the fiber is $\langle(\mathbb{Q}[x_1,\ldots,x_m]/I_K \otimes \Lambda(u_1,\ldots,u_m),d)\rangle$. In [2], Backelin studied fibrations $\langle R/I\rangle \to \langle R\rangle$ where R is a polynomial ring and I is an ideal generated by monomial relations. His idea was to consider the increasing sequence of ideals $I_1 \subseteq \cdots \subseteq I_m$ defined by $I_j = I \cap \mathbb{Q}[x_1,\ldots,x_j]$. The quotient map $R \to R/I$ can then be decomposed :

$$R \xrightarrow{p_1} R/I_1 \xrightarrow{p_2} R/I_2 \xrightarrow{p_3} \cdots \xrightarrow{p_m} R/I_m = R/I.$$

Backelin studied the maps p_i and proved the following theorem.

Theorem 7.16 (Backelin). *Applying the functor $\langle - \rangle$ to the maps $p_j \colon R/I_{j-1} \to R/I_j$ gives us maps $\langle p_j \rangle \colon \langle R/I_j \rangle \to \langle R/I_{j-1} \rangle$ whose homotopy fiber F_j has the rational homotopy type of a wedge of spheres $\vee_j S^{d_j}$ (possibly zero spheres).*

Let's apply Backelin results to the fibration we are interested in : $\mathcal{Z}_K \to DJ(K) \to (\mathbb{C}P^\infty)^n$.

Lemma 7.17. *We have :* $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} = \infty$ *if and only if its homotopy Lie algebra* $\pi_*(\Omega DJ(K)) \otimes \mathbb{Q}$ *contains a free Lie subalgebra on two generators.*

Proof. If $\pi_*(\Omega DJ(K)) \otimes \mathbb{Q}$ contains a free Lie subalgebra on two generators, then obviously $DJ(K)$ is rationally hyperbolic.

To prove the converse, let's suppose that $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} = \infty$. Looking at the models, the map $DJ(K) \to (\mathbb{C}P^\infty)^m$ is the composition of the fibrations $\langle R/I_p \rangle \to \langle R/I_{p-1} \rangle$ for $p = 1, \ldots, m$.

$$\begin{tikzcd}
 & F_1 \arrow[d] & F_2 \arrow[d] & \cdots & F_m \arrow[d] & \\
 & \langle R/I_1 \rangle \arrow[d] & \langle R/I_2 \rangle \arrow[d] & \cdots & \langle R/I_m \rangle \arrow[d] \arrow[r, equal] & DJ(K) \\
(\mathbb{C}P^\infty)^m \arrow[r, equal] & \langle R \rangle & \langle R/I_1 \rangle & \cdots & \langle R/I_{m-1} \rangle &
\end{tikzcd}$$

By Backelin's result, each fiber F_i is rationally a wedge of spheres.

Let's prove that there exists a F_i such that F_i is a wedge of two spheres or more. If this is not the case, then each F_i is either a point or a sphere. In that case, the long exact sequence in rational homotopy of the first fibration shows that $\dim \pi_*(\langle R/I_1 \rangle) \otimes \mathbb{Q} < \infty$. Using this, the second fibration shows that $\dim \pi_*(\langle R/I_2 \rangle) \otimes \mathbb{Q} < \infty$. Repeating the process until the mth fibration gives the proof that $\dim \pi_*(\langle R/I_K \rangle) \otimes \mathbb{Q} = \dim \pi_*(\langle R/I_m \rangle) \otimes \mathbb{Q} < \infty$, or in other words, that $\dim \pi_*(DJ(K)) \otimes \mathbb{Q} < \infty$, which is a contradiction.

So, we can define k as the largest i such that F_i is a wedge of two spheres or more. We have a fibration $F_k \to \langle R/I_k \rangle \to \langle R/I_{k-1} \rangle$. By corollary 1.27, the homotopy Lie algebra $\pi_*(\Omega \langle R/I_k \rangle) \otimes \mathbb{Q}$ contain the homotopy Lie algebra $\pi_*(\Omega F_k) \otimes \mathbb{Q}$, which is a free Lie algebra on two or more generators $\mathbb{L}(a_1, \ldots, a_r)$.

If $k = m$, the proposition is proved. If $k < m$, let's show that $\pi_*(\Omega \langle R/I_{k+1} \rangle) \otimes \mathbb{Q}$ contains also a free Lie algebra on two generators.

The space $\langle R/I_{k+1}\rangle$ fits in a fibration

$$F_{k+1} \to \langle R/I_{k+1}\rangle \to \langle R/I_k\rangle.$$

where F_{k+1} is a sphere or a point. The long exact sequence in homotopy of that fibration shows that there exists $n \in \mathbb{N}$ such that $\pi_{>n}(\Omega\langle R/I_{k+1}\rangle)\otimes\mathbb{Q} = \pi_{>n}(\Omega\langle R/I_k\rangle)\otimes\mathbb{Q}$. Then $(\pi_{>n}(\Omega\langle R/I_{k+1}\rangle)\otimes\mathbb{Q})\cap\mathbb{L}(a_1,\ldots,a_r) = (\pi_{>n}(\Omega\langle R/I_k\rangle)\otimes\mathbb{Q})\cap\mathbb{L}(a_1,\ldots,a_r)$ is a free Lie algebra and contains a free Lie algebra on two generators $\mathbb{L}(a',b')$. So, $\pi_*(\Omega\langle R/I_{k+1}\rangle)\otimes\mathbb{Q}$ contains $\mathbb{L}(a',b')$. We can repeat the same proof for $k+2,\ldots,m$. □

Finally, most of the work is done and we can now state and prove the main theorem of this section.

Theorem 7.18. *Let $\mathcal{A}$ be a complex coordinate subspace arrangement. The following conditions are equivalent :*

1. *the complement space $M(\mathcal{A})$ is rationally hyperbolic,*
2. *there exists an injective map $\mathbb{L}(u,v) \to \pi_*(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$, where $\mathbb{L}(u,v)$ denotes a free Lie algebra on two generators.*

Proof. As in the proof of theorem 6.10, we can suppose that $\mathcal{A}$ does not contain any hyperplane. Lemmas 3.24 and 7.17 prove that (1) implies (2). The converse statement is obvious. □

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Index

arrangement of hyperplanes, 14
Arvola's presentation, 20

Betti numbers, 36
bigrading of minimal models, 4

central arrangement, 14
cheese cutting problem, vii, 35
commutative differential graded algebra, 3
complement space, 13
complex
  arrangement, 14
  braid arrangement, 23
coordinate
  subspace, 14
  subspace arrangement, 14

Davis-Januszkiewicz space, 32
deleted arrangement, 14, 35
deletion-restriction theorem, 45
dichotomy, 6

elliptic space, 6
Euler characteristic, 44

formal
  cdga, 5
  space, 5
free commutative algebra, 3
fundamental group, 19

geometric lattice, 16

holonomy map, 8, 9
homotopy Lie algebra, 7
hyperbolic space, 6

independant subset, 18

lattice, 15
linear part of a morphism, 4

Mœbius function, 46
minimal
  model, 3
  Sullivan algebra, 3
moment-angle complex, 32

nilpotent, 6

Orlik-Solomon algebra, 23

Poincaré
  duality algebra, 64
  polynomial, 36

rank
  function, 16
  of a subspace arrangement, 16
rational
  homotopy equivalence, 2
  homotopy type, 1
  model, 23
rationalization, 1
real arrangement, 14
restricted arrangement, 14, 15, 35

separator, 45

Stanley-Reisner ring, 32
subspace arrangement, 13
Sullivan
    algebra, 3
    model, 3, 4

wedge of spheres, 55