Statistical comparison of Path-Complete Lyapunov Functions: a Discrete-Event Systems perspective

Raphael M. Jungers *

* *ICTEAM, UCLouvain, Louvain-la-Neuve, Belgium.*
raphael.jungers@uclouvain.be

Abstract: The goal of this paper is to advertise the tight links between the theory of Discrete-Event Systems and Path-Complete Lyapunov Functions. These are algebro-combinatorial stability criteria for hybrid systems, whose meta-parameters are an automaton and a template of candidate Lyapunov functions.
To do this, we analyse a phenomenon recently observed in the literature, namely the *statistical ordering of Path-Complete Lyapunov Functions*, which is by far not well understood yet: Path-Complete Lyapunov functions can be endowed with a preorder structure with respect to their performance. It has been recently shown that the preorder corresponding to relative worst case performance can be characterized with tools from automata theory, but the poset corresponding to *statistical relative performance* has remained elusive. We advocate for a Discrete-Event Systems approach to this problem and provide preliminary results in this direction.

Keywords: Path-complete Lyapunov functions, Stability, Switched Systems, Automata theory

1. INTRODUCTION

Lyapunov theory remains one of the most powerful tools to control complex systems and in particular Hybrid Systems Alur (2015); Antsaklis et al. (1995); Goebel et al. (2009); Tabuada (2009). Indeed, other classical control approaches, such as frequency domain techniques or analytic ones are not able to tackle such complex dynamical systems, due to their nonlinearity, the presence of discrete or combinatorial components in the definition of their behaviour, or to swiching phenomena, which are intrinsic to many modern engineered systems. This is even more the case in view of the non-standard specifications arising in emerging technologies, which more often than not take the form of logical statements. For this reason, Lyapunov techniques have been an active research topic throughout the last decades Ahmadi et al. (2014); Branicky (1998); Chen et al. (2021); Parrilo (2000).

Path-complete Lyapunov techniques constitute a general framework to automatically generate stability criteria for hybrid systems, which can be efficiently checked by standard optimization solvers (Semi-Definite Programming Lee and Dullerud (2006); Parrilo (2000), Linear Programs Blanchini et al. (2008); Jungers and Athanasopoulos (2021), or even more recently Neural Networks Debauche et al. (2023b). They have been introduced in 2014 Ahmadi et al. (2014) in order to understand, classify, and generalize several Lyapunov criteria that had been proposed in the previous decades in the literature on hybrid and switched systems. Formally, they are composed of two elements:

⋆ RJ is a FNRS honorary Research Associate. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 864017 - L2C.

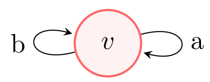


Fig. 1. The graph G_0 corresponding to the Common Lyapunov Function. Path-complete stability criteria may be regarded as discrete-event systems, where the discrete states represent regions of the actual (continuous) state-space of the dynamical system, and the system jumps from one region to another along time. Since these regions are bounded, such an abstract discrete-event system delivers a proof of stability, as soon as corresponding regions are found.

First, a *labeled graph* (that is, an automaton), that provides a symbolic description of the algebraic equations, with one edge per equation; Second, a *template*, that is, a set of candidate Lyapunov functions, on which it is assumed that numerical solvers can efficiently solve optimization problems.

Path-complete Lyapunov functions provide an agile general setting for deriving stability criteria: the automaton encoding the algebraic inequalities constitutes a flexible tool to model the criterion, and in theory, it can be made more complex (by adding nodes, or edges) in order to increase the performance of the criterion (the *performance* being here the ability of the criterion to prove stability for large classes of systems). However, even though, intuitively, the more complex the automaton, the better the criterion will be, the precise relation in terms of performance between two given graphs is not well understood. In Philippe and Jungers (2019), a criterion was given in order to compare two graphs in an absolute meaning: a procedure is given, allowing to compare two graphs from their sole topological structure, and decide whether one is absolutely better than the other (see Theorem 8 below). Despite this result, it has been shown later that the ordering is more complicated in practice. Indeed, in practice, one needs to define a template, on which the candidate Lyapunov functions are sought (that is, the domain of the optimization program). As it turns out, the ordering becomes more intricate *when taking into account the particular template used*, and in recent years this *template-dependent ordering* has been progressively understood for some templates (polytopic templates Debauche et al. (2021), or templates closed under addition such as quadratic Lyapunov functions Debauche et al. (2023a)).

In this paper, a slightly different point of view is taken, namely to analyse the *average* performance of path-complete Lyapunov functions. Indeed, it has been reported that some pairs of graphs are not comparable in the sense described above, even when the actual template used is specified, but are still comparable in an average sense: that is, even in the case where Graph G_1 can provide a better stability criterion than Graph G_2 on some particular systems, and conversely Graph G_2 can provide a better stability criterion than Graph G_1 on some other systems, it may well be that Graph G_1 performs less well than G_2 for *most* systems (see Example 3 for a similar situation). This consideration has much practical interest, since, in practice, one might not know which criterion will perform better in advance, and it would thus make more sense to utilize the criteria that perform better most often. In addition, we believe that understanding the reason why some programs perform often better than others could help to design techniques for more general purposes than the pure stability of hybrid systems, by generalizing the ideas developed for this problem. Finally, we think that this problem is interesting from a purely theoretical point of view: understanding the structure of optimization problems that perform well on average is a difficult, but important challenge from a mathematical perspective. This is especially true in our view in the era of machine learning, where general purpose algorithms are deployed in situations where little is known on the algebraic structure of the problem beforehand.

As the reader will see, many concepts explaining the quality of the performance of path-complete Lyapunov techniques bear similarities with discrete-event techniques. This is because a path-complete graph may be seen as a *discrete abstraction* of the system, where the behaviour of the dynamics is quotiented by a relation on the trajectories satisfying some Lyapunov stability inequalities. That is, the path-complete criterion 'forgets' about the true system, but only remembers that the trajectory is contained in a succession of bounded sets. For this reason, we believe that the discipline of discrete-event systems has much to bring to the still widely open research topic of path-complete Lyapunov techniques.

The rest of the manuscript is as follows: in Section 2 we formalize the concepts described above. In Section 3 we present preliminary results which show how one can explain the statistical ordering in some particular cases. We end with a brief conclusion, where we try to identify further directions.

2. TECHNICAL PRELIMINARIES

2.1 The system

Even though the concept of path-complete Lyapunov functions encompasses more general situations, in this paper we restrict our attention to discrete-time switched linear systems, for the sake of simplicity of exposition. These systems are described by

$$x(k+1) \;=\; A_{\sigma(k)}(x(k)), \tag{1}$$

where the state $x \in \mathbb{R}^n$ at each time $k \in \mathbb{N}$, and the switching signal $\sigma : \mathbb{N} \to \{1, \ldots, M\}$ regulates the switching between the M subsystems of the indexed set $\mathcal{M} := \{A_i : i = 1, \ldots, M\} \subset (\mathbb{R}^{n\times n})$. In our setting, $\sigma : \mathbb{N} \to \{1, \ldots, M\}$ is supposed to be an exogenous and unpredictable input. For an overview of this setting, we refer to Jungers (2009). Also, we restrict our attention to the *stability* problem.

Definition 1. The system (1) is said to be *Globally Asymptotically Stable* (in short: *stable*) if for any initial state $x_0 \in \mathbb{R}^n$, and any switching signal σ, one has

$$\lim_{k\to\infty} x(k) = 0.$$

2.2 Path-complete Lyapunov criteria

A directed and labeled graph on the alphabet $\mathcal{M}$ is a couple $G = (S, E)$ where S denotes the set of nodes, and $E \subseteq S \times S \times \mathcal{M}$ is the set of directed edges labeled by an element of the alphabet [1]. The following graph property of *path-completeness* introduced in Ahmadi et al. (2014) guarantees that the set of equations corresponding to G (as explained in (2) below) will provide a stability criterion [2].

Definition 2. (Path-complete graph). Given $M \in \mathbb{N}$, a graph $G = (S, E)$ on the alphabet $\mathcal{M} = \{A_1, \ldots, A_M\}$ is *path-complete* if, for any $K \geq 1$ and any possible switching signal of length $K : \sigma = (j_1 \ldots j_K) \in \mathcal{M}^K$, there exists a *path* [3] $(w_1, j_1, w_2, j_2, \ldots, j_K, w_{K+1})$.

Path-complete Lyapunov functions are stability criteria where the graph parameterizes in an abstract way a set of algebraic equations. The unknown variables in these equations are functions, which are represented in the graph by its vertices. These functions may not be arbitrary, but must satisfy the property of candidate Lyapunov functions, which we define now.

Definition 3. (Candidate Lyapunov function). A *candidate Lyapunov function* is a positive-definite homogeneous function from $\mathbb{R}^n$ to $\mathbb{R}_+$:

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}^n; \quad V(x) = 0 \Longleftrightarrow x = 0;$$
$$V(\lambda x) = |\lambda| V(x) \quad \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

[1] In this text, by a small abuse of notation, we assimilate a matrix $A \in \mathcal{M}$, and the character A used to label the edges.

[2] If the switched system is *unconstrained,* that is, any possible sequence of mode is allowed by the dynamics of the system, then path-completeness is equivalent to an automaton being *universal* in Theoretical Computer Science. However, we introduce this specific term for the sake of generalizability to other families of switched systems.

[3] Given a graph $G = (V, E)$, a *path* in G is an alternating sequence of nodes and labels $\pi = (w_1, a_1, w_2, a_2, \ldots, a_{n-1}, w_n)$ such that $\forall k \in [1, \ldots, n-1]$, $(w_k, w_{k+1}, a_k) \in E$. The *label sequence* of path π is $L(\pi) := a_1 \ldots a_{n-1}$.

Definition 4. (Path-complete Lyapunov Function).
Given a switched system $\mathcal{M} = \{A_1, \dots A_M\} \subset \mathbb{R}^{n\times n}$, a *path-complete Lyapunov function* (PCLF) for $\mathcal{M}$ is a pair $(G = (S, E), V_S)$ where G is a path-complete graph on $\mathcal{M}$, and $V_S := \{V_s : s \in S\}$ is a set of candidate Lyapunov functions such that the following inequalities are satisfied:

$$\forall (a, b, i) \in E,\ \forall x \in \mathbb{R}^n : V_b(A_i x) \ \le\ V_a(x). \qquad (2)$$

If this is the case, we say that V_S is *admissible for G and $\mathcal{M}$*.

Theorem 5. (Stability result). Ahmadi et al. (2014)

Given any $\mathcal{M} = \{A_i \mid i \in \{1, \dots , M\}\} \subset \mathbb{R}^{n\times n}$ and any path-complete graph G with labels in $\mathcal{M}$, if there exists an admissible set of candidate Lyapunov functions, then system (1) is *marginally stable*, that is, all the trajectories are bounded. By homogeneity of the dynamics, if there is an admissible set of candidate Lyapunov functions for[4] $\mathcal{M}/r$ for some $r < 1$, then the system is globally asymptotically stable, and the trajectories satisfy

$$x(t) \ \approx O(r^t).$$

For the proof, the link with the so-called *joint spectral radius,* and further discussion, we refer to Ahmadi et al. (2014).

2.3 Problem definition

Unfortunately, for a given set of matrices, not all path-complete criteria will manage to prove stability. In fact, the stability problem is widely believed to be undecidable, and it is certainly an extremely hard algorithmic problem, see (Jungers, 2009, theorem 2.6). For an intuitive understanding of this, think of the simplest criterion represented in Figure 1, together with the template of (square root of) positive definite quadratic functions. This path-complete criterion corresponds to the celebrated *Common Quadratic Lyapunov Function* (see also Example 3). This particular stability analysis technique has been studied in depth in the last three decades, and it is well understood right now that it allows to prove stability only for a restricted class of linear switched systems (see Kozyakin (1990); Olshevsky and Tsitsiklis (2008); Shorten and Narendra (1998)). Indeed not all stable linear switched systems enjoy a common quadratic Lyapunov function (even though all *linear systems* do). For more advanced semi-algebraic stability criteria and an analysis of their limitation, see Ahmadi and Jungers (2014); Gurvits (1995).

Thus, for a finite stability criterion as described above, one can only hope for the criterion to work on some subset of stable switched systems. Accordingly, in order to evaluate the performance of a particular criterion, we introduce a series of definitions. In the first one, we restrict our attention to the performance of a particular criterion on a particular given system.

Definition 6. Given a particular set of matrices $\mathcal{M}$, and a particular path-complete criterion (defined by a graph G and a template $\mathcal{T}$), we define the *performance index of $(G, \mathcal{T})$ on the set $\mathcal{M}$,* which we note $\rho_{G,\mathcal{T}}(\mathcal{M})$, as the infimum real t such that graph G, together with template $\mathcal{T}$, has an admissible solution for $\mathcal{M}/t$.

[4] We note $\mathcal{M}/r$ for $\{A/r : A \in \mathcal{M}\}$

It is easily seen that the performance index $\rho_{G,\mathcal{T}}(\mathcal{M})$ actually provides an upper bound on the asymptotic rate of growth of all trajectories in System (1). Thus, **the smallest the performance index $\rho_{G,\mathcal{T}}(\mathcal{M})$, the better is the combination of graph G and template $\mathcal{T}$ for analysing the stability of $\mathcal{M}$.**

We briefly recall the comparison problem previously developed in the literature, in order to make clearer the problem that we are interested in:

Definition 7. (Absolute order relation between graphs).
Consider two path-complete graphs G_1 and G_2 with labels in $\mathcal{M}$. We say that G_1 has less good performance than G_2, which we note $G_1 \le G_2$, if

$$\forall \mathcal{M} = \{A_1, \dots , A_M\} \subset \mathbb{R}^{n\times n}, \forall \text{ template } \mathcal{T},$$
$$\rho_{G_1,\mathcal{T}}(\mathcal{M}) \ge \rho_{G_2,\mathcal{T}}(\mathcal{M}).$$

The next theorem provides a characterization of the above ordering:

Theorem 8. Philippe and Jungers (2019) Consider two graphs $G_1(S_1, E_1)$, $G_2(S_2, E_2)$ with labels in the same set $\{1, \dots , M\}$. One has $G_1 \le G_2$ if and only if G_1 *Simulates* G_2, that is, if there exists a function $f : S_2 \to S_1$, such that for every edge $e_2 = (v, v', l) \in E_2$, there is an edge $e_1 = (f(v), f(v'), l) \in E_1$.

We are now able to introduce the main subject of our study.

Problem 1. (Statistical order relation between graphs).
Let us consider two path-complete graphs G_1 and G_2 with labels in $\{1, \dots , M\} \subset \mathbb{R}^{n\times n}$, and a template $\mathcal{T}$. Assume that one samples sets of matrices according to some measure μ on $\mathbb{R}^{n\times n\times M}$. We say that G_1 has *statistically* worse performance than G_2 with respect to μ, which we note $G_1 \le_{\mu,\mathcal{T}} G_2$ if

$$\mathbb{P}(\rho_{G_1,\mathcal{T}}(\mathcal{M}) < \rho_{G_2,\mathcal{T}}(\mathcal{M})) \le \mathbb{P}(\rho_{G_2,\mathcal{T}}(\mathcal{M}) < \rho_{G_1,\mathcal{T}}(\mathcal{M})).$$

The focus of the present paper is to estimate, in view of the graphs G_1, G_2, and the chosen template $\mathcal{T}$, the above probabilities.

Example 2. Consider the Graph

$$G_0 = (\{v\}, \{(v, v, 1), (v, v, 2)\})$$

depicted in Figure 1. It is straightforward from Definition 4 that this graph, with a single unknown function $V(x)$, represents the *Common Lyapunov Function.* It is easily seen from Theorem 8 that $G_0 \le G$ for any other graph G with labels in $\{a, b\}$. Indeed, there is an obvious simulation relation from G to G_0. This is in accordance to our intuition that, when the template is fixed, asking for a common Lyapunov function is the most stringent property allowing to prove stability.

Example 3. Consider now the graph G_1 in Figure 2. By Example 2, we have that $G_0 \le G_1$. However, the question of whether $G_1 \le_\mu G_0$ is less clear.That is, is there a nonzero probability that, for a given set of matrices, one would have

$$\rho_{G_1,\mathcal{T}}(\mathcal{M}) < \rho_{G_0,\mathcal{T}}(\mathcal{M})?$$

It turns out that this is the case. More precisely, it is reported in Debauche et al. (2023a) that, in practical experiments, *when the template of quadratic Lyapunov functions is used,* the graph G_1 provides a tighter upper bound in about 17% of the time. No explanation is given

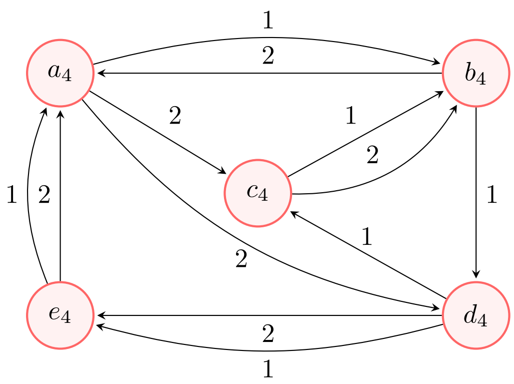


Fig. 2. The graph $G_1 = (S_1, E_1)$ turns out to have statistically better performance than G_0 : It is reported in Debauche et al. (2023a) that, out of 10000 sampled systems $\mathcal{M}_i$: $i = 1, \ldots, 10000$, it outperforms the graph G_0 for 1668 samples, while the 8332 remaining samples ended up in a draw. It is the purpose of this paper to understand these numbers from the sole analysis of the graph topology.

in Debauche et al. (2023a), nor anywhere in the literature to the best of our knowledge, of why G_1 has a better performance in average.

3. RESULTS

In this section, we will show two preliminary results that help to understand the statistical ordering in particular cases. We start first with a result on the particular case of the recently introduced generalized De Bruijn graphs, in order to gain some intuition on the problem. Then we provide an interpretation of the ordering in terms of language and automata theory.

3.1 Generalized De Bruijn graphs and importance of the template

In this subsection, we study a generalization of De Bruijn graphs which provide among the most efficient path-complete Lyapunov criteria in practice. This generalization has been recently introduced in Della Rossa and Jungers (2023) and it provides a large family of graphs, called *Generalized De Bruijn graphs.* These graphs exhibit interesting properties in terms of performance (in the sense defined above) and have an interpretation in terms of 'memory' and 'look-ahead'. For the sake of space constraints, we do not remind here all these properties. It may be useful for the reader to observe Figure 3 while parsing the definition.

Definition 9. (Generalized De Bruijn graphs). Consider a *level* $K \in \mathbb{N}$, and fix $i, j \in \mathbb{N}$ such that $i + j = K$. The *De Bruijn graph of memory* i *and look-ahead* j, denoted by $G_{[i,j]} = (S_{[i,j]}, E_{[i,j]})$ is defined as follows. In the next equation, x (resp. y) represents an arbitrary word of length $i-1$ (resp. $j-1$). Also, for a word $w \in \mathcal{M}^i$, and $1 \leq k \leq l \leq i$, we note $w_{|k\ldots l}$ for the subword made of the characters with indices k to l.

$$S_{[i,j]} := \{[w, w'] \in \mathcal{M}^i \times \mathcal{M}^j\};$$
$$([w_1, w_1'], [w_2, w_2'], h) \in E_{[i,j]}$$

iff

$$\begin{cases} w_1' = hx & \text{if } j \neq 0, \\ w_2 = yh & \text{if } i \neq 0, \\ w_{1|2\ldots i} = w_{2|1,\ldots i-1} & \text{if } i \geq 2 \\ w'_{1|2\ldots j} = w'_{2|1\ldots j-1} & \text{if } j \geq 2. \end{cases} \tag{3}$$

The De Bruijn graph $G_{[i,j]}$ can be interpreted as encoding a 'past' memory, and a 'look-ahead' memory, that record, respectively, the i last characters, and the j next characters, of any signal that can be read on the graph. In Figure 3, one may verify that, in any node $v = [w, w']$ in the graph $G_{i,j}$, the word w encodes the i last symbols read before entering the vertex v, and the word w' encodes the j next symbols that will be read after leaving vertex v, independently of the path chosen.

As it turns out, the statistical order between two graphs depends on the chosen template. Indeed, it is reported in Della Rossa and Jungers (2023) that, when using the template $\mathcal{D}$ of diagonal quadratic Lyapunov functions [5], the different De Bruijn graphs of level 2 do not show the same performance. In fact, it is empirically observed that $G_{[2,0]} <_{\mu,\mathcal{D}} G_{[1,1]}$ and $G_{[0,2]} <_{\mu,\mathcal{D}} G_{[1,1]}$ (with μ the uniform Lebesgue measure). However, we now show that all the De Bruijn graphs of a same level are equivalent if the template is closed under composition with the dynamics [6]. Because this relation only holds for certain templates, following Debauche et al. (2023a), we introduce the following restriction of Definition 7 to a particular template.

Definition 10. (Template-dependent order relation). Consider two path-complete graphs G_1 and G_2 with labels in $\mathcal{M}$, and a template $\mathcal{T}$. We say that G_1 has less good performance than G_2 with Template $\mathcal{T}$, which we note $G_1 \leq_{\mathcal{T}} G_2$, if

$$\forall \mathcal{M} = \{A_1, \ldots, A_M\} \subset \mathbb{R}^{n \times n},$$
$$\rho_{G_1,\mathcal{T}}(\mathcal{M}) \geq \rho_{G_2,\mathcal{T}}(\mathcal{M}).$$

The following theorem shows that the De Bruijn graphs of a given level are actually equivalent if the template is closed under composition.

Theorem 11. Consider a template $\mathcal{T}$ closed under composition. Then the generalized De Bruijn graphs as described in Definition 9 satisfy the following properties:

$$\begin{aligned} G_{[i,j]} &\leq_{\mathcal{T}} G[i-1, j+1] \\ G[i-1, j+1] &\leq_{\mathcal{T}} G_{[i,j]}. \end{aligned} \tag{4}$$

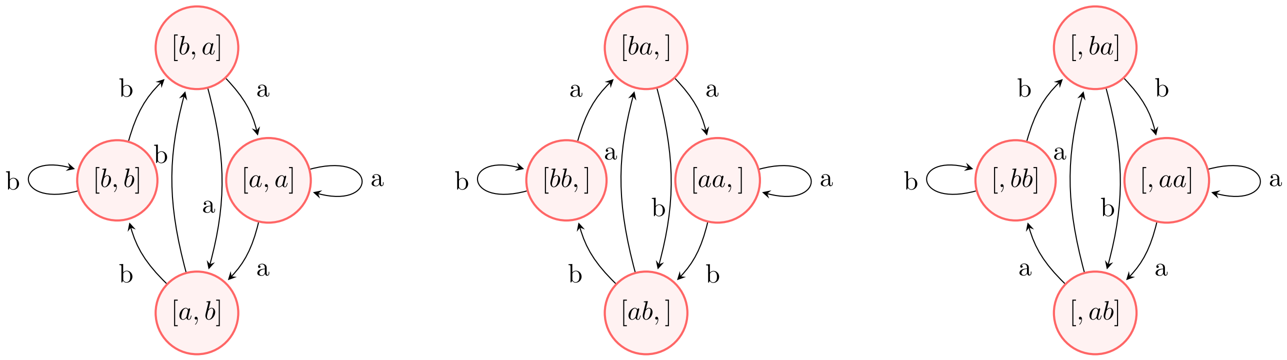
A proof can be found in the online long version of the present paper.

Corollary 12. Consider a template $\mathcal{T}$ closed under composition, and an arbitrary system $\mathcal{M}$. Then, all the De Bruijn graphs of identical level k have the same performance index.

Proof 4. By the theorem above, for any two De Bruijn graphs $G_{[i,j]}$ and $G_{[i',j']}$ such that $i + j = i' + j'$, one

[5] The template $\mathcal{D}$ is the set of functions of the form $W(x) = a_1 x_1^2 + \cdots + a_n x_n^2$ with $a_i \geq 0$.

[6] A Template $\mathcal{T}$ is closed under composition with the dynamics if for every possible A_i describing a mode of the system, and every candidate Lyapunov function $W(\cdot) \in \mathcal{T}$, one has $W(A_i \cdot) \in \mathcal{T}$ and $W(A_i^{-1} \cdot) \in \mathcal{T}$. We assume throughout the paper that A_i are invertible; we leave the case of non-invertible matrices to a further journal version.



(a) The graph $G_{[1,1]}$. (b) The graph $G_{[2,0]}$. (c) The graph $G_{[0,2]}$.

Fig. 3. The three De Bruijn graphs of level $K = 2$.

can apply Theorem 11 several times to show that both relations in (4) hold.

In conclusion, this subsection was motivated by the above mentioned statistical observations from Della Rossa and Jungers (2023), where different De Bruijn graphs of the same level (and thus, same number of nodes and edges) seem to have different performance. We have shown that this differences in performance are not only due to intrinsic properties of the De Bruijn graphs, but also to the used template. Indeed, if the template of quadratic Lyapunov functions had been used, no significant difference would have been observed between the different De Bruijn graphs of identical level, since they are all equivalent on every single system. Indeed, observe that quadratic functions are closed under linear operators, and thus Theorem 11 applies.

3.2 A language-theoretic criterion

In this subsection, we provide a partial explanation of why the topology of an automaton may impact the performance of its related path-complete stability criterion. The explanation relies on language theoretic properties of the automaton, and can be seen as a relaxation of the simulation criterion of Theorem 8. Indeed, in the latter theorem, one requires a relation between the nodes of both automata such that *every single word* that corresponds to a path in G_2 also corresponds to a path in G_1, which is given by the simulation relation. In the theorem below, we show that in fact, when a particular set of matrices is given, this condition is only necessary for the words that matter, that is, the words that lead to tight inequalities in Program (2). We first define properly these words [7].

Definition 13. Consider a set of matrices $\mathcal{M}$, a path-complete graph $G(S, E)$, and a template $\mathcal{T}$. We say that a set Π of paths in G is a *support set* if the graph

$$G_\Pi(S, \{(v, v', L(\pi)) : \pi \in \Pi \text{is a path from v to v'}\})$$

has the same performance index as G.

[7] In this section, we generalize the path-complete criteria to graphs with words (that is, sequences of labels longer than one) on the edges. The inequality corresponding to an edge $(v, v', l_1 \dots l_k)$ with label $l_1 \dots l_k$ is simply

$$V_{v'}(A_{l_k} \dots A_{l_1} \cdot) \leq V_v(\cdot).$$

That is, a support set of G is a subset of the paths in G that is sufficient to obtain the performance index $\rho_{G,\mathcal{T}}(\mathcal{M})$: if one erases all the other constraints from the corresponding optimization problem, still, one obtains the same optimal solution. Note that this property depends on the particular set of matrices $\mathcal{M}$ as well as on the template $\mathcal{T}$.

Theorem 14. Consider any graphs G_1, G_2, a template $\mathcal{T}$, and a set of matrices $\mathcal{M}$. If there exists a support set Π of G_2 and a *partial simulation* $f_\Pi : V_2 \to V_1$ such that

$$\forall \pi \in \Pi,\ \pi = (w_1, a_1, w_2, a_2, \dots, a_{n-1}, w_n),$$

there exists a path in G_1 :

$$\pi' = (f_\Pi(w_1), a_1, f_\Pi(w_2), a_2, \dots, a_{n-1}, f_\Pi(w_n)),$$

then,

$$\rho_{G_2,\mathcal{T}}(\mathcal{M}) \leq \rho_{G_1,\mathcal{T}}(\mathcal{M}).$$

A proof can be found in the online long version of the present paper.

The above theorem relies on a relaxed notion of simulation, which we call *pseudo-simulation,* and is restricted to a specific set of paths, rather than the full set of edges. The theorem provides an intuitive explanation of why a given graph G_2 might be statistically better than another graph G_1: the reason is that, even though they might not have a proper simulation relation, there might well exist a pseudo-simulation *for most of the sets of paths* Π, which prevents G_1 to have a lower performance index than G_2 as soon as any support set of G_2 is in one of these sets of paths. Unfortunately, Theorem 11 only provides a sufficient condition, and so it does not allow to predict the precise probability that $\rho_{G_2,\mathcal{T}}(\mathcal{M}) \leq \rho_{G_1,\mathcal{T}}(\mathcal{M})$, for a randomly sampled $\mathcal{M}$. (Note that, for achieving this, one should also understand the probability for a given set Π to be a support set, which seems a challenge.) Nevertheless, it provides an intuitive explanation of why an automaton is 'often' better than another one, and we believe that it is an important milestone on the way to estimate the relative performance of two path-complete criteria.

Example 5. (Example 3 continued) Let us consider again the graph in Figure 2 compared with the Common Lyapunov Function represented in Graph G_0 in Fig. 1. As we have seen, it is clearly impossible that $\rho_{G_0,\mathcal{T}}(\mathcal{M}) < \rho_{G_1,\mathcal{T}}(\mathcal{M})$. The opposite strict inequality on the other hand is possible; however, as shown in Theorem 14, this strict inequality will not hold if any support set of G_0 is simulated in G_1. Observe that, for a given set of matrices $\mathcal{M}$, if every support set of G_0 contains (v_0, a, v_0), or (v_0, b, v_0), there cannot be a pseudo-simulation from G_0 to G_1 in the sense of Theorem 14, because G_1 does not contain any self-loop. This suggests that G_0 can have the same performance as G_1 only if there is a nontrivial support set for G_0, with all paths of lengths larger than one.

The above example is in accordance with our intuition that small graphs are expected to have worse performance than large graphs. Indeed, in a large graph G, which does not contain any small cycles, the graph G_0 can have equal performance as G only for a set of matrices such that there is a support set for G_0 with only large paths. In practice, large paths represent long products of matrices. Since $\rho_{G_0,\mathcal{T}}(\mathcal{M})^t$ is a (often loose) upper bound on the norm of products of length t, long paths tend to correspond to non-tight constraints, and the probability that there is a support set with only long paths is low. Thus, so does the probability that G_0 has the same performance index as a given (non-trivial) large graph.

4. CONCLUSION

In this paper, we have introduced the problem of statistical comparison of path-complete functions. Formally, this amounts to compare the volume of two semi-algebraic sets (parameterized by the dimension). This problem is extremely hard, and probably undecidable to the best of our knowledge. However, we have shown that in our case, the algebraic inequalities describing the sets have a restricted and well-defined combinatorial structure, which allows to give intuition on the above mentioned order. The results provided in this paper only provide a partial intuition, since they only provide comparison relations in some particular cases. We hope to obtain a more complete characterization of the statistical ordering by relating the sampling measure μ with the measure of support sets allowing to establish a comparison in Theorem 14.

5. ACKNOWLEDGEMENTS

The author wishes to thank the reviewers for their helpful comments.

REFERENCES

Ahmadi, A.A. and Jungers, R.M. (2014). On complexity of lyapunov functions for switched linear systems. *IFAC Proceedings Volumes*, 47(3), 5992–5997.

Ahmadi, A.A., Jungers, R.M., Parrilo, P.A., and Roozbehani, M. (2014). Joint spectral radius and path-complete graph lyapunov functions. *SIAM Journal on Control and Optimization*, 52(1), 687–717.

Alur, R. (2015). *Principles of cyber-physical systems*. MIT press.

Antsaklis, P., Kohn, W., Nerode, A., and Sastry, S. (1995). *Hybrid systems II*, volume 999. Springer Science & Business Media.

Blanchini, F., Miani, S., et al. (2008). *Set-theoretic methods in control*, volume 78. Springer.

Branicky, M.S. (1998). Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on automatic control*, 43(4), 475–482.

Chen, S., Fazlyab, M., Morari, M., Pappas, G.J., and Preciado, V.M. (2021). Learning lyapunov functions for hybrid systems. In *Proceedings of the 24th International Conference on Hybrid Systems: Computation and Control*, 1–11.

Debauche, V., Della Rossa, M., and Jungers, R. (2023a). Characterization of the ordering of path-complete stability certificates with addition-closed templates. In *Proceedings of the 26th ACM International Conference on Hybrid Systems: Computation and Control*, 1–10.

Debauche, V., Della Rossa, M., and Jungers, R.M. (2021). Template-dependent lifts for path-complete stability criteria and application to positive switching systems. volume 54, 151–156. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021.

Debauche, V., Edwards, A., Jungers, R.M., and Abate, A. (2023b). Stability analysis of switched linear systems with neural lyapunov functions. *arXiv preprint arXiv:2312.08190*.

Della Rossa, M. and Jungers, R.M. (2023). Multiple lyapunov functions and memory: A symbolic dynamics approach to systems and control. *arXiv preprint arXiv:2307.13543*.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2009). Hybrid dynamical systems. *IEEE control systems magazine*, 29(2), 28–93.

Gurvits, L. (1995). Stability of discrete linear inclusion. *Linear algebra and its applications*, 231, 47–85.

Jungers, R. (2009). *The joint spectral radius: theory and applications*, volume 385. Springer Science & Business Media.

Jungers, R.M. and Athanasopoulos, N. (2021). Safety guarantees for hybrid systems. In *Encyclopedia of Systems and Control*, 2005–2012. Springer.

Kozyakin, V.S. (1990). Algebraic unsolvability of problem of absolute stability of desynchronized systems. *Automation and Remote Control*, 51(6), 754–759.

Lee, J.W. and Dullerud, G.E. (2006). Uniform stabilization of discrete-time switched and markovian jump linear systems. *Automatica*, 42(2), 205–218.

Olshevsky, A. and Tsitsiklis, J.N. (2008). On the nonexistence of quadratic lyapunov functions for consensus algorithms. *IEEE Transactions on Automatic Control*, 53(11), 2642–2645.

Parrilo, P.A. (2000). *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. California Institute of Technology.

Philippe, M. and Jungers, R. (2019). A complete characterization of the ordering of path-complete methods. *HSCC 19: 22nd ACM International Conference on Hybrid Systems: Computation and Control*, 138–146.

Shorten, R. and Narendra, K. (1998). On the stability and existence of common lyapunov functions for stable linear switching systems. In *Proceedings of the 37th IEEE Conference on Decision and Control (Cat. No. 98CH36171)*, volume 4, 3723–3724. IEEE.

Tabuada, P. (2009). *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media.

6. APPENDIX: PROOF OF THEOREM 11

Proof 6. Consider an arbitrary set of matrices $\mathcal{M}$, and an *optimal* solution

$$\{V_v : v \in \{([w_1, w_2] : w_1 \in \mathcal{M}^i, w_2 \in \mathcal{M}^j)\}\}$$

for $G_{[i,j]}$, that is, an admissible solution for the equations corresponding to $G_{[i,j]}$, with the set of

matrices $\mathcal{M}/(\rho_{G_{[i,j]},\mathcal{T}}(\mathcal{M}))$. In the following, we will define a solution for the graph $G[i-1, j+1]$. (Note that we have assumed that the solution $\{V_v\}$ exists for $\rho_{G_{[i,j]},\mathcal{T}}(\mathcal{M})$, while in practice, this quantity being defined as an infimum, a solution is only guaranteed for $\rho_{G_{[i,j]},\mathcal{T}}(\mathcal{M}) + \epsilon$. However, by constructing a solution below for an arbitrary ϵ, we show that the infimum in Definition 6 is the same for $G[i-1, j+1]$ and $G[i, j]$.)

These equations are of the form

$$V_{[w_1 p, w_2 r]}(A_p \cdot) \le V_{[s w_1, p w_2]}(\cdot). \tag{5}$$

We can define a solution for the graph $G[i-1, j+1]$ as follows:

$$\begin{array}{c} \forall w_1 \in \mathcal{M}^{(i-1)}, w_2 \in \mathcal{M}^{(j)}, \forall p \in \mathcal{M}, \\ V'_{[w_1, p w_2]}(\cdot) := V_{[w_1 p, w_2]}(A_p \cdot). \end{array} \tag{6}$$

One can check that the solution $\{V'\}$ is satisfiable for the graph $G[i-1, j+1]$. Indeed, take any edge $([r w_1, p a w_2], [w_1 p, a w_2 s], p)$ in $G[i-1, j+1]$, for some $w_1 \in \mathcal{M}^{(i-2)}, w_2 \in \mathcal{M}^{(j-1)},\ r, p, s \in \mathcal{M}$. One has

$$\begin{array}{rcl} V'_{[w_1 p, a w_2 s]}(A_p \cdot) & = & V_{[w_1 p a, w_2 s]}(A_a A_p \cdot) \\ & \le & V_{[r w_1 p, a w_2]}(A_p \cdot) \\ & = & V'_{[r w_1, p a w_2]}(\cdot), \end{array}$$

where the first and last equalities are from (6), and the middle inequality is from (5).

One can define a solution for $G[i+1, j-1]$ in a very similar way as we just found a solution for $G[i-1, j+1]$, and the proof is done.

The crux of the proof is Equation 6, where we explicitely build a solution for $G[i-1, j+1]$ from a solution for $G[i, j]$. Observe that the assumption that the template is closed under composition is crucial for this.

7. APPENDIX: PROOF OF THEOREM 14

Proof 7. (sketch) The proof is a simple consequence of classical properties of convex optimization. It relies on the observation that for a given graph and its corresponding optimization problem, all the paths represent constraints in the polar of the optimization problem; that is, they constitute valid constraints for any feasible solution. The hypothesis of the theorem then essentially assumes that all the constraints corresponding to the support set of G_2 are inherent in G_1, which guarantees that its performance cannot be better.