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Fixed-charge transportation on a path: optimization, LP formulations and separation

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Abstract

The fixed-charge transportation problem is an interesting problem in its own right. This paper further motivates its study by showing that it is both a special case and a strong relaxation of the big-bucket multi-item lot-sizing problem. We then provide a polyhedral analysis of the polynomially solvable special case in which the associated bipartite graph is a path.

We give a *O(n^3)*-time optimization algorithm and two *O(n^2)*-size linear programming extended formulation. We describe a new class of inequalities that we call "path-modular" inequalities. We give two distinct proofs of their validity. The first one is direct and crucially relies on sub- and super-modularity of an associated set function. The second proof is by showing that the projection of one of the extended linear programming formulations onto the original variable space yields exactly the polyhedron described by the path- modular inequalities. Thus we also show that these inequalities suffice to describe the convex hull of the set of feasible solutions. We finally report on computational experiments comparing extended LP formulation, valid inequalities separation and a standard MIP solver.

Keywords: mixed-integer programming, lot-sizing, transportation, convex hull, extended formulation.

MSC Classification: 68Q25, 90C11, 90C27, 90C35, 90B05, 90B06

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1 Introduction

In the fixed charge transportation problem (FCT), we are given a set of depots $i \in I$ each with a quantity of available items C_i, and a set of clients $j \in J$ each with a maximum demand D_j. For each depot-client pair (i, j), both the unit profit $q_{i,j}$ of transporting one unit from the depot to the client is known, together with the fixed charge $g_{i,j}$ of transportation along that arc. The goal is to find a profit-maximizing transportation program. Problem FCT can therefore be expressed as the following mixed-integer linear program:

$$\max \quad \sum_{i \in I} \sum_{j \in J} (q_{i,j} w_{i,j} - g_{i,j} v_{i,j}), \tag{1}$$

$$\sum_{j \in J} w_{i,j} \leq C_i, \qquad \forall i \in I \tag{2}$$

$$\sum_{i \in I} w_{i,j} \leq D_j, \qquad \forall j \in J \tag{3}$$

$$0 \leq w_{i,j} \leq \min(C_i, D_j) v_{i,j}, \qquad \forall i \in I, j \in J, \tag{4}$$

$$v \in \{0, 1\}^{I \times J}, \tag{5}$$

where $w_{i,j}$ is a variable representing the amount transported from depot i to client j and $v_{i,j}$ is the associated binary setup variable.

In this description, the role of clients and depots are interchangeable. Indeed, this problem can be modelled as a bipartite graph in which nodes are either depots or clients and edges between a depot and a client exist if the client can be served from that depot. A standard variant (and indeed a special case) is that in which the demand of each client must be satisfied, in which case the unit profit is usually replaced by a unit cost.

Problem FCT can be considered as a basic problem in supply chain management in its own right, and is also a special fixed-charge network flow problem. However surprisingly few polyhedral results are known for FCT. When there is only one client or one depot, FCT reduces to a single node flow set for which the (lifted) cover and reverse cover inequalities have been described and shown to be effective [PvRW85, vRW87, GNS99]. Note that this also implies that FCT is NP-Hard. The flow structure of FCT is similar to that of the Capacitated Facility Location (CFL) problem, but this last problem has fixed cost for opening depots (nodes) as opposed to transportation (edges). Known valid inequalities are essentially flow cover type inequalities [Aar98, CFLP00]. As discussed in Section 2, FCT also appears to have strong ties to the multi-item big-bucket lot-sizing problem (BBLS). This actually constitutes the initial motivation of this paper.

In this paper, we study the polynomially solvable special case of FCT in which depots and clients are arranged in an alternating manner, and can only serve/be served by their one or two immediate neighbours (FCTP). From a graphical perspective, the associated bipartite graph is a path. This special case FCTP is also a relaxation of FCT, and it can therefore be hoped that polyhedral results for FCTP will be helpful in solving the general case.

Let $[k,l]$ denote $\{k, k+1, \ldots, l\}$. For FCTP, we assume that we have $n+1$ depots and clients (or nodes) in total, indexed by the set $[0,n]$. We index depot-client pairs $(i-1,i)$ (or edges) by $i \in [1,n]$. Depots are represented by even nodes while clients are represented by odd nodes. The problem FCTP can be formulated as the following mixed-integer program:

$$\max \quad \sum_{i=1}^{n} p_j x_j - \sum_{i=1}^{n} f_j y_j, \tag{6}$$

$$x_i + x_{i+1} \leq a_i, \qquad \forall i \in [0,n], \tag{7}$$

$$x_0 = x_{n+1} = 0 \tag{8}$$

$$0 \leq x_i \leq \min(a_{i-1}, a_i) y_i, \qquad \forall i \in [1,n], \tag{9}$$

$$y \in \{0,1\}, \qquad \forall i \in [1,n],, \tag{10}$$

where x_i is the amount transported between $i-1$ and i, y_i is the setup variable associated with x_i, and p_i and f_i are respectively the unit profit and the fixed cost of transportation between $i-1$ and i. The dummy variables x_0 and x_{n+1} are introduced for notational convenience. We denote the set of feasible solutions to (7)–(10) by X^{FCTP}.

Program (7)–(10) can easily be recast into the framework introduced by Conforti et al. [CDSEW09] by operating the change of variables $x'_i = x_i$ for i odd and $x'_i = -x_i$ for i even. Recall that in this framework, variables are represented by nodes and constraints of the type $x - y \geq b$ are represented by a directed arc (x,y). Then arcs (x'_i, x'_{i-1}) and (x'_i, x'_{i+1}) for i even represent constraints (7) while arcs (y_i, x'_i) for i odd and (x'_i, y_i) for i even represent constraints (9). Constraints (8) and non-negativity for x can be modelled using additional dummy integer variables (nodes).

In this graph, the subgraph associated to the nodes representing continuous variables is a simple path. Therefore, it trivially only contains a polynomial number of subpaths, so that FCTP admits a compact extended formulation and is therefore polynomially solvable. Parts of Section 5 and Section 6 can be seen as a specialization for FCTP of the general results of Conforti et al. [CDSEW09] concerning the structure of extreme points and extended formulations. In addition, in the special case of FCTP we are able to describe a dedicated combinatorial optimization algorithm and to project the extended formulation onto the original variable space.

Recently, using the same framework, Di Summa and Wolsey [DSW10] aim at studying the mixed-integer set in which continuous nodes are linked by a bidirected path. This model subsumes FCTP. However they are only able to characterize the convex hull of the set for two special cases that do not subsume FCTP.

The rest of the paper is organized as follows.

We show in Section 2 that FCT is a special case of BBLS, but also that FCT is a strong relaxation of BBLS when modelled using the facility location reformulation [KB77]. In Section 3 we study the max-flow variant of FCTP (no fixed cost and a profit of 1 for all depot-client pairs). In particular, we give a simple optimization algorithm and prove basic properties of its solution that will be useful in the following sections. In Section 4, we introduce a new class of valid inequalities for FCTP that we call the "path-modular inequalities". We also give an $\mathcal{O}(n^3)$ separation algorithm. In Section 5 we characterize extreme points of X^{FCTP}, directly leading to a $\mathcal{O}(n^3)$ dynamic program for solving FCTP. In Section 6 we give two linear-programming extended formulations with $\mathcal{O}(n^2)$ nonnegative variables, $\mathcal{O}(n^2)$ constraints and $\mathcal{O}(n^2)$ non-zero coefficients. The first one is more compact, and we can characterize the projection of the second one onto the original variable space. We show that the equalities we obtain are the "path-modular inequalities", thereby showing that they are sufficient to describe the convex hull of X^{FCTP}. We report on computational experiments in solving instances of FCTP using the different formulations introduced in Section 7. Section 8 shows that the path-modular inequalities introduced in this paper can sometimes explain a substantial number of facets of the fixed-charge transportation problem on complete bipartite graph (FCT). We conclude by discussing future research on the topic.

Notation. Throughout this paper we use the following notation: $[k,l] = \{k, k+1, \dots, l\}$, $N = [1,n]$, E and O denote the even and odd integers respectively, $\underline{e}_i$ denotes the unit vector with component i equal to 1 and all others components equal to 0, $\underline{1}$ denotes the all ones vector, and $\tilde{y}$ denotes the vector y with odd components being complemented (i.e. replaced by $1 - y_i$).

2 Fixed charge transportation and big-bucket multi-item lot-sizing

In its simplest form, the big-bucket multi-item lot-sizing problem (BBLS) involves given demands $D_{i,t} \geq 0$ for each of a collection of products $i \in I$ and in each time period $t \in \{1, \dots, T\}$ that must be satisfied through production. Production must take place before demand is due. Production of product i in period t implies a fixed cost $f_{i,t}$ and a variable unit cost $p_{i,t}$. Finally, the total number of items produced in any time period t cannot exceed a given amount C_t.

The most natural way to formulate this problem as a mixed-integer program is to define variables x^i_t representing the number of products i produced in period t and associated binary setup variables y^i_t. The problem is then formulated as

$$\min \quad \sum_{i \in I} \sum_{t=1}^{T} (p^i_t x^i_t + f^i_t y^i_t), \tag{11}$$

$$\sum_{t=1}^{k} x^i_t \geq \sum_{t=1}^{k} D^i_t \qquad \forall i \in I, k = 1, \dots, T-1 \tag{12}$$

$$\sum_{t=1}^{T} x^i_t = \sum_{t=1}^{T} D^i_t \qquad \forall i \in I \tag{13}$$

$$\sum_{i \in I} x^i_t \leq C_t \qquad t = 1, \dots, T \tag{14}$$

$$0 \leq x^i_t \leq \min(\sum_{k=t}^{T} D^i_k, C_t) y^i_t \qquad \forall i \in I, t = 1, \dots, T, \tag{15}$$

$$y^i_t \in \{0, 1\}, \qquad \forall i \in I, t = 1, \dots, T. \tag{16}$$

Consider an instance in which the demand is zero for all periods except the last one, the capacity C_1 of period 1 is larger than the total demand $\sum_{i \in I} D_{i,T}$ and where $f_{i,1} = 0$ for all $i \in I$. Then eliminating variables $x_{i,1}$ by substitution using the equality constraints (13), we obtain:

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{t=2}^{T}((p_t^i - p_1^i)x_t^i + f_t^i y_t^i) + \text{constant}, & \\
& \sum_{t=2}^{T} x_t^i \le D_T^i & \forall i \in I, \\
& \sum_{i\in I} x_t^i \le C_t & t = 2,\dots,T, \\
& 0 \le x_t^i \le \min(D_T^i, C_t) y_t^i & \forall i \in I, t = 2,\dots,T, \\
& y_t^i \in \{0,1\}, & \forall i \in I, t = 2,\dots,T.
\end{aligned}
$$

This last problem is exactly a fixed charge transportation problem with set of depots $\{2,\dots,T\}$ and set of clients I. Therefore FCT is a special case of BBLS and one has little hope to solve BBLS efficiently if one cannot solve FCT efficiently.

Furthermore, we now show that, besides being a special case of BBLS, FCT is also a strong relaxation of BBLS. Improving the formulation (12)-(16), the stronger facility location reformulation [KB77] of BBLS is obtained by defining variables $z_{t,k}^i$ representing the amount of item i produced in period t to satisfy demand in period k:

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{t=1}^{T}(\sum_{k=t}^{T} p_t^i z_{t,k}^i + f_t^i y_t^i), & & (17) \\
& \sum_{t=1}^{k} z_{t,k}^i = D_k^i & \forall i \in I, k = 1,\dots,T, & \quad (18) \\
& \sum_{i\in I}\sum_{k=t}^{T} z_{t,k}^i \le C_t & t = 1,\dots,T, & \quad (19) \\
& 0 \le z_{t,k}^i \le \min(C_t, D_k^i) y_t^i, & \forall i \in I, t = 1,\dots,T, k = t,\dots,T, & \quad (20) \\
& y_t^i \in \{0,1\}, & \forall i \in I, t = 1,\dots,T. & \quad (21)
\end{aligned}
$$

Firstly, because $\sum_{i\in I}\sum_{t=1}^{T}\sum_{k=t}^{T} z_{t,k}^i$ sums to the constant $\sum_{i\in I}\sum_{k=1}^{T} D_k^i$ by constraints (18), we can replace the equality constraint (18) by a $\le$ constraint if we replace p_t^i by $p_t^i - M$ in the objective (with M large enough). This will not modify the set of optimal solutions.

Secondly, we replace variables y_t^i by variables $y_{t,k}^i$ together with constraints $y_{t,k}^i = y_{t,k+1}^i$ for all $k < T$.

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{t=1}^{T}(\sum_{k=t}^{T}((p_t^i - M)z_{t,k}^i + f_t^i y_{t,T}^i), & \\
& \sum_{t=1}^{k} z_{t,k}^i \le D_k^i & \forall i \in I, k = 1,\dots,T, \\
& \sum_{i\in I}\sum_{k=t}^{T} z_{t,k}^i \le C_t & t = 1,\dots,T, \\
& 0 \le z_{t,k}^i \le \min(C_t, D_k^i) y_{t,k}^i, & \forall i \in I, t = 1,\dots,T, k = t,\dots,T, \\
& y_{t,k}^i \in \{0,1\}, & \forall i \in I, t = 1,\dots,T, k = t,\dots,T, \\
& y_{t,k}^i = y_{t,k+1}^i & \forall i \in I, t = 1,\dots,T, k = t,\dots,T-1,
\end{aligned}
$$

Relaxing the last constraint, we obtain exactly a fixed-charge transportation problem with depots $\{1,\dots,T\}$ and clients $I \times \{1,\dots,T\}$ in which depot t cannot serve client (i,k) if $t > k$ (i.e. the bipartite graph is not complete unless backlogging was allowed). This shows that BBLS is essentially equivalent to FCT with setup costs on family of arcs instead of individual arcs.

These tight links between BBLS and FCT motivate the hope that a better understanding of the polyhedral structure of the convex hull of solutions to FCT may help in solving BBLS more efficiently.

3 Building blocks: the max-flow variant

In this section, we study the max-flow variant of FCTP that is formulated as:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} x_j, & \\
& (7) - (8) & \\
& 0 \le x_i, & \forall i \in [1,n],
\end{aligned}
$$

We will refer to this problem as MF-FCTP. The various technical results of this section will be key in proving some results in the subsequent sections.

In this max-flow variant, the simple algorithm that takes each variable x_i in turn for $i \in [1,n]$ in turn and assigns to it the maximum possible value is correct. A backward pass then outputs an optimal dual solution. This is formalized in the next algorithm and proposition.

$\text{LEX}(n, a)$

1 $u \leftarrow 0,\ x_0 \leftarrow 0$
2 **for** $i \leftarrow 1$ **to** n
3   **do** $x_i \leftarrow \min(a_{i-1} - x_{i-1}, a_i)$
4 $l \leftarrow n + 1$
5 **for** $i \leftarrow n$ **to** 1
6   **do if** $x_i = a_i$
7   **then** $u_i \leftarrow 1$
8             $l \leftarrow i$
9   **else if** $l - i$ is even
10   **then** $u_i \leftarrow 1$
11   **else** $u_{i-1} \leftarrow 1$
12 **return** x, u

Proposition 1 *Algorithm* LEX *outputs an optimal primal-dual solution to MF-FCTP.*

Proof. The vector x is primal feasible throughout the algorithm. The vector u is dual feasible at termination as for each $i \in [1, n]$, at least one of u_{i-1} and u_i is assigned the value 1. It remains therefore to show that they are complementary.

The only case in which $x_j + x_{j+1} < a_j$ is when x_{j+1} is assigned the value a_{j+1} at Line 3 for $i = j+1$. This first implies that $x_j < a_j$ and second that in the backward pass, Lines 7-8 are executed for $i = j+1$ and line 11 is executed for $i = j$. Therefore $u_j = 0$.

There are two cases in which $u_{j-1} + u_j > 1$. The first is when Line 7 is executed both for $i = j-1$ and $i = j$. But then $a_{j-1} + a_j = x_{j-1} + x_j \leq a_{j-1}$ and therefore $x_j = 0$. The second case is when Line 10 is executed for $i = j$ and Line 7 is executed for $i = j-1$. But then $x_{j-1} = a_{j-1}$ and therefore $x_j = 0$. ■

Corollary 2 *Algorithm* LEX *outputs a lexicographic optimal solution x: this solution x maximizes $\sum_{i=1}^{k} x_i$ among all the optimal solutions for all k (or equivalently minimizes $\sum_{i=k}^{n} x_i$ for all k).*

Proof. For $k < n$, $\text{LEX}(k, a)$ assigns the same values to $x_1, \ldots, x_k$ as $\text{LEX}(n, a)$. It remains to note that by Proposition 1 $\text{LEX}(k, a)$ outputs a solution that maximizes $\sum_{i=1}^{k} x_i$. ■

Let us define the following set of values for $i \in [1, n]$ and $j \in [0, n+1]$:

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } j = i, \\ \min(a_{i-1} - \alpha_{i-1,j}, a_i) & \text{if } j < i, \\ \min(a_i - \alpha_{i+1,j}, a_{i-1}) & \text{if } j > i, \end{cases}$$

assuming $\alpha_{0,0} = \alpha_{n+1,n+1} = 0$. LEX precisely outputs $x_i = \alpha_{i,0}$. Let REV-LEX be the same algorithm as LEX but starting from $x_{n+1} = 0$ and working in decreasing order of i. REV-LEX yields another optimal solution, but one that is lexicographically reverse. The value for x output by REV-LEX is $\alpha_{i,n+1}$.

For each $j \in [1, n]$, one could also apply LEX (resp. REV-LEX) to the subinterval $[j+1, n]$ (resp. $[1, j-1]$). These runs would exactly output values $x_i = \alpha_{i,j}$. We illustrate this notation by the following example.

Example 1 *We consider an instance with* $n = 6$ *and* $a = (5, 8, 6, 5, 7, 6, 3)$. *Then* α *is the following matrix:*

$$\alpha = \begin{pmatrix} 5 & 0 & 5 & 2 & 5 & 2 & 5 & 3 \\ 3 & 6 & 0 & 6 & 1 & 6 & 2 & 5 \\ 3 & 0 & 5 & 0 & 5 & 0 & 4 & 1 \\ 2 & 5 & 0 & 5 & 0 & 5 & 1 & 4 \\ 5 & 2 & 6 & 2 & 6 & 0 & 6 & 3 \\ 1 & 3 & 0 & 3 & 0 & 3 & 0 & 3 \end{pmatrix}$$

All columns of the matrix constitute feasible solutions, but only the first and last columns, output by LEX *and* REV-LEX *respectively, maximize* $\sum_{i=1}^{n} x_i$. *Row* i *holds all the possible different values for* x_i *contructed by these variants of* LEX.

The following proposition will be key in proving the validity of the path-modular inequalities in Section 4. It shows that all elements to the left (or to the right) of a diagonal element can be ordered as follows:

- $\min(a_{i-1}, a_i) = \alpha_{i,i-1} \geq \alpha_{i,i-3} \geq \ldots \geq \alpha_{i,0 \text{ or } 1} \geq \alpha_{i,1 \text{ or } 0} \geq \ldots \geq \alpha_{i,i-2} \geq \alpha_{i,i} = 0$,
- $\min(a_{i-1}, a_i) = \alpha_{i,i+1} \geq \alpha_{i,i+3} \geq \ldots \geq \alpha_{i,n \text{ or } n+1} \geq \alpha_{i,n+1 \text{ or } n} \geq \ldots \geq \alpha_{i,i+2} \geq \alpha_{i,i} = 0$.

Proposition 3 *For all* $i \in [1, n]$ *and* $j, k \in [0, n+1]$,

(i) if $i - j \in O$ *and* $k < j \leq i$, *then* $\alpha_{i,j} \geq \alpha_{i,k}$,

(ii) if $i - j \in O$ and $k > j \geq i$, then $\alpha_{i,j} \geq \alpha_{i,k}$,

(iii) if $i - j \in E$ and $k < j \leq i$, then $\alpha_{i,j} \leq \alpha_{i,k}$,

(iv) if $i - j \in E$ and $k > j \geq i$, then $\alpha_{i,j} \leq \alpha_{i,k}$.

Before proving the result, we illustrate this on an example and prove a lemma.

Example 1 [Continued] *On the second row of α, starting from position $(2, 3)$, we walk to the right skipping one element at each step, then walk back to the diagonal element through the skipped elements to obtain $(6, 6, 5, 2, 1, 0)$. This is indeed a decreasing sequence.*

Lemma 4 *1. For $j < i$, $\alpha_{i,j}$ is*

(a) monotonously increasing with a_i,

(b) monotonously increasing with a_{i-t} for $t \in [1, i - j + 1] \cap O$,

(c) monotonously decreasing with a_{i-t} fort $\in [1, i - j + 1] \cap E$,

(d) independent of a_t for $t > i$ or $t < j - 1$.

2. For $j > i$, $\alpha_{i,j}$ is

(a) monotonously increasing with a_{i-1},

(b) monotonously increasing with a_{i+t} for $t \in [0, n - i] \cap E$,

(c) monotonously decreasing with a_{i+t} for $t \in [0, n - i] \cap O$,

(d) independent of a_t with $t < i - 1$.

Proof. We start with trivial observations:

- O1: $\min(a, b)$ is increasing with a or b if a does not depend on b and conversely,
- O2: $a - \min(b, a)$ is increasing in a if b is independant of a.
- O3: A decreasing function of an increasing (resp. decreasing) function is decreasing (resp. increasing).

Let us fix j. $\alpha_{j+1,j} = \min(a_j, a_{j+1})$ which is increasing with a_j and a_{j+1} by O1. $\alpha_{j+2,j} = \min(a_{j+1} - \alpha_{j+1,j}, a_{j+2})$ is increasing with a_{j+2} by O1, increasing with a_{j+1} by O2 and decreasing with a_j by O3. $\alpha_{j+3,j} = \min(a_{j+2} - \alpha_{j+2,j}, a_{j+3})$ is increasing with a_{j+3} by O1, increasing with a_{j+2} by O2, decreasing with a_{j+1} by O3 and increasing with a_j by O3.

Iterating this argument yields 1(a), 1(b) and 1(c). 1(d) follows directly from the definition of α. Cases 2 are similar to case 1. ∎

Proof of Proposition 3. Case (i). Let the vector a^j and a^k be defined as follows:

$$a^j_l = \begin{cases} 0 & \text{if } l = k-1 \text{ or } l = j-1, \\ a_l & \text{otherwise} \end{cases} \qquad a^j_l = \begin{cases} 0 & \text{if } l = k-1, \\ a_l & \text{otherwise} \end{cases}$$

so that a^k is obtained from a^j by increasing the $j-1^{th}$ component. Observe that $\alpha_{i,j}$ is the value for x_i output by $\text{Lex}(n, a^j)$ as the algorithm sets $x_j = 0$. Similarly $\alpha_{i,k}$ is the value for x_i output by $\text{Lex}(n, a^k)$. It remains to note that Lemma 4 case 1.c applies so that $\alpha_{i,j} \geq \alpha_{i,k}$. The other cases are similar. ∎

4 Path-modular inequalities

In this section we derive a new family of inequalities that we call path-modular inequalities. The proof that they are valid for FCTP relies on submodularity and supermodularity properties of the following set function. For a given set $S \subseteq [1,n]$ and vector $a \in \mathbb{R}^{n+1}_+$, let $\phi(S,a)$ be defined as

$$\begin{aligned} \phi(S,a) = \max \quad & \sum_{i=1}^{n} x_j, \\ & (7)-(8) \\ & x_i \geq 0, \qquad \forall i \in S, \\ & x_i = 0, \qquad \forall i \in N \setminus S. \end{aligned}$$

For notational convenience, we will sometimes omit the argument a in $\phi(S,a)$ when we unambiguously refer to the original input data of the problem.

Let $\rho_i(S) = \phi(S+i) - \phi(S)$ be the increment function of i at S. Clearly $\rho_i(S)$ is always nonnegative, but it is neither globally submodular ($\rho_i(S) \geq \rho_i(T)$ for all $S \subset T, i \notin T$) nor supermodular ($\rho_i(S) \leq \rho_i(T)$ for all $S \subset T, i \notin T$), as illustrated by the next example.

Example 1 [Continued] $\rho_1(\{2\}) = \phi(\{1,2\}) - \phi(\{2\}) = 8-6 = 2$. $\rho_1(\{2,3\}) = \phi(\{1,2,3\}) - \phi(\{2,3\}) = 11-6 = 5$. *Therefore* $\rho_1(\{2,3\}) > \rho_1(\{2\})$ *which is a supermodular-type property: opening edge* 1 *increases* ϕ *more when edge* 3 *is already open.*

But $\rho_1(\emptyset) = \phi(\{1\}) - \phi(\emptyset) = 5-0 = 5$. *Therefore* $\rho_1(\{2\}) < \rho_1(\emptyset)$ *which is submodular-like.*

We now define the path-modular inequalities. Let L be a subinterval $[l, l']$ of N, let $L = O_L \cup E_L$ be the partition of L into odd and even numbers, and let $\mathcal{L} = (j_1, j_2, \ldots, j_{|L|})$

be a permutation of L. Let $O_L^{j_k} = \{j_1, \dots, j_k\} \cap O_L$ and let $E_L^{j_k} = \{j_1, \dots, j_{k-1}\} \cap E_L$. We call the the following inequality the $(l, l', \mathcal{L})$-path-modular inequality:

$$\sum_{i \in L} x_i \leq \phi(O_L) + \sum_{i \in E_L} \rho_i(O_L \cup E_L^i \setminus O_L^i) y_i - \sum_{i \in O_L} \rho_i(O_L \cup E_L^i \setminus O_L^i)(1 - y_i) \tag{22}$$

The following proposition essentially characterizes certain vectors y at which a given path-modular inequality is tight.

Proposition 5 *Let an interval $L = [l, l']$ and a permutation $\mathcal{L} = (j_1, j_2, \dots, j_{|L|})$ of L be given. For each $k \in [0, |L|]$, there exists a point $(x^k, y^k) \in X^{FCTP}$ which is tight for the corresponding $(l, l', \mathcal{L})$-path-modular inequality, satisfying*

$$y^k = \sum_{i \in O_L} \underline{e}_i \bigtriangleup \sum_{1}^{k} \underline{e}_{j_k}$$

and at which the inequality reduces to $\sum_{i \in Y^k} x_i \leq \phi(Y^k)$ where $Y^k = \{i \in N : y_i^k = 1\}$.

Proof. The proof is by induction on k. For $k = 0$, $Y^0 = O_L$ and the path modular inequality reduces to $\sum_{i \in O_L} x_i \leq \phi(O_L)$. By definition of ϕ, there exists x such that this inequality is satisfied at equality.

So let us assume that the proposition is true for $k-1$ and $k > 0$. If j_k is even, then the inequality reduces to $\sum_{i \in Y^k} x \leq \phi(Y^{k-1}) + \rho_{j_k}(Y^{k-1}) = \phi(Y^{k-1} + j_k) = \phi(Y^k)$. If j_k is odd, then the inequality reduces to $\sum_{i \in Y^k} x \leq \phi(Y^{k-1}) - \rho_{j_k}(Y^{k-1} - j_k) = \phi(Y^k - j_k) = \phi(Y^k)$. By definition of ϕ, there exists x such that this inequality is satisfied at equality. ■

In particular the previous proposition shows that all path-modular inequalities are tight at points with exactly either all odd or all even edges open. Before proving the validity of the path-modular inequalities we look at a few examples.

Example 1 [Continued] *Let $L = [1, 6]$ and consider the ordering $\mathcal{L} = (2, 4, 6, 1, 3, 5)$. Then the path-modular inequality is*

$$\begin{aligned}
\sum_{i=1}^{6} x_i \leq\ & \phi(\{1,3,5\}) + \rho_2(\{1,3,5\}) y_2 + \rho_4(\{1,2,3,5\}) y_4 + \rho_6(\{1,2,3,4,5\}) y_6 \\
& - \rho_1(\{2,3,4,5,6\})(1 - y_1) - \rho_3(\{2,4,5,6\})(1 - y_3) - \rho_5(\{2,4,6\})(1 - y_5) \\
\leq\ & 16 + 1y_2 + 1y_4 + 1y_6 - 3(1 - y_1) - 0(1 - y_3) - 2(1 - y_5) \\
=\ & 11 + 3y_1 + y_2 + y_4 + 2y_5 + y_6
\end{aligned}$$

and is facet-defining. This inequality is tight at points satisfying $y = (1,0,1,0,1,0)$, $(1,1,1,0,1,0)$, $(1,1,1,1,1,0)$, $(1,1,1,1,1,1)$, $(0,1,1,1,1,1)$, $(0,1,0,1,1,1)$, $(0,1,0,1,0,1)$.

With the different ordering $(4,3,6,2,1,5)$*, one obtains*

$$\begin{aligned}\sum_{i=1}^{6} x_i &\leq \phi(\{1,3,5\}) + \rho_4(\{1,3,5\})y_4 + \rho_3(\{1,4,5\})(1-y_3) + \rho_6(\{1,4,5\})y_6 \\ &\qquad + \rho_2(\{1,4,5,6\})y_2 - \rho_1(\{2,4,5,6\})(1-y_1) - \rho_5(\{2,4,6\})(1-y_5) \\ &\leq 16 + 0y_4 - 4(1-y_3) + 3y_6 + 3y_2 - 2(1-y_1) - 2(1-y_5) \\ &= 8 + 2y_1 + 3y_2 + 4y_3 + 0y_4 + 2y_1 + 3y_6\end{aligned}$$

which is valid but not facet-defining. This inequality is tight at points satisfying $y = (1,0,1,0,1,0)$, $(1,0,1,1,1,0)$, $(1,0,0,1,1,0)$, $(1,0,0,1,1,1)$, $(1,1,0,1,1,1)$, $(0,1,0,1,1,1)$, $(0,1,0,1,0,1)$.

In total, there are 720 orderings but there are only 9 distinct facets for the interval L=[1,6] (see Appendix A).

The next lemma characterizes $\rho_i(N-i)$ or what can be gained in ϕ by opening arc i when all the other arcs are open. The result then easily generalizes to $\rho_i(S)$ for general S by introducing new notation only.

Lemma 6 $\rho_i(N-i) = \min(\alpha_{i,0}, \alpha_{i,n+1})$.

Proof. Clearly the following decomposition holds: $\phi(N-i,a) = \phi([1,i-1],a) + \phi([i+1,n],a)$. By Proposition 1 and definition of α we obtain $\phi(N-i,a) = \sum_{k=1}^{i-1}\alpha_{k,0} + \sum_{k=i+1}^{n+1}\alpha_{k,n}$.

More generally, fixing $x_i = k$ decomposes the problem into two intervals $[1,i-1]$ and $[i+1,n]$, so we can also write:

$$\phi(N,a) = \max_k\{\phi([1,i-1], a - k\underline{e}_{i-1}) + k + \phi([i+1,n], a - k\underline{e}_{i+1})\}. \tag{23}$$

Let us characterize $\phi([1,i-1], a-k\underline{e}_{i-1})$ as a function of k by assuming that it is computed using Algorithm Lex. This is indeed possible because $[1,i-1]$ is an interval. Algorithm Lex will assign the same values to $x_1 = \alpha_{1,0}, \ldots, x_{i-2} = \alpha_{i-2,0}$ for all values of k. It is then a simple observation that, by definition of α and Lex, x_{i-1} will be assigned the value

$$x_{i-1} = \begin{cases}\alpha_{i-1,0} \text{ if } k \leq a_{i-1} - \alpha_{i-1,0} \\ \alpha_{i-1,0} - (k - a_{i-1} + \alpha_{i-1,0}) \text{ if } k > a_{i-1} - \alpha_{i-1,0}\end{cases}$$

Similarly for $\phi([i+1,n], a - k\underline{e}_{i+1})$ computed by Rev-Lex: it is equal to the constant $\sum_{k=i}^{n}\alpha_{k,n+1}$ for $k \leq a_i - \alpha_{i+1,n+1}$ and then decreases with k. Therefore the maximum of the right hand-side is attained at $k = \min(a_{i-1} - \alpha_{i-1,0}, a_i - \alpha_{i+1,n+1})$, for a value of $\phi(N,a) = \phi(N-i,a) + k = \phi(N-i,a) + \min(a_{i-1} - \alpha_{i-1,0}, a_i - \alpha_{i+1,n+1})$. Therefore

$$\begin{aligned}
\rho_i(N-i) &= \min(a_{i-1} - \alpha_{i-1,0}, a_i - \alpha_{i+1,n+1}) \\
&= \min(a_{i-1} - \alpha_{i-1,0}, a_i, a_{i-1}, a_i - \alpha_{i+1,n+1}) \\
&= \min(\min(a_{i-1} - \alpha_{i-1,0}, a_i), \min(a_{i-1}, a_i - \alpha_{i+1,n+1})) \\
&= \min(\alpha_{i,0}, \alpha_{i,n+1})
\end{aligned}$$

■

To generalize the proposition to $\rho_i(S)$, we need the following observation: suppose $i, j \notin S$ and $j < i$. Then $\rho_i(S)$ does not depend on whether $k \in S$ for $k < j$. The situation is similar for $j > i$ and $k > j$.

Observation 1 *Let $l_i(S) = \max_j\{j : j < i, j \notin S\}$ and $r_i(S) = \min_j\{j : j > i, j \notin S\}$. Then $\rho_i(S) = \rho_i(T)$ if $l_i(S) = l_i(T)$ and $r_i(S) = r_i(T)$.*

The next proposition follows.

Proposition 7 *$\rho_i(S) = \min(\alpha_{i,l_i(S)}, \alpha_{i,r_i(S)})$.*

We are now ready to prove the main technical result of this section.

Proposition 8 *[Sub- and Supermodularity properties of ϕ] Let $S \subset T \subseteq N$ and $i \in N \setminus T$ be given.*

(i) $\rho_i(T) \geq \rho_i(S)$ if the two following conditions (a) and (b) hold:

a. $l_i(S) = l_i(T)$ or $l_i(S) - i$ is even,

b. $r_i(S) = r_i(T)$ or $r_i(S) - i$ is even.

(ii) $\rho_i(T) \leq \rho_i(S)$ if the two following conditions (a) and (b) hold:

a. $l_i(S) = l_i(T)$ or $l_i(S) - i$ is odd,

b. $r_i(S) = r_i(T)$ or $r_i(S) - i$ is odd.

Proof. Because $S \subseteq T$, we have that $l_i(T) \leq l_i(S)$ and $r_i(T) \geq r_i(S)$.

Case (i). Either $\alpha_{i,l_i(T)} = \alpha_{i,l_i(S)}$ because $l_i(S) = l_i(T)$ or Case (iii) of Proposition 3 applies so that $\alpha_{i,l_i(T)} \geq \alpha_{i,l_i(S)}$ holds. Similarly, either $\alpha_{i,r_i(T)} = \alpha_{i,r_i(S)}$ because $l_i(S) = l_i(T)$

or case (iv) of Proposition 3 applies so that $\alpha_{i,r_i(T)} \geq \alpha_{i,r_i(S)}$. Then $\rho_i(T) \geq \rho_i(S)$ follows by Proposition 7.

Case (ii). Either $\alpha_{i,l_i(T)} = \alpha_{i,l_i(S)}$ because $l_i(S) = l_i(T)$ or Case (i) of Proposition 3 applies so that $\alpha_{i,l_i(T)} \leq \alpha_{i,l_i(S)}$ holds. Similarly, either $\alpha_{i,r_i(T)} = \alpha_{i,r_i(S)}$ because $l_i(S) = l_i(T)$ or case (ii) of Proposition 3 applies so that $\alpha_{i,r_i(T)} \leq \alpha_{i,r_i(S)}$. Then $\rho_i(T) \leq \rho_i(S)$ follows by Proposition 7. ■

The following corollary specializes Proposition 8 to exactly what is needed to prove the validity of the path-modular inequalities. Essentially, the corollary tells us that ϕ exhibits supermodularity when we keep on opening edges of the same parity as the edge i under consideration. Conversely, it tells us also that ϕ exhibits submodularity when we keep on opening edges of the opposite parity compared to the edge i under consideration. Note that, broadly speaking, this is also the case for flow cover inequalities: in that case each edge is at an odd distance from any other one, so that the associated max-flow function is completely submodular.

Corollary 9 *Let $S \subset T \subseteq N$ and $i \in N \setminus T$ be given.*

(i) If $(T \setminus S) \subseteq E$ and $i \in E$, then $\rho_i(T) \geq \rho_i(S)$.

(ii) If $(T \setminus S) \subseteq O$ and $i \in O$, then $\rho_i(T) \geq \rho_i(S)$.

(iii) If $(T \setminus S) \subseteq E$ and $i \in O$, then $\rho_i(T) \leq \rho_i(S)$.

(iv) If $(T \setminus S) \subseteq O$ and $i \in E$, then $\rho_i(T) \leq \rho_i(S)$.

Proof. All four statements are special cases of Proposition 8.

(i) If $l_i(S)$ is odd then $l_i(S) = l_i(T)$ because $l_i(S)$ cannot be in $T \setminus S$. If $l_i(S)$ is even, then $l_i(S) - i$ is even. Therefore condition (i.a) of Proposition 8 applies. Similarly condition (i.b) applies as well, so that $\rho_i(T) \geq \rho_i(S)$.

(ii) similar to case (i).

(iii) If $l_i(S)$ is odd then $l_i(S) = l_i(T)$ because $l_i(S)$ cannot be in $T \setminus S$. If $l_i(S)$ is even, then $l_i(S) - i$ is odd. Therefore condition (ii.a) of Proposition 8 applies. Similarly condition (ii.b) applies as well, so that $\rho_i(T) \leq \rho_i(S)$.

(iv) similar to case (iii). ■

We are now ready to prove the validity of the path-modular inequalities.

Proposition 10 *The path-modular inequalities* (22) *are valid for X^{FCTP}.*

Proof. Let a feasible point $(x, y) \in X^{FCTP}$ be given with $Y = \{i \in N : y_i = 1\}$. Let $L \subseteq N$ be a set of consecutive integers and let a permutation $\mathcal{L} = (j_1, j_2, \ldots, j_{|L|})$ of L be given. We show that the point (x, y) satisfies the corresponding path-modular inequality.

Let $\bar{E}_L^i = E_L^i \cap Y$ and $\bar{O}_L^i = O_L^i \cap Y$. The following relations hold

$$
\begin{aligned}
&\sum_{i \in L} x_i \leq \phi(Y) \\
&\qquad = \phi(O_L) - \sum_{i \in O_L \setminus Y} \rho_i(O_L \cup \bar{E}_L^i \setminus \bar{O}_L^i) + \sum_{i \in E_L \cap Y} \rho_i(O_L \cup \bar{E}_L^i \setminus \bar{O}_L^i) \\
&\qquad \leq \phi(O_L) - \sum_{i \in O_L \setminus Y} \rho_i(O_L \cup \bar{E}_L^i \setminus O_L^i) + \sum_{i \in E_L \cap Y} \rho_i(O_L \cup \bar{E}_L^i \setminus O_L^i) \\
&\qquad \leq \phi(O_L) - \sum_{i \in O_L \setminus Y} \rho_i(O_L \cup E_L^i \setminus O_L^i) + \sum_{i \in E_L \cap Y} \rho_i(O_L \cup E_L^i \setminus O_L^i) \\
&\qquad = \phi(O_L) - \sum_{i \in O_L} \rho_i(O_L \cup E_L^i \setminus O_L^i)(1 - y_i) + \sum_{i \in E_L} \rho_i(O_L \cup E_L^i \setminus O_L^i) y_i
\end{aligned}
$$

where the first inequality is by definition of ϕ, the first equality is by definition of the increment function ρ and the fact that the set argument of each term (ordered according to the order $(j_1, j_2, \ldots, j_{|L|})$) differs from the previous term by exactly the element being incremented or decremented, the second inequality is an application of Corollary 9 (ii) for the first term and (iv) for the second term, the third inequality is an application of Corollary 9 (iii) for the first term and (i) for the second term, and finally the last equality holds because the added terms in the sum are null. ■

We will prove in Section 6 that the path-modular inequalities are sufficient to describe the convex hull of X^{FCTP} by projecting an extended formulation. We now turn to the question of separating a point (x^*, y^*) with $x_i^* \geq 0$ and $y_I^* \leq 1$ for all $i \in N$ from the polyhedron defined by the path-modular inequalities. For fixed L, this reduces to choose the ordering $\mathcal{L}$.

Consider first the difference between two path-modular inequalities that differ only by the ordering of two adjacent even indices i and j. Observe that interchanging two adjacent indices of the same parity leaves the coefficients of all other variables and the right-hand side of the inequality unchanged. When i is before j the coefficients of y_i and y_j are respectively $\rho_i(S)$ and $\rho_j(S + i)$ for some $S \subseteq N$. If the order is reversed, the coefficients are respectively $\rho_i(S + j)$ and $\rho_j(S)$. Note that in both inequalities, the sum of the coefficients of y_i and y_j is equal to $\phi(S + i + j) - \phi(S)$ and is thus the same. Therefore the difference between the coefficients of y_i in both inequalities which is $\rho_i(S) - \rho_i(S + j)$, and that of y_j which is $\rho_j(S + i) - \rho_j(S)$ have opposite values. Therefore, to maximize the violation, we must order the variables with adjacent even indices in decreasing order of y_i. Similarly, we must order the variables with adjacent odd indices in increasing order of y_i.

Consider now two path-modular inequalities that differ only by the ordering of two adjacent indices i and j of different parity. Observe that that the coefficients of variables corresponding to all other indices remain unchanged. Assume i is even and j is odd. When i is before j, the coefficients of y_i and $(1-y_j)$ are respectively $\rho_i(S)$ and $\rho_j(S+i-j)$ for some $S \subseteq N$. When j is before i, the coefficients of y_i and $(1-y_j)$ are respectively $\rho_i(S-j)$ and $\rho_j(S-j)$. Note that in both inequalities, the difference of the coefficients of y_i and $1-y_j$ is equal to $\phi(S+i-j)-\phi(S)$ and is thus constant. Therefore the difference between the coefficients of y_i in both inequalities which is $\rho_i(S)-\rho_i(S-j)$, and that of $(1-y_j)$ which is $\rho_j(S+i-j)-\rho_j(S-j)$ have the same value. Therefore, to maximize the violation, we must order i before j if $y_i+y_j \geq 1$.

Recall that $\tilde{y}$ denotes the vector obtained by complementing odd components of y. We have proved the following result.

Proposition 11 *Given a point (x^*, y^*) and $l, l' \in N$, the permutation $\mathcal{L}$ maximizes the violation of the $(l, l', \mathcal{L})$ path-modular inequality if $\mathcal{L}$ sorts $\tilde{y}$ in decreasing order.*

Note also that this ordering is independent of L and thus needs to be computed once for all L.

Computing all $\alpha_{i,j}$ can be done in $\mathcal{O}(n^2)$ by n calls to Lex and n calls to Rev-Lex. Finally, for each L and given the ordering $\mathcal{L}$, computing each coefficient of the path-modular inequality can be done in constant time provided $l_i(O_L \cup E_L^i \setminus O_L^i)$ and $r_i(O_L \cup E_L^i \setminus O_L^i)$ are available. Because, for $L=[k,p]$,

$$l_i(O_L \cup E_L^i \setminus O_L^i) = \max\left(k, l_i(O_N \cup E_N^i \setminus O_N^i)\right)$$

and

$$r_i(O_L \cup E_L^i \setminus O_L^i) = \min\left(p, r_i(O_N \cup E_N^i \setminus O_N^i)\right)$$

computing these can be done once for all L in $\mathcal{O}(n^2)$ operations. To summarize, we have proved the following result:

Proposition 12 *Separating a most violated path-modular inequalities can be done in $\mathcal{O}(n^3)$ time.*

In practice, it is unlikely that path-modular inequalities for very large intervals L will be necessary. So it is interesting to consider separating "short" path-modular inequalities. In fact, a similar analysis yields the following result.

Proposition 13 *Separating all path-modular inequalities of size at most k can be done in $\mathcal{O}(n^2 + k^2 n)$ time.*

5 Extreme solutions and Optimization

In this section, we characterize the extreme solutions of FCTP. This directly leads to an optimization algorithm that runs in $\mathcal{O}(n^3)$ time.

Observe first that FCTP is equivalent to a fixed charge network flow problem in the following graph. There is one node i for $i \in [0, n]$, plus one source node s and a sink node t. There are capacitated arcs (s, i) for i even and (i, t) for i odd. The capacity is a_i. There are uncapacitated arcs $(i, i-1)$ for all $i \in [2, n] \cap E$ and uncapacitated arcs $(i, i+1)$ for all $i \in [0, n-1] \cap E$. It is well-known that extreme points of such models are solutions that do not imply cycles of arcs that are neither at their lower or upper bounds [AMO93].

Observe that when a flow variable x_i is zero, then FCTP decomposes into two smaller problems of type FCTP. More generally, an extreme solution of FCTP decomposes into r regeneration intervals $[i_k, i_{k+1}]$ delimited by regeneration edges $0 = i_0, i_1, i_2, \ldots, i_{r-1}, i_r = n+1$ such that $x_{i_k} = 0$ for $k = 0, \ldots, r$ and $x_i > 0$ otherwise. A regeneration interval $[j, l]$ can therefore be seen as the capacitated network flow model depicted in Figure 5 for $j = 0$ and $l = 6$, and for which there is a positive flow on all arcs.

In this network, the only capacitated arcs are (s, i) and (j, t). Therefore all uncapacitated arcs $(i, i+1)$ or $(i, i-1)$ are basic, as is the arc (t, s). Hence, exactly one arc (s, i) or (i, t) is basic (i.e. not necessarily at capacity a_i) in extreme solutions. To summarize, for a regenation interval $[j, l]$ with basic arc $k \in [j, l-1]$, the values of x_i are:

$$x_i = \begin{cases} a_{i-1} - a_{i-2} + a_{i-3} \ldots a_{j+1} & \text{if } i < k \\ a_i - a_{i+1} + a_{i+2} \ldots a_l & \text{if } i \geq k \end{cases} = \begin{cases} \bar{\alpha}_{i,j} & \text{if } i < k \\ \bar{\alpha}_{i,l} & \text{if } i \geq k \end{cases}$$

where

$$\bar{\alpha}_{i,j} = \begin{cases} 0 & \text{if } j = i, \\ a_{i-1} - \bar{\alpha}_{i-1,j} & \text{if } j < i, \\ a_i - \bar{\alpha}_{i+1,j} & \text{if } j > i, \end{cases}$$

defined for $i, j \in [0, n+1]$. These values x_i constitute a piece of a feasible solution if constraints (7) are satisfied for $i \in [j, l-1]$. We will denote by F the set of triples (j, k, l) that lead to such feasible pieces of solution.

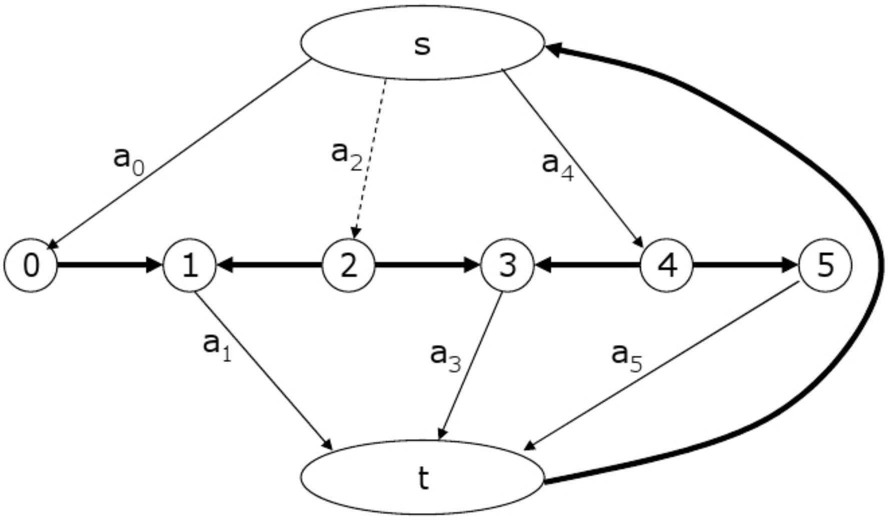

Figure 1: The capacitated network flow model for the regeneration interval $[0, 6]$. Capacities are shown for each arc, if any. Thick arcs are uncapacitated and always basic. One of the other arc, $(s, 3)$ in this example, is basic.

Example 1 [Continued] *$\bar{\alpha}_{i,j}$ for $i, j \in [0, n+1]$ is the following matrix*

$$\bar{\alpha} = \begin{pmatrix} 0 & 5 & -3 & 3 & -2 & 5 & -1 & 2 \\ 5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\ 3 & 8 & 0 & 6 & 1 & 8 & 2 & 5 \\ 3 & -2 & 6 & 0 & 5 & -2 & 4 & 1 \\ 2 & 7 & -1 & 5 & 0 & 7 & 1 & 4 \\ 5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\ 1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 \\ 2 & -3 & 5 & -1 & 4 & -3 & 3 & 0 \end{pmatrix}$$

Row i holds a superset of possible values of x_i at extreme points.

For the same regeneration interval $[j, l]$ with basic arc k, we could alternatively define x' as

$$x'_i = \begin{cases} \alpha_{i,j} & \text{if } i < k \\ \alpha_{i,l} & \text{if } i \geq k \end{cases}$$

In this case, the solution is feasible only if $x'_{k-1} + x'_k \leq a_k$, because by construction of α the other constraints are feasible. But note that x and x' will be identical whenever $(j, k, l) \in F$ as this condition guarantees that x_i is assigned the value $a_{i-1} - x_{i-1}$ and not a_i in Line 4 of Algorithm Lex. If $(j, k, l) \notin F$, then x' might be feasible and x not, but $x'_i = 0$ for some i, and it can therefore be decomposed into two or more regeneration intervals.

This characterization directly leads to a dynamic programming algorithm for solving FCTP. Computing all $\alpha_{i,j}$ requires $\mathcal{O}(n^2)$ operations. For each regeneration interval $[j, l]$, determining the optimal basic arc k and associated profit can be done performed $\mathcal{O}(n)$ operations. A standard $\mathcal{O}(n^2)$ recursion for optimally decomposing the set $[0, n]$ into regeneration intervals completes the algorithm. Therefore we have the following proposition:

Proposition 14 *An optimal solution of FCTP can be computed in $\mathcal{O}(n^3)$ time.*

Expressing this dynamic program as a linear program and taking its dual naturally yields a linear programming extended formulation for FCTP counting $\mathcal{O}(n^3)$ constraints and variables. However in the next section we give other extended formulations that are more compact and easier to project on the original variable space.

6 Compact linear programming formulations

In this section, we give two extended linear programming formulations for FCTP of size $\mathcal{O}(n^2)$ variables and constraints. They can be be seen as specializing the results of Conforti et al. [CWZar]. In addition, we are able to explicitely project one of them onto the original space of variables. We show that the inequalities we obtain by projection are exactly the path-modular inequalities.

The first formulation we describe crucially depends on the fact that in extreme solutions $x_i = \alpha_{i,j}$ for some j. In the second formulation, we use the fact that $x_i = \bar{\alpha}_{i,j}$ for some j. The second formulation has more variables and constraints because they are more distinct values $\bar{\alpha}_{i,j}$ than distinct $\alpha_{i,j}$ for each i. However it has a nicer structure and is easier to project.

For each i, let $0 = \beta_{i,0} < \beta_{i,1} < \beta_{i,2} < \ldots < \beta_{i,m_i}$ denote the distinct values taken by $\alpha_{i,j}$. Defining the binary variable $z_{i,j}$ that takes value 1 if x_i is at least $\beta_{i,j}$ and 0 otherwise, we introduce the following constraints:

$$y_i \geq z_{i,1} \qquad \forall i \in N, \tag{24}$$

$$x_i = \sum_{j=1}^{m_i} (\beta_{i,j} - \beta_{i,j-1}) z_{i,j} \qquad \forall i \in N, \tag{25}$$

$$z_{i,1} \geq z_{i,2} \geq \ldots \geq z_{i,m_i}, \qquad \forall i \in N, \tag{26}$$

$$z_{i,j} \in \{0,1\}, \qquad \forall i \in N, j \in [1, m_i]. \tag{27}$$

This model restricts the continuous variable x_i to take a finite set of values. However, as all extreme points of X^{FCTP} are feasible in this restriction and X^{FCTP} is bounded, the restriction will have the same convex hull as X^{FCTP}.

Note first that constraints (7) for $i = 0$ and $i = n$ are satisfied by solutions of (25)–(27) by construction of $\beta_{i,j}$. That constraints (7) for $i \in [1, n-1]$ can be reformulated as

$$z_{i,j} + z_{i+1,\gamma(i,j)} \leq 1, \qquad \forall i \in N \setminus \{n\}, j \in G(i), \tag{28}$$

where $\gamma(i,j) = \min_k \{\beta_{i,j} + \beta_{i+1,k} > a_i\}$ and $G(i) = \{j : \gamma(i, j-1) \neq \gamma(i,j)\}$ (restricting to $G(i)$ is not necessary for validity but eliminates redundant constraints).

Proposition 15 *The feasible points of* (7),(25)–(27) *and* (25)–(28) *are identical.*

Proof. If a point (x, z) satisfies (7),(25)–(27), then it also satisfies (28) as $z_{i,j} = 1$ implies $x_i \geq \beta_{i,j}$.

If a point (x, z) satisfies (25)–(27),(28), then it also satisfies $z_{i,j} + z_{i+1,k} \leq 1$ for all i, j, k such that $\beta_{i,j} + \beta_{i+1,k} > a_i$. The result follows as x_i and x_{i+1} can only take values $0, \beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,m_i}$ and $0, \beta_{i+1,1}, \beta_{i+1,2}, \ldots, \beta_{i+1,m_{i+1}}$ respectively. ■

Proposition 16 *The LP relaxation of* (24)–(28) *is a linear programming extended formulation for FCTP of size* $\mathcal{O}(n^2)$ *variables and constraints. The result still holds if one adds constraints of the type* $y_i - y_k \leq d$ *with* i *and* k *of the same parity.*

Proof. By the change of variables $z'_{i,j} = -z_{i,j}$ and $y'_i = -y_i$ for i even and $z'_{i,j} = z_{i,j}$ and $y'_i = y_i$ for i odd, each constraint (24),(26),(28) and $y_i - y_k \leq d$ with i and k of the same parity becomes a bound on a difference of two variables. The matrix associated to this modified constraint system is therefore a network flow matrix. The result follows. ■

We now describe a similar but different extended formulation that turns out to be easier to project. We start by a few observations.

Observation 2 *For fixed* j, $\bar{\alpha}_{i,j}$ *is the unique solution of the following system of linear equations:* $x_{i-1} + x_i = a_i$ *for* $i \in [0, n]$ *and* $x_j = 0$. *Therefore the two following properties hold*

$$\bar{\alpha}_{i,j} + \bar{\alpha}_{i+1,j} = a_i, \qquad \forall i \in [0, n], j \in [0, n+1], \tag{29}$$

$$\bar{\alpha}_{i,j} = \begin{cases} \bar{\alpha}_{i,0} + \bar{\alpha}_{0,j} & i \in E \\ \bar{\alpha}_{i,0} - \bar{\alpha}_{0,j} & i \in O \end{cases} \tag{30}$$

Let $(j_0, j_1, \ldots, j_{\bar{m}})$ be a permutation of a subset of $[0, n+1]$ that removes duplicate entries and sorts $\{\bar{\alpha}_{1,j}\}_{j=0}^{n+1}$ in increasing order: $\bar{\alpha}_{i,j_0} < \bar{\alpha}_{i,j_1} < \ldots < \bar{\alpha}_{i,j_{\bar{m}}}$. For fixed $i \in [1, n]$, it follows from (30) that the same permutation removes duplicate entries and sorts $\{\bar{\alpha}_{i,j}\}_{j=0}^{n+1}$ in increasing (resp. decreasing) order for i odd (resp. even). Defining

$$\bar{\beta}_{i,k} = \begin{cases} \bar{\alpha}_{i,j_k} & \text{if } i \in [1, n] \cap O,\ k \in [0, \bar{m}], \\ \bar{\alpha}_{i,j_{k-1}} & \text{if } i \in [1, n] \cap E,\ k \in [1, \bar{m}+1]. \end{cases}$$

and $\gamma_k = \bar{\beta}_{1,k} - \bar{\beta}_{1,k-1} > 0$ for $k \in [1, \bar{m}]$, (30) implies that

$$\bar{\beta}_{i,k} - \bar{\beta}_{i,k-1} = \gamma_k, \text{for } i \in [1, n] \cap O,\ k \in [1, \bar{m}],$$

$$\bar{\beta}_{i,k} - \bar{\beta}_{i,k+1} = \gamma_k, \text{for } i \in [1, n] \cap E,\ k \in [1, \bar{m}].$$

Example 2 [Continued] *A possible permutation* $(j_0, j_1, \ldots, j_{\bar{m}})$ *removing duplicates and sorting* $\{\bar{\alpha}_{1,j}\}_{j=0}^{n+1}$ *in increasing order is* $(1,3,7,0,6,4,2)$*, resulting in*

$$\{\bar{\beta}_{i,k}\}_{\substack{i\in[1,n]\\ k\in[0,\bar{m}]}} = \begin{pmatrix} 0 & 2 & 3 & 5 & 6 & 7 & 8 & \\ & 8 & 6 & 5 & 3 & 2 & 1 & 0 \\ -2 & 0 & 1 & 3 & 4 & 5 & 6 & \\ & 7 & 5 & 4 & 2 & 1 & 0 & -1 \\ 0 & 2 & 3 & 5 & 6 & 7 & 8 & \\ & 6 & 4 & 3 & 1 & 0 & -1 & -2 \end{pmatrix}$$

and $\gamma = (2,1,2,1,1,1)$.

Consider now the following formulation in which the intended meaning is that $z_{i,k} = 1$ if x_i takes a value at least $\bar{\beta}_{i,k}$ and 0 otherwise.

$$x_i = \bar{\beta}_{i,0} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} \qquad \forall i \in O, \tag{31}$$

$$x_i = \bar{\beta}_{i,\bar{m}+1} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} \qquad \forall i \in E, \tag{32}$$

$$y_i \geq z_{i,k} \qquad \forall i \in N, k \in [1,\bar{m}] : \bar{\beta}_{i,k} > 0 \tag{33}$$

$$z_{i,k} + z_{i+1,k} \leq 1, \qquad \forall i \in N, k \in [1,\bar{m}] \tag{34}$$

$$z_{i,k} = 0, \qquad \forall i \in N, k \in [1,\bar{m}] : \bar{\beta}_{i,k} > \min(a_{i-1}, a_i). \tag{35}$$

$$z_{i,k} = 1, \qquad \forall i \in N, k \in [1,\bar{m}] : \bar{\beta}_{i,k} \leq 0 \tag{36}$$

$$z_{i,k} \geq 0, \qquad \forall i \in N, k \in [1,\bar{m}]. \tag{37}$$

There are two main differences between this last formulation and the formulation (24)–(28). The first one is that constraints (34) are separable in k. The second one is that we have replaced constraints (24),(26) by the weaker constraints (33). These two modifications make the projection of the extended formulation on the (x,y) variable space easier.

Proposition 17 *Formulation* (31)-(37) *is a linear programming extended formulation of* $conv(X^{FCTP})$.

Proof. We first show that constraints (7) are implied by (31),(32),(34). Suppose i is odd (the other case is similar). By (29) and definition of γ, we know that $\beta_{i,0} + \beta_{i+1,\bar{m}+1} =$

$(\beta_{i+1,\bar{m}} + \beta_{i,\bar{m}}) + (\beta_{i,0} - \beta_{i,\bar{m}}) = a_i - \sum_{k=1}^{\bar{m}} \gamma_k$. Therefore we can write

$$\begin{aligned} x_i + x_{i+1} =& \bar{\beta}_{i,0} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} + \bar{\beta}_{i+1,\bar{m}} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i+1,k} \\ =& a_i - \sum_{k=1}^{\bar{m}} \gamma_k + \sum_{k=1}^{\bar{m}} \gamma_k (z_{i,k} + z_{i+1,k}) \\ \leq& a_i - \sum_{k=1}^{\bar{m}} \gamma_k + \sum_{k=1}^{\bar{m}} \gamma_k = a_i \end{aligned}$$

We now show that constraints (9) are implied by (31)-(33),(36)-(37). Indeed, because of (36) and the fact that $\bar{\beta}_{i,k} = 0$ for some k, (31) can be rewritten as $x_i = \sum_{k=1:\bar{\beta}_{i,k}>0}^{\bar{m}} \gamma_k z_{i,k}$. Because of (37) and $\gamma_k > 0$ for all k, $x_i > 0$ if and only if $z_{i,k} > 0$ for some k. This implies $y_i > 0$ using (33). Hence (31)-(37) is a correct formulation of FCTP.

It remains to see that, using the same argument as the one used in the proof of proposition 16, the constraint matrix associated to (33)-(34) is a network flow matrix and therefore extreme points of (31)-(37) have y and z integer-valued. ■

A linear programming extended formulation automatically leads to a characterization of the convex hull of the solution in the original space of variables through projection. Indeed, testing if a point (x^*, y^*) satisfying $y^* \leq \underline{1}$ belongs to $conv(X^{FCTP})$ is equivalent to testing whether the following LP in variables z is feasible:

$$\begin{aligned} \max \quad & 0, \\ & \sum_{k \in K_i} \gamma_k z_{i,k} = x_i^* \qquad \forall i \in N, && (\Delta_i) \\ & z_{i,k} \leq y_i^* \qquad \forall i \in N, k \in K_i, && (\delta_{i,k}) \\ & z_{i,k} + z_{i+1,k} \leq 1, \qquad \forall i \in [1, n-1], k \in K_i \cap K_{i+1} && (\rho_{i,k}) \\ & z_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i, \end{aligned}$$

where $K_i = \{k \in [1, \bar{m}] : 0 < \bar{\beta}_{i,k} \leq \min(a_{i-1}, a_i)\}$. Through LP duality, this is equivalent to testing whether the following LP is bounded:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \rho_{i,k}, \\ & \gamma_k \Delta_i + \delta_{i,k} + \rho_{i-1,k} + \rho_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i \\ & \rho_{i,k} = 0, \qquad \forall i \notin [1, n-1] \text{ or } k \notin K_i \cap K_{i+1}, \\ & \delta, \rho \geq 0, \end{aligned}$$

Dividing the constraint by γ_k and rescaling $\delta_{i,k}$ and $\rho_{i,k}$ by γ_k, one obtains the equivalent LP

$$\min \qquad \sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} \gamma_k y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k}, \tag{38}$$

$$\Delta_i + \delta_{i,k} + \rho_{i-1,k} + \rho_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i \tag{39}$$

$$\rho_{i,k} = 0, \qquad \forall i \notin [1, n-1] \text{ or } k \notin K_i \cap K_{i+1}, \tag{40}$$

$$\delta, \rho \geq 0, \tag{41}$$

This is true if (x^*, y^*) satisfies

$$\sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} \gamma_k y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k} \geq 0$$

for all extreme rays of the cone associated to the last constraint system. We will characterize sufficient inequalities to describe the polyhedron $conv(X^{FCTP})$ by characterizing these extreme rays. Because $x^*, y^*, \gamma, \delta, \rho \geq 0$, extreme rays with negative cost satisfy $\Delta \leq 0$. Hence we can normalize rays by assuming $\Delta_i \geq -1$ for all $i \in N$.

A first observation is that $\Delta_i < 0$ for consecutive i's (otherwise the ray is the sum of two other rays). The following result is less trivial.

Proposition 18 *The matrix associated to the constraint system* (39) *is totally unimodular.*

Proof. Variable $\delta_{i,k}$ appears only in one constraint and can therefore be neglected. We prove the result by proving that for any subset J of the columns of the matrix B under consideration, there exists a partition (J^-, J^+) of J such that $\sum_{j \in J^+} b_{i,j} - \sum_{j \in J^-} b_{i,j} \in \{-1, 0, 1\}$ for all rows i.

If none of the variables Δ_i belong to J then the remaining matrix satisfies the consecutive ones property and is TU. So we can assume the contrary and consider columns associated to Δ_i belonging to J in increasing order of i: $i_1 < i_2 < \ldots$. We assign Δ_{i_1} to J^+. Then we assign Δ_{i_j} to the same partition as $\Delta_{i_{j-1}}$ if the parity of i_j and i_{j-1} are different and to the other partition if the parity is the same (in particular, consecutive columns are assigned to the same partition).

Note that having partitioned columns associated to Δ, the problem becomes separable in k. Consider $\rho_{i',k}$ belonging to J and let $j^<$ be the highest index such that $i_{j^<} \leq i'$ and let $j^>$ be the lowest index such that $i_{j^>} > i'$. In other words, $i_{j^<}$ and $i_{j^>}$ are the two closest columns Δ_i before and after i' belonging to J. At least one of them exists. We assign $\rho_{i,k}$ to

- the same partition as $\Delta_{i_{j^<}}$ if $i_{j^<}$ and i' have a different parity,

- the opposite partition to that of $\Delta_{i_{j^<}}$ if $i_{j^<}$ and i' have the same parity,
- the same partition as $\Delta_{i_{j^>}}$ if i' and $i_{j^>}$ have the same parity,
- the opposite partition to that of $\Delta_{i_{j^>}}$ if i' and $i_{j^>}$ have a different parity.

Observe that these rules cannot be contradictory in case both $j^<$ and $j^>$ exist because of the choosen partitioning of Δ_i.

We claim that this partitioning satisfies the desired property for row (39) for any given i, k. If Δ_i does not belong to j, then $\rho_{i-1,k}$ and $\rho_{i,k}$ are assigned to opposite partitions and the property holds. If Δ_i belongs to j, then $\rho_{i-1,k}$ and $\rho_{i,k}$ are both assigned to the opposite partition to that of Δ_i and again the property holds. ■

Corollary 19 *Normalized extreme rays of negative cost of* (38)-(41) *satisfy* $\Delta_i = -1$ *if* $i \in [l, l']$ *and* 0 *otherwise for some* $l, l' \in N$.

Therefore we can assume without loss of generality Δ fixed accordingly, and analyze optimal solutions of the LP (38)-(41) under this assumption. The following characterization will be sufficient for our purposes.

Proposition 20 *For given* (x^*, y^*) *and fixed* Δ *according to Corollary 19, the set of optimal solutions to* (38)-(41) *is the same for all vectors* y^* *that admit the same ordering when sorted in decreasing order of* $\tilde{y}^*$. *When* y^* *is such that this ordering is unique, the optimal solution is unique as well.*

Proof. Observe that for fixed Δ, the LP (38)-(41) is separable in k. Furthermore, for given each k, the problem is a problem of the form:

$$\min \quad \sum_{i=1}^{n} y_i^* \delta_i + \sum_{i=1}^{n-1} \rho_k, \tag{42}$$

$$\delta_1 + \rho_1 \geq 1, \tag{43}$$

$$\delta_i + \rho_{i-1} + \rho_i \geq 1, \qquad \forall i \in [2, n-1], \tag{44}$$

$$\delta_n + \rho_n \geq 1, \tag{45}$$

$$\rho_i = 0, \qquad \forall i \in Q, \tag{46}$$

$$\delta, \rho \geq 0, \tag{47}$$

where Q can be choosen to represent constraints (40). From Proposition 18, we know that optimal solutions can be assumed to be integral. From a graphical perspective, we have an undirected path of which each node i must be covered either by itself at cost y_i^* or by one of its incident edges at cost 1. Note that as $0 \leq y_i^* \leq 1$, we can assume that in optimal solutions all inequalities (43)-(45) will be tight. Indeed, if inequality i is not tight and $\Delta_i > 0$, we can decrease it by 1 without increasing the objective. If inequality i is not tight and $\Delta_i = 0$, then ρ_{i-1} or ρ_i is positive. Suppose ρ_i. Then we can decrease ρ_i by 1 and increase y_{i+1}^* by 1 without increasing the objective.

Therefore the same problem can be modelled as the more classical perfect matching problem in the following bipartite graph. The node set is $V = I \cup I'$ where $I = I' = [1, \bar{n}]$ and we index nodes in I (resp. I') by i (resp. i'). The edge set E includes edges (i, i') if $i = i'$ with cost y_i^* and edges $(i, i+1)$ and $(i', i'+1)$ for all $i, i' \in [1, \bar{n}-1] \setminus Q$ with cost $\frac{1}{2}$. Note that this graph is bipartite but not under the usual partition $E \subseteq I \times I'$. The two problems are equivalent because if edge $(i, i+1)$ is in the matching, then the edge $(i', i'+1)$ for $i' = i$ must also be in the matching and together they cost 1.

Elementary cycles in this bipartite graph are of the form $i, i+1, \ldots, j, j', j'-1, \ldots, i'$ and can therefore be unambigously denoted by $C_{i,j}$ for $i < j$. Consider a given perfect matching M of the graph (V, E) just defined. $C_{i,j}$ is an alternating cycle with respect to M if and only if the four following conditions hold:

(i) M does not contain an edge (k, k') with $i < k < j$,

(ii) if either (i, i') or (j, j') but not both belongs to M, then i and j are of the same parity,

(iii) if either both (i, i') and (j, j') or none of them belong to M, then i and j are of different parity.

(iv) there is no $k \in Q$ with $i \leq k < j$.

Let $C_{i,j}$ be such an alternating cycle and consider the perfect matching M' obtained by taking the symetric difference $M' = M \triangle C_{i,j}$. Denoting the cost of matching M by $c(M)$, the structure of the graph implies that:

$$c(M') = C(M) + \begin{cases} 1 - y_i^* - y_j^* & \text{if both } (i,i') \text{ and } (j,j') \text{ belong to } M, \\ y_i^* + y_j^* - 1 & \text{if neither } (i,i') \text{ nor } (j,j') \text{ belong to } M, \\ y_i^* - y_j^* & \text{if } (j,j') \text{ belongs to } M \text{ but not } (i,i'), \\ y_j^* - y_i^* & \text{if } (i,i') \text{ belongs to } M \text{ but not } (j,j'). \end{cases}$$

Combining this with the characterization of an alternating cycle above, we obtain that the set of optimal solutions (matchings) to (42)-(47) will be the same for all vectors y^* that admit the same orderings when sorted in decreasing order of $\tilde{y}^*$. When this ordering is unique the optimal matching is unique as well as taking the symetric difference with any elementary alternating cycle will strictly increase its cost.

For fixed Δ, subproblems k of (38)-(41) will only differ by Q in constraint (46). Hence the same is true for optimal solutions of (38)-(41). ■

Corollary 21 *Together with bounds $x_i \geq 0$ and $y_i \leq 1$ for $i \in N$, the following family of valid inequalities is sufficient to describe the convex hull of X^{FCTP}:*

$$\sum_{i=l}^{l'} x_i \leq \tau(\mathcal{L}) + \sum_{i=l}^{l'} \sigma(i, \mathcal{L}) y_i, \tag{48}$$

where $l, l' \in N$, $l \leq l'$, $\mathcal{L}$ is a permutation of $[l, l']$ and $\tau(\mathcal{L}) = \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k}$ and $\sigma(i, \mathcal{L}) = \sum_{k \in K_i} \gamma_k \delta_{i,k}$ for the unique optimal solution (δ, ρ) of (38)-(41) obtained when $\mathcal{L}$ is the unique permutation that sorts $\tilde{y}^*$ in decreasing order and Δ is fixed at $\Delta_i = -1$ for $i \in [l, l']$ and 0 otherwise.

We know prove that there is a one-to-one correspondence between this last family of valid inequalities and the path-modular inequalities, thereby proving the main result of this section.

Proposition 22 *Together with bounds $x_i \geq 0$ and $y_i \leq 1$ for $i \in N$, the path-modular inequalities are sufficient to describe the convex hull of X^{FCTP}.*

Proof. Let $l, l' \in N$, $l \leq l'$ and a permutation $\mathcal{L}$ of $[l, l']$ be given. Consider any of the $|l' - l + 2|$ points (x^k, y^k) of Proposition 5 for the ordering $(j_1, j_2, \ldots, j_n) = \mathcal{L}$. Such a point lies on the boundary of $conv(X^{FCTP})$, and therefore a separation algorithm that maximizes the violation will output a valid inequality that is tight at this point. Consider now the LP (38)-(41) with Δ fixed at $\Delta_i = -1$ for $l \leq i \leq l'$ and 0 otherwise. This LP actually determines a valid inequality of the form $\sum_{i=l}^{l'} x_i \leq \pi_0 + \sum_{i=l}^{l'} \pi_i y_i$ maximizing the violation. Now observe that the permutation $\mathcal{L}$ sorts $\tilde{y}^k$ in decreasing order for any k. Therefore the corresponding inequality (48) is tight at (x^k, y^k) for any k. By Proposition 5, this is also the case for the path-modular inequality associated to $(l, l', \mathcal{L})$. As these $|l' - l + 2|$ points are affinely independent, these two inequalities are identical. ■

7 Computational Experiments

We report statistics on solving FCTL instances of various sizes using four different algorithms:

- MIPSOLV: a standard off-the-shelve mip solver (XPRESS-MP 20.00) using the standard formulation (7)-(10).
- CUT10: a cut-and-branch approach using the same solver and initial formulation as MIPSOLV, but separating path-modular of size at most 10 at the root node. For each interval L of length at most 10, the most violated path-modular inequality is computed, and added to the system if the violation is positive.
- CUT20: same as CUT10, but with path-modular inequalities of size at most 20.
- XFORM: the same solver using the extended formulation (24)-(28).

The instances are generated as follows:

- a_i are drawn from the uniform distribution $U(5, 10)$,
- p_i are drawn from the uniform distribution $U(0.95, 1.05)$,
- f_i are drawn from the uniform distribution $U(\frac{a_i p_i}{22}, \frac{a_i p_i}{18})$.

The goal is to generate fairly hard instances, with "long" chains of consecutive open edges in optimal solutions. We generated instances of size 200, 300, 400, 500 and 600, with 10 instances in each group, with a time limit of 600 seconds and the standard settings of the optimizer. Using a more aggressive cutting plane strategy yields essentially identical results.

The following table gives the average time and number of nodes needed to solve the instances to optimality. For the instances of size 600, we report the final gap for MIPSOLV instead of the solution time, as it was unable to terminate before the time limit in many cases.

	solution times				nb. of nodes			
n	MIPSOLV	CUT10	CUT20	XFORM	MIPSOLV	CUT10	CUT20	XFORM
200	0.71	0.75	1.43	0.42	359	33	4	1
300	8.90	1.71	3.12	0.67	6000	95	9	1
400	27.95	2.94	5.40	0.92	15100	245	11	1
500	68.91	4.05	8.26	1.14	29600	190	5.4	1
600	0.01%	11.15	12.81	1.55	80000	1991	11	1

The extended formulation is most efficient, even for small instances. This can be explained by the fact that the number of variables and constraints grows in practice linearly with the size of the problem, and not quadratically as could be expected from the worst-case bound.

For the instances considered, the extended formulation counts only 5 or 6 times more variables and constraints compared to the standard formulation, whatever the size of the instance.

Short path-modular inequalities close most of the gap. From a total computing time point of view, CUT10 is more efficient than CUT20 for all instances (except 2 of size 600) and more efficient than MIPSOLV for large instances.

XPRESS-MP is able to close most of the gap using built-in cutting planes. The situation is essentially the same for all instances: the gap of the LP relaxation is initially about 3%, and is reduced by 95% to 0.015% at the root node. But closing the remaining gap by branching takes exponentially more nodes and time when size grows.

8 Facets of small instances

Appendix A lists all facets of Example 1 and give their construction as path-modular inequalities.

We also enumerated all facets of an instance of a fixed charge transportation problem (2)-(5) of size 3×2 with data $C = [4, 5, 3]$ and $D = [8, 6]$. There are 136 facets, of which 19 are trivial (one of the inequalities (2)-(5)). Seven facets correspond to flow cover or path-modular inequalities of length 2 (they are identical in this case). Six facets are facets of a single-node flow relaxation of size 3 (flow cover, lifted flow cover or other inequalities). Eight facets are path-modular inequalities of size 3 and three facets are path-modular inequalities of size 4.

The other 93 facets have edges with non-zero coefficients that are not stars (flow covers) or paths (path-modular inequalities).

In view of this, we can hope that the path modular inequalities can be helpful in solving fixed-charge transportation problems on bipartite graphs more efficiently.

9 Conclusion

This paper is a polyhedral analysis of the Fixed Charge Transportation problem on Paths (FCTP). The aim of this study is to improve our ability to solve the Fixed Charge Transportation problem on general bipartite graphs, and the even more general Big Bucket Lot-Sizing problem.

For FCTP, we characterize extreme points, we give a $\mathcal{O}(n^3)$ algorithm and two compact linear programming extended formulations. We describe a new family of valid inequalities that are shown to describe the convex hull of the solutions to FCTP, and that can be separated in $\mathcal{O}(n^3)$ time. We also report computational results showing that the extended formulation is most efficient for solving instances of FCTP because not too many additional variables and constraints are necessary in practice. These experiments also illustrate that "short" path-modular inequalities are usually sufficient to obtain excellent bounds. Finally, we show through an example that a substantial number of facet-defining inequalities for FCT are path-modular inequalities of path relaxations.

This work can be pursued in many directions. Among the 93 unexplained facets discussed in Section 8, 4 seem to generalize path-modular inequalities for cycles. Indeed, they have coefficients 0 or 1 for variables x_i, and their support corresponds to a cycle of size 4 (instead of a path for path-modular inequalities). Moreover the coefficients of variables y_i are exactly of the form (22) for some permutation of the edges, with the set functions ϕ and ρ_i naturally interpreted for cycles. This would complete the study of fixed charge transportation on graphs of node degree at most 2.

This in turns raises the question of generalizing path-modular inequalities to other graph structures. In particular, it would be nice to be able to describe a family of inequalities that subsumes path-modular (paths) and simple flow-cover inequalities (stars).

Aniother question is whether path-modular inequalities can be adapted to deal with capacities on the edges and/or setup variables on the nodes. From the point of view of [CDSEW09], a generalization of our work would be to study mixing sets linked by a path of which each edge is arbitrarily directed.

From a computational point of view, we would like to improve the running time of the separation algorithm for path-modular inequalities. In particular one could try to exploit the similarity between two inequalities associated to intervals differiing by one element. If one wants to use the present work to better solve general fixed charge transportation problems, the crucial step will be to determine path relaxations that generate violated path-modular inequalities. This is not a trivial task.

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Appendices

A Facets of Example 1

Table 1 gives the convex hull of the instance of Example 1 used throughout the paper: $n = 6$ and $a = (5, 8, 6, 5, 7, 6, 3)$. We also indicate how to obtain each one as a path-modular inequality, by giving the interval L and the ordering. Note that in all the facets of this example, the ordering starts by the even numbers and ends with the odd numbers. This means that these facets are all tight at a point where all edges are open. There are instances of FCTP for which path-modular inequalities are facets and do not satisfy this property.

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														L	*ordering*
$+x_1$						$-5y_1$						$\leq$	0	$[1,1]$	1
	$+x_2$						$-6y_2$					$\leq$	0	$[2,2]$	2
		$+x_3$						$-5y_3$				$\leq$	0	$[3,3]$	3
			$+x_4$						$-5y_4$			$\leq$	0	$[4,4]$	4
				$+x_5$						$-6y_5$		$\leq$	0	$[5,5]$	5
					$+x_6$						$-3y_6$	$\leq$	0	$[6,6]$	6
				$+x_5$	$+x_6$					$-3y_5$		$\leq$	3	$[5,6]$	$6,5$
$+x_1$	$+x_2$					$-2y_1$	$-3y_2$					$\leq$	3	$[1,2]$	$2,1$
			$+x_4$	$+x_5$					$-1y_4$	$-2y_5$		$\leq$	4	$[4,5]$	$4,5$
			$+x_4$	$+x_5$	$+x_6$				$-4y_4$	$-2y_5$		$\leq$	4	$[4,6]$	$6,4,5$
		$+x_3$	$+x_4$									$\leq$	5	$[3,4]$	$4,3$
	$+x_2$	$+x_3$					$-1y_2$					$\leq$	5	$[2,3]$	$2,3$
$+x_1$	$+x_2$	$+x_3$				$-2y_1$	$-1y_2$	$-3y_3$				$\leq$	5	$[1,3]$	$2,3,1$
		$+x_3$	$+x_4$	$+x_5$				$-4y_3$		$-2y_5$		$\leq$	5	$[3,5]$	$4,3,5$
		$+x_3$	$+x_4$	$+x_5$	$+x_6$			$-1y_3$		$-2y_5$		$\leq$	8	$[3,6]$	$4,6,3,5$
$+x_1$	$+x_2$	$+x_3$	$+x_4$			$-2y_1$	$-1y_2$		$-2y_4$			$\leq$	8	$[1,4]$	$2,4,1,3$
	$+x_2$	$+x_3$	$+x_4$	$+x_5$			$-2y_2$			$-2y_5$		$\leq$	9	$[2,5]$	$4,2,5,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$		$-2y_1$	$-2y_2$	$-3y_3$		$-2y_5$		$\leq$	9	$[2,6]$	$4,6,2,5,3$
	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$		$-5y_2$			$-2y_5$		$\leq$	9	$[1,5]$	$4,2,3,1,5$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$		$-2y_1$	$-1y_2$		$-1y_4$	$-5y_5$		$\leq$	9	$[1,5]$	$2,4,5,1,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$		$-2y_1$	$-2y_2$			$-5y_5$		$\leq$	9	$[1,5]$	$4,2,5,1,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-2y_1$	$-1y_2$	$-1y_3$	$-1y_4$	$-2y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$2,4,6,3,1,5$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-2y_1$	$-2y_2$	$-1y_3$		$-2y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$4,2,6,3,1,5$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-2y_1$	$-1y_2$	$-1y_3$	$-2y_4$	$-2y_5$		$\leq$	11	$[1,6]$	$6,2,4,3,1,5$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-3y_1$	$-1y_2$		$-1y_4$	$-2y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$2,4,6,1,5,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-2y_1$	$-1y_2$		$-1y_4$	$-3y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$2,4,6,5,1,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-3y_1$	$-2y_2$			$-2y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$4,2,6,1,5,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-3y_1$	$-1y_2$		$-2y_4$	$-2y_5$		$\leq$	11	$[1,6]$	$6,2,4,1,5,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-2y_1$	$-2y_2$			$-3y_5$	$-1y_6$	$\leq$	11	$[1,6]$	$4,2,6,5,1,3$
$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$-3y_1$	$-3y_2$			$-2y_5$		$\leq$	11	$[1,6]$	$6,4,2,1,5,3$

Table 1: facets of instance of Example 1 and their construction as path-modular inequalities

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