LAC

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ACTION ACCESSIBLE AND WEAKLY ACTION REPRESENTABLE VARIETIES OF ALGEBRAS

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MAIN GOAL

We prove that the converse of the implication

weakly action rep. category $\Rightarrow$ action accessible category

is not true for varieties of non-associative algebras over a field.

Counterexamples:

$\mathrm{Nil}_k(\mathrm{Lie})$ and $\mathrm{Solv}_t(\mathrm{Lie})$.

$k \geq 3$ $\qquad$ $t \geq 2$

PRELIMINARIES

- $\mathscr{C}$ semi-abelian category (pointed + Barr-exact + protomodular + finite coproducts)
- Split extension of B by X:

$$0 \to X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \to 0$$

with $\alpha\beta = id_B$ and $(X,k) = \ker\alpha$.

- Split extensions by X:

$$\mathrm{SplExt}(-,X) : \mathscr{C}^{op} \to \mathrm{Set}$$

where $\mathrm{SplExt}(B,X)$ is the set of iso classes of split ext. of B by X.

- Internal actions on X:

$\mathrm{Act}(-,X): \mathscr{C}^{op} \longrightarrow \mathrm{Set}$

<u>Remark</u> $\mathrm{Act}(-,X) \cong \mathrm{SplExt}(-,X)$.

- Action representable category

(Borceux, Janelidze, Kelly, 2005)

$\forall X \in \mathscr{C}, \exists \underbrace{[X]}_{\text{ACTOR OF } X} \in \mathscr{C}$ such that

$\mathrm{SplExt}(-,X) \cong \mathrm{Hom}(-,[X])$.

<u>Examples</u>

- $\mathscr{C} = \mathrm{Grp} \rightsquigarrow [X] = \mathrm{Aut}(X)$;
- $\mathscr{C} = \mathrm{Lie} \rightsquigarrow [X] = \mathrm{Der}(X)$;
- $\mathscr{C}$ abelian $\rightsquigarrow [X] = 0$.

Remark

The notion of A.R. category is quite restrictive.

Theorem (García-Martínez, Tsishyn, Van der Linden, Vienne, 2021) Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$ ($\mathbb{F}$ infinite, $\mathrm{char}(\mathbb{F}) \neq 2$). If $\mathcal{V}$ is A.R., then $\mathcal{V} = \mathsf{AbAlg}$ or $\mathcal{V} = \mathsf{Lie}$.

Remark

A non-associative algebra X is a vector space endowed with a bilinear multiplication

$$X \times X \longrightarrow X$$
$$(x, y) \longmapsto xy.$$

A variety $\mathcal{V}$ is a class of algebras which satisfies a set of identities.

Remark

Any variety $\mathcal{V}$ is a semi-abelian category.

- Weakly action rep. category (Janelidze, 2022): $\forall X \in \mathcal{C}, \exists T = T(X) \in \mathcal{C}$ and $\tau : \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}(-, T)$

  $\leadsto$ weak representation of $\mathrm{SplExt}(-, X)$

- Acting morphism: $(\varphi : B \to T) \in \mathrm{Im}(\tau_B)$

Examples

- $\mathcal{C}$ A.R. $\leadsto T = [X]$.
- $\mathcal{C} = \mathbf{Assoc} \leadsto T = \mathrm{Bim}(X)$ (Janelidze, 2022).
- $\mathcal{C} = \mathbf{Leib} \leadsto T = \mathrm{Bider}(X)$ (Cigoli, M., Montoli, 2023).

Proposition (Janelidze, 2022)

Weakly act. rep. $\Rightarrow$ $\underbrace{\text{Act. accessible}}$ (*).

every object in $\mathrm{SplExt}(X)$ is "accessible", i.e., it has a morphism into a "faithful" or "subterminal" object.

Example

If $\mathscr{C}$ is A.R., then

$$0 \to X \to [X] \ltimes X \rightleftarrows [X] \to 0$$

is a terminal object in $\mathrm{SplExt}(X)$.

Question

Does the converse of $(*)$ holds?

Theorem (Gray, 2025)

Let $\mathscr{C}$ A.R., $\mathbb{X} \subseteq \mathscr{C}$ $\underbrace{\text{BIRKHOFF}}_{}$.

full reflective closed under subobjects and quotients/images

Suppose $\exists$

$$\begin{tikzcd}
S \arrow[r, tail, "m'"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\equiv"] & B' \arrow[d, tail, "u'"] \arrow[dr, dotted, "v_{u'}"] & \\
B \arrow[r, tail, "u"] \arrow[rr, dotted, bend right, "v_u"'] & D \arrow[r, tail, "v"] & {[X]}
\end{tikzcd}$$

$um = u'm'$

$X \in \mathbb{X}$

$m, m' \in \mathbb{X}$

$u, u' \in \mathscr{C}$

such that:

(1) (m, m') cannot be amalgamated in $\mathbb{X}$;

(2) the split ext with kernel X corresp. to v_u and $v_{u'}$ are in $\mathbb{X}$.

Then $\mathbb{X}$ is NOT weakly act. rep..

DEF Two monomorphisms

$$B \overset{m}{\longleftarrowtail} S \overset{m'}{\rightarrowtail} B'$$

have an amalgam in $\mathcal{C}$ if $\exists\, B \overset{u}{\rightarrowtail} D \overset{u'}{\longleftarrowtail} B'$ in $\mathcal{C}$ such that

$$\begin{tikzcd} S \arrow[r, tail, "m'"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "="] & B' \arrow[d, tail, "u'"] \\ B \arrow[r, tail, "u"'] & D \end{tikzcd}.$$

Remark

$\mathcal{C}$ A.R. $\Rightarrow$ $\mathcal{C}$ has the A.P..

Example (Gray, 2025)

- $\mathcal{C} = \mathsf{Grp}$, $\mathbb{X} = \mathsf{Sol}_t(\mathsf{Grp})$, $t \geq 3$.

$\exists\, S$ abelian, B 2-nilpotent and $m, m' : S \rightarrowtail B$ which cannot be amalgamated in $\mathbb{X}$.

Thus, $\mathbb{X}$ is not weakly act rep.

SUBVARIETIES OF $\mathcal{L}ie$

- $\mathcal{N}il_k(\mathcal{L}ie)$, $k \geq 3$
$(\mathcal{L}^0 = \mathcal{L}, \mathcal{L}^k = [L, L^{k-1}] = 0)$.
- $\mathcal{S}ol_t(\mathcal{L}ie)$, $t \geq 2$
$(\mathcal{L}^{(0)} = \mathcal{L}, \mathcal{L}^{(t)} = [L^{(t-1)}, L^{(t-1)}] = 0)$.

Remark (Brox, García-Martínez, M, Van der Linden, Vienne, 2025)

$\mathcal{N}il_2(\mathcal{L}ie)$ is weakly act. rep.
with

$$T(X) = \{ f \in End(X) \mid \dots \\ \dots f([x,y]) = [f(x), y] = 0 \}$$

and $f * g = 0$.

Theorem (García-Martínez, M, 2025)

$\mathrm{Nil}_k(\mathrm{Lie})$ and $\mathrm{Sol}_t(\mathrm{Lie})$ are not weakly act. rep., for any $k \geq 3$ and $t \geq 2$.

Proof

- $S = \mathbb{F}\{a, b\}$ abelian.
- $B = \mathbb{F}\{x, a, b\}$, $[x, a] = b$.
- $B' = \mathbb{F}\{y, a, b\}$, $[y, b] = a$.
- $B \cong B' \cong \mathfrak{h}_3 \rightsquigarrow$ Heisenberg algebra
- Suppose $\exists D \in \mathrm{Sol}_t(\mathrm{Lie})$ such that

$$\begin{tikzcd} S \arrow[r, tail, "i'"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "="] & B' \arrow[d, tail, "\exists u'"] \\ B \arrow[r, tail, "\exists u"'] & D \end{tikzcd}$$

- Let $P = \langle u(B), u'(B') \rangle \leq D$.
- S is an ideal of B and B' $\Rightarrow$

  $\Rightarrow$ $V = u(S) = u'(S)$ is an ideal of P $\Rightarrow$

  $\Rightarrow$ we have the adjoint map

$$ad : P \longrightarrow Der(V)$$
$$p \longmapsto [p, -]$$

where $Der(\underbrace{V}_{\text{abelian 2-dim.}}) \cong gl(2, \mathbb{F})$.

- P solvable $\Rightarrow$ $ad(P)$ solvable.
- $ad(P) = \langle ad_{u(x)}, ad_{u'(x)} \rangle$

$$\cong \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle \cong \underbrace{sl(2, \mathbb{F})}_{\text{SIMPLE}}.$$

Thus, m and m' cannot be amalgamated in $\mathrm{Sol}_t(\mathcal{Lie})$ (and $\mathrm{Nil}_k(\mathcal{Lie})$).

<u>Remark</u> $\mathcal{Lie}$ has the A.P.

- $\mathcal{L} = \mathfrak{gl}(3, \mathbb{F})$.
- $\psi : B \rightarrowtail L$

$$\begin{cases} x \longmapsto -e_{23} \\ a \longmapsto e_{12} \\ b \longmapsto e_{13}. \end{cases} \rightsquigarrow \psi(B) \cong \mathfrak{h}_3$$

$$\psi(B) \supseteq \psi(S) = \psi'(S) \subseteq \psi'(B')$$

- $\psi' : B' \rightarrowtail L$

$$\begin{cases} y \longmapsto -e_{32} \\ a \longmapsto e_{12} \\ b \longmapsto e_{13}. \end{cases} \rightsquigarrow \psi'(B') \cong \mathfrak{h}_3$$

- $X \cong \mathbb{F}^3$ abelian $\rightsquigarrow [X] \cong L$.
- $\nu = \mathrm{id}_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{L} \cong [X]$.

· The split extension of B by X corresp. to $r\circ\psi=\psi : B \rightarrowtail L$ is

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, tail, "i_2"] \arrow[d, phantom, "\cong" rotate=90] & B \ltimes_\psi X \arrow[r, shift left, two heads, "\pi_1"] \arrow[d, phantom, "="] & B \arrow[l, shift left, "i_1"] \arrow[r] & 0 \\
& \mathbb{F}\{e_1,e_2,e_3\} & \mathbb{F}\{x,a,b,e_1,e_2,e_3\} & &
\end{tikzcd}$$

where

$$[x,a]=b,\ [x,e_3]=-e_2,$$
$$[a,e_2]=[b,e_3]=e_1.$$

One has

- $B\psi = B\ltimes_\psi X = \mathbb{F}\{x,a,b,e_1,e_2,e_3\}$
- $[B\psi, B\psi] = \mathbb{F}\{b,e_1,e_2\}$
- $[B\psi,[B\psi,B\psi]] = \mathbb{F}\{e_1\}$
- $[B\psi,[B\psi,[B\psi,B\psi]]] = 0.$

Thus $B\psi \in \mathrm{Nil}_3(\mathrm{Lie}) \Longrightarrow$

$\Longrightarrow B\psi \in \mathrm{Sol}_2(\mathrm{Lie}).$

In a similar way
$B'_{\psi'} \in \mathrm{Nil}_3(\mathrm{Lie}) \Longrightarrow$
$\Longrightarrow B'_{\psi'} \in \mathrm{Sol}_2(\mathrm{Lie})$.
Thus, $B_\psi, B'_{\psi'} \in \mathrm{Nil}_k(\mathrm{Lie})$
and $B_\psi, B'_{\psi'} \in \mathrm{Sol}_t(\mathrm{Lie})$,
for any $k \geq 3$ and $t \geq 2$. $\square$

Remark

Gray's example can be refined in order to prove that $\mathrm{Nil}_k(\mathrm{Grp})$ $(k \geq 3)$ and $\mathrm{Sol}_2(\mathrm{Grp})$ are not weakly act. rep..

Open question What about $\mathrm{Nil}_2(\mathrm{Grp})$?