Random normal matrices: eigenvalue correlations near a hard wall

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July 25, 2024

Abstract

We study pair correlation functions for planar Coulomb systems in the pushed phase, near a ring-shaped impenetrable wall. We assume coupling constant $\Gamma = 2$ and that the number n of particles is large. We find that the correlation functions decay slowly along the edges of the wall, in a narrow interface stretching a distance of order $1/n$ from the hard edge. At distances much larger than $1/\sqrt{n}$, the effect of the hard wall is negligible and pair correlation functions decay very quickly, and in between sits an interpolating interface that we call the "semi-hard edge".

More precisely, we provide asymptotics for the correlation kernel $K_n(z, w)$ as $n \to \infty$ in two microscopic regimes (with either $|z - w| = \mathcal{O}(1/\sqrt{n})$ or $|z - w| = \mathcal{O}(1/n)$), as well as in three macroscopic regimes (with $|z - w| \asymp 1$). For some of these regimes, the asymptotics involve oscillatory theta functions and weighted Szegő kernels.

AMS Subject Classification (2020): 41A60, 60B20, 60G55.
Keywords: Random normal matrices, correlation kernel, pushed phase, hard wall, theta functions.

1 Introduction: the pushed phase Coulomb gas

Hard edge boundary conditions are well-known in the theory of Hermitian random matrices, where they are, for instance, associated with the Bessel kernel, see [65, 32, 41]. In dimension two, the study of one-component plasmas near a hard wall is likewise of interest and variants appear in for example [47, 48, 62, 58, 34, 1, 15, 16, 36, 41, 37, 35, 60, 61, 30, 11, 12, 20, 49, 31, 33]. As in the majority of these works, we take the coupling constant to be $\Gamma = 2$, or equivalently, we consider eigenvalues of random normal matrices (see [28, 66]).

In [47], Jancovici considers two versions of the Ginibre ensemble,[1] the first being the usual "soft edge" ensemble. In the second model a wall is placed along the boundary of the droplet (nowadays this is called a "soft/hard edge"). One of Jancovici's motivations for studying a Coulomb system near a hard wall is its interest for describing an electrolyte near a colloidal wall or an electrode plate.

In both cases it was shown that the pair correlation functions along the edge decay very slowly compared with the situation in the bulk. More precisely, they decay only as an inverse power of the distance along the edge. The heuristic is that the screening cloud that surrounds a particle sitting near the edge is prevented by the external field and/or the wall from being rotationally symmetric, which gives the particle plus cloud system a nonvanishing electrical dipole moment. This dipole moment is not strongly localized and causes long-range correlations along the edge [47].

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[1] Ginibre matrices are almost surely non-normal. Nevertheless, their eigenvalues are distributed in the same way as the eigenvalues of random normal matrices with the Gaussian potential $|z|^2$.

In the soft edge-case, Jancovici's results were first generalized to the elliptic Ginibre ensemble in [43] and later to quite general potentials in the paper [14], where a leading term for macroscopic correlations near the outer boundary component of the droplet is found and expressed in terms of a Szegő kernel. The paper [10] provides corresponding results near the boundary of a ring-shaped spectral gap, where some additional oscillations (depending on the number of particles) enter the picture. Microscopically, the oscillations are expressed in terms of the Jacobi theta function, which enters the subleading term of the microscopic density near the soft edge; macroscopically, the pair correlations along the boundary involve an oscillatory Szegő kernel. The heuristic is that there is an additional uncertainty concerning the number of particles that fall near each of the two boundary components of the spectral gap, and this manifests itself in terms of some oscillations. It is observed in [10] that the subleading term of the microscopic density near the edge is closely related to fluctuations of linear statistics.

The goal of the present work is to adapt the above results to the setting where a hard wall is placed inside the bulk of a plasma. This is known as a "pushed phase" [36, 37, 35] or as a "hard edge" [61, 11], and is quite different from the "critical phase", i.e., the soft/hard edge in Jancovici's original work. In the pushed phase, the plasma undergoes two transitions, at distances $\mathcal{O}(1/\sqrt{n})$ and $\mathcal{O}(1/n)$ respectively, from the hard edge.

Before we continue, it is expedient to briefly recall the Coulomb gas model. We consider Coulomb systems $\{z_j\}_{j=1}^n$ in the complex plane $\mathbb{C}$ subjected to an appropriate confining potential Q, where the Hamiltonian is

$$H_n = \sum_{i \neq j} \log \frac{1}{|z_i - z_j|} + n \sum_{j=1}^n Q(z_j),$$

i.e. it is the sum of the logarithmic interaction energy and the energy of interaction with the external field. The probability law of the system is given by the Gibbs measure

$$d\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{n! Z_n} e^{-H_n(z_1, \ldots z_n)} \prod_{j=1}^n \frac{d^2 z_j}{\pi} \tag{1.1}$$

where Z_n is the normalizing constant.

Classically, one approximates the random measure $\frac{1}{n}\sum_{j=1}^n \delta_{z_j}$ by a continuous unit charge distribution $\sigma = \sigma[Q]$. This distribution is precisely given by Frostman's equilibrium measure associated with the potential Q, which is the unique minimizer σ of the "Q-energy"

$$\nu \mapsto I_Q[\nu] = \iint_{\mathbb{C}^2} \log \frac{1}{|z-w|}\, d\nu(z)\, d\nu(w) + \int_{\mathbb{C}} Q\, d\nu$$

among all compactly supported unit measures ν. The support $S = \operatorname{supp} \sigma$ is called the droplet in potential Q. (The measures $\frac{1}{n}\sum_{j=1}^n \delta_{z_j}$ converge, in a probabilistic and weak sense, to σ as $n \to \infty$, see e.g. [39, 7].)

If Q is smooth in a neighbourhood of the droplet, then by Frostman's theorem (see [59]), σ is absolutely continuous with respect to the area measure $d^2 z$ and takes the form $d\sigma(z) = \Delta Q(z) \chi_S(z) \frac{d^2 z}{\pi}$ where $\Delta Q := \frac{1}{4}(Q_{xx} + Q_{yy})$ is the usual Laplacian divided by four and χ_S is the indicator function of S. For example, if $Q(z) = |z|^2$ we obtain the Ginibre ensemble for which $S = \{z : |z| \le 1\}$ and $d\sigma(z) = \chi_S(z) \frac{d^2 z}{\pi}$.

We now fix a suitable (smooth) potential. In this paper we assume the rotational symmetry $Q(z) = Q(|z|)$ and that the droplet is a disc S. We also fix an open subset G of the interior of S

which we take to be an annulus $r_1 < |z| < r_2$ and modify the potential inside G to $+\infty$, i.e. we put

$$\tilde{Q}(z) = \begin{cases} Q(z), & z \in \mathbb{C} \setminus G, \\ +\infty, & z \in G. \end{cases} \tag{1.2}$$

Replacing Q by $\tilde{Q}$ has a drastic effect even at the level of the equilibrium measure. Indeed, the equilibrium measure $\tilde{\sigma}$ associated with the potential $\tilde{Q}$ puts zero mass in the gap G, and the restriction $\sigma|_G$ is swept to a measure ν_G supported on the boundary of G via a construction known as balayage [59]. In short, the measure ν_G has the same total mass as $\sigma|_G$, is supported on the boundary of G and takes the form $\nu_G = c_1\nu_1 + c_2\nu_2$ where ν_j denotes the arclength measure along the circle $|z| = r_j$ and c_j are certain constants, given in (2.4) below. The balayage measure ν_G and the restriction $\sigma|_G$ have furthermore identical logarithmic potentials in the complement $\mathbb{C} \setminus \overline{G}$.

The equilibrium measure $\tilde{\sigma}$ can now be written

$$\tilde{\sigma} = \sigma|_{\tilde{S}} + \nu_G,$$

where $d\sigma = \Delta Q(z)\frac{d^2z}{\pi}$ and $\tilde{S} = S \setminus G$ is the droplet associated with the potential $\tilde{Q}$.

On the level of the Coulomb system $\{z_j\}_{j=1}^n$, the picture is that most of the particles that originally occupied the gap get "swept" to a thin interface in $\tilde{S}$ near the boundary ∂G, at a distance of $\mathcal{O}(1/n)$; following [11] we call this the "hard edge regime". If we denote this interface by E_n then the corresponding random measure $\frac{1}{n}\sum_{z_j \in E_n} \delta_{z_j}$ approximates the singular part ν_G of $\tilde{\sigma}$.

The 1-particle density of the pushed system is of order of magnitude n^2 in the hard edge regime. Further inside the bulk of $\tilde{S}$, at distances much larger than $1/\sqrt{n}$ from ∂G, the effect of the hard wall becomes negligible and the 1-particle density is, to a first order approximation, given by n times the equilibrium density $n\Delta Q(z)\frac{d^2z}{\pi}$ of the unconstrained ensemble (associated with the potential Q). In between these regimes sits a transitional regime which we call the "semi-hard edge", following [11].

As we already mentioned, a different kind of wall, known as a soft/hard wall or a critical phase, is obtained by placing the hard wall along the boundary of the droplet S. This corresponds to setting $G = \mathbb{C} \setminus S$ in (1.2). In this case the local statistics near the wall, in a $\mathcal{O}(1/\sqrt{n})$-interface, is affected, but the equilibrium measure and the droplet are unchanged, and no hard-edge regime is present.

In studying hard walls, we must face the difficulty that the wall might cause long-range correlations merely due to its "symmetry breaking". A main insight in the forthcoming work [38] is that, for "general" potentials, symmetry breaking is avoided precisely if the hard wall is placed along the boundary of the droplet associated with the potential Q/τ where $0 < \tau < 1$ is a fixed suitable constant. In other words, the hard wall must be placed along the boundary of a mass-τ droplet associated with Q (cf. [53] for more about τ-droplets). If this is done, some first order universality results can be proven; however, we emphasize that much more refined asymptotic results, such as the ones we obtain below, remain currently out of reach in this generality. Hard walls which break the symmetry are so far studied mostly at the level of the equilibrium measure, see the works [1, 31, 33].

For rotationally symmetric pushed phase models (including Coulomb gases in $\mathbb{R}^d$ and Yukawa gases) more work has been done. In [35] it is proven that the weighted logarithmic energy $I_Q[\tilde{\sigma}]$ of the equilibrium measure exhibits a third-order phase transition as the wall crosses the critical phase and enters the pushed phase. In the planar case ($d = 2$), a large n-expansion of the free-energy is proved in [30], allowing for the case of annular spectral gaps. The third order phase transition is reflected in the leading coefficient of that expansion. In the work [61], Seo finds the leading order microscopic one-point density near a hard wall and proves it to be universal for a class of rotationally symmetric ensembles (which may have an additional logarithmic singularity along the hard edge).

Partition functions with hard edges are related with so-called *large gap (or hole) probabilities*; this topic has a long history, see e.g. [40, 5, 25], and has found important recent applications to physics, see e.g. [52]. Disc counting statistics are considered in [11, 12], and [20] gives a functional limit theorem for certain smooth radially symmetric linear statistics in the hard edge regime, near the outer boundary of the droplet.

It is worth noting that a different type of hard-edge ensemble, corresponding to the external potential $Q = 0$ and a hard wall outside some compact set Σ bounded by a Jordan curve has attracted some recent interest (e.g. [49]). In this case the entire system will tend to occupy the hard wall regime, i.e. the portion of Σ which has distance $\mathcal{O}(1/n)$ to the boundary of Σ. In a way, this case is simpler than the kind of hard walls studied in the present paper, since there is essentially just one regime (no "bulk" to interact with). In the case when Σ is an elliptic disc (possibly with an added logarithmic singularity along the hard edge) correlations are studied in [3, 56], following the earlier work [67] on truncated unitary matrices.

The present work gives a comprehensive study of correlations near a ring-shaped hard wall in the hard and semi-hard regimes on the microscopic and macroscopic levels. We shall consider the class of underlying rotationally symmetric potentials of the form

$$Q(z) = |z|^{2b} + \frac{2\alpha}{n} \log \frac{1}{|z|} \tag{1.3}$$

where $b > 0$ and $\alpha > -1$. These are sometimes called model Mittag-Leffler potentials, and they give rise to the droplets $S = \{z : |z| \leq b^{-1/2b}\}$. The corresponding point-processes (1.1) are well studied and are sometimes called model Mittag-Leffler ensembles, see [6, 18, 29, 22] and the references there.

For the potentials (1.3) and an annular spectral gap $G = \{r_1 < |z| < r_2\}$ with $0 < r_1 < r_2 < b^{-1/2b}$ we provide a full asymptotic picture of the correlations associated with the potential $\tilde{Q}$, cf. Figures 1 and 2 for illustrations.

In the rest of this paper, we will use the symbol Q to denote the pushed-phase Mittag-Leffler potential (above denoted $\tilde{Q}$) which is $+\infty$ in G, and we will write μ for the equilibrium measure of Q and $S = \operatorname{supp} \mu = \{|z| \leq b^{-1/2b}\} \setminus G$ for the pushed-phase droplet, see Figure 2.

2 Description of the model

The k-point correlation functions $\{R_{n,k} : \mathbb{C}^k \to [0, +\infty)\}_{k=1}^n$ associated with the point process $\{z_j\}_{j=1}^n$ in (1.1) are defined such that

$$\mathbb{E}_n[f(z_1, \ldots, z_k)] = \frac{(n-k)!}{n!} \int_{\mathbb{C}^k} f(z_1, \ldots, z_k)\, R_{n,k}(z_1, \ldots, z_k) \prod_{j=1}^k \frac{d^2 z_j}{\pi}$$

holds for all continuous and compactly supported functions f on $\mathbb{C}^k$. The point process (1.1) is determinantal, meaning that all correlation functions exist and that there exists a correlation kernel $K_n(z, w)$ such that

$$R_{n,k}(z_1, \ldots, z_n) = \det(K_n(z_i, z_j))_{i,j=1}^k, \qquad \text{for all } n \in \mathbb{N}_{>0},\ k \in \{1, 2, \ldots, n\},\ z_1, \ldots, z_k \in \mathbb{C}.$$

It should be noted that while the correlation functions are uniquely defined, a correlation kernel $K_n(z, w)$ is only defined up to a multiplicative "cocycle", i.e., a function of the form $c_n(z, w) = g_n(z)\overline{g_n(w)}$ where g_n is a unimodular function. We fix K_n uniquely by taking $K_n(z, w)$ to be the reproducing kernel of the n-dimensional subspace of L^2 consisting of weighted polynomials

$p(z)e^{-nQ(z)/2}$ where p is a holomorphic polynomial of degree at most $n-1$. This K_n is called the *canonical* correlation kernel and is used without exception in the following; see (2.5) for an explicit formula.

We shall study large n asymptotics for pair correlations $K_n(z,w)$ when z,w are close to a hard wall. However, before specializing to our setting, it is convenient to recall a few well known asymptotic results in other regimes, such as the bulk and soft edge regimes. Our discussion is far from exhaustive and we refer to the recent survey [23] for more background.

If z_0 is a regular point in the bulk (i.e. the interior of S) then [19] the rescaled kernels

$$\mathrm{K}_n(\mathrm{z},\mathrm{w}) = \frac{1}{n\Delta Q(z_0)} K_n\Big(z_0 + \frac{\mathrm{z}}{(n\Delta Q(z_0))^{1/2}}, z_0 + \frac{\mathrm{w}}{(n\Delta Q(z_0))^{1/2}}\Big), \qquad \mathrm{z},\mathrm{w}\in\mathbb{C},$$

converge as $n\to+\infty$ (after multiplication by suitable cocycles $c_n(\mathrm{z},\mathrm{w})$) to the Ginibre kernel $e^{\mathrm{z}\overline{\mathrm{w}}-|\mathrm{z}|^2/2-|\mathrm{w}|^2/2}$, and where we recall that ΔQ denotes the standard Laplacian divided by four, i.e. $\Delta Q = (Q_{xx}+Q_{yy})/4$.

In the soft edge case, the large n behavior of $K_n(z,w)$ is well understood when z,w are close to the "outer boundary" of S, provided that this is an everywhere regular Jordan curve and that $\Delta Q > 0$ along the curve. By the outer boundary, we mean the component ∂U of ∂S where U (throughout) denotes the unbounded component of $\mathbb{C}\setminus S$. In this case, the leading order asymptotics of $K_n(z,w)$ is found in e.g. [42, 50, 63, 15, 46] for the case when z,w are at a distance of order $n^{-1/2}$ from the outer boundary ∂U and such that $|z-w| = \mathcal{O}(n^{-1/2})$.[2] The papers [54, 10] provide subleading corrections to the microscopic kernel near an outer boundary, and [10] also gives such correction terms near the boundary of a ring-shaped spectral gap and relates them to fluctuations of linear statistics.

As we already noted in Section 1, two points z,w near a (smooth) outer soft edge ∂U are strongly correlated even if $|z-w|\asymp 1$,[3] i.e. even if z,w lie at a macroscopic distance of each other. (By contrast, if z,w are in the bulk, $K_n(z,w)$ gets exponentially small as $n\to+\infty$ with $|z-w|\asymp 1$, and $K_n(z,w)$ has also a Gaussian decay in the distance $|z-w|$ for fixed n). A formula for long-range correlations along the outer boundary ∂U of the droplet is computed in [14]. As noticed in [10], the situation is more subtle when z,w are in a $n^{-1/2}$ neighborhood of the boundary of a (soft) spectral gap that is not simply connected: in this case, the asymptotics of $K_n(z,w)$ are oscillatory and described in terms of Jacobi θ-functions (if $|z-w| = \mathcal{O}(n^{-1/2})$) or weighted Szegő kernels (if $|z-w|\asymp 1$).

Some further related results for correlations along the boundary are found in [4] (higher dimensional elliptic Ginibre ensemble) and [27] (lemniscate ensembles).

Scaling limits have also been investigated in other situations than the bulk and soft edge regimes, see e.g. [2, 18] near root-type bulk singularities, [15, 16, 60, 24, 51] near singular boundary points (such as cusp-like singularities and so-called local droplets), [44, 3, 9, 26] for bandlimited point processes, and [47, 34, 58, 15, 16, 17] near soft/hard edges and other sorts of soft/hard boundary conditions.

Much less is known about pair correlations near the hard edge for pushed-phase ensembles (i.e. with a non-constant potential). Indeed, to our knowledge, the only hitherto recorded results appear in the paper [61], where large n asymptotics for $K_n(z,w)$ is studied the microscopic regime where z,w satisfy $|z-w| = \mathcal{O}(n^{-1})$ and are in a n^{-1} neighborhood of ∂U (with ∂U being a hard edge),

[2] $f(n) = \mathcal{O}(g(n))$ means $f(n)\le Cg(n)$ for all large enough n and $C>0$ is independent of n.
[3] $f(n)\asymp g(n)$ means $C_1 g(n)\le f(n)\le C_2 g(n)$ for all large enough n and $C_2\ge C_1>0$ are independent of n.

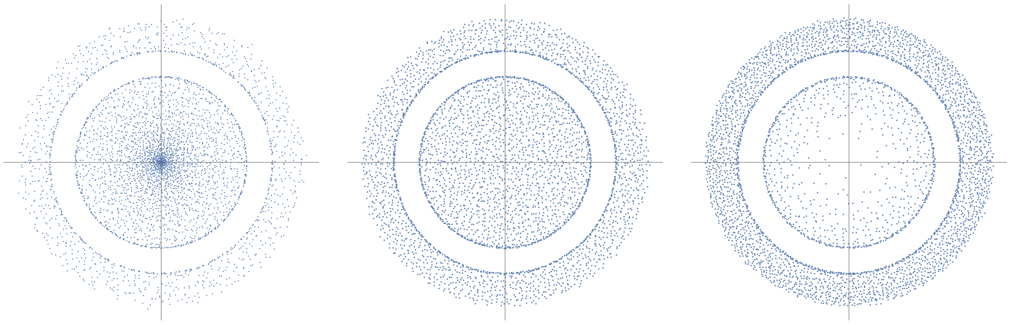

Figure 1: Illustration of the point process (2.2) with $n = 4096$, $r_1 = \frac{3}{5}b^{-\frac{1}{2b}}$, $r_2 = \frac{4}{5}b^{-\frac{1}{2b}}$, $\alpha = 0$ and the indicated values of b.

in the case of rotation-invariant Q (which may have a weak logarithmic singularity along the hard edge).

Recall that the droplet associated with the Mittag-Leffler potential (1.3) is given by $\{|z| \leq b^{-1/2b}\}$. We now fix numbers r_1, r_2 with $0 < r_1 < r_2 < b^{-1/2b}$ and redefine that potential to be $+\infty$ in the gap (or hard wall) $G := \{z : |z| \in (r_1, r_2)\}$, i.e. we set

$$Q(z) = \begin{cases} |z|^{2b} - \frac{2\alpha}{n}\log|z|, & \text{if } |z| \notin (r_1, r_2), \\ +\infty, & \text{if } |z| \in (r_1, r_2), \end{cases} \qquad b > 0,\ \alpha > -1. \tag{2.1}$$

In this work, we focus on the corresponding point process:

$$\frac{1}{n!\mathcal{Z}_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)} \prod_{j=1}^{n} \frac{d^2 z_j}{\pi}. \tag{2.2}$$

The method of balayage in [12] shows that the equilibrium measure associated with (2.1) is

$$\mu(d^2 z) = 2b^2 r^{2b-1} \chi_{[0,r_1] \cup [r_2, b^{-\frac{1}{2b}}]}(r) dr \frac{d\theta}{2\pi} + \sigma_1 \delta_{r_1}(r) dr \frac{d\theta}{2\pi} + \sigma_2 \delta_{r_2}(r) dr \frac{d\theta}{2\pi}, \tag{2.3}$$

where $z = re^{i\theta}$, $r > 0$, $\theta \in (-\pi, \pi]$ and

$$\sigma_1 = \sigma_\star - b r_1^{2b}, \qquad \sigma_2 = b r_2^{2b} - \sigma_\star, \qquad \sigma_\star := \frac{r_2^{2b} - r_1^{2b}}{2\log(\frac{r_2}{r_1})}. \tag{2.4}$$

The droplet is given by $S = \{z : |z| \in [0, r_1] \cup [r_2, b^{-\frac{1}{2b}}]\}$. We write $U = \{z : |z| > b^{-\frac{1}{2b}}\}$ and $G = \{z : r_1 < |z| < r_2\}$ for the unbounded and bounded components of $\mathbb{C} \setminus S$ respectively, and refer to G as the "spectral gap", or "hard wall". Note also that G is not simply connected.

Thus $\partial U = \{|z| = b^{-\frac{1}{2b}}\}$ is a soft edge while $\partial G = \partial S \setminus \partial U = \{|z| = r_1\} \cup \{|z| = r_2\}$ is the union of two hard edges (see also Figure 1). The quantity $\sigma_1 > 0$ represents the proportion of the points swept out from G which accumulate near $\{|z| = r_1\}$, $\sigma_2 > 0$ is the proportion of the points accumulating near $\{|z| = r_2\}$, and $\sigma_\star = \mu(\{z : |z| \leq r_1\})$.

Since Q is rotation-invariant, the canonical correlation kernel of (2.2) is given by

$$K_n(z,w) = e^{-\frac{n}{2}Q(z)}e^{-\frac{n}{2}Q(w)}\sum_{j=1}^{n}\frac{z^{j-1}\overline{w}^{j-1}}{h_j}, \tag{2.5}$$

where

$$\begin{aligned} h_j &:= \|z^{j-1}e^{-\frac{n}{2}Q(z)}\|^2_{L^2(\mathbb{C},d^2z/\pi)} = \int_{\mathbb{C}} |z|^{2j-2}e^{-nQ(z)}\frac{d^2z}{\pi} \\ &= \frac{1}{bn^{\frac{j+\alpha}{b}}}\bigg(\gamma(\tfrac{j+\alpha}{b}, nr_1^{2b}) - \gamma(\tfrac{j+\alpha}{b}, nr_2^{2b}) + \Gamma(\tfrac{j+\alpha}{b})\bigg), \qquad j = 1,2,\ldots,n, \end{aligned} \tag{2.6}$$

and the gamma functions γ, Γ are defined by

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt, \qquad \gamma(a,z) := \int_0^z t^{a-1}e^{-t}dt. \tag{2.7}$$

We recall some asymptotic properties of γ in Appendix B.

Our main results are asymptotic formulas for $K_n(z,w)$ as $n \to +\infty$ for five regimes where $z, w \in S$ are "close" to ∂G. We will explore (i) the so-called "hard edge regime", i.e. when z, w are $\mathcal{O}(n^{-1})$ away from ∂G, and (ii) the "semi-hard edge regime", i.e. when z, w are at a distance of order $n^{-1/2}$ from ∂G. (If z, w are slightly further away from ∂G, for example if z, w are at a distance of order $\sqrt{\log n}/\sqrt{n}$ from ∂G, then the asymptotics of $K_n(z,w)$ are no longer affected by the hard wall and we recover the bulk regime [11]. This regime is not included here as it is standard by now.) The semi-hard edge regime was recently discovered in [11] in the study of a problem of counting statistics, but this regime has not yet been explored at the level of the correlation kernel. This regime is genuinely different from the bulk regime and from the hard edge regime.

For both the hard and semi-hard edge regimes, we study the asymptotics of $K_n(z,w)$ in the microscopic (i.e. when $z \approx w$) and the macroscopic (i.e. when $|z-w| \asymp 1$) cases. We find that the asymptotics of $K_n(z,w)$ oscillates in n in the hard edge regime, but not in the semi-hard edge regime. In the microscopic (or "diagonal") case, the oscillations are described in terms of the Jacobi θ function, while in the macroscopic (or "off-diagonal") case they are described in terms of weighted Szegő kernels. Our main results are stated in Section 3 and can be summarized as follows (see also Figure 2):

1. Theorem 3.1 establishes the asymptotics of $K_n(z,w)$, up to and including the fourth term of order $\sqrt{n}$, when $|z-w| = \mathcal{O}(\frac{1}{n})$ with z, w being both at a distance $\mathcal{O}(\frac{1}{n})$ from ∂G,
2. Theorem 3.5 establishes the asymptotics of $K_n(z,w)$, up to and including the second term of order $\sqrt{n}$, when $|z-w| = \mathcal{O}(\frac{1}{\sqrt{n}})$ with z, w being both at a distance $\asymp \frac{1}{\sqrt{n}}$ from ∂G,
3. Theorem 3.11 establishes the leading order asymptotics of $K_n(z,w)$ when z, w are on different sides of G and are both at a distance $\mathcal{O}(\frac{1}{n})$ from ∂G,
4. Theorem 3.13 establishes the leading order asymptotics of $K_n(z,w)$ when $|z-w| \asymp 1$ with z, w being on the same side of G and being both at a distance $\mathcal{O}(\frac{1}{n})$ from ∂G,
5. Theorem 3.15 establishes an upper bound on $K_n(z,w)$ when $|z-w| \asymp 1$ with z, w being on the same side of G and being both at a distance $\asymp \frac{1}{\sqrt{n}}$ from ∂G.

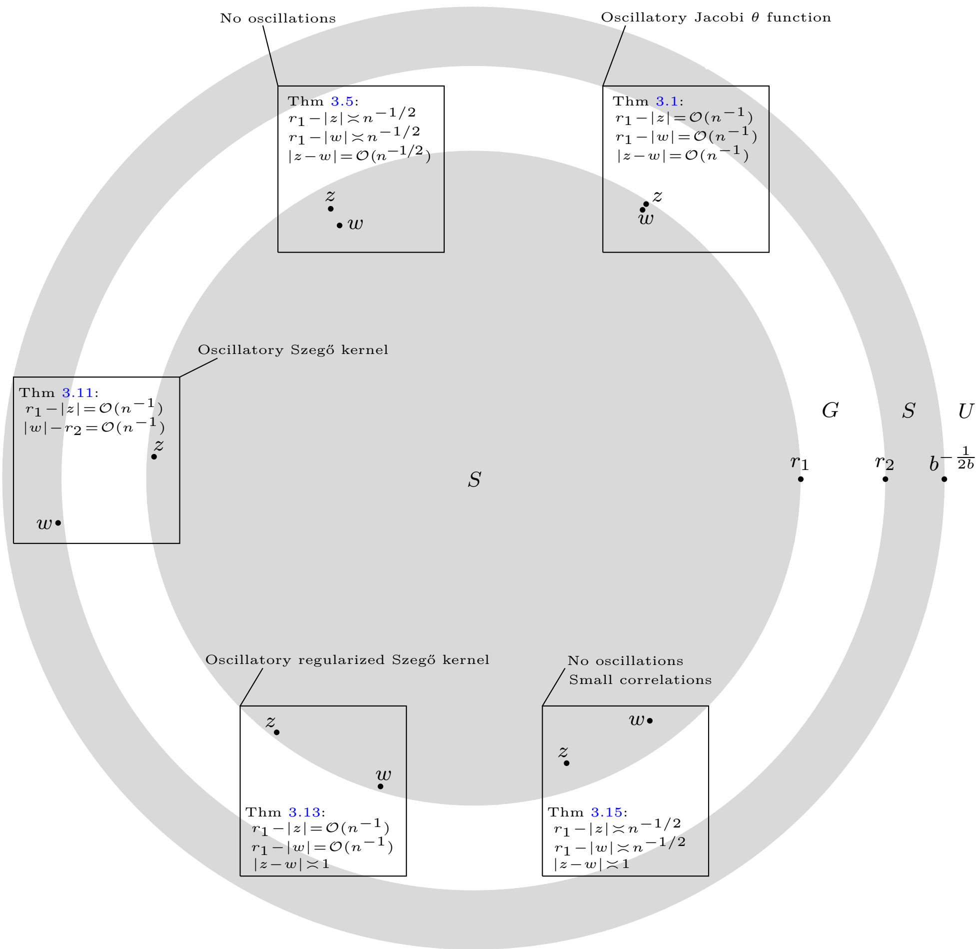

Figure 2: Summary of our main results on the asymptotics of $K_n(z,w)$.

3 Main results

Our first main result is an asymptotic formula for $K_n(z,w)$ in the microscopic regime $|z-w|=\mathcal{O}(n^{-1})$ when both z and w are in the hard edge regime. This asymptotic formula contains oscillations that are described in terms of the Jacobi θ function. We recall that this function is defined by

$$\theta(z;\tau) := \sum_{\ell=-\infty}^{+\infty} e^{2\pi i\ell z} e^{\pi i \ell^2 \tau}, \qquad z\in\mathbb{C}, \qquad \tau \in i(0,+\infty), \tag{3.1}$$

and satisfies $\theta(-z;\tau)=\theta(z;\tau)$ and $\theta(z+1;\tau)=\theta(z;\tau)$. Other properties of θ can be found in e.g. [57, Chapter 20]. The statement of Theorem 3.1 also involves Euler's gamma constant $\gamma_{\mathrm{E}}\approx 0.5772$, as well as the exponential integral E_1 and the complementary error function erfc (see e.g. [57, eqs

6.2.1 and 7.2.2]), which we recall are defined by

$$E_1(x) := \int_x^{+\infty} \frac{e^{-t}}{t} dt, \qquad \mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt. \tag{3.2}$$

Theorem 3.1. ("r_1 hard edge case") *Let $t_1, t_2 \geq 0$ and $\beta \in \mathbb{R}$ be fixed, and define*

$$z = r_1\Big(1 - \frac{t_1}{\sigma_1 n}\Big)e^{i\beta}, \qquad w = r_1\Big(1 - \frac{t_2}{\sigma_1 n}\Big)e^{i\beta}. \tag{3.3}$$

As $n \to +\infty$,

$$K_n(z,w) = C_1 n^2 + C_2\, n \log n + \Big(C_3 + \frac{\sigma_1}{r_1^2} e^{-t_1-t_2}\mathcal{F}_n\Big)n + C_4\sqrt{n} + \mathcal{O}(n^{\frac{2}{5}}), \tag{3.4}$$

where $\mathcal{F}_n$ is given by

$$\mathcal{F}_n = \frac{(\log\theta)'\Big(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\log(\sigma_2/\sigma_1)}{2\log(r_2/r_1)}; \frac{\pi i}{\log(r_2/r_1)}\Big) + \log(\sigma_2/\sigma_1)}{2\log(r_2/r_1)}, \tag{3.5}$$

and σ_1, σ_2 and $\sigma_\star$ are defined in (2.4). If $(t_1, t_2) \neq (0,0)$, the constants $C_1, \ldots, C_4$ (constants with respect to n) are given by

$$\begin{aligned}
& C_1 = \sigma_1^2 \frac{1 - e^{-t_1-t_2}(1+t_1+t_2)}{r_1^2(t_1+t_2)^2}, \\
& C_2 = \frac{b^2 r_1^{2b-2}}{2}, \\
& C_3 = -b^2 r_1^{2b-2}\bigg(E_1(t_1+t_2) + \gamma_{\mathrm{E}} + \log\bigg(\frac{b r_1^{2b}(t_1+t_2)}{\sigma_1\sqrt{2\pi}}\bigg)\bigg) \\
& \quad + \frac{\sigma_1}{r_1^2(t_1+t_2)^3}\bigg\{\frac{(t_1+t_2)^2(t_1^2+t_2^2)}{2e^{t_1+t_2}} + \Big(2t_1t_2 - \frac{b^2 r_1^{2b}}{\sigma_1}(t_1+t_2)(t_1^2+t_2^2)\Big)\Big(1 - \frac{1+t_1+t_2}{e^{t_1+t_2}}\Big)\bigg\}, \\
& C_4 = \sqrt{2} b^2 r_1^{b-2}(1 - 2br_1^{2b}\frac{t_1+t_2}{\sigma_1})\mathcal{I},
\end{aligned} \tag{3.6}$$

and $\mathcal{I} \in \mathbb{R}$ is given by

$$\mathcal{I} = \int_{-\infty}^{+\infty} \bigg\{\frac{y\, e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2 + \frac{1}{2}\Big]\bigg\}dy \approx -0.81367. \tag{3.7}$$

If $t_1 = t_2 = 0$, then one simply needs to let $t_1 + t_2 \to 0$ in the above formulas for $C_1, \ldots, C_4$; more precisely, for $t_1 = t_2 = 0$ the constants $C_1, \ldots, C_4$ are given by

$$C_1 = \frac{\sigma_1^2}{2r_1^2}, \qquad C_2 = \frac{b^2 r_1^{2b-2}}{2}, \qquad C_3 = \frac{b^2 r_1^{2b-2}}{2}\log\Big(\frac{2\pi\sigma_1^2}{b^2 r_1^{2b}}\Big), \qquad C_4 = \sqrt{2} b^2 r_1^{b-2}\mathcal{I}.$$

Remark 3.2. *If either $t_1 < 0$ or $t_2 < 0$, then either $z \in G$ or $w \in G$ and $K_n(z,w)$ is trivially 0 by (2.1) and (2.5); this is why we restricted ourselves to $t_1, t_2 \geq 0$ in the statement of Theorem 3.1. Taking $t_1 = t_2$ in Theorem 3.1 yields precise asymptotics for the one-point function $R_{n,1}\big(r_1 e^{i\beta}(1 - \frac{t_1}{\sigma_1 n})\big)$ in the hard edge regime. For $t_1 = t_2$, C_1 in Theorem 3.1 also appeared in [67, eq (21) with $L = 1$] in the context of truncated unitary matrices, and for general t_1, t_2 the quantity C_1 also appeared when considering perturbations of rank L to Hermitian matrices, see e.g. [23, eq (2.64)],*

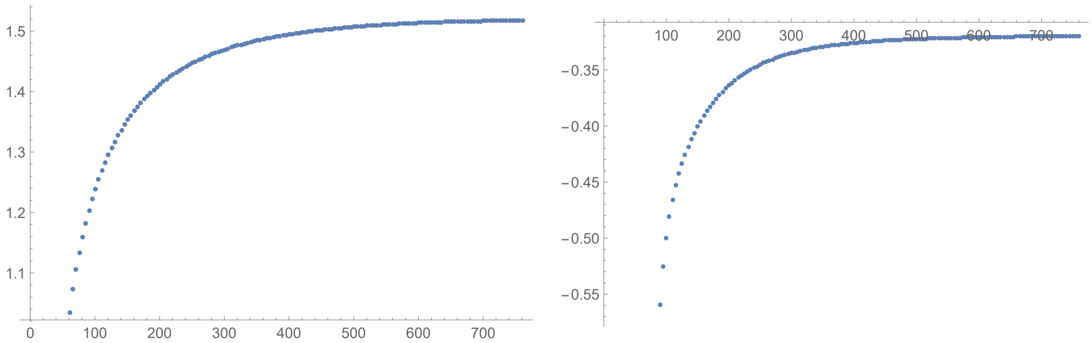

Figure 3: Numerical confirmations of Theorems 3.1 (left) and 3.5 (right). For both pictures, $b = 1.3$, $\alpha = 1.26$, $r_1 = 0.42b^{-\frac{1}{2b}}$, $r_2 = 0.67b^{-\frac{1}{2b}}$. Left: the function $n \mapsto \frac{1}{\log n}\big[K_n(z,w) - \big(C_1 n^2 + C_2\, n \log n + \big(C_3 + \frac{\sigma_1}{r_1^2} e^{-t_1-t_2}\mathcal{F}_n\big)n + C_4\sqrt{n}\big)\big]$, with $z = r_1(1 - \frac{t_1}{\sigma_1 n})$, $w = r_1(1 - \frac{t_2}{\sigma_1 n})$, $t_1 = 0.21$, $t_2 = 0.45$, and C_1, C_2, C_3, C_4 as in (3.6). This function seems to approach a constant as $n \to +\infty$, which is consistent with Theorem 3.1, and also suggests that the term $\mathcal{O}(n^{\frac{2}{5}})$ in (3.4) is actually $\mathcal{O}(\log n)$. Right: the function $n \mapsto K_n(z,w) - \big(C_1 n + C_2\sqrt{n}\big)$, with $z = r_1\big(1 - \frac{\mathfrak{s}_1}{b r_1^b \sqrt{2n}}\big)$, $w = r_1\big(1 - \frac{\mathfrak{s}_2}{b r_1^b \sqrt{2n}}\big)$, $\mathfrak{s}_1 = 1.21$, $\mathfrak{s}_2 = 1.45$, and C_1, C_2 as in (3.10). This function seems to approach a constant as $n \to +\infty$, which is consistent with Theorem 3.5.

[45, Section 2.2 and 3.1] and the references therein. (For truncated unitary matrices, the situation is somewhat simpler because there is no bulk and the droplet is connected. Therefore, the asymptotics for the correlation kernel are expected to be of a simpler form than (3.4), i.e. without oscillations, and without terms of order $n \log n$ – see also the discussion [13, end of Section 1].)

For comparison, in the soft edge setting, the asymptotics of the one-point function near a spectral gap take the following form: for $z = e^{i\beta}(r_1 + \frac{t}{\sqrt{n\Delta Q(r_1)}})$, we have

$$R_{n,1}(z) = \tilde{C}_1 n + \sqrt{n}(\tilde{C}_2 + e^{-2t^2}\tilde{\mathcal{F}}_n) + \mathcal{O}((\log n)^4), \qquad \text{as } n \to +\infty,$$

where $\tilde{C}_1, \tilde{C}_2$ are independent of n and $\tilde{\mathcal{F}}_n$ is independent of t and of order 1, see [10, Theorem 1.9].

Remark 3.3. *In this paper we focus on the case where the hard wall G lies entirely in the interior of $\{z : |z| \leq b^{-\frac{1}{2b}}\}$. We do not cover the simpler case where the hard wall is of the form $\tilde{G} = \{z : |z| > r_1\}$; in this case we expect that a similar formula as (3.4) holds but with $\mathcal{F}_n = 0$ (there should be no oscillations since the droplet is then given by $\{z : |z| \leq r_1\}$, which is a connected set).*

This expectation is supported by the works [61]. Consider a radially symmetric potential Q whose associated equilibrium measure is $\mu(d^2z) = 2\Delta Q(z)\,\chi_{[\rho_1,r_1]}(r) r\,dr\frac{d\theta}{2\pi} + \sigma_1\delta_{r_1}(r)dr\frac{d\theta}{2\pi}$ $(z = re^{i\theta})$, supported on a connected set of the form $\tilde{S} = \{z : |z| \in [\rho_1, r_1]\}$ with $r_1 > \rho_1 \geq 0$. Let $p \in \partial\tilde{U} = \{z : |z| = r_1\}$. Under general conditions on Q, it is proved in [61, Theorem 2.1] that, as $n \to +\infty$,

$$K_n\Big(r_1 e^{i\beta}\Big[1 - \frac{t_1}{n\sigma_1}\Big], r_1 e^{i\beta}\Big[1 - \frac{t_2}{n\sigma_1}\Big]\Big) = \frac{n^2\sigma_1^2}{r_1^2}\int_0^1 t e^{-x(t_1+t_2)}dx + o(n^2) = C_1 n^2 + o(n^2).$$

Comparing the above with (3.4), we conclude that the leading order behavior of $K_n\big(r_1 e^{i\beta}(1 - \frac{t_1}{\sigma_1 n}), r_1 e^{i\beta}(1 - \frac{t_2}{\sigma_1 n})\big)$ in the two situations $G = \{z : |z| \in (r_1, r_2)\}$ (considered here) and $\tilde{G} = \{z : |z| \in (r_1, +\infty)\}$ (covered in [61]) are identical.

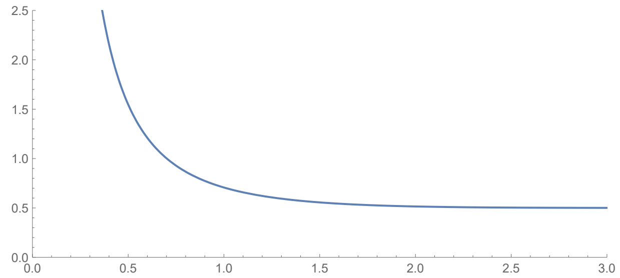

Figure 4: The semi-hard edge density profile $\rho(x)$.

Remark 3.4. *For two dimensional point processes in the multi-component regime (i.e. when S consists of several disjoint components), the Jacobi θ function is known to also describe large gap fluctuations [30], smooth linear statistics and microscopic correlations for ensembles with soft edges [10], and disk counting statistics for ensembles with hard edges [12]. It is still an open problem to establish the emergence of the theta function for two-dimensional point processes that are not rotation-invariant; such ensembles include the so-called lemniscate ensemble (see e.g. [27]) and the elliptic Ginibre point process with a large point charge [21].*

Our next theorem on the asymptotics of $K_n(z,w)$ concerns the semi-hard edge regime in the microscopic regime $|z-w| = \mathcal{O}(n^{-1/2})$. Let $\tilde{\Delta}Q(r_1) := \lim_{r\to r_1, r<r_1} \Delta Q(r) = b^2 r_1^{2b-2}$.

Theorem 3.5. ("r_1 semi-hard edge case") *Let $\mathfrak{s}_1, \mathfrak{s}_2 > 0$, $\beta \in \mathbb{R}$ be fixed parameters, and define*

$$z = r_1\Big(1 - \frac{\mathfrak{s}_1}{br_1^b\sqrt{2n}}\Big)e^{i\beta}, \qquad w = r_1\Big(1 - \frac{\mathfrak{s}_2}{br_1^b\sqrt{2n}}\Big)e^{i\beta} = \bigg(r_1 - \frac{\mathfrak{s}_2}{(2\tilde{\Delta}Q(r_1))^{\frac{1}{2}}\sqrt{n}}\bigg)e^{i\beta}. \tag{3.8}$$

As $n \to +\infty$,

$$K_n(z,w) = C_1 n + C_2\sqrt{n} + \mathcal{O}(1), \tag{3.9}$$

where

$$\begin{aligned} C_1 &= 2\tilde{\Delta}Q(r_1)\int_{-\infty}^{+\infty} \frac{e^{-\frac{(y+\mathfrak{s}_1)^2+(y+\mathfrak{s}_2)^2}{2}}}{\sqrt{\pi}\mathrm{erfc}(y)}dy, \\ C_2 &= b\frac{(2\tilde{\Delta}Q(r_1))^{\frac{1}{2}}}{r_1}\int_{-\infty}^{+\infty} \frac{e^{-\frac{(y+\mathfrak{s}_1)^2+(y+\mathfrak{s}_2)^2}{2}}}{\sqrt{\pi}\mathrm{erfc}(y)}\bigg\{\frac{(10y^2-2)e^{-y^2}}{3\sqrt{\pi}\mathrm{erfc}(y)} + 5y - \frac{10y^3}{3} + \frac{\mathfrak{s}_1+\mathfrak{s}_2}{b} - y\frac{\mathfrak{s}_1^2+\mathfrak{s}_2^2}{2b} \\ &\qquad - 2y^2(\mathfrak{s}_1+\mathfrak{s}_2) + \frac{2b-3}{b}\frac{\mathfrak{s}_1^3+\mathfrak{s}_2^3}{6}\bigg\}dy. \end{aligned} \tag{3.10}$$

Remark 3.6. *In the statement of Theorem 3.5, note that $\mathfrak{s}_1, \mathfrak{s}_2 > 0$. The case $\mathfrak{s}_1 = \mathfrak{s}_2 = 0$ is very different and is covered as a special case of Theorem 3.1.*

Remark 3.7. *The constant C_1 in (3.9) is universal for general potentials in the semi-hard regime: if things are defined appropriately, the associated "microscopic semi-hard correlation kernel"*

$$\tilde{K}_n(\mathfrak{s}_1, \mathfrak{s}_2) := \frac{1}{2n\tilde{\Delta}Q(r_1)}K_n\Big(r_1 - \tfrac{\mathfrak{s}_1}{\sqrt{2n\tilde{\Delta}Q(r_1)}}, r_1 - \tfrac{\mathfrak{s}_2}{\sqrt{2n\tilde{\Delta}Q(r_1)}}\Big)$$

satisfies

$$\tilde{K}_n(\mathfrak{s}_1, \mathfrak{s}_2) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(y+\mathfrak{s}_1)^2+(y+\mathfrak{s}_2)^2}{2}}}{\sqrt{\pi}\mathrm{erfc}(y)} dy + o(1), \quad \text{as } n \to +\infty. \tag{3.11}$$

For convenience, precise conditions implying the universality (3.11) are discussed in Appendix A.

Passing to the limit as $n \to \infty$ in (3.11) with $\mathfrak{s}_2 = \mathfrak{s}_1 = x$, we obtain the microscopic semi-hard edge one-point density profile

$$\rho(x) := \int_{-\infty}^{+\infty} \frac{e^{-(y+x)^2}}{\sqrt{\pi}\mathrm{erfc}(y)} dy, \qquad (x > 0), \tag{3.12}$$

which is believed to be universal at any kinds hard walls in the semi-hard regime, see Appendix A. The density profile $\rho(x)$ is depicted in Figure 4.

Remark 3.8. *In the paper [58], the authors consider two independent plasmas whose droplets are annuli which are adjacent to each other, and where the common boundary between the plasmas is an impermeable membrane, i.e. it is a soft/hard edge for each of the plasmas.*

Interestingly, a density profile for the $\mathcal{O}(1/\sqrt{n})$-interface about the edge is computed in [58, Eq. (3.4)]. The profile bears a formal resemblance to (3.12), and also to the soft/hard edge plasma function studied in [47, 15, 16].

Our next three theorems on the asymptotics of $K_n(z, w)$ treat some macroscopic regimes when $|z - w| \asymp 1$.

For comparison purposes, we first briefly recall the results from [14, 10] about asymptotics for $K_n(z, w)$ in some macroscopic regimes for ensembles with soft edges. When z, w are far from each other but both close to ∂U, the asymptotics of $K_n(z, w)$ involves the (unweighted) Szegő kernel $S^U_{\text{soft}}(z, w)$, which is the reproducing kernel for the Hardy space $H^2_0(U)$ of all holomorphic functions on U vanishing at infinity, supplied with the norm $\|f\|^2_{H^2_0(U)} := \int_{\partial U} |f|^2 \, |dz|$, see [14]. On the other hand, when z, w are far from each other and both close to a (soft) spectral gap of the form $G = \{z : |z| \in (r_1, r_2)\}$, there is no convergence towards a limiting kernel [10]: for example, if z and w are close to ∂G but lie on different regions of $\mathbb{C} \setminus G$ (i.e. $|z| - r_1 \asymp n^{-1/2}$ and $|w| - r_2 \asymp n^{-1/2}$, or vice-versa), the asymptotics are described by the oscillatory Szegő kernel

$$\mathcal{S}^G_{\text{soft}}(z, w; n) := \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} \frac{(z\bar{w})^\ell}{\frac{{r_1}^{2\ell+1-2x}}{\sqrt{\Delta Q(r_1)}} + \frac{r_2^{2\ell+1-2x}}{\sqrt{\Delta Q(r_2)}}}, \tag{3.13}$$

where $x = x(n) := n \int_{|z| \le r_1} \mu(d^2 z) - \lfloor n \int_{|z| \le r_1} \mu(d^2 z) \rfloor \in [0, 1)$ and μ is the equilibrium measure associated with Q. $\mathcal{S}^G_{\text{soft}}(z, w; n)$ is an analytic function of $z, \bar{w} \in G$ which is also well-defined when $|z| - r_1 \asymp n^{-1/2}$ and $|w| - r_2 \asymp n^{-1/2}$ (or vice-versa). This function is also the reproducing kernel for the weighted Hardy space $H^2(G; n)$ consisting of all analytic functions on G such that

$$\|f\|^2_{H^2(G;n)} := \int_{\partial G} |f(z)|^2 |z|^{-2x(n)} (\Delta Q(z))^{-\frac{1}{2}} \, |dz| < \infty. \tag{3.14}$$

The situation when z, w are far from each other but are on the same side of G gets more complicated because the series (3.13) is divergent if $z, w \in \partial G$ with $|z| = |w|$. For $z = r_1 e^{i\theta_1}$ and $w = r_1 e^{i\theta_2}$, the weighted Szegő kernel appearing in the asymptotics of $K_n(z, w)$ turns out to be

$$\mathcal{S}^G_{\mathrm{soft}}(r_1e^{i\theta_1},r_1e^{i\theta_2};n) := \frac{1}{2\pi}\frac{\sqrt{\Delta Q(r_1)}}{r_1^{1-2x}}\bigg(\frac{1}{e^{i(\theta_1-\theta_2)}-1} + \sum_{\ell=0}^{\infty}\frac{e^{i(\theta_1-\theta_2)\ell}}{1+\sqrt{\frac{\Delta Q(r_1)}{\Delta Q(r_2)}}(\frac{r_2}{r_1})^{2\ell-2x+1}} - \sum_{\ell=-\infty}^{-1}\frac{e^{i(\theta_1-\theta_2)\ell}}{1+\sqrt{\frac{\Delta Q(r_2)}{\Delta Q(r_1)}}(\frac{r_1}{r_2})^{2\ell-2x+1}}\bigg). \tag{3.15}$$

As noticed in [10], the series in (3.15) can be recognized as the Abel limit of (3.13); more precisely, $\mathcal{S}^G_{\mathrm{soft}}(r_1e^{i\theta_1},r_1e^{i\theta_2};n)$ in (3.15) is equal to $\lim_{r\searrow r_1}\mathcal{S}^G_{\mathrm{soft}}(re^{\theta_1},re^{\theta_2};n)$, where $\mathcal{S}^G_{\mathrm{soft}}(re^{\theta_1},re^{\theta_2};n)$ is as in (3.13) with $z=re^{i\theta_1}$ and $w=re^{i\theta_2}$.

It is apriori not clear at all whether analogues to the above results from [14, 10] exist near a hard edge. For example, $\Delta Q(r_1)$ appears in (3.13), (3.14) and (3.15), but in our setting $\{|z|=r_1\}$ is a hard edge and $\Delta Q(r_1)$ does not even make sense. It therefore came as a surprise to us that our findings completely mimic the results of [10]: the main difference is that the quantities $\sqrt{\Delta Q(r_1)}$ and $\sqrt{\Delta Q(r_2)}$ appearing in (3.13)–(3.15) should here be replaced by σ_1/r_1 and σ_2/r_2, respectively. More precisely, we have the following definition.

Definition 3.9. The oscillatory Szegő kernel $\mathcal{S}^G_{\mathrm{hard}}(z,w;n)$ associated with $G=\{r_1<|z|<r_2\}$ and n is defined for $z,w\in G$ by

$$\mathcal{S}^G_{\mathrm{hard}}(z,w;n) := \frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\frac{(z\bar{w})^\ell}{\frac{r_1^{2(\ell+1-x)}}{\sigma_1}+\frac{r_2^{2(\ell+1-x)}}{\sigma_2}}. \tag{3.16}$$

This is an analytic function of $z,\bar{w}\in G$. This function is also well-defined as long as $|z\overline{w}|$ lies in a compact subset of (r_1^2,r_2^2); for example, it is well-defined for $|z|=r_1$ and $|w|=r_2$ (and vice-versa, it is also well-defined for $|z|=r_2$ and $|w|=r_1$). For $|z|=|w|=r_1$, $z\neq w$, the above series does not converge absolutely. Nevertheless, for $\theta_1\neq\theta_2$, the Abel limit $\lim_{r\searrow 1}\mathcal{S}^G_{\mathrm{hard}}(re^{\theta_1},re^{\theta_2};n)$ exists, we denote it by $\mathcal{S}^G_{\mathrm{hard}}(r_1e^{i\theta_1},r_1e^{i\theta_2};n)$, and is equal to

$$\mathcal{S}^G_{\mathrm{hard}}(r_1e^{i\theta_1},r_1e^{i\theta_2};n) = \frac{1}{2\pi}\frac{\sigma_1}{r_1^{2(1-x)}}\bigg(\frac{1}{e^{i(\theta_1-\theta_2)}-1}+\sum_{\ell=0}^{+\infty}\frac{e^{i(\theta_1-\theta_2)\ell}}{1+\frac{\sigma_1}{\sigma_2}(\frac{r_2}{r_1})^{2(\ell+1-x)}} - \sum_{\ell=-\infty}^{-1}\frac{e^{i(\theta_1-\theta_2)\ell}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_1}{r_2})^{2(\ell+1-x)}}\bigg). \tag{3.17}$$

Because one had to take the Abel limit to make sense of $\mathcal{S}^G_{\mathrm{hard}}(r_1e^{i\theta_1},r_1e^{i\theta_2};n)$, we call $\mathcal{S}^G_{\mathrm{hard}}(r_1e^{i\theta_1}, r_1e^{i\theta_2};n)$ an oscillatory "regularized" Szegő kernel, see also Figure 2.

Remark 3.10. *The kernel (3.16) is the reproducing kernel for the weighted Hardy space $H^2(G;n)$ of all analytic functions g on G satisfying*

$$\|f\|^2_{H^2(G;n)} := \int_{\{|z|=r_1\}}|f(z)|^2\frac{r_1^{1-2x(n)}}{\sigma_1}|dz| + \int_{\{|z|=r_2\}}|f(z)|^2\frac{r_2^{1-2x(n)}}{\sigma_2}|dz| < \infty. \tag{3.18}$$

We next state our results on the asymptotics of $K_n(z,w)$ in the macroscopic regime when z and w are in different regions of $\mathbb{C}\setminus G$.

Theorem 3.11. ("r_1-r_2 hard edge case") *Let $t_1,t_2\geq 0$ and $\theta_1,\theta_2\in\mathbb{R}$ be fixed, and define*

$$z=r_1\Big(1-\frac{t_1}{\sigma_1 n}\Big)e^{i\theta_1}, \qquad w=r_2\Big(1+\frac{t_2}{\sigma_2 n}\Big)e^{i\theta_2}, \qquad t_1,t_2\geq 0,\ \ \theta_1,\theta_2\in\mathbb{R}. \tag{3.19}$$

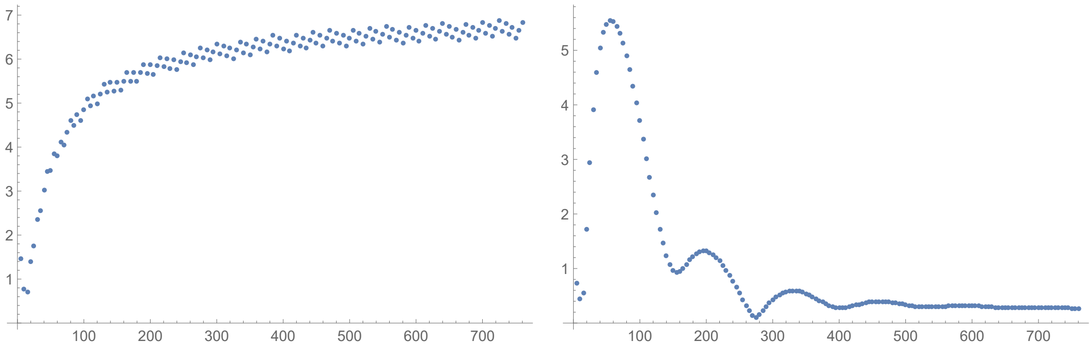

Figure 5: Numerical confirmations of Theorems 3.11 (left) and 3.13 (right). For both pictures, $b = 1.3$, $\alpha = 1.26$, $r_1 = 0.42b^{-\frac{1}{2b}}$, $r_2 = 0.67b^{-\frac{1}{2b}}$. Left: the function $n \mapsto \big|\big[K_n(z,w) - 2\pi n \cdot \mathcal{S}^G_{\mathrm{hard}}(z,w;n) \cdot e^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}(r_1r_2)^{-x}e^{-t_1-t_2}\big|$, with $z = r_1(1-\frac{t_1}{\sigma_1 n})e^{i\theta_1}$, $w = r_2(1+\frac{t_2}{\sigma_2 n})e^{i\theta_2}$, $t_1 = 0.21$, $t_2 = 0.45$, $\theta_1 = 0$, $\theta_2 = 0.312$. This function seems to grow very slowly as $n \to +\infty$, which is consistent with Theorem 3.11. Right: the function $n \mapsto \frac{1}{\sqrt{n}}\big|K_n(z,w) - 2\pi n \cdot \mathcal{S}^G_{\mathrm{hard}}(r_1e^{\theta_1}, r_1e^{\theta_2};n) \cdot e^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}r_1^{-2x}e^{-t_1-t_2}\big|$, with $z = r_1(1-\frac{t_1}{\sigma_1 n})e^{i\theta_1}$, $w = r_1(1-\frac{t_2}{\sigma_1 n})e^{i\theta_2}$, $t_1 = 0.91$, $t_2 = 1.45$, $\theta_1 = 0$, $\theta_2 = 0.312$. This function seems to approach a constant as $n \to +\infty$, which is consistent with Theorem 3.13, and also suggests that the term $\mathcal{O}(\sqrt{n \log n})$ in (3.22) is actually $\mathcal{O}(\sqrt{n})$.

As $n \to +\infty$, we have

$$K_n(z,w) = 2\pi n \cdot \mathcal{S}^G_{\mathrm{hard}}(z,w;n) \cdot e^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}(r_1r_2)^{-x}e^{-t_1-t_2} + \mathcal{O}((\log n)^2),$$

where $j_\star := n\sigma_\star - \alpha$, and $x = x(n)$ is given by

$$x = j_\star - \lfloor j_\star \rfloor = n\sigma_\star - \alpha - \lfloor n\sigma_\star - \alpha \rfloor \in [0,1). \tag{3.20}$$

Remark 3.12. *The analogue of Theorem 3.11 near soft edges is given by [10, Corollary 1.14] and is as follows: for $z = \big(r_1 + \frac{t}{\sqrt{n\Delta Q(r_1)}}\big)e^{i\theta_1}$, $w = \big(r_2 + \frac{s}{\sqrt{n\Delta Q(r_2)}}\big)e^{i\theta_2}$ with $s,t \in \mathbb{R}$, we have*

$$K_n(z,w) = \sqrt{2\pi n} \cdot \mathcal{S}^G_{\mathrm{soft}}(z,w;n) \cdot e^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}(r_1r_2)^{-x}e^{-t^2-s^2} + \mathcal{O}((\log n)^3),$$

where $\mathcal{S}^G_{\mathrm{soft}}(z,w;n)$ is given by (3.13), $j_\star := n\int_{|z|\le r_1}\mu(d^2z)$ and x and μ are as in (3.13).

We next consider the macroscopic regime where z and w are on different sides of G.

Theorem 3.13. ("r_1-r_1 hard edge case") *Let $t_1, t_2 \ge 0$ and $\theta_1 \ne \theta_2 \in \mathbb{R}$ be fixed, and define*

$$z = r_1\Big(1 - \frac{t_1}{\sigma_1 n}\Big)e^{i\theta_1}, \qquad w = r_1\Big(1 - \frac{t_2}{\sigma_1 n}\Big)e^{i\theta_2}. \tag{3.21}$$

As $n \to \infty$, we have

$$K_n(z,w) = 2\pi n \cdot \mathcal{S}^G_{\mathrm{hard}}(r_1e^{\theta_1}, r_1e^{\theta_2};n) \cdot e^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}r_1^{-2x}e^{-t_1-t_2} + \mathcal{O}(\sqrt{n\log n}), \tag{3.22}$$

where $j_\star := n\sigma_\star - \alpha$, $x = j_\star - \lfloor j_\star \rfloor$.

Remark 3.14. *The analogue of Theorem 3.13 near a soft edge is given by [10, Corollary 1.14] and is as follows: for* $z = \big(r_1 + \frac{t}{\sqrt{n\Delta Q(r_1)}}\big)e^{i\theta_1}$, $w = \big(r_1 + \frac{s}{\sqrt{n\Delta Q(r_2)}}\big)e^{i\theta_2}$ *with* $s, t \in \mathbb{R}$, $\theta_1 \neq \theta_2$, *we have*

$$K_n(z, w) = \sqrt{2\pi n} \cdot \mathcal{S}^G_{\mathrm{soft}}(r_1 e^{i\theta_1}, r_1 e^{i\theta_2}; n) \cdot e^{i\lfloor j_\star \rfloor (\theta_1 - \theta_2)} r_1^{-2x} e^{-t^2 - s^2} + \mathcal{O}((\log n)^5),$$

where $\mathcal{S}^G_{\mathrm{soft}}(r_1 e^{i\theta_1}, r_1 e^{i\theta_2}; n)$ *is given by* (3.15), $j_\star = n \int_{|z| \leq r_1} \mu(d^2 z)$, *and* x *and* μ *are as in* (3.13).

Our last result shows that $K_n(z, w)$ is small whenever z and w are in the semi-hard edge regime and $|z - w| \asymp 1$.

Theorem 3.15. ("r_1-r_1 semi-hard edge case") *Let* $\mathfrak{s}_1, \mathfrak{s}_2 > 0$ *and* $\theta_1 \neq \theta_2 \in \mathbb{R}$ *be fixed, and define*

$$z = r_1 \Big(1 - \frac{\mathfrak{s}_1}{b r_1^b \sqrt{2n}}\Big) e^{i\theta_1}, \qquad w = r_1 \Big(1 - \frac{\mathfrak{s}_2}{b r_1^b \sqrt{2n}}\Big) e^{i\theta_2}.$$

As $n \to +\infty$,

$$K_n(z, w) = \mathcal{O}(1). \tag{3.23}$$

Remark 3.16. *The estimate* (3.23) *can be probably be strengthened with more efforts. In fact, in the setting of Theorem 3.15, we believe that* $K_n(z, w) = \mathcal{O}(n^{-K})$ *holds as* $n \to \infty$ *for any fixed* $K > 0$.

Outline of the paper. In Section 4, we obtain large n estimates for h_j valid uniformly in different ranges of $j \in \{1, \ldots, n\}$. These estimates will be useful for the proofs of each of our five theorems. The proof of Theorem 3.11 is given in Section 5, the proofs of Theorems 3.1 and 3.13 are given in Section 6, and the proofs of Theorems 3.5 and 3.15 are given in Section 7.

4 Asymptotics of h_j

Recall that h_j is defined in (2.6). Following [29, 30], we introduce the following quantities: for $j = 1, \ldots, n$ and $\ell = 1, 2$, we define

$$a_j := \frac{j + \alpha}{b}, \qquad \lambda_{j,\ell} := \frac{b n r_\ell^{2b}}{j + \alpha}, \qquad \eta_{j,\ell} := (\lambda_{j,\ell} - 1)\sqrt{\frac{2(\lambda_{j,\ell} - 1 - \log \lambda_{j,\ell})}{(\lambda_{j,\ell} - 1)^2}}. \tag{4.1}$$

Let $\epsilon > 0$ be a small constant independent of n. Define

$$\begin{aligned} & j_{\ell,-} := \lceil \tfrac{b n r_\ell^{2b}}{1+\epsilon} - \alpha \rceil, && j_{\ell,+} := \lfloor \tfrac{b n r_\ell^{2b}}{1-\epsilon} - \alpha \rfloor, && \ell = 1, 2, \\ & j_{0,-} := 1, && j_{0,+} := M', && j_{3,-} := n + 1, \end{aligned} \tag{4.2}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$, $\lfloor x \rfloor$ denotes the largest integer $\leq x$, and M' is a large but fixed constant. We take ϵ sufficiently small such that

$$\frac{b r_1^{2b}}{1 - \epsilon} < \frac{b r_2^{2b}}{1 + \epsilon}, \qquad \text{and} \qquad \frac{b r_2^{2b}}{1 - \epsilon} < 1. \tag{4.3}$$

The quantity $\sigma_\star$, which we recall is defined in (2.4), will appear naturally in our analysis. It is easy to check that $\sigma_\star \in (b r_1^{2b}, b r_2^{2b})$. For technical reasons, we also assume $\epsilon > 0$ is small enough so that

$$\frac{b r_1^{2b}}{1 - \epsilon} < \sigma_\star < \frac{b r_2^{2b}}{1 + \epsilon}. \tag{4.4}$$

Recall also that $j_\star := n\sigma_\star - \alpha$, and note that

$$\frac{a_j(\eta_{j,2}^2 - \eta_{j,1}^2)}{2} = 2(j_\star - j)\log\frac{r_2}{r_1}. \tag{4.5}$$

Let M be such that $M'\sqrt{\log n} \leq M \leq n^{\frac{1}{10}}$, and define

$$g_{\ell,-} := \left\lceil \frac{bnr_\ell^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \right\rceil, \qquad g_{\ell,+} := \left\lfloor \frac{bnr_\ell^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \right\rfloor, \qquad \ell = 1,2,$$

and

$$M_{j,k} := \sqrt{n}(\lambda_{j,k} - 1), \qquad \text{for all } k \in \{1,2\} \text{ and } j \in \{g_{k,-},\ldots,g_{k,+}\}. \tag{4.6}$$

In this section, M can be arbitrary within the range $M \in [M'\sqrt{\log n}, n^{\frac{1}{10}}]$, but we already mention that in Section 5 and Subsection 6.2 we will take $M = M'\sqrt{\log n}$, in Subsection 6.1 we will take $M = n^{\frac{1}{10}}$, and in Section 7 we will take $M = M'\log n$. For $k = 1,2$ and $g_{k,-} \leq j \leq g_{k,+}$, we also define

$$\chi_{j,k}^- := \chi_{(-\infty,0)}(M_{j,k}), \qquad \chi_{j,k}^+ := \chi_{(0,+\infty)}(M_{j,k}).$$

We start with the following useful lemma.

Lemma 4.1. *As $n \to +\infty$ with $M' \leq j \leq n$, we have*

$$\frac{n^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} = \frac{\sqrt{n}\, e^{\frac{n}{b}(j/n - j/n\log(\frac{j/n}{b}))}}{\sqrt{2\pi}} \left(\frac{b}{j/n}\right)^{\frac{\alpha}{b}} \sqrt{\frac{j/n}{b}} \left(1 + \frac{-b^2+6b\alpha-6\alpha^2}{12bj/n}\frac{1}{n} + \mathcal{O}(j^{-2})\right). \tag{4.7}$$

As $n \to +\infty$ with $g_{k,-} \leq j \leq g_{k,+}$, $k = 1,2$, we have

$$\begin{aligned}
\frac{n^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} &= e^{r_k^{2b}(1-\log(r_k^{2b}))n} e^{M_{j,k} r_k^{2b}\log(r_k^{2b})\sqrt{n}} e^{-r_k^{2b}(\frac{1}{2}+\log(r_k^{2b}))M_{j,k}^2} \frac{r_k^b\sqrt{n}}{\sqrt{2\pi}} \\
&\times \Bigg(1 + \frac{M_{j,k}}{6}(-3+5M_{j,k}^2 r_k^{2b} + 6M_{j,k}^2 r_k^{2b}\log(r_k^{2b}))\frac{1}{\sqrt{n}} + \frac{1}{n}\Bigg\{ r_k^{4b} M_{j,k}^6 \left(\frac{25}{72} + \frac{5\log(r_k^{2b})}{6} + \frac{(\log(r_k^{2b}))^2}{2}\right) \\
&- \frac{3}{2} r_k^{2b} M_{j,k}^4 (1+\log(r_k^{2b}))\Bigg\} + \frac{125+450\log(r_k^{2b})+540\log^2(r_k^{2b})+216\log^3(r_k^{2b})}{1296} r_k^{6b} M_{j,k}^9 n^{-3/2} \\
&+ \mathcal{O}\left(\frac{1+M_{j,k}^2}{n} + \frac{1+M_{j,k}^7}{n^{3/2}} + \frac{1+M_{j,k}^{12}}{n^2}\right)\Bigg).
\end{aligned} \tag{4.8}$$

Proof. Formula (4.7) directly follows from the well-known large j asymptotics of $\Gamma(\frac{j+\alpha}{b})$ (see e.g. [57, 5.11.1]), and (4.8) directly follows from the large j asymptotics of $\Gamma(\frac{j+\alpha}{b})$ and the fact that, by (4.1) and (4.6), we have

$$\frac{j}{n} = -\frac{\alpha}{n} + \frac{br_k^{2b}}{1+\frac{M_{j,k}}{\sqrt{n}}}, \qquad g_{k,-} \leq j \leq g_{k,+},\ k = 1,2.$$

$\square$

Lemma 4.2. *M' can be chosen sufficiently large such that the following asymptotics hold uniformly for $M'\sqrt{\log n} \leq M \leq n^{\frac{1}{10}}$.*

1. Let j be fixed. As $n \to +\infty$, we have

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1 + \mathcal{O}(n^{\frac{j+\alpha}{b}-1}e^{-nr_1^{2b}})\Big). \tag{4.9}$$

2. As $n \to +\infty$ with $M' \leq j \leq j_{1,-}$, we have

$$\begin{aligned} h_j^{-1} &= \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1 + \mathcal{O}(n^{-\frac{1}{2}}e^{-nr_1^{2b}\frac{\epsilon-\log(1+\epsilon)}{1+\epsilon}}\Big) \\ &= b\frac{\sqrt{n}\, e^{\frac{n}{b}(j/n - j/n\log(\frac{j/n}{b}))}}{\sqrt{2\pi}}\Big(\frac{b}{j/n}\Big)^{\frac{\alpha}{b}}\sqrt{\frac{j/n}{b}}\Big(1 + \frac{-b^2+6b\alpha-6\alpha^2}{12bj/n}\frac{1}{n} + \mathcal{O}(j^{-2})\Big). \end{aligned} \tag{4.10}$$

3. As $n \to +\infty$ with $j_{1,-} \leq j \leq g_{1,-}$, we have

$$\begin{aligned} h_j^{-1} &= \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1 + \mathcal{O}(n^{-100})\Big) \\ &= b\frac{\sqrt{n}\, e^{\frac{n}{b}(j/n - j/n\log(\frac{j/n}{b}))}}{\sqrt{2\pi}}\Big(\frac{b}{j/n}\Big)^{\frac{\alpha}{b}}\sqrt{\frac{j/n}{b}}\Big(1 + \frac{-b^2+6b\alpha-6\alpha^2}{12bj/n}\frac{1}{n} + \mathcal{O}(n^{-2})\Big). \end{aligned} \tag{4.11}$$

4. As $n \to +\infty$ with $g_{1,-} \leq j \leq g_{1,+}$, we have

$$\begin{aligned} h_j^{-1} &= \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\frac{1}{\frac{1}{2}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})}\Big(1 + \frac{(5M_{j,1}^2 r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})\sqrt{n}} + \frac{\chi_{j,1}^-}{n}\Big\{\frac{25r_1^{4b}M_{j,1}^6}{72} \\ &+ \frac{3r_1^{2b}M_{j,1}^4}{2}\Big\} - \chi_{j,1}^-\frac{125r_1^{6b}M_{j,1}^9}{1296n^{3/2}} + \mathcal{O}\Big(\frac{1+M_{j,1}^2}{n} + \frac{1+|M_{j,1}^7|}{n^{3/2}} + \frac{1+M_{j,1}^{12}}{n^2}\Big)\Big) \\ &= b\frac{e^{r_1^{2b}(1-\log(r_1^{2b}))n}e^{M_{j,1}r_1^{2b}\log(r_1^{2b})\sqrt{n}}e^{-r_1^{2b}(\frac{1}{2}+\log(r_1^{2b}))M_{j,1}^2}r_1^b\sqrt{n}}{\frac{1}{2}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})\sqrt{2\pi}} \\ &\times\Big(1 + \frac{1}{\sqrt{n}}\Big\{\frac{(5M_{j,1}^2r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} + M_{j,1}^3r_1^{2b}\log(r_1^{2b}) - \frac{M_{j,1}}{2} + \frac{5M_{j,1}^3r_1^{2b}}{6}\Big\} \\ &+ \frac{\frac{r_1^{4b}}{2}(\log(r_1^{2b}))^2M_{j,1}^6 - 2r_1^{2b}\log(r_1^{2b})M_{j,1}^4}{n} + \frac{r_1^{6b}(\log(r_1^{2b}))^3M_{j,1}^9}{6n^{3/2}} \\ &+ \mathcal{O}\Big(\frac{1+M_{j,1}^2+M_{j,1}^6\chi_{j,1}^+}{n} + \frac{1+|M_{j,1}^7|+M_{j,1}^9\chi_{j,1}^+}{n^{3/2}} + \frac{1+M_{j,1}^{12}}{n^2}\Big)\Big). \end{aligned} \tag{4.12}$$

5. As $n \to +\infty$ with $g_{1,+} \leq j \leq j_{1,+}$, we have

$$\begin{aligned} h_j^{-1} &= \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\sqrt{n}\, e^{\frac{n}{b}(br_1^{2b} - j/n + j/n\log(\frac{j/n}{br_1^{2b}}))}\frac{\sqrt{2\pi}}{\sqrt{b\, j/n}}\Big(\frac{j/n}{br_1^{2b}}\Big)^{\frac{\alpha}{b}}(j/n - br_1^{2b}) \\ &\times\Big(1 + \frac{12b^3r_1^{2b}j/n + 6b((j/n)^2-(br_1^{2b})^2)\alpha + (b^2+6\alpha^2)(j/n-br_1^{2b})^2}{12bj/n(j/n-br_1^{2b})^2n} \\ &- \frac{2b^4r_1^{4b}}{n^2(j/n-br_1^{2b})^4} + \frac{10b^6r_1^{6b}}{n^3(j/n-br_1^{2b})^6} + \mathcal{O}\Big(\frac{1}{n^2(j/n-br_1^{2b})^3} + \frac{1}{n^4(j/n-br_1^{2b})^8}\Big)\Big) \end{aligned}$$

$$= \frac{n(j/n - br_1^{2b})}{r_1^{2\alpha}} e^{n(r_1^{2b} - 2j/n\log(r_1))} \Bigg(1 + \frac{b^2 r_1^{2b} + (j/n - br_1^{2b})\alpha}{n(j/n - br_1^{2b})^2} - \frac{2b^4 r_1^{4b}}{n^2(j/n - br_1^{2b})^4}$$
$$+ \frac{10 b^6 r_1^{6b}}{n^3(j/n - br_1^{2b})^6} + \mathcal{O}\Big(\frac{1}{n^2(j/n - br_1^{2b})^3} + \frac{1}{n^4(j/n - br_1^{2b})^8}\Big)\Bigg). \tag{4.13}$$

6. *As $n \to +\infty$ with $j_{1,+} \leq j \leq \lfloor j_\star \rfloor$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \sqrt{n}\, e^{\frac{n}{b}(br_1^{2b} - j/n + j/n\log(\frac{j/n}{br_1^{2b}}))} \frac{\sqrt{2\pi}}{\sqrt{b\,j/n}} \frac{\big(\frac{j/n}{br_1^{2b}}\big)^{\frac{\alpha}{b}}}{\frac{1}{j/n - br_1^{2b}} + \frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{br_2^{2b} - j/n}}$$
$$\times \Bigg(1 + \frac{12b^3 r_1^{2b} j/n + 6b((j/n)^2 - (br_1^{2b})^2)\alpha + (b^2 + 6\alpha^2)(j/n - br_1^{2b})^2}{12bj/n(j/n - br_1^{2b})^2 n}$$
$$+ \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{n} + \frac{1}{n^2}\Big)\Bigg)$$
$$= \frac{n}{r_1^{2\alpha}} \frac{e^{n(r_1^{2b} - 2j/n\log(r_1))}}{\frac{1}{j/n - br_1^{2b}} + \frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{br_2^{2b} - j/n}} \Bigg(1 + \frac{b^2 r_1^{2b} + (j/n - br_1^{2b})\alpha}{n(j/n - br_1^{2b})^2} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{n} + \frac{1}{n^2}\Big)\Bigg). \tag{4.14}$$

7. *As $n \to +\infty$ with $\lfloor j_\star \rfloor + 1 \leq j \leq j_{2,-}$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \sqrt{n}\, e^{\frac{n}{b}(br_2^{2b} - j/n + j/n\log(\frac{j/n}{br_2^{2b}}))} \frac{\sqrt{2\pi}}{\sqrt{b\,j/n}} \frac{\big(\frac{j/n}{br_2^{2b}}\big)^{\frac{\alpha}{b}}}{\frac{1}{br_2^{2b} - j/n} + \frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{j/n - br_1^{2b}}}$$
$$\times \Bigg(1 + \frac{12b^3 r_2^{2b} j/n + 6b((j/n)^2 - (br_2^{2b})^2)\alpha + (b^2 + 6\alpha^2)(j/n - br_2^{2b})^2}{12bj/n(j/n - br_2^{2b})^2 n}$$
$$+ \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{n} + \frac{1}{n^2}\Big)\Bigg)$$
$$= \frac{n}{r_2^{2\alpha}} \frac{e^{n(r_2^{2b} - 2j/n\log(r_2))}}{\frac{1}{br_2^{2b} - j/n} + \frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{j/n - br_1^{2b}}} \Bigg(1 + \frac{b^2 r_2^{2b} + (j/n - br_2^{2b})\alpha}{n(j/n - br_2^{2b})^2} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{n} + \frac{1}{n^2}\Big)\Bigg). \tag{4.15}$$

8. *As $n \to +\infty$ with $j_{2,-} \leq j \leq g_{2,-}$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \sqrt{n}\, e^{\frac{n}{b}(br_2^{2b} - j/n + j/n\log(\frac{j/n}{br_2^{2b}}))} \frac{\sqrt{2\pi}}{\sqrt{b\,j/n}} \left(\frac{j/n}{br_2^{2b}}\right)^{\frac{\alpha}{b}} (br_2^{2b} - j/n)$$
$$\times \Bigg(1 + \frac{12b^3 r_2^{2b} j/n + 6b((j/n)^2 - (br_2^{2b})^2)\alpha + (b^2 + 6\alpha^2)(j/n - br_2^{2b})^2}{12bj/n(j/n - br_2^{2b})^2 n}$$
$$+ \mathcal{O}\Big(\frac{1}{n^2(j/n - br_2^{2b})^4}\Big)\Bigg)$$
$$= \frac{n(br_2^{2b} - j/n)}{r_2^{2\alpha}} e^{n(r_2^{2b} - 2j/n\log(r_2))} \Bigg(1 + \frac{b^2 r_2^{2b} + (j/n - br_2^{2b})\alpha}{n(j/n - br_2^{2b})^2} + \mathcal{O}\Big(\frac{1}{n^2(j/n - br_2^{2b})^4}\Big)\Bigg). \tag{4.16}$$

9. *As $n \to +\infty$ with $g_{2,-} \leq j \leq g_{2,+}$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \frac{1}{1 - \frac{1}{2}\mathrm{erfc}(-\frac{M_{j,2} r_2^b}{\sqrt{2}})} \Bigg(1 + \frac{(2 - 5M_{j,2}^2 r_2^{2b}) e^{-\frac{r_2^{2b} M_{j,2}^2}{2}}}{3\sqrt{2\pi} r_2^b (2 - \mathrm{erfc}(-\frac{M_{j,2} r_2^b}{\sqrt{2}}))\sqrt{n}} + \frac{\chi_{j,2}^+}{n} \Bigg\{ \frac{25 r_2^{4b} M_{j,2}^6}{72}$$

$$+\frac{3r_2^{2b}M_{j,2}^4}{2}\bigg\} - \chi_{j,2}^{+}\frac{125r_2^{6b}M_{j,2}^9}{1296n^{3/2}} + \mathcal{O}\bigg(\frac{1+M_{j,2}^2}{n} + \frac{1+|M_{j,2}^7|}{n^{3/2}} + \frac{1+M_{j,2}^{12}}{n^2}\bigg)\bigg)$$

$$= b\frac{e^{r_2^{2b}(1-\log(r_2^{2b}))n}e^{M_{j,2}r_2^{2b}\log(r_2^{2b})\sqrt{n}}e^{-r_2^{2b}(\frac{1}{2}+\log(r_2^{2b}))M_{j,2}^2}r_2^b\sqrt{n}}{(1-\frac{1}{2}\mathrm{erfc}(-\frac{M_{j,2}r_2^b}{\sqrt{2}}))\sqrt{2\pi}}$$

$$+\frac{\frac{r_2^{4b}}{2}(\log(r_2^{2b}))^2M_{j,2}^6 - 2r_2^{2b}\log(r_2^{2b})M_{j,2}^4}{n} + \frac{r_2^{6b}(\log(r_2^{2b}))^3M_{j,2}^9}{6n^{3/2}}$$

$$+\mathcal{O}\bigg(\frac{1+M_{j,2}^2+M_{j,2}^6\chi_{j,2}^{-}}{n} + \frac{1+|M_{j,2}^7|+M_{j,2}^9\chi_{j,2}^{-}}{n^{3/2}} + \frac{1+M_{j,2}^{12}}{n^2}\bigg)\bigg). \tag{4.17}$$

10. *As $n \to +\infty$ with $g_{2,+} \leq j \leq j_{2,+}$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1+\mathcal{O}(n^{-100})\Big)$$

$$= b\frac{\sqrt{n}\,e^{\frac{n}{b}(j/n - j/n\log(\frac{j/n}{b}))}}{\sqrt{2\pi}}\left(\frac{b}{j/n}\right)^{\frac{\alpha}{b}}\sqrt{\frac{j/n}{b}}\bigg(1+\frac{-b^2+6b\alpha-6\alpha^2}{12bj/n}\frac{1}{n} + \mathcal{O}(n^{-2})\bigg). \tag{4.18}$$

11. *As $n \to +\infty$ with $j_{2,+} \leq j \leq n$, we have*

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\bigg(1+\mathcal{O}(n^{-\frac{1}{2}}e^{-nr_2^{2b}\frac{-\epsilon-\log(1-\epsilon)}{1-\epsilon}})\bigg)$$

$$= b\frac{\sqrt{n}\,e^{\frac{n}{b}(j/n - j/n\log(\frac{j/n}{b}))}}{\sqrt{2\pi}}\left(\frac{b}{j/n}\right)^{\frac{\alpha}{b}}\sqrt{\frac{j/n}{b}}\bigg(1+\frac{-b^2+6b\alpha-6\alpha^2}{12bj/n}\frac{1}{n} + \mathcal{O}(n^{-2})\bigg). \tag{4.19}$$

Proof. Note from (2.6) and (4.1) that

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\bigg(\frac{\gamma(\frac{j+\alpha}{b},nr_1^{2b})}{\Gamma(\frac{j+\alpha}{b})} - \frac{\gamma(\frac{j+\alpha}{b},nr_2^{2b})}{\Gamma(\frac{j+\alpha}{b})}+1\bigg)^{-1}$$

$$= \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(a_j)}\bigg(\frac{\gamma(a_j,\lambda_{j,1}a_j)}{\Gamma(a_j)} - \frac{\gamma(a_j,\lambda_{j,2}a_j)}{\Gamma(a_j)}+1\bigg)^{-1}, \qquad j=1,2,\ldots,n. \tag{4.20}$$

For fixed j, Lemma B.1 implies that $\gamma(\frac{j+\alpha}{b},nr_k^{2b}) = \Gamma(\frac{j+\alpha}{b}) + \mathcal{O}(n^{\frac{j+\alpha}{b}-1}e^{-nr_k^{2b}})$ as $n \to \infty$, and (4.9) follows. Now we turn to (4.10). By Lemma B.3 (i),

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1+\mathcal{O}(n^{-\frac{1}{2}}e^{-\frac{a_j\eta_{j,1}^2}{2}}) + \mathcal{O}(n^{-\frac{1}{2}}e^{-\frac{a_j\eta_{j,2}^2}{2}})\Big) = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1+\mathcal{O}(n^{-\frac{1}{2}}e^{-\frac{a_j\eta_{j,1}^2}{2}})\Big)$$

as $n \to \infty$, $M' \leq j \leq j_{1,-}$. Moreover,

$$\frac{a_j\eta_{j,1}^2}{2} = n\frac{j/n}{b}\bigg(\frac{br_1^{2b}}{j/n} - 1 - \log\Big(\frac{br_1^{2b}}{j/n}\Big)\bigg) + \mathcal{O}(1) \geq nr_1^{2b}\frac{\epsilon - \log(1+\epsilon)}{1+\epsilon} + \mathcal{O}(1)$$

as $n \to \infty$, $M' \leq j \leq j_{1,-}$, which implies the first line in (4.10). The second line in (4.10) then directly follows from Lemma 4.1. Now we prove (4.11). By (4.20) and Lemmas B.2 and B.3 (i),

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\bigg(\frac{1}{\frac{1}{2}\mathrm{erfc}(-\eta_{j,1}\sqrt{\frac{a_j}{2}})} + \mathcal{O}(n^{-\frac{1}{2}}e^{-\frac{1}{2}a_j\eta_{j,1}^2})\bigg)$$

as $n \to +\infty$ with $j_{1,-} \le j \le g_{1,-}$. On the other hand,

$$-\eta_{j,1}\sqrt{\frac{a_j}{2}} \le -\frac{Mr_1^2}{\sqrt{2}} + \mathcal{O}\Big(\frac{M^2}{\sqrt{n}}\Big), \qquad \text{as } n \to +\infty,\ j_{1,-} \le j \le g_{1,-},$$

which implies the first line in (4.11) provided M' is chosen large enough (recall that $M'\sqrt{\log n} \le M$). The second line in (4.11) then directly follows from Lemma 4.1. Now we prove (4.12). By Lemmas B.2 and B.3 (i), as $n \to +\infty$ with $g_{1,-} \le j \le g_{1,+}$, we have

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \frac{1 + \mathcal{O}(e^{-\frac{1}{2}\frac{a_j}{2}(\eta_{j,2}^2 - \eta_{j,1}^2)})}{\frac{1}{2}\mathrm{erfc}(-\eta_{j,1}\sqrt{\frac{a_j}{2}}) - \frac{e^{-\frac{1}{2}a_j\eta_{j,1}^2}}{\sqrt{2\pi a_j}}\big(c_0(\eta_{j,1}) + \frac{c_1(\eta_{j,1})}{a_j} + \mathcal{O}(\frac{1}{n^2})\big)}.$$

The above can be expanded in terms of $M_{j,1}$ using (4.1) and (4.6), namely using

$$\eta_{j,1} = (\lambda_{j,1} - 1)\sqrt{\frac{2(\lambda_{j,1} - 1 - \log \lambda_{j,1})}{(\lambda_{j,1} - 1)^2}}, \qquad \lambda_{j,1} = 1 + \frac{M_{j,1}}{\sqrt{n}}, \qquad j = \frac{br_1^{2b}n}{\lambda_{j,1}} - \alpha.$$

Using also

$$\frac{e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} = \begin{cases} -\frac{r_1^b M_{j,1}}{\sqrt{2}} - \frac{1}{\sqrt{2}r_1^b M_{j,1}} + \mathcal{O}(M_{j,1}^{-3}), & \text{as } M_{j,1} \to -\infty, \\ \mathcal{O}(e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}), & \text{as } M_{j,1} \to +\infty, \end{cases}$$

we obtain

$$\begin{aligned} &\frac{1}{\frac{1}{2}\mathrm{erfc}(-\eta_{j,1}\sqrt{\frac{a_j}{2}}) - \frac{e^{-\frac{1}{2}a_j\eta_{j,1}^2}}{\sqrt{2\pi a_j}}\big(c_0(\eta_{j,1}) + \frac{c_1(\eta_{j,1})}{a_j} + \mathcal{O}(\frac{1}{n^2})\big)} \\ &\quad = \frac{1}{\frac{1}{2}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})}\bigg\{1 + \frac{(5M_{j,1}^2 r_1^{2b} - 2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b \mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})\sqrt{n}} + \frac{\chi_{j,1}^-}{n}\bigg(\frac{25r_1^{4b}M_{j,1}^6}{72} + \frac{3r_1^{2b}M_{j,1}^4}{2}\bigg) \\ &\qquad - \chi_{j,1}^- \frac{125r_1^{6b}M_{j,1}^9}{1296n^{3/2}} + \mathcal{O}\bigg(\frac{1 + M_{j,1}^2}{n} + \frac{1 + |M_{j,1}^7|}{n^{3/2}} + \frac{1 + M_{j,1}^{12}}{n^2}\bigg)\bigg)\bigg\} \end{aligned}$$

as $n \to +\infty$ uniformly for $g_{1,-} \le j \le g_{1,+}$, and the first expansion in (4.12) follows. The second expansion in (4.12) then follows from Lemma 4.1. Now we prove (4.13). By (4.4), (4.20), Lemma B.2 and Lemma B.3 (i), we have

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})} \frac{1}{\frac{1}{2}\mathrm{erfc}(-\eta_{j,1}\sqrt{a_j/2}) - \frac{e^{-\frac{a_j\eta_{j,1}^2}{2}}}{\sqrt{2\pi a_j}}(c_0(\eta_{j,1}) + \frac{c_1(\eta_{j,1})}{a_j})}\Big(1 + \mathcal{O}\big(n^{-2} + e^{-\frac{a_j}{2}(\eta_{j,2}^2 - \eta_{j,1}^2)}\big)\Big),$$

and the first expansion in (4.13) follows from a long but direct computation using (4.1), (B.4) and (B.5). The second expansion in (4.13) then follows from Lemma 4.1. Now we prove (4.14) and (4.15). Using Lemma B.3 (i)–(ii), we find

$$h_j^{-1} = \frac{bn^{\frac{j+\alpha}{b}}}{\Gamma(\frac{j+\alpha}{b})}\Big(1 + \mathcal{O}(n^{-2})\Big)$$

$$\times \frac{1}{\frac{e^{-\frac{a_j \eta_{j,1}^2}{2}}}{\sqrt{2\pi}}\left(\frac{1}{1-\lambda_{j,1}}\frac{1}{\sqrt{a_j}} + \frac{1+10\lambda_{j,1}+\lambda_{j,1}^2}{12(\lambda_{j,1}-1)^3}\frac{1}{a_j^{3/2}}\right) - \frac{e^{-\frac{a_j \eta_{j,2}^2}{2}}}{\sqrt{2\pi}}\left(\frac{-1}{\lambda_{j,2}-1}\frac{1}{\sqrt{a_j}} + \frac{1+10\lambda_{j,2}+\lambda_{j,2}^2}{12(\lambda_{j,2}-1)^3}\frac{1}{a_j^{3/2}}\right)}$$

as $n \to +\infty$ with $j_{1,+} \le j \le j_{2,-}$. Then (4.14) and (4.15) follow from a long but direct computation using (4.1), (4.5) and Lemma 4.1.

The proofs of (4.16), (4.17), (4.18) and (4.19) are similar to that of (4.13), (4.12), (4.11) and (4.10), respectively, so we omit them. $\square$

5 Proof of Theorem 3.11

In this section z and w are given by

$$z = r_1\Big(1 - \frac{t_1}{\sigma_1 n}\Big)e^{i\theta_1}, \qquad w = r_2\Big(1 + \frac{t_2}{\sigma_2 n}\Big)e^{i\theta_2}, \qquad t_1, t_2 \ge 0, \theta_1, \theta_2 \in \mathbb{R}. \tag{5.1}$$

Let us rewrite (2.5) as $K_n(z,w) = \sum_{j=1}^n F_j$, where

$$F_j := e^{-\frac{n}{2}Q(z)}e^{-\frac{n}{2}Q(w)}\frac{z^{j-1}\overline{w}^{j-1}}{h_j} = e^{-\frac{n}{2}|z|^{2b}}e^{-\frac{n}{2}|w|^{2b}}|z|^{\alpha}|w|^{\alpha}\frac{z^{j-1}\overline{w}^{j-1}}{h_j}. \tag{5.2}$$

We will obtain the asymptotics of the summand F_j as $n \to +\infty$ for several regimes of the indice j; such splitting can also be found in e.g. [40, 5, 25] in the context of large gap probabilities. In this section we take $M = M'\sqrt{\log n}$. The following lemma is proved using Lemma 4.2.

Lemma 5.1. *Let $\delta \in (0, \frac{1}{100})$ be fixed. M' can be chosen sufficiently large and independently of δ such that the following hold.*

1. *Let j be fixed. As $n \to +\infty$, we have*
$$F_j = \mathcal{O}(e^{-\frac{n}{2}(1-\delta)(r_1^{2b}+r_2^{2b})}). \tag{5.3}$$
2. *As $n \to +\infty$ with $M' \le j \le j_{1,-}$, we have*
$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_1}). \tag{5.4}$$
3. *As $n \to +\infty$ with $j_{1,-} \le j \le g_{1,-}$, we have*
$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_1}). \tag{5.5}$$
4. *As $n \to +\infty$ with $g_{1,-} \le j \le g_{1,+}$, we have*
$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_1}). \tag{5.6}$$
5. *As $n \to +\infty$ with $g_{1,+} \le j \le j_{1,+}$, we have*
$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})(\sigma_\star - \frac{br_1^{2b}}{1-\epsilon})}). \tag{5.7}$$
6. *As $n \to +\infty$ with $j_{1,+} \le j \le \lfloor j_\star \rfloor$, we have*
$$F_j = \frac{n}{r_1 r_2}\frac{(\frac{r_1}{r_2})^{j_\star - j}e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{j/n-br_1^{2b}} + \frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{br_2^{2b}-j/n}}e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}\Big(1+\mathcal{O}\big(n^{-1}\big)\Big). \tag{5.8}$$

7. *As $n \to +\infty$ with $\lfloor j_\star \rfloor + 1 \le j \le j_{2,-}$, we have*

$$F_j = \frac{n}{r_1 r_2} \frac{(\frac{r_1}{r_2})^{j-j_\star} e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{br_2^{2b}-j/n} + \frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}} e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}} \Big(1+\mathcal{O}(n^{-1})\Big). \tag{5.9}$$

8. *As $n \to +\infty$ with $j_{2,-} \le j \le g_{2,-}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})(\frac{br_2^{2b}}{1+\epsilon}-\sigma_\star)}). \tag{5.10}$$

9. *As $n \to +\infty$ with $g_{2,-} \le j \le g_{2,+}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{5.11}$$

10. *As $n \to +\infty$ with $g_{2,+} \le j \le j_{2,+}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{5.12}$$

11. *As $n \to +\infty$ with $j_{2,+} \le j \le n$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{5.13}$$

Proof. (5.3) directly follows from (4.9) and (5.2). Let us now prove (5.4). By (4.10) and (5.2), as $n \to +\infty$ with $M' \le j \le j_{1,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[\frac{1}{b}(j/n - j/n\log(\frac{j/n}{b})) - \frac{1}{2}r_1^{2b} - \frac{1}{2}r_2^{2b} + j/n\log(r_1r_2) \bigg]\bigg\}\bigg).$$

It is easy to check that the function $[0, br_1^b r_2^b] \in y \mapsto y - y\log(\frac{y}{b}) + y\log(r_1^b r_2^b)$ is increasing. Since $j/n \le br_1^{2b}$, we thus have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[r_1^{2b} - r_1^{2b}\log(r_1^{2b}) - \frac{1}{2}r_1^{2b} - \frac{1}{2}r_2^{2b} + br_1^{2b}\log(r_1r_2) \bigg]\bigg\}\bigg),$$

which is (5.4) (recall the definitions of σ_1 and $\sigma_\star$ in (2.4)). The proof of (5.5) is identical (but uses (4.11) instead of (4.10)). We now turn to the proof of (5.6). By (4.12) and (5.2), as $n \to +\infty$ with $g_{1,-} \le j \le g_{1,+}$ we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta/2)\bigg[r_1^{2b} - r_1^{2b}\log(r_1^{2b}) - \frac{1}{2}r_1^{2b} - \frac{1}{2}r_2^{2b} + j/n\log(r_1r_2) \bigg]\bigg\}\bigg).$$

Since $j/n = br_1^{2b} + \mathcal{O}(n^{-\frac{1}{2}})$,

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[r_1^{2b} - r_1^{2b}\log(r_1^{2b}) - \frac{1}{2}r_1^{2b} - \frac{1}{2}r_2^{2b} + br_1^{2b}\log(r_1r_2) \bigg]\bigg\}\bigg),$$

and (5.6) follows. Now we prove (5.7). By (4.13) and (5.2), as $n \to +\infty$ with $g_{1,+} \le j \le j_{1,+}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[r_1^{2b} - 2j/n\log(r_1) - \frac{1}{2}r_1^{2b} - \frac{1}{2}r_2^{2b} + j/n\log(r_1r_2) \bigg]\bigg\}\bigg).$$

Recall from (4.4) that $\sigma_\star > \frac{br_1^{2b}}{1-\epsilon}$. Since $j/n \leq \frac{br_1^{2b}}{1-\epsilon} + \mathcal{O}(n^{-1})$ by (4.2), we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[-\frac{r_2^{2b}-r_1^{2b}}{2} + \frac{br_1^{2b}}{1-\epsilon}\log(\frac{r_2}{r_1})\bigg]\bigg\}\bigg),$$

which is (5.7). Now we prove (5.8). As $n \to +\infty$ with $j_{1,+} \leq j \leq \lfloor j_\star \rfloor$, we have

$$\begin{aligned}
&e^{n(r_1^{2b}-2j/n\log(r_1))}e^{-\frac{n}{2}|z|^{2b}}e^{-\frac{n}{2}|w|^{2b}}|z|^\alpha|w|^\alpha z^{j-1}\overline{w}^{j-1} \\
&= e^{-\frac{n}{2}(r_2^{2b}-r_1^{2b}-2j/n\log(\frac{r_2}{r_1}))}e^{(j-1)i(\theta_1-\theta_2)}e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}}e^{-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}r_1^{\alpha-1}r_2^{\alpha-1}\big(1+\mathcal{O}(n^{-1})\big)
\end{aligned}$$

Now (5.8) follows directly from (4.14), (5.2) and the identity (recall $j_\star = n\sigma_\star - \alpha$)

$$\frac{r_2^\alpha}{r_1^\alpha}e^{-\frac{n}{2}(r_2^{2b}-r_1^{2b}-2j/n\log(\frac{r_2}{r_1}))} = \Big(\frac{r_2}{r_1}\Big)^{-(j_\star-j)}.$$

The proofs of (5.9), (5.10), (5.11), (5.12) and (5.13) are similar to that of (5.8), (5.7), (5.6), (5.5) and (5.4), respectively, so we omit them. □

Theorem 5.2. *Let $z, w, t_1, t_2, \theta_1, \theta_2$ be as in (5.1). As $n \to +\infty$, we have*

$$\begin{aligned}
K_n(z,w) &= \frac{n}{r_1 r_2}e^{-t_1-t_2}e^{i(\theta_1-\theta_2)\lfloor j_\star \rfloor} \\
&\times \left\{ \sum_{j=0}^{+\infty} \frac{(\frac{r_1}{r_2})^{j+x}e^{(j+1)i(\theta_2-\theta_1)}}{\frac{1}{\sigma_1}+\frac{(\frac{r_1}{r_2})^{2(j+x)}}{\sigma_2}} + \sum_{j=0}^{+\infty} \frac{(\frac{r_1}{r_2})^{j+1-x}e^{ji(\theta_1-\theta_2)}}{\frac{1}{\sigma_2}+\frac{(\frac{r_1}{r_2})^{2(j+1-x)}}{\sigma_1}} \right\} + \mathcal{O}((\log n)^2),
\end{aligned}$$

where $\sigma_\star, \sigma_1, \sigma_2$ are given by (2.4), $j_\star = n\sigma_\star - \alpha$, and $x = x(n)$ is given by (3.20).

Proof. By Lemma 5.1,

$$K_n = \sum_{j=j_{1,+}}^{j_{2,-}} F_j + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty$$

for some $c > 0$. Furthermore, by (5.8) and (5.9),

$$\begin{aligned}
\sum_{j=j_{1,+}}^{\lfloor j_\star \rfloor} F_j &= \sum_{j=j_{1,+}}^{\lfloor j_\star \rfloor} \frac{n}{r_1 r_2} \frac{(\frac{r_1}{r_2})^{j_\star-j}e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{j/n-br_1^{2b}}+\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{br_2^{2b}-j/n}} e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}\Big(1+\mathcal{O}\big(n^{-1}\big)\Big), \\
\sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} F_j &= \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} \frac{n}{r_1 r_2} \frac{(\frac{r_1}{r_2})^{j-j_\star}e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{br_2^{2b}-j/n}+\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}} e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}\Big(1+\mathcal{O}\big(n^{-1}\big)\Big).
\end{aligned}$$

For $j - \lfloor j_\star \rfloor \geq M' \log n$, we have $(\frac{r_1}{r_2})^{j-\lfloor j_\star} = \mathcal{O}(n^{-200})$ provided that M' is chosen large enough. We infer that

$$\begin{aligned}
\sum_{j=j_{1,+}}^{\lfloor j_\star \rfloor} F_j &= \mathcal{O}(n^{-100}) + \sum_{j=\lfloor j_\star \rfloor - M'\log n}^{\lfloor j_\star \rfloor} \frac{n}{r_1 r_2} \frac{(\frac{r_1}{r_2})^{j_\star-j}e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{j/n-br_1^{2b}}+\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{br_2^{2b}-j/n}} e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}\Big(1+\mathcal{O}\big(n^{-1}\big)\Big), \\
\sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} F_j &= \mathcal{O}(n^{-100}) + \sum_{j=\lfloor j_\star \rfloor+1}^{\lfloor j_\star \rfloor + M'\log n} \frac{n}{r_1 r_2} \frac{(\frac{r_1}{r_2})^{j-j_\star}e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{br_2^{2b}-j/n}+\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}} e^{-(j/n-br_1^{2b})\frac{t_1}{\sigma_1}-(br_2^{2b}-j/n)\frac{t_2}{\sigma_2}}\Big(1+\mathcal{O}\big(n^{-1}\big)\Big).
\end{aligned}$$

provided that M' is chosen sufficiently large. For $|j-\lfloor j_\star\rfloor| \leq M'\log n$, we have $j/n = \sigma_\star + \mathcal{O}(\frac{\log n}{n})$, and thus

$$\sum_{j=j_{1,+}}^{\lfloor j_\star \rfloor} F_j = \mathcal{O}(n^{-100}) + \frac{n}{r_1 r_2} e^{-t_1-t_2} \sum_{j=\lfloor j_\star\rfloor - M'\log n}^{\lfloor j_\star \rfloor} \frac{(\frac{r_1}{r_2})^{j_\star - j} e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{\sigma_1} + \frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{\sigma_2}} \Big(1+\mathcal{O}\big(\frac{\log n}{n}\big)\Big),$$
$$\sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} F_j = \mathcal{O}(n^{-100}) + \frac{n}{r_1 r_2} e^{-t_1-t_2} \sum_{j=\lfloor j_\star\rfloor +1}^{\lfloor j_\star \rfloor + M'\log n} \frac{(\frac{r_1}{r_2})^{j - j_\star} e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{\sigma_2} + \frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{\sigma_1}} \Big(1+\mathcal{O}\big(\frac{\log n}{n}\big)\Big).$$

We can rewrite this as

$$\sum_{j=j_{1,+}}^{\lfloor j_\star \rfloor} F_j = \mathcal{O}\big((\log n)^2\big) + \frac{n}{r_1 r_2} e^{-t_1-t_2} \sum_{j=\lfloor j_\star\rfloor - M'\log n}^{\lfloor j_\star \rfloor} \frac{(\frac{r_1}{r_2})^{j_\star - j} e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{\sigma_1} + \frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{\sigma_2}},$$
$$\sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} F_j = \mathcal{O}\big((\log n)^2\big) + \frac{n}{r_1 r_2} e^{-t_1-t_2} \sum_{j=\lfloor j_\star\rfloor +1}^{\lfloor j_\star \rfloor + M'\log n} \frac{(\frac{r_1}{r_2})^{j - j_\star} e^{(j-1)i(\theta_1-\theta_2)}}{\frac{1}{\sigma_2} + \frac{(\frac{r_1}{r_2})^{2(j - j_\star)}}{\sigma_1}}.$$

In the above sums appearing on the right-hand sides, we can replace $M'\log n$ by $+\infty$ at the cost of an error $\mathcal{O}(n^{-100})$; this error can in turn be absorbed in $\mathcal{O}\big((\log n)^2\big)$. We then find the claim after changing indices. $\square$

A computation using (3.16) gives

$$\frac{1}{2\pi r_1 r_2}\left\{ \sum_{j=0}^{+\infty} \frac{(\frac{r_1}{r_2})^{j+x} e^{(j+1)i(\theta_2-\theta_1)}}{\frac{1}{\sigma_1} + \frac{(\frac{r_1}{r_2})^{2(j+x)}}{\sigma_2}} + \sum_{j=0}^{+\infty} \frac{(\frac{r_1}{r_2})^{j+1-x} e^{ji(\theta_1-\theta_2)}}{\frac{1}{\sigma_2} + \frac{(\frac{r_1}{r_2})^{2(j+1-x)}}{\sigma_1}} \right\}$$
$$= (r_1 r_2)^{-x} \mathcal{S}^G_{\mathrm{hard}}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}; n) = (r_1 r_2)^{-x} \mathcal{S}^G_{\mathrm{hard}}(z,w;n) + \mathcal{O}(n^{-1}),$$

and Theorem 3.11 follows.

6 Proofs of Theorems 3.1 and 3.13

In this section z and w are given by

$$z = r_1\Big(1-\frac{t_1}{\sigma_1 n}\Big)e^{i\theta_1}, \qquad w = r_1\Big(1-\frac{t_2}{\sigma_1 n}\Big)e^{i\theta_2}, \qquad t_1,t_2 \geq 0, \theta_1,\theta_2 \in \mathbb{R}, \tag{6.1}$$

and the parameters $t_1, t_2, \theta_1, \theta_2$ are independent of n. At this point, $\theta_1,\theta_2 \in \mathbb{R}$ can be arbitrary, but we already mention that in Section 6.1 below, we will focus on the case $\theta_1 = \theta_2$, and in Section 6.2 we will consider the case $\theta_1 \neq \theta_2 \mod 2\pi$. As in (5.2) we write $K_n(z,w) = \sum_{j=1}^n F_j$, where

$$F_j := e^{-\frac{n}{2}Q(z)} e^{-\frac{n}{2}Q(w)} \frac{z^{j-1}\overline{w}^{j-1}}{h_j} = e^{-\frac{n}{2}|z|^{2b}} e^{-\frac{n}{2}|w|^{2b}} |z|^\alpha |w|^\alpha \frac{z^{j-1}\overline{w}^{j-1}}{h_j}. \tag{6.2}$$

Here M can be arbitrary within the range $M \in [M'\sqrt{\log n}, n^{\frac{1}{10}}]$ (the values of M will be more restricted in Subsections 6.1 and 6.2 below). The following lemma is proved using Lemma 4.2.

Lemma 6.1. *Let $\delta \in (0, \frac{1}{100})$ be fixed. M' can be chosen sufficiently large and independently of δ such that the following hold.*

1. *Let j be fixed. As $n \to +\infty$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)r_1^{2b}}). \tag{6.3}$$

2. *As $n \to +\infty$ with $M' \leq j \leq j_{1,-}$, we have*

$$F_j = \mathcal{O}(e^{-nr_1^{2b}(1-\delta)\frac{\epsilon-\log(1+\epsilon)}{1+\epsilon}}). \tag{6.4}$$

3. *As $n \to +\infty$ with $j_{1,-} \leq j \leq g_{1,-}$, we have*

$$F_j = \mathcal{O}(n^{-100}). \tag{6.5}$$

4. *As $n \to +\infty$ with $g_{1,-} \leq j \leq g_{1,+}$, we have*

$$\begin{aligned} F_j = e^{i(j-1)(\theta_1-\theta_2)}\sqrt{n}\frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{M_{j,1}^2 r_1^{2b}}{2}}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})}\bigg(1 + \frac{1}{\sqrt{n}}\bigg\{\frac{(5M_{j,1}^2 r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \\ + \frac{M_{j,1}}{6}\Big(5M_{j,1}^2 r_1^{2b} - 3 + 6br_1^{2b}\frac{t_1+t_2}{\sigma_1}\Big)\bigg\} \\ + \mathcal{O}\bigg(\frac{1+M_{j,1}^2+M_{j,1}^6\chi_{j,1}^+}{n} + \frac{1+M_{j,1}^7+M_{j,1}^9\chi_{j,1}^+}{n^{3/2}} + \frac{1+M_{j,1}^{12}}{n^2}\bigg)\bigg). \end{aligned} \tag{6.6}$$

5. *As $n \to +\infty$ with $g_{1,+} \leq j \leq j_{1,+}$, we have*

$$\begin{aligned} F_j = e^{i(j-1)(\theta_1-\theta_2)}n\frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}(j/n-br_1^{2b})}{r_1^2}\bigg(1 + \frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\ + \frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2) + \frac{b^2r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\} - \frac{2b^4r_1^{4b}}{n^2(j/n-br_1^{2b})^4} \\ + \frac{10b^6r_1^{6b}}{n^3(j/n-br_1^{2b})^6} + \mathcal{O}\Big(\frac{1}{n^2(j/n-br_1^{2b})^3} + \frac{1}{n^4(j/n-br_1^{2b})^8}\Big)\bigg). \end{aligned} \tag{6.7}$$

6. *As $n \to +\infty$ with $j_{1,+} \leq j \leq \lfloor j_\star \rfloor$, we have*

$$\begin{aligned} F_j = e^{i(j-1)(\theta_1-\theta_2)}\frac{n}{r_1^2}\frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{\frac{1}{j/n-br_1^{2b}} + \frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{br_2^{2b}-j/n}}\bigg(1 + \frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\ + \frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2) + \frac{b^2r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{n} + \frac{1}{n^2}\Big)\bigg). \end{aligned} \tag{6.8}$$

7. *As $n \to +\infty$ with $\lfloor j_\star \rfloor + 1 \leq j \leq j_{2,-}$, we have*

$$\begin{aligned} F_j = e^{i(j-1)(\theta_1-\theta_2)}\frac{n}{r_1^2}\frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}(\frac{r_1}{r_2})^{2(j-j_\star)}}{\frac{1}{br_2^{2b}-j/n} + \frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}}\bigg(1 + \frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\ + \frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2) + \frac{b^2r_2^{2b}+(j/n-br_2^{2b})\alpha}{(j/n-br_2^{2b})^2}\bigg\} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{n} + \frac{1}{n^2}\Big)\bigg). \end{aligned} \tag{6.9}$$

8. *As $n \to +\infty$ with $j_{2,-} \leq j \leq g_{2,-}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})(\frac{br_2^{2b}}{1+\epsilon}-\sigma_\star)}). \tag{6.10}$$

9. *As $n \to +\infty$ with $g_{2,-} \leq j \leq g_{2,+}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{6.11}$$

10. *As $n \to +\infty$ with $g_{2,+} \leq j \leq j_{2,+}$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{6.12}$$

11. *As $n \to +\infty$ with $j_{2,+} \leq j \leq n$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{6.13}$$

Proof. (6.3) follows from (4.9) and (6.2). Let us now prove (6.4). By (4.10) and (6.2), as $n \to +\infty$ with $M' \leq j \leq j_{1,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[\frac{1}{b}(j/n - j/n\log(\frac{j/n}{b})) - r_1^{2b} + 2j/n\log(r_1)\bigg]\bigg\}\bigg).$$

It is easy to check that the function $y \mapsto y - y\log(\frac{y}{b}) + y\log(r_1^{2b})$ is increasing for $y \in [0, br_1^{2b}]$. Since $j/n \leq \frac{br_1^{2b}}{1+\epsilon} + \mathcal{O}(n^{-1})$, we thus have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ -nr_1^{2b}(1-\delta)\frac{\epsilon - \log(1+\epsilon)}{1+\epsilon}\bigg\}\bigg),$$

which is (6.4). Now we prove (6.5). By (4.11) and (6.2), as $n \to +\infty$ with $j_{1,-} \leq j \leq g_{1,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[\frac{j/n - j/n\log(\frac{j/n}{b})}{b} - r_1^{2b} + 2j/n\log(r_1)\bigg]\bigg\}\bigg).$$

Since $[0, br_1^{2b}] \in y \mapsto y - y\log(\frac{y}{b}) + y\log(r_1^{2b})$ is increasing and $j/n \leq \frac{br_1^{2b}}{1+\frac{M}{\sqrt{n}}} + \mathcal{O}(n^{-1})$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ -(1-\delta)\frac{M^2 r_1^{2b}}{2}\bigg\}\bigg).$$

Since $M \geq M'\sqrt{\log n}$, (6.5) holds provided that M' is chosen sufficiently large. Now we prove (6.6). As $n \to +\infty$ with $g_{1,-} \leq j \leq g_{1,+}$, we have

$$\begin{aligned}
&e^{r_1^{2b}(1-\log(r_1^{2b}))n} e^{M_{j,1} r_1^{2b}\log(r_1^{2b})\sqrt{n}} e^{-r_1^{2b}(\frac{1}{2}+\log(r_1^{2b}))M_{j,1}^2} e^{-\frac{n}{2}|z|^{2b}} e^{-\frac{n}{2}|w|^{2b}} |z|^\alpha |w|^\alpha z^{j-1}\overline{w}^{j-1} \\
&= e^{i(j-1)(\theta_1-\theta_2)} e^{-\frac{1}{2}M_{j,1}^2 r_1^{2b}} r_1^{-2}\bigg\{ 1 - \frac{bM_{j,1}r_1^{2b}}{\sqrt{n}}\Big(2M_{j,1}^2\log r_1 - \frac{t_1+t_2}{\sigma_1}\Big) \\
&+ \frac{1}{n}\bigg(2b^2 r_1^{4b}(\log r_1)^2 M_{j,1}^6 + 2br_1^{2b}(1 - br_1^{2b}\frac{t_1+t_2}{\sigma_1})\log(r_1)M_{j,1}^4\bigg) - \frac{r_1^{6b}(\log(r_1^{2b}))^3 M_{j,1}^9}{6n^{3/2}} \\
&+ \mathcal{O}\bigg(\frac{1+M_{j,1}^2}{n} + \frac{1+M_{j,1}^7}{n^{3/2}} + \frac{1+M_{j,1}^{12}}{n^2}\bigg)\bigg\}.
\end{aligned}$$

Now (6.6) follows directly from (4.12) and (6.2). The proofs of (6.7), (6.8) and (6.9) follow in a similar way, using (4.13), (4.14) and (4.15), respectively.

Let us now prove (6.10). By (4.16) and (6.2), as $n \to +\infty$ with $j_{2,-} \leq j \leq g_{2,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\Big[r_2^{2b} - 2j/n \log r_2 - r_1^{2b} + 2j/n \log(r_1) \Big] \bigg\} \bigg).$$

Since $j/n = \frac{br_2^{2b}}{1+\epsilon}$, (6.10) follows. The proof of (6.11) is similar, and uses the fact that for $g_{2,-} \leq j \leq g_{2,+}$, we have $j/n = br_2^{2b} + \mathcal{O}(n^{-1/2})$. The proofs of (6.12) and (6.13) are also similar, and uses the fact that for $g_{2,+} \leq j \leq n$, we have $\frac{1}{b}(j/n - j/n \log(\frac{j/n}{b})) - r_1^{2b} + 2j/n \log(r_1) \leq -2\log(\frac{r_2}{r_1})\sigma_2$. $\square$

Lemma 6.2. *As $n \to +\infty$, we have*

$$K_n(z,w) = S_4 + S_5 + S_6 + S_7 + \mathcal{O}(n^{-100}),$$

where

$$S_4 := \sum_{j=g_{1,-}}^{g_{1,+}} F_j, \qquad S_5 := \sum_{j=g_{1,+}+1}^{j_{1,+}} F_j, \qquad S_6 := \sum_{j=j_{1,+}+1}^{\lfloor j_\star \rfloor} F_j, \qquad S_7 := \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} F_j.$$

Furthermore, as $n \to +\infty$ we have

$$\begin{aligned}
S_4 = {} & \sum_{j=g_{1,-}}^{g_{1,+}} e^{i(j-1)(\theta_1-\theta_2)} \sqrt{n} \frac{\sqrt{2}\, br_1^{b-2} e^{-\frac{M_{j,1}^2 r_1^{2b}}{2}}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \Bigg(1 + \frac{1}{\sqrt{n}} \bigg\{ \frac{(5M_{j,1}^2 r_1^{2b} - 2) e^{-\frac{r_1^{2b} M_{j,1}^2}{2}}}{3\sqrt{2\pi} r_1^b \mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \\
& + \frac{M_{j,1}}{6} \Big(5M_{j,1}^2 r_1^{2b} - 3 + 6br_1^{2b} \frac{t_1+t_2}{\sigma_1} \Big) \bigg\} \\
& + \mathcal{O}\bigg(\frac{1 + M_{j,1}^2 + M_{j,1}^6 \chi_{j,1}^+}{n} + \frac{1 + M_{j,1}^7 + M_{j,1}^9 \chi_{j,1}^+}{n^{3/2}} + \frac{1 + M_{j,1}^{12}}{n^2} \bigg) \Bigg), \qquad (6.14) \\
S_5 = {} & \sum_{j=g_{1,+}+1}^{j_{1,+}} e^{i(j-1)(\theta_1-\theta_2)} n \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}} (j/n - br_1^{2b})}{r_1^2} \Bigg(1 + \frac{1}{n} \bigg\{ \frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\
& + \frac{b}{2\sigma_1^2}(1-2b) r_1^{2b}(t_1^2+t_2^2) + \frac{b^2 r_1^{2b} + (j/n - br_1^{2b})\alpha}{(j/n - br_1^{2b})^2} \bigg\} - \frac{2b^4 r_1^{4b}}{n^2 (j/n - br_1^{2b})^4} \\
& + \frac{10 b^6 r_1^{6b}}{n^3 (j/n - br_1^{2b})^6} + \mathcal{O}\Big(\frac{1}{n^2 (j/n - br_1^{2b})^3} + \frac{1}{n^4 (j/n - br_1^{2b})^8} \Big) \Bigg), \qquad (6.15) \\
S_6 = {} & \sum_{j=j_{1,+}+1}^{\lfloor j_\star \rfloor} e^{i(j-1)(\theta_1-\theta_2)} \frac{n}{r_1^2} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{\frac{1}{j/n - br_1^{2b}} + \frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{br_2^{2b} - j/n}} \bigg(1 + \frac{1}{n} \bigg\{ \frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\
& + \frac{b}{2\sigma_1^2}(1-2b) r_1^{2b}(t_1^2+t_2^2) + \frac{b^2 r_1^{2b} + (j/n - br_1^{2b})\alpha}{(j/n - br_1^{2b})^2} \bigg\} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star - j)}}{n} + \frac{1}{n^2} \Big) \bigg), \qquad (6.16) \\
S_7 = {} & \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} e^{i(j-1)(\theta_1-\theta_2)} \frac{n}{r_1^2} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}} (\frac{r_1}{r_2})^{2(j-j_\star)}}{\frac{1}{br_2^{2b} - j/n} + \frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n - br_1^{2b}}} \bigg(1 + \frac{1}{n} \bigg\{ \frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\
& + \frac{b}{2\sigma_1^2}(1-2b) r_1^{2b}(t_1^2+t_2^2) + \frac{b^2 r_2^{2b} + (j/n - br_2^{2b})\alpha}{(j/n - br_2^{2b})^2} \bigg\} + \mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{n} + \frac{1}{n^2} \Big) \bigg). \qquad (6.17)
\end{aligned}$$

Proof. This is a straightforward consequence of Lemma 6.1. $\square$

Lemma 6.3. *As $n \to +\infty$,*

$$S_7 = \frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2 e^{t_1+t_2}} \sum_{j=0}^{+\infty} \frac{e^{ij(\theta_1-\theta_2)}}{1+\frac{\sigma_1}{\sigma_2}(\frac{r_2}{r_1})^{2(j+1-x)}} + \mathcal{O}\big((\log n)^2\big),$$

where $x = x(n)$ is given by (3.20).

Proof. By (6.17), as $n \to +\infty$,

$$S_7 = \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} e^{i(j-1)(\theta_1-\theta_2)} \frac{n}{r_1^2} \frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}(\frac{r_1}{r_2})^{2(j-j_\star)}}{\frac{1}{br_2^{2b}-j/n}+\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}} + \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}} \mathcal{O}\bigg(\Big(\frac{r_1}{r_2}\Big)^{2(j-j_\star)}\bigg).$$

The last sum is $\mathcal{O}(\log n)$. For the first sum, thanks to $(\frac{r_1}{r_2})^{2(j-j_\star)}$ appearing in the numerator, the upper bound of summation $j_{2,-}$ can be replaced by $\lfloor j_\star \rfloor + M' \log n$ at the cost of an error $\mathcal{O}(\log n)$. We thus have

$$S_7 = \sum_{\lfloor j_\star \rfloor+1}^{\lfloor j_\star \rfloor+M'\log n} e^{i(j-1)(\theta_1-\theta_2)} \frac{n}{r_1^2} \frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}(\frac{r_1}{r_2})^{2(j-j_\star)}}{\frac{1}{br_2^{2b}-j/n}+\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{j/n-br_1^{2b}}} + \mathcal{O}\big(\log n\big).$$

For $|j - \lfloor j_\star \rfloor| \leq M' \log n$, we have $j/n = \sigma_\star + \mathcal{O}(\frac{\log n}{n})$, and thus

$$S_7 = \frac{ne^{-t_1-t_2}}{r_1^2} \sum_{\lfloor j_\star \rfloor+1}^{\lfloor j_\star \rfloor+M'\log n} e^{i(j-1)(\theta_1-\theta_2)} \frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{\frac{1}{\sigma_2}+\frac{(\frac{r_1}{r_2})^{2(j-j_\star)}}{\sigma_1}} + \mathcal{O}\big((\log n)^2\big).$$

In the above sum, we can replace $\lfloor j_\star \rfloor + M' \log n$ by $+\infty$ at the cost of an error $\mathcal{O}(\log n)$ (again thanks to $(\frac{r_1}{r_2})^{2(j-j_\star)}$ appearing in the numerator). Then the claim follows from a simple shift of the indice of summation. $\square$

Lemma 6.4. *As $n \to +\infty$,*

$$\begin{aligned} S_6 = & -\frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2 e^{t_1+t_2}} \sum_{j=0}^{+\infty} \frac{e^{-i(j+1)(\theta_1-\theta_2)}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_2}{r_1})^{2(j+x)}} \\ & + \sum_{j=j_{1,+}+1}^{\lfloor j_\star \rfloor} e^{i(j-1)(\theta_1-\theta_2)} \frac{n(j/n-br_1^{2b})}{r_1^2 e^{(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}} \bigg(1+\frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2) \\ & + \frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2) + \frac{b^2 r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\} + \mathcal{O}\Big(\frac{1}{n^2}\Big)\bigg) + \mathcal{O}\big((\log n)^2\big). \end{aligned}$$

Proof. Let us split $S_6 = S_6^{(1)} + S_6^{(2)}$, where

$$S_6^{(1)} = \sum_{j=j_{1,+}+1}^{\lfloor j_\star \rfloor-(M'+1)\log n} F_j, \qquad S_6^{(2)} = \sum_{j=\lfloor j_\star \rfloor-M'\log n}^{\lfloor j_\star \rfloor} F_j.$$

Using (6.16) and a similar analysis as in the proof of Lemma 6.3, we obtain

$$S_6^{(2)} = \mathcal{O}\big((\log n)^2\big) + \frac{n}{r_1^2}e^{-t_1-t_2} \sum_{j=\lfloor j_\star \rfloor-M'\log n}^{\lfloor j_\star \rfloor} \frac{e^{i(j-1)(\theta_1-\theta_2)}}{\frac{1}{\sigma_1}}$$

$$+\frac{n}{r_1^2}e^{-t_1-t_2}\sum_{j=\lfloor j_\star\rfloor-M'\log n}^{\lfloor j_\star\rfloor}\left\{\frac{e^{i(j-1)(\theta_1-\theta_2)}}{\frac{1}{\sigma_1}+\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{\sigma_2}}-\frac{e^{i(j-1)(\theta_1-\theta_2)}}{\frac{1}{\sigma_1}}\right\} \tag{6.18}$$

as $n\to+\infty$. The last term in (6.18) is easily shown to be

$$-\frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2e^{t_1+t_2}}\sum_{j=0}^{+\infty}\frac{e^{-i(j+1)(\theta_1-\theta_2)}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_2}{r_1})^{2(j+x)}}+\mathcal{O}(\log n)$$

as $n\to+\infty$. For the first sum in (6.18), since $|j-\lfloor j_\star\rfloor|\leq M'\log n$, we can replace σ_1 by $j/n-br_1^{2b}$ at the cost of an error $\mathcal{O}((\log n)^2)$ in the asymptotics of $S_6^{(2)}$; we thus find

$$\begin{aligned}
S_6^{(2)}&=\mathcal{O}\big((\log n)^2\big)+\frac{n}{r_1^2}e^{-t_1-t_2}\sum_{j=\lfloor j_\star\rfloor-M'\log n}^{\lfloor j_\star\rfloor}\frac{e^{i(j-1)(\theta_1-\theta_2)}}{\frac{1}{j/n-br_1^{2b}}}-\frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2e^{t_1+t_2}}\sum_{j=0}^{+\infty}\frac{e^{-i(j+1)(\theta_1-\theta_2)}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_2}{r_1})^{2(j+x)}}\\
&=\mathcal{O}\big((\log n)^2\big)-\frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2e^{t_1+t_2}}\sum_{j=0}^{+\infty}\frac{e^{-i(j+1)(\theta_1-\theta_2)}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_2}{r_1})^{2(j+x)}}\\
&+\sum_{j=\lfloor j_\star\rfloor-M'\log n}^{\lfloor j_\star\rfloor}e^{i(j-1)(\theta_1-\theta_2)}\frac{n}{r_1^2}\frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{\frac{1}{j/n-br_1^{2b}}}\bigg(1+\frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha)-\frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2)\\
&+\frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2)+\frac{b^2r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\}+\mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{n}+\frac{1}{n^2}\Big)\bigg)
\end{aligned}$$

For $j_{1,+}+1\leq j\leq\lfloor j_\star\rfloor-(M'+1)\log n$, we can replace $(\frac{r_1}{r_2})^{2(j_\star-j)}$ by 0 in the asymptotics (6.8) of F_j at the cost of an error $\mathcal{O}(n^{-100})$ (provided M' is chosen large enough). We thus have

$$\begin{aligned}
S_6^{(1)}&=\sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor-(M'+1)\log n}e^{i(j-1)(\theta_1-\theta_2)}\frac{n}{r_1^2}\frac{e^{-(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{\frac{1}{j/n-br_1^{2b}}}\bigg(1+\frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha)-\frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2)\\
&+\frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2)+\frac{b^2r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\}+\mathcal{O}\Big(\frac{(\frac{r_1}{r_2})^{2(j_\star-j)}}{n}+\frac{1}{n^2}\Big)\bigg),
\end{aligned}$$

The claim follows after combining the above two expansions. □

As a direct consequence of Lemmas 6.2, 6.3 and 6.4, we obtain the following.

Lemma 6.5. *As $n\to+\infty$,*

$$S_5+S_6+S_7=S_{567}+\frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2e^{t_1+t_2}}\mathcal{Q}_n+\mathcal{O}\big((\log n)^2\big),$$

where

$$\begin{aligned}
S_{567}&=\sum_{j=g_{1,+}+1}^{\lfloor j_\star\rfloor}e^{i(j-1)(\theta_1-\theta_2)}\frac{n(j/n-br_1^{2b})}{r_1^2e^{(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}\bigg(1+\frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha)-\frac{j/n}{2\sigma_1^2}(t_1^2+t_2^2)\\
&+\frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2)+\frac{b^2r_1^{2b}+(j/n-br_1^{2b})\alpha}{(j/n-br_1^{2b})^2}\bigg\}-\frac{2b^4r_1^{4b}}{n^2(j/n-br_1^{2b})^4}\\
&+\frac{10b^6r_1^{6b}}{n^3(j/n-br_1^{2b})^6}+\mathcal{O}\Big(\frac{1}{n^2(j/n-br_1^{2b})^3}+\frac{1}{n^4(j/n-br_1^{2b})^8}\Big)\bigg),
\end{aligned} \tag{6.19}$$

$$\mathcal{Q}_n := \sum_{j=0}^{+\infty} \frac{e^{ij(\theta_1-\theta_2)}}{1+\frac{\sigma_1}{\sigma_2}(\frac{r_2}{r_1})^{2(j+1-x)}} - \sum_{j=0}^{+\infty} \frac{e^{-i(j+1)(\theta_1-\theta_2)}}{1+\frac{\sigma_2}{\sigma_1}(\frac{r_2}{r_1})^{2(j+x)}}, \tag{6.20}$$

and $x = x(n)$ is given by (3.20).

6.1 Proof of Theorem 3.1

In this subsection, we take $\theta_1 = \theta_2$ and $M = n^{\frac{1}{10}}$.

Lemma 6.6. *For $\theta_1 = \theta_2$, we have*

$$\mathcal{Q}_n = \frac{(\log\theta)'(n\sigma_\star - \alpha + \frac{\log(\frac{\sigma_2}{\sigma_1}\frac{r_2}{r_1})}{2\log\frac{r_2}{r_1}}; \frac{\pi i}{\log\frac{r_2}{r_1}}) + (2x-1)\log\frac{r_2}{r_1} + \log\frac{\sigma_2}{\sigma_1}}{2\log\frac{r_2}{r_1}}, \tag{6.21}$$

where $x = x(n)$ is given by (3.20).

Proof. For $y \in \mathbb{R}$, $\rho \in (0,1)$ and $a > 0$, define

$$\Theta(y;\rho,a) = y(y-1)\log(\rho) + y\log(a) + \sum_{j=0}^{+\infty}\log\left(1 + a\,\rho^{2(j+y)}\right) + \sum_{j=0}^{+\infty}\log\left(1 + a^{-1}\rho^{2(j+1-y)}\right). \tag{6.22}$$

It is proved in [30, Lemma 3.25] that

$$\Theta(y;\rho,a) = \frac{1}{2}\log\left(\frac{\pi a\rho^{-\frac{1}{2}}}{\log(\rho^{-1})}\right) + \frac{(\log a)^2}{4\log(\rho^{-1})} - \sum_{j=1}^{+\infty}\log(1-\rho^{2j}) + \log\theta\left(y + \frac{\log(a\rho)}{2\log(\rho)}\bigg|\frac{\pi i}{\log(\rho^{-1})}\right),$$

where θ is the Jacobi theta function defined in (3.1). By taking the derivative with respect to y in the above identity, by replacing (y,ρ,a) by (x,ρ^{-1},a^{-1}), we get

$$\sum_{j=0}^{+\infty}\frac{1}{1+a^{-1}\rho^{2(j+1-x)}} - \sum_{j=0}^{+\infty}\frac{1}{1+a\rho^{2(j+x)}} = \frac{(\log\theta)'(x + \frac{\log(a\rho)}{2\log\rho}; \frac{\pi i}{\log\rho}) + (2x-1)\log\rho + \log a}{2\log\rho}. \tag{6.23}$$

The claim follows from (6.23) with $\rho = \frac{r_2}{r_1} > 1$ and $a = \frac{\sigma_2}{\sigma_1}$, and from the facts that $x = n\sigma_\star - \alpha - \lfloor n\sigma_\star - \alpha\rfloor$ and $\theta(z+1|\tau) = \theta(z|\tau)$. ☐

To expand S_{567}, we will need the following Riemann sum lemma.

Lemma 6.7. *(taken from [30, Lemma 3.4]) Let A, a_0, B, b_0 be bounded function of $n \in \{1,2,\ldots\}$, such that*

$$a_n := An + a_0 \qquad \textit{and} \qquad b_n := Bn + b_0$$

are integers. Assume also that $B - A$ is positive and remains bounded away from 0. Let f be a function independent of n, and which is $C^4([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}])$ for all $n \in \{1,2,\ldots\}$. Then as $n \to +\infty$, we have

$$\sum_{j=a_n}^{b_n} f(\tfrac{j}{n}) = n\int_A^B f(y)dy + \frac{(1-2a_0)f(A) + (1+2b_0)f(B)}{2}$$

$$
+\frac{(-1+6a_0-6a_0^2)f'(A)+(1+6b_0+6b_0^2)f'(B)}{12n}+\frac{(-a_0+3a_0^2-2a_0^3)f''(A)+(b_0+3b_0^2+2b_0^3)f''(B)}{12n^2}
$$

$$
+\mathcal{O}\Big(\frac{\mathfrak{m}_{A,n}(f''')+\mathfrak{m}_{B,n}(f''')}{n^3}+\sum_{j=a_n}^{b_n-1}\frac{\mathfrak{m}_{j,n}(f'''')}{n^4}\Big), \tag{6.24}
$$

where, for a given function g continuous on $[\min\{\frac{a_n}{n},A\},\max\{\frac{b_n}{n},B\}]$,

$$
\mathfrak{m}_{A,n}(g):=\max_{y\in[\min\{\frac{a_n}{n},A\},\max\{\frac{a_n}{n},A\}]}|g(y)|,\qquad \mathfrak{m}_{B,n}(g):=\max_{y\in[\min\{\frac{b_n}{n},B\},\max\{\frac{b_n}{n},B\}]}|g(y)|,
$$

and for $j\in\{a_n,\ldots,b_n-1\}$, $\mathfrak{m}_{j,n}(g):=\max_{y\in[\frac{j}{n},\frac{j+1}{n}]}|g(y)|$.

It turns out that S_{567} has oscillatory asymptotics as $n\to+\infty$. To handle these oscillations, we follow [29] and introduce the following quantities:

$$
\theta_-^{(n,M)}:=g_--\Big(\frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\Big)=\Big\lceil\frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\Big\rceil-\Big(\frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\Big),
$$

$$
\theta_+^{(n,M)}:=\Big(\frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\Big)-g_+=\Big(\frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\Big)-\Big\lfloor\frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\Big\rfloor.
$$

Note that $\theta_-^{(n,M)},\theta_+^{(n,M)}\in[0,1)$ are oscillatory but remain bounded as $n\to+\infty$. Recall that in this section, we take $\theta_2=\theta_1$.

Lemma 6.8. *If $(t_1,t_2)\neq(0,0)$, then as $n\to+\infty$,*

$$
S_{567}=D_1n^2+D_2\,n\log n+D_3^{(n,M)}n+D_4^{(n,M)}\sqrt{n}+\mathcal{O}\Big(M^4+\frac{n}{M^6}+\frac{\sqrt{n}}{M}\Big), \tag{6.25}
$$

$$
D_1:=\sigma_1^2\frac{1-e^{-t_1-t_2}\big(1+t_1+t_2\big)}{r_1^2(t_1+t_2)^2},\qquad D_2:=\frac{b^2r_1^{2b-2}}{2},
$$

$$
D_3^{(n,M)}:=-b^2r_1^{2b-2}\Big(E_1\big(t_1+t_2\big)+\gamma_{\mathrm{E}}+\log(br_1^{2b}\frac{t_1+t_2}{\sigma_1}M)\Big)-\frac{b^2M^2r_1^{4b-2}}{2}
$$

$$
+\frac{(1-2x)\sigma_1}{2r_1^2e^{t_1+t_2}}+\frac{\sigma_1}{r_1^2(t_1+t_2)^3}\bigg\{\frac{(t_1+t_2)^2(t_1^2+t_2^2)}{2e^{t_1+t_2}}
$$

$$
+\Big(2t_1t_2-b^2r_1^{2b}\frac{t_1+t_2}{\sigma_1}(t_1^2+t_2^2)\Big)\Big(1-\frac{1+t_1+t_2}{e^{t_1+t_2}}\Big)\bigg\}-\frac{b^2}{M^2r_1^2}+\frac{5b^2}{2M^4r_1^{2+2b}},
$$

$$
D_4^{(n,M)}:=\frac{b^2r_1^{4b}\big(br_1^{2b}\frac{t_1+t_2}{\sigma_1}-3\big)}{3r_1^2}M^3+\frac{br_1^{2b}\big(2\theta_+^{(n,M)}-1-2b+2b^2r_1^{2b}\frac{t_1+t_2}{\sigma_1}\big)}{2r_1^2}M+\frac{(2\alpha+2\theta_+^{(n,M)}-1)b}{2r_1^2M}.
$$

If $t_1=t_2=0$, then (6.25) holds with

$$
D_1=\frac{\sigma_1^2}{2r_1^2},\qquad D_2=\frac{b^2r_1^{2b-2}}{2},
$$

$$
D_3^{(n,M)}=-\frac{b^2M^2r_1^{4b-2}}{2}+\frac{(\frac{1}{2}-x)\sigma_1}{r_1^2}-b^2r_1^{2b-2}\log\frac{Mbr_1^{2b}}{\sigma_1}-\frac{b^2}{M^2r_1^2}+\frac{5b^2}{2M^4r_1^{2+2b}},
$$

$$
D_4^{(n,M)}=-b^2r_1^{4b-2}M^3+\frac{br_1^{2b}\big(2\theta_+^{(n,M)}-1-2b\big)}{2r_1^2}M+\frac{(2\alpha+2\theta_+^{(n,M)}-1)b}{2r_1^2M}.
$$

Proof. By (6.19) with $\theta_1 = \theta_2$, as $n \to +\infty$ we have

$$S_{567} = n \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} f_1(j/n) + \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} f_2(j/n) + \frac{1}{n} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{r_1^2} \frac{-2b^4 r_1^{4b}}{(j/n - br_1^{2b})^3}$$
$$+ \frac{1}{n^2} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}} 10b^6}{r_1^{2-6b}(j/n - br_1^{2b})^5} + \mathcal{O}\Big(\frac{1}{n} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{1}{(j/n - br_1^{2b})^2} + \frac{1}{n^3} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{1}{(j/n - br_1^{2b})^7} \Big), \tag{6.26}$$

where $f_1(x) := \frac{x - br_1^{2b}}{r_1^2 e^{(x - br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}$ and

$$f_2(x) := \frac{x - br_1^{2b}}{r_1^2 e^{(x - br_1^{2b})\frac{t_1+t_2}{\sigma_1}}} \Big(\frac{t_1 + t_2}{\sigma_1}(1-\alpha) + \frac{b(1-2b)r_1^{2b} - x}{2\sigma_1^2}(t_1^2 + t_2^2) + \frac{b^2 r_1^{2b} + (x - br_1^{2b})\alpha}{(x - br_1^{2b})^2} \Big).$$

Since $g_{1,+} = \lfloor \frac{bnr_1^{2b}}{1 - \frac{M}{\sqrt{n}}} - \alpha \rfloor$, for the error term in (6.26) we have the following estimate:

$$\frac{1}{n} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{1}{(j/n - br_1^{2b})^2} + \frac{1}{n^3} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{1}{(j/n - br_1^{2b})^7}$$
$$= \mathcal{O}\Big(\int_{M/\sqrt{n}}^1 \frac{dx}{x^2} + \frac{1}{n^2} \int_{M/\sqrt{n}}^1 \frac{dx}{x^7} \Big) = \mathcal{O}\Big(\frac{\sqrt{n}}{M} + \frac{n}{M^6} \Big).$$

Since f_1 has no pole at br_1^{2b} and f_2 has a pole of order 1 at br_1^{2b}, long but direct calculations using Lemma 6.7 with $A = \frac{br_1^{2b}}{1 - \frac{M}{\sqrt{n}}}$, $a_0 = 1 - \alpha - \theta_+^{(n,M)}$, $B = \sigma_\star$ and $b_0 = -\alpha - x$ show that

$$n \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} f_1(j/n) = d_1 n^2 + (d_{2,1}M^2 + d_{2,2})n + (d_{3,1}M^3 + d_{3,2}M)\sqrt{n} + \mathcal{O}(M^4),$$
$$\sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} f_2(j/n) = e_1 n \log n + (e_{2,1}\log M + e_{2,2})n + (e_{3,1}M + e_{3,2}M^{-1})\sqrt{n} + \mathcal{O}(M^2),$$
$$\frac{1}{n} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}}}{r_1^2} \frac{-2b^4 r_1^{4b}}{(j/n - br_1^{2b})^3} = -\frac{b^2 n}{r_1^2 M^2} + \mathcal{O}\Big(\frac{\sqrt{n}}{M} \Big),$$
$$\frac{1}{n^2} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} \frac{e^{-(j/n - br_1^{2b})\frac{t_1+t_2}{\sigma_1}} 10b^6}{r_1^{2-6b}(j/n - br_1^{2b})^5} = \frac{5b^2 n}{2r_1^{2+2b}M^4} + \mathcal{O}\Big(\frac{\sqrt{n}}{M^3} \Big),$$

as $n \to +\infty$, for some explicit coefficients $d_1, d_{2,1}, d_{2,2}, d_{3,1}, d_{3,2}, e_1, e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}$ independent of M that we do not write down. We then find the claim by combining the above two asymptotic formulas and simplifying (using primitives). □

To expand S_4, we will need the following lemma.

Lemma 6.9. *(Taken from [30, Lemma 3.10]) Let $h \in C^3(\mathbb{R})$ and $k = 1$. As $n \to +\infty$, we have*

$$\sum_{j=g_{k,-}}^{g_{k,+}} h(M_{j,k}) = br_k^{2b} \int_{-M}^{M} h(t)dt \, \sqrt{n} - 2br_k^{2b} \int_{-M}^{M} th(t)dt + \Big(\frac{1}{2} - \theta_{k,-}^{(n,M)} \Big) h(M) + \Big(\frac{1}{2} - \theta_{k,+}^{(n,M)} \Big) h(-M)$$

$$+\frac{1}{\sqrt{n}}\bigg[3br_k^{2b}\int_{-M}^{M}t^2h(t)dt+\bigg(\frac{1}{12}+\frac{\theta_{k,-}^{(n,M)}(\theta_{k,-}^{(n,M)}-1)}{2}\bigg)\frac{h'(M)}{br_k^{2b}}-\bigg(\frac{1}{12}+\frac{\theta_{k,+}^{(n,M)}(\theta_{k,+}^{(n,M)}-1)}{2}\bigg)\frac{h'(-M)}{br_k^{2b}}\bigg]$$
$$+\mathcal{O}\bigg(\frac{1}{n^{3/2}}\sum_{j=g_{k,-}+1}^{g_{k,+}}\Big((1+|M_{j,1}|^3)\tilde{\mathfrak{m}}_{j,n}(h)+(1+M_{j,1}^2)\tilde{\mathfrak{m}}_{j,n}(h')+(1+|M_{j,1}|)\tilde{\mathfrak{m}}_{j,n}(h'')+\tilde{\mathfrak{m}}_{j,n}(h''')\Big)\bigg), \tag{6.27}$$

where, for $\tilde{h}\in C(\mathbb{R})$ *and* $j\in\{g_{k,-}+1,\ldots,g_{k,+}\}$, *we define* $\tilde{\mathfrak{m}}_{j,n}(\tilde{h}):=\max_{x\in[M_{j,k},M_{j-1,k}]}|\tilde{h}(x)|$.

Lemma 6.10. *As* $n\to+\infty$,

$$S_4=E_3^{(M)}n+E_4^{(M)}\sqrt{n}+\mathcal{O}\Big(M^4+\frac{M^9}{\sqrt{n}}+\frac{M^{14}}{n^{3/2}}\Big),$$
$$E_3^{(M)}:=br_1^{2b}\int_{-M}^{M}h_1(t)dt,$$
$$E_4^{(M)}:=br_1^{2b}\int_{-M}^{M}(h_2(t)-2th_1(t))dt+\bigg(\frac{1}{2}-\theta_-^{(n,M)}\bigg)h_1(M)+\bigg(\frac{1}{2}-\theta_+^{(n,M)}\bigg)h_1(-M),$$

where

$$h_1(y):=\frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{r_1^{2b}}{2}y^2}}{\sqrt{\pi}\mathrm{erfc}(-\frac{r_1^by}{\sqrt{2}})},$$
$$h_2(y):=\frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{r_1^{2b}}{2}y^2}}{\sqrt{\pi}\mathrm{erfc}(-\frac{r_1^by}{\sqrt{2}})}\Bigg(\frac{y}{6}\Big(5y^2r_1^{2b}-3+6br_1^{2b}(t_1+t_2)\Big)+\frac{e^{-\frac{r_1^{2b}}{2}y^2}(5y^2r_1^{2b}-2)}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{r_1^by}{\sqrt{2}})}\Bigg).$$

Proof. By (6.14), as $n\to+\infty$ we have

$$S_4=\sqrt{n}\sum_{j=g_{1,-}}^{g_{1,+}}h_1(M_{j,1})+\sum_{j=g_{1,-}}^{g_{1,+}}h_2(M_{j,1})+\mathcal{O}\bigg(\sum_{j=g_{1,-}}^{g_{1,+}}\Big(\frac{1+M_{j,1}^3}{\sqrt{n}}+\frac{1+M_{j,1}^8}{n}+\frac{1+M_{j,1}^{13}}{n^{3/2}}\Big)\bigg).$$

Since $\sum_{j=g_{1,-}}^{g_{1,+}}1=\mathcal{O}(M\sqrt{n})$, the above error term can be rewritten as $\mathcal{O}(M^4+\frac{M^9}{\sqrt{n}}+\frac{M^{14}}{n^{3/2}})$. We then use Lemma 6.9 to expand the two sums involving h_1 and h_2, and the claim follows. □

Recall that $\mathcal{I}$ is define by (3.7). Define also

$$\mathcal{I}_1=\int_{-\infty}^{+\infty}\bigg\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\chi_{(0,+\infty)}(y)\bigg[y+\frac{y}{2(1+y^2)}\bigg]\bigg\}dy, \tag{6.28}$$
$$\mathcal{I}_2=\int_{-\infty}^{+\infty}\bigg\{\frac{y^3e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\chi_{(0,+\infty)}(y)\bigg[y^4+\frac{y^2}{2}-\frac{1}{2}\bigg]\bigg\}dy, \tag{6.29}$$
$$\mathcal{I}_3=\int_{-\infty}^{+\infty}\bigg\{\bigg(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\bigg)^2-\chi_{(0,+\infty)}(y)\bigg[y^2+1\bigg]\bigg\}dy, \tag{6.30}$$
$$\mathcal{I}_4=\int_{-\infty}^{+\infty}\bigg\{\bigg(\frac{y\,e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\bigg)^2-\chi_{(0,+\infty)}(y)\bigg[y^4+y^2-\frac{3}{4}\bigg]\bigg\}dy. \tag{6.31}$$

Lemma 6.11. *If* $(t_1,t_2)\neq(0,0)$, *then as* $n\to+\infty$,

$$S_4+S_{567}=C_1n^2+C_2\,n\log n+\bigg(\tilde{C}_3+\frac{(1-2x)\sigma_1}{2\pi r_1^2e^{t_1+t_2}}\bigg)n$$

$$+\tilde{C}_4\sqrt{n}+\mathcal{O}\Big(M^4+\frac{M^9}{\sqrt{n}}+\frac{M^{14}}{n^{3/2}}+\frac{n}{M^6}+\frac{\sqrt{n}}{M}\Big), \tag{6.32}$$

where C_1, C_2 are as in the statement of Theorem 3.1 and $\tilde{C}_3, \tilde{C}_4$ are given by

$$\begin{aligned}
\tilde{C}_3 &= 2b^2 r_1^{2b-2}\mathcal{I}_1 - \frac{b^2 r_1^{2b-2}}{2}\log\Big(\frac{2}{r_1^{2b}}\Big) - b^2 r_1^{2b-2}\Big(E_1(t_1+t_2)+\gamma_{\mathrm{E}}+\log(br_1^{2b}\frac{t_1+t_2}{\sigma_1})\Big)\\
&+\frac{\sigma_1}{r_1^2(t_1+t_2)^3}\bigg\{\frac{(t_1+t_2)^2(t_1^2+t_2^2)}{2e^{t_1+t_2}}\\
&\qquad +\Big(2t_1t_2-\frac{b^2r_1^{2b}}{\sigma_1}(t_1+t_2)(t_1^2+t_2^2)\Big)\Big(1-\frac{1+t_1+t_2}{e^{t_1+t_2}}\Big)\bigg\},\\
\tilde{C}_4 &= \frac{\sqrt{2}b^2 r_1^{b-2}}{3}\Big\{\big[15-6br_1^{2b}\frac{t_1+t_2}{\sigma_1}\big]\mathcal{I}-10\,\mathcal{I}_2-2\,\mathcal{I}_3+10\,\mathcal{I}_4\Big\}.
\end{aligned}$$

If $t_1=t_2=0$, then (6.32) holds with

$$\begin{aligned}
&C_1=\frac{\sigma_1^2}{2r_1^2}, \qquad C_2=\frac{b^2r_1^{2b-2}}{2}, \qquad \tilde{C}_3=2b^2r_1^{2b-2}\mathcal{I}_1+\frac{b^2r_1^{2b-2}}{2}\log\Big(\frac{\sigma_1^2}{2b^2r_1^{2b}}\Big),\\
&\tilde{C}_4=\frac{\sqrt{2}b^2r_1^{b-2}}{3}\Big\{15\,\mathcal{I}-10\,\mathcal{I}_2-2\,\mathcal{I}_3+10\,\mathcal{I}_4\Big\}.
\end{aligned}$$

Proof. Combining Lemmas 6.8 and 6.10, we infer that as $n\to+\infty$,

$$S_4+S_{567}=D_1n^2+D_2\,n\log n+(D_3^{(n,M)}+E_3^{(M)})n+(D_4^{(M)}+E_4^{(M)})\sqrt{n}+\mathcal{O}\Big(M^4\Big).$$

On the other hand, as $n\to+\infty$,

$$\begin{aligned}
&(D_3^{(n,M)}+E_3^{(M)})n=\bigg(\tilde{C}_3+\frac{(1-2x)\sigma_1}{2r_1^2e^{t_1+t_2}}\bigg)n+\mathcal{O}\Big(\frac{n}{M^2}\Big),\\
&(D_4^{(M)}+E_4^{(M)})\sqrt{n}=\tilde{C}_4\sqrt{n}+\mathcal{O}\Big(\frac{\sqrt{n}}{M}\Big),
\end{aligned}$$

and the claim follows. ☐

Since $M=n^{\frac{1}{10}}$, we have $M^4=\frac{M^9}{\sqrt{n}}=\frac{M^{14}}{n^{3/2}}=\frac{n}{M^6}=\frac{\sqrt{n}}{M}=n^{\frac{2}{5}}$, and thus the $\mathcal{O}$-term in (6.32) can be written as $\mathcal{O}(n^{\frac{2}{5}})$. Hence, combining Lemmas 6.2, 6.5 and 6.11 yields

$$K_n(z,w)=C_1n^2+C_2\,n\log n+\bigg(\tilde{C}_3+\frac{(\frac{1}{2}-x+\mathcal{Q}_n)\sigma_1}{r_1^2e^{t_1+t_2}}\bigg)n+\tilde{C}_4\sqrt{n}+\mathcal{O}(n^{\frac{2}{5}}).$$

Furthermore, by (3.5) and Lemma 6.6, we have $\frac{1}{2}-x+\mathcal{Q}_n=\mathcal{F}_n$, where $\mathcal{F}_n$ is given by (3.5). Finally, using the following identities (proved in [11, Lemma 2.10])

$$\mathcal{I}_1=\frac{\log(2\sqrt{\pi})}{2}, \qquad \mathcal{I}_3=\mathcal{I}, \qquad \mathcal{I}_4=\mathcal{I}_2-\mathcal{I}, \tag{6.33}$$

we obtain that $\tilde{C}_3=C_3$ and $\tilde{C}_4=C_4$, where C_3 and C_4 are as in the statement of Theorem 3.1. This finishes the proof of Theorem 3.1.

6.2 Proof of Theorem 3.13

In this subsection we assume that $\theta_1 \neq \theta_2 \mod 2\pi$, and we take $M = M'\sqrt{\log n}$. By (6.19) we have

$$S_{567} = \frac{n}{r_1^2} \sum_{j=g_{1,+}+1}^{\lfloor j_\star \rfloor} e^{i(j-1)(\theta_1-\theta_2)} \frac{j/n - br_1^{2b}}{e^{(j/n-br_1^{2b})\frac{t_1+t_2}{\sigma_1}}} \left(1 + \mathcal{O}\left(\frac{1}{n(j/n-br_1^{2b})^2}\right)\right) = \frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2} \sum_{\ell=1}^{N} c_\ell d_\ell$$

where $N := \lfloor j_\star \rfloor - g_{1,+}$, $d_\ell := p^\ell$, $p := e^{-i(\theta_1-\theta_2)}$, and c_ℓ satisfies

$$\begin{aligned} c_\ell = {} & \frac{-\ell/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n}}{e^{(-\ell/n-br_1^{2b}+\frac{\lfloor j_\star \rfloor+1}{n})\frac{t_1+t_2}{\sigma_1}}} \Bigg(1 + \frac{1}{n}\bigg\{\frac{t_1+t_2}{\sigma_1}(1-\alpha) - \frac{-\ell/n+\frac{\lfloor j_\star \rfloor+1}{n}}{2}\frac{t_1^2+t_2^2}{\sigma_1^2} \\ & + \frac{b}{2\sigma_1^2}(1-2b)r_1^{2b}(t_1^2+t_2^2) + \frac{b^2r_1^{2b} + (-\ell/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n})\alpha}{(-\ell/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n})^2}\bigg\} - \frac{2b^4r_1^{4b}}{n^2(-\ell/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n})^4} \\ & + \frac{10b^6r_1^{6b}}{n^3(-\ell/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n})^6} + \mathcal{O}\left(\frac{1}{n^2(\ell/n-\sigma_1)^3} + \frac{1}{n^4(\ell/n-\sigma_1)^8}\right)\Bigg) \end{aligned} \tag{6.34}$$

as $n \to +\infty$. Using summation by parts, we get

$$S_{567} = \frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2}\left(c_N D_N - c_1 D_0 - \frac{c_N - c_1}{1-p} + \sum_{\ell=1}^{N-1}(c_{\ell+1}-c_\ell)\frac{p^{\ell+1}}{1-p}\right), \tag{6.35}$$

where $D_\ell := \sum_{j=0}^{\ell} d_j$. Note that

$$\frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2}\left(-c_1D_0 - \frac{-c_1}{1-p}\right) = \frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2}\frac{p}{1-p}\frac{\sigma_1}{e^{t_1+t_2}}(1+\mathcal{O}(n^{-1})), \tag{6.36}$$

$$\frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2}\left(c_ND_N - \frac{c_N}{1-p}\right) = \mathcal{O}(M\sqrt{n}), \tag{6.37}$$

where for (6.37) we have used that $-N/n - br_1^{2b} + \frac{\lfloor j_\star \rfloor+1}{n} = \mathcal{O}(M/\sqrt{n})$. Let us now prove that

$$\left|\frac{ne^{i\lfloor j_\star \rfloor(\theta_1-\theta_2)}}{r_1^2}\sum_{\ell=1}^{N-1}(c_{\ell+1}-c_\ell)\frac{p^{\ell+1}}{1-p}\right| = \mathcal{O}\left(\frac{\sqrt{n}}{M}\right). \tag{6.38}$$

From (6.34) we deduce that

$$c_{\ell+1} - c_\ell = c_\ell\left(\frac{1+(\ell/n-\sigma_1)\frac{t_1+t_2}{\sigma_1}}{(\ell/n-\sigma_1)n} + \mathcal{O}\left(\frac{1}{n^2(\ell/n-\sigma_1)^3}\right)\right), \qquad \text{as } n \to +\infty. \tag{6.39}$$

Using (6.39) and then (6.34), we get

$$\begin{aligned} & \sum_{\ell=1}^{N-1}(c_{\ell+1}-c_\ell)\frac{p^{\ell+1}}{1-p} \\ & = \frac{1}{n}\frac{p}{1-p}\sum_{\ell=1}^{N-1}c_\ell\frac{1+(\ell/n-\sigma_1)\frac{t_1+t_2}{\sigma_1}}{\ell/n-\sigma_1}p^\ell + \sum_{\ell=1}^{N-1}\mathcal{O}\left(\frac{1}{n^2(\ell/n-\sigma_1)^2}\right) \\ & = \frac{1}{n}\frac{p}{1-p}\sum_{\ell=1}^{N-1}\tilde{c}_\ell p^\ell + \mathcal{O}\left(\frac{1}{M\sqrt{n}}\right), \qquad \tilde{c}_\ell := \frac{1+(\ell/n-\sigma_1)\frac{t_1+t_2}{\sigma_1}}{e^{(\ell/n-\sigma_1)\frac{t_1+t_2}{\sigma_1}}}. \end{aligned} \tag{6.40}$$

Summing again by parts, we get

$$\sum_{\ell=1}^{N-1}\tilde{c}_\ell p^\ell = \tilde{c}_{N-1}D_{N-1} - \tilde{c}_1 D_0 - \frac{\tilde{c}_{N-1}-c_1}{1-p} + \sum_{\ell=1}^{N-2}(\tilde{c}_{\ell+1}-\tilde{c}_\ell)\frac{p^{\ell+1}}{1-p} = \mathcal{O}(1),$$

where we have used that $\tilde{c}_{\ell+1}-\tilde{c}_\ell = \mathcal{O}(n^{-1})$ uniformly for $\ell = 1,\dots,N-2$. Substituting the above in (6.40), this proves (6.38). Substituting now (6.36), (6.37) and (6.38) in (6.35), we get

$$S_{567} = \frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}}{r_1^2}\frac{p}{1-p}\frac{\sigma_1}{e^{t_1+t_2}} + \mathcal{O}\big(M\sqrt{n}\big). \tag{6.41}$$

Now we analyze S_4. By (6.14), we have

$$S_4 = \sqrt{n}\sum_{j=g_{1,-}}^{g_{1,+}} p^{-(j-1)}\hat{c}_j = p^{2-g_{1,-}}\sqrt{n}\sum_{\ell=1}^{\hat{N}} p^{-\ell}\hat{c}'_\ell, \qquad \text{as } n\to+\infty,$$

where $\hat{N} := g_{1,+}-g_{1,-}+1$ and $\hat{c}_j, \hat{c}'_\ell$ satisfy

$$\begin{aligned}\hat{c}_j = {} & \frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{M_{j,1}^2r_1^{2b}}{2}}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})}\Bigg(1+\frac{1}{\sqrt{n}}\Bigg\{\frac{(5M_{j,1}^2r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \\ & + \frac{M_{j,1}}{6}\Big(5M_{j,1}^2r_1^{2b}-3+6br_1^{2b}(t_1+t_2)\Big)\Bigg\} \\ & + \mathcal{O}\bigg(\frac{1+M_{j,1}^2+M_{j,1}^6\chi_{j,1}^+}{n}+\frac{1+M_{j,1}^7+M_{j,1}^9\chi_{j,1}^+}{n^{3/2}}+\frac{1+M_{j,1}^{12}}{n^2}\bigg)\Bigg), \\ \hat{c}'_\ell := {} & \hat{c}_j, \quad \text{where } j=\ell+g_{1,-}-1.\end{aligned}$$

Summing by parts, we get

$$\sum_{\ell=1}^{\hat{N}} p^{-\ell}\hat{c}'_\ell = \hat{c}'_{\hat{N}}\hat{D}_{\hat{N}} - \hat{c}'_1\hat{D}_0 - \frac{\hat{c}'_{\hat{N}}-\hat{c}'_1}{1-p^{-1}} + \sum_{\ell=1}^{\hat{N}-1}(\hat{c}'_{\ell+1}-\hat{c}'_\ell)\frac{p^{-\ell-1}}{1-p^{-1}}$$

where $\hat{D}_\ell := \sum_{j=0}^{\ell}p^{-j}$. Since $\hat{c}'_1 = \mathcal{O}(e^{-M^2})$, $\hat{c}'_{\hat{N}} = \mathcal{O}(1)$, $\hat{c}'_{\ell+1}-\hat{c}'_\ell = \mathcal{O}(n^{-\frac{1}{2}})$ and $\hat{N} = \mathcal{O}(M\sqrt{n})$, we get $\sum_{\ell=1}^{\hat{N}} p^{-\ell}\hat{c}'_\ell = \mathcal{O}(M)$, and thus

$$S_4 = \mathcal{O}(M\sqrt{n}), \qquad \text{as } n\to+\infty. \tag{6.42}$$

Combining (6.41), (6.42) and Lemmas 6.2 and 6.5 yields

$$K_n(z,w) = \frac{ne^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}\sigma_1}{r_1^2e^{t_1+t_2}}\bigg(\frac{1}{p^{-1}-1}+\mathcal{Q}_n\bigg) + \mathcal{O}(M\sqrt{n}). \tag{6.43}$$

Using (3.17) and (6.20), we infer that

$$\frac{\sigma_1}{r_1^2}\bigg(\frac{1}{p^{-1}-1}+\mathcal{Q}_n\bigg) = 2\pi\cdot r_1^{-2x}\cdot\mathcal{S}^G_{\mathrm{hard}}(r_1e^{\theta_1},r_1e^{\theta_2};n). \tag{6.44}$$

Substituting (6.44) in (6.43) yields

$$K_n(z,w) = 2\pi n\cdot\mathcal{S}^G_{\mathrm{hard}}(r_1e^{\theta_1},r_1e^{\theta_2};n)\cdot\frac{e^{i\lfloor j_\star\rfloor(\theta_1-\theta_2)}}{e^{t_1+t_2}}r_1^{-2x} + \mathcal{O}(M\sqrt{n}).$$

Since $M = M'\sqrt{\log n}$, this finishes the proof of Theorem 3.13.

7 Proofs of Theorems 3.5 and 3.15

In this section z and w are given by

$$z = r_1\Big(1 - \frac{s_1}{\sqrt{n}}\Big)e^{i\theta_1}, \qquad w = r_1\Big(1 - \frac{s_2}{\sqrt{n}}\Big)e^{i\theta_2}, \qquad s_1, s_2 > 0, \theta_1, \theta_2 \in \mathbb{R}, \tag{7.1}$$

and $s_1, s_2, \theta_1, \theta_2$ are independent of n. At this point $\theta_1, \theta_2 \in \mathbb{R}$ can be arbitrary (in Subsection 7.1 we will take $\theta_2 = \theta_1$, and in Subsection 7.2 we will take $\theta_2 \neq \theta_1 \mod 2\pi$). We emphasize that s_1, s_2 are strictly positive (the case $s_1 = s_2 = 0$ is very different and covered in Section 6). The parameters s_1, s_2 in (7.1) are related to $\mathfrak{s}_1, \mathfrak{s}_2$ in (3.8) via $s_j = \mathfrak{s}_j/(\sqrt{2}br_1^b)$, $j = 1, 2$. As in (5.2) we write $K_n(z, w) = \sum_{j=1}^n F_j$, where

$$F_j := e^{-\frac{n}{2}Q(z)}e^{-\frac{n}{2}Q(w)}\frac{z^{j-1}\overline{w}^{j-1}}{h_j} = e^{-\frac{n}{2}|z|^{2b}}e^{-\frac{n}{2}|w|^{2b}}|z|^\alpha|w|^\alpha\frac{z^{j-1}\overline{w}^{j-1}}{h_j}. \tag{7.2}$$

In this section we take $M = M'\log n$. The following lemma is proved using Lemma 4.2.

Lemma 7.1. *Let $\delta \in (0, \frac{1}{100})$ be fixed. M' can be chosen sufficiently large and independently of δ such that the following hold.*

1. *Let j be fixed. As $n \to +\infty$, we have*

$$F_j = \mathcal{O}(e^{-n(1-\delta)r_1^{2b}}). \tag{7.3}$$

2. *As $n \to +\infty$ with $M' \le j \le j_{1,-}$, we have*

$$F_j = \mathcal{O}(e^{-nr_1^{2b}(1-\delta)\frac{\epsilon-\log(1+\epsilon)}{1+\epsilon}}). \tag{7.4}$$

3. *As $n \to +\infty$ with $j_{1,-} \le j \le g_{1,-}$, we have*

$$F_j = \mathcal{O}(n^{-100}). \tag{7.5}$$

4. *As $n \to +\infty$ with $g_{1,-} \le j \le g_{1,+}$, we have*

$$\begin{aligned} F_j = e^{i(j-1)(\theta_1-\theta_2)}\sqrt{n}&\frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{1}{2}r_1^{2b}(M_{j,1}^2-2bM_{j,1}(s_1+s_2)+2b^2(s_1^2+s_2^2))}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \\ &\times\Bigg(1 + \frac{1}{\sqrt{n}}\Bigg\{\frac{(5M_{j,1}^2r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} - \frac{M_{j,1}}{2} + s_1 + s_2 + r_1^{2b}\Bigg[\frac{5}{6}M_{j,1}^3 \\ &- bM_{j,1}^2(s_1+s_2) + bM_{j,1}\frac{s_1^2+s_2^2}{2} + \frac{b^2(2b-3)}{3}(s_1^3+s_2^3)\Bigg]\Bigg\} + \mathcal{O}\Big(\frac{1+M_{j,1}^6}{n}\Big)\Bigg). \end{aligned} \tag{7.6}$$

5. *As $n \to +\infty$ with $g_{1,+} \le j \le j_{1,+}$, we have*

$$F_j = \mathcal{O}(n^{-100}). \tag{7.7}$$

6. *As $n \to +\infty$ with $j_{1,+} \le j \le \lfloor j_\star \rfloor$, we have*

$$F_j = \mathcal{O}\Big(e^{-(1-\delta)\frac{\epsilon br_1^{2b}}{1-\epsilon}(s_1+s_2)\sqrt{n}}\Big). \tag{7.8}$$

7. As $n \to +\infty$ with $\lfloor j_\star \rfloor + 1 \leq j \leq j_{2,-}$, we have

$$F_j = \mathcal{O}\Big(e^{-(1-\delta)\sigma_1(s_1+s_2)\sqrt{n}} \Big). \tag{7.9}$$

8. As $n \to +\infty$ with $j_{2,-} \leq j \leq g_{2,-}$, we have

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})(\frac{br_2^{2b}}{1+\epsilon}-\sigma_\star)}). \tag{7.10}$$

9. As $n \to +\infty$ with $g_{2,-} \leq j \leq g_{2,+}$, we have

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{7.11}$$

10. As $n \to +\infty$ with $g_{2,+} \leq j \leq j_{2,+}$, we have

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{7.12}$$

11. As $n \to +\infty$ with $j_{2,+} \leq j \leq n$, we have

$$F_j = \mathcal{O}(e^{-n(1-\delta)\log(\frac{r_2}{r_1})\sigma_2}). \tag{7.13}$$

Proof. (7.3) directly follows from (4.9) and (7.2). Let us now prove (7.4). By (4.10) and (7.2), as $n \to +\infty$ with $M' \leq j \leq j_{1,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[\frac{1}{b}(j/n - j/n\log(\frac{j/n}{b})) - r_1^{2b} + 2j/n\log(r_1) \bigg] \bigg\} \bigg).$$

Since the function $[0, br_1^{2b}] \in y \mapsto y - y\log(\frac{y}{b}) + y\log(r_1^{2b})$ is increasing, and since $j/n \leq \frac{br_1^{2b}}{1+\epsilon} + \mathcal{O}(n^{-1})$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ -nr_1^{2b}(1-\delta)\frac{\epsilon - \log(1+\epsilon)}{1+\epsilon} \bigg\} \bigg),$$

which is (7.4). Now we prove (7.5). By (4.11) and (7.2), as $n \to +\infty$ with $j_{1,-} \leq j \leq g_{1,-}$, we have

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ n(1-\delta)\bigg[\frac{j/n - j/n\log(\frac{j/n}{b})}{b} - r_1^{2b} + 2j/n\log(r_1) + \frac{s_1+s_2}{\sqrt{n}}(br_1^{2b} - j/n) \bigg] \bigg\} \bigg).$$

For $j_{1,-} \leq j \leq g_{1,-}$, the above exponential attains its maximum at $j = g_{1,-} = n(\frac{br_1^{2b}}{1+\frac{M}{\sqrt{n}}} + \mathcal{O}(n^{-1}))$, and therefore

$$F_j = \mathcal{O}\bigg(\exp\bigg\{ -(1-\delta)\bigg[\frac{M^2 r_1^{2b}}{2} - Mbr_1^{2b}(s_1+s_2) \bigg] \bigg\} \bigg).$$

Since $M = M'\log n$ and s_1, s_2 are fixed, (7.5) holds provided M' is chosen large enough. Now we prove (7.6). As $n \to +\infty$ with $g_{1,-} \leq j \leq g_{1,+}$, we have

$$e^{r_1^{2b}(1-\log(r_1^{2b}))n} e^{M_{j,1}r_1^{2b}\log(r_1^{2b})\sqrt{n}} e^{-r_1^{2b}(\frac{1}{2}+\log(r_1^{2b}))M_{j,1}^2} e^{-\frac{n}{2}|z|^{2b}} e^{-\frac{n}{2}|w|^{2b}} |z|^\alpha |w|^\alpha z^{j-1}\overline{w}^{j-1}$$
$$= e^{i(j-1)(\theta_1-\theta_2)} e^{-\frac{1}{2}r_1^{2b}(M_{j,1}^2 - 2bM_{j,1}(s_1+s_2) + 2b^2(s_1^2+s_2^2))} r_1^{-2}$$

$$\times \left\{1+\frac{1}{\sqrt{n}}\left[s_1+s_2+br_1^{2b}\left(\frac{b(2b-3)}{3}(s_1^3+s_2^3)+M_{j,1}\frac{s_1^2+s_2^2}{2}-M_{j,1}^2(s_1+s_2)-2M_{j,1}^3\log(r_1)\right)\right] + \mathcal{O}\Big(\frac{1+M_{j,1}^6}{n}\Big)\right\}.$$

Now (7.6) follows directly from a computation using (4.12) and (7.2). Now we prove (7.7). Using (4.13) and (7.2), as $n \to +\infty$ with $g_{1,+} \leq j \leq j_{1,+}$ we obtain

$$\begin{aligned} F_j &= \mathcal{O}\Big(\exp\Big\{-\sqrt{n}(1-\delta)(j/n - br_1^{2b})(s_1+s_2)\Big\}\Big) \\ &= \mathcal{O}\Big(\exp\Big\{-br_1^{2b}(s_1+s_2)M\Big\}\Big). \end{aligned}$$

Since $s_1, s_2 > 0$ are fixed and $M = M' \log n$, we can choose M' large enough so that (7.7) holds. The proofs of (7.8)–(7.13) are similar to the proof of (7.7) and are omitted. □

Lemma 7.2. *As $n \to +\infty$, we have*

$$K_n(z,w) = S_4 + \mathcal{O}(1),$$

where

$$\begin{aligned} S_4 := \sum_{j=g_{1,-}}^{g_{1,+}} & e^{i(j-1)(\theta_1-\theta_2)}\sqrt{n}\frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{1}{2}r_1^{2b}(M_{j,1}^2-2bM_{j,1}(s_1+s_2)+2b^2(s_1^2+s_2^2))}}{\sqrt{\pi}\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} \\ & \times \Bigg(1+\frac{1}{\sqrt{n}}\Bigg\{\frac{(5M_{j,1}^2r_1^{2b}-2)e^{-\frac{r_1^{2b}M_{j,1}^2}{2}}}{3\sqrt{2\pi}r_1^b\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})} - \frac{M_{j,1}}{2}+s_1+s_2+r_1^{2b}\Bigg[\frac{5}{6}M_{j,1}^3 \\ & - bM_{j,1}^2(s_1+s_2)+bM_{j,1}\frac{s_1^2+s_2^2}{2}+\frac{b^2(2b-3)}{3}(s_1^3+s_2^3)\Bigg]\Bigg\}\Bigg), \end{aligned} \tag{7.14}$$

Proof. By Lemma 7.1, as $n \to +\infty$ we have

$$K_n(z,w) = S_4 + \mathcal{O}\Bigg(\sum_{j=g_{1,-}}^{g_{1,+}}\sqrt{n}\frac{e^{-\frac{1}{2}r_1^{2b}(M_{j,1}^2-2bM_{j,1}(s_1+s_2))}}{\mathrm{erfc}(-\frac{M_{j,1}r_1^b}{\sqrt{2}})}\frac{1+M_{j,1}^6}{n}\Bigg) + \mathcal{O}(n^{-50}), \tag{7.15}$$

Since $s_1, s_2 > 0$ are fixed, the function $x \mapsto \frac{e^{-\frac{1}{2}r_1^{2b}(x^2-2bx(s_1+s_2))}}{\mathrm{erfc}(-xr_1^b/\sqrt{2})}$ is $\mathcal{O}(e^{-c|x|})$ as $x \to \pm\infty$ (for some $c > 0$). Lemma 6.9 then implies that the $\mathcal{O}$-terms in (7.15) remain bounded as $n \to +\infty$, and the claim follows. □

7.1 Proof of Theorem 3.5

Here we take $\theta_2 = \theta_1$. Recall also that s_1, s_2 in (7.1) are related to $\mathfrak{s}_1, \mathfrak{s}_2$ in (3.8) via $s_j = \mathfrak{s}_j/(\sqrt{2}br_1^b) = \mathfrak{s}_j/(\tilde{\Delta}Q(r_1))^{\frac{1}{2}}$, $j = 1,2$.

Using (7.14), we write $S_4 = \sum_{j=g_{1,-}}^{g_{1,+}}(\sqrt{n}\,h_1(M_{j,1}) + h_2(M_{j,1}))$, where

$$h_1(x) = \frac{\sqrt{2}\,br_1^{b-2}e^{-\frac{1}{2}((\frac{r_1^b x}{\sqrt{2}}-\mathfrak{s}_1)^2+(\frac{r_1^b x}{\sqrt{2}}-\mathfrak{s}_2)^2)}}{\sqrt{\pi}\mathrm{erfc}(-\frac{xr_1^b}{\sqrt{2}})},$$

$$h_2(x) = h_1(x)\Big\{ \frac{(5x^2 r_1^{2b} - 2)e^{-\frac{r_1^{2b}x^2}{2}}}{3\sqrt{2\pi} r_1^b \operatorname{erfc}(-\frac{x r_1^b}{\sqrt{2}})} - \frac{x}{2} + \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{\sqrt{2}\, b r_1^b} + r_1^{2b}\Big[\frac{5}{6}x^3 - x^2 \frac{\mathfrak{s}_1 + \mathfrak{s}_2}{\sqrt{2} r_1^b} + x \frac{\mathfrak{s}_1^2 + \mathfrak{s}_2^2}{4b r_1^{2b}} + \frac{2b-3}{6\sqrt{2}\, b} \frac{\mathfrak{s}_1^3 + \mathfrak{s}_2^3}{r_1^{3b}} \Big] \Big\}.$$

Since $\mathfrak{s}_1, \mathfrak{s}_2 > 0$ are fixed, $h_1(x), h_2(x) = \mathcal{O}(e^{-c|x|})$ as $x \to \pm\infty$ for some $c > 0$. Since $M = M' \log n$, by Lemma 6.9, we obtain

$$\begin{aligned} S_4 &= b r_1^{2b} \int_{-M}^{M} h_1(t) dt\, n + b r_1^{2b} \int_{-M}^{M} (h_2(t) - 2t h_1(t)) dt \sqrt{n} + \mathcal{O}(1), \\ &= b r_1^{2b} \int_{-\infty}^{+\infty} h_1(t) dt\, n + b r_1^{2b} \int_{-\infty}^{+\infty} (h_2(t) - 2t h_1(t)) dt \sqrt{n} + \mathcal{O}(1). \end{aligned}$$

The claim now follows from a simple change of variables. This finishes the proof of Theorem 3.5.

7.2 Proof of Theorem 3.15

Here we take $\theta_1 \neq \theta_2 \mod 2\pi$. Let $p := e^{-i(\theta_1 - \theta_2)}$. By (7.14),

$$S_4 = \sqrt{n} \sum_{j=g_{1,-}}^{g_{1,+}} p^{-(j-1)} c_j = p^{2-g_{1,-}} \sqrt{n} \sum_{\ell=1}^{N} p^{-\ell} c'_\ell,$$

where $N := g_{1,+} - g_{1,-} + 1$, and c_j, c'_ℓ are defined by

$$\begin{aligned} c_j &:= \frac{\sqrt{2}\, b r_1^{b-2} e^{-\frac{1}{2} r_1^{2b}(M_{j,1}^2 - 2b M_{j,1}(s_1+s_2) + 2b^2(s_1^2+s_2^2))}}{\sqrt{\pi} \operatorname{erfc}(-\frac{M_{j,1} r_1^b}{\sqrt{2}})} \\ &\quad \times \bigg(1 + \frac{1}{\sqrt{n}} \Big\{ \frac{(5M_{j,1}^2 r_1^{2b} - 2)e^{-\frac{r_1^{2b} M_{j,1}^2}{2}}}{3\sqrt{2\pi} r_1^b \operatorname{erfc}(-\frac{M_{j,1} r_1^b}{\sqrt{2}})} - \frac{M_{j,1}}{2} + s_1 + s_2 + r_1^{2b} \Big[\frac{5}{6} M_{j,1}^3 \\ &\quad - b M_{j,1}^2 (s_1 + s_2) + b M_{j,1} \frac{s_1^2 + s_2^2}{2} + \frac{b^2(2b-3)}{3}(s_1^3 + s_2^3) \Big] \Big\} \bigg), \\ c'_\ell &:= c_j, \quad \text{where } j = \ell + g_{1,-} - 1. \end{aligned}$$

Summing by parts, we get

$$\sum_{\ell=1}^{N} p^{-\ell} c'_\ell = c'_N \hat{D}_N - c'_1 \hat{D}_0 - \frac{c'_N - c'_1}{1 - p^{-1}} + \sum_{\ell=1}^{N-1} (c'_{\ell+1} - c'_\ell) \frac{p^{-\ell-1}}{1 - p^{-1}}$$

where $\hat{D}_\ell := \sum_{j=0}^{\ell} p^{-j}$. Note that $c'_1 = \mathcal{O}(e^{-cM})$, $c'_N = \mathcal{O}(e^{-cM^2})$, and $c_{j+1} - c_j = \mathcal{O}(n^{-\frac{1}{2}}) c_j$, $c_j = \mathcal{O}(e^{-c|M_{j,1}|})$. Hence, using several times summation by parts, we get

$$\sum_{\ell=1}^{N} p^{-\ell} c'_\ell \lesssim e^{-cM} + \frac{1}{\sqrt{n}} \sum_{\ell=1}^{N-1} p^{-\ell} c'_\ell \lesssim e^{-cM} + \frac{1}{n^{\frac{21}{2}}} \sum_{\ell=1}^{N-21} p^{-\ell} c'_\ell \lesssim \frac{M}{n^{10}},$$

where we have used that $N = \mathcal{O}(M\sqrt{n})$ for the last equality. Hence $S_4 = \mathcal{O}(\frac{M}{n^{10}})$. By Lemma 7.2, we thus have $K_n(z, w) = \mathcal{O}(1)$, which is the statement of Theorem 3.15.

A The semi-hard 1-point function and its universality

In the following, we consider any radially symmetric potential V which is smooth near a point $r_1 > 0$ in the bulk of the droplet $S = S[V]$; we also assume that $\Delta V(r_1) > 0$. We redefine V by placing a hard wall along the circle $r = r_1$, say

$$Q(z) = \begin{cases} V(z), & |z| \le r_1, \\ +\infty, & |z| > r_1. \end{cases}$$

Let $K_n(z, w)$ be the associated correlation kernel, i.e., the reproducing kernel of the space

$$\mathcal{W}_n^h = \{p \cdot e^{-\frac{1}{2}nQ} : p \text{ is a holomorphic polynomial with } \deg p \le n-1\} \subset L^2(\mathbb{C}, dA).$$

We write $R_n(z) = K_n(z, z)$ for the 1-point function.

Let us write $\tilde{K}_n(z, w)$ for the correlation kernel of the rescaled process pertaining to the blow-up map $z \mapsto \sqrt{2n\Delta V(r_1)} \cdot (r_1 - z)$, i.e.

$$\tilde{K}_n(z, w) = \frac{1}{2n\Delta V(r_1)} K_n\Big(r_1 - \frac{z}{\sqrt{2n\Delta V(r_1)}}, r_1 - \frac{w}{\sqrt{2n\Delta V(r_1)}}\Big). \tag{A.1}$$

Our goal is to study the asymptotics of this rescaled "semi-hard edge kernel" for z, w in a compact subset of the right half-plane: $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$.

Remark A.1. *In [16] the following slightly different rescaling was used:*

$$\hat{K}_n(z, w) = \frac{1}{n\Delta V(r_1)} K_n\Big(r_1 + \frac{z}{\sqrt{n\Delta V(r_1)}}, r_1 + \frac{w}{\sqrt{n\Delta V(r_1)}}\Big). \tag{A.2}$$

Below we recall some theory for the limit of $\hat{K}_n$, and when we translate back to $\tilde{K}_n$, we use:

$$\tilde{K}_n(z, w) = \frac{1}{2}\hat{K}_n(-\frac{z}{\sqrt{2}}, -\frac{w}{\sqrt{2}}), \qquad \hat{K}_n(z, w) = 2\tilde{K}_n(-\sqrt{2}z, -\sqrt{2}w). \tag{A.3}$$

By [16] we know that there is a sequence of cocycles $c_n(z, w)$ such that the kernels $c_n(z, w)\hat{K}_n(z, w)$ converge locally uniformly for z, w in the left half-plane to a limit kernel $\hat{K}(z, w)$ of the form

$$\hat{K}(z, w) = G(z, w)\Psi(z, w) \qquad \operatorname{Re} z < 0, \operatorname{Re} w < 0, \tag{A.4}$$

where $G(z, w) = e^{z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}$ is the "Ginibre kernel" while $\Psi(z, w)$ is a function analytic in z and $\bar{w}$ and well-defined for $\operatorname{Re} z < 0$, $\operatorname{Re} w < 0$. Also, the limit kernel gives rise to a solution $\hat{R}(z)$ of the Ward equation

$$\bar{\partial}\hat{C}(z) = \hat{R}(z) - 1 - \Delta \log \hat{R}(z), \qquad \operatorname{Re} z < 0, \tag{A.5}$$

where

$$\hat{R}(z) = \hat{K}(z, z) = \Psi(z, z), \qquad \hat{C}(z) = \int_{\operatorname{Re} w < 0} \frac{|\hat{K}(z, w)|^2}{\hat{K}(z, z)} \frac{1}{z - w}\, dA(w).$$

(If $\hat{R}(z) > 0$ for one z, then $\hat{R}(z) > 0$ everywhere in the left half-plane, see [16] for details.)

By standard balayage arguments [11], we know that $\hat{R}$ blows up at the hard edge (this is not surprising since $K_n(z, z)$ grows like n^2 in the hard edge regime), i.e.

$$\lim_{\operatorname{Re} z \to 0} \hat{R}(z) = +\infty. \tag{A.6}$$

It is also useful to note that when $\operatorname{Re} z \to -\infty$ we have, by well-known bulk asymptotics [8],

$$\lim_{\operatorname{Re} z \to -\infty} \hat{R}(z) = 1. \tag{A.7}$$

The rotational symmetry of the ensemble also easily implies that $\hat{R}$ is vertical translation invariant (see [15] for a detailed proof of this)

$$\hat{R}(z+it) = \hat{R}(z), \qquad \forall t \in \mathbb{R}. \tag{A.8}$$

This is equivalent to saying that the analytic function Ψ in (A.4) takes the form $\Psi(z,w) = \Phi(z+\bar{w})$ (see e.g. [15]) where $\Phi(z)$ is an analytic function of one variable, defined for $\operatorname{Re} z < 0$.

Thus we have

$$\hat{K}(z,w) = G(z,w)\Phi(z+\bar{w}), \qquad \hat{R}(z) = \Phi(z+\bar{z}). \tag{A.9}$$

We now have the following result, which is implicit in the analysis in [16].

Proposition A.2. *There is a unique analytic function $\Phi(z)$ for $\operatorname{Re} z < 0$ which satisfies the Ward equation (A.5) with boundary conditions (A.6) and (A.7). This function is given by*

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(z-t)^2}}{\frac{1}{2}\operatorname{erfc}\frac{t}{\sqrt{2}}}\, dt, \qquad (\operatorname{Re} z < 0). \tag{A.10}$$

Proof. That (A.10) solves the Ward equation is shown in [16, Theorem 8], and (A.6) and (A.7) are likewise clear for the solution (A.10). In the other direction, given an arbitrary solution Φ there exists, by [16, Theorem 8], an interval $I \subset \mathbb{R}$ such that Φ has the structure

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_I \frac{e^{-\frac{1}{2}(z-t)^2}}{\frac{1}{2}\operatorname{erfc}\frac{t}{\sqrt{2}}}\, dt.$$

The conditions $\Phi(x) \to 1$ as $x \to -\infty$ and $\Phi(x) \to +\infty$ as $x \to 0$ easily implies that we must have $I = \mathbb{R}$. $\square$

It follows that the limit kernel $\hat{K}(z,w)$ is uniquely given by

$$\hat{K}(z,w) = e^{z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(z+\bar{w}-t)^2}}{\frac{1}{2}\operatorname{erfc}\frac{t}{\sqrt{2}}}\, dt, \qquad \operatorname{Re} z < 0,\ \operatorname{Re} w < 0.$$

Using (A.3), we then get for z and w in the right half-plane $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$ that

$$\begin{aligned} \tilde{K}(z,w) &= \frac{1}{2} e^{\frac{1}{2} z\bar{w} - \frac{1}{4}|z|^2 - \frac{1}{4}|w|^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\frac{z+\bar{w}}{\sqrt{2}}+t)^2}}{\frac{1}{2}\operatorname{erfc}\frac{t}{\sqrt{2}}}\, dt \\ &= e^{\frac{1}{2} z\bar{w} - \frac{1}{4}|z|^2 - \frac{1}{4}|w|^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{4}(z+\bar{w}+2t)^2}}{\operatorname{erfc} t}\, dt. \end{aligned} \tag{A.11}$$

If we choose $\mathfrak{s}_1, \mathfrak{s}_2$ real and positive this gives

$$\tilde{K}(\mathfrak{s}_1, \mathfrak{s}_2) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(t+\mathfrak{s}_1)^2 + (t+\mathfrak{s}_2)^2}{2}}}{\sqrt{\pi}\operatorname{erfc} t} dt. \tag{A.12}$$

Universality for non-rotation invariant potentials. Note that the above arguments work not only for rotationally symmetric potentials, but for arbitrary potentials and scaling limits $\tilde{K}$ which are vertical translation invariant.

Indeed, consider a smooth potential V (not necessarily rotation invariant), fix any point r_1 and define Q as follows:

$$Q(z) = \begin{cases} V(z), & z \in D, \\ +\infty, & z \notin D, \end{cases}$$

where $D \subset \operatorname{int} S[V]$ is such that ∂D is smooth and passes through r_1. Assume that $\Delta V(r_1) > 0$. Let K_n be the correlation kernel associated to Q, and define $\tilde{K}_n, \hat{K}_n$ as in (A.1) and (A.2). By [16] we know that each subsequential limit $\hat{K}$ of the rescaled kernels $\hat{K}_n$ satisfy Ward's equation (A.5) as well as the boundary conditions (A.6) and (A.7). Hence if $\hat{K}$ is vertical translation invariant, i.e., $\hat{K}(z+it, z+it) = \hat{K}(z,z)$ for all $t \in \mathbb{R}$, [16, Theorem 8] implies that $\hat{K}$ is uniquely given by (A.9),(A.10). In other words, the kernel $\tilde{K}$ in (A.11) is universal among translation invariant scaling limits.

B Uniform asymptotics of the incomplete gamma function

To analyze the large n behavior of $K_n(z,w)$, we will use the asymptotics of $\gamma(a,z)$ in various regimes of the parameters a and z. These asymptotics are available in the literature and are summarized in the following lemmas.

Lemma B.1. *([57, formula 8.11.2]). Let a be fixed. As $z \to \infty$,*

$$\gamma(a,z) = \Gamma(a) + \mathcal{O}(z^{a-1}e^{-z}).$$

Lemma B.2. *([64, Section 11.2.4]). The following hold:*

$$\frac{\gamma(a,z)}{\Gamma(a)} = \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{a/2}) - R_a(\eta), \qquad R_a(\eta) := \frac{e^{-\frac{1}{2}a\eta^2}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} g(u)du, \tag{B.1}$$

where erfc *is given by* (3.2), $g(u) := \frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u+i\eta}$,

$$\lambda = \frac{z}{a}, \quad \eta = (\lambda-1)\sqrt{\frac{2(\lambda - 1 - \log\lambda)}{(\lambda-1)^2}}, \quad u = -i(t-1)\sqrt{\frac{2(t-1-\log t)}{(t-1)^2}}, \tag{B.2}$$

and the principal branch is used for the roots. In particular, $t \in \mathcal{L} := \{\frac{\theta}{\sin\theta}e^{i\theta} : -\pi < \theta < \pi\}$ for $u \in \mathbb{R}$ and $\eta \in \mathbb{R}$ for $\lambda > 0$. Furthermore, as $a \to +\infty$, uniformly for $z \in [0,\infty)$,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\sum_{j=0}^{\infty}\frac{c_j(\eta)}{a^j}, \tag{B.3}$$

where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in (0,\infty)$). The coefficients $c_0(\eta)$ and $c_1(\eta)$ are explicitly given by (see [64, p. 312])

$$c_0(\eta) = \frac{1}{\lambda-1} - \frac{1}{\eta}, \qquad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}. \tag{B.4}$$

Recall that erfc satisfies [57, 7.12.1]

$$\operatorname{erfc}(y) = \frac{e^{-y^2}}{\sqrt{\pi}}\left(\frac{1}{y} - \frac{1}{2y^3} + \frac{3}{4y^5} - \frac{15}{8y^7} + \mathcal{O}(y^{-9})\right), \qquad \text{as } y \to +\infty, \tag{B.5}$$

and $\operatorname{erfc}(-y) = 2 - \operatorname{erfc}(y)$. By combining Lemma B.2 with the above large y asymptotics of $\operatorname{erfc}(y)$, we get the following.

Lemma B.3. *Let η be as in (B.2).*

(i) Let $\delta > 0$ be fixed. As $a \to +\infty$, uniformly for $\lambda \geq 1 + \delta$,

$$\frac{\gamma(a, \lambda a)}{\Gamma(a)} = 1 + \frac{e^{-\frac{a\eta^2}{2}}}{\sqrt{2\pi}}\left(\frac{-1}{\lambda - 1}\frac{1}{\sqrt{a}} + \frac{1 + 10\lambda + \lambda^2}{12(\lambda-1)^3}\frac{1}{a^{3/2}} + \mathcal{O}(a^{-5/2})\right).$$

(ii) As $a \to +\infty$, uniformly for λ in compact subsets of $(0, 1)$,

$$\frac{\gamma(a, \lambda a)}{\Gamma(a)} = \frac{e^{-\frac{a\eta^2}{2}}}{\sqrt{2\pi}}\left(\frac{-1}{\lambda - 1}\frac{1}{\sqrt{a}} + \frac{1 + 10\lambda + \lambda^2}{12(\lambda-1)^3}\frac{1}{a^{3/2}} + \mathcal{O}(a^{-5/2})\right).$$

Conflict of interest statement. The authors have no conflict of interest to disclose.

Data availability statement. There is no data associated to this work.

Acknowledgement.

CC acknowledges support from the Swedish Research Council, Grant No. 2021-04626. We are grateful to Sung-Soo Byun for useful remarks.

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