Synchronization Approached Through the Lens of Primitivity

François Gonze* Balázs Gerencsér† Raphaël M. Jungers*[1]
francois.gonze@uclouvain.be gerencser.balazs@renyi.mta.hu raphael.jungers@uclouvain.be
* ICTEAM Institute, UCLouvain, Louvain-la-Neuve, Belgium
† Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Introduction

Because of its links with synchronization and reachability of consensus, primitivity has recently attracted attention in the engineering community. Moreover, it allows to approach the *Černý Conjecture*, one of the most famous open problems in automata theory. In this work, we tackle the problem of generating sets of large matrices with the primitivity property. We propose a methodology to generate such sets and show its performance on numerical experiments.

1 From primitive sets to synchronizing automata

In matrix theory, a set of matrices with nonnegative entries is called *primitive* if there exists a positive product of matrices of this set. If a binary matrix has no zero row and no zero column, we call it *NZRC*. Furthermore, a set of binary row-stochastic matrices (*letters*) is called a *synchronizing automaton* if there exists a product of matrices (*word*) of this set with a positive column. This product is called a *synchronizing word*.

Blondel et al. [1] showed how the length of a shortest positive product composed from a primitive set of NZRC matrices is linked with the length of synchronizing words:

Theorem 1 (Theorem 17 in [1])**.** *For any primitive set of NZRC matrices*

$$M = \{A_1, \dots, A_m\} \subset \{0,1\}^{n\times n}$$

there exist two synchronizing automata

$$M_B = \{B_1, \dots, B_t\},\ M_C = \{C_1, \dots, C_u\},$$

such that

$$\forall 1 \le s \le t,\ \exists l \in \{1,\dots,m\} : B_s \le A_l \text{ (entrywise)} \tag{1}$$

$$\forall 1 \le s \le u,\ \exists l \in \{1,\dots,m\} : C_s \le A_l^{\intercal} \text{ (entrywise).} \tag{2}$$

Thus, for two synchronizing words w_B and w_C of M_B and M_C, we can obtain a positive product of matrices from M partly by choosing the matrices based on the majorating property ensured by (1) and (2). The length of the product is at most the sum of the lengths of these words plus $n-1$.

In the context of this theorem, the *Černý Conjecture*, which states that any synchronizing automaton of size n has a synchronizing word of length at most $(n-1)^2$ (see [2]), would imply that any primitive set of NZRC matrices has a positive product of length at most $2(n-1)^2 + n - 1$, whereas any shortest positive product of higher length would provide a counterexample. This is why the constructive approach of primitive sets is really interesting. Also, an upper bound for the length of the shortest positive product of a set of NZRC matrices is obtained from the best proven bound of Černý Conjecture (see [1, 3]).

2 Construction of primitive sets of matrices

In order to generate primitive sets of NZRC matrices with long shortest primitive product, we need conditions guaranteeing primitivity, while minimizing the number of positive entries. The method we propose is to take two random permutation matrices, and to change a "0" entry into a "1". This construction guarantees primitivity with high probability if the size is a prime number, and a minimal number of positive entries. Indeed, primitivity is equivalent to the set being irreducible and having no block-permutation structure. First, a set of two random permutations is irreducible with high probability. In addition, a second theorem allow us to say that for prime numbers, there is either no block-permutation structure or that the blocks are of size 1:

Theorem 2. *If an irreducible set of permutation matrices has a block-permutation structure, then all the blocks have the same size.*

Since our construction is not compatible with blocks of size 1, the set must be primitive.

We compared the probability of being primitive and the length of the shortest product of the matrices obtained using this method with other random NZRC sets generation methods. For both tests, it appears that our method provides a substantial improvement.

References

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