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Quantization of solvable symplectic symmetric spaces

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Résumé en français

L'histoire des interactions entre géométrie et physique s'écrit depuis maintenant bien longtemps. Avec l'avènement de la relativité générale et de la physique quantique, le vingtième siècle a vu apparaître un nombre particulièrement impressionnant de nouvelles idées au carrefour de ces deux domaines. Parmi celles-ci, la "quantification de Weyl" des observables sur l'espace des phases d'une particule libre a dès le début posé les jalons de la compréhension des relations profondes entre géométrie de Poisson et algèbres quantiques. Prenant racine dans cette théorie, deux objets se sont révélés d'une importance capitale lors des dernières décennies: les produits de Moyal et de Weyl, respectivement en quantification par déformation formelle et non formelle. La notion de point milieu apparaît naturellement dans les définitions de la quantification et du produit de Weyl, amenant à considérer les espaces symétriques symplectiques possédant une "application point milieu", qui forment la clef de voûte de notre travail.

Une conjecture formulée en 1994 par Weinstein établit un lien fort entre associativité et géométrie en identifiant en un certain sens la partie oscillante du noyau à trois points d'un produit associatif non formel avec l'aire symplectique d'un "triangle double", c'est à dire d'un triangle dont les côtés ont pour milieu trois points donnés. Le but de cette thèse est d'améliorer la compréhension de la structure des espaces symétriques symplectiques résolubles et d'établir les outils nécessaires à une construction inductive de quantifications de ces espaces pour lesquelles la conjecture est vérifiée.

Dans un premier temps, nous établissons que les espaces symétriques qui admetttent une application point milieu sont exactement ceux pour lequels l'application exponentielle est un difféomorphisme global en chaque point, et donnons une généralisation du théorème de Dixmier-Saito sur les groupes de Lie exponentiels. Nous montrons alors que l'hypothèse que chaque triangle possède un unique triangle double caractérise exactement les espaces symétriques résolubles exponentiels.

Nous nous intéressons ensuite aux espaces symétriques symplectiques, et

introduisons une notion adaptée de réduction symplectique et une construction inverse de "double extension", analogues à celles de Dardié, Medina et Revoy pour les groupes de Lie symplectiques. Un rôle capital est tenu par ce que nous appellons les "systèmes primitifs", qui sont canoniquement induits par les systèmes triples de Lie symplectiques et permettent la construction de T^*-extensions *à la* Bordemann.

Nous poursuivons par l'étude de l'ingrédient principal de notre méthode de quantification (la projection Severa), une application définie sur le groupoïde des paires d'un espace symétrique symplectique et à valeurs dans son fibré cotangent. Nous prouvons que cette projection jouit d'importantes propriétés d'invariance, qu'elle commute avec la réduction symplectique, qu'elle est un symplectomorphisme global dans le cas résoluble exponentiel et établissons une relation avec l'aire des triangles doubles.

Nous terminons cette thèse en effectuant des calculs explicites pour une nouvelle classe d'exemples. En utilisant notre produit et le travail récent de Bieliavsky et Gayral, nous donnons des formules de déformation universelles pour les algèbres de vecteurs lisses des actions tempérées fortement continues de certains groupes de Lie symplectiques non kählériens sur les algèbres de Fréchet.

Introduction

Quantization is an old problem, aiming at relating two major pieces of contemporary physics, namely, classical and quantum mechanics.

Different physical theories have different scopes and length scales. Classical mechanics is concerned with physical systems at a macroscopic scale, while quantum mechanics was designed to account for the behaviour of microscopic objects. Mathematically, a classical system, or at least its kinematical part, is characterized by a phase space modelled as a symplectic manifold or, more generally, as a Poisson manifold. The observables of such a system, such as the speed or the energy, are simply functions on the phase space. On the other hand, again from the kinematical point of view, a quantum system is the datum of a Hilbert space, usually the (bounded) observables are the (bounded) self-adjoint operators on that Hilbert space, and their eigenvalues should be the measurable quantities.

The two structures described above are radically different in nature. Nevertheless, they share a crucial characteristic in that the sets of their observables both have a natural *Poisson algebra* structure. For the classical systems, the latter is given by the Poisson bracket on (smooth) functions on the phase space together with the commutative product of functions, whereas for the quantum systems, it is defined by the commutator $[A, B] = AB - BA$ of operators, which is easily seen to satisfy the Jacobi identity and the Leibniz rule for the usual (noncommutative) composition of operators.

It was very soon realized that the behaviour of a quantum system should tend to that of a classical one as the number of particles becomes larger. A different concept of classical limit, which seems to have emerged later and is nowadays taken as a postulate in many quantization programmes, is that we should recover a classical system when the *action scale* of the quantum system tends to zero, that scale being given by the (reduced) Planck constant $\hbar$[1]. The actual way in which the classical system is recovered, that is, the precise notion of limit used, is highly dependent on the quantization theory at hand, and is of

[1] See [Wei91] for a comment and some references about this.

prime importance as part of its definition.

With that in mind, the idea of quantization is then to reverse the process, that is, from a classical system, and probably with the help of some *additional* structure, to build a quantum one of which the former is sort of a shadow [Rie98]. The two Poisson algebras should somehow correspond, in that we should be able to recover, in the limit $\hbar \to 0$, the commutative product of functions from the noncommutative composition of operators, and the classical Poisson bracket $\{a, b\}$ from $\frac{i}{\hbar}$ times the commutator $[A, B]$.

Weyl quantization

The best known and most studied case of quantization of a classical mechanical system is of course that of a system with n unconstrained degrees of freedom: the configuration space $\mathbb{R}^n$. The literature on this subject is vast and goes back to the very early days of quantum mechanics. Among the quantization methods that emerged through time, a pioneering one is the *Weyl quantization* [Wey27, Wey28, Wey31, Gro46, Moy49, Wig32][2].

Its modern form goes as follows [Poo66, Fol89, Won98]. The classical phase space is $\mathbb{R}^{2n} = \{(q, p) \mid q, p \in \mathbb{R}^n\}$ with its canonical symplectic form $\Omega = \sum_{k=1}^{n} dq^k \wedge dp_k$, and its quantum counterpart is given by the Hilbert space $L^2(\mathbb{R}^n)$ of (equivalence classes of) square integrable functions on the configuration space $\mathbb{R}^n$. To a classical observable a in, say, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to begin with, one associates a *Weyl operator* $\hat{A} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{A}u(q) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle q-q' | p\rangle} a\left(\frac{q+q'}{2}, p\right) u(q') dq' dp \tag{1}$$

of which a is called the symbol ($\langle \cdot \mid \cdot \rangle$ is the Euclidian scalar product on $\mathbb{R}^n$). For $a \in L^1(\mathbb{R}^{2n})$, the operator $\hat{A}$ is continuous for the induced L^2-norm on $\mathcal{S}(\mathbb{R}^n)$, so that it can be extended to a bounded operator $\hat{A} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Usually, though, L^2 boundedness is too restrictive, and the correspondance is extended to much larger classes of symbols and $\hat{A}$ is acting on (tempered) distributions [Fol89].

The operators corresponding to two symbols a, b in $\mathcal{S}(\mathbb{R}^{2n})$ can be composed, and it turns out that the composition is again a Weyl operator, whose corresponding symbol $a *^{\mathrm{W}} b$, defined by

$$a *^{\mathrm{W}} b(x) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}(\Omega(x,y)+\Omega(y,z)+\Omega(z,x))} a(y) b(z) dy dz, \tag{2}$$

[2] See [Sch07] for a very nice review of how Hermann Weyl contributed to the early 'new' quantum mechanics around 1925.

is called the *Weyl product* of a and b. This product was first written by von Neumann around 1930 [vN31, vN55], see [HD09] for a short history of its development.

The Weyl quantization is a typical example where more structure —that is, more of the "additional" structure referred to above— is displayed in the quantization maps than in the algebras obtained, so that, as was pointed out again recently in [Haw08], the latter should be more natural objects than the former. As an example, looking at (1) and (2), we see that the classical system comes in under two different guises. In the first one it is seen as the cotangent bundle of the configuration space. In particular, in it is endowed with a particular (real) *polarization*, that is, an integrable lagrangian smooth distribution in the tangent bundle of the phase space, here given by the vectors tangent to the fibers. In the second one, this polarization does not appear, and the phase space is simply treated as $\mathbb{R}^{2n}$ with its standard symplectic form.

Deformation quantization

The progressive understanding that the algebra structure of quantized operators could be formulated on the phase space using formula (2) led to the now well-known programme of *deformation quantization*.

The Weyl product (2) may be expanded as a formal series in $\hbar$ to give the so-called *Moyal product*:

$$u *^{\mathrm{M}} v = uv + \frac{\hbar}{2i}\{u,v\} + \sum_{k=2}^{\infty} \frac{1}{k!}\left(\frac{\hbar}{2i}\right)^k \sum_{\substack{i_1\dots i_k \\ j_1\dots j_k}} \Omega^{i_1 j_1}\dots\Omega^{i_k j_k}\, \partial_{i_1\dots i_k} u\, \partial_{j_1\dots j_k} v, \quad (3)$$

where Ω^{ij} are the components of the inverse matrix of Ω. This product was actually introduced by Groenewold in 1946 [Gro46], before the celebrated paper of Moyal of 1949 [Moy49] where only an antisymmetrized version of it appeared.

The Moyal product is associative but not commutative, is a deformation of the usual commutative product uv, and its commutator is moreover a deformation of $\frac{\hbar}{i}\{u,v\}$. This observation was formalized and turned into a *deformation quantization programme* of general Poisson manifolds in [Vey75, BFF$^+$78], in which the following definition appeared.

Definition. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. A* formal deformation *of the algebra $A = C^\infty(M)$, or equivalently a* star-product $*$ *on A, is defined to be a map*

$$* : A \times A \to A[\hbar]$$

$$(u,v) \mapsto \sum_{k=0}^{\infty} \hbar^k c_k(u,v)$$

satisfying:

1. *formal associativity: for all* $p \geq 0$

$$\sum_{k+l=p} [c_k(c_l(u,v),w)) - c_k(u,c_l(v,w))] = 0,$$

2. $c_0(u,v) = uv$,

3. $c_1(u,v) - c_1(v,u) = \{u,v\}$,

4. *each map* $c_k : A \times A \to A$ *should be a bidifferential operator.*

Various existence theorems appeared for star-products, first on symplectic manifolds [DWL83, Gut83, OMY91, Fed94], and subsequently for general real C^∞ Poisson structures on manifolds, for which the questions of existence and equivalence were completely settled by Kontsevich in 1997 [Kono3].

In the mean time, another deformation quantization programme was launched by Rieffel [Rie89], in which the objects obtained after deformation were actual functions, and some precise notion of convergence was defined.

Recall that on the algebra of continuous complex-valued functions $C(M)$ there is a natural involution given by complex conjugation $u \mapsto u^* := \bar{u}$. In that context, a Poisson bracket will be requested to be *real* in the sense that $\{u^*, v^*\} = \{u,v\}^*$ for all $u, v \in C(M)$. Let $C_c^\infty(M)$ (resp. $C_\infty(M)$) denote the space of smooth functions with compact support (resp. of continuous functions which vanish at infinity).

Definition. *A* strict deformation quantization *of* M *in the direction of* $\{\cdot,\cdot\}$ *is the data of a dense* $*$*-subalgebra* A *of* $C_\infty(M)$ *which is carried into itself by the Poisson bracket together with an open interval* I *of real numbers containing* 0, *and for each* $\hbar \in I$, *an associative product* $*_\hbar$, *an involution* $^{*_\hbar}$, *and a* C^**-norm* $\|\cdot\|_\hbar$ *(for* $*_\hbar$ *and* $^{*_\hbar}$*) on* A, *which for* $\hbar = 0$ *are the original pointwise product, complex conjugation involution, and supremum norm, such that:*

1. *the completions of* A *for the various* C^**-norms form a continuous field of* C^**-algebras over* I [3],

2. *for every* $u, v \in A$, $\|(u *_\hbar v - v *_\hbar u)/i\hbar - \{u,v\}\|_\hbar$ *converges to* 0 *as* $\hbar$ *goes to* 0.

[3] See [Lan98, Definition 1.2.1] for a definition and more about that.

A major success and generalization of that programme was obtained in 1993 by Rieffel [Rie93], who showed that a formula similar to (2), for a general constant Poisson bracket on $\mathbb{R}^d$, defined a strict deformation quantization, and could be used to provide a strict deformation quantization (in a generalized sense) of *any* C^*-algebra with an action of $\mathbb{R}^d$. The deformation of $\mathbb{R}^d$ thus provides what is called a *universal deformation formula*.

In [Rie98], Rieffel asked if such universal deformation formulæ could exist for more general Lie groups than $\mathbb{R}^d$. At the formal level, such formulæ have been found by Giaquinto and Zhang [GZ98] for the actions of the affine group $ax + b$ of the real line, in the context of Drinfel'd twists for bi-algebras. A generalization of this result for the actions of a large class of Kahlerian Lie groups is ongoing [BBD$^+$]. At the non-formal level, explicit formulæ have been found for the actions of some non-abelian solvable Lie groups, though not in a C^*-algebraic setting, see [BBM05, BBM07]. Recently, in a series of papers to appear [BG11a, BG11b], a framework for universal deformation formulæ for the actions of all negatively curved Kahlerian Lie groups has been successfully developed, both in the Fréchet and C^*-algebraic contexts. An extension of universal deformation formulæ to abelian Lie supergroups has also been performed in [BdGT10, dG11].

Midpoints and exponential Poisson manifolds

Another very important structure which appears in (1) and is somewhat hidden in (2), is coming from the map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (q, q') \mapsto \frac{q+q'}{2}$ which associates to q and q' their *midpoint* $\frac{q+q'}{2}$. This map is actually part of a global diffeomorphism, which we will call the *Weyl map*,

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n : (q, q') \mapsto \left(\frac{q+q'}{2}, q - q' \right) \tag{4}$$

sending two points to a pair consisting of their midpoint and the vector joining them. The notion of midpoint seems to be central to the understanding of the Weyl quantization, as evidenced by its appearence in two generalizations of the latter:

- Landsman [Lan98, Lan93] gave a strict deformation quantization, in a more general sense adapted from that of Rieffel [Rie93, Rie98], of the cotangent bundle of any Riemannian manifold Q. He defined a generalization of the Weyl operators (1) using, on the one hand, an analogue of the map (4) given in terms of the exponential map of the Levi-Cività connection, and on the other hand, some cut-off function to discard from the integral those pairs of points which are not joined by a unique minimizing geodesic.

It seems that the only hope for extracting a symbol calculus from these Weyl operators is that the generalized Weyl map be a global diffeomorphism from $Q \times Q$ to TQ.

- Wildberger [Wil05] gave a quantization of all locally compact Abelian groups, with a formula very similar to (2), under the condition that *multiplication by 2 be an automorphism*, meaning exactly that the notion of "half" a point, and thus of midpoint, should be well-defined.

There is a related notion, and a conjecture, that were expressed by Weinstein in [Wei91] after the examination of a number of examples of quantization.

Recall that a *Lie groupoid* is a manifold Γ together with two submersions $s, t : \Gamma \to M$ on a base manifold M, an embedding $1 : M \to \Gamma$, a diffeomorphism $\iota : \Gamma \to \Gamma$ and a smooth map $m : \Gamma_2 \to \Gamma : (g, h) \mapsto m(g, h) = gh$ defined on the space Γ_2 of pairs $(g, h) \in \Gamma \times \Gamma$ such that $s(g) = t(h)$. The axioms that these objects should satisfy are those that make Γ a small category in which all morphisms are isomorphisms, when viewing them respectively as the arrows, the source and target maps, the unit map, the inversion and the multiplication.

A *symplectic groupoid* is a Lie groupoid Γ together with a symplectic form, such that the graph $\{(g, h, gh) \mid (g, h) \in \Gamma_2\}$ of the multiplication is Lagrangian in $\Gamma \times \Gamma \times \overline{\Gamma}$, where $\overline{\Gamma}$ is Γ with the opposite symplectic form. The base manifold of a symplectic groupoid acquires a Poisson structure for which the source and target maps are Poisson and anti-Poisson maps, respectively.

The converse problem of building a symplectic groupoid whose base manifold is a given Poisson manifold is called the *integration problem* (and was completely solved by Crainic and Fernandes in 2002 [CF02]). When a Poisson manifold is integrable by a symplectic groupoid Γ, then the latter has a covering which is also a symplectic groupoid and for which the fibers of the source map are simply connected. That cover is then called *the s-simply connected symplectic groupoid* of P.

Definition. *A Poisson manifold P is of* exponential type *if its s-simply connected symplectic groupoid Γ exists and admits a diffeomorphism to T^*P which is symplectic and is $\mathbb{Z}_2$-equivariant with respect to the inversion in Γ and multiplication by -1 in T^*P.*

Weinstein made the *"conjecture that the property of exponential type is equivalent to the existence of a good quantization of the Poisson algebra $C^\infty(P)$ by a family (not just formal) of deformed multiplications on the same space."* He then added that, of course, the notion of good quantization should be made more precise and that the property of exponential type may have to be refined.

In the case of a symplectic manifold M, an integrating symplectic groupoid always exists, in the form of the *pair groupoid* $M \times \overline{M}$ where, as before, $\overline{M}$ is M with the opposite symplectic form. Recall that in that case, the multiplication is $(x,y).(y,z) = (x,z)$ and the inversion $\iota(x,y) = (y,x)$. The corresponding s-simply connected symplectic groupoid is the *fundamental groupoid* $\Pi(M)$, comprised of the homotopy classes of paths in M with fixed end-points, with the concatenation as multiplication. For M to be of exponential type, there should thus exist a $\mathbb{Z}_2$-equivariant symplectomorphism from $\Pi(M)$ to T^*M or, when M is simply connected, from $M \times \overline{M}$ to T^*M.

Quantization by groupoids and double triangles

The conjecture that we mentioned above was actually formulated as part of a broader programme. Around the end of the 1980', Karasev [Kar86, Kar89], Weinstein [CDW87, Wei91] and Zakrzewski [Zak90] independently invented symplectic groupoids and sketched a strategy to quantize Poisson manifolds. A basic idea was that to quantize such a manifold, one should actually quantize a symplectic groupoid integrating it.

More precisely, given a symplectic manifold Γ and some additional structure, geometric quantization produces a vector space (of polarized sections of a line bundle over Γ). When Γ is a symplectic groupoid integrating a Poisson manifold M, that vector space should acquire an algebra structure making it a quantization of M, along the lines of the following ideal dictionary:

Symplectic world	*Linear world*
Symplectic manifold M	Vector space
Opposite symplectic manifold $\overline{M}$	Dual vector space
Cartesian product	Tensor product
Lagrangian submanifold	Vector

Being a lagrangian submanifold of $\overline{\Gamma} \times \overline{\Gamma} \times \Gamma$, the graph of the groupoid multiplication should then quantize to an element of $A^* \otimes A^* \otimes A$ representing a bilinear map $A \otimes A \to A$ (part of the "ideal" character of the above dictionary lies in the absence of precise directions as to which completion of the algebraic tensor product should be used). If the correspondance is precise enough, the associativity of the groupoid multiplication should imply that of the new bilinear map. In the same way, the inversion should define an involution on A and the unit manifold Γ_0 should define a unit element in A.

In the quest for examples of such a quantization, Weinstein [Wei94] studied a particular class of symplectic manifolds, those having what he called a *com-*

plete symmetric structure, and proved a beautiful theorem about the symplectic geometry of the graph of the groupoid multiplication. We recall this now.

The main working example in any quantization programme is that of a cotangent bundle, so that it is natural to look first for symplectic manifolds M such that their pair groupoid $M \times \overline{M}$ is symplectomorphic to some cotangent bundle T^*N by a map ϕ. The lagrangian submanifolds L of the latter, when diffeomorphically projectable on the base N, would then correspond to some "WKB" functions (see [BW97] for more information on this) $e^{iS_L/\hbar}$ on N, possibly multiplied by a half-density, where the phase S_L is such that $L = dS_L(N)$. In order to end up with an algebra A of functions on the initial manifold M, we could further impose that $N = M$.

Finally, if we want the constant unit function and the complex conjugation to remain respectively the unit and the involution for the deformed product, we should also ask that $\phi(x,x) = 0_x$ and $\phi(x,y) = -\phi(y,x)$. In other words, ϕ should send the unit manifold of $M \times \overline{M}$ to the unit manifold of T^*M, and should act similarly for the inversion.

In particular, $\phi^{-1}(T^*_xM)$ is a lagrangian submanifold of $M \times \overline{M}$ which is preserved by the inversion and only intersects the diagonal at (x,x), hence it is the graph of a (local) involutive symplectomorphism s_x having x as its unique fixed point. This leads Weinstein to the following definition.

Definition. *A* symmetric structure *on a symplectic manifold (M,ω) is a family $\{s_x\}_{x\in M}$ of symplectic involutions for which x is the unique fixed point of s_x.*

By the cotangent coordinate theorem [Wei71, Theorem 7.1], there is a (unique) symplectomorphism ϕ from a neighbourhood of the diagonal in $M \times \overline{M}$ to a neighbourhood of the zero section in T^*M taking the graph of each s_x to the fiber T^*_xM.

Definition. *A symmetric structure is called* complete *if ϕ is a global symplectomorphism from $M \times \overline{M}$ to T^*M.*

Given a symmetric structure, the graph of the groupoid multiplication of $\Gamma = M \times \overline{M}$ may be identified with a lagrangian submanifold L of $T^*(M^3)$ through

$$\Gamma \times \Gamma \times \overline{\Gamma} \xrightarrow{\phi\times\phi\times\phi} T^*M \times T^*M \times \overline{T^*M} \xrightarrow{\mathrm{Id}\times\mathrm{Id}\times-\mathrm{Id}} T^*(M \times M \times M).$$

Weinstein proved that the graph of the groupoid multiplication may in fact be viewed as the graph of the differential of a function $S(x,y,z)$ on M^3 given by the symplectic area of a triangle for which x, y, z are the midpoints of the edges.

Theorem. *The lagrangian submanifold L is the graph of dS, where S is the multiple-valued function defined as follows on the subset $U \subset M \times M \times M$ consisting of those*

triples (x,y,z) for which $s_x s_y s_z$ has a fixed point. For each such fixed point A, $S(x,y,z)$ takes the value $-\int_\Sigma \omega$, where Σ is any surface in M bounded by a triangle consisting of paths from A to $B = s_x A$ to $C = s_y B$ to $A = s_z B$ which are symmetric under s_x, s_y, and s_z respectively.

The significance of this theorem for quantization is as follows. By the Schwartz kernel theorem, any bilinear continuous product on $C^\infty(M)$ may be written in the form

$$f * g(x) = \int_{M\times M} K(x,y,z) f(y) g(z) dy dz. \tag{5}$$

The quantization by groupoids programme amouts to the ansatz that the kernel K be of the form $K_\hbar(x,y,z) = \hbar^{-\dim M} A_\hbar(x,y,z) e^{iS(x,y,z)/\hbar}$, where the phase S is, as above, a function on M^3 such that the graph of its differential is the graph of the groupoid multiplication (recall the idea exposed at the beginning of this section, that a lagrangian submanifold should correspond to an element of the algebra, applied here to the graph of groupoid multiplication), and $A_\hbar$ is some smooth amplitude of the form $A_\hbar(x,y,z) = a_0(x,y,z) + \hbar a_1(x,y,z) + \dots$

Now the above theorem asserts that *the phase S of the product* (5) *should be the symplectic area of a double triangle*, where by a double triangle we mean a surface as in the theorem. Using a heuristic argument, Weinstein even conjectured a form for the amplitude, in which a_0 has an explicit formula, namely $a_0(x,y,z) = (-\det(\mathrm{Id} - (s_x s_y s_z)_{*_A}))^{-1/2}$, and all a_i for $i \geq 1$ are zero (but then added that this last assertion was probably wrong in general).

Symplectic symmetric spaces

A natural place where symmetric structures may be found is in *symplectic symmetric spaces*. Those spaces were first studied by P. Bieliavsky in [Bie95, BCG97], and are the natural analogues in symplectic geometry of the more customary Riemannian symmetric spaces.

Definition. *A* symplectic symmetric space *consists in a symplectic manifold (M,ω) together with a family of symplectic involutions $\{s_x\}_{x\in M}$ (called the symmetries) such that x is an isolated (not necessarily unique) fixed point of s_x, and such that $s_x(s_y z) = s_{(s_x y)}(s_x z)$ for all $x,y,z \in M$.*

Such a space has a unique symplectic connection for which the symmetries are affine maps and, when connected, is a homogeneous space of its automorphism group G, in such a way that the stabilizer K of any point lies between the fixed point set G^σ of an involution σ on the one hand, and its connected component containing the identity, $(G^\sigma)_0$, on the other hand. In other words, it

is a symmetric space in the usual sense [Hel78, KN96b]. Conversely, a usual symmetric space G/K endowed with a G-invariant symplectic form is a symplectic symmetric space in the above sense (see Chapter 1 for details).

When quantizing a manifold endowed with a group action, it is natural to ask that the quantization be equivariant with respect to the group action, in which case we speak of an invariant quantization. In what follows, we will always implictly look for quantizations of symplectic symmetric spaces which are invariant under the some transitive group of automorphisms.

Some symplectic symmetric spaces have been quantized by P. Bieliavsky [Bie02, BM01, Bie07], by completely explicit formulæ of the form (5), where the phase $S(x,y,z)$ is indeed equal to the area of a geodesic triangle whose edges have x,y,z as midpoints. However, the method used to obtain these formulæ was completely different from what was sketched above, and came from the $*$-representation theory of solvable Lie groups [AC85, AC90, ACL95] in formal deformation quantization.

To be more precise, the first class of such spaces that was quantized using this method, was that of *elementary solvable symplectic symmetric spaces* [Bie02]. Those spaces are the semi-direct products of two lagrangian abelian symmetric subspaces, and were meant to form the first step in an inductive construction of all solvable symmetric spaces through semi-direct products by flat factors. They are best described using symplectic triples $(\mathfrak{g},\sigma,\Omega)$, that is, involutive Lie algebras $(\mathfrak{g},\sigma)$ endowed with a 2-cocycle Ω vanishing on $\mathfrak{k}$ and non-degenerate on $\mathfrak{m}\times\mathfrak{m}$, where $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ is the decomposition in $+1$ and -1 eigenspaces for σ.

Definition. *An* elementary solvable symplectic triple *is a symplectic triple* $(\mathfrak{g},\sigma,\Omega)$ *where* $\mathfrak{g}$ *is a semi-direct product* $\mathfrak{g}=\mathfrak{a}\ltimes\mathfrak{b}$ *of two abelian Lie algebras stabilized by* σ *and* Ω *is a coboundary.*

This implies that $\mathfrak{b}=\mathfrak{l}\oplus\mathfrak{k}$ with $\mathfrak{l}\subset\mathfrak{m}$, and that $\mathfrak{a}$ and $\mathfrak{l}$ are two lagrangians in involution. To an elementary solvable symplectic triple, we associate a symplectic symmetric space $M=G/K$ where G is the connected and simply connected Lie group integrating $\mathfrak{g}$ and K is its connected Lie subgroup with Lie algebra $\mathfrak{k}$, with the G-invariant symplectic form whose value at eK is Ω.

A G-invariant quantization is then obtained through the following sequence of observations. First, since Ω is exact, G acts in a strongly hamiltonian way, and moreover there is a global Darboux chart for M. In this chart, the Moyal product $*^{\mathrm{M}}$ is *covariant* [ACMP83] in the sense that $[\lambda_X,\lambda_Y]_{*^{\mathrm{M}}}=\frac{\hbar}{i}\{\lambda_X,\lambda_Y\}$ for all $X,Y\in\mathfrak{g}$, where the λ_X, $X\in\mathfrak{g}$, are the momentum functions. This allows to define a representation $\rho:\mathfrak{g}\to\mathrm{End}(C^\infty(M)[[\hbar]]):X\mapsto[\lambda_X,\cdot]_{*^{\mathrm{M}}}$ by derivations of $*^{\mathrm{M}}$. This approach to representations has been applied to $SL(2,\mathbb{R})$ and the conformal groups in [BP02, BP04].

The main point is then to show that there exists a intertwiner T sending ρ to the representation $X \mapsto X^*$ by fundamental vector fields, so that the image of the Moyal product under this intertwiner is an associative product which is *invariant* under G instead of covariant. Applying the same intertwiner to the Weyl product yields an integral formula for the product, which may then be studied analytically.

In our case, an intertwiner is defined as follows. Consider $\mathcal{F}$ the partial Fourier transform in the $\mathfrak{l}$ variables and identify $\mathfrak{l}^*$ with $\mathfrak{a}$ via the symplectic form. Let $\xi \in \mathfrak{k}^*$ be an element such that $\Omega = \delta\xi$, denote $\eta : \mathfrak{a} \to \mathrm{End}(\mathfrak{b})$ the action of $\mathfrak{a}$ on $\mathfrak{b}$, and define $\phi : \mathfrak{a} \to \mathfrak{a}$ by

$$\Omega(\phi(a), l) = \xi(\sinh(\eta(a))l),$$

for all $a \in \mathfrak{a}$ and $l \in \mathfrak{l}$. Now, define the map $\phi_\hbar(a, a') = (a, \frac{1}{\hbar}\phi(\hbar a'))$ and finally the intertwiner T by

$$T = \mathcal{F}^{-1} \circ \phi_\hbar^* \circ \mathcal{F}.$$

As a remark, notice that is is shown in [Bie02] that the map ϕ is a global diffeomorphism if and only if, for some base point $o \in M$, each point $x \in M$ has a unique "midpoint" $\frac{x}{2}$ such that $s_{\frac{x}{2}} o = x$. This provides again evidence for the importance of the notion of midpoint in quantization.

When a symplectic symmetric space M admits a *simply transitive* actions of a subgroup S of automorphisms (i.e., a transitive and free action), any invariant quantization of M may be pulled-back to S as a left-invariant quantization. In that case, any sufficienty nice action of S on a manifold M' yields, at least formally, a deformation of the algebra of functions on that manifold. That approach to finding universal deformation formulæ for Lie group actions has been successfully pursued in [BBM05, BBM07, BBD$^+$, BG11a].

Aim of the thesis

The starting point of this thesis is our observation that the intertwiner T can be recast in the form

$$T = \mathcal{F}^{\omega/\hbar} \circ \varphi^* \circ \mathcal{F}^{\omega/\hbar}$$

where $\mathcal{F}^{\omega/\hbar}$ is the symplectic Fourier transform and $\varphi : M \to T_o M$ is a linear projection from M seen as a coadjoint orbit to its tangent space at some point o, and that this formula: 1) makes sense for any symplectic symmetric space having a global Darboux chart, and 2) is invertible if φ is so.

The aim of the present work is three-fold:

1. *to develop that reinterpretation*,
2. in order *to understand the geometry of the products* we just described, and
3. *to generalize them to large classes of solvable symplectic symmetric spaces*,

the ultimate goals being essentially the understanding of the three following points which we will barely touch upon:

1. the $*$-representation theoretic meaning of the intertwiner, and its relation to the $*$-exponential and the twisted convolution;
2. the interplay between the associativity of the product and the geometry of the space, in particular, the identification of some intrinsic properties a phase and an amplitude should satisfy in order for the corresponding product to be associative; and
3. the way in which this fits into the quantization by groupoids programme, as this would on the one hand shed some light on the geometry and meaning of these products, and on the other hand provide explicit and non-trivial test cases for that general strategy.

The central object that we would like to emphasize in this thesis is the coadjoint orbit projection, which should not be confused with another notion of orbit projection (on $\mathfrak{k}^*$ and not $\mathfrak{m}^*$) used in representation theory [DHV84].

In pursuing our three main goals, we will be led to spend more time studying the spaces than their quantization, and to develop some structural aspects of symmetric spaces and symplectic symmetric spaces that, beside for their own sake, are of crucial interest in quantization.

Results

We briefly expose here the results that we obtained. We refer the reader to the introduction of each chapter for more background and motivation concerning their respective subjects.

Midpoints and exponentiality

As we mentioned earlier in this introduction, the notion of midpoint seems to be very much related to the existence of a non-formal deformation quantization by functions on the initial phase space. Therefore, in Chapter 2, we study the notion of *midpoint map* on a general symmetric space, building on earlier work by Qian [Qia97].

Definition. *Let (M,s) be a symmetric space. A smooth map $\gamma : M \times M \to M$ is called a* midpoint map *if for all $x \in M$, $\gamma^{-1}(x) = \mathrm{Graph}(s_x)$.*

A sufficient condition for the existence of a midpoint map is that the symmetric space be exponential in the following strong sense (see the introduction to Chapter 2 for a short review of other notions of exponentiality for Lie groups and symmetric spaces).

Definition. *A symmetric space (M,s) is called* exponential *if for all $x \in M$ the exponential map at x for the canonical connection is a diffeomorphism from T_xM onto M.*

Recall that a pair (G,σ) is a *symmetric pair* if G is a Lie group and σ an involutive automorphism of G. Given such a pair, we denote G^σ the set of fixed points of σ, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the decomposition of the Lie algebra $\mathfrak{g}$ of G in $+1$ and -1 eigenspaces of σ. Realizing a symmetric space M as a homogeneous space G/G^σ for a symmetric pair (G,σ), the exponentiality of M implies the following property.

Definition. *A symmetric pair (G,σ) is said* to have property $(\mathfrak{m})$ *if G/G^σ is connected and simply connected and if no* $\operatorname{ad} X$, *$X \in \mathfrak{m}$, has non-zero purely imaginary eigenvalues.*

Our first main result for this chapter is the following characterization of exponential symmetric spaces, generalizing the well-known characterization of exponential solvable Lie groups proved by Dixmier [Dix57] and Saito [Sai57].

Theorem (2.2.20)**.** *Let (M,s) be a symmetric space. Then the following conditions are equivalent:*

1. *M is connected and admits a midpoint map,*
2. *M is exponential,*
3. *there exists a symmetric pair (G,σ) having property $(\mathfrak{m})$ and such that $M \cong G/G^\sigma$,*
4. *every symmetric pair (G,σ) such that $M \cong G/G^\sigma$, has property $(\mathfrak{m})$.*

Moreover, in case any of these four conditions is satisfied,

i. *if (M,s) is a solvable symmetric space, then there exists a symmetric pair (G,σ) such that $G/G^\sigma \cong M$ with G a solvable exponential Lie group,*
ii. *if (M,s) is a semisimple symmetric space, then there exists a symmetric pair (G,σ) such that $G/G^\sigma \cong M$ with G semisimple and σ a Cartan involution.*

For the last two assertions, a symmetric space is said to be solvable (resp. semisimple) if its transvection group (see Section 1.4) is solvable (resp. semisimple).

Using the same technique, we then investigate the symmetric spaces in which each triangle with vertices, say, x, y, z, gives rise to a unique "double triangle" having x, y, z as midpoints of its edges. A symmetric space will have that property if and only if the following map is a diffeomorphism.

Definition. *Let (M, s, γ) be a symmetric space with midpoint map. The map $\gamma_3 : M \times M \times M \to M \times M \times M$ is defined by*

$$\gamma_3(x, y, z) = (\gamma(z, x), \gamma(x, y), \gamma(y, z))$$

for all $x, y, z \in M$.

Our second main result is the following.

Theorem (2.3.12)**.** *Let (M, s, γ) be a connected symmetric space with midpoint map. Then γ_3 is a diffeomorphism if and only if (M, s) is a solvable symmetric space.*

As a first step towards the realization of Weinstein's programme for symplectic symmetric spaces, we may want to restrict ourselves to the spaces for which the phase S is everywhere defined and single-valued. The last theorem above tells us that we should then consider the class of exponential solvable symmetric spaces.

Structure theory

In Chapter 3, we introduce the basis for a structure theory for solvable symplectic symmetric spaces.

Our main results in this chapter consist in a definition and a procedure. The first being that of a *primitive structure* for a Lie triple system and the second being a *reduction procedure* for symplectic symmetric spaces combining the Marsden-Weinstein-Meyer reduction and the theory of normal symmetric subspaces. These definition and procedure are inspired from, and are natural analogues of the corresponding notions for symplectic Lie groups, developed by Dardié, Medina and Revoy in [MR91, DM96a].

Definition (3.3.2)**.** *Let $\mathfrak{m}$ be a vector space. A* primitive system *on $\mathfrak{m}$ is a trilinear product $(\cdot, \cdot, \cdot) : \mathfrak{m}^3 \to \mathfrak{m}$ such that, denoting $L_{x,y} z = [x, y, z] = (x, y, z) - (y, x, z)$,*

1. $(x, y, z) = (z, y, x)$,
2. $L_{u,v}(x, y, z) = (L_{u,v} x, y, z) + (x, L_{u,v} y, z) + (x, y, L_{u,v} z)$,

3. $(L_{u,v}x, y, z) + (x, y, L_{u,v}z) = ((x, u, z), y, v) - (u, y, (x, v, z))$,

for all $x, y, z, u, v \in \mathfrak{m}$.

Every primitive system $(\cdot,\cdot,\cdot)$ gives rise to a Lie triple system $[\cdot,\cdot,\cdot]$ through antisymmetrization in the first two variables: $[x, y, z] = (x, y, z) - (y, x, z)$. Conversely, we show that any symplectic symmetric space canonically has such a structure on each of its tangent spaces:

Proposition (3.3.7)**.** *Let* $(\mathfrak{m}, [\cdot,\cdot,\cdot], \Omega)$ *be a symplectic Lie triple system, that is, a Lie triple system* $\mathfrak{m}$ *with a closed non-degenerate 2-form* Ω *such that*

$$\Omega(L_{x,y}u, v) + \Omega(u, L_{x,y}v) = 0$$

for all $x, y, u, v \in \mathfrak{m}$*. Then there is a canonical primitive structure* $(\cdot,\cdot,\cdot)$ *on* $\mathfrak{m}$*, defined by*

$$\Omega((x, y, z), t) = -\Omega(x, [t, y, z]) = \Omega([t, y, x], z)$$

for all $x, y, z, t \in \mathfrak{m}$.

This allows to provide the following notion of T^**-extension* for symplectic symmetric spaces, originally introduced by Bordemann for non-associative algebras with a non-degenerate bilinear form [Bor97], and then adapted to metrised Lie triple systems by Lin *et al.* [LWD09]. This construction also appears as a particular case of a so-called "double extension" procedure in the theory of symplectic Lie groups (see Section 3.2) [DM96a].

Theorem (3.3.10)**.** *Let* $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ *be a Lie triple system, and consider its cotangent space* $T^*\mathfrak{m} = \mathfrak{m} \oplus \mathfrak{m}^*$ *with the standard symplectic structure* Ω *defined by*

$$\Omega((x, \alpha), (y, \beta)) = \alpha(y) - \beta(x)$$

for all $x, y \in \mathfrak{m}$ *and* $\alpha, \beta \in \mathfrak{m}^*$*. Then there is a one-to-one correspondance between the structures of symplectic Lie triple system on* $(T^*\mathfrak{m}, \Omega)$ *in which* $\mathfrak{m}^*$ *is an ideal and* $T^*\mathfrak{m}/\mathfrak{m}^*$ *is isomorphic to* $\mathfrak{m}$*, and the primitive structures on* $\mathfrak{m}$ *together with a 3-cocycle* $C \in C^3(\mathfrak{m}, \mathfrak{m}^*)$ *(see Section 1.4) such that*

$$C_{x,y,z}t = C_{x,y,t}z \tag{6}$$

for all $x, y, z, t \in \mathfrak{m}$.

If $(\cdot,\cdot,\cdot)$ *is such a primitive structure and* $C : (x, y, z) \mapsto C_{x,y,z}$ *an element of* $C^3(\mathfrak{m}, \mathfrak{m}^*)$ *satisfying* (6)*, denoting* $\lambda_{x,y} = (x, y, \cdot)$ *and* $L_{x,y} = [x, y, \cdot] = \lambda_{x,y} - \lambda_{y,x}$*, the*

corresponding Lie triple system is defined by

$$[(x,\alpha),(y,\beta),(z,\gamma)] = ([x,y,z], C_{x,y,z} + \beta \circ \lambda_{z,x} - \alpha \circ \lambda_{z,y} - \gamma \circ L_{x,y})$$

for all $x,y,z \in \mathfrak{m}$ *and* $\alpha,\beta,\gamma \in \mathfrak{m}^*$. *It is called the* T^*-extension *of the primitive system* $(\mathfrak{m},(\cdot,\cdot,\cdot))$ *by the 3-cocycle* C. *The extension is said to be* split *if* $C = 0$.

Theorem (3.3.12)**.** *Every symplectic Lie triple system having a lagrangian ideal (resp. and such that there is a complementary lagrangian subsystem) is isomorphic to a* T^**-extension (resp. a split* T^**-extension).*

We next turn to the reduction procedure that we mentioned. The idea is that if N is a normal subspace of a symmetric space M (see Section 1.4) through some base point o, then there exists an "orthogonal" subspace $N^\perp$ of M through o, with the property that the tangent spaces at o of N and $N^\perp$ are the symplectic orthogonal of each other in T_oM. Then $N^\perp/(N^\perp \cap N)$ can be shown to have a structure of symplectic symmetric space which is both a symplectic and a symmetric reduction of M.

Proposition (3.4.4)**.** *Let* (M,s,ω,o) *be a pointed connected and simply connected symplectic symmetric space, and* N *be a closed normal subspace of* M. *Let* G *be (the central extension by* ω *of) the automorphism group of* (M,s,ω), *acting in a strongly Hamiltonian manner on* M *with momentum map* J *(see Section 1.3). Let* σ *be the conjugation by* s_o *in* G, *and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *be the associated decomposition. Let* $\mathfrak{n}$ *be the subspace of* $\mathfrak{m}$ *corresponding to* T_oN, *and let* $\mathfrak{h}$ *be the smallest ideal of* $\mathfrak{g}$ *containing* $\mathfrak{n}$, *defined by* $\mathfrak{h} = \mathfrak{n} + [\mathfrak{m},\mathfrak{n}]$. *Denote* $\iota_\mathfrak{h} : \mathfrak{h} \to \mathfrak{g}$ *the inclusion,* $J^\mathfrak{h} = \iota_\mathfrak{h}^* \circ J : M \to \mathfrak{h}^*$ *the corresponding projection of* J, *and* μ *the image of* o *by* $J^\mathfrak{h}$. *Then*

$$N^\perp = \left(J^\mathfrak{h}\right)^{-1}(\mu)$$

is a closed symmetric subspace of M *such that* $T_o(N^\perp) = (T_oN)^\perp$. *It is independent of the choice of* J, *and we call it the* orthogonal subspace *to* N *at* o *in* M.

Theorem (3.4.5)**.** *Let* (M,s,ω,o) *be a pointed connected and simply connected symplectic symmetric space, and let* N *be a closed normal symmetric subspace. Then* $M' = N^\perp/(N^\perp \cap N)$ *is a symplectic symmetric space with symplectic form* ω' *such that*

$$\pi^*\omega' = \iota^*\omega$$

where $\pi : N^\perp \to M'$ *is the natural projection, and* $\iota : N^\perp \to M$ *the natural inclusion.*

When a symmetric space M has a simply transitive action of a subgroup of its automorphism group, then the latter becomes a symplectic Lie group (that is,

it has a left-invariant symplectic form, see Section 3.2). We end the chapter by showing that, under some hypothesis, the group and the symmetric space can be both reduced in such a way that the reduced group still acts simply transitively on the reduced space.

Definition. *A* combined algebra *is a Lie admissible algebra* $(\mathfrak{m}, \diamond)$ *such that* $L_x = x\diamond\cdot$ *is a derivation of* $[\cdot,\cdot,\cdot]^\diamond$*, for all* $x \in \mathfrak{m}$*.*

If moreover, $\mathfrak{m}$ *is endowed with a symplectic form* Ω *which is* L_x*-invariant, for all* $x \in \mathfrak{m}$*:*

$$\Omega(x \diamond y, z) + \Omega(y, x \diamond z) = 0, \quad \textit{for all } x, y, z \in \mathfrak{m},$$

then $(\mathfrak{m}, \diamond, \Omega)$ *is said to be a* symplectic combined algebra.

Proposition (3.4.12)**.** *Let* $(\mathfrak{m}, \diamond, \Omega)$ *be a symplectic combined algebra. Let* $\mathfrak{n}$ *be a two-sided ideal of* $(\mathfrak{m}, \diamond)$*. Then,*

1. $\mathfrak{m} \diamond \mathfrak{n}^\perp \subset \mathfrak{n}^\perp$ *and* $\mathfrak{n} \diamond \mathfrak{n}^\perp = 0$.

As a consequence, with $\mathfrak{m}' = \mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$*, and* $\diamond'$ *(resp.* Ω'*) the image of* $\diamond$ *(resp.* Ω*) in* $\mathfrak{m}'$*,*

2. $(\mathfrak{m}', \diamond', \Omega')$ *is a well-defined symplectic combined algebra,*
3. $(\mathfrak{m}', [\cdot,\cdot]^{\diamond'}, \Omega')$ *is the symplectic reduction of* $(\mathfrak{m}, [\cdot,\cdot]^\diamond, \Omega)$ *by* $\mathfrak{n}$*, and*
4. $(\mathfrak{m}', [\cdot,\cdot,\cdot]^{\diamond'}, \Omega')$ *is the symplectic reduction of* $(\mathfrak{m}, [\cdot,\cdot,\cdot]^\diamond, \Omega)$ *by* $\mathfrak{n}$*.*

Hence when a symplectic combined algebra has a two-sided ideal, the structure of the corresponding (locally) simply transitive group of automorphism is somehow similar to that of the space on which it acts. This is in general not the case since, for example, all Hermitian symmetric spaces of the non-compact type have a solvable simply transitive group of automorphism (a normal j-algebra, see [Pya69]), whereas they are themselves semisimple so that they cannot be reduced.

In general, when performing a reduction of a symplectic Lie triple system by an isotropic ideal $\mathfrak{n}$, the orthogonal subspace $\mathfrak{n}^\perp$ is not an ideal itself, nor is it complemented by a subspace $\mathfrak{a}$ so as to have a semi-direct product $\mathfrak{a} \ltimes \mathfrak{n}^\perp$. When it is the case, we speak of a *double extension* of $\mathfrak{n}^\perp/\mathfrak{n}$ by $\mathfrak{a}$, and if moreover, $\mathfrak{n}$ is central in $\mathfrak{m}$, we speak of a classical double extension (Definitions 3.5.1 and 3.5.2), adapting thereby the analoguous notions for symplectic Lie algebras [MR91, DM96a].

We prove that large classes of spaces may be described by repeated application of such (classical) double extensions. Our first result is an analogue of

a theorem proved in [LWD09] for metrised Lie triple systems. The analogue in the context of symplectic Lie algebras is only known for Kahler Lie algebras [DM96b], where the complex structure canonically gives the desired structure.

Theorem (3.5.4)**.** *Every completely solvable symplectic Lie triple system is the result of a sequence of double extensions by one-dimensional primitive systems, starting from the zero-dimensional system.*

Our second result is analoguous to the symplectic Lie algebra case [MR91].

Theorem (3.5.5)**.** *Every nilpotent symplectic Lie triple system is the result of a sequence of classical double extensions starting from the zero-dimensional system.*

Projection

In Chapter 4, we begin the actual study of the projection.

Let (M, s, ω) be a connected and simply connected symplectic symmetric space, let $o \in M$ be some base point, and let G be the central extension of its automorphism group by ω_o (see Section 1.3).

Definition (4.1.5 and 4.1.6)**.** *The* canonical map $\phi : M \times M \to TM$ *is defined by*

$$\phi(x,y) = \phi_x(y) = J_{*x}^{-1}\left(\frac{J(y) - J(s_x y)}{2}\right).$$

for any G-equivariant momentum map J. If γ is a midpoint map on M, the Ševera projection $\check{\phi} : M \times M \to T^*M$ *is defined by*

$$\check{\phi}(x,y) = \omega_{\gamma(x,y)}\left(J_{*\gamma(x,y)}^{-1}(J(y) - J(x)), \cdot\right)$$

for any G-equivariant momentum map J.

We show that the above definitions make sense, and are in fact independent of the group G chosen amongst (the appropriate central extensions of) the transitive subgroups of automorphisms stabilized by σ (the conjugation $g \mapsto \sigma(g) = s_o g s_o$).

We then study its behaviour under morphisms.

Proposition (4.3.1)**.** *Let M, M' be two connected and simply connected symplectic symmetric spaces, let ϕ, ϕ' be the corresponding canonical maps, and let $f : M \to M'$ be a morphism of symplectic symmetric spaces. For each $x \in M$, denote $p_x : T_{f(x)}M' \to f_{*_x}(T_xM)$ the projector on $f_{*_x}(T_xM)$ parallel to $f_{*_x}(T_xM)^\perp$, and $\left(f_{*_x}\right)^{-1}$ the left inverse of f_{*_x} defined by $\left(f_{*_x}\right)^{-1}\Big|_{f_{*_x}(T_xM)} \circ p_x$. Then*

$$\phi = f^*\phi',$$

where $(f^*\phi')(x,y) = \left(f_{*_x}\right)^{-1} \circ \phi'(f(x), f(y))$.

This implies that the projection is natural under isomorphisms, inclusions of subspaces, and direct products (Corollaries 4.3.6, 4.3.7 and 4.3.8). This also proves the following covariance result.

Corollary (4.3.9)**.** *Let* (M,s) *be a connected and simply connected symmetric space such that there exists a symplectic form* ω *such that* (M,s,ω) *is a symplectic symmetric space. For each such* ω*, denote* ϕ^ω *the corresponding canonical map. Then, for each* $A \in \mathrm{Aut}(M,s)$*, we have*

$$A^*\phi^\omega = \phi^{A^*\omega}.$$

We then show that the projection really reflects the structure of the symplectic symmetric space in the sense that, when performing a symplectic reduction, one obtains the canonical map for the reduced space by restriction and linear projection from the canonical map of the initial space.

Corollary (4.4.2)**.** *Let* (M,s,ω,o) *be a pointed, connected and simply connected symplectic symmetric space, and let* N *be a closed connected normal subspace through* o*. Denote* C *the congruence of* $N^\perp$ *corresponding to* $N^\perp \cap N$*. Denote* $\phi_o : M \to T_oM$ *(resp.* $\phi'_{o'} : M' \to T_{o'}M'$*) the canonical projection of* M *at* o *(resp. of* $M' = N^\perp/(N^\perp \cap N)$ *at* o'*, the image of* o *in* M'*). Denote* $\pi_{\mathfrak{n}} : \mathfrak{m} \to \mathfrak{m}/\mathfrak{n}$ *the projection, and* $i : N^\perp \to M$ *the inclusion. Then the map* $\pi_{\mathfrak{n}} \circ \phi_o \circ i$

1. *takes values in* $\mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$,
2. *only depends on the equivalence classes of* C*, and*
3. *passes to the quotient* M' *as* $\phi'_{o'}$.

Concerning the regularity of the canonical projection and the relation with the exponentiality, we have the following result.

Theorem (4.5.1)**.** *Let* (M,s,ω) *be a connected and simply connected symplectic symmetric space. Then the Ševera projection* $\check{\phi} : M \times \overline{M} \to T^*M$ *is everywhere a local diffeomorphism if and only if* M *is exponential. If, moreover,* M *is semisimple or solvable, then* $\check{\phi}$ *is a global diffeomorphism.*

Since Ševera proved in [Ševera06] that his projection is a symplectomorphism, this gives a wealth of examples of complete symmetric structures in the sense of Weinstein.

By merely combining the above result of Ševera and others of Qian [Qia97], we observe that the Ševera projection completely determines the differential of the area of the double triangle.

Theorem (4.6.14). *Let (M, s, γ) be a symmetric space with midpoint map such that γ_3 is a diffeomorphism. Then the Ševera projection $\phi : M \times M \to T^*M$ and the associated Liouville 1-form $\theta = \check{\phi}^*\alpha$ are such that, for all $x, y, z \in M$, and identifying $(T^*M)^3$ with $T^*(M^3)$,*

$$\phi \times \phi \times \phi((x,y),(y,z),(z,x)) = ((u,v,w), dS(u,v,w))$$
$$i_{\mathcal{M}}^*(-\theta \times \theta \times \theta) = i_{\mathcal{M}}^* \circ (\gamma \times \gamma \times \gamma)^*(dS),$$

where $u = \gamma(x,y)$, $v = \gamma(y,z)$ and $w = \gamma(z,x)$, $\mathcal{M}$ is the graph of the groupoid multiplication and S is the Weinstein phase function.

We conclude by giving in Section 4.7 an equivalent definition of the Ševera projection in terms of the fundamental groupoid of a symplectic symmetric space. This definition is cleaner, explicitly canonical, and promising concerning the geometric content of the projection.

A new class of examples

In the Chapters 5 and 6 we quantize explicitly a new class of examples based on elementary normal Lie algebras [Pya69, AP97], using the canonical projection.

We compute the automorphism group of the symmetric pairs associated to the examples (Proposition 5.4.3), as well as their orbits in the space of invariant symplectic forms (Proposition 5.4.6).

We then provide global Darboux charts and compute the canonical projection in these charts (Propositions 5.5.1 and 5.5.3). With this, we compute the twist T, show that it intertwines the $*$-representation with the representation by fundamental vector fields, and compute also the explicit form of the product:

Theorem (5.8.1). *The product $*$ on $(M, s, \omega = \omega^{\lambda,\lambda w})$ defined by*

$$f * g = (T_{\epsilon/2})^{-1}\left(T_{\epsilon/2} f *^M_{\epsilon/2} T_{\epsilon/2} g\right)$$

is an $\mathrm{Aut}(M, s, \omega^{\lambda,\lambda w})$-invariant associative WKB product in the sense of [Bie02] given, in the Darboux coordinates $I^w : (\mathfrak{m}, \Omega^{\lambda,0}) \to (M, \omega^{\lambda,\lambda w})$, by

$$f * g(x_0) = \frac{1}{(\pi\epsilon)^{\dim M}} \int_{M\times M} e^{\frac{2i}{\epsilon} S(x_0,x_1,x_2)} A(x_0,x_1,x_2) f(x_1) g(x_2) d^{\Omega}x_1 d^{\Omega}x_2$$

where the phase S and the amplitude A are given by

$$S(x_0,x_1,x_2) = \lambda \bigoplus_{0,1,2} \Big(\Omega_V(\cosh(a_2 - a_0)v_1, \cosh(a_0 - a_1)v_2)$$
$$- \frac{1}{2}\sinh(2(a_1 - a_2))z_0$$

$$-\Big(\sinh(a_1-a_2)-\frac{1}{2}\sinh(2(a_1-a_2))\Big)\Omega_V(w,v_0)\Big)$$
$$A(x_0,x_1,x_2)=\cosh\left(2(a_2-a_1)\right)\cosh\left(a_2-a_1\right)^{2n-2}$$

Moreover the assignment $\Omega \mapsto *^{\Omega}$ *is equivariant with respect to the actions of* $\mathrm{Aut}(M,s)$.

We finally compute the area of the double triangle, and conclude that the phase we obtained in the previous theorem is indeed equal to that area.

Chapter 6, the last chapter, is devoted to showing that the new framework developed in [BG11a] applies in our case, so that the product obtained above makes sense as an oscillatory integral for very large classes of symbols, and provides a *universal deformation formula* for the strongly continuous tempered actions of some simply transitive Lie groups of automorphisms of the spaces quantized in Theorem 5.8.1.

We essentially follow, and modify where necessary, the arguments in [BG11a]. The main point of this chapter is to indicate how the methods developed in Chapters 2, 3 and 4 could be used to show that the central conditions of *temperedness* and *admissibility* in the above paper should hold in a quite general context.

Chapter 1

Preliminaries

1.1 Conventions

Unless the contrary is mentioned, and without explicitly defining it, the Lie algebra of a Lie group is always denoted by the lower case gothic letter corresponding to the uppercase Latin letter denoting the group. Thus, for example, the Lie algebra of a Lie group F, G, H, or K is denoted respectively $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, or $\mathfrak{k}$. The (vector space) dual of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is denoted $\mathfrak{g}^*$, and the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{K}$ is written as $(\alpha, X) \mapsto \langle \alpha, X \rangle$ or $\alpha(X)$. A Lie group G acts on itself by *conjugation*

$$\mathrm{C}_g(h) = ghg^{-1},$$

on $\mathfrak{g}$ through the *adjoint action*

$$\mathrm{Ad}_g(X) = (\mathrm{C}_g)_{*_e}(X),$$

and on $\mathfrak{g}^*$ through the *coadjoint action*

$$\mathrm{Ad}^*_g(\alpha) = \alpha \circ \mathrm{Ad}_{g^{-1}}.$$

We will often drop the parentheses and write $\mathrm{C}_g\, h$, $\mathrm{Ad}_g\, X$ and $\mathrm{Ad}^*_g\, \alpha$. The corresponding *Lie algebra adjoint and coadjoint actions* are

$$\mathrm{ad}_X\, Y = \mathrm{ad}(X)Y = [X, Y]$$

and

$$\mathrm{ad}^*_X\, \alpha = \mathrm{ad}^*(X)\alpha = -\alpha \circ \mathrm{ad}_X.$$

Let M be a manifold, $x \in M$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The tangent space to M at x is

denoted by T_xM and its dual by T_x^*M. If $f : M \to \mathbb{K}$ is a smooth function, its differential at x is denoted by $(f_*)_x$ or f_{*x}, or $(df)_x$ in the context of differential forms. The set of smooth functions on M is denoted $C^\infty(M)$ if $\mathbb{K} = \mathbb{R}$, and $C^\infty_{\mathbb{C}}(M)$ if $\mathbb{K} = \mathbb{C}$. Unless explicitly mentioned, the base field is $\mathbb{R}$. The action on f of a vector field X on M is written $X.f$, and its value at x is $(X.f)(x) = X_x.f = f_{*x}(X_x)$. The *Lie derivative* of a tensor α on M in the direction X is written $L_X\alpha$. Recall *Cartan's formula* for the Lie derivative of a differential form:

$$L_X\alpha = d\iota_X\alpha + \iota_X d\alpha,$$

where d is the exterior derivative, and ι_X the contraction with the vector field X, defined on n-forms by

$$\iota_X\alpha(X_1,\dots,X_{n-1}) = \alpha(X, X_1,\dots,X_n).$$

1.2 Symplectic actions

A 2-form ω on M is said to be *symplectic* if it is closed and non-degenerate, and (M,ω) is then called a *symplectic manifold*. By non-degeneracy, a smooth function f on (M,ω) generates a *Hamiltonian vector field* X_f through the formula

$$df = \iota_{X_f}\omega.$$

This defines a *Poisson bracket* on M by

$$\{f,g\} = -\omega(X_f, X_g) = -X_g.f = X_f.g,$$

that is, a Lie algebra structure on $C^\infty(M)$ satsifying the Leibniz identity $\{fg,h\} = f\{g,h\} + \{f,h\}g$. A function f such that $df = \iota_X\omega$ for some vector field X, is called a *Hamiltonian function* of X.

If G is a Lie group acting on (M,ω), the action $G \times M \to M : (g,x) \mapsto g \cdot x$ is said to be *symplectic* if it preserves the 2-form ω:

$$g^*\omega = \omega$$

for all $g \in G$. The action on M gives rise to the left-regular representation on $C^\infty(M)$, defined by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

Hence the *fundamental vector fields* X^* of the action, defined by

$$X^*.f = \frac{d}{dt}\exp(tX)\cdot f\Big|_{t=0}$$

for all $X \in \mathfrak{g}$, form a Lie algebra representation by vector fields on M, in the sense that $[X^*, Y^*] = [X, Y]^*$ for all $X, Y \in \mathfrak{g}$. These vector fields preserve the symplectic form: $L_{X^*}\omega = 0$, so that by Cartan's formula,

$$0 = d\iota_{X^*}\omega + \iota_{X^*}d\omega = d\iota_{X^*}\omega,$$

that is, $\iota_{X^*}\omega$ is a closed 1-form. If, for all $X \in \mathfrak{g}$, the fundamental vector field X^* is a Hamiltonian vector field for some Hamiltonian function $-\lambda_X$, so that

$$X_{\lambda_X} = -X^*,$$

then the action is called *Hamiltonian*. In that case, the vector field $-X^* = X_{\lambda_X}$ is said to be the *Hamiltonian vector field*, and λ_X the *Hamiltonian function*, associated to $X \in \mathfrak{g}$. If, moreover, the map $X \mapsto \lambda_X$ is a Lie algebra morphism from $\mathfrak{g}$ to $C^\infty(M)$, that is, if

$$\{\lambda_X, \lambda_Y\} = \lambda_{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$, then the action is called *strongly Hamiltonian*.

For a Hamiltonian action, we may collect all Hamiltonian functions of the action into a map $J : M \to \mathfrak{g}^*$ by

$$\langle J(x), X\rangle = \lambda_X(x)$$

for all $x \in M$ and $X \in \mathfrak{g}$. It is a *momentum map* in the sense that

$$d\langle J(x), X\rangle = \iota_{(-X^*)}\omega.$$

If the action is strongly Hamiltonian and G is connected, then J is *G-equivariant* in the sense that

$$J(g\cdot x) = \mathrm{Ad}^*_g J(x).$$

Hence the orbit of x in M is sent by J equivariantly onto the coadjoint orbit of $J(x)$ in $\mathfrak{g}^*$, so that $J : G\cdot x \to \mathrm{Ad}^*_G J(x)$ is a surjective submersion. If the action of G is transitive, we call (M, ω) a *homogeneous symplectic space* for G.

The "Hamiltonian vector fields map" $A : \mathfrak{g} \to \mathrm{Ham}(M) \subset \Gamma(TM)$ may be seen as an element of $\Gamma(TM)\otimes\mathfrak{g}^*$. Similarly the differential dJ of a momentum map $J : M \to \mathfrak{g}^*$ may be seen as an element of $\Gamma(T^*M)\otimes\mathfrak{g}^*$. With $\omega^\flat : \Gamma(TM) \to \Gamma(T^*M)$ the symplectic isomorphism defined by $\omega^\flat(X) = \iota_X\omega$, the relation defining a

momentum map is thus

$$dJ = (\omega^\flat \otimes \mathrm{Id})(A).$$

If J and $\tilde{J}$ are two momentum maps, we must then have $d(J - \tilde{J}) = 0$. If M is connected, this shows that they differ by a constant $\eta \in \mathfrak{g}^*$. If moreover they are G equivariant, then

$$\mathrm{Ad}^*_g J(x) + \eta = J(g \cdot x) + \eta = \tilde{J}(g \cdot x) = \mathrm{Ad}^*_g \tilde{J}(x) = \mathrm{Ad}^*_g J(x) + \mathrm{Ad}^*_g \eta,$$

so that η must be invariant under Ad^*_G. Conversely if J is a G-equivariant momentum map and if $\eta \in \mathfrak{g}^*$ is a fixed point of Ad^*_G, then $J + \eta$ is clearly a G-equivariant momentum map. Thus the set of G-equivariant momentum maps for a Hamiltonian action on a connected manifold is an affine subspace of $\mathfrak{g}^*$ parallel to the vector space of fixed points of Ad^*_G.

1.3 Homogeneous symplectic spaces

Let (M, ω) be a homogeneous symplectic space for a connected Lie group G. Denote $\alpha : G \times M \to M : (g, x) \mapsto \alpha(g, x) = \alpha_g(x) = \alpha^x(g) = g \cdot x$ the action, and choose some base point $o \in M$. Then the symplectic form ω induces a closed left-invariant 2-form on G through the formula $\omega_G = (\alpha^o)^* \omega$. Since it is closed and left-invariant, the value of ω_G at the identity is thus a 2-cocycle Ω (for the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{R}$):

$$\Omega([X, Y], Z) + \Omega([Y, Z], X) + \Omega([Z, X], Y) = 0 \tag{1.1}$$

for all $X, Y, Z \in \mathfrak{g}$. We may then perform the central extension $\mathfrak{h}$ of $\mathfrak{g}$ by Ω, which we will identify with $\mathfrak{g} \times \mathbb{R}$ with the bracket

$$[(X, t), (X', t')] = ([X, X'], \Omega(X, X')).$$

Such a Lie algebra central extension always exists, but it is not necessarily the case for the corresponding Lie group central extension. We want to show that when M is connected and simply connected, the Lie algebra central extension always lifts to a Lie group central extension.

Recall first the following result of K.-H. Neeb.

Theorem 1.3.1 ([Nee96, Theorem II.4 (i,iv)])**.** *Let $\mathfrak{g}$ be a Lie algebra, $\Omega \in \Lambda^2(\mathfrak{g}^*)$ a two-cocycle on $\mathfrak{g}$, and $\mathfrak{h}$ the central extension of $\mathfrak{g}$ by Ω. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and denote $\lambda = (0, 1) \in \mathfrak{h}^*$. Then there exists a central*

extension H of G corresponding to the Lie algebra extension $\mathfrak{h}$ of $\mathfrak{g}$ if and only if $\pi_1(G)$ fixes the functional $\lambda \in \mathfrak{h}^*$.

Proposition 1.3.2. *If M is connected and simply connected, then the Lie algebra central extension always lifts to a Lie group central extension.*

Proof. For a Lie group L, denoting $q_L : \tilde{L} \to L$ its universal covering, we may identify $\pi_1(L)$ with $\ker q_L$. Let K be the stabilizer of $o \in M$ in G. Since G is connected and M is simply connected, K is connected. Denote K' the connected subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$. We have $M \cong G/K \cong \tilde{G}/K'$.

Since M is simply connected, the long exact sequence for the fibration $K \to G \to M$

$$\cdots \to \pi_2(M) \to \pi_1(K) \xrightarrow{f} \pi_1(G) \to \pi_1(M) \to \cdots.$$

implies that $f : \pi_1(K) \to \pi_1(G)$ is surjective. But the latter map can be seen as follows: from the inclusion $\mathfrak{k} \hookrightarrow \mathrm{Lie}(G)$, Lie's third theorem yields unique morphisms $j_G : \tilde{K} \to G$ and $j_{\tilde{G}} : \tilde{K} \to \tilde{G}$ making the following diagram commute

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{j_{\tilde{G}}} & \tilde{G} \\ {\scriptstyle q_K}\downarrow & \searrow^{j_G} & \downarrow{\scriptstyle q_G} \\ K & \xhookrightarrow{i} & G \end{array}$$

and the map $j_{\tilde{G}}|_{\ker q_K}$ realizes f.

The image $j_{\tilde{G}}(\tilde{K})$ is a connected subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$, hence it coincides with K'. And since $f = j_{\tilde{G}}|_{\ker q_K}$ is surjective, we obtain that $\pi_1(G) = \mathrm{Im}(f) \subset K'$.

Now let $Y \in \mathfrak{k} = \mathrm{Lie}(K')$. As $\mathfrak{k} = \ker \Omega$, we get

$$(\mathrm{ad}^*(Y)\lambda)(X,t) = -\lambda([Y,X],0) = 0$$

for all $(X,t) \in \mathfrak{h}$. Since K' is connected, this implies that itself, and *a fortiori* $\pi_1(G)$, stabilize λ, so that Theorem 1.3.1 yields the result. □

Recall the famous result of Kostant [Kos70], saying that if G is a connected and simply connected Lie group, then every strongly Hamiltonian homogeneous G-space is a covering of a coadjoint orbit in $\mathfrak{g}^*$. The above proposition proves a slighty modified version of this result that we will need in Chapter 4, where we focus more on the space than on the group, that is, we do not ask that the action of the original group be Hamiltonian nor that the latter be simply connected, but instead ask that the space be connected and simply connected and allow the group to be centrally extended to obtain such an action.

Before stating the result, recall that a coadjoint orbit $\mathcal{O}$ of a Lie group G is the orbit of a point $o \in \mathfrak{g}^*$ under the coadjoint action:

$$\mathcal{O} = \mathrm{Ad}^*_G\, o = \{o \circ \mathrm{Ad}_{g^{-1}} \mid g \in G\}.$$

It is, by definition, a homogeneous space, and it is canonically endowed with a symplectic structure defined as follows. The fundamental vector fields at $g \in G$ for the left action of G on itself —that is, the right-invariant vector fields—, span the tangent space T_gG. Hence, if $x = \mathrm{Ad}^*_g\, o$, their images through the coadjoint action at x span $T_x\mathcal{O}$. They are given by

$$\begin{aligned} X^*_x &= \frac{d}{dt}\,\mathrm{Ad}^*_{e^{tX}g}\, o\Big|_{t=0} \\ &= \frac{d}{dt}\left(\mathrm{Ad}^*_g\, o\right) \circ \mathrm{Ad}_{e^{-tX}}\Big|_{t=0} \\ &= -x \circ \mathrm{ad}_X\,. \end{aligned}$$

Now consider the following bilinear form on $\mathfrak{g}$:

$$(X, Y) \mapsto \langle x, [X, Y]\rangle = -\langle X^*_x, Y\rangle = \langle Y^*_x, X\rangle.$$

The last two equalities show that the bilinear form vanishes whenever X or Y has a vanishing fundamental vector field at $x \in \mathcal{O}$. Thus, it may be seen as a bilinear form $\omega^{\mathcal{O}}_x$ on $T_x\mathcal{O}$, defined by

$$\omega^{\mathcal{O}}_x(X^*_x, Y^*_x) = \langle x, [X, Y]\rangle. \tag{1.2}$$

This form is obviously antisymmetric, and it is non-degenerate since, if $X \in \mathfrak{g}$ is such that $\omega^{\mathcal{O}}_x(X^*_x, Y^*_x) = 0$ for all $Y \in \mathfrak{g}$, then we have $-\langle X^*_x, Y\rangle = 0$ for all $Y \in \mathfrak{g}$, that is, $X^*_x = 0$. It remains to see the closedness. For this, notice that the 2-form $x \mapsto \omega^{\mathcal{O}}_x$ is G-invariant. Indeed, one computes that $\mathrm{Ad}^*_g\, X^*_x = (\mathrm{Ad}_g\, X)^*_{\mathrm{Ad}^*_g\, x}$, which implies

$$\begin{aligned} \omega^{\mathcal{O}}_{\mathrm{Ad}^*_g\, x}\left(\mathrm{Ad}^*_g\, X^*_x, \mathrm{Ad}^*_g\, Y^*_x\right) &= \left\langle \mathrm{Ad}^*_g\, x, [\mathrm{Ad}_g\, X, \mathrm{Ad}_g\, Y]\right\rangle \\ &= \langle x, [X, Y]\rangle \\ &= \omega^{\mathcal{O}}_x(X^*_x, Y^*_x). \end{aligned}$$

This proves that $\omega^{\mathcal{O}}$ is a smooth 2-form, and that its closedness is equivalent to the identity (1.1) at some base point $o \in \mathcal{O}$. But this last condition follows directly from the definition (1.2) and from the Jacobi identity in $\mathfrak{g}$.

The 2-form we just built is called the *Kirillov-Kostant-Souriau symplectic form*,

or *canonical symplectic form* on a coadjoint orbit.

Corollary 1.3.3. *Every connected and simply connected homogeneous symplectic space is a covering of a coadjoint orbit.*

Proof. Let (M, ω) be a connected and simply connected homogeneous symplectic space for a connected Lie group G. Fix some base point $o \in M$ and denote Ω the corresponding cocycle defined in the introduction of this section. By Proposition 1.3.2, there exists a Lie group central extension H of G by Ω corresponding to the Lie algebra central extension $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}Z$. Then H acts on M through the action of G composed with the morphism $H \to G$ given by the quotient by the Lie subgroup generated by Z. Denote $\lambda \in \mathfrak{h}^*$ the element such that $\lambda(X, z) = z$ for all $(X, z) \in \mathfrak{h}$, and define a map $J : M \to \mathfrak{h}^*$ by $J(o) = \lambda$ and $J(g \cdot o) = \mathrm{Ad}^*_g \lambda$. This is well-defined since, as we saw in the proof of Proposition 1.3.2, the stabilizer of o also stabilizes λ. Actually the Lie algebras of the stabilizers of o and λ are both equal to $\ker\Omega$, so that J is everywhere a local diffeomorphism. And J is indeed a momentum map since at o, we have

$$\begin{aligned}
\langle (dJ)_x(Y^*), X\rangle &= \left\langle \frac{d}{dt} J\left(e^{-tY}\cdot o\right)\Big|_{t=0}, X \right\rangle \\
&= \frac{d}{dt}\left\langle J(o), e^{t\,\mathrm{ad}_Y} X\right\rangle\Big|_{t=0} \\
&= -\langle \lambda, [X, Y]\rangle \\
&= -\omega_o(X^*, Y^*) \\
&= -i_{X^*}\omega(Y^*)
\end{aligned}$$

for all $X, Y \in \mathfrak{h}$, and since by equivariance, this implies the result at any point $x \in M$. □

1.4 Symmetric spaces

In this section, we briefly introduce the basic notions related to symmetric spaces. For a more general treatment, we refer the reader to the fundamental books of Loos [Loo69], Kobayashi and Nomizu [KN96a, KN96b], and Helgason [Hel78]. The last two books mentionned above mainly treat symmetric spaces in terms of affine connections and involutive Lie algebras, whereas the first emphasizes a multiplication map generalizing that of a Lie group. We will take the latter perspective as starting point.

This section is divided into four parts. The first one introduces the concept and various equivalent definitions of symmetric spaces. The second exposes the notion of normal symmetric subspace, and describes the associated quotient

symmetric space. The third is concerned with ways of realizing symmetric spaces as subsets of their automorphism groups. The last part focuses on the infinitesimal counterpart of symmetric spaces, namely the Lie triple systems, and presents their basic structure theory.

Equivalent definitions

As mentionned above, we start with the definition of O. Loos. The proofs for all statements made here may be found in [Loo69].

Definition 1.4.1. *A* symmetric space *is a manifold M together with a smooth map $s : M \times M \to M : (x, y) \mapsto s(x, y) = s_x y$ such that*

1. *x is an isolated fixed point of s_x,*
2. *$s_x^2 = \mathrm{Id}$,*
3. *$s_x s(y, z) = s(s_x y, s_x z)$,*

for all $x, y, z \in M$. It is denoted (M, s), or simply M if there is no possible confusion.

A morphism $(M, s) \to (M', s')$ of symmetric spaces is a smooth map $f : M \to M'$ such that $f(s(x, y)) = s'(f(x), f(y))$ for all $x, y \in M$.

When some base point o is chosen, (M, s, o) is called a pointed symmetric space. A morphism between pointed symmetric spaces is asked to send the base point of the first on that of the second.

The first two conditions formalize the idea of central symmetry. Indeed, $s_x^2 = \mathrm{Id}$ together with the fact that $s_x x = x$ imply that $(s_x)_{*x}$ is an involutive linear transformation of $T_x M$, so that it is diagonalizable with eigenvalues $+1$ and -1. The condition that x be an *isolated* fixed point of s_x then states that only -1 eigenvalues appear. Relaxing the isolated condition leads to the study of *reflexion spaces* as in [Loo67]. The last condition asserts that each s_x is an automorphism of (M, s).

There are two distinguished groups canonically attached to a symmetric space (M, s): the *automorphism group* $\mathrm{Aut}(M, s)$, that is, the group of all automorphisms of (M, s), and the *transvection group* G, which is the group generated by the products of an even number of symmetries:

$$G = \left\langle s_x s_y \mid x, y \in G \right\rangle.$$

By point 3. in Definition 1.4.1, if ϕ is an automorphism of (M, s), then $\phi s_x \phi^{-1} = s_{\phi(x)}$ for all $x \in M$, so that G is normal in $\mathrm{Aut}(M, s)$. Choose now some base point $o \in M$ and, remembering that $(s_o)^2 = \mathrm{Id}$, consider the inner automorphism

$$\sigma : \mathrm{Aut}(M, s) \to \mathrm{Aut}(M, s) : \phi \mapsto s_o \phi s_o.$$

It follows that σ is involutive, and since G is normal, that it stabilizes G as well.

There are corresponding infinitesimal notions. A vector field X on M is called a *derivation* if

$$X_{s(x,y)} = (\partial_1 s)_{(x,y)}(X_x) + (\partial_2 s)_{(x,y)}(X_y).$$

Denoting s as a product $x \cdot y = s(x,y)$ and keeping the same notation for the partial differentials $X_x \cdot y = (\partial_1 s)_{(x,y)}(X_x)$ and $x \cdot X_y = (\partial_2 s)_{(x,y)}(X_y)$, this reads

$$X_{x\cdot y} = X_x \cdot y + x \cdot X_y.$$

Actually, defining for $X \in T_xM$, $Y \in T_yM$ a new product $X * Y = X \cdot y + x \cdot Y$, yields a canonical symmetric space structure on TM. A vector field $X : M \to TM$ is then a derivation if and only if it is a morphism from M to TM.

Proposition 1.4.2. *The set* $\mathrm{Der}(M,s)$ *of derivations of* M *is a Lie algebra of complete vector fields.*

The tangent space T_oM at o may be identified with a subspace of $\mathfrak{d} = \mathrm{Der}(M,s)$ as follows. One checks that the map $X \mapsto s_{o*} \circ X \circ s_o$ is an involutive automorphism σ of $\mathfrak{d}$. Let $\mathfrak{d} = \mathfrak{d}_+ \oplus \mathfrak{d}_-$ be the decomposition in (± 1)-eigenspaces of σ. Then the map which, to each $X \in T_oM$, associates the vector field

$$x \mapsto \tilde{X}_x = \frac{1}{2} X \cdot (o \cdot x)$$

gives a linear isomorphism from T_oM onto $\mathfrak{d}_-$, whose inverse is simply evaluation at o. Moreover, one has $\mathfrak{d}_+ = \{X \in \mathfrak{d} \mid D(o) = 0\}$.

On the (-1)-eigenspace of an involution on any Lie algebra, there is a special structure of *Lie triple system*. Let $\mathfrak{g}$ be a Lie algebra, and σ an involutive automorphism of $\mathfrak{g}$. Denote $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ the associated decomposition in (± 1)-eigenspaces $\mathfrak{g}_\pm = \frac{\mathrm{Id} \pm \sigma}{2}(\mathfrak{g})$. Then we have

$$[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+ \qquad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_- \qquad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

In particular, $[[\mathfrak{g}_-, \mathfrak{g}_-], \mathfrak{g}_-] \subset \mathfrak{g}_-$, and one readily checks that this trilinear product gives $\mathfrak{g}_-$ the structure of the following definition.

Definition 1.4.3. *A* Lie triple system *is a vector space* $\mathfrak{m}$ *together with a trilinear map* $[\cdot,\cdot,\cdot] : \mathfrak{m}^3 \to \mathfrak{m}$ *satisfying the following properties for all* $x, y, z, u, v, w \in \mathfrak{m}$*:*

1. *Antisymmetry:* $[x,x,y] = 0$,
2. *Jacobi:* $[x,y,z] + [y,z,x] + [z,x,y] = 0$,

3. *Derivation:* $[x, y[u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$.

A vector subspace $\mathfrak{n} \subset \mathfrak{m}$ *is a* (Lie triple) subsystem *if* $[\mathfrak{n}, \mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$, *and an* ideal *if* $[\mathfrak{n}, \mathfrak{m}, \mathfrak{m}] \subset \mathfrak{n}$. *A* morphism *of Lie triple systems is a linear map* $f : \mathfrak{m} \to \mathfrak{m}'$ *such that*

$$f([x, y, z]) = [f(x), f(y), f(z)]'$$

for all $x, y, z \in \mathfrak{m}$. *A* derivation *of a Lie triple system is a linear map* $D : \mathfrak{m} \to \mathfrak{m}$ *such that*

$$D[x, y, z] = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]$$

for all $x, y, z \in \mathfrak{m}$.

Remark 1.4.4. The "derivation" identity above thus tells that the linear maps $L_{x,y} : \mathfrak{m} \to \mathfrak{m} : z \mapsto [x, y, z]$ are all derivations of $\mathfrak{m}$. They are called the *inner derivations* of $\mathfrak{m}$. And the "Jacobi" identity is termed like this because in the example of $\mathfrak{d}_-$ above, it boils down to the usual Jacobi identity for $\mathfrak{d}$. These conventions are perfectly acceptable since we will see that every Lie triple system arises in this way. However, it should be noted that in a more general context, these names do not reflect the real nature of the equations, since the Jacobi identity in Lie algebras reflects the fact that the left multiplication ad_x is a derivation of the bracket $[\cdot, \cdot]$, so that our derivation identity really is a Jacobi identity for the bracket $[\cdot, \cdot, \cdot]$. There exist generalizations of Lie algebras, called n-Lie algebras, and of Lie triple systems, called n-ary Leibniz algebras, where analogues of our derivation identity actually play the role of the Jacobi identity. See [Fil85, Kas87] for the former, and [CLP02] for the latter. ○

As we just mentioned, by a construction called the *standard embedding*, any Lie triple system has the form given above, that is, is the (-1)-eigenspace of an involution on a Lie algebra. The construction proceeds as follows. Let $\mathfrak{m}$ be a Lie triple system, and let $\mathfrak{h}$ be the span of the linear transformations $L_{x,y} : \mathfrak{m} \to \mathfrak{m} : z \mapsto [x, y, z]$. Then their direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ becomes a Lie algebra with the bracket defined by

$$\begin{aligned} [a, b] &= ab - ba \\ [a, x] &= -[x, a] = ax \\ [x, y] &= L_{x,y} \end{aligned}$$

for $x, y \in \mathfrak{m}$ and $a, b \in \mathfrak{h}$. The fact that the commutator $[a, b]$ is again in $\mathfrak{h}$ comes from the derivation identity above, which can be rewritten as

$$[L_{x,y}, L_{s,t}] = L_{L_{x,y}s,t} + L_{s,L_{x,y}t}.$$

Moreover, there is an involutive automorphism σ on $\mathfrak{g}$ given by $\sigma(a+x) = a-x$ for all $a \in \mathfrak{h}$ and $x \in \mathfrak{m}$. And $\mathfrak{m}$ is the (-1)-eigenspace of the involution σ on the Lie algebra $\mathfrak{g}$, as claimed.

Remark 1.4.5. We have just seen that the (-1)-eigenspaces of involutions on Lie algebras exhaust the examples of Lie triples systems. But on the other hand, one may check that any Lie algebra $\mathfrak{g}$, with ternary bracket given by $[x,y,z] = \frac{1}{4}[[x,y],z]$, is a Lie triple system. Hence it should be of the form above. It is indeed so, for the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the canonical involution $\sigma_{\mathfrak{g}}(x,y) = (y,x)$.

Now if σ is an involution *on* $\mathfrak{g}$, then its (-1)-eigenspace is a Lie triple system, for the same ternary bracket $[x,y,z] = [[x,y],z]$. This shows that any Lie triple system my be included as a subsystem in a Lie algebra seen as a Lie triple system, by the map $\mathfrak{m} \to \mathfrak{g} : x \mapsto 2x$. We will encounter the corresponding phenomenon for symmetric spaces with the quadratic representation, in the third part of this section, on realizations of symmetric spaces as subsets of their automorphism groups. ○

Returning to the symmetric space (M,s) and its derivation algebra $\mathfrak{d}$, we thus see that $\mathfrak{m} = T_oM$ acquires, through the linear isomorphism with $\mathfrak{d}_-$, a structure of Lie triple system. It is called *the Lie triple system of* M (more precisely, of (M,s,o)). This correspondance is functorial.

Proposition 1.4.6. *Let $f : M \to M'$ be a morphism of pointed symmetric spaces. Then $f_* : TM \to TM'$ is a morphism of Lie triple systems.*

Let us define the map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by

$$\nabla_X Y = XY + \frac{1}{2} X \cdot Y.$$

Then one may check that ∇ is a connection (we will identify connections and covariant derivatives), that its torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ vanishes, and that its curvature tensor at o is

$$R(X,Y)Z = -[X,Y,Z]$$

and is covariant constant. Moreover, it is invariant under all symmetries, and is complete. It is called the *canonical connection* of M.

This motivates the second definition.

Definition 1.4.7. *An* affine symmetric space *is a connected manifold M with a complete torsion-free connection such that its curvature tensor is covariant constant.*

Given a manifold M with an affine connection, we may define a local multiplication on a neighbourhood N of the diagonal in $M \times M$, by

$$x \cdot \mathrm{Exp}_x(v) = \mathrm{Exp}_x(-v).$$

It can be shown that $T = 0$ and $\nabla R = 0$ imply that M is a locally symmetric space, and the completeness yields the following equivalence.

Theorem 1.4.8. *Let M be a connected manifold. Then M is an affine symmetric space if and only if it is a symmetric space.*

We now prepare for the third approach to symmetric spaces, which is the classical one.

Proposition 1.4.9. *Let M (as a symmetric space) be generated by finitely many connected components. Then,*

1. *the automorphism group* $\mathrm{Aut}(M,s)$ *is a finite dimensional Lie transformation group of M with Lie algebra* $\mathrm{Der}(M,s)$*;*
2. *the transvection group G is a Lie subgroup of* $\mathrm{Aut}(M,s)$ *with countably many components, and its Lie algebra is generated by $\mathfrak{d}_-$.*

When, moreover, M is connected, G is the smallest transitive group of automorphisms which is stabilized by σ.

Remark 1.4.10. When the space is not connected, the transvection group is not transitive in general. Since it is transitive on any connected component, it will be transitive on the whole space if and only if it is so on the discrete set of its connected components. The space $M = \mathbb{Z}$ with symmetries $s_n m = 2n - m$ is an example where the transvection group (and even the group generated by all the symmetries) is not transitive, since any product of symmetries $s_{n_1} \dots s_{n_k}$ applied to the point 0 yields an *even* integer $2(n_1 - n_2 + \dots + (-1)^{k+1} n_k)$. A sufficient condition for this not to happen is the existence, for each pair of points (x,y), of a "midpoint" z such that $s_z x = y$, since then $s_z s_x$ is an element of the transvection group sending x to y. The notion of *midpoint map*, associating to each pair of points a unique midpoint, will be the subject of Chapter 2. ○

Assume (M,s,o) is a pointed connected symmetric space. Then its transvection group G is a transitive Lie transformation group of M. Denote H the stabilizer in G of o: $H = \{g \in G \mid g \cdot o = o\}$. It is a closed Lie subgroup of G, and M is diffeomorphic to the quotient manifold G/H. Consider now the map $\sigma : G \to G : g \mapsto s_o g s_o$. Since $(s_o)^2 = \mathrm{Id}$, it follows that it is an involutive automorphism of G, whose fixed points set we denote G^σ. The latter is also a

closed Lie subgroup of G, and it is the commutant of s_o in G. If $g \in H$, then $s_o \circ g = s_{g \cdot o} \circ g = g \circ s_o$, that is, $g \in G^\sigma$. Hence $H \subset G^\sigma$.

Conversely, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of σ. Then $\mathfrak{k}$ is the Lie algebra of G^σ and, for $X \in \mathfrak{k}$,

$$s_o(e^{tX} \cdot o) = s_o \cdot e^{tX} \cdot s_o s_o o = e^{tX} \cdot o$$

so that $e^{tX} \cdot o = o$ for all $t \in \mathbb{R}$ since o is an isolated fixed point of s_o. Hence $(G^\sigma)_0 \subset H$.

This motivates the last definition.

Definition 1.4.11. *A* homogeneous symmetric space *is a triple (G, H, σ) where G is a connected Lie group endowed with an involutive automorphism σ, and H is a Lie subgroup of G such that $(G^\sigma)_0 \subset H \subset G^\sigma$, where $G^\sigma = \{g \in G \mid \sigma(g) = g\}$. When the situation is clear, the corresponding quotient G/H is also called a homogeneous symmetric space.*

Given a homogeneous symmetric space $M = G/H$, we may define a map $s : M \times M \to M$ by

$$s_{g_1 H} g_2 H = g_1 \sigma(g_1^{-1} g_2) H,$$

and check that s satisfies all the axioms of a symmetric space, yielding the last equivalence:

Theorem 1.4.12. *Let M be a connected manifold. Then M is a homogeneous symmetric space if and only if it is a symmetric space.*

We have thus seen three equivalent (when restricting to connected manifolds) notions of symmetric space.

Let now (M, s, o) be a pointed connected and simply connected symmetric space, let G be its transvection group and with σ the involution associated to o. Then the involutive Lie algebra $(\mathfrak{g}, \sigma)$, with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, is such that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$ and $\mathfrak{k}$ acts faithfully on $\mathfrak{m}$.

Conversely, given an involutive Lie algebra $(\mathfrak{g}, \sigma)$, one builds the connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$ and integrates σ as a Lie group involution. It may be shown that the fixed point subgroup of an involution of a simply connected Lie group is always connected [Loo69, IV, Theorem 3.4], so that $(G^\sigma)_0 = G^\sigma$ and to the pair (G, σ) corresponds one and only one homogeneous symmetric space $M = G/G^\sigma$.

In fact, one has the following characterization:

Theorem 1.4.13. *There is a bijection between the isomorphism classes of involutive*

Lie algebras $(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \sigma)$ *such that* $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$ *and* $\mathfrak{k}$ *acts faithfully on* $\mathfrak{m}$*, and the isomorphism classes of connected and simply connected symmetric spaces.*

Remark 1.4.14. We will interchangeably work in the different pictures: (M, s), (M, ∇), (G, σ, K), (G, σ), $(\mathfrak{g}, \sigma)$, and $\mathfrak{m}$. For pointed connected and simply connected symmetric spaces, these are all equivalent.

Nevertheless, different viewpoints are best adapted to different problems. For example, if we are concerned with general geometric properties of M, working in (M, s) or $\mathfrak{m}$ might be the most natural. However, if we deal with additional structures on M having specific invariance properties, we might want to keep track of this invariance by specifying explicitly a transitive σ-invariant group of automorphisms G, or its Lie algebra $\mathfrak{g}$. Similarly, if we want to embed M in some larger, simpler space, then these transitive groups and their (dual) Lie algebras are natural candidates, as we shall see farther in this section, and in Chapter 4, for example. ○

Before turning to quotients of symmetric spaces by normal subspaces, we mention an important theorem that we will use repeatedly throughout the text.

Theorem 1.4.15 ([Loo69, IV, Theorem 4.6])**.** *Let M be a symmetric space and G its transvection group. Then there exists a covering $\hat{G}$ of G and an involutive automorphism σ of $\hat{G}$ compatible with the covering projection such that*

$$M \cong \hat{G}/\hat{G}^{\sigma}.$$

This theorem allows us to only consider symmetric pairs (G, σ) and spaces of the form G/G^{σ}, instead of the triples (G, H, σ) and the spaces G/H with $(G^{\sigma})_0 \subset H \subset G^{\sigma}$.

Quotients

There is for symmetric spaces a notion playing the role of normal subgroups for Lie groups: that of *congruence*, developed by Loos in [Loo69].

Recall that an equivalence relation in a set M is a subset $R \subset M \times M$ which

1. contains the diagonal $\Delta(M) = \{(x, x) \mid x \in M\}$,
2. is symmetric: $(x, y) \in R$ implies $(y, x) \in R$, and
3. is transitive: $x, y \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

Definition 1.4.16. *Let M be a symmetric space. Then an equivalence relation $R \subset M \times M$ is a* congruence *if it is a symmetric subspace of $M \times M$ with the direct product structure. An equivalence class of a congruence is called a* normal subspace.

Loos proves the following correspondance.

Theorem 1.4.17. *Let R be a congruence.*

1. *The equivalence classes of R are symmetric subspaces of M. If N is the equivalence class of a point $o \in M$, then $\mathfrak{n} = T_oN$ is an ideal of the Lie triple system $\mathfrak{m} = T_oM$.*

2. *Let $M = G/H$ be connected with base point o, let $\mathfrak{n} \subset \mathfrak{m}$ be an ideal, and let N be the corresponding connected symmetric subspace of M. Then there exists exactly one congruence such that the equivalence class of o is N.*

This result is then used to give the set M/R of equivalence classes of a congruence, the structure of a symmetric space.

Theorem 1.4.18. 1. *Let $M = G/H$ be connected, R be a congruence with countably many components, N the equivalence class of o, and $L = \{g \in G \mid g \cdot o \in N\}$. Then L is a Lie subgroup of G, stable under σ, and such that $gl\sigma(l)^{-1}g^{-1} \in L$ for all $l \in L$ and $g \in G$.*

2. *R is closed if and only if N is closed if and only if L is closed. In this case, M/R can be made in a natural way into a symmetric space such that the canonical map $M \to M/R$ is a submersion and a locally trivial fiber bundle.*

Note that if G' is a closed σ-invariant normal Lie subgroup of G, then $G'/(G' \cap H)$ is a normal subspace. But in general the transvection group G' of a normal subspace N of M is not a normal subgroup of G. At the level of the Lie algebras, this comes from the fact that only $\mathfrak{n} + [\mathfrak{m}, \mathfrak{n}]$ (and not $\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$) is an ideal in $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$.

Realizations

Let (G, σ) be a symmetric pair with G connected, denote $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the corresponding decomposition of the Lie algebra in $+1$ and -1 eigenspaces of σ, respectively. We may define three subsets of G locally realizing the associated symmetric space. The first is simply

$$\exp \mathfrak{m}.$$

For the second, define an action α of G on itself by

$$\alpha_g(h) := gh\sigma(g)^{-1}. \tag{1.3}$$

We may check that we indeed have $\alpha_g\alpha_{g'}h = \alpha_{gg'}h$ and $\alpha_e h = h$ for all $g, g', h \in G$, and define the second subset as

$$M_G := \alpha_G e = \{g\sigma(g)^{-1} \mid g \in G\},$$

the orbit of e under α. And finally, the third subset is

$$P := \{g \in G \mid \sigma(g) = g^{-1}\}.$$

Since for $X \in \mathfrak{m}$, we have the equality $\sigma\exp(X) = \exp(-X)$ and thus also $\exp(X) = \exp(X/2)\sigma(\exp(X/2))^{-1}$, we obtain the inclusion $\exp\mathfrak{m} \subset M_G$. Then if $p = g\sigma(g)^{-1} \in M_G$, obviously $\sigma(p) = \sigma(g)g^{-1} = p^{-1}$, so that $M_G \subset P$. Hence we have

$$\exp\mathfrak{m} \subset M_G \subset P.$$

Define in G the map $s(g,h) = s_g h = gh^{-1}g$. This is the canonical symmetric structure on a Lie group, coming from the symmetric pair $(G \times G, \sigma_G)$ where $\sigma_G(g,h) = (h,g)$. Now, we may see that s restricts to M_G and P as well-defined maps $M_G \times M_G \to M_G$ and $P \times P \to P$ respectively, thus giving them the structure of symmetric space. Indeed, for M_G we have

$$s_{g\sigma(g)^{-1}}h\sigma(h)^{-1} = \left(g\sigma(g^{-1}h)\right)\sigma\left(g\sigma(g^{-1}h)\right)^{-1} \in M_G,$$

and for P, if $p, q \in P$ we have

$$\sigma(s_p q) = \sigma(pq^{-1}p) = \left(pq^{-1}p\right)^{-1} = (s_p q)^{-1} \in P.$$

Moreover, we have a diffeomorphism

$$G/G^\sigma \to M_G : gG^\sigma \mapsto g\sigma(g)^{-1}$$

sending the canonical symmetric structure $s_{gG^\sigma}hG^\sigma = g\sigma(g^{-1}h)G^\sigma$ on G/G^σ to the map we just defined on M_G, so that our diffeomorphism is in fact an isomorphism of symmetric spaces. In other words, every symmetric space G/G^σ is realized in that way as a symmetric subspace M_G of $G \times G/G$.

Finally, M_G is thus a symmetric subspace of P, both of them having the same tangent space at e, namely $\mathfrak{m}$. As G is connected, M_G also is, and by [Loo69, III, Corollary of Proposition 1.2 and Lemma 1.3] we obtain that M_G is just the connected component of P containing e. Actually, the action α defined in equation (1.3) sends P to P since, for $h \in P$, we have

$$\sigma(\alpha_g(h)) = \sigma(gh\sigma(g)^{-1}) = \sigma(g)\sigma(h)g^{-1} = (\alpha_g(h))^{-1}.$$

Thus P decomposes as a disjoint union of orbits of α, the orbit of e being M_G.

Example 1.4.19. Consider the symmetric pair $(SL_2(\mathbb{R}), \sigma)$ with $\sigma(g) = (g^t)^{-1}$, so that

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Then

$$\begin{aligned} \exp\mathfrak{m} &= \left\{ \exp\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} \cosh(r) + \frac{a\sinh(r)}{r} & \frac{b\sinh(r)}{r} \\ \frac{b\sinh(r)}{r} & \cosh(r) - \frac{a\sinh(r)}{r} \end{pmatrix} \middle| a, b \in \mathbb{R}, r = \sqrt{a^2+b^2} \right\}, \end{aligned}$$

$$\begin{aligned} M_G &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \middle| ad - bc = 1 \right\} \\ &= \left\{ \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \middle| ad - bc = 1 \right\} \end{aligned}$$

and

$$\begin{aligned} P &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad - bc = 1 \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \middle| ad - b^2 = 1 \right\}. \end{aligned}$$

In this case, we have $\exp\mathfrak{m} = M_G$, in accordance with the fact that the symmetric pair considered here is Riemannian of noncompact type, hence has a global polar decomposition. And M_G is only one sheet of the two-sheeted hyperboloid P. ◁

Example 1.4.20. Consider the symmetric pair (G, σ), where G is the semidirect product $\mathbb{R} \ltimes_\alpha \mathbb{C}$ with the homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathbb{C})$ given by the formula $\alpha(t)c = e^{it}c$, $t \in \mathbb{R}$, $c \in \mathbb{C}$, and the involutive automorphism $\sigma(t, c) = (-t, \bar{c})$ of the group G. The group G is the universal covering of the 2-dimensional special Euclidian group $SO(2) \ltimes \mathbb{R}^2$.

The product in G is given by $(t, c)(t', c') = (t + t', c + e^{it}c')$ so that the inverse is $(t, c)^{-1} = (-t, -e^{-it}c) = \sigma(t, -e^{it}\bar{c})$. The Lie algebra is $\mathbb{R} \times \mathbb{C}$ with commutators

$[(T,C),(T',C')] = (0, i(TC' - T'C))$. We thus have

$$\exp \mathfrak{m} = \left\{ \exp(T,A) = \left(T, \frac{e^{iT}-1}{iT} A \right) \middle| \, T, A \in \mathbb{R} \right\},$$

$$M_G = \{ g\sigma(g)^{-1} \mid g \in G \} = \{ (2t, c - e^{2it}\bar{c}) \mid t \in \mathbb{R}, c \in \mathbb{C} \}$$

and

$$\begin{aligned} P &= \{ (t,c) \mid (-t,\bar{c}) = (-t, -e^{-it}c), t \in \mathbb{R}, c \in \mathbb{C} \} \\ &= \{ (t,c) \mid t \in \mathbb{R}, c \in \mathbb{C}, \bar{c} = -e^{-it}c \}. \end{aligned}$$

Here $M_G = P$. But $\exp \mathfrak{m} \neq M_G$, the missing set in $\exp \mathfrak{m}$ being

$$\{ (2k\pi, ia) \mid k \in \mathbb{N}_*, a \in \mathbb{R}_* \}.$$

◁

The notions presented above will mostly be used in Sections 2.2 and 2.3.

Lie triple systems

In this last part, we present some basic structure theory of Lie triple systems, mainly due to Jacobson [Jac49] and Lister [Lis52]. We then describe a cohomology theory due to Yamaguti [Yam60], part of which will be used in Section 3.3. There exists another cohomology theory for Lie triple systems, due to Harris [Har61], which we won't use nor describe here. We refer the reader to the mentioned papers for the proofs of all statements below.

Definition 1.4.21. *An embedding E of a Lie triple system $\mathfrak{m}$ in a Lie algebra $\mathfrak{g}$ is an injective linear map $\mathfrak{m} \to \mathfrak{g} : x \mapsto x^E$ such that*

1. $[x,y,z]^E = [[x^E, y^E], z^E]$, *and*
2. *the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{m}^E$ is $\mathfrak{g}$.*

When there is an embedding of $\mathfrak{m}$ into $\mathfrak{g}$, $\mathfrak{g}$ is called an enveloping Lie algebra *of* $\mathfrak{m}$.

In the following, we shall always consider the Lie triples systems $\mathfrak{m}$ as embedded in their standard embedding $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ described in the first part of this section.

Recall that an ideal $\mathfrak{n}$ in a Lie triple system $\mathfrak{m}$ is a vector subspace such that $[\mathfrak{n}, \mathfrak{m}, \mathfrak{m}] \subset \mathfrak{n}$. Given an ideal $\mathfrak{n}$, consider the vector space quotient $\mathfrak{m}/\mathfrak{n} = \{ x + \mathfrak{n} \mid x \in \mathfrak{m} \}$ and define a bracket on it by $[x+\mathfrak{n}, y+\mathfrak{n}, z+\mathfrak{n}] = [x,y,z] + \mathfrak{n}$, then $\mathfrak{m}/\mathfrak{n}$ becomes a Lie triple system, the quotient of $\mathfrak{m}$ by $\mathfrak{n}$. The mapping $x \mapsto x + \mathfrak{n}$

is a morphism of $\mathfrak{m}$ onto $\mathfrak{m}/\mathfrak{n}$ with kernel $\mathfrak{n}$. Conversely, if $\mathfrak{n}$ is the kernel of a morphism $f : \mathfrak{m} \to \mathfrak{m}'$, then $f([x,y,z]) = [f(x), f(y), f(z)] = 0$ whenever $x \in \mathfrak{n}$, so that $\mathfrak{n}$ is an ideal. Hence, this is the right notion of ideal in this context.

The starting point for the structure theory is the following observation: if $\mathfrak{n}$ is an ideal of a Lie triple system $\mathfrak{m}$, then $\mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$ is an ideal of $\mathfrak{n} + [\mathfrak{m},\mathfrak{n}]$, which is itself an ideal of $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$, namely the ideal generated by $\mathfrak{n}$.

If $\mathfrak{n}$ is an ideal of $\mathfrak{m}$, define $\mathfrak{n}^{(1)} = [\mathfrak{m},\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{n}^{(k)} = [\mathfrak{m}, \mathfrak{n}^{(k-1)}, \mathfrak{n}^{(k-1)}]$.

Lemma 1.4.22. *For each k, $\mathfrak{n}^{(k)}$ is an ideal in $\mathfrak{m}$. Thus, $\mathfrak{n} \supset \mathfrak{n}^{(1)} \supset \cdots \supset \mathfrak{n}^{(k)}$.*

Definition 1.4.23. *An ideal $\mathfrak{n}$ in a Lie triple system $\mathfrak{m}$ is* solvable in $\mathfrak{m}$ *if there is a positive integer k for which $\mathfrak{n}^{(k)} = 0$.*

Lemma 1.4.24. *Let $\mathfrak{n}$ and $\mathfrak{u}$ be ideals in a Lie triple system $\mathfrak{m}$. Then,*

1. *if $\mathfrak{u} \subset \mathfrak{n}$ and $\mathfrak{n}$ is solvable, so is $\mathfrak{u}$,*
2. *if $\mathfrak{n}^{(k)}$ is solvable, so is $\mathfrak{n}$,*
3. *if $\mathfrak{n}$ and $\mathfrak{u}$ are solvable, so is $\mathfrak{n} + \mathfrak{u}$.*

This allows us to form $\mathcal{R}(\mathfrak{m}) = \sum_\alpha \mathfrak{n}_\alpha$, the sum of all solvable ideal $\mathfrak{n}_\alpha$ in $\mathfrak{m}$. Since it is solvable, it is the unique maximal solvable ideal in $\mathfrak{m}$.

Definition 1.4.25. *The* radical $\mathcal{R}(\mathfrak{m})$ *is the unique maximal solvable ideal whose existence is guaranteed by Lemma 1.4.24. If $\mathcal{R}(\mathfrak{m}) = 0$, then $\mathfrak{m}$ is* semisimple. *If* $\dim \mathfrak{m} \neq 1$ *and $\mathfrak{m}$ has no non-trivial ideal, then $\mathfrak{m}$ is* simple.

Theorem 1.4.26. *If $\mathcal{R}$ is the radical of a Lie triple system $\mathfrak{m}$, then $\mathfrak{m}/\mathcal{R}$ is semisimple, and if $\mathfrak{n}$ is an ideal in $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{n}$ is semisimple, then $\mathfrak{n} \supset \mathcal{R}$.*

The following results establish the connections between the radical of a Lie triple system $\mathfrak{m}$, and the radical of an enveloping Lie algebra of $\mathfrak{m}$, and give the rough structure of semisimple Lie triple systems.

Theorem 1.4.27. *Any enveloping Lie algebra of a solvable Lie triple system is solvable, and if a Lie triple system has some solvable enveloping Lie algebra, it is solvable. Any enveloping Lie algebra of a semisimple Lie triple system is semisimple.*

Theorem 1.4.28. *If $\mathfrak{m}$ is a semisimple Lie triple system, then*

1. *$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$, where $\mathfrak{m}_i$ is a simple ideal in $\mathfrak{m}$, for $i = 1, \ldots, n$.*
2. *$\mathfrak{m}^{(1)} = \mathfrak{m}$,*
3. *any derivation is inner.*

Theorem 1.4.29. *Let $\mathfrak{m}$ be a simple Lie triple system. Then either $\mathfrak{m}$ is a simple Lie algebra seen as a Lie triple system, or $\mathfrak{m}$ is the (-1)-eigenspace of an involution of a simple Lie algebra.*

There is Levi decomposition for Lie triple systems:

Theorem 1.4.30. *If $\mathfrak{m}$ is a Lie triple system, then there exists a semisimple subsystem $\mathfrak{s}$ such that $\mathfrak{m} = \mathfrak{s} \oplus \mathcal{R}(\mathfrak{m})$.*

A notion of nilpotency for Lie triple systems was developed by N. Hopkins in [Hop95].

Definition 1.4.31. *Let $\mathfrak{m}$ be a Lie triple system. If $\mathfrak{n}$ is an ideal of $\mathfrak{m}$, then $\mathfrak{n}^0 = \mathfrak{n}$ and $\mathfrak{n}^{n+1} = [\mathfrak{n}^n, \mathfrak{m}, \mathfrak{n}] + [\mathfrak{n}, \mathfrak{m}, \mathfrak{n}^\perp]$, $n \geq 0$, is a lower central series for $\mathfrak{n}$. We say that $\mathfrak{n}$ is* $\mathfrak{m}$-nilpotent *if $\mathfrak{n}^n = 0$ for some n.*

Moreover, $\mathfrak{m}$ is said to be nilpotent *if it is $\mathfrak{m}$-nilpotent.*

N. Hopkins showed that there is a unique nilradical $\mathcal{N}(\mathfrak{m})$ (the maximal nilpotent ideal of $\mathfrak{m}$) and showed that the nilradical of the standard embedding of $\mathfrak{m}$ is the standard embedding of the nilradical of $\mathfrak{m}$:

$$\mathcal{N}(\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]) = \mathcal{N}(\mathfrak{m}) + [\mathcal{N}(\mathfrak{m}), \mathcal{N}(\mathfrak{m})],$$

which implies the following equivalence.

Proposition 1.4.32. *A Lie triple system is nilpotent if and only if its standard embedding is nilpotent.*

We now turn to modules for Lie triple systems, and to an associated cohomology defined by Yamaguti in [Yam60] (see also [LWD09]).

Definition 1.4.33. *A vector space V is a* module *for a Lie triple system $\mathfrak{m}$ if on the direct product $\mathfrak{m} \oplus V$, there is a structure of Lie triple system such that*

1. *$\mathfrak{m}$ is a subsystem,*
2. *$[x, y, z]$ lies in V whenever one of x, y and z lies in V, and*
3. *$[x, y, z] = 0$ whenever any two of x, y and z are in V.*

The $\mathfrak{m}$-modules may be described in the following way. Let V be a vector space. A triple bracket $[\cdot, \cdot, \cdot]$ on $\mathfrak{m} \oplus V$ that statisfies the three requirements of the definition of $\mathfrak{m}$-module must be such that, for all $x, y, z \in \mathfrak{m}$ and $v, w \in V$,

$$[x, y, z] = [x, y, z]_{\mathfrak{m}}$$
$$[v, x, y] = \theta(x, y)v$$

$$[x,y,v] = \epsilon(x,y)v$$
$$[v,w,x] = [v,x,w] = 0$$

for some bilinear map $\theta : \mathfrak{m} \times \mathfrak{m} \to \mathrm{End}(V)$ and (the opposite of) its antisymmetrization $\epsilon : \mathfrak{m} \times \mathfrak{m} \to \mathrm{End}(V)$ defined by $\epsilon(x,y) = \theta(y,x) - \theta(x,y)$. Then one checks that V is a $\mathfrak{m}$-module if and only if θ is such that

$$\theta(z,t)\theta(x,y) - \theta(y,t)\theta(x,z) - \theta(x,[y,z,t]) + \epsilon(y,z)\theta(x,t) = 0, \tag{1.4}$$
$$\theta(z,t)\epsilon(x,y) - \epsilon(x,y)\theta(z,t) + \theta([x,y,z],t) + \theta(z,[x,y,z]) = 0. \tag{1.5}$$

Remark 1.4.34. Any Lie triple system $\mathfrak{m}$ is itself a $\mathfrak{m}$-module, with $\theta(x,y)z = R_{x,y}z = [z,x,y]$ the right multiplication by x,y, and $\epsilon(x,y) = L_{x,y}$. The corresponding extension $\mathfrak{m} \oplus \mathfrak{m}$ defined above is the Lie triple system of the tangent bundle of the symmetric space associated to $\mathfrak{m}$. ○

Let V be a $\mathfrak{m}$-module defined by a linear map θ. Denote $C^0(\mathfrak{m},V) = V$ and $C^n(\mathfrak{m},V)$, $n \geq 1$, the vector spaces spanned by all the n-linear maps $f : \mathfrak{m} \times \cdots \times \mathfrak{m} \to V$ such that

$$f(x_1,x_2,\ldots,x_{n-3},x,x,x_n) = 0,$$
$$f(x_1,x_2,\ldots,x_{n-3},x,y,z) + f(x_1,x_2,\ldots,x_{n-3},y,z,x) + f(x_1,x_2,\ldots,x_{n-3},z,x,y) = 0.$$

Define then the coboundary operators $\delta : C^n(\mathfrak{m},V) \to C^{n+2}(\mathfrak{m},V)$ by

$$\delta f(x_1,x_2) = \theta(x_1,x_2)f$$

for all $f \in C^0(\mathfrak{m},V)$,

$$\begin{aligned}
&\delta f(x_1,x_2,\ldots,x_{2n+1}) \\
&\quad = \theta(x_{2n},x_{2n+1})f(x_1,x_2,\ldots,x_{2n-1}) - \theta(x_{2n-1},x_{2n+1})f(x_1,x_2,\ldots,x_{2n-2},x_{2n}) \\
&\qquad\quad + \sum_{k=1}^{n}(-1)^{n+k}\epsilon(x_{2k-1},x_{2k})f(x_1,x_2,\ldots,\hat{x}_{2k-1},\hat{x}_{2k},\ldots,x_{2n+1}) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1,x_2,\ldots,\hat{x}_{2k-1},\hat{x}_{2k},\ldots,[x_{2k-1},x_{2k},x_j],\ldots,x_{2n+1})
\end{aligned} \tag{1.6}$$

for all $f \in C^{2n-1}(\mathfrak{m},V)$, $n \geq 1$, and

$$\begin{aligned}
&\delta f(y,x_1,x_2,\ldots,x_{2n+1}) \\
&\quad = \theta(x_{2n},x_{2n+1})f(y,x_1,x_2,\ldots,x_{2n-1}) - \theta(y,x_{2n-1},x_{2n+1})f(x_1,x_2,\ldots,x_{2n-2},x_{2n})
\end{aligned}$$

$$+\sum_{k=1}^{n}(-1)^{n+k}\epsilon(x_{2k-1},x_{2k})f(x_1,x_2,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,x_{2n+1})$$
$$+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1,x_2,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,[x_{2k-1},x_{2k},x_j],\dots,x_{2n+1}) \quad (1.7)$$

for all $f \in C^{2n}(\mathfrak{m},V)$, $n \geq 1$, where the presence of a hat means that the variable must be omitted.

One then proves [Yam60] that $\delta^2 = 0$ on all cochain spaces. As usual, we denote $Z^n(\mathfrak{m},V)$ the space of cocycles $\operatorname{Ker}\delta|_{C^n(\mathfrak{m},v)}$, $B^n(\mathfrak{m},V)$ the space of coboundaries $\operatorname{Im}\delta|_{C^{n-2}(\mathfrak{m},V)}$, and $H^n(\mathfrak{m},V) = Z^n(\mathfrak{m},V)/B^n(\mathfrak{m},V)$ the associated cohomology space.

1.5 Symplectic symmetric spaces

Definition 1.5.1 ([BCG97]). *A* symplectic symmetric space (M,s,ω) *is a symmetric space* (M,s) *together with a symplectic form* ω *on* M *which is invariant under all symmetries:*

$$s_x^*\omega = \omega \quad \textit{for all } x \in M.$$

A morphism *of symplectic symmetric spaces is a morphism of symmetric spaces which is also a symplectomorphism.*

By definition, the automorphism group of a symplectic symmetric space (M,s,ω) is thus the intersection of the automorphism group of (M,s) and the symplectomorphism group of (M,ω):

$$\operatorname{Aut}(M,s,\omega) = \operatorname{Aut}(M,s) \cap \operatorname{Symp}(M,\omega).$$

As a result, since being a symplectomorphism is defined by an equality, it is a closed subgroup of $\operatorname{Aut}(M,s)$, hence a Lie transformation group of M.

The automorphism group contains all the symmetries and, in particular, the transvection group. Hence, when M is connected, the automorphism group is transitive.

The canonical symmetric space connection is the unique torsion-free symplectic connection which is invariant under all symmetries. It may be computed alternatively using the following formulæ [BCG97]:

$$(\nabla_X Y)_x = \frac{1}{2}\,[X, Y + s_{x*}Y]_x$$

or

$$\omega_x(\nabla_X Y, Z) = X_x(\omega(Y + s_{x*}Y, Z)).$$

The infinitesimal characterizations of symplectic symmetric spaces are

1. *symplectic triples* $(\mathfrak{g}, \sigma, \Omega)$, where $(\mathfrak{g}, \sigma)$ is an involutive Lie algebra, and Ω is a scalar 2-cocycle vanishing on $\mathfrak{k}$ and non-degenerate on $\mathfrak{m} \times \mathfrak{m}$;

2. *symplectic Lie triple systems* $(\mathfrak{m}, \Omega)$, where $\mathfrak{m}$ is a Lie triple system and Ω is a non-degenerate linear 2-form on $\mathfrak{m}$ such that

$$\Omega\left(L_{x,y}u, v\right) + \Omega\left(u, L_{x,y}v\right) = 0$$

for all $x, y, u, v \in \mathfrak{m}$.

Chapter 2

Midpoint maps and exponentiality

As mentioned in the Introduction, midpoints appear quite often in quantization problems, be it in the quantization maps or in the formulæ defining a symbol calculus. In this chapter, we investigate the notion of midpoint map, associating to a pair of points x, y a unique point z such that $s_z x = y$.

Midpoints in symmetric spaces were already studied by Nobusawa [Nob74] for finite symmetric spaces, by Qian [Qia97] for general manifolds and symmetric spaces, and by Lawson and Lim [LL04b, LL04a] for symmetric sets (symmetric spaces without a manifold structure), motivated by the study of the geometric and spectral means on sets of matrices. Midpoints appear in other contexts like metric spaces [Hor09], or Bol loops [FK10, Rob66].

A related notion is that of exponential symmetric space. There exist essentially three definitions of exponentiality, first developed for Lie groups, and more recently adapted to symmetric spaces. They require respectively the exponential map 1) to be surjective, 2) to have dense image, and 3) to be a diffeomorphism. The third notion above was the first considered for Lie groups, and was fully characterized independently by Dixmier [Dix57] and Saito [Sai57]. The other two, less restrictive, notions were then approached by authors like Borel, Wüstner, Hofmann, Neeb, Sacksteder and Đoković (see [Wü01, ĐH97] for two surveys and more references), still in the Lie group case. Recently, a number of their results were generalized to symmetric spaces by Rozanov [Roz09]. One purpose of this chapter is to generalize the results of Dixmier and Saito about the stronger notion of exponentiality.

We show that for connected spaces, strong exponentiality is equivalent to the existence of a midpoint map, and further equivalent to an algebraic condition

on the spectrum of the adjoint representation.

Using this result, we then study the existence and uniqueness of "double triangles", which are triples of points such that their three midpoints form a given triple of points. We characterize the spaces for which existence and uniqueness hold as the (strongly) exponential symmetric spaces which have a solvable transvection group, thereby clarifying the class of spaces for which the conjecture of Weinstein about quantization and double triangles may hold without change (see the Introduction).

2.1 Midpoint maps

In this section, we introduce for a general symmetric space the notion of a midpoint map, and study how the properties of the exponential map and the existence of a midpoint map relate. We exhibit some possible obstructions for the exponential map to be a diffeomorphism. After studying the notion of exponential symmetric space, we will see in Section 2.2 that these obstructions never occur.

Following Qian [Qia97], let us introduce a first, coarse, notion of midpoint map on a manifold. This will already impose strong topological conditions.

Definition 2.1.1 (Qian, *op. cit.*, Definition 2.1.1)**.** *Let M be a differential manifold. A smooth map $\gamma : M \times M \to M$ is called* a weak midpoint map *if γ satisfies*

1. $\gamma(x,x) = x$,
2. $\gamma(x,y) = \gamma(y,x)$,

for every $x, y \in M$.

One may check that such a map could not be defined on the circle S^1, for example, because it would not be continuous around antipodal points. Actually this is very general for compact manifolds, as shows the following proposition whose proof is based on a recursive use of Künneth's theorem.

Proposition 2.1.2 (Qian, *op. cit.*, Proposition 2.1.7)**.** *Let M be a connected manifold with a weak midpoint map γ. Then the nonzero-order $\mathbb{Z}_2$-homologies of M are trivial. In particular, there exists no midpoint map on any connected compact manifold without boundary.*

In the symmetric space case, we may improve the above proposition of Qian:

Proposition 2.1.3. *Let (M,s) be a connected symmetric space, and γ be a weak midpoint map on M. Then M is homeomorphic to a Euclidean space.*

Proof. In [Koh65, Theorem 1], it is shown that any connected symmetric space G/H is fibered over a compact Riemannian symmetric space $K/(K \cap H)$ with fibers homeomorphic to a Euclidean space. Denote N the compact Riemannian symmetric subspace of M so defined, and set $\gamma' : N \times N \to N$ such that $\gamma' = \pi \circ \gamma|_{N\times N}$ where $\pi : M \to N$ is the projection on the base of the fibration. Then we easily see that γ' is a midpoint map of N as a differential manifold, which is impossible unless N is a point, by Proposition 2.1.2. That concludes the proof. □

On a symmetric space M —actually on any affine manifold—, locally there is a canonical midpoint map given by the geodesic midpoint $(x, \mathrm{Exp}_x(v)) \mapsto \mathrm{Exp}_x(v/2)$. Hence on symmetric spaces we will ask that the midpoint map really extend the geodesic midpoint. More precisely, $y \mapsto \gamma_x(y) := \gamma(x,y)$ should be the global inverse of the map $m \mapsto s^x(m) := s_m x$.

Definition 2.1.4. *Let (M,s) be a symmetric space. A smooth map $\gamma : M \times M \to M$ is called* a midpoint map, *if $\gamma^{-1}(m)$ is the graph* $\mathrm{Graph}(s_m) = \{(x, s_m x) \mid x \in M\}$ *of s_m, for all $m \in M$.*

This is indeed equivalent to what we wanted.

Lemma 2.1.5. *Let (M,s) be a symmetric space, and $\gamma : M \times M \to M$ be a smooth map. Define $\gamma_x, s^x : M \to M$ by $\gamma_x(y) = \gamma(x,y)$ and $s^x(m) = s(m,x)$ for all $x, y \in M$. Then γ is a midpoint map if and only if for all $x \in M$, s^x is invertible and $\gamma_x = (s^x)^{-1}$.*

Proof. Assume γ is a midpoint map and let $x \in M$. Then for all $y \in M$, $s_{\gamma(x,y)}x = y$ since $(x,y) \in \mathrm{Graph}(s_{\gamma(x,y)})$, hence $s^x\gamma_x y = s_{\gamma(x,y)} = y$; and for all $m \in M$, $\gamma_x s^x m = \gamma(x, s_m x) = m$. Thus s^x is invertible and $\gamma_x = (s^x)^{-1}$.

Conversely, assume s^x is invertible and $\gamma_x = (s^x)^{-1}$ for all $x \in M$. Then if $x, y, m \in M$ are such that $\gamma(x,y) = m$, we have $y = \gamma_x^{-1}(m) = s^x m = s_m x$, so $(x,y) \in \mathrm{Graph}(s_m)$ and $\gamma^{-1}(x) \subset \mathrm{Graph}(s_m)$. For the reverse inclusion, note that for all $x, m \in M$, $\gamma(x, s_m x) = \gamma_x(s^x m) = m$. Hence γ is a midpoint map. □

Lemma 2.1.6. *Let M be a symmetric space. If there exists a midpoint map on M, then it is unique.*

Proof. Let γ, γ' be two such midpoint maps, and let $x, y \in M$. Since $(x,y) \in \gamma^{-1}(\gamma(x,y))$ and since

$$\gamma^{-1}(\gamma(x,y)) = \mathrm{Graph}(s_{\gamma(x,y)}) = \gamma'^{-1}(\gamma(x,y)),$$

we have $\gamma'(x,y) = \gamma(x,y)$. □

Definition 2.1.4 implies at once that a midpoint map is in particular a weak midpoint map: if $\gamma(x,y) = m$ then $y = s_m x$, so that $x = s_m y$ and $\gamma(y,x) = \gamma(y, s_m y) = m = \gamma(x,y)$. And obviously $\gamma(x,x) = x$. Moreover, midpoint maps are symmetry-invariant: if $\gamma(x,y) = m$ then $s_z y = s_z s_m x = s_{s_z m} s_z x$ so that

$$\gamma(s_z x, s_z y) = s_z m = s_z \gamma(x,y).$$

The three properties of γ we just mentioned were obtained from the corresponding three properties of s, and actually similar computations show that we have an equivalence between two sets of axioms. For this, first note that in a symmetric space with midpoint map, not only x is an isolated fixed point of s_x, but it is the only one.

Lemma 2.1.7. *Let (M,s,γ) be a symmetric space with midpoint map. Then x is the only fixed point of s_x, for all $x \in M$.*

Proof. Let $x \in M$ and assume $y \in M$ is such that $s_x y = y$. Then $x = \gamma(y, s_x y) = \gamma(y,y) = y$. □

This very simple observation that the "isolated" condition is not needed is without doubt the reason why symmetric spaces with midpoint maps may be studied in other categories than that of smooth manifolds without losing any of the basic properties of these spaces (non-symmetric reflection spaces do not appear, for example), as was done for finite symmetric spaces by Nobusawa and collaborators [Nob74, Nob77, Nag79] and by Lawson and Lim in a broad algebraic context [LL04b].

Lemma 2.1.8. *Let M be a differential manifold and $s, \gamma : M \times M \to M$ two smooth maps. Define γ_x, s_x and s^x as above. Then (M,s,γ) is a symmetric space with midpoint map if and only if*

1. $s(x,x) = x$,
2. $s_x^2 = \mathrm{Id}$,
3. $s_x s(y,z) = s(s_x y, s_x z)$,
4. $\gamma^{-1}(x) = \mathrm{Graph}(s_x)$

for all $x,y,z \in M$, if and only if

1. $\gamma(x,x) = x$,
2. $\gamma(x,y) = \gamma(y,x)$,
3. $s_x \gamma(y,z) = \gamma(s_x y, s_x z)$,

4. $s^{-1}(x) = \mathrm{Graph}(\gamma_x)$

for all $x, y, z \in M$.

Concerning the exponential map, the uniqueness of the fixed points of the symmetries obviously implies the uniqueness of geodesics joining two points.

Lemma 2.1.9. *Let* (M, s, γ) *be a symmetric space with midpoint map. Then for each* $x \in M$*, the geodesics through* x *only intersect at* x*, that is,* Exp_x *is injective.*

Proof. Assume $x, y \in M$ are joined by two geodesics. Denote z and z' the two respective geodesic midpoints. Then $(x, y) \in \mathrm{Graph}(s_z) = \gamma^{-1}(z)$ and $(x, y) \in \mathrm{Graph}(s_{z'}) = \gamma^{-1}(z')$ so that $z = z'$. Replacing successively x and y by z, and iterating the process, we obtain that all iterated geodesic midpoints of x and y coincide for the two geodesics. As those points are dense in their respective geodesics, the latter must coincide entirely. □

In fact, the exponential map is a diffeomophism onto its image.

Lemma 2.1.10. *Let* (M, s, γ) *be a symmetric space with midpoint map. Then for all* $x \in M$, Exp_x *is a diffeomorphism from* T_xM *onto its image.*

Proof. Let U be an open neighbourhood of $0 \in T_xM$ such $\mathrm{Exp}_x|_U$ is a diffeomorphism onto its image. Now s^x is a diffeomorphism and we have $s^x \mathrm{Exp}_x(v) = \mathrm{Exp}_x(2v)$ for all $v \in T_xM$. Thus denoting d_λ the multiplication by $\lambda \in \mathbb{R}$ in T_xM, $\mathrm{Exp}_x|_{2U} = s^x \circ \mathrm{Exp}_x|_U \circ d_{1/2}$ is a composition of diffeomorphisms, so it is one too. By induction, we get the result. □

An interesting question is to know whether the exponential map of a connected symmetric space with midpoint map is also surjective. Put another way, could any two points of such a space be joined by a unique geodesic? As we will see in Section 2.2, this is indeed always the case, but the proof really uses Lie theory whereas the problem seems to be of a purely geometric nature. It would be interesting to find a geometric proof of this fact. Here is one possible obstruction, which we can rule out for solvable symmetric spaces, but again using Lie theory.

Definition 2.1.11. *For any odd integer* n*, define the* n-points space (M_n, s, γ) *as the 0-dimensional symmetric space* $M = \{x_0, \ldots, x_{n-1}\}$ *with symmetries given by* $s_{x_i} x_j = x_{2i-j \mod n}$*, and midpoint map* $\gamma(x_i, x_j) = x_{(i+j)/2}$ *if* i *and* j *have the same parity, and* $\gamma(x_i, x_j) = x_{(i+j+n)/2 \mod n}$ *if* i *and* j *have different parity.*

These spaces may be realized as evenly spaced points on the circle, with symmetries induced by the usual symmetric structure on S^1, and midpoint map given for points x_i and x_j as the only x_k lying in the intersection of the

circle and the bisector of the line segment joining x_i to x_j. This is well-defined since the spaces have an odd number of points, and thus no antipodal pairs of points. They form the simplest class of *finite* symmetric spaces with midpoints, as studied by Nobusawa and his collaborators [Nob74]. The transvection group of M_n is the abelian torsion group $\mathbb{Z}_n$, and it has a simply transitive action on it.

The reason why these spaces are interesting for us is that their midpoint map has a strange behaviour. In metric spaces, where midpoints are defined as being at equal and minimal distance of two given points x_0 and x_1, one may build a geodesic by taking their midpoint $x_{1/2}$, and then take $x_{1/4}$ and $x_{3/4}$ the midpoints of $(x_0, x_{1/2})$ and $(x_{1/2}, x_1)$ respectively, and continue inductively to build the dyadic segment joining x_0 to x_1. The closure of this dyadic line gives the desired geodesic. For this to work, the essential property is that for any two points x and y, the sequence $(y_k)_{k\in\mathbb{N}}$ defined by $y_0 = y$ and $y_k = \gamma(x, y_{k-1})$ for all $k \geq 1$, converges to x.

The spaces at hand are prototypical of the opposite situation. Indeed we may see from the formula defining the midpoint map, that $\gamma(x_i, x_j)$ is equal to x_i or x_j if and only if $i = j$. Taking into account the finiteness of the space, this implies that starting from two different points x and y, the sequence (y_k) is a loop passing through y and not through x.

Hence if there existed an injective morphism of an n-points space into a *connected* symmetric space with midpoint map (M, s, γ), then the exponential map of the latter couldn't be surjective since for any two points x and y in the image of the morphism, the sequence (y_n) would loop through a finite number of points while if they were joined by a geodesic, the sequence would clearly converge to x. We have thus proved the following proposition.

Proposition 2.1.12. *Let (M, s, γ) be a connected symmetric space with midpoint map. If there exists an injective morphism of the n-points space into M, then there exist points in M not joined by a geodesic.*

Recall the following result due to H. Wang [Wan56] (see also Mostow [Mos54]) on the fundamental group of solvmanifolds, that is, of manifolds on which solvable Lie groups of transformations act transitively.

Definition 2.1.13. *A* $\mathfrak{S}$-group *is a topological group L isomorphic with an extension, by a finitely generated discrete abelian group H, of a torsion free nilpotent Lie group whose component group is finitely generated. If H is torsion free, then L is called a* strongly torsion free $\mathfrak{S}$-group.

Theorem 2.1.14 ([Wan56])**.** *A discrete group can be embedded in a connected and simply connected solvable Lie group (can be realized as the fundamental group of a solvmanifold) if and only if it is a strongly torsion free $\mathfrak{S}$-group.*

This result gives a first indication of why solvable symmetric spaces with midpoint maps might possibly all have a surjective exponential map.

Proposition 2.1.15. *Assume there exists an injective morphism of a n-space into a connected symmetric space with midpoint map (M, s, γ). Then M cannot be a solvable symmetric space.*

Proof. Assume M is a solvable symmetric space. Then there exists a symmetric pair (G, σ) such that G is a connected and simply connected solvable Lie group and $G/G^\sigma \cong M$. If there were an injective morphism of an n-space into M, it would also yield an injection of $\mathbb{Z}_n$ in G, contradicting Theorem 2.1.14. □

2.2 Exponential symmetric spaces

In this section, we are interested in characterizing the symmetric spaces whose exponential map is a global diffeomorphism. We will see in Theorem 2.2.20 that this is the case if and only if the symmetric space is connected and admits a midpoint map, and will further obtain a result generalizing to symmetric spaces the characterization of exponential solvable Lie groups. We build on what has been said in Section 1.4 about realizations of symmetric spaces as subsets of Lie groups.

The following two results are essentially standard, see [Loo69], [Ben84], [Roz09] for example.

Lemma 2.2.1. *Let (G, σ) be a symmetric pair, H a subgroup of G and $\pi : G \to G/H$ the canonical projection. If there exists a smooth section $s : G/H \to G$ of π, then the multiplication $m : s(G/H) \times H \to G$ is a diffeomorphism onto.*

Proof. Assume there exists a smooth section $s : G/H \to G$ of π, and let $g \in G$. We have $\pi \circ s = \mathrm{Id}$, and g and $s(\pi(g))$ lie in the same fibre of π. Consequently, $p = s(\pi(g)) \in s(G/H) \subset G$ and $h = s(\pi(g))^{-1} g \in H$ are such that $m(p, h) = g$, and the map $g \mapsto (p, h)$ is smooth. For the other way, let $p = s(g'H) \in s(G/H)$ and $h \in H$. Then the map m is smooth and $g = m(p, h)$ is such that $s(\pi(g)) = s(\pi(s(g'H))) = s(g'H) = p$ and $s(\pi(g))^{-1} g = h$ so that we indeed have two smooth maps inverse of each other. □

Proposition 2.2.2. *Let (G, σ) be a symmetric pair. Denote G^σ, M_G and P as above, $\pi : G \to G/G^\sigma$ the canonical projection, m the multiplication in G, $\iota : \mathfrak{m} \to \mathfrak{g}$ the inclusion, and $\gamma : G \to G : g \mapsto \alpha_g e = g\sigma(g)^{-1}$. Then the following conditions are equivalent:*

1. *the map $\pi \circ \exp \circ \iota$ is a diffeomorphism from $\mathfrak{m}$ onto G/G^σ;*

2. *the map* $\gamma \circ \exp \circ \iota$ *is a diffeomorphism from* $\mathfrak{m}$ *onto* M_G*;*

3. *G admits a global decomposition as* $(\exp \mathfrak{m}).G^\sigma$*, more precisely, the map* $m \circ ((\exp \circ \iota) \times \mathrm{Id})$ *is a diffeomorphism from* $\mathfrak{m} \times G^\sigma$ *onto G.*

If moreover, the set $P \cap G^\sigma = \{k \in G^\sigma \,|\, k^2 = e\}$ *is reduced to* $\{e\}$*, then the above conditions are also equivalent to:*

4. *the map* $\exp \circ \iota$ *is a diffeomorphism from* $\mathfrak{m}$ *onto P;*

Proof. 1. $\Leftrightarrow$ 2. As we saw, the map $f : G/G^\sigma \to M_G : gG^\sigma \mapsto g\sigma(g)^{-1}$ (such that $f \circ \pi = \gamma$) is a diffeomorphism onto.

1. $\Rightarrow$ 3. The given diffeomorphism shows that $\exp \circ \iota$ is a diffeomorphism from $\mathfrak{m}$ onto $\exp \mathfrak{m}$, and defines a smooth section $s = \exp \circ (\pi \circ \exp \circ \iota)^{-1} : G/G^\sigma \to \exp \mathfrak{m} \subset G$. By Lemma 2.2.1, this yields a global decomposition $G = \exp \mathfrak{m} \times G^\sigma$, which together with $\exp \circ \iota$ gives the awaited map.

3. $\Rightarrow$ 1. Clear.
Now assume that the set $P \cap G^\sigma = \{k \in G^\sigma \,|\, k^2 = e\}$ is reduced to $\{e\}$.

2. $\Rightarrow$ 4. Denote $\alpha : \mathfrak{m} \to M_G$ the diffeomorphism $\gamma \circ \exp \circ i$. Let $p \in P$. Then $p^2 = p\sigma(p)^{-1} \in M_G$ so that $p^2 = \alpha(X) = \exp(2X)$ for some unique $X \in \mathfrak{m}$. As $\gamma(p) = \gamma(\exp(X))$ we must have $p = \exp(X)k$ for some $k \in G^\sigma$. But then $\exp(X)k = p = \sigma(p)^{-1} = k^{-1}\exp(X)$. So $\exp(\mathrm{Ad}(k)X)k^2 = \exp(X)$ which, applying $\alpha^{-1} \circ \gamma$ on both sides, implies that $\mathrm{Ad}(k)X = X$. But this gives $\exp(X)k^2 = \exp(X)$ hence $k^2 = e$ or equivalently $k \in P \cap G^\sigma$. By hypothesis, $k = e$ and thus $p \in \exp \mathfrak{m}$. We have thus seen that $P = \exp \mathfrak{m} = M_G$, and α is the required diffeomorphism.

4. $\Rightarrow$ 2. Since P is diffeomorphic to $\mathfrak{m}$, it is connected. As discussed above, M_G is a symmetric subspace of P, which is a connected symmetric subspace of G viewed as a symmetric space. Since M_G and P have the same tangent space at e, they must then be equal [Loo69, III, Corollary of Proposition 1.2 and Lemma 1.3]. □

Note that whenever G^σ has a compact subgroup, it contains in particular a compact one-parameter subgroup, hence a non-trivial $k \in G^\sigma$ such that $k^2 = e$, so that P might be disconnected. This is the case in Example 1.4.19 where $P \cap G^\sigma = \{I, -I\}$ and P has two connected components, namely the orbits of I and $-I$ under the action α defined in (1.3). But beware that it is not always so, it could be that $P \cap G^\sigma$ be disconnected and P only comprise one orbit, see Example 2.2.9.

Definition 2.2.3. *A symmetric pair (G, σ) is called* exponential *if any of the first three equivalent conditions of Proposition 2.2.2 is satisfied.*

Definition 2.2.4. *A symmetric space M is called* exponential *if there exists a point $o \in M$ such that the exponential map Exp_o at o of the canonical connection is a diffeomorphism from T_oM onto M.*

It is clear that these definitions characterize the same spaces:

Proposition 2.2.5. *Let M be a symmetric space. The following conditions are equivalent:*

1. *M is exponential;*
2. *Exp_x is a diffeomorphism from T_xM onto M for every $x \in M$;*
3. *M is connected and there exists a point $o \in M$ and a connected transitive group G of automorphisms of M stabilized by the involution $\sigma(g) = s_o g s_o$, such that (G, σ) is exponential;*
4. *every symmetric pair (G, σ) such that the associated symmetric space G/G^σ is isomorphic to M, is exponential.*

Proof. 1. $\Rightarrow$ 2. Let o be the point at which Exp_o is a diffeomorphism, and let $x \in M$. Let $x \in \mathrm{Aut}(M, s)$ such that $x = g \cdot o$. Then $\mathrm{Exp}_x = g \circ \mathrm{Exp}_o \circ g_{*_x}^{-1}$, so that Exp_x is also a diffeomorphism onto.

2. $\Rightarrow$ 1. Evident.

1. $\Leftrightarrow$ 4. Let (G, σ) be a symmetric pair such that G/G^σ is isomorphic to M. As an isomorphism is an affinity, we may without loss of generality identify M with G/G^σ and o with eG^σ. We may as well identify $\mathfrak{m}$ with $T_{eG^\sigma} G/G^\sigma$ through π_{*_e}. Then as Exp_{eG^σ} is precisely the map $\pi \circ \exp \circ i$, we have that Exp_{eG^σ} is a diffeomorphism from $\mathfrak{m}$ onto G/G^σ if and only if (G, σ) is exponential.

4. $\Rightarrow$ 3. Let G be the transvection group of M. It is connected since M is connected. By [Loo69, IV, Theorem 4.6] there is a covering group G' and an involutive automorphism σ of G' such that G'/G'^σ is isomorphic to M. Now (G', σ) must be exponential, by hypothesis.

3. $\Rightarrow$ 1. Note that since G is connected and G/G^σ is simply connected, G^σ is connected. Hence M, which is isomorphic to G/H with $(G^\sigma)_0 \subset H \subset G^\sigma$, must be isomorphic to G/G^σ. We may then proceed as in the proof of 1. $\Leftrightarrow$ 4.

□

When the exponential map of a symmetric space is a diffeomorphism onto, the geodesic midpoints of course provide a midpoint map, as we show just below. The rest of this section will then be devoted to showing the converse of this proposition, namely that every connected symmetric space with midpoint map is in fact exponential.

Proposition 2.2.6. *Let M be an exponential symmetric space. Then there exists a midpoint map on M.*

Proof. Let $x, y \in M$, and $c : \mathbb{R} \to M$ be the geodesic $c(t) = \mathrm{Exp}_x(t \mathrm{Exp}_x^{-1}(y))$. Define $\gamma(x,y) = c(1/2)$. Now, since $c(0) = x$ and $c(1) = y$, it is clear that $\gamma(x,y) = \gamma(y,x)$, that if $y = x$ then $\gamma(x,y) = \gamma(x,x) = x$, and that $s_{\gamma(x,y)}x = y$. Moreover, since the map $TM \to M \times M : (x,X) \mapsto (x, \mathrm{Exp}_x(X))$ is a diffeomorphism, γ is indeed smooth. □

Recall the definition of the equivalent notion for Lie groups, that of exponential Lie group.

Definition 2.2.7. *A Lie group G is said to be* exponential *if the exponential map from its Lie algebra $\mathfrak{g}$ to G is injective.*

It is well known [BCD+72, Chap I, Thm 2.1 and Rem 2.9] that the previous definition implies that G is solvable, that the exponential map is a diffeomorphism from $\mathfrak{g}$ onto G, and that the adjoint action of the Lie algebra $\mathfrak{g}$ has no purely imaginary eigenvalues. In fact, if G is a connected and simply connected solvable Lie group, there is a well-known characterization, stated in Proposition 2.2.8. To state it, let us recall the notion of roots of a solvable Lie algebra.

Let $\mathfrak{g}$ be a real solvable Lie algebra. Then [Puk67, Bou98, Chap. 1, §5] there exists a sequence of ideals $\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{0\}$ of $\mathfrak{g}$ such that

1. $\dim \mathfrak{g}_j/\mathfrak{g}_{j+1} \leq 2$, for each $0 \leq i \leq n-1$,
2. if $\dim \mathfrak{g}_j/\mathfrak{g}_{j+1} = 2$, the representation induced by the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}_j/\mathfrak{g}_{j+1}$ is irreducible.

To each one-dimensional quotient $\mathfrak{g}_j/\mathfrak{g}_{j+1}$ corresponds a linear form γ on $\mathfrak{g}$ given by

$$\mathrm{ad}(X)Y = \gamma(X)Y \mod \mathfrak{g}_{j+1}$$

for any $Y \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$. Similarly, for each two-dimensional quotient $\mathfrak{g}_j/\mathfrak{g}_{j+1}$, there exist two vectors $Y, Z \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$ linearly independent modulo $\mathfrak{g}_{j+1}$ such that the

adjoint action is given by

$$\begin{aligned} \mathrm{ad}(X)Y &= \alpha(X)Y - \beta(X)Z \mod \mathfrak{g}_{j+1} \\ \mathrm{ad}(X)Z &= \beta(X)Y + \alpha(X)Z \mod \mathfrak{g}_{j+1} \end{aligned}$$

for some linear forms α and β on $\mathfrak{g}$. In other words, in the complexified Lie algebra, we have

$$\mathrm{ad}(X)(Y \pm iZ) = (\alpha \pm i\beta)(X)(Y \pm iZ) \mod \mathfrak{g}_{j+1}.$$

The linear forms $\alpha \pm i\beta, \gamma$ for the different quotients $\mathfrak{g}_j/\mathfrak{g}_{j+1}$ are called *the roots of* $\mathfrak{g}$. Up to their order, they are indenpendent of the choices made, they all vanish on the derived ideal of $\mathfrak{g}$, and for a fixed $X \in \mathfrak{g}$ the set of their values on X are the eigenvalues of $\mathrm{ad}(X)$.

Proposition 2.2.8 ([Dix57, Sai57, BCD$^+$72])**.** *Let G be a connected and simply connected solvable Lie group. Then the following conditions are equivalent:*

1. *G is exponential,*
2. *the exponential map is surjective,*
3. *the exponential map is a diffeomorphism,*
4. *all the roots of $\mathfrak{g}$ are of the form $(1+ib)\phi(X)$ where $\phi \in \mathfrak{g}^*$ and $b \in \mathbb{R}$,*
5. *no $\operatorname{ad} X$, $X \in \mathfrak{g}$, has non-zero purely imaginary eigenvalues.*

In particular, a connected and simply connected *completely solvable* —that is, with only real eigenvalues for the adjoint action— or nilpotent Lie group is exponential.

Example 2.2.9. Let G be a Lie group and consider the symmetric pair $(G \times G, \sigma_G)$ with $\sigma_G(g,h) = (h,g)$. We have $\mathfrak{m} = \{(X,-X) \mid X \in \mathfrak{g}\}$ so that

$$\exp \mathfrak{m} = \{(\exp X, \exp -X) \mid X \in \mathfrak{g}\}.$$

On the other hand, $(g,h)\sigma(g,h)^{-1} = (gh^{-1}, hg^{-1})$ for all $g,h \in G$, so that

$$\begin{aligned} M_{G\times G} &= \{(gh^{-1}, hg^{-1}) \mid g,h \in G\} \\ &= \{(g,g^{-1}) \mid g \in G\} \end{aligned}$$

and

$$P = \{(g,h) \in G \times G \mid \sigma(g,h) = (g,h)^{-1}\}$$

$$= \{(g, g^{-1}) \mid g \in G\}.$$

Thus we always have $M_{G\times G} = P$, even if $P \cap (G \times G)^{\sigma_G} \neq \{0\}$, and $(G \times G, \sigma_G)$ is exponential if and only if G is exponential as a Lie group. ◁

When G is exponential, we have the following proposition of Y. Benoist.

Proposition 2.2.10 ([Ben84, Proposition 2.1])**.** *Let (G, σ) be a symmetric pair, with G exponential. Then,*

1. *the exponential map is a diffeomorphism from $\mathfrak{g}^\sigma$ onto G^σ;*
2. *the exponential map is a diffeomorphism form $\mathfrak{m}$ onto P;*
3. *the multiplication m is a diffeomorphism from $P \times G^\sigma$ onto G.*

To compute the existence of a midpoint map, the following characterization may be useful. It was proved in the language of twisted subgroups of involutive (not necessarily Lie) groups, in [LL04b], which itself cites many references in the Bol loops literature.

Proposition 2.2.11 ([LL04b, Theorem 7.4])**.** *Let (G, σ) be a symmetric pair, and denote G^σ and M_G as above, $M = G/G^\sigma$ and $\pi : G \to M$ the canonical projection. The following conditions are equivalent:*

1. *M admits a midpoint map,*
2. *there exists a smooth section of π with values in $M_G \subset G$,*
3. *the polar map $M_G \times G^\sigma \to G : (p, k) \mapsto p.k$ is a global diffeomorphism.*

Proof. 1. $\Rightarrow$ 2. Assume M has a midpoint map γ. Denote $r : M_G \to M_G$ the "square root" diffeomorphism given by the map $M \to M : gG^\sigma \to \gamma(eG^\sigma, gG^\sigma)$ seen through the isomorphism $f : M \to M_G : gK \mapsto g\sigma(g)^{-1}$. We then have $r^{-1}(g\sigma(g)^{-1}) = s_{g\sigma(g)^{-1}}e = g\sigma(g)^{-1}.e.g\sigma(g)^{-1}$, that is, $r^{-1}(p) = p^2$ for all $p \in M_G$. But this is equal to $f \circ \pi|_{M_G}$ since if $p \in M_G$ we have $\sigma(p) = p^{-1}$ so that $f(\pi(p)) = f(pG^\sigma) = p\sigma(p)^{-1} = p^2 = r^{-1}(p)$. Thus defining $s : M \to G$ by $s = r \circ f$, we have $f(\pi(s(gG^\sigma))) = r^{-1}(r(f(gG^\sigma))) = f(gG^\sigma)$. Hence we have defined a smooth section of π with values in M_G.

2. $\Rightarrow$ 3. This follows from Lemma 2.2.1.

3. $\Rightarrow$ 1. Assume the multiplication $M_G \times G^\sigma \to G$ is a diffeomorphism. We then have a diffeomorphism $t : M_G \to M$ such that $(f \circ t)(p) = p^2$. Define $\gamma_o : M \to M$ by $\gamma_o = (t \circ f)^{-1}$. Then $\gamma_o^{-1} = s^o$ since $\gamma_o^{-1}(gG^\sigma) = t(g\sigma(g)^{-1}) = g\sigma(g)^{-1}G^\sigma = s_{gG^\sigma}eG^\sigma$. Defining $\gamma_{gG^\sigma}(x) = g\gamma_o(g^{-1}x)$ we also have $\gamma_x^{-1} = s^x$ for all $x \in M$. Thus by Lemma 2.1.5, M has a midpoint map.

□

Example 2.2.12. Let (G,σ) be the symmetric pair of Example 1.4.20. Then the symmetric space G/G^σ admits no midpoint map.

Indeed, we compute for general $p = (2t, c - e^{2it}\bar{c}) \in P$ and $k = (0,d) \in G^\sigma$ ($d \in \mathbb{R}$) that

$$pk = (2t, (a + \cos(2t)(d-a) - \sin(2t)b) + i(b + \cos(2t)b + \sin(2t)(d-a))).$$

Thus, when $2t = \pi$ we get $pk = (\pi, (2a-d)+i.0)$ for all $a,b,d \in \mathbb{R}$, so that the polar map is not surjective. Hence by Proposition 2.2.11, G/G^σ admits no midpoint map. ◁

We now come to the link between the existence of a midpoint map and the exponentiality. Recall first the following fact about the exponential map of a symmetric space.

Proposition 2.2.13 ([Hel78, Theorem 4.1])**.** *Let (G,σ) be a symmetric pair, and for $g \in G$ denote $\tau(g)$ the map $hG^\sigma \mapsto ghG^\sigma$ from G/G^σ onto itself. The differential of the exponential map from $\mathfrak{m}$ to G/G^σ is given by*

$$\mathrm{Exp}_{*X} = \tau(\exp X)_{*eG^\sigma} \circ \frac{\sinh \mathrm{ad}\, X}{\mathrm{ad}\, X}$$

where the function $z \mapsto \frac{\sinh(z)}{z}$ is interpreted as the sum of the absolutely convergent series $\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k+1)!}$.

Definition 2.2.14. *A symmetric pair (G,σ) is said* to have property $(\mathfrak{m})$ *if G/G^σ is connected and simply connected, and if no $\mathrm{ad}\, X$, $X \in \mathfrak{m}$, has non-zero purely imaginary eigenvalues.*

We want to unveil the fact that the property $(\mathfrak{m})$, the existence of a midpoint map, and the exponentiality are three equivalent properties. We first show that the existence a midpoint map implies property $(\mathfrak{m})$.

Proposition 2.2.15. *Let (M,s,γ) be a connected symmetric space with midpoint map and let (G,σ) be a symmetric pair such that $M \cong G/G^\sigma$. Then (G,σ) has property $(\mathfrak{m})$.*

Proof. By Proposition 2.1.3, G/G^σ is connected and simply connected. And by Lemma 2.1.10, the exponential map of G/G^σ is everywhere a local diffeomorphism, thus for all $X \in \mathfrak{m}$ we have

$$\det\left(\frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X}\right) \neq 0.$$

But the eigenvalues of that map are of the form $\frac{\sinh(\lambda)}{\lambda}$ with λ an eigenvalue of $\operatorname{ad} X$, and this is easily seen to vanish if and only if $\lambda = ik\pi$ for some $k \in \mathbb{Z}_*$, so that concludes the proof. □

To see that property $(\mathfrak{m})$ implies exponentiality, we start by showing the result for solvable and semisimple symmetric pairs.

Proposition 2.2.16. *Let (G, σ) be a solvable symmetric pair with property $(\mathfrak{m})$. Then (G, σ) is exponential.*

Proof. Consider the transvection group G' of $M = G/G^\sigma$, and σ' an involution such that $M \cong G'/H$ for some closed subgroup H with $(G'^{\sigma})_0 \subset H \subset G'^{\sigma}$. Denote $\pi : \tilde{G}' \to G'$ the universal covering of G', and still σ' the involution of G' lifted to $\tilde{G}'$. Then $\tilde{G}'$ acts on M with stabilizer $\pi^{-1}(H)$ and we have $(\tilde{G}'^{\sigma'})_0 \subset \pi^{-1}(H) \subset \tilde{G}'^{\sigma'}$. But the fixed point subgroup of an involution of a simply connected Lie group is always connected [Loo69, IV, Theorem 3.4], so $\pi^{-1}(H) = \tilde{G}'^{\sigma'}$ and $M \cong \tilde{G}'/\tilde{G}'^{\sigma'}$.

We saw in Proposition 2.2.5 that if two symmetric pairs (G, σ) and (G', σ') are such that $G/G^\sigma \cong G'/G'^{\sigma'}$, then they are simultaneously exponential or not. Thus we may suppose that G is the universal cover $\tilde{G}'$ of the transvection group of M.

Now since $\mathfrak{g}$ is solvable and is the transvection Lie algebra, we see that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ is contained in the nilradical of $\mathfrak{g}$ so that for all $X \in \mathfrak{k}$, $\operatorname{ad} X$ has all eigenvalues equal to zero. By Lie's theorem, the complexification of $\mathfrak{g}$ may be represented by upper triangular matrices, and the representatives of $\mathfrak{k}$ must then be strictly upper triangular. This shows that the eigenvalues of a general element $X = X_\mathfrak{k} + X_\mathfrak{m} \in \mathfrak{g}$ are equal to those of $X_\mathfrak{m} \in \mathfrak{m}$, which cannot be purely imaginary. Hence G is exponential by Proposition 2.2.8, and consequently G/G^σ is exponential by Proposition 2.2.10. □

We may be slightly more precise with the following result.

Proposition 2.2.17. *Let M be a solvable exponential symmetric space. Then its transvection group G is exponential and, if σ is the conjugation by s_o for some base point $o \in M$, then $M \cong G/G^\sigma$.*

Proof. We have seen in the proof of Proposition 2.2.16 that the universal cover $\tilde{G}$ of G is exponential, that $\tilde{G}^{\tilde{\sigma}}$ is connected, and that $M \cong \tilde{G}/\tilde{G}^{\tilde{\sigma}}$.

By [Loo69, Corollary, p. 171] [1], all we have to show is that $Z(\tilde{G}) \cap \tilde{G}^{\tilde{\sigma}}$ is reduced to a point, since then we will have $G = \tilde{G}$ and $G^\sigma = \tilde{G}^{\tilde{\sigma}}$.

[1] **Corollary.** *Let M be a simply connected symmetric space with transvection group G, and let $\tilde{G}$ be its universal cover. Then $G \cong \tilde{G}/Z(\tilde{G})^\sigma$ where $Z(\tilde{G})^\sigma = \{z \in Z(\tilde{G}) \mid \sigma(g) = g\}$.*

Let $z \in Z(\tilde{G}) \cap G^\sigma$. Since $\tilde{G}$ is exponential, there exists a unique $Z \in \mathfrak{g}$ such that $z = e^Z$. Since z is in the center of $\tilde{G}$, we have

$$e^Z = g e^Z g^{-1} = e^{\mathrm{Ad}_g Z},$$

so that $Z = \mathrm{Ad}_g Z$ by exponentiality, for all $g \in \tilde{G}$. For the same reason, this implies that X is in the center of $\mathfrak{g}$.

But z is also in $\tilde{G}^{\tilde{\sigma}}$, so that $Z \in \mathfrak{k}$. Since $\mathfrak{g}$ is the transvection algebra, the action of $\mathfrak{k}$ on $\mathfrak{m}$ is effective, and the only central element in $\mathfrak{k}$ is 0. Hence $z = e$, and that concludes the proof. □

Proposition 2.2.18. *Let (G, σ) be a semisimple symmetric pair with property $(\mathfrak{m})$. Then G/G^σ is a Riemannian symmetric space of the noncompact type. In particular, (G, σ) is exponential, and if G acts effectively on G/G^σ, σ is a Cartan involution.*

Proof. Let K' be the largest normal subgroup of G contained in G^σ. Then $(G/K')/(G^\sigma/K') \cong G/G^\sigma$, G/K' acts effectively, and by Proposition 2.2.5 we may replace G by G/K' to prove the exponentiality.

Observe that now $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$ since if it were not the case, then $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ would be a proper ideal of $\mathfrak{g}$, and its orthogonal for the Killing form B would be a non-trivial ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$, contradicting the preceding paragraph.

Let θ be a Cartan involution commuting with σ ([Ber57], [Loo69, p. 153, Theorem 2.1]). We have the decompositions in eigenspaces

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{m} \quad \text{for } \sigma \\ &= \mathfrak{h} \oplus \mathfrak{n} \quad \text{for } \theta \\ &= \mathfrak{k}_\mathfrak{h} \oplus \mathfrak{k}_\mathfrak{n} \oplus \mathfrak{m}_\mathfrak{h} \oplus \mathfrak{m}_\mathfrak{n}, \end{aligned}$$

where the spaces of the last line are the obvious intersections of those of the first two lines. Now if $0 \neq T \in \mathfrak{m}_\mathfrak{h}$, then T is in $\mathfrak{h}$ so $\mathrm{ad}\, T$ is a skewsymmetric matrix for the scalar product $B_\theta(X, Y) = -B(X, \theta Y)$. Since the adjoint representation is faithful, $\mathrm{ad}\, T$ has non-zero purely imaginary eigenvalues, which is impossible. Hence $\mathfrak{m}_\mathfrak{h} = \{0\}$. But then $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}_\mathfrak{n}, \mathfrak{m}_\mathfrak{n}] = \mathfrak{k}_\mathfrak{h}$, so that $\mathfrak{k}_\mathfrak{n} = 0$. We have thus shown that $\sigma = \theta$.

Since σ is a Cartan involution on G and G is connected, the global polar decomposition for Riemannian symmetric spaces of the noncompact type [Hel78, Chapter VI, Theorem 1.1] implies that (G, σ) is exponential. □

Using these two special cases, we may now prove the general result by induction on the dimension of the radical. The argument is inspired from [Roz09].

Theorem 2.2.19. *Let (G,σ) be a symmetric pair with property $(\mathfrak{m})$. Then (G,σ) is exponential.*

Proof. Let R be the radical of G. If R is 0-dimensional, the theorem reduces to Proposition 2.2.18, so let us assume the contrary.

Reasoning as in the proof of Proposition 2.2.16, we may assume that G is simply connected, so that we can find a global Levi decomposition $G = S \ltimes R$, with S semisimple and R the radical, both simply connected.

Let N be a minimal σ-invariant normal subgroup of G. It is contained in R, and is connected since otherwise its connected component containing e would be preserved by σ and, G being connected, by conjugation by G. Similarly, N is Abelian, since otherwise the commutant would be a smaller connected σ-invariant normal subgroup. As a connected subgroup of a simply connected solvable Lie group, N is also closed and simply connected ([Mal45]).

Let $\hat{G} = G/N$, let $\pi : G \to \hat{G}$ be the natural projection, $\hat{\sigma}$ the induced involution and $\pi' : G/G^\sigma \to \hat{G}/\hat{G}^{\hat{\sigma}}$ the natural morphism. Since G/G^σ is connected and simply connected, and N/N^σ is connected, $\hat{G}/\hat{G}^{\hat{\sigma}}$ is connected and simply connected. Moreover, the eigenvalues of $\operatorname{ad}\mathfrak{m}/\mathfrak{m}'$ are to be found amongst those of $\operatorname{ad}\mathfrak{m}$, so that $(\hat{G},\hat{\sigma})$ has property $(\mathfrak{m})$ and, by the inductive assumption, is exponential.

The map π' realizes G/G^σ, which is simply connected, as a fiber bundle over the contractible space $\hat{G}/\hat{G}^{\hat{\sigma}}$, thus the fiber bundle is trivial and the fiber N/N^σ is also simply connected.

Let $x \in G/G^\sigma$. Then there exists a unique $X \in \hat{\mathfrak{m}}$ such that $\operatorname{Exp}_{e\hat{G}^{\hat{\sigma}}} X = \pi'(x)$. Denote $F \subset G$ the complete preimage through π of $\{\exp tX \mid t \in \mathbb{R}\}$. It is a σ-invariant subgroup isomorphic to $\exp\langle X'\rangle \ltimes N$ for some $X' \in \mathfrak{m}$ such that $\pi_* X' = X$. Since N/N^σ is connected and simply connected, $F/F^\sigma = F/N^\sigma$ also is, and thus (F,σ) inherits property $(\mathfrak{m})$ from (G,σ). Since F is solvable, F/F^σ is then exponential by Proposition 2.2.16, so that there exists a unique $X'' \in \mathfrak{f} \cap \mathfrak{m}$ such that $\operatorname{Exp}_{eG^\sigma} X'' = x$.

We thus have a bijection $\operatorname{Exp}_{eG^\sigma}$ between $\mathfrak{m}$ and G/G^σ, and property $(\mathfrak{m})$ shows that it is everywhere a local diffeomorphism, hence a global diffeomorphism. That concludes the proof. □

We are now in a position to state the main result of this section.

Theorem 2.2.20. *Let (M,s) be a symmetric space. Then the following conditions are equivalent:*

1. *M is connected and admits a midpoint map,*
2. *M is exponential,*

3. *there exists a symmetric pair* (G,σ) *having property* $(\mathfrak{m})$ *and such that* $M \cong G/G^\sigma$,

4. *every symmetric pair* (G,σ) *such that* $M \cong G/G^\sigma$, *has property* $(\mathfrak{m})$.

Moreover, in case any of these four conditions is satisfied,

i. *if* (M,s) *is a solvable symmetric space, then its transvection group* G *is exponential and* $M \cong G/G^\sigma$ *with* σ *the conjugation by* s_o *for some* $o \in M$,

ii. *if* (M,s) *is a semisimple symmetric space, then it is a Riemannian symmetric space of the noncompact type.*

Proof. If M admits a midpoint map, let (G,σ) be a symmetric pair such that $G/G^\sigma \cong M$. Then by Proposition 2.2.15, the latter has property $(\mathfrak{m})$, so that 1. implies 3. Moreover, 3. and 4. are equivalent by Proposition 2.2.5.

But if there exists a symmetric pair (G,σ) having property $(\mathfrak{m})$ and such that $G/G^\sigma \cong M$, by Theorem 2.2.19, G/G^σ is exponential, which by Proposition 2.2.5 is equivalent to M being exponential. Thus 3. implies 2.

Finally, if M is exponential, then M admits a midpoint map, by Proposition 2.2.6. Hence 2. implies 1.

Now, if all three conditions are satisfied, i. follows from Propositions 2.2.16 and 2.2.17, and ii. from Proposition 2.2.18. □

Points 2–4 of this theorem constitute a generalization to symmetric spaces of Proposition 2.2.8 for solvable exponential Lie groups, since we saw in Example 2.2.9 that a symmetric pair $(G\times G,\sigma_G)$ is exponential if and only if the Lie group G is exponential. Again, it would be interesting to have a unified geometric proof of the equivalence between the existence of a midpoint map and the exponentiality. And similarly for the equivalence between exponentiality and property $(\mathfrak{m})$ for symmetric pairs. Indeed, on the one hand the proof of the polar decomposition for Riemannian symmetric spaces of the noncompact type uses the facts that 1) they are complete Riemannian manifolds, so that any two points are joined by a geodesic and the exponential map is surjective, and 2) there is a polar decomposition for $GL(n,\mathbb{C})$, so that it is also injective. And on the other hand, the proof, by Dixmier for example [Dix57], of Proposition 2.2.8 about the exponentiality of a Lie group uses the decomposition of a solvable Lie algebra into a sequence of ideals of codimensions 1 or 2, and an inductive resolution of differential equations. Those are completely orthogonal techniques, whereas there should be a general, unified way to prove these results. The approach of Moskowitz and Sacksteder [MS03, Theorem 5.8] using differential equations on a general Lie group seems promising for a generalization to symmetric spaces.

We now state a few corollaries of Theorem 2.2.20.

Corollary 2.2.21. *Let (M,s) be an exponential symmetric space. Then every connected symmetric subspace is also exponential.*

Proof. Let N be a connected symmetric subspace of M, let $x \in N$ and denote $N_E = \mathrm{Exp}_x(T_xN) \subset N$. Since Theorem 2.2.20 implies that Exp_x is a diffeomorphism, we only have to prove that its restriction to T_xN is surjective on N. But N_E is open in N since the exponential map is nowhere singular. And it is also closed in N since Exp_x is a diffeomorphism and T_xN is closed in T_xM. Hence $N_E = N$. □

Corollary 2.2.22. *Let (M,s,γ) be a connected symmetric space with midpoint map. The intersection of any two connected symmetric subspaces passing through a point $x \in M$ is connected.*

Proof. By Corollary 2.2.21, the two subspaces are the images through the diffeomorphism Exp_x of two vector subspaces of T_xM, whose intersection must be connected. □

In terms of midpoint maps, Corollary 2.2.21 reads as follows.

Corollary 2.2.23. *Let (M,s,γ) be a connected symmetric space with midpoint map, and N a symmetric subspace. Then N admits a midpoint map if and only if $\gamma|_{N\times N} \subset N$ if and only if N is connected.*

Proof. Assume N admits a midpoint map. Then it is connected since it is exponential. Moreover, the midpoint map must coincide with $\gamma|_{N\times N}$ since it is unique, by Lemma 2.1.6. Conversely, assume N is connected and let $x,y \in N$. Then there exists a unique geodesic c from x to y. If c were not entirely contained in N, then c and N would constitute two connected symmetric subspaces of M with a disconnected intersection (actually discrete since N is totally geodesic), which is impossible by Corollary 2.2.22. Thus the geodesic midpoint of x and y is contained in N, that is, $\gamma|_{N\times N} \subset N$. Now if the latter condition is satisfied, it is clear from the definition that the restriction is a midpoint map. □

Corollary 2.2.24. *Let (M,s,γ) be a connected symmetric space with midpoint map, and $x \in M$. If a connected symmetric subspace N is stabilized by s_x, then $x \in N$.*

Proof. Since N is connected, γ restricts to N as a midpoint map. Let $y \in N$. Then $s_xy \in N$, so that $x = \gamma(y, s_xy)$ is also in N. □

Note that the above corollary is not true when N is not connected, nor when M has no midpoint map, as seen by considering $\mathbb{R}$ with subspace $\mathbb{Z}$ and point $1/2$ in the first case, and the equator of the 2-sphere stabilized by the North pole in the second case.

Corollary 2.2.25. *Let (M,s,γ) be a symmetric space with midpoint map, (N,s') a symmetric space, and $f : M \to N$ a surjective morphism. Then N has a midpoint map if and only if $f^{-1}(x)$ is connected, for some $x \in N$.*

Proof. Since M is connected and f is surjective and continuous, N is connected. The morphism f makes M a fiber bundle above N with general fiber isomorphic to $f^{-1}(x)$, so that the latter is connected if and only if N is simply connected. This proves that if $f^{-1}(x)$ is not connected, then N has no midpoint map. Assume now that it is connected. Let (G,σ) be a symmetric pair such that $G/G^\sigma \cong M$. By Theorem 2.2.20, it has property $(\mathfrak{m})$ since M has a midpoint map. Then there exists a closed normal σ-invariant subgroup G' of G such that $G'/G'^\sigma \cong f^{-1}(x)$, and in this notation, we have $N \cong (G/G')/(G^\sigma/G'^\sigma)$. Now, every eigenvalue of $\operatorname{ad} X$ for $X \in \mathfrak{m}/\mathfrak{m}'$ is also an eigenvalue of $\operatorname{ad} Y$ for some $Y \in \mathfrak{m}$. Hence the symmetric pair $(G/G',\sigma)$ has property $(\mathfrak{m})$, and by Theorem 2.2.20, N has a midpoint map. □

Example 2.2.26. Let $M = \mathbb{R}$ with standard symmetries $s_x y = 2x - y$ and midpoint map $\gamma(x,y) = (x+y)/2$. Then $N = \mathbb{Z}$ is a disconnected symmetric subspace, and $\gamma(0,1) = 1/2 \notin N$, in accordance with Corollary 2.2.23. Moreover, the map $f : M \to S^1 : x \mapsto x \mod 1$ is a surjective morphism of symmetric spaces, and S^1 admits no midpoint map, in accordance with Corollary 2.2.25. ◁

2.3 Double triangles

To conclude this chapter, we study the notion of *double triangle* constructed on three points x,y,z in a symmetric space: a geodesic triangle whose midpoints of the edges are x,y,z. Such a triangle may not exist, nor be unique. We shall determine the symmetric spaces with midpoint map for which all double triangles exist and are unique. We will then describe the geometric meaning of the Ševera projection by showing its relation with the phase function of Weinstein [Wei94, Qia97] given by the area of the double triangle.

Definition 2.3.1 (Qian, *op. cit.*)**.** *Let (M,s,γ) be a symmetric space with midpoint map. The map $\gamma_3 : M \times M \times M \to M \times M \times M$ is defined by*

$$\gamma_3(x,y,z) = (\gamma(z,x), \gamma(x,y), \gamma(y,z))$$

for all $x,y,z \in M$.

All double triangles will exist and be unique if and only if γ_3 is a diffeomorphism. When (x,y,z) is sufficiently close to the diagonal, there always exists a unique double triangle in a neighbourhood of (x,y,z). But globally, γ_3 may

be nor injective, nor surjective. And if the geodesics joining two points are not unique —thus if the space has no midpoint map—, the triangle is not completely defined by its vertices.

The existence of a double triangle is tantamount to the existence of a point A such that $s_x s_y s_z A = A$ —that point is a vertex of the double triangle. A way to consider the notion of double triangle is the following. If a symmetric space (M, s) has a midpoint map γ, then for each $x, y \in M$, the map $M \to M : A \mapsto z = \gamma(A, s_x s_y A)$ is well-defined. This means that every elementary transvection, i.e. every $s_x s_y$, may at any given point A, be uniquely represented as a single symmetry through a third point z: $s_x s_y A = s_z A$, where $z = \gamma(A, s_x s_y A)$. By induction, one shows that this implies the same result for every transvection element. Now to say that all double triangles exist and are unique is equivalent to saying that the map $A \mapsto z$ is surjective and injective, for every choice of x and y. In other words, we ask that for every elementary transvection $s_x s_y$ and for every z, there exist a unique point A such that z is "the midpoint of the action of $s_x s_y$" on A. In a sense, the spaces for which γ_3 is a diffeomorphism are as close as possible to affine spaces, where any vector can be placed so that its "midpoint" lies at some given point.

To illustrate the various possibilities that may appear, let us consider the following four examples in two dimensions: the flat Euclidean space, the sphere, the hyperbolic plane, and the non-flat solvable symplectic symmetric space.

Example 2.3.2. Consider the flat Euclidean space $\mathbb{R}^{2n}$ with symmetries $s_x y = 2x - y$ and midpoint map $\gamma(x, y) = (x + y)/2$. Then for $x, y, z \in \mathbb{R}^{2n}$, the vertices of the associated double triangle are the points which complete the parallelograms built on x, y, z, y, z, x and z, x, y respectively. More precisely, the point A of the above definition is $y + (x - y) + (z - y) = x - y + z$. In this case, the double triangle always exists and is unique. Moreover, the symplectic area of such a triangle equals four times the area of the simple triangle. ◁

Example 2.3.3. Consider the sphere S^2 with symmetry s_x at x given by the axial symmetry of axis $0\vec{x}$. Consider x, y, and z the intersections with the sphere of the positive axes of an orthogonal basis of $\mathbb{R}^3$. Then *any* point on the sphere is a vertex of a double triangle of x, y, z. Indeed, consider what happens in a plane containing two of the three axes. The symmetry around the point not in the plane amounts to a central symmetry around the origin in the plane. And the other two symmetries are axial symmetries with perpendicular axes, and thus compose to a rotation of angle π around the origin. Hence the composition of the three is the identity. That is, $s_x s_y s_z A = A$ for any A on the sphere. Nevertheless, when considering geodesic edges of length less than π, all these double triangles have the same area, namely two times the area of the small triangle x, y, z. ◁

Example 2.3.4. It was shown in [Tuy99, dMRT08, dMROdA04] that not every triangle x, y, z in the hyperbolic plane admits a double triangle, so that γ_3 is not a diffeomorphism. Seeing the hyperbolic plane as the $x_3 > 0$ sheet of the two-sheeted hyperboloid $(x_3)^2 - (x_1)^2 - (x_2)^2 = 1$ in $\mathbb{R}^3$, the condition for the existence of a double triangle is that $\det(xyz)^2 < 1$, where xyz denotes the 3×3 matrix whose columns are the vectors x, y and z.

This condition is related to the fact that, in the hyperbolic plane, the areas of geodesic triangles are globally bounded by a constant, whereas the operation of taking the double triangle more than doubles the areas.

An interpretation of the determinant condition when $x = (0, 0, 1)$ is that the area of the orthogonal projection of the triangle x, y, z on the x_1x_2-plane should be less than 1. That flat area may in turn be seen as a symplectic area in the tangent space at $x = (0, 0, 1)$ to the hyperbolic plane. That linear projection will be the central subject of Chapter 4 and will be seen to be related to the area of the double triangle in general symplectic symmetric spaces. ◁

Example 2.3.5. Consider $M = \mathbb{R}^2$ with symmetries

$$s_{(a,l)}(a', l') = (2a - a', 2\cosh(2(a - a'))l - l')$$

and midpoint map $\gamma((a, l), (a', l')) = ((a + a')/2, (l + l')/2\cosh(a + a'))$. Then the double triangle on any three points always exists and is unique. Given three points $x_i = (a_i, l_i)$, $i = 1, 2, 3$, the unique point A such that $s_{x_1}s_{x_2}s_{x_3}A = A$ is given by

$$\begin{aligned} A = (&a_1 - a_2 + a_3, \\ &\cosh(2(a_1 - a_2))l_3 - \cosh(2(a_3 - a_1))l_2 + \cosh(2(a_2 - a_3))l_1). \end{aligned}$$

As we can see in the expression of the symmetries, the a part is flat, and the l part is twisted, but not too much: it is still linear in l'. As we will see below, this is a general feature of solvable symmetric spaces with midpoint maps, which allows to find a unique double triangle as a solution of a linear equation. ◁

The following proposition asserts that, when γ_3 is a diffeomorphism, the geodesic surface bounded by the double triangle built on a triple of points contained in a connected subspace, is itself entirely contained in that subspace.

Proposition 2.3.6. *Let (M, s, γ) be a connected symmetric space with midpoint map. If γ_3 is a diffeomorphism and if N is a connected symmetric subspace of M, then $\gamma_3|_{N\times N\times N}$ is a diffeomorphism $N^3 \to N^3$.*

Proof. Since N is connected, by Corollary 2.2.23, γ restricts to N as a midpoint

map, so that $\gamma_3(N \times N \times N) \subset N \times N \times N$. The proof is then the same as that of Corollary 2.2.21, replacing Exp_x by γ_3. □

Proposition 2.3.7. *Let (M, s, γ) be a connected semisimple symmetric space with midpoint map, then γ_3 is not surjective.*

Proof. To represent M, by Theorem 2.2.20 it is sufficient to consider (G, θ) a Riemannian symmetric pair of the non-compact type. Then there exists a symmetric subspace of M isomorphic to the hyperbolic plane [Hel78, Chapter IX], which by Example 2.3.4 does not have all double triangles (i.e., its γ_3 is not surjective). Proposition 2.3.6 then implies that M does not have all double triangles either. □

Theorem 2.3.8. *Let (M, s, γ) be a connected solvable symmetric space with midpoint map. Then γ_3 is a diffeomorphism.*

Proof. By Theorem 2.2.20, it is equivalent to show the result for all solvable exponential symmetric pairs (G, σ). We will prove the result by induction on the dimension of G. So let (G, σ) be an exponential solvable symmetric pair. Without loss of generality, by the same theorem we may assume that G is an exponential Lie group.

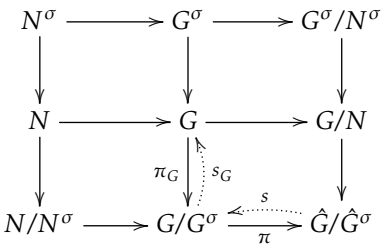
If G is 1-dimensional, it is isomorphic to $\mathbb{R}$ with the usual symmetries $s_x y = 2x - y$. Hence, $\delta_3(x, y, z) = (y - z + x, z - x + y, x - y + z)$ is a smooth inverse of γ_3.

Assume now that the dimension of G is greater than 1, and let N be a minimal σ-invariant normal subgroup of G. As in the proof of Theorem 2.2.19, we can see that N is a closed abelian connected and simply connected normal subgroup, such that N/N^σ is also connected and simply connected.

Denote $\hat{G} = G/N$. Since $\hat{G}/\hat{G}^\sigma$ is contractible, the canonical morphism $\pi : G/G^\sigma \to \hat{G}/\hat{G}^\sigma$ (which is a fibre bundle) admits a smooth global section s [Ste51, pp. 25, 55–56]. Moreover by Proposition 2.2.11, since G/G^σ is exponential, there is a smooth global section s_G of $\pi_G : G \to G/G^\sigma$ with values in $M_G = \{g\sigma(g)^{-1} \mid g \in G\} \subset G$.

To summarize, we have the following commutative diagram, where the arrows on the two upper lines are group morphisms, and those on the lower line

are symmetric space morphisms:



Thus the composition $s_G \circ s$ gives a map $\hat{G}/\hat{G}^\sigma \to G$, and for each $x \in \hat{G}/\hat{G}^\sigma$ the set $\{s_G(s(x)).n \mid n \in N\}$ is mapped through π_G onto the fiber $\pi^{-1}(x)$.

Moreover, since N is abelian connected and simply connected, it contains no element whose square is the identity, whence by Proposition 2.2.2, the set $N' = \{n \in N \mid \sigma(n) = n^{-1}\}$ is such that the composition $N' \hookrightarrow N \to N/N^\sigma$ is a diffeomorphism, and it is a Lie subgroup of N.

With the help of the above constructions, we may uniquely represent any element $x \in G/G^\sigma$ as $x = gnG^\sigma$ where $g = s_G(s(\pi(x)))$ and n is the unique element of N' such that $gnG^\sigma = x$. Note that n depends smoothly on x since g does and since $g^{-1}x \in \ker(\pi) = N/N^\sigma$ which is diffeomorphic to N' through π_G.

Let now $x_1, x_2, x_3 \in G/G^\sigma$. Then, by the induction assumption, there exists a unique $x' \in \hat{G}/\hat{G}^\sigma$ such that

$$s_{\pi(x_3)} s_{\pi(x_2)} s_{\pi(x_1)} x' = x',$$

and x' is a smooth function of x_1, x_2, x_3. We are looking for an $x \in G/G^\sigma$ depending smoothly on x_1, x_2, x_3 and such that

$$s_{x_3} s_{x_2} s_{x_1} x = x. \tag{2.1}$$

By uniqueness of x', we must have $\pi(x) = x'$ so that we may write $x = gnG^\sigma$ with $g = s_G(s(x'))$ and for a unique $n \in N'$. Writing $x_i = g_i G^\sigma$ for some $g_i \in G$, we have

$$\begin{aligned} s_{x_3} s_{x_2} s_{x_1}(gnG^\sigma) &= s_{x_3} s_{x_2} g_1 \sigma(g_1^{-1}) gn) G^\sigma \\ &= s_{x_3} g_2 \sigma(g_2^{-1} g_1) g_1^{-1} gnG^\sigma \\ &= \underbrace{g_3 \sigma(g_3^{-1} g_2) g_2^{-1} g_1 \sigma(g_1^{-1} g)}_{=:\tilde{g}} \sigma(n) G^\sigma \\ &= \tilde{g} \sigma(n) G^\sigma. \end{aligned} \tag{2.2}$$

But by definition of x' and g, we have $\tilde{g} G^\sigma = gn'' G^\sigma$ for a unique $n'' \in N'$. As we

saw at the end of the last paragraph, the latter depends smoothly on $\tilde{g}$ and g, and thus in x_1, x_2, x_3 also. Hence there further exists a unique $k \in G^\sigma$ such that $\tilde{g} = gn''k$, and the right hand side in equation (2.2) is equal to

$$g\,\mathrm{C}_k(\sigma(n))n''G^\sigma$$

(note again again that $k = n''^{-1}g^{-1}\tilde{g}$ is a smooth function of x_1, x_2, x_3). This will be equal to gnG^σ if and only if $nG^\sigma = \mathrm{C}_k(\sigma(n))n''G^\sigma$. But since $k \in G^\sigma$ and $n \in N'$ we have $\sigma\,\mathrm{C}_k(\sigma(n)) = \mathrm{C}_k(n) = (\mathrm{C}_k(\sigma(n)))^{-1}$ so that $\mathrm{C}_k(\sigma(n)) = \mathrm{C}_k(n^{-1}) \in N'$. Thus the equality (2.1) becomes $n\,\mathrm{C}_k(n) = n''$. Replacing the n's by their log, we get

$$(\mathrm{Id} + \mathrm{Ad}_k)(n) = n''.$$

Now G is exponential so $k = e^X$ for some $X \in \mathfrak{k}$, and $\mathrm{Ad}_k\, n = e^{\mathrm{ad}\, X} n$ will never be equal to $-n$ since $\mathrm{ad}\, X$ has no non-zero purely imaginary eigenvalue (actually all its eigenvalues are zero since it is nilpotent). Hence $\mathrm{Id} + \mathrm{Ad}_k$ is invertible and is polynomial in $\mathrm{ad}\, X$, so that its inverse also is. We thus have a unique solution $n = (\mathrm{Id} + \mathrm{Ad}_k)^{-1} n''$ which is smooth in x_1, x_2, x_3, and consequently a unique solution $x = s_G(s(x'))n$ of the equation $s_{x_3} s_{x_2} s_{x_1} x = x$ with the same property. Hence γ_3 is invertible and has a smooth inverse. As γ_3 itself is smooth, we have a diffeomorphism. □

Remark 2.3.9. Notice that the proof does not really rely on the fact that we looked at *triangles*. Indeed, everything still holds for polygons with any odd number of vertices, so that the obviously defined maps γ_{2n+1} for $n \geq 0$ are all diffeomorphisms as soon as the symmetric space is solvable exponential.

In terms of the transvection group G of M, this may be rephrased as follows:

Corollary 2.3.10. *Let (M, s, γ) be an exponential solvable symmetric space with transvection group G. Then for any $g \in G$ and any $x \in M$, there exists a unique $y \in M$ such that $g \cdot y = s_x y$.*

For the polygons with an even number of vertices, in general the "double polygon" does not exist. In the flat $\mathbb{R}^n$ case for example, there exists an x such that $s_{x_2} s_{x_1} x = x$ if and only if $x_1 = x_2$, and in this case any x is suitable. Similarly, a given quadrangle has a "double" if and only if it is a parallelogram, and in this case any point is a vertex of such a double.

In terms of the transvection group, the existence of such a double in every case would mean that any element $g \in G$ has a fixed point in M. In the flat case, only the identity in the transvection group has a fixed point, from which we recover the example of the previous paragraph: a quadrangle (x_1, x_2, x_3, x_4)

is a parallelogram if and only if the corresponding element $s_{x_1}s_{x_2}s_{x_3}s_{x_4}$ of the transvection group is the identity.

Proposition 2.3.11. *Let (M,s,γ) be a symmetric space with midpoint map. If, for $x,y \in M$, there exists $z \in M$ such that $s_x s_y z = z$, then $x = y$.*

Proof. Let $x,y,z \in M$ be such that $s_x s_y z = z$. Then $s_x z = s_y z$ so that $x = \gamma(z, s_x z) = \gamma(z, s_y z) = y$. □

This means that among the elementary transvections, only the identity has a fixed point. ○

Combining the above propositions, we arrive at a characterization of those symmetric spaces for which all double triangles exist and are unique.

Theorem 2.3.12. *Let (M,s,γ) be a connected symmetric space with midpoint map. Then γ_3 is a diffeomorphism if and only if (M,s) is a solvable symmetric space.*

Proof. Let (G,σ) be a symmetric pair such that $G/G^\sigma \cong M$. As usual we may assume that G is connected and simply connected. Now there exists a global Levi decomposition $G = S \ltimes R$, with S and R stable under σ, S semisimple and R the radical. Thus S/S^σ is a semisimple symmetric subspace of M.

Assume S/S^σ is not reduced to a point. Then Proposition 2.3.7 implies that it does not have all double triangle, and hence by Proposition 2.3.6 the same is true for M so that γ_3 is not a diffeomorphism. The converse is the content of Proposition 2.3.8. □

This last result is the main reason for the presence of the word "solvable" in the title of this thesis. Indeed, our quantization procedure is based on the Ševera projection and on the fact that it is invertible, so that we must restrict ourselves to symmetric spaces with midpoint maps. Furthermore, the quantization formulæ that we obtain are expressed in terms of the area of the double triangle, which needs γ_3 to be a diffeomorphism, so that we end up with solvable symmetric spaces with midpoint map. On the other hand, since all of them always have unique double triangles, it leaves room for the possibility that such associative quantization formulæ hold for the whole class of solvable (symplectic) symmetric spaces with midpoint map.

To finish this chapter, we just mention a direct consequence of the above theorem.

Proposition 2.3.13. *Let (M,s,γ) be a connected symmetric space with midpoint map, (N,s') a symmetric space and $f : M \to N$ a surjective morphism. Assume γ_3 is a diffeomorphism and $f^{-1}(x')$ is a connected symmetric subspace of M, for some $x' \in N$. Denote γ' the induced midpoint map on N. Then γ'_3 is a diffeomorphism $N^3 \to N^3$.*

Proof. Since $f^{-1}(x')$ is connected, N indeed has midpoint map, by Corollary 2.2.25. By Theorem 2.3.12 since γ_3 is a diffeomorphism, M is a solvable symmetric space, and since f is surjective, N is solvable as well. As the latter admits a midpoint map, γ_3 is thus a diffeomorphism using again the same theorem. □

Chapter 3

Structure of symplectic symmetric spaces

It was proved in [BCG97] that a connected and simply connected symplectic symmetric space admits, up to isomorphism, one and only one decomposition into a direct sum of indecomposable (that is, not the direct sum of two normal subspaces) non-flat factors and a flat one. Just as for Lie groups, the indecomposable factors fall into three categories: the semisimple, the solvable, and the nor semisimple nor solvable ones.

Due to the existence of much structure theory for semisimple Lie groups and Lie algebras, the semisimple symplectic symmetric spaces may be studied very precisely, and have indeed been classified in [Bie95, Bie98b].

The nor semisimple nor solvable symplectic symmetric spaces are semi-direct products of a semisimple subspace and a solvable normal subspace. Hence, in order to study the non-semisimple symmetric spaces, more should be known about the other constituents of the above semi-direct products, namely, the solvable spaces.

Not much is known about solvable symplectic symmetric spaces. Their study was initiated in [Bie95], a few infinite classes of examples are known [Bie02, Bie07, CGRS09], and they have been completely classified in dimension 2 and 4 [Bie98a]. But no general structure theory is known.

The goal of this chapter is to introduce some new and very simple tools, which are adapted to both the symplectic and the symmetric structures. They are inspired from corresponding tools existing for the symplectic Lie groups and, perhaps surprisingly, many of the notions and results there have very precise analogues in our context, although unlike in the non-symplectic case it is not clear to what extent symplectic Lie groups and symplectic symmetric spaces

have a large intersection.

The chapter is structured as follows. In Section 3.1, we describe an algebraic point of view on subgroups of the automorphism group of a symmetric space which act simply transitively, and show that these subgroups are naturally endowed with a "primitive" of the Lie bracket, given essentially by the symmetric space connection. In Section 3.2, we present the basics of symplectic Lie groups, and introduce some notions and results which will have analogues for symplectic symmetric spaces. In Section 3.3, we introduce the notion of primitive system and show that a symplectic Lie triple system canonically has such a structure. We then show that this notion is precisely the one needed to characterize the symplectic Lie triple system structures on cotangent bundles of Lie triple systems. In Section 3.4, we use the tool of the previous section to start studying a notion of symplectic reduction for symplectic symmetric spaces. Finally, in Section 3.5, we define a notion of double extension for symplectic Lie triple systems, and express the completely solvable spaces in terms of this notion.

3.1 Simply transitive groups of automorphisms

We mention that most of the content of this section is also contained in Section 3 of the article [Bor90] of M. Bordemann.

Let $(\mathfrak{g}, \sigma)$ be an involutive Lie algebra with associated decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ in $+1$ and -1 eigenspaces.

Definition 3.1.1. *A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is called* simply transitive *if the projection of $\mathfrak{s}$ on $\mathfrak{m}$ parallel to $\mathfrak{k}$ is a linear isomorphism.*

If G is a Lie group with Lie algebra $\mathfrak{g}$, and K is a subgroup of G with Lie algebra $\mathfrak{k}$, the simply transitive subalgebras of $\mathfrak{g}$ correspond to subgroups of G which have a locally simply transitive action on G/K.

Proposition 3.1.2. *The set of simply transitive subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ is in bijection with the set of linear maps $k : \mathfrak{m} \to \mathfrak{k}$ such that*

$$k([k(x), y] + [x, k(y)]) = [x, y] + [k(x), k(y)] \tag{3.1}$$

for all $x, y \in \mathfrak{m}$. The correspondance is given by $k \mapsto \mathfrak{s}_k = \{x + k(x) \mid x \in \mathfrak{m}\}$.

Proof. A linear subspace $\mathfrak{s}$ of $\mathfrak{g}$ is isomorphically projected on $\mathfrak{m}$ parallel to $\mathfrak{k}$ if and only if $\mathfrak{s} = \{x + k(x) \mid x \in \mathfrak{m}\}$ for some linear map $k : \mathfrak{m} \to \mathfrak{k}$, and such a k uniquely determines $\mathfrak{s}$. Moreover, $\mathfrak{s}$ will be a subalgebra of $\mathfrak{g}$ if and only if for all

$x, y \in \mathfrak{m}$, $[x + k(x), y + k(y)]$ may be written as $z + k(z)$ for some $z \in \mathfrak{m}$. Using the facts that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, this means that

$$\begin{aligned}[x + k(x), y + k(y)] &= [k(x), y] + [x, k(y)] + [x, y] + [k(x), k(y)] \\ &= [k(x), y] + [x, k(y)] + k([k(x), y] + [x, k(y)])\end{aligned}$$

which yields the condition. □

Let $\mathfrak{s}$ be a simply transitive subalgebra of $\mathfrak{g}$ with corresponding map $k : \mathfrak{m} \to \mathfrak{k}$. For each $x \in \mathfrak{m}$, we will denote $x_{\mathfrak{s}}$ the element $x + k(x)$ of $\mathfrak{s}$.

Example 3.1.3. Let $\mathfrak{h}$ be a Lie algebra (such as the Lie algebra of vector fields on a manifold M) and define $\mathfrak{g} = \mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$ with $\mathbb{C}$-linearly extended Lie bracket, and $\sigma(x + iy) = x - iy$. Then σ is an involutive automorphism of $\mathfrak{g}$ and $\mathfrak{k} = \mathfrak{h} \oplus \{0\}$, $\mathfrak{m} = \{0\} \oplus i\mathfrak{h}$. Let $J : \mathfrak{h} \to \mathfrak{h}$ be an anti-involutive linear map (such as an almost complex structure on M) and define $k : \mathfrak{m} \to \mathfrak{k}$ by $k(ix) = J(x)$ for all $x \in \mathfrak{h}$. Then the condition (3.1) on k coincides with the integrability of the almost complex structure J. Indeed, $\mathfrak{s}_k$ is a subalgebra of $\mathfrak{g}$ if and only if

$$k([J(x), iy] + [ix, J(y)]) = [ix, iy] + [J(x), J(y)]$$

if and only if

$$J([J(x), y] + [x, J(y)]) = -[x, y] + [J(x), J(y)]$$

that is, if and only if the Nijenhuis tensor of J is zero. ◁

Definition 3.1.4. *We may define on* $\mathfrak{m}$ *a bilinear product* $\diamond$ *by*

$$x \diamond y = [k(x), y]$$

for $x, y \in \mathfrak{m}$. *The corresponding product on* $\mathfrak{s}$ *is denoted* $\circ$ *:*

$$x_{\mathfrak{s}} \circ y_{\mathfrak{s}} = (x \diamond y)_{\mathfrak{s}} = [k(x), y]_{\mathfrak{s}}.$$

Proposition 3.1.5. *For all* $U, V \in \mathfrak{s}$, *we have*

$$[U, V] = U \circ V - V \circ U.$$

Proof. Let $x, y \in \mathfrak{m}$. We have, using (3.1) at the second step,

$$\begin{aligned}[x_{\mathfrak{s}}, y_{\mathfrak{s}}] &= [x + k(x), y + k(y)] \\ &= ([k(x), y] + [x, k(y)])_{\mathfrak{s}}\end{aligned}$$

$$= x_{\mathfrak{s}} \circ y_{\mathfrak{s}} - y_{\mathfrak{s}} \circ x_{\mathfrak{s}}. \qquad \square$$

Definition 3.1.6. *Define the* associator *of a bilinear product $\cdot$ on a vector space $\mathfrak{g}$ by*

$$(U, V, W)^{\cdot} = (U \cdot V) \cdot W - U \cdot (V \cdot W)$$

and its antisymmetrization in the first two variables

$$[U, V, W]^{\cdot} = (U, V, W)^{\cdot} - (V, U, W)^{\cdot}.$$

Recall the following fact.

Proposition 3.1.7. *A bilinear product $\cdot$ on a vector space $\mathfrak{g}$ gives rise to a Lie bracket through antisymmetrization if and only if*

$$[U, V, W]^{\cdot} + [V, U, W]^{\cdot} + [W, U, V]^{\cdot} = 0$$

for all $U, V, W \in \mathfrak{g}$. In case the condition is verified, $(\mathfrak{g}, \cdot)$ is said to be a Lie admissible algebra.

Proof. Write the Jacobi identity and group the terms by associators. $\square$

Two important cases are the *associative algebras* where $(U, V, W)^{\cdot} = 0$ and the *left-symmetric algebras* where $[U, V, W]^{\cdot} = 0$. The left-symmetric algebras will be of importance in Section 3.2 on symplectic Lie groups. Here, the algebra $(\mathfrak{s}, \circ)$ is nor associative nor left-symmetric as long as the symmetric space is not flat:

Proposition 3.1.8. *With the notation of Definition 3.1.4, the identity*

$$[x, y, z]^{\diamond} = [[x, y], z]$$

(and thus $[x_{\mathfrak{s}}, y_{\mathfrak{s}}, z_{\mathfrak{s}}]^{\circ} = ([[x, y], z])_{\mathfrak{s}}$) holds for all $x, y, z \in \mathfrak{m}$. In particular, the operator $L_x : \mathfrak{m} \to \mathfrak{m} : y \mapsto x \diamond y$ is a derivation of $[\cdot, \cdot, \cdot]^{\diamond}$, for all $x \in \mathfrak{m}$.

Proof. We have, using the Jacobi identity at the third step and condition (3.1) at the fourth,

$$\begin{aligned}
[x, y, z]^{\diamond} &= ((x \diamond y) \diamond z - x \diamond (y \diamond z)) - ((y \diamond x) \diamond z - y \diamond (x \diamond z)) \\
&= ([k([k(x), y]), z] - [k(x), [k(y), z]]) \\
&\quad - ([k([k(y), x]), z] - [k(y), [k(x), z]]) \\
&= [k([k(x), y] + [x, k(y)]), z] - [[k(x), k(y)], z] \\
&= [[x, y], z].
\end{aligned}$$

The second assertion follows from the facts that $L_x = \operatorname{ad} k(x)$ and $k : \mathfrak{m} \to \mathfrak{k}$, $\operatorname{ad}\mathfrak{k}$ being the derivation algebra of the Lie triple system which, by the first assertion, is equal to $[\cdot,\cdot,\cdot]^\diamond$. □

Remark 3.1.9. In particular, each simply transitive subalgebra yields a *primitive* $(\cdot,\cdot,\cdot)^\diamond$ of the canonical Lie triple system on $\mathfrak{m}$, in the sense that

$$[[x,y],z] = (x,y,z)^\diamond - (y,x,z)^\diamond.$$

Well-known cases of trilinear maps giving rise to Lie triple systems through such an antisymmetrization are the Jordan triple systems. However, here $(\cdot,\cdot,\cdot)^\diamond$ has no reason to be a Jordan triple system. For example, in general, $(x,y,z)^\diamond \neq (z,y,x)^\diamond$.

We will define in Section 3.3 another notion of primitive of a Lie triple system, when the latter has an invariant symplectic form. These two notions in general do not coincide, although they do in some cases. ○

Now consider *any* bilinear product $\diamond$ on a vector space $\mathfrak{m}$. Then by analogy with the above case, we could *define* a trilinear product $[\cdot,\cdot,\cdot]$ on $\mathfrak{m}$ as

$$\begin{aligned}
[x,y,\cdot] &= [x,y,\cdot]^\diamond \\
&= (x,y,\cdot)^\diamond - (y,x,\cdot)^\diamond \\
&= (x\diamond y - y\diamond x)\diamond\cdot - (x\diamond(y\diamond\cdot) - y\diamond(x\diamond\cdot)) \\
&= L_{[x,y]^\diamond} - [L_x, L_y],
\end{aligned}$$

where we denoted $[x,y]^\diamond = x\diamond y - y\diamond x$ and $L_x y = x\diamond y$.

Thus $\diamond$ defines a trilinear and a bilinear product, $[\cdot,\cdot,\cdot]^\diamond$ and $[\cdot,\cdot]^\diamond$ respectively. It turns out that, for these products to form a Lie triple system together with a simply transitive subalgebra of derivations, it is necessary and sufficient that $\diamond$ be Lie admissible and that the operators L_x be derivations of the triple product.

Proposition 3.1.10. *There is a bijection between the set of Lie triple systems together with a simply transitive subalgebra of the derivations Lie algebra, and the set of Lie admissible algebras such that left multiplication is a derivation of the antisymmetrization in the two first variables of the associator.*

Proof. Indeed, let $(\mathfrak{m},\diamond)$ be a Lie admissible algebra. Then by Proposition 3.1.7, $[\cdot,\cdot,\cdot]^\diamond$ satisfies the Jacobi identity. If, moreover, L_x is a derivation of $[\cdot,\cdot,\cdot]^\diamond$ for all $x \in \mathfrak{m}$, then so are $L_{[x,y]^\diamond}$ and $[L_x, L_y]$, hence

$$L_{x,y} = [x,y,\cdot]^\diamond = L_{[x,y]^\diamond} - [L_x, L_y]$$

is a derivation of $[\cdot,\cdot,\cdot]^\diamond$. Hence the latter is a Lie triple system. And $\mathfrak{s} = \{(x, L_x) \mid x \in \mathfrak{m}\}$ is a simply transitive Lie subalgebra of $\mathfrak{g} = \mathfrak{m} \oplus \operatorname{Der}([\cdot,\cdot,\cdot]^\diamond)$.

Conversely, let $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ be a Lie triple system and $\mathfrak{s}$ be a simply transitive subalgebra of the Lie algebra of derivations $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ with $\mathfrak{k} = \mathrm{Der}([\cdot,\cdot,\cdot])$. Then, by Proposition 3.1.2, $\mathfrak{s}$ corresponds to some map $k : \mathfrak{m} \to \mathfrak{k}$ satisfying equation (3.1). Set, as in Definition 3.1.4, $x \diamond y = [k(x), y]$ for all $x, y \in \mathfrak{m}$. By Proposition 3.1.8, this implies that $[\cdot,\cdot,\cdot]^\diamond$ is Lie-admissible and that L_x is a derivation thereof, for all $x \in \mathfrak{m}$.

These two constructions are obviously inverse of each other, which concludes the proof. □

Proposition 3.1.11. *Let $(\mathfrak{g}, \sigma)$ be a solvable involutive Lie algebra such that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$, and let $\mathfrak{s} \subset \mathfrak{g}$ be a simply transitive subalgebra. Then $\mathfrak{s}$ is exponential if and only if $(\mathfrak{g}, \sigma)$ is exponential.*

Proof. Since $\mathfrak{g}$ is solvable and $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$, $\mathfrak{k}$ is contained in the nilradical of $\mathfrak{g}$. By Lie's theorem, there exists a basis of $\mathfrak{g}_{\mathbb{C}}$ such that the complexified adjoint representation is by upper triangular matrices. This shows that if $x \in \mathfrak{m}$ and $y \in \mathfrak{k}$, then $\operatorname{ad} x$ and $\operatorname{ad}(x + y)$ have the same eigenvalues.

Since the elements of $\mathfrak{s}$ have the form $x + k(x)$, $x \in \mathfrak{m}$, for some linear map $k : \mathfrak{m} \to \mathfrak{k}$, we see that $\operatorname{ad}(x + k(x))$ and $\operatorname{ad}(x)$ have the same eigenvalues, for all $x \in \mathfrak{m}$. Hence the result. □

Example 3.1.12. As an illustration of Proposition 3.1.10, let us compute all the Lie triple systems for which the Lie algebra $ax + b$ of affine motions of the real line is a simply transitive subalgebra of derivations. This already shows some interesting phenomenon.

Proposition 3.1.13. *Let $(\langle H, E\rangle, \diamond)$ be a Lie admissible algebra such that the left multiplication is a derivation of $[\cdot,\cdot,\cdot]^\diamond$. Then $\diamond$ has one of the following forms:*

	$H \diamond H$	$H \diamond E$	$E \diamond H$	$E \diamond E$	$[H,E,H]$	$[H,E,E]$
$I_{c,d}$	$\frac{d^2-6d+4}{d+2}H + cE$	dE	$(d-2)E$	0	0	0
I'_e	$2H$	0	$-2E$	eH	0	0
$II_{c,d}$	cE	dE	$(d-2)E$	0	$-(d^2-6d+4)E$	0
III_e	0	0	$-2E$	eH	$-4E$	$2eH$

The forms $I_{c,d}$ and I'_e have a flat Lie triple system, $II_{c,d}$ are isomorphic to $(\mathfrak{so}(1,1) \ltimes \mathbb{R}^2)/\mathbb{R}$, and III_e are isomorphic to $\mathfrak{su}(1,1)/\mathfrak{u}(1)$.

Proof. Write $L_H = H \diamond \cdot$ and $L_E = H \diamond \cdot$ as matrices

$$L_H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad L_E = \begin{pmatrix} b & e \\ d-2 & f \end{pmatrix}$$

constrained by the relation $L_H E - L_E H = 2E$, then compute

$$\begin{aligned} L_{H,E} &= L_{[H,E]} - [L_H, L_E] \\ &= \begin{pmatrix} (4-d)b + ec & (d+2-a)e - bf + b^2 \\ -d^2 + (6+a)d - 4 + 2a + c(f-b) & 2f - ce + (d-2)b \end{pmatrix} \end{aligned}$$

and solve the derivation identity for $a, b, c, d, e, f \in \mathbb{R}$. □

Notice that when $d \to 0$, $\mathrm{II}_{c,d}$ tends to the flat Lie triple system, but that there is no way to reach the latter starting from III_e and letting $e \to 0$. Hence any contraction of the symmetric space $SU(1,1)/U(1)$ preserving a simply transitive action of $ax + b$ must tend to the non-abelian completely solvable space $(SO(1,1) \ltimes \mathbb{R}^2)/\mathbb{R}$, and only then, it may be contracted further to the flat space.

This is exactly the route that was followed in [BDS09] by Bieliavsky, Detournay and Spindel to quantize the hyperbolic plane. They started from a $(ax+b)$-invariant quantization of $(SO(1,1) \ltimes \mathbb{R}^2)/\mathbb{R}$, which was obtained in [BM01] and is itself a deformation of the flat Moyal-Weyl product, and then solved some complicated evolution equation to somehow "decontract" the space towards the hyperbolic plane. What the above computations suggest is that, when preserving the $(ax+b)$-invariance, this was essentially the only possible route to follow. ◁

3.2 Symplectic Lie groups

Definition 3.2.1 ([Chu74, LM88])**.** *A* symplectic Lie group $(G, \tilde{\omega})$ *is a Lie goup* G *endowed with a left-invariant symplectic form* $\tilde{\omega}$*.*

Being left-invariant, $\tilde{\omega}$ is completely determined by its value $\omega = \tilde{\omega}_e$ at the identity, and the closedness condition reads

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0 \tag{3.2}$$

for all $x, y, z \in \mathfrak{g}$. A Lie algebra together with a non-degenerate 2-form satisfying (3.2) will be called a *symplectic Lie algebra*.

On a symplectic Lie group, there is a canonical *left-invariant flat torsion-free connection*. To define it, recall that, using the Leibniz rule, any connection on a Lie group may be defined by its value on left-invariant vector fields. If the connection is left-invariant, the result is also a left-invariant vector field, so that its definition amounts to the data of any bilinear map $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proposition 3.2.2. *Let* $(G, \tilde{\omega})$ *be a connected symplectic Lie group, and* $\omega = \tilde{\omega}_e$*. The*

following formula defines a left-invariant flat torsion-free connection on G:

$$\omega(\nabla_x y, z) = \omega([x,z],y) \tag{3.3}$$

for all $x, y, z \in \mathfrak{g}$.

Proof. The connection ∇ is indeed torsion-free since, for any $x, y, z \in \mathfrak{g}$, the cocycle condition for ω implies

$$\begin{aligned}\omega(\nabla_x y - \nabla_y x, z) &= \omega([x,z],y) - \omega([y,z],x)\\ &= \omega([x,y],z)\end{aligned}$$

so that $\nabla_x y - \nabla_y x = [x,y]$. And for $x, y, z, t \in \mathfrak{g}$, the Jacobi identity yields

$$\begin{aligned}\omega(\nabla_x\nabla_y z - \nabla_y\nabla_x z, t) &= \omega([x,t],\nabla_y z) - \omega([y,t],\nabla_x z)\\ &= \omega(z,[y,[x,t]]) - \omega(z,[x,[y,t]])\\ &= \omega([[x,y],t],z)\\ &= \omega(\nabla_{[x,y]}z,t)\end{aligned}$$

so that $\nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z = 0$, and ∇ is indeed flat. □

In the spirit of the previous section, the canonical connection just defined may be seen as a non-associative product $\cdot$ on $\mathfrak{g}$ which is Lie admissible and, more precisely, *left-symmetric*, that is, its associator is symmetric in the first two variables.

Definition 3.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra. A* left-symmetric structure *on* $\mathfrak{g}$ *is a bilinear product* $\cdot : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *such that*

1. $x\cdot y - y\cdot x = [x,y]$,
2. $(x,y,z)^{\cdot} = (y,x,z)^{\cdot}$,

for all $x, y, z \in \mathfrak{g}$, *and where* $(\cdot,\cdot,\cdot)^{\cdot}$ *is the associator of Definition 3.1.6.*

Proposition 3.2.4. *Let* $(G,\tilde{\omega})$ *be a connected symplectic Lie group. The bilinear product* $\cdot : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *defined by*

$$x\cdot y = \nabla_x y,$$

where ∇ *is defined in equation* (3.3), *is a left-symmetric structure on* $\mathfrak{g}$.

In particular, $\mathfrak{g}$ *is a simply transitive subalgebra of* $\mathfrak{gl}(n,\mathbb{R}) \ltimes \mathbb{R}^n$, *that is, G acts locally simply transitively on* $\mathbb{R}^n$ *by affine transformations, with* $n = \dim G$.

Proof. The first condition in Definition 3.2.3 comes from the fact that ∇ is torsion-free:

$$x \cdot y - y \cdot x = \nabla_x y - \nabla_y x = [x, y],$$

and the second condition from the fact that it is flat:

$$\begin{aligned}(x,y,z)^{\cdot} - (y,x,z)^{\cdot} &= (x \cdot y) \cdot z - x \cdot (y \cdot z) \\ &\quad - (y \cdot x) \cdot z + y \cdot (x \cdot z) \\ &= \nabla_{x \cdot y - y \cdot x} z - [\nabla_x, \nabla_y] z \\ &= 0.\end{aligned}$$

Hence $\cdot$ is indeed a left-symmetric structure on $\mathfrak{g}$.

The second assertion is a direct consequence of Proposition 3.1.10, taking into account the fact that the antisymmetrization of the associator in the first two variables is zero, so that the resulting Lie triple system also has vanishing triple bracket. □

In general, G is not globally simply transitive on $\mathbb{R}^n$. For this, it is necessary and sufficient that the flat torsion-free connection ∇ be complete, a condition which was characterized as follows, by D. Segal [Seg92]. See also W. Goldman and M. Hirsch [GH86].

Proposition 3.2.5 ([Seg92])**.** *Let $\mathfrak{g}$ be a n-dimensional Lie algebra, and $\cdot$ a left-symmetric structure on $\mathfrak{g}$. Denote $\nabla_x y = x \cdot y$ the corresponding flat torsion-free connection on the connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$. Denote $R_y : \mathfrak{g} \to \mathfrak{g} : x \mapsto x \cdot y$ the right multiplication by y, for all $y \in \mathfrak{g}$. Then following conditions are equivalent:*

1. *∇ is complete, so that G acts simply transitively on $\mathbb{R}^n$,*
2. *the linear transformation $\mathrm{Id}_{\mathfrak{g}} + R_y : \mathfrak{g} \to \mathfrak{g}$ is bijective, for all $y \in \mathfrak{g}$,*
3. *R_y has no eigenvalue in $\mathbb{R} \setminus \{0\}$, for all $y \in \mathfrak{g}$,*
4. *$\mathfrak{g}$ is right nil, that is, R_y is a nilpotent linear transformation, for all $y \in \mathfrak{g}$.*

What is interesting in this result is that a global geometric condition (simply transitivity, or completeness of the connection) is completely encoded in some algebraic conditions. We do not know of a corresponding result in the general case of a locally simply transitive subgroup of automorphisms of a *non-flat* symmetric space.

Example 3.2.6. Consider the $ax+b$ symplectic Lie algebra $(\mathfrak{g},\omega)$, where $\mathfrak{g}=\langle H,E\rangle$ with $[H,E]=2E$ and $\omega(H,E)=2$. Then the canonical left-symmetric product defined in Proposition 3.2.4 is given by

$$H\cdot H=-2H \qquad E\cdot H=-2E$$
$$H\cdot E=0 \qquad E\cdot E=0.$$

Hence the right-multiplication by H is not a nilpotent linear transformation of $\mathfrak{g}$, and the associated flat connection is not complete. ◁

The canonical flat torsion-free connection on a Lie group induced by a left-invariant symplectic form is a powerful tool for the study of symplectic Lie groups. Among the results obtained through its use, the following is of prime importance. We will give an analogue of it for symplectic symmetric spaces in Section 3.4.

Theorem 3.2.7 ([DM92b, DM96a])**.** *Let (G,ω) be a symplectic Lie group. If H is a closed normal subgroup of G, then $R=H^{\perp}/(H_0\cap H^{\perp})$ is a reduced symplectic Lie group of (G,ω) and the canonical sequence*

$$0\to H_0\cap H^{\perp}\to H^{\perp}\to R\to 0$$

is an exact sequence of Lie groups with affine structures coming from that of (G,ω).

Notice that if, moreover, H is connected and $\mathfrak{h}\subset\mathfrak{h}^{\perp}$, then the sequence of the proposition becomes

$$0\to H\to H^{\perp}\to R=H^{\perp}/H\to 0.$$

Using a result of J. Helmstetter [Hel79] stating that a Lie algebra $\mathfrak{g}$ admitting a left-symmetric structure has a derived ideal $D\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ different from $\mathfrak{g}$, J.-M. Dardié and A. Medina prove further the following result.

Theorem 3.2.8 ([DM96a])**.** *Let (G,ω) be a connected and simply connected symplectic Lie group. Then G contains a non-trivial closed connected normal subgroup, isomorphic to a vector group and such that either G is a semi-direct product of H by $H^{\perp}$, the action of $H^{\perp}$ being by symplectomorphisms, or $H\subset H^{\perp}$, and in this case we have the exact sequence of affine groups*

$$0\to H\to H^{\perp}\to R=H^{\perp}/H\to 0$$

and the canonical principal fibration

$$H^{\perp}\xrightarrow{i}G\xrightarrow{p}G/H^{\perp},$$

where $H^\perp$, $G/H^\perp$, and the maps i and p, are affine.

As a reverse process of that of reduction, A. Medina and Ph. Revoy defined in [MR91] the notion of *symplectic double extension* of a symplectic Lie group. They notably proved that every nilpotent symplectic Lie group is the result of a finite sequence of such extensions starting from the group reduced to a point.

Definition 3.2.9 ([MR91])**.** *A symplectic Lie group (G, ω) is a* (classical) symplectic double extension *of a symplectic Lie group (W, ω') if G contains a 1-dimensional closed central subgroup H such that the reduced symplectic group $H^\perp/H$ is isomorphic to (W, ω').*

In order to recover a larger class of symplectic Lie groups as double extensions of smaller ones, J.-M. Dardié and A. Medina [DM96a] generalized that notion as follows, in terms of Lie algebras.

Definition 3.2.10. *A symplectic Lie algebra $(\mathfrak{g}, \omega)$ is a* symplectic double extension *of a symplectic Lie algebra $(\mathfrak{g}', \omega')$ if $\mathfrak{g}$ contains an isotropic ideal $\mathfrak{h}$ such that $\mathfrak{h}^\perp$ is also an ideal, $\mathfrak{h}^\perp/\mathfrak{h}$ is isomorphic to $\mathfrak{g}'$ as a symplectic Lie algebra, and the canonical sequence of Lie algebras*

$$0 \to \mathfrak{h}^\perp/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}^\perp \to 0$$

is split.

They characterized these extensions in terms of cohomological data and of left-symmetric structures.

A particular case of double extension is when $\mathfrak{g}' = 0$, and was studied independently by Boyom [Boy93] under the name of cotangent extension problem. In that case, $\mathfrak{g}$ is a semi-direct product of a lagrangian subalgebra $\mathfrak{a}$ by an abelian lagrangian ideal $\mathfrak{h} = \mathfrak{a}^*$. Given a Lie algebra $\mathfrak{a}$, there is a bijection between the left symmetric structures on $\mathfrak{a}$ and the structures of symplectic Lie algebra on the symplectic vector space $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, \omega_{\text{can}})$ for which $\mathfrak{a}$ is a subalgebra and $\mathfrak{a}^*$ a (necessarily abelian) ideal.

3.3 Primitive systems

In this section, we show that there is a perfect analogue for symplectic Lie triple systems of the structure theory of symplectic Lie groups exposed in Section 3.2. Namely, we introduce a notion of primitive of a Lie triple system, playing the same role as left-symmetric structures for Lie algebras. We then show that every symplectic Lie triple system canonically admits such a primitive structure.

Conversely, we show that a primitive structure on a Lie triple system induces on its "cotangent space" a structure of symplectic Lie triple system.

The canonical primitive structure introduced here will then be used in Section 3.4 to study the notion of symplectic reduction of symplectic symmetric spaces.

Lemma 3.3.1. *Let* $(\cdot,\cdot,\cdot) : \mathfrak{m}^3 \to \mathfrak{m}$ *be a trilinear product on a vector space* $\mathfrak{m}$*. Denote* $[\cdot,\cdot,\cdot]$ *its antisymmetrization in the first two variables:* $[x,y,z] = (x,y,z) - (y,x,z)$*, and define* $L_{x,y}z = [x,y,z]$*. Assume that*

1. $(x,y,z) = (z,y,x)$*,*
2. $L_{x,y}$ *is a derivation of* $(\cdot,\cdot,\cdot)$*,*

for all $x,y,z \in \mathfrak{m}$*. Then* $(\mathfrak{m},[\cdot,\cdot,\cdot])$ *is a Lie triple system.*

Proof. We must check the three defining properties of a Lie triple system (see Definition 1.4.3). The antisymmetry comes from the definition of $[\cdot,\cdot,\cdot]$. The Jacobi identity follows directly from the symmetry in the first and last variables:

$$\begin{aligned}[x,y,z]+[y,z,x]+[z,x,y] &= (x,y,z)-(y,x,z)\\ &\quad +(y,x,z)-(z,y,x)\\ &\quad +(z,x,y)-(x,z,y)\\ &= 0.\end{aligned}$$

Finally, the derivation identity follows by antisymmetrizing of our second assumption:

$$\begin{aligned}L_{x,y}[u,v,w] &= (L_{x,y}u,v,w)+(u,L_{x,y}v,w)+(u,v,L_{x,y}w)\\ &\quad -(L_{x,y}v,u,w)-(v,L_{x,y}u,w)-(v,u,L_{x,y}w)\\ &= [L_{x,y}u,v,w]+[u,L_{x,y}v,w]+[u,v,L_{x,y}w].\end{aligned}$$

That concludes the proof. □

Definition 3.3.2. *Let* $\mathfrak{m}$ *be a vector space. A* primitive system *on* $\mathfrak{m}$ *is a trilinear product* $(\cdot,\cdot,\cdot) : \mathfrak{m}^3 \to \mathfrak{m}$ *such that, denoting* $m_{x,z}y = (x,y,z)$*, and* $L_{x,y}$ *as in Lemma 3.3.1,*

1. $(x,y,z) = (z,y,x)$*,*
2. $m_{L_{u,v}x,z} + m_{x,L_{u,v}z} = [L_{u,v}, m_{x,z}]$*,*
3. $m_{L_{u,v}x,z} + m_{x,L_{u,v}z} = m_{m_{x,z}u,v} - m_{u,m_{x,z}v}$*,*

for all $x, y, z, u, v \in \mathfrak{m}$.

A morphism of primitive systems in a linear map $\phi : \mathfrak{m} \to \mathfrak{m}'$ *such that*

$$(\phi(x), \phi(y), \phi(z))' = \phi((x, y, z))$$

for all $x, y, z \in \mathfrak{m}$.

Notice that, denoting $\lambda_{x,y}z = (x, y, z) = m_{x,z}y$ and using condition 1, the conditions 2 and 3 of Definition 3.3.2 may be rewritten as

2'. $L_{u,v} \circ \lambda_{x,y} = \lambda_{L_{u,v}x,y} + \lambda_{x,L_{u,v}y} + \lambda_{x,y} \circ L_{u,v}$,

3'. $\lambda_{L_{u,v}x,y} + \lambda_{x,y} \circ L_{u,v} = \lambda_{v,y} \circ \lambda_{x,u} - \lambda_{u,y} \circ \lambda_{x,v}$.

In particular, conditions 1 and 2 of Definition 3.3.2 are just rewrites of conditions 1 and 2 of Lemma 3.3.1. That proves the following proposition, justifying the name of our notion:

Proposition 3.3.3. *Let* $(\mathfrak{m}, (\cdot,\cdot,\cdot))$ *be a primitive system. Then* $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ *is a Lie triple system, with* $[x, y, z] = (x, y, z) - (y, x, z)$ *for all* $x, y, z \in \mathfrak{m}$.

Definition 3.3.4. *Let* $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ *be a Lie triple system. A* primitive structure *on* $\mathfrak{m}$ *is a primitive system on* $\mathfrak{m}$ *such that*

$$(x, y, z) - (y, x, z) = [x, y, z] \tag{3.4}$$

for all $x, y, z \in \mathfrak{m}$. *Conversely,* $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ *is said to be the* associated Lie triple system *of* $(\mathfrak{m}, (\cdot,\cdot,\cdot))$.

Definition 3.3.5. *Let* $(\mathfrak{m}, (\cdot,\cdot,\cdot)$ *be a primitive system, and* $\mathfrak{n}$ *a vector subspace of* $\mathfrak{m}$. *Then* $\mathfrak{n}$ *is said to be a* left- (resp. center-, right-) ideal *of* $\mathfrak{m}$ *if*

$$(\mathfrak{n}, \mathfrak{m}, \mathfrak{m}) \subset \mathfrak{n}$$

(resp. $(\mathfrak{m}, \mathfrak{n}, \mathfrak{m}) \subset \mathfrak{n}$, $(\mathfrak{m}, \mathfrak{m}, \mathfrak{n}) \subset \mathfrak{n}$*). It is said to be an* ideal *if it is a left-, center- and right-ideal.*

Replacing the $\mathfrak{n}$ *on the right-hand side of the above equations by* 0, *we get the notions of* left-, center-, and right-central subsystems, *and that of* central subsystem.

Obviously, by symmetry, a subspace of a primitive system is a left-ideal (resp. is left-central) if and only if it is a right-ideal (resp. it is right-central).

Remark 3.3.6. This definition may seem to be "the wrong way", as usually in an algebra $\mathfrak{m}$, a subspace $\mathfrak{n}$ is said to be a left-ideal if it is stabilized by the left-action: $\mathfrak{m} \cdot \mathfrak{n} \subset \mathfrak{n}$. We chose the other convention, mainly to have a simple denomination for what is called here a center-ideal. ○

Notice that if $\mathfrak{n}$ is an ideal of the primitive structure $\mathfrak{m}$, then it is also an ideal of the associated Lie triple system, and we may check that $\mathfrak{m}/\mathfrak{n}$ inherits a quotient primitive structure from $\mathfrak{m}$.

The following proposition defines an analogue for symplectic Lie triple systems of the canonical affine connection on a symplectic Lie group (compare Proposition 3.2.2).

Proposition 3.3.7. *Let $(\mathfrak{m}, [\cdot,\cdot,\cdot], \Omega)$ be a symplectic Lie triple system, that is, a Lie triple system $\mathfrak{m}$ with a closed non-degenerate 2-form Ω such that*

$$\Omega(L_{x,y}u, v) + \Omega(u, L_{x,y}v) = 0$$

for all $x, y, u, v \in \mathfrak{m}$. Then there is a canonical primitive structure $(\cdot,\cdot,\cdot)$ on $\mathfrak{m}$, defined by

$$\Omega((x,y,z),t) = -\Omega(x,[t,y,z]) = \Omega([t,y,x],z) \tag{3.5}$$

for all $x, y, z, t \in \mathfrak{m}$.

Before passing to the proof, notice that the antisymmetry of Ω and its invariance under $L_{x,y}$ imply that the map $(z,t) \mapsto \Omega(L_{x,y}z, t)$ is *symmetric*:

$$\Omega(L_{x,y}z, t) = -\Omega(z, L_{x,y}t) = \Omega(L_{x,y}t, z). \tag{3.6}$$

Proof. We have, using successively the definition (3.5), the Jacobi identity, and equation (3.6),

$$\begin{aligned}\Omega((x,y,z)-(y,x,z),t) &= \Omega([t,y,x]-[t,x,y],z)\\ &= \Omega([x,y,t],z)\\ &= \Omega([x,y,z],t)\end{aligned}$$

for all $x, y, z, t \in \mathfrak{m}$, so that $(x,y,z)-(y,x,z) = [x,y,z]$ for all $x, y, z \in \mathfrak{m}$.

Now, we check the conditions 1, 2 and 3 of Definition 3.3.2. We have, using the definition (3.5), equation (3.6), and (3.5) again,

$$\begin{aligned}\Omega((x,y,z),t) &= \Omega([t,y,x],z)\\ &= \Omega([t,y,z],x)\\ &= \Omega((z,y,x),t)\end{aligned}$$

for all $x, y, z, t \in \mathfrak{m}$, which proves the first condition. Then, using the fact that

$L_{x,y}$ is a derivation of $[\cdot,\cdot,\cdot]$ at the second step, we compute

$$\begin{aligned}\Omega((x,[u,v,y],z),t) &= \Omega([t,[u,v,y],x],z)\\ &= \Omega([u,v,[t,y,x]]-[[u,v,t],y,x]-[t,y,[u,v,x]],z)\\ &= -\Omega([t,y,x],[u,v,z])-\Omega((x,y,z),[u,v,t])\\ &\quad -\Omega(([u,v,x],y,z),t)\\ &= \Omega(-(x,y,[u,v,z])+[u,v,(x,y,z)]-([u,v,x],y,z),t)\end{aligned}$$

for all $x,y,z,t,u,v\in\mathfrak{m}$. That yields the second condition. And finally, we have, using two times the definition (3.5),

$$\begin{aligned}\Omega((u,v,(x,y,z)),t) &= \Omega([t,v,u],(x,y,z))\\ &= \Omega(z,[[t,v,u],y,x]).\end{aligned}$$

Using this computation at the first and fourth steps, and the fact that $L_{x,y}$ is a derivation of $[\cdot,\cdot,\cdot]$ at the second step, we get

$$\begin{aligned}\Omega((u,y,(x,v,z)),t) &= \Omega(z,[[t,y,u],v,x])\\ &= \Omega(z,[t,y,[u,v,x]]-[u,[t,y,v],x]-[u,v,[t,y,x]])\\ &= \Omega(z,[t,y,[u,v,x]]+[[t,y,v],u,x]\\ &\qquad +[[t,y,x],u,v]-[[t,y,x],u,v])\\ &= \Omega(t,([u,v,x],y,z))+\Omega(t,(v,y,(x,u,z)))\\ &\qquad +\Omega((x,y,(v,u,z)),t)-\Omega((x,y,(u,v,z)),t)\\ &= \Omega(-([u,v,x],y,z)+(v,y,(x,u,z))-(x,y,[u,v,z]),t)\end{aligned}$$

for all $x,y,z,t,u,v\in\mathfrak{m}$, which is the third condition. That concludes the proof. □

Lemma 3.3.8. *Let* $(\mathfrak{m},\Omega)$ *be a symplectic Lie triple system, and* $\mathfrak{n}$ *be an isotropic ideal. Then* $[\mathfrak{n},\mathfrak{n},\mathfrak{m}]=[\mathfrak{n},\mathfrak{m},\mathfrak{n}]=[\mathfrak{m},\mathfrak{n},\mathfrak{n}]=0$. *In particular,* $\mathfrak{n}$ *is abelian.*

Proof. For all $x,t\in\mathfrak{m}$ and $y,z\in\mathfrak{n}$, we have, since $\mathfrak{n}\subset\mathfrak{n}^{\perp}$,

$$\Omega([x,y,z],t)=-\Omega(z,[x,y,t])\subset\Omega(\mathfrak{n}^{\perp},\mathfrak{n})=0,$$

hence $[\mathfrak{m},\mathfrak{n},\mathfrak{n}]=0$ since Ω is non-degenerate.

By the Jacobi identity and the antisymmetry of the triple bracket, this implies $[\mathfrak{n},\mathfrak{n},\mathfrak{m}]\subset[\mathfrak{m},\mathfrak{n},\mathfrak{n}]+[\mathfrak{n},\mathfrak{m},\mathfrak{n}]=0$. □

Theorem 3.3.9. *Let* $(\mathfrak{m}, [\cdot,\cdot,\cdot])$ *be a Lie triple system, and consider its cotangent space* $T^*\mathfrak{m} = \mathfrak{m} \oplus \mathfrak{m}^*$ *with the standard symplectic structure* Ω *defined by*

$$\Omega((x,\alpha),(y,\beta)) = \alpha(y) - \beta(x)$$

for all $x, y \in \mathfrak{m}$ *and* $\alpha, \beta \in \mathfrak{m}^*$*. Then there is a one-to-one correspondance between the structures of symplectic Lie triple system on* $(T^*\mathfrak{m}, \Omega)$ *in which* $\mathfrak{m}$ *is a subsystem and* $\mathfrak{m}^*$ *an ideal, and the primitive structures on* $\mathfrak{m}$*.*

If $(\cdot,\cdot,\cdot)$ *is such a primitive structure, denoting* $\lambda_{x,y} = (x,y,\cdot)$ *and* $L_{x,y} = [x,y,\cdot] = \lambda_{x,y} - \lambda_{y,x}$*, the corresponding Lie triple system is defined by*

$$[(x,\alpha),(y,\beta),(z,\gamma)] = ([x,y,z], \beta\circ\lambda_{z,x} - \alpha\circ\lambda_{z,y} - \gamma\circ L_{x,y}) \tag{3.7}$$

for all $x, y, z \in \mathfrak{m}$ *and* $\alpha, \beta, \gamma \in \mathfrak{m}^*$*. It is called the* split T^*-extension *of the primitive system* $(\mathfrak{m}, (\cdot,\cdot,\cdot))$*.*

Proof. Assume $(T^*\mathfrak{m}, \Omega)$ has a structure of symplectic Lie triple system in which $\mathfrak{m}$ is a subsystem and $\mathfrak{m}^*$ an ideal.

Since $\mathfrak{m}^*$ is an isotropic ideal, by Lemma 3.3.8, it is such that the triple product of three elements of $T^*\mathfrak{m}$ vanishes whenever any two of them lie in $\mathfrak{m}^*$. As a consequence, the triple product $[\cdot,\cdot,\cdot]$ on $T^*\mathfrak{m}$ is entirely determined by its value on $\mathfrak{m}$, and by two bilinear maps $\theta, \epsilon : \mathfrak{m}\times\mathfrak{m} \to \mathrm{End}(\mathfrak{m}^*)$ such that

$$[(x,\alpha),(y,\beta),(z,\gamma)] = ([x,y,z], \theta(x,z)\beta - \theta(y,z)\alpha + \epsilon(x,y)\gamma) \tag{3.8}$$

for all $x, y, z \in \mathfrak{m}$ and $\alpha, \beta, \gamma \in \mathfrak{m}^*$.

Now, by the Jacobi identity, we get

$$\begin{aligned} 0 &= [(0,\alpha),(y,0),(z,0)] + [(y,0),(z,0),(0,\alpha)] + [(z,0),(0,\alpha),(y,0)] \\ &= (0, -\theta(y,z)\alpha + \epsilon(y,z)\alpha + \theta(z,y)\alpha) \end{aligned}$$

so that $\epsilon(x,y) = \theta(x,y) - \theta(y,x)$ for all $x, y \in \mathfrak{m}$.

On the other hand, the invariance of the symplectic form yields

$$\begin{aligned} 0 &= \Omega\left(L_{(x,0),(y,0)}(0,\gamma),(t,0)\right) + \Omega\left((0,\gamma), L_{(x,0),(y,0)}(t,0)\right) \\ &= (\theta(x,y)\gamma - \theta(y,x)\gamma)(t) + \gamma([x,y,t]) \end{aligned}$$

so that

$$\epsilon(x,y)\gamma = -\gamma\circ L_{x,y} \tag{3.9}$$

for all $x, y \in \mathfrak{m}$ and $\gamma \in \mathfrak{m}^*$. In the same way,

$$\begin{aligned} 0 &= \Omega\left(L_{(0,\alpha),(y,0)}(z,0),(t,0)\right) + \Omega\left((z,0),L_{(0,\alpha),(y,0)}(t,0)\right) \\ &= -(\theta(y,z)\alpha)\,t + (\theta(y,t)\alpha)\,z \end{aligned} \tag{3.10}$$

for all $x, y, z, t \in \mathfrak{m}$ and $\alpha \in \mathfrak{m}^*$.

Finally, using the fact that $L_{(x,\alpha),(y,\beta)}$ is a derivation of $[\cdot,\cdot,\cdot]$, we get the two conditions

$$\epsilon(u,v) \circ \theta(x,y) - \theta(x, L_{u,v}y) = \theta(v,y) \circ \theta(x,u) - \theta(u,y) \circ \theta(x,v) \tag{3.11}$$

$$[\epsilon(u,v), \theta(x,y)] = \theta(L_{u,v}x, y) + \theta(x, L_{u,v}y). \tag{3.12}$$

Defining a trilinear product $(x,y,z) = \lambda_{x,y}z$ on $\mathfrak{m}$ by

$$\alpha \circ \lambda_{x,y} = \theta(y,x)\alpha,$$

we observe that the equations 3.10, 3.11 and 3.12 are altogether equivalent to the conditions 1, 2 and 3 of Definition 3.3.2, and that equation 3.9 means that $(\cdot,\cdot,\cdot)$ is a primitive structure on $\mathfrak{m}$.

Conversely, assume that $(\cdot,\cdot,\cdot)$ is a primitive structure on $\mathfrak{m}$ and denote $\lambda_{x,y}z = (x,y,z)$ for all $x, y, z \in \mathfrak{m}$. Define on $T^*\mathfrak{m}$ the product

$$[(x,\alpha),(y,\beta),(z,\gamma)] = ([x,y,z], \beta \circ \lambda_{z,x} - \alpha \circ \lambda_{z,y} - \gamma \circ L_{x,y}).$$

It is antisymmetric in the first two variables. For the Jacobi identity, the equation 3.4 for the primitive system implies that the $\mathfrak{m}^*$ part of the cyclic sum of $[(x,\alpha),(y,\beta),(z,\gamma)]$ is given by

$$\begin{aligned} &\beta \circ \lambda_{z,x} - \alpha \circ \lambda_{z,y} - \gamma \circ L_{x,y} \\ &+\gamma \circ \lambda_{x,y} - \beta \circ \lambda_{x,z} - \alpha \circ L_{y,z} \\ &+\alpha \circ \lambda_{y,z} - \gamma \circ \lambda_{y,x} - \beta \circ L_{z,x} = 0. \end{aligned}$$

And for the derivation identity, we have, dropping the $\mathfrak{m}$ part after the first equality,

$$\begin{aligned} &\left([L_{(x,\alpha),(y,\beta)}, L_{(s,\sigma),(t,\tau)}] - L_{L_{(x,\alpha),(y,\beta)}(s,\sigma),(t,\tau)} - L_{(s,\sigma),L_{(x,\alpha),(y,\beta)}(t,\tau)}\right)(z,\gamma) \\ &\quad = \Big[(x,\alpha),(y,\beta),([s,t,z], \tau \circ \lambda_{z,s} - \sigma \circ \lambda_{z,t} - \gamma \circ L_{s,t})\Big] \\ &\qquad - \Big[(s,\sigma),(t,\tau),([x,y,z], \beta \circ \lambda_{z,x} - \alpha \circ \lambda_{z,y} - \gamma \circ L_{x,y})\Big] \\ &\qquad - \Big[([x,y,s], \beta \circ \lambda_{s,x} - \alpha \circ \lambda_{s,y} - \sigma \circ L_{x,y}),(t,\sigma),(z,\gamma)\Big] \end{aligned}$$

$$
\begin{aligned}
&-\Big[(s,\sigma),([x,y,t],\beta\circ\lambda_{t,x}-\alpha\circ\lambda_{t,y}-\tau\circ L_{x,y}),(z,\gamma)\Big]\\
&=\beta\circ\lambda_{L_{s,t}z,x}-\alpha\circ\lambda_{L_{s,t}z,y}-\tau\circ\lambda_{z,s}\circ L_{x,y}+\sigma\circ\lambda_{z,t}\circ L_{x,y}+\gamma\circ L_{s,t}\circ L_{x,y}\\
&\quad-\tau\circ\lambda_{L_{x,y}z,s}+\sigma\circ\lambda_{L_{x,y}z,t}+\beta\circ\lambda_{z,x}\circ L_{s,t}-\alpha\circ\lambda_{z,y}\circ L_{s,t}-\gamma\circ L_{x,y}\circ L_{s,t}\\
&\quad-\tau\circ\lambda_{z,L_{x,y}s}+\beta\circ\lambda_{s,x}\circ\lambda_{z,t}-\alpha\circ\lambda_{s,y}\circ\lambda_{z,t}-\sigma\circ L_{x,y}\circ\lambda_{z,t}+\gamma\circ L_{L_{x,y}s,t}\\
&\quad-\eta\circ\lambda_{t,x}\circ\lambda_{z,s}+\alpha\circ\lambda_{t,y}\circ\lambda_{z,s}+\tau\circ\ L_{x,y}\circ\lambda_{z,s}+\sigma\circ\lambda_{z,,L_{x,y}t}+\gamma\circ L_{s,L_{x,y}t}\\
&=0
\end{aligned}
$$

where the sum of the terms in α vanishes by condition 3 in Definition 3.3.2, as does the sum of the terms in β. The terms in γ vanish because of the derivation identity in $\mathfrak{m}$, and those in σ and τ vanish by condition 2 in Definition 3.3.2. Hence we indeed have a Lie triple system.

The fact that it is symplectic follows from the symmetry of the primitive system in the first and last variables:

$$
\begin{aligned}
&\Omega\left(L_{(x,\alpha),(y,\beta)}(s,\sigma),(t,\tau)\right)+\Omega\left((s,\sigma),L_{(x,\alpha),(y,\beta)}(t,\tau)\right)\\
&\quad=\left(\beta\circ\lambda_{s,x}t-\alpha\circ\lambda_{s,y}t-\sigma\circ L_{x,y}t\right)-\tau(L_{x,y}s)\\
&\qquad+\sigma(L_{x,y}t)-\left(\beta\circ\lambda_{t,x}s-\alpha\circ\lambda_{t,y}s-\tau\circ L_{x,y}s\right)\\
&\quad=0.
\end{aligned}
$$

This concludes the proof. □

Actually, we saw in Section 1.4 (see Definition 1.4.33) that, given a Lie triple system $\mathfrak{m}$, a vector space V is a $\mathfrak{m}$-module if and only if $\mathfrak{m}\oplus V$ is a Lie triple system with bracket given by (3.8) for two bilinear maps $\theta,\epsilon:\mathfrak{m}\times\mathfrak{m}\to V$ satisfying (3.11) and (3.12), and $\epsilon(x,y)=\theta(x,y)-\theta(y,x)$ for all $x,y\in\mathfrak{m}$. Hence, a primitive structure on $\mathfrak{m}$ is just a $\mathfrak{m}$-module structure on $\mathfrak{m}^*$ with the additional requirements (3.9) and (3.10).

By introducing a 3-cocycle $C\in C^3(\mathfrak{m},\mathfrak{m}^*)$ for the $\mathfrak{m}$-module $\mathfrak{m}^*$ (see Section 1.4), we may get rid of the semi-direct product restriction, to obtain all (non-split) extensions of $\mathfrak{m}$ by $\mathfrak{m}^*$:

Theorem 3.3.10. *Let $(\mathfrak{m},[\cdot,\cdot,\cdot])$ be a Lie triple system, and consider its cotangent space $T^*\mathfrak{m}=\mathfrak{m}\oplus\mathfrak{m}^*$ with the standard symplectic structure Ω defined by*

$$\Omega((x,\alpha),(y,\beta))=\alpha(y)-\beta(x)$$

for all $x,y\in\mathfrak{m}$ and $\alpha,\beta\in\mathfrak{m}^*$. Then there is a one-to-one correspondance between the structures of symplectic Lie triple system on $(T^*\mathfrak{m},\Omega)$ in which $\mathfrak{m}^*$ an ideal and $T^*\mathfrak{m}/\mathfrak{m}^*$ is isomorphic to $\mathfrak{m}$, and the primitive structures on $\mathfrak{m}$ together with a

3-cocycle $C \in C^3(\mathfrak{m}, \mathfrak{m}^*)$ *such that*

$$C_{x,y,z}t = C_{x,y,t}z \tag{3.13}$$

for all $x, y, z, t \in \mathfrak{m}$.

If $(\cdot,\cdot,\cdot)$ *is such a primitive structure and* $C : (x, y, z) \mapsto C_{x,y,z}$ *an element of* $C^3(\mathfrak{m}, \mathfrak{m}^*)$ *satisfying* (3.13)*, denoting* $\lambda_{x,y} = (x, y, \cdot)$ *and* $L_{x,y} = [x, y, \cdot] = \lambda_{x,y} - \lambda_{y,x}$*, the corresponding Lie triple system is defined by*

$$[(x, \alpha), (y, \beta), (z, \gamma)] = ([x, y, z], C_{x,y,z} + \beta \circ \lambda_{z,x} - \alpha \circ \lambda_{z,y} - \gamma \circ L_{x,y}) \tag{3.14}$$

for all $x, y, z \in \mathfrak{m}$ *and* $\alpha, \beta, \gamma \in \mathfrak{m}^*$*. It is called the* T^*-extension *of the primitive system* $(\mathfrak{m}, (\cdot,\cdot,\cdot))$ *by the 3-cocycle* C*.*

The notion of T^*-extension was first introduced in [Bor97] for non-associative algebras with an invariant bilinear form. It was later extended in [LWD09] to *metrised* Lie triple systems. A Lie triple system is said to be metrised if it is endowed with an invariant symmetric non-degenerate bilinear 2-form. It is known [LWD09] that every finite-dimensional nilpotent metrised Lie triple system over an algebraically closed field is the T^*-extension of an appropriate quotient system. It would be interesting to investigate the question in the case of symplectic Lie triple systems.

Lemma 3.3.11. *Let* $(\mathfrak{m}, \Omega)$ *be symplectic Lie triple system, and* $\mathfrak{n}$ *a lagrangian ideal. Then* $\mathfrak{n}$ *is an ideal for the canonical primitive structure on* $\mathfrak{m}$*.*

Proof. For $x, t \in \mathfrak{n}$ and $y, z \in \mathfrak{m}$ we have

$$\Omega((x, y, z), t) = -\Omega(x, [t, y, z]) = 0$$

so that $(\mathfrak{n}, \mathfrak{m}, \mathfrak{m}) \subset \mathfrak{n}$. Now for $y \in \mathfrak{n}$ and $x, z \in \mathfrak{m}$,

$$(x, y, z) = [x, y, z] + (y, x, z) \subset [\mathfrak{m}, \mathfrak{n}, \mathfrak{m}] + (\mathfrak{n}, \mathfrak{m}, \mathfrak{m}) \subset \mathfrak{n}$$

so that $(\mathfrak{m}, \mathfrak{n}, \mathfrak{m}) \subset \mathfrak{n}$. □

In particular, $\mathfrak{m}/\mathfrak{n}$ inherits a primitive structure, so that $T^*(\mathfrak{m}/\mathfrak{n})$ is a symplectic Lie triple system. Actually, we have the following theorem.

Theorem 3.3.12. *Every symplectic Lie triple system having a lagrangian ideal (resp. and such that there is a complementary lagrangian subsystem) is isomorphic to a* T^**-extension (resp. a split* T^**-extension).*

Proof. Let $(\mathfrak{m}, \Omega)$ be a symplectic Lie triple system and let $(\cdot,\cdot,\cdot)$ be its canonical primitive structure. Assume $\mathfrak{n}$ is a lagrangian ideal. Then $\mathfrak{m}/\mathfrak{n}$ is a quotient Lie triple system.

Choose in $\mathfrak{m}$ a lagrangian vector subspace $\mathfrak{m}'$ supplementary to $\mathfrak{n}$, and fix an isomorphism $\mathfrak{m}' \cong \mathfrak{m}/\mathfrak{n}$. The symplectic form Ω then allows to identify $\mathfrak{n}$ with $\mathfrak{m}'^*$ in such a way that Ω transfers to $T^*\mathfrak{m}'$ as the standard symplectic form. As a consequence, the symplectic Lie triple structure of $\mathfrak{m}$ transfers to $T^*\mathfrak{m}'$ in such a way that $\mathfrak{m}^*$ is an ideal and $T^*\mathfrak{m}'/\mathfrak{m}'^*$ is isomorphic to $\mathfrak{m}'$. Hence the result follows from Theorem 3.3.10.

If there exists a lagrangian Lie triple subsystem supplementary to $\mathfrak{n}$, then taking $\mathfrak{m}'$ to be this subsystem and applying Theorem 3.3.9 yields the second assertion. □

Let us illustrate these results with a few examples.

Example 3.3.13. Every two-dimensional solvable Lie triple system has a one-dimensional (lagrangian) ideal, and any complementary subspace, being one-dimensional and thus abelian, is a subsystem, so we are in the context of Theorem 3.3.9. Let $\mathfrak{m} = \langle H \rangle$. The set of primitive structures on $\mathfrak{m}$ is parametrized by a real numer a: $(H,H,H)_a = \lambda^a_{H,H} = aH$. And the set of 3-cocycles C is reduced to zero, by antisymmetry.

According to equation (3.14), the resulting Lie triple systems on $T^*\mathfrak{m}$ are such that the only non-zero brackets involving H and H^* are

$$[H, H^*, H]_a = H^* \circ \lambda^a_{H,H} = aH^* \tag{3.15}$$

and its antisymmetrization $[H^*, H, H] = -aH^*$, and the symplectic form is given by $\Omega(H, H^*) = -H^*(H) = -1$.

Moreover, for any non-zero real number b, the isomorphism $\phi : T^*\mathfrak{m} \to T^*\mathfrak{m}$ given by $\phi(H) = bH$ and $\phi(H^*) = \frac{1}{b}H^*$ is such that $[\phi(H), \phi(H^*), \phi(H)]_a = b^2 a \phi(H^*)$, and $\phi^*\Omega = \Omega$, so that all Lie triple systems $[\cdot,\cdot,\cdot]_a$ with a of the same sign are isomorphic. Notice that all these isomorphisms come from isomorphisms ϕ' of $\mathfrak{m}$ given by $\phi'(H) = bH$, so that the isomorphism classes of Lie triple structures on $T^*\mathfrak{m}$ compatible with the canonical symplectic form are in one-to-one correspondance with the isomorphism classes of primitive structures on $\mathfrak{m}$.

Moreover, sending H^* on cH^*, for c ranging in the non-zero real numbers, yields every possible symplectic form starting from the canonical one, without changing the triple structure (see equation (3.15)). Thus we recover the fact [Bie98a] that, up to isomorphism, the only two-dimensional solvable symplectic

Lie triple systems are defined by

$$[H,H^*,H]_+ = H^* \qquad [H,H^*,H]_0 = 0 \qquad [H,H^*,H]_- = -H^*$$

and $\Omega(H,H^*) = -1$. ◁

Example 3.3.14. Going through the lists of Proposition 1.1, Proposition 2.3 and Proposition 4.3 of [Bie98a], we observe that every indecomposable four-dimensional non-semisimple symplectic Lie triple system has a lagrangian ideal, and is thus isomorphic to a T^*-extension of a two-dimensional primitive system as in Theorem 3.3.10. The two-dimensional primitive systems involved all come from symplectic Lie triple systems, and are the three solvable systems of Example 3.3.13, and the three simple systems $\mathfrak{su}(2)/\mathfrak{u}(1)$, $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(2)$ and $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(1,1)$.

As an example, consider the only non-isomorphic four-dimensional solvable symplectic Lie triple systems $\mathfrak{m}$ such that the derived ideal $[\mathfrak{m},\mathfrak{m},\mathfrak{m}]$ is not abelian, given by Proposition 2.3 of [Bie98a]. They are given in terms of involutive Lie algebras. Translating in terms of Lie triple systems, we get, in a basis $\mathfrak{m} = \langle H,V,W,E\rangle$, the following forms:

1. For $\epsilon = \pm 1$,

$$\begin{array}{lll} [H,V,H] = -\epsilon V & [H,W,H] = -\epsilon W & [H,E,H] = -4\epsilon E \\ [H,V,W] = \epsilon E & [H,W,V] = -\epsilon E & [V,W,H] = -2\epsilon E \end{array}$$

   with $\Omega(H,E) = -a$, $\Omega(V,W) = -a$ and $\Omega(H,V) = -a\eta$ for $a \in \mathbb{R}_0^+$ and $\eta = 0,1$.

2. For $\epsilon = \pm 1$,

$$\begin{array}{lll} [H,V,H] = -\epsilon V & [H,E,H] = -\epsilon E & [H,V,V] = W \\ [V,W,H] = -E & [H,V,W] = E & \end{array}$$

   with $\Omega(H,E) = -a\epsilon$, $\Omega(V,W) = -a$ and $\Omega(H,V) = -a\eta$ for $a \in \mathbb{R}_0$ and $\eta \in \mathbb{R}$.

We would like to see these systems as T^*-extensions of $\mathfrak{m}' = \langle H,V\rangle$ by identifying $\langle W,E\rangle$ with $\mathfrak{m}'^*$. To this end, we perform a change of variables so that $\mathfrak{m}'$ and $\mathfrak{m}'^*$ are lagrangian and the symplectic form is normalized, and then compute the primitive system $(\cdot,\cdot,\cdot)$ and the 3-cocycle C using the equation (3.14).

1. Replacing V by $V - \eta E$, so that $\Omega(H,V) = 0$ and the only change in the brackets is $[H,V,H] = -\epsilon V + \eta(4-\epsilon)E$, and then identifying W with $\frac{1}{a}V^*$

and E with $\frac{1}{a}H^*$, we get

$$[H,V,H] = -\epsilon V + \frac{\eta}{a}(4-\epsilon)H^* \quad [H,V^*,H] = -\epsilon V^* \quad [H,H^*,H] = -4\epsilon H^*$$
$$[H,V,V^*] = \epsilon H^* \quad [H,V^*,V] = -\epsilon H^* \quad [V,V^*,H] = -2\epsilon H^*,$$

and $\Omega(H,H^*) = -1$, $\Omega(V,V^*) = -1$ as required. Thus the only nonvanishing components of the 3-cocycle are

$$C(H,V,H) = \frac{\eta}{a}(4-\epsilon)H^* = -C(V,H,H).$$

For the primitive structure, the equations $[H,V^*,H] = \epsilon H^* = V^* \circ \lambda_{H,H}$ and $[H,H^*,H] = -4\epsilon H^* = H^* \circ \lambda_{H,H}$ yield $(H,H,H) = \epsilon V - 4\epsilon H$. Proceeding similarly, we get

$$(H,H,H) = \epsilon V - 4\epsilon H \qquad (V,H,H) = -\epsilon V \qquad (H,V,H) = -2\epsilon V$$

as only non-vanishing components, up to symmetry in the first and third variables.

Notice that when $\epsilon = -1$, the operator $x \mapsto [x,H,H]$ has negative eigenvalues, meaning that the Lie triple system $T^*\mathfrak{m}'$ is not exponential. On the contrary, when $\epsilon = 1$, the system is completely solvable.

When $\eta = 0$, the extension is split, and when moreover $\epsilon = 1$ we recover the usual four-dimensional Ricci-type symplectic symmetric space described in [Bie07]. With $\eta = 1$ and $\epsilon = 1$, we get a particular case of a class of spaces which will be studied and quantized in Chapter 5.

2. Replacing V by $V - \eta\epsilon E$ and then identifying W with $\frac{1}{a}V^*$ and E with $\frac{\epsilon}{a}H^*$, we get

$$[H,V,H] = -\epsilon V \qquad [H,H^*,H] = -\epsilon H^* \qquad [H,V,V] = \frac{1}{a}V^*$$
$$[V,V^*,H] = -H^* \qquad [H,V,V^*] = H^*,$$

with $\Omega(H,H^*) = -1$ and $\Omega(V,V^*) = -1$. The non-vanishing components of the 3-cocycle are

$$C(H,V,V) = \frac{1}{a}V^* = -C(V,H,V)$$

and those of the primitive structure are

$$(H,H,H) = -H \qquad (H,V,H) = -\epsilon V.$$

◁

3.4 Symplectic reduction

In this section, we show that the notion of symplectic reduction makes sense in the category of symplectic symmetric spaces. This sheds some light on the structure of those spaces. The solvable ones, for example, are not well-known. They have been classified in dimension 2 and 4 in [Bie98a] and some infinite classes of examples are known. These include the ESET's [Bie02] and those corresponding to elementary normal j-algebras [Bie07], which we will describe in Chapter 5. Other classes include extrinsic symplectic symmetric spaces [CGRS09], that is, spaces embedded in some symplectic vector space and for which the symmetries are defined as ordinary reflection along the tangent space and identity along the orthogonal subspace. It has been proved that these spaces are always solvable [Sch10], so that it would be interesting to see if the symplectic reduction also applies in this more constrained case.

We start with an algebraic version of symplectic reduction, in terms of Lie triple systems. We then provide a geometric version, in terms of momentum maps in the spirit of the classical Marsden-Weinstein reduction.

Proposition 3.4.1. *Let $(\mathfrak{m}, \Omega)$ be a symplectic Lie triple system, and let $\mathfrak{n}$ be an ideal of $\mathfrak{m}$. Denote $(\cdot,\cdot,\cdot)$ the canonical primitive structure on $\mathfrak{m}$ (see Proposition 3.3.7), and $\mathfrak{n}^\perp$ the symplectic orthogonal $\mathfrak{n}^\perp = \{x \in \mathfrak{m} \mid \Omega(x, \mathfrak{n}) = 0\}$ of $\mathfrak{n}$. Then,*

1. $(\mathfrak{n}^\perp, \mathfrak{m}, \mathfrak{m}) \subset \mathfrak{n}^\perp$,
2. $(\mathfrak{m}, \mathfrak{n}, \mathfrak{n}^\perp) = 0$ *and* $(\mathfrak{n}^\perp, \mathfrak{m}, \mathfrak{n}) = 0$.

In particular,

3. $\mathfrak{n}^\perp$ *is a Lie triple subsystem of* $\mathfrak{m}$,
4. $\mathfrak{n}^\perp \cap \mathfrak{n}$ *is a central subsystem of* $\mathfrak{n}^\perp$,
5. $[\mathfrak{m}, \mathfrak{m}, \mathfrak{n}^\perp] \subset \mathfrak{n}^\perp$.

Moreover, $\mathfrak{n}^\perp$ is an ideal of the Lie triple system $\mathfrak{m}$ if and only if $\mathfrak{n}^\perp$ is a center-ideal of the primitive system $\mathfrak{m}$ if and only if $[\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{m}] = 0$.

Proof. Consider the first assertion. Since $\mathfrak{n}$ is an ideal of the Lie triple system, we have for all $x \in \mathfrak{n}^\perp$, $y, z \in \mathfrak{m}$ and $t \in \mathfrak{n}$,

$$\Omega((x, y, z), t) = -\Omega(x, [t, y, z]) \subset \Omega(\mathfrak{n}^\perp, \mathfrak{n}) = 0, \tag{3.16}$$

so that $(\mathfrak{n}^\perp, \mathfrak{m}, \mathfrak{m}) \subset \mathfrak{n}^\perp$. This proves 1. and in particular 3. Using the symmetry in the first and third variables, this also gives $(\mathfrak{m}, \mathfrak{m}, \mathfrak{n}^\perp) \subset \mathfrak{n}^\perp$, which implies 5.

For the second assertion, we have, for all $x, t \in \mathfrak{m}$, $y \in \mathfrak{n}$ and $z \in \mathfrak{n}^\perp$,

$$\Omega((x,y,z),t) = \Omega([t,y,x],z) \subset \Omega(\mathfrak{n}, \mathfrak{n}^\perp) = 0.$$

Since Ω is non-degenerate, we get $(\mathfrak{m}, \mathfrak{n}, \mathfrak{n}^\perp) = 0$. In the same way, and using assertion 5., we get $\Omega((\mathfrak{n}^\perp, \mathfrak{m}, \mathfrak{n}), \mathfrak{m}) = \Omega([\mathfrak{m}, \mathfrak{m}, \mathfrak{n}^\perp], \mathfrak{n}) = 0$, so that $(\mathfrak{n}, \mathfrak{m}, \mathfrak{n}^\perp) = 0$. This proves 2. and in particular 4.

For the question of when $\mathfrak{n}^\perp$ is an ideal, by 1. and the symmetry of the primitive system, we see that $\mathfrak{n}^\perp$ is a Lie triple ideal if and only if it is a primitive center-ideal, and the equality $\Omega((\mathfrak{m}, \mathfrak{n}^\perp, \mathfrak{m}), \mathfrak{n}) = \Omega(\mathfrak{m}, [\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{m}])$ shows that this is equivalent to $[\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{m}] = 0$. □

Before presenting the reduction, let us consider the case of a symplectic ideal.

Proposition 3.4.2. *Let $(\mathfrak{m}, \Omega)$ be a symplectic Lie triple system, and let $\mathfrak{n}$ be a symplectic ideal of $\mathfrak{m}$. Then,*

1. *$\mathfrak{m}$ is a semi-direct product $\mathfrak{n}^\perp \ltimes \mathfrak{n}$,*
2. *the only possibly non-zero brackets involving $\mathfrak{n}^\perp$ and $\mathfrak{n}$ are*

$$[\mathfrak{n}^\perp, \mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n} \qquad\qquad [\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{n}] \subset \mathfrak{n}$$

   and we have

$$[[\mathfrak{n}^\perp, \mathfrak{n}^\perp, \mathfrak{n}^\perp], \mathfrak{n}, \mathfrak{n}] = 0,$$

3. *if $\mathfrak{n}^\perp$ is semisimple, then $\mathfrak{m}$ is a direct product $\mathfrak{n}^\perp \oplus \mathfrak{n}$,*
4. *in particular, if the radical of $\mathfrak{m}$ is symplectic, then $\mathfrak{m} = \mathfrak{n}^\perp \oplus \mathfrak{n}$ is a Levi decomposition which is a direct product.*

Proof. 1. This is a rephrasing of the fact that $\mathfrak{n}^\perp$ is a subsystem.

2. By Lemma 3.4.1, the only non-trivial brackets involving $\mathfrak{n}$ and $\mathfrak{n}^\perp$ are $[\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{m}] \subset \mathfrak{n}$. But $[\mathfrak{m}, \mathfrak{m}, \mathfrak{n}^\perp] \subset \mathfrak{n}^\perp$ and $\mathfrak{m} = \mathfrak{n}^\perp \oplus \mathfrak{n}$ as vector spaces, so that $[[\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{n}^\perp] = 0$ and there remains only $[\mathfrak{n}, \mathfrak{n}^\perp, \mathfrak{n}] \subset \mathfrak{n}$.

   Using the derivations identity, we get for $x, y, z \in \mathfrak{n}^\perp$ and $u, v \in \mathfrak{n}$,

$$\begin{aligned}[[x,y,z],u,v] &= [x,y,[z,u,v]] - [z,[x,y,u],v] - [z,u,[x,y,v]] \\ &= 0 - 0 - 0 = 0\end{aligned}$$

since for the first term $[\mathfrak{n}^\perp, \mathfrak{n}^\perp, [\mathfrak{n}^\perp, \mathfrak{n}, \mathfrak{n}]] \subset [\mathfrak{n}^\perp, \mathfrak{n}^\perp, \mathfrak{n}] = 0$ and for the second and last terms $[\mathfrak{n}^\perp, \mathfrak{n}^\perp, \mathfrak{n}] = 0$.

3.,4. These follow directly from the previous point and the fact that $\mathfrak{n}^\perp = [\mathfrak{n}^\perp, \mathfrak{n}^\perp, \mathfrak{n}^\perp]$. Point 4. was first obtained in [Bie95, Proposition 4.1.6].

□

We thus get the following reduction theorem for symplectic Lie triple systems.

Theorem 3.4.3. *Let (m, Ω) be a symplectic Lie triple system, and let $\mathfrak{n}$ be an ideal of $\mathfrak{m}$. Then $\mathfrak{m}' = \mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$ is naturally a symplectic Lie triple system with symplectic form Ω' such that*

$$\pi^*\Omega' = \iota^*\Omega \tag{3.17}$$

where $\pi : \mathfrak{n}^\perp \to \mathfrak{m}'$ is the natural projection, and $\iota : \mathfrak{n}^\perp \to \mathfrak{m}$ the natural inclusion.

Proof. By Proposition 3.4.1, $\mathfrak{n}^\perp$ is a Lie triple system, and $\mathfrak{n}^\perp \cap \mathfrak{n}$ is an ideal of $\mathfrak{n}^\perp$. Hence $\mathfrak{m}' = \mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$ has a canonical structure of Lie triple system.

Moreover, $\mathfrak{n}^\perp \cap \mathfrak{n}$ is exactly the kernel of the restriction of Ω to $\mathfrak{n}^\perp$, so that $\mathfrak{m}'$ bears a canonical symplectic form Ω' satisfying equation (3.17). □

This implies a corresponding theorem for connected and simply connected symplectic symmetric spaces. However, we will now give a global, more geometric version of symplectic reduction for those spaces.

We start by defining an appropriate orthogonal subspace to a normal subspace.

Proposition 3.4.4. *Let (M, s, ω, o) be a pointed connected and simply connected symplectic symmetric space, and N be a closed normal subspace of M. Let G be (the central extension by ω of) the automorphism group of (M, s, ω), acting in a strongly Hamiltonian manner on M with momentum map J (see Section 1.3). Let σ be the conjugation by s_o in G, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the associated decomposition. Let $\mathfrak{n}$ be the subspace of $\mathfrak{m}$ corresponding to T_oN, and let $\mathfrak{h}$ be the smallest ideal of $\mathfrak{g}$ containing $\mathfrak{n}$, defined by $\mathfrak{h} = \mathfrak{n} + [\mathfrak{m}, \mathfrak{n}]$. Denote $\iota_\mathfrak{h} : \mathfrak{h} \to \mathfrak{g}$ the inclusion, $J^\mathfrak{h} = \iota_\mathfrak{h}^* \circ J : M \to \mathfrak{h}^*$ the corresponding projection of J, and μ the image of o by $J^\mathfrak{h}$. Then*

$$N^\perp = \left(J^\mathfrak{h}\right)^{-1}(\mu)$$

is a closed symmetric subspace of M such that $T_o(N^\perp) = (T_oN)^\perp$. It is independent of the choice of J, and we call it the orthogonal subspace *to N at o in M.*

Proof. We first show that $J^{\mathfrak{h}}$ is G-equivariant. Since $\mathfrak{h}$ is normal, $\iota_{\mathfrak{h}}$ is Ad_G-equivariant. As J is by definition G-equivariant, we see that $J^{\mathfrak{h}}$ also is.

Now consider the set

$$G_\mu^{\mathfrak{h}} = \{g \in G \mid J^{\mathfrak{h}}(g \cdot o) = \mu\}.$$

By the G-equivariance of $J^{\mathfrak{h}}$, this is just equal to the stabilizer of $\mu \in \mathfrak{h}^*$ in G, whence it is a closed subgroup. Moreover, it is transitive on $N^\perp$ since G is, and it is σ-invariant since, if $g \in G_\mu^{\mathfrak{h}}$,

$$J^{\mathfrak{h}}(\sigma(g) \cdot o) = J^{\mathfrak{h}}(s_o g s_o o) = \mathrm{Ad}^*(s_o) J^{\mathfrak{h}}(g \cdot o) = \mathrm{Ad}^*(s_o) J^{\mathfrak{h}}(o) = J^{\mathfrak{h}}(s_o o) = \mu.$$

All this implies that $N^\perp = G_\mu^{\mathfrak{h}} \cdot o$ is a closed symmetric subspace of M.

To see that $T_o(N^\perp) = (T_o N)^\perp$, first recall that $(J_*)_o$ is an isomorphism bewteen $T_o M$ and $\mathrm{ad}^*_{\mathfrak{g}} J(o)$, so that we may work in the coadjoint orbit. Notice also that

$$J_*\left(T_o(N^\perp)\right) = \mathrm{ad}^*_{\mathfrak{g}_\mu^{\mathfrak{h}}} J(o) \qquad\qquad J_*(T_o N) = \mathrm{ad}^*_{\mathfrak{h}} J(o),$$

and that $J(o)|_{\mathfrak{h}} = \mu$ and $\mathrm{ad}^*_{\mathfrak{g}_\mu^{\mathfrak{h}}} \mu = 0$. Now if $X \in \mathfrak{g}_\mu^{\mathfrak{h}}$ and $Y \in \mathfrak{h}$, since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, we have

$$\begin{aligned} \omega_o^{\mathcal{O}}(\mathrm{ad}^*_X J(o), \mathrm{ad}^*_Y J(o)) &= \langle J(o), [X, Y] \rangle \\ &= \langle \mu, [X, Y] \rangle \\ &= -\langle \mathrm{ad}^*_X \mu, Y \rangle \\ &= 0. \end{aligned}$$

Hence we indeed have $T_o(N^\perp) = (T_o N)^\perp$.

It remains to see that $N^\perp$ is independent of J. But this comes from the fact that any other momentum map differs from J by an element of $\mathfrak{g}^*$ which is Ad^*_G-invariant, so that all the constructions above remain unchanged. □

Theorem 3.4.5. *Let (M, s, ω, o) be a pointed connected and simply connected symplectic symmetric space, and let N be a closed normal symmetric subspace. Then $M' = N^\perp/(N^\perp \cap N)$ is a symplectic symmetric space with symplectic form ω' such that*

$$\pi^* \omega' = \iota^* \omega$$

where $\pi : N^\perp \to M'$ is the natural projection, and $\iota : N^\perp \to M$ the natural inclusion.

Proof. Since $N^\perp$ is a symmetric subspace of M and N is normal in M, it follows

that $N^\perp \cap N$ is normal in $N^\perp$. It is moreover closed since N is closed. Hence the quotient $M' = N^\perp/(N^\perp \cap N)$ can be made in a natural way into a symmetric space such that the canonical map $\pi : N^\perp \to M'$ is a submersion and a locally trivial fibre bundle (see Section 1.4). Now for $o \in N^\perp \cap N$, and denoting $\mathfrak{n}$ and $\mathfrak{n}^\perp$ the Lie triple systems of N and $N^\perp$ respectively, the restriction of $\Omega = \omega_o$ to $\mathfrak{n}^\perp$ has kernel $\mathfrak{n}^\perp \cap \mathfrak{n}$. Hence its projection Ω' to $\mathfrak{m}' = \mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$ is well-defined, closed, and non-degenerate, and it satisfies $(\pi_*)^*\Omega' = (\iota_*)^*\Omega$, which concludes the proof. □

For a solvable Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, one can define the *weights of* $\mathfrak{g}$ *in* V in the same manner as we did for the roots of $\mathfrak{g}$ (see Section 2.2), except for the replacement of the sequence of ideals of $\mathfrak{g}$ by a Jordan-Hölder sequence of the $\mathfrak{g}$-module V. On then has the following definition, generalizing that of an exponential Lie algebra.

Definition 3.4.6 ([BCD$^+$72])**.** *A $\mathfrak{g}$-module V is said to be* of exponential type *if the weights of $\mathfrak{g}$ in V are of the form $(1 + ib)\phi$ for ϕ a real linear form on V and $b \in \mathbb{R}$.*

Lemma 3.4.7 ([BCD$^+$72])**.** *Let G be an exponential Lie group (see Definition 2.2.7), and let $\rho : G \to \mathrm{End}(V)$ be a representation of exponential type. Then the stabilizer of any element $v \in V$ is connected. In particular, any coadjoint orbit of G is connected and simply connected.*

Corollary 3.4.8. *Let (M, s, ω, o) be a pointed connected and simply connected symplectic symmetric space whose transvection group G is exponential. Let N be a normal symmetric subspace of M. Then $N^\perp$ is connected.*

Proof. This follows from the fact that $G^{\mathfrak{h}}_\mu$, in the proof of Proposition 3.4.4, is the stabilizer of μ in G, which is exponential. Hence, by Lemma 3.4.7, $G^{\mathfrak{h}}_\mu$ is connected and thus $N^\perp$ as well. □

Lemma 3.4.7 implies that an exponential symplectic symmetric space is not only a covering of, but is equivariantly diffeomorphic to a coadjoint orbit of an exponential Lie group.

Proposition 3.4.9. *Let (M, s, ω, o) be a pointed connected and simply connected symplectic symmetric space whose transvection group G is exponential, and let $\tilde{G} = G \times \mathbb{R}E$ be its central extension by ω_o, defined in Section 1.3. Then M is G-equivariantly diffeomorphic to the coadjoint orbit of $E^* \in \tilde{\mathfrak{g}}^*$.*

Proof. It was seen in Proposition 1.3.3 that the momentum map J for the action of $\tilde{G}$ is a smooth G-equivariant covering. But $\tilde{G}$ is exponential so that all its coadjoint orbits are simply connected. Hence J is a diffeomorphism. □

Proposition 3.4.10. *Let (M, s, ω, o) be a pointed symplectic symmetric space, and let M' be its reduction by some closed normal subspace N through o. If M is exponential and N is connected, then M' is exponential.*

Proof. By Theorem 2.2.20, M is connected and admits a midpoint map. But Corollary 3.4.8 says that $N^\perp$ is connected, hence, by Corollary 2.2.23 $N^\perp$ has a midpoint map, and by Corollary 2.2.22, $N^\perp \cap N$ is connected. Finally, Corollary 2.2.25 implies that $M' = N^\perp/(N^\perp \cap N)$ has a midpoint map, and is thus exponential. □

If (M, s, ω) is a symplectic symmetric space, and S a simply transitive group of automorphisms, then S becomes a symplectic Lie group. We have at our disposal reduction procedures for both M and S. The following proposition shows that for "good" subspaces, the reduction can be carried out simultaneously for both structures, while retaining the (locally) simply transitiveness of the action. It is most easily and naturally formulated in the context of Proposition 3.1.10.

Definition 3.4.11. *A* combined algebra *is a Lie admissible algebra $(\mathfrak{m}, \diamond)$ such that $L_x = x \diamond \cdot$ is a derivation of $[\cdot,\cdot,\cdot]^\diamond$, for all $x \in \mathfrak{m}$.*

If moreover, $\mathfrak{m}$ is endowed with a symplectic form Ω which is L_x-invariant, for all $x \in \mathfrak{m}$:

$$\Omega(x \diamond y, z) + \Omega(y, x \diamond z) = 0, \quad \text{for all } x, y, z \in \mathfrak{m},$$

then $(\mathfrak{m}, \diamond, \Omega)$ is said to be a symplectic combined algebra.

Remark that in a symplectic combined algebra, the invariance algebra of Ω is in general strictly between the one generated by the $L_{x,y}$'s and $\mathrm{Der}([\cdot,\cdot,\cdot]^\diamond)$. In other words, the symplectic form on the associated connected and simply connected symmetric space is at least invariant under the transvection group, but not necessarily under the whole automorphism group. The invariance group contains at least the smallest σ-invariant group of automorphisms containing the simply transitive group of automophisms.

Proposition 3.4.12. *Let $(\mathfrak{m}, \diamond, \Omega)$ be a symplectic combined algebra. Let $\mathfrak{n}$ be a two-sided ideal of $(\mathfrak{m}, \diamond)$. Then,*

1. $\mathfrak{m} \diamond \mathfrak{n}^\perp \subset \mathfrak{n}^\perp$ *and* $\mathfrak{n} \diamond \mathfrak{n}^\perp = 0$.

As a consequence, with $\mathfrak{m}' = \mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$, and $\diamond'$ (resp. Ω') the image of $\diamond$ (resp. Ω) in $\mathfrak{m}'$,

2. *$(\mathfrak{m}', \diamond', \Omega')$ is a well-defined symplectic combined algebra,*
3. *$(\mathfrak{m}', [\cdot,\cdot]^{\diamond'}, \Omega')$ is the symplectic reduction of $(\mathfrak{m}, [\cdot,\cdot]^\diamond, \Omega)$ by $\mathfrak{n}$, and*

4. $(\mathfrak{m}', [\cdot,\cdot,\cdot]^{\diamond'}, \Omega')$ *is the symplectic reduction of* $(\mathfrak{m}, [\cdot,\cdot,\cdot]^{\diamond}, \Omega)$ *by* $\mathfrak{n}$.

Proof. Since Ω is L_x-invariant and $\mathfrak{n}$ is such that $\mathfrak{m} \diamond \mathfrak{n} \subset \mathfrak{n}$, we have for $x \in \mathfrak{m}$, $y \in \mathfrak{n}^\perp$ and $z \in \mathfrak{n}$,

$$\Omega(x \diamond y, z) = -\Omega(y, x \diamond z) \subset \Omega(\mathfrak{n}^\perp, \mathfrak{n}) = 0$$

so that $\mathfrak{m} \diamond \mathfrak{n}^\perp \subset \mathfrak{n}^\perp$. Using the fact that $\mathfrak{n} \diamond \mathfrak{m} \subset \mathfrak{n}$, the same computation with $x \in \mathfrak{n}$, $y \in \mathfrak{n}^\perp$ and $z \in \mathfrak{m}$, shows that $\mathfrak{n} \diamond \mathfrak{n}^\perp = 0$. This proves 1.

Since $\mathfrak{m} \diamond \mathfrak{n}^\perp \subset \mathfrak{n}^\perp$, $\mathfrak{n}^\perp$ is in particular a subalgebra, of which $\mathfrak{n}$ is still a two-sided ideal, so that $(\mathfrak{m}', \diamond')$ is a well-defined algebra. The fact that it is a combined algebra follows directly. The kernel of Ω restricted to $\mathfrak{n}^\perp \times \mathfrak{n}^\perp$ being $\mathfrak{n}^\perp \cap \mathfrak{n}$, Ω' is indeed non-degenerate, and since $\mathfrak{n}^\perp \cap \mathfrak{n}$ is a two-sided ideal in $\mathfrak{n}^\perp$, Ω' is L_x-invariant as well. This proves 2.

Since $\mathfrak{n}$ is a two-sided ideal for $\diamond$, it is also an ideal for $[\cdot,\cdot]^\diamond$ and for $[\cdot,\cdot,\cdot]^\diamond$, so that Theorems 3.2.7 and 3.4.3 apply. This proves 3. and 4. □

Remark 3.4.13. Notice that a combined algebra $(\mathfrak{m}, \diamond)$ such that the associated Lie triple system is simple never has any non-trivial two-sided ideals, since otherwise those ideals would also be ideals of the Lie triple system, and so would be trivial. Nevertheless, they may have left-ideals for example, reflecting the fact that semisimple symmetric spaces may have solvable simply transitive subgroups of automorphisms, as is the case for all bounded symmetric domains [Pya69]. ○

Example 3.4.14. An important example, which will be studied in great detail in Chapter 5 is that of *split-solvable Ricci-type symplectic symmetric spaces*, studied in [CGR99] and in [Bie07]. We will call them *elementary normal spaces*, because of their relation with elementary normal j-algebras [Pya69].

The Lie algebra of the transvection group is a semi-direct product of two commuting copies of the Heisenberg Lie algebra on a symplectic vector space (V, Ω_V) by a one-dimensional Lie algebra acting by dilations:

$$\mathfrak{g} = \mathfrak{a} \ltimes (\mathfrak{h} \oplus \mathfrak{h}) \qquad \mathfrak{a} = \mathbb{R}H \qquad \mathfrak{h} = V \oplus \mathbb{R}E$$

with non-zero brackets given by

$$\begin{aligned}
[H, (v + lE, v' + l'E)] &= (v + 2lE, -v' - 2l'E) \\
[(v + lE, 0), (v' + l'E, 0)] &= (\Omega_V(v, v')E, 0) \\
[(0, v + lE), (0, v' + l'E)] &= (0, \Omega_V(v, v')E).
\end{aligned}$$

The involution σ on $\mathfrak{g}$ is defined by

$$\sigma(aH + (h, h')) = -aH + (h', h)$$

for all $a \in \mathbb{R}$ and $h, h' \in \mathfrak{h}$.

For $h \in \mathfrak{h}$, we will denote $h_1 = (h, 0)$, $h_2 = (0, h)$, $h_\mathfrak{m} = \frac{1}{2}(h, -h)$ and $h_\mathfrak{k} = \frac{1}{2}(h, h)$.

The corresponding Lie triple system $\mathfrak{m}$ has the non-zero brackets:

$$[H, v_\mathfrak{m}, H] = -v_\mathfrak{m} \qquad [H, E_\mathfrak{m}, H] = -4E_\mathfrak{m}$$
$$[H, v_\mathfrak{m}, v'_\mathfrak{m}] = \frac{1}{2}\Omega_V(v, v')E_\mathfrak{m} \qquad [v_\mathfrak{m}, v'_\mathfrak{m}, H] = -\Omega_V(v, v')E_\mathfrak{m}$$

(plus their antisymmetrization).

There subalgebra $\mathfrak{s} = \mathfrak{a} + (\mathfrak{h} \oplus \{0\})$ is a simply transitive subalgebra of $(\mathfrak{g}, \sigma)$. It corresponds to a linear map $k : \mathfrak{m} \to \mathfrak{k}$ given by

$$k(H) = 0 \qquad k(v_\mathfrak{m}) = v_\mathfrak{k} \qquad k(E_\mathfrak{m}) = E_\mathfrak{k},$$

which yields the following combined algebra structure on $\mathfrak{m}$:

$$H \diamond H = 0 \qquad v_\mathfrak{m} \diamond H = -v_\mathfrak{m} \qquad E_\mathfrak{m} \diamond H = -2E_\mathfrak{m}$$
$$H \diamond v_\mathfrak{m} = 0 \qquad v_\mathfrak{m} \diamond v'_\mathfrak{m} = \frac{1}{2}\Omega_V(v, v')E_\mathfrak{m} \qquad E_\mathfrak{m} \diamond v_\mathfrak{m} = 0$$
$$H \diamond E_\mathfrak{m} = 0 \qquad v_\mathfrak{m} \diamond E_\mathfrak{m} = 0 \qquad E_\mathfrak{m} \diamond E_\mathfrak{m} = 0.$$

Now observe that any subspace $\mathfrak{n} = W_\mathfrak{m} \oplus \mathbb{R}E_\mathfrak{m}$, with $W \subset V$ a vector subspace, is a two-sided ideal for the above structure, and may thus be used to reduce the combined algebra. With $W = 0$, the orthogonal subspace is $\mathfrak{n}^\perp = V_\mathfrak{m} \oplus \mathbb{R}E_\mathfrak{m}$. It is an ideal, and the (Lie triple system) reduction is isomorphic to the flat space $V_\mathfrak{m}$.

When $\mathfrak{m}$ is seen as a Lie algebra with the bracket $[\cdot, \cdot]^\diamond$, the same holds: $E_\mathfrak{m}$ is an ideal, and the (symplectic Lie algebra) reduction is isomorphic to the abelian Lie algebra $V_\mathfrak{m}$. ◁

3.5 Double extension

In this section, we introduce the notions of *classical double extension* and of *double extension* for symplectic Lie triple systems, as analogues of the corresponding notions for symplectic Lie algebras. We show that any nilpotent (resp. completely solvable) symplectic Lie triple system is the result of a sequence of classical double extensions (resp. of double extensions) by one-dimensional primitive systems starting from the zero-dimensional space.

Definition 3.5.1. *Let $\mathfrak{m}, \mathfrak{m}'$ be two symplectic Lie triple systems. Then $\mathfrak{m}$ is said to be a* classical double extension *of $\mathfrak{m}'$ if $\mathfrak{m}$ has a one-dimensional central ideal $\mathfrak{n}$ such that $\mathfrak{n}^\perp$ is an ideal of $\mathfrak{m}$ and $\mathfrak{n}^\perp/\mathfrak{n}$ is isomorphic to $\mathfrak{m}'$.*

Definition 3.5.2. *Let $\mathfrak{m}, \mathfrak{m}'$ be two symplectic Lie triple systems, and $\mathfrak{a}$ a primitive system. Then $\mathfrak{m}$ is said to be a* double extension *of $\mathfrak{m}'$ if $\mathfrak{m}$ has an isotropic ideal $\mathfrak{n}$ such that $\mathfrak{n}^\perp$ is also an ideal and the canonical sequence of Lie triple systems*

$$0 \to \mathfrak{n}^\perp/\mathfrak{n} \to \mathfrak{m}/\mathfrak{n} \to \mathfrak{m}/\mathfrak{n}^\perp \to 0$$

is split.

In that case, if $\mathfrak{a}$ is a supplementary subsystem to $\mathfrak{n}^\perp$, then 1) $\mathfrak{a}$ has a primitive structure, 2) $\mathfrak{m}$ is isomorphic with the semi-direct product $\mathfrak{a} \ltimes (\mathfrak{m}' \oplus \mathfrak{a}^*)$ where $\mathfrak{m}' = \mathfrak{n}^\perp/\mathfrak{n}$ and $\mathfrak{m}' \oplus \mathfrak{a}^*$ is a central extension of $\mathfrak{m}'$ by $\mathfrak{a}^*$, and 3) $\mathfrak{m}$ is said to be a double extension of $\mathfrak{m}'$ by $\mathfrak{a}$.

A Lie triple system is said to be *completely solvable* if it is solvable and the linear maps $R_x : \mathfrak{m} \to \mathfrak{m} : y \mapsto [y, x, x]$ only have real positive eigenvalues, for all $x \in \mathfrak{m}$, and *exponential* if no R_x has real strictly negative eigenvalues. This is equivalent to requiring the standard embedding to be a completely solvable (resp. exponential) Lie algebra.

Lemma 3.5.3. *Let $\mathfrak{m}$ be a completely solvable Lie triple system. Then $\mathfrak{m}$ has a one-dimensional (isotropic) ideal $\mathfrak{n}$. Moreover, if $\mathfrak{m}$ is symplectic, then $\mathfrak{n}^\perp$ is also an ideal of $\mathfrak{m}$.*

Proof. Consider the standard embedding $\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$. By definition, since $\mathfrak{m}$ is solvable, $\mathfrak{g}$ is solvable. Let $x \in \mathfrak{m}$. Since $R_x = (\operatorname{ad} x)^2|_{\mathfrak{m}}$ only has real positive eigenvalues, the same holds for $(\operatorname{ad} x)^2 : \mathfrak{g} \to \mathfrak{g}$ (see Lemma 4.5.4 for a precise proof), and thus $\operatorname{ad} x$ has no purely imaginary eigenvalues. As in the proof of Proposition 2.2.16, the eigenvalues of $\operatorname{ad} x$ for a general $x \in g$ are those of its $\mathfrak{m}$ part, and $\mathfrak{g}$ is thus completely solvable.

Let $\mathfrak{h}$ be a minimal ideal of $\mathfrak{g}$. Then $\mathfrak{h}$ is one-dimensional, and it is not contained in $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ since $\mathfrak{g}$ is a minimal embedding of $\mathfrak{m}$. Now, $\mathfrak{n} = (\mathfrak{h} + \sigma\mathfrak{h}) \cap \mathfrak{m}$ is a one-dimensional ideal of $\mathfrak{m}$.

We have moreover $[\mathfrak{k}, \mathfrak{n}] = 0$ since $\mathfrak{n}$ is a one-dimensional representation space of the nilpotent Lie algebra $\mathfrak{k}$. Hence, if $\mathfrak{m}$ is symplectic, we get

$$\Omega([\mathfrak{m}, \mathfrak{m}, \mathfrak{m}], \mathfrak{n}) = \Omega(\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}, \mathfrak{n}]) = 0,$$

which proves that $[\mathfrak{m}, \mathfrak{m}, \mathfrak{m}] \subset \mathfrak{n}^\perp$ and, in particular, that $\mathfrak{n}^\perp$ is an ideal of $\mathfrak{m}$. □

Theorem 3.5.4. *Every completely solvable symplectic Lie triple system is the result of a sequence of double extensions by one-dimensional primitive systems, starting from the zero-dimensional system.*

Proof. Let $\mathfrak{m}$ be a completely solvable Lie triple system. By Lemma 3.5.3, $\mathfrak{m}$ has a one-dimensional ideal $\mathfrak{n}$. Since it is of codimension one, any supplementary subspace $\mathfrak{a} \subset \mathfrak{m}$ is a Lie triple subsystem, and $\mathfrak{m}$ is thus the semi-direct product $\mathfrak{a} \ltimes \mathfrak{n}^\perp$, where $\mathfrak{n}^\perp$ is isomorphic to a central extension of $\mathfrak{n}^\perp/\mathfrak{n}$ by $\mathfrak{n} \cong \mathfrak{a}^*$ (see point 4 of Proposition 3.4.1). Hence $\mathfrak{m}$ is a double extension of $\mathfrak{n}^\perp/\mathfrak{n}$ by $\mathfrak{a}$.

Observe now that $\mathfrak{n}^\perp/\mathfrak{n}$ is again completely solvable, so that the result follows by induction. □

Theorem 3.5.5. *Every nilpotent symplectic Lie triple system is the result of a sequence of classical double extensions starting from the zero-dimensional system.*

Proof. Let $\mathfrak{m}$ be a nilpotent Lie triple system. Recall from the Introduction that this implies that its standard embedding is nilpotent. Hence $\mathfrak{m}$ has a one-dimensional ideal $\mathfrak{n}$, which is thus central, and $\mathfrak{m}$ is isomorphic the a central extension of $\mathfrak{m}/\mathfrak{n}$ by $\mathfrak{n}$.

As in the proofs of Lemma 3.5.3 and Theorem 3.5.4, $\mathfrak{n}^\perp$ is also an ideal, and $\mathfrak{n}^\perp/\mathfrak{n}$ is complemented by a subspace $\mathfrak{a}$ of $\mathfrak{m}/\mathfrak{n}$. Hence $\mathfrak{m}$ is isomorphic to the central extension by $\mathfrak{n}$ of the semi-direct product $\mathfrak{a} \ltimes (\mathfrak{n}^\perp/\mathfrak{n})$. □

Example 3.5.6. The elementary normal spaces of Example 3.4.14 are all completely solvable. Choosing a basis $e_1, \ldots, e_n$ of a lagrangian subspace of $V_\mathfrak{m}$, one may successively reduce $\mathfrak{m}$ by $E_\mathfrak{m}$, e_n, e_{n-1}, $\ldots$, e_1 to get the zero-dimensional Lie triple system. ◁

Chapter 4

Ševera projection

In this chapter, we study the main object of this thesis, namely a map which associates to a pair of points in a symplectic symmetric space a point in its cotangent bundle. This covector is obtained through a linear projection from a coadjoint orbit realization of the space.

In Section 4.1, we give its definition in terms of coadjoint orbits of (a central extension of) the automorphism group, and we show in Section 4.2 that we obtain the same map when considering transitive σ-invariant subgroups of the automorphism group. In Section 4.3, we show that the projection is natural under isomorphisms, inclusions of subspaces, and direct products. We investigate in Section 4.4 the behaviour of the projection under the reduction procedure introduced in Chapter 3, and continue in Section 4.5 with the study of conditions under which the projection is a local or a global diffeomorphism. In the last two sections, we first present an important interpretation of the projection in terms of the differential of the Weinstein phase function, and then conclude with an equivalent definition in terms of the fundamental groupoid which provides an interesting geometric point of view on this object.

4.1 Definition

In this section, we define the *canonical map* $M \times M \to TM$ and the *Ševera projection* $M \times M \to T^*M$ of a symplectic symmetric space realized as a coadjoint orbit.

Let (M, s, ω) be a connected and simply connected symplectic symmetric space, and let $G = \mathrm{Aut}(M, s, \omega)$ be its automorphism group. Fixing a base point o in M defines an involutive automorphism σ of G by $\sigma(g) = s_o g s_o$, and the symplectic form ω on M induces a 2-cocycle Ω on $\mathfrak{g} = \mathrm{Lie}(G)$ such that $\ker \Omega = \mathfrak{k}$

and $\Omega|_{\mathfrak{m}\times\mathfrak{m}}$ is non-degenerate, where $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ is the canonical decomposition associated with the differential of σ at the identity in G, which we also denote σ.

Taking for G a central extension of $\mathrm{Aut}(M,s,\omega)$ by Ω if needed (see Proposition 1.3.2), we may assume that Ω is exact. Any $\xi\in\mathfrak{g}^*$ such that $\Omega = -\delta\xi = \xi([\cdot,\cdot])$ then yields a G-equivariant momentum map $J : M \to \mathfrak{g}^*$ defined by $J(o) = \xi$, which is a submersion. Such a map is indeed well-defined since we saw in the proof of Proposition 1.3.2 that the stabilizer of o also stabilizes ξ. Moreover, σ allows us to identify $\mathfrak{k}^*$ with $\mathfrak{m}^\perp$ and $\mathfrak{m}^*$ with $\mathfrak{k}^\perp$, and defines an involution on $\mathfrak{g}^*$, which we also denote σ, by

$$\langle\sigma(l),X\rangle = \langle l,\sigma(X)\rangle, \quad \forall l\in\mathfrak{g}^*, X\in\mathfrak{g}.$$

This yields the decomposition $\mathfrak{g}^* = \mathfrak{k}^*\oplus\mathfrak{m}^*$ in eigenspaces of σ.

More generally, any point x in M defines such an involution σ_x of $\mathfrak{g}^*$ as the dual of the differential at e of the involution $g\mapsto s_x g s_x$ on G.

Let J be any G-equivariant momentum map for M.

Lemma 4.1.1. *We have $J\circ s_x = \sigma_x\circ J$.*

Proof. By definition, for all $l\in\mathfrak{g}^*$ and $X\in\mathfrak{g}$,

$$\langle\sigma_x(l),X\rangle = \left\langle l, \frac{d}{dt}s_x e^{tX}s_x\Big|_{t=0}\right\rangle = \langle \mathrm{Ad}^*(s_x)l, X\rangle,$$

that is, $\sigma_x = \mathrm{Ad}^*(s_x)$. Thus by G-equivariance of J,

$$J\circ s_x = \mathrm{Ad}^*(s_x)\circ J = \sigma_x\circ J.$$

□

Note that the projection π_x on the (-1)-eigenspace of σ_x on $\mathfrak{g}^*$ parallel to the $(+1)$-eigenspace may be written as

$$\pi_x = \frac{\mathrm{Id}-\sigma_x}{2}.$$

Lemma 4.1.2. *We have $T_{J(x)}J(M) = \mathrm{Im}\,\pi_x$.*

Proof. The previous lemma clearly implies that $T_{J(x)}J(M)\subset \mathrm{Im}\,\pi_x$ since for $X\in T_xM$ we have

$$\sigma_x J_* X = \sigma_{x*}J_*X = J_* s_{x*}X = -J_*X.$$

But $T_{J(x)}J(M) = J_*(T_xM)$, J is an submersion, and the action of G on M induces a linear isomorphism between T_xM and the (-1)-eigenspace of $g\mapsto s_x g s_x$, which is isomorphic to $\mathrm{Im}\,\pi_x$, so $T_{J(x)}J(M)$ and $\mathrm{Im}\,\pi_x$ have equal dimension. □

Lemma 4.1.3. *The map $x \mapsto \sigma_x$ is G-equivariant in the sense that*

$$\mathrm{Ad}^*(g) \circ \sigma_x = \sigma_{g\cdot x} \circ \mathrm{Ad}^*(g)$$

for all $g \in G$ and $x \in M$. As a consequence, the same stands for the map $x \mapsto \pi_x$.

Proof. By definition of σ_x and as the symmetries are preserved by the automorphisms, we have

$$\begin{aligned}\langle (\mathrm{Ad}^*(g) \circ \sigma_x) l, X\rangle &= \left\langle \sigma_x l, \mathrm{Ad}(g^{-1})X \right\rangle \\ &= \left\langle l, \mathrm{Ad}(s_x)\,\mathrm{Ad}(g^{-1})X \right\rangle \\ &= \left\langle l, \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(s_{g\cdot x})X \right\rangle \\ &= \left\langle \left(\sigma_{g\cdot x} \circ \mathrm{Ad}^*(g)\right) l, X \right\rangle\end{aligned}$$

for all $l \in \mathfrak{g}^*$, $g \in G$ and $X \in \mathfrak{g}$. □

Lemma 4.1.4. *The map $J_{*x}^{-1} \circ \pi_x \circ J$ is well-defined, and independent of the G-equivariant momentum map J.*

Proof. The fact that the map is well-defined follows from Lemma 4.1.2. Now to see that it is independent of J, let $\tilde{J}$ be another G-equivariant momentum map. Then, as recalled in Section 1.3, we must have $\tilde{J} = J + \eta$ for some $\eta \in \mathfrak{g}^*$ fixed by the coadjoint action. But this implies that $\sigma_x(\eta) = \mathrm{Ad}^*(s_x)\eta = \eta$. Thus $\pi_x(\eta) = 0$ and $\pi_x \circ \tilde{J} = \pi_x \circ J$. □

We may thus make the following definition.

Definition 4.1.5. *The* canonical map $\phi : M \times M \to TM$ *is defined by*

$$\phi(x,y) = \phi_x(y) = \left(J_{*x}^{-1} \circ \pi_x \circ J\right)(y),$$

for any G-equivariant momentum map J.

In view of Lemma 4.1.1 this map may also be written

$$\phi(x,y) = \phi_x(y) = J_{*x}^{-1}\left(\frac{J(y) - J(s_x y)}{2}\right).$$

We may thus as well define a map on the union of the graphs of the symmetries by $\phi'(s_x y, y) = J_{*x}^{-1}(J(y) - J(s_x y)) = 2\phi(x,y)$. For this map to be defined on the whole $M \times M$, we need to have well-defined *midpoint map* $\gamma : M \times M \to M$ as defined in Section 2.1. We may also use the symplectic form on M to arrive in the cotangent bundle instead of the tangent bundle.

Definition 4.1.6. *If γ is a midpoint map on M, the* Ševera projection *$\check{\phi} : M \times M \to T^*M$ is defined by*

$$\check{\phi}(x,y) = \omega_{\gamma(x,y)}\left(J^{-1}_{*\gamma(x,y)}(J(y) - J(x)), \cdot\right)$$

for any G-equivariant momentum map J.

This map was introduced to me a few years ago by Pierre Bieliavsky in the form of a linear projection from a (two-dimensional symplectic symmetric space viewed as a) coadjoint orbit to $\mathfrak{m}^*$ parallel to $\mathfrak{k}^*$. I then realized that it played a central role in defining an invariant deformation quantization of some solvable symplectic symmetric spaces, as will be presented in Chapter 5, and proceeded to generalize it to all symplectic symmetric spaces and study its properties. We realized later that this projection had already been defined in the short but interesting paper of Ševera in 2006 [Ševera06] and in its 1999 preprint [Ševera99].

Proposition 4.1.7. *Let $x, y \in M$. The canonical map has the following properties.*

(i) The map ϕ is G-equivariant for the diagonal action of G on $M \times M$ and the usual action on TM.

(ii) If there exists a point $m \in M$ such that $s_m x = y$, then

$$\phi(y,x) = s_{m*}\phi(x,y).$$

(iii) $\phi(x, s_x y) = -\phi(x,y)$.

(iv) $\phi_{x*x} = \mathrm{Id}_{T_xM}$.

Proof. (i) This follows from the Ad^*-equivariance of J, the fact that Ad^* is a linear action, and from Lemma 4.1.3:

$$\begin{aligned}
\phi_{g\cdot x}(g\cdot y) &= \left(J^{-1}_{*g\cdot x} \circ \pi_{g\cdot x} \circ J\right)(g\cdot y) \\
&= \left(J^{-1}_{*g\cdot x} \circ \pi_{g\cdot x} \circ \mathrm{Ad}^*(g) \circ J\right)(y) \\
&= \left(J^{-1}_{*g\cdot x} \circ \mathrm{Ad}^*(g) \circ \pi_x \circ J\right)(y) \\
&= g_* \cdot \left(J^{-1}_{*x} \circ \pi_x \circ J\right)(y) \\
&= g_* \cdot \phi_x(y),
\end{aligned}$$

where we denote $g_* \cdot$ the action of g on TM.

(ii) This follows from (i) with $g = s_m$.

(iii) This follows from (i) with $g = s_x$ and from the fact that $s_{x*x} = -\mathrm{Id}_{T_xM}$.

(iv) As π_x is a projection and $J_{*x}(T_xM) = \mathrm{Im}\,\pi_x$, we have

$$\phi_{x*x} = J_{*x}^{-1} \circ \pi_x \circ J_{*x} = J_{*x}^{-1} \circ J_{*x} = \mathrm{Id}_{T_xM}\,.$$

□

These basic properties are shared —at least locally around the diagonal— with the inverse of the exponential map for the canonical connection: $(x, y) \mapsto \mathrm{Exp}_x^{-1}(y)$. But, as we shall see in Section 4.6, these two maps are different, since the projection defines moreover a symplectomorphism from $M \times \overline{M}$ to T^*M, which is not the case for the exponential map.

Example 4.1.8. For the flat symplectic symmetric space $\mathbb{R}^{2n}$ with symmetries $s_x y = 2x - y$ and constant symplectic form ω, identifying $\mathbb{R}^{2n}$ with its dual, the Ševera projection is given by

$$\check{\phi}(x, y) = \left(\frac{x+y}{2}, y - x\right).$$

Hence it looks very much like the map defining the usual Weyl quantization (see equation (4) in the Introduction). The difference is that the Weyl map occurs *inside* the phase space $\mathbb{R}^{2n}$. More precisely, it sends a pair of points q, q' in the configuration space $\mathbb{R}^n$ to the point $(\frac{q+q'}{2}, q' - q)$ in the phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. Thus the Ševera projection may, with some identifications, be seen here as the Weyl map for $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ seen as cotangent bundle of $\mathbb{R}^{2n}$, which fits the idea that its goal is to define a quantization of the symplectic groupoid integrating the initial symplectic manifold. ◁

4.2 Subgroups

Let now G' be a subgroup of G, containing the transvection group of M and stabilized by $\sigma : g \mapsto s_o g s_o$, and let Ω' be the 2-cocycle on $\mathfrak{g}'$ induced by the choice of base point o. Let $\iota : \mathfrak{g}' \to \mathfrak{g}$ be the inclusion, and $\iota^* : \mathfrak{g}^* \to \mathfrak{g}'^*$ be the dual map given by $\langle \iota^*(l), X\rangle = \langle l, \iota(X)\rangle$ for all $l \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. It is well known that the composition of a G-equivariant momentum map J with ι^* yields a G'-equivariant momentum map $J' = \iota^* \circ J$ (see Lemma 4.2.2 (i)).

Lemma 4.2.1. *If $\xi \in \mathfrak{g}^*$ is such that $\Omega = \delta\xi$, then*

$$\Omega' = \delta(\iota^*\xi).$$

In particular, Ω' is still exact.

Proof. By definition of Ω and Ω', and as G' is a subgroup of G, we have $\Omega' = \iota^*\Omega$. But again because G' is a subgroup of G, the inclusion ι is a homomorphism, so $\iota^* \circ \delta = \delta \circ \iota^*$. □

That G' is stabilized by σ implies that it is also stabilized by all the involutions $C_{g'} \circ \sigma \circ C_{g'^{-1}}$ for $g' \in G'$. These involutions on $\mathfrak{g}'$ induce by transposition involutions on $\mathfrak{g}'^*$ which only depend on $x = g' \cdot o$, and which we denote by σ'_x. These in turn allow us to define the projections $\pi'_x = (\mathrm{Id} - \sigma'_x)/2$ on the (-1)-eigenspace of σ'_x.

Lemma 4.2.2. *The restriction ι^* has the following properties:*

(i) If $\xi \in \mathfrak{g}^*$ is such that $\delta\xi = \Omega$, and J and J' are the G- and G'-equivariant momentum maps respectively, such that $J(o) = \xi$ and $J'(o) = \iota^*\xi$, then

$$\iota^* \circ J = J',$$

(ii) $\iota^* \circ \pi_x = \pi'_x \circ \iota^*$, for all $x \in M$,

(iii) $\iota^*|_{\mathrm{Im}\,\pi_x}$ is an isomorphism between $\mathrm{Im}\,\pi_x$ and $\mathrm{Im}\,\pi'_x$, for all $x \in M$.

Proof. (i) This is a consequence of the fact that ι is a homomorphism[1] and that G' acts transitively. If $x = g' \cdot o$ with $g \in G'$, we have

$$J'(x) = \mathrm{Ad}^*(g')J'(o) = \mathrm{Ad}^*(g')\iota^*\xi = \iota^* \mathrm{Ad}^*(i(g'))\xi = \iota^* J(i(g') \cdot o) = \iota^* J(x).$$

(ii) This follows directly from the fact that σ preserves $\mathfrak{g}'$ in $\mathfrak{g}$, which implies $\iota \circ \sigma = \sigma \circ \iota$.

(iii) Point (ii) implies that $\iota^*(\mathrm{Im}\,\pi_x) \subset \mathrm{Im}\,\pi'_x$. But for any G- and G'-equivariant momentum maps J and J', we have $J_*(T_xM) = \mathrm{Im}\,\pi_x$ and $J'_*(T_xM) = \mathrm{Im}\,\pi'_x$ since they are submersions (G and G' are transitive). Thus they have equal dimension.

□

Corollary 4.2.3. *For any $\xi \in \mathfrak{g}^*$ such that $\delta\xi = \Omega$, let J' be the G'-equivariant momentum map such that $J'(o) = \iota^*\xi$. Then the map ϕ' from $M \times M$ to TM given by*

$$\phi'(x, y) = \phi'_x(y) = \left(J'^{-1}_{*x} \circ \pi'_x \circ J'\right)(y)$$

coincides with the canonical map ϕ. In particular, it is $\mathrm{Aut}(M, s, \omega)$-equivariant.

[1] We denote by ι both the group inclusion $G' \hookrightarrow G$ and its differential at the identity $\mathfrak{g}' \hookrightarrow \mathfrak{g}$.

We could reformulate this corollary as follows. Let $\tilde{G}'$ be a subgroup of $\tilde{G} = \mathrm{Aut}(M, s, \omega)$ containing the transvection group and stabilized by σ. Let $\tilde{\Omega}$ (resp. $\tilde{\Omega}'$) be the 2-form on $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{g}}'$) corresponding to ω on M, let $G = \tilde{G} \times \mathbb{R}E$ (resp. $G' = \tilde{G}' \times \mathbb{R}E'$) be the one-dimensional central extension of $\tilde{G}'$ by $\tilde{\Omega}'$, and let $\xi = E^* \in \mathfrak{g}^*$ (resp. $\xi' = E'^* \in \mathfrak{g}'^*$). Then ξ' is clearly the restriction of ξ to $\mathfrak{g}'$, so the above corollary applies and we get the following result:

Corollary 4.2.4. *Let (M, s, ω) be a connected and simply connected symplectic symmetric space, and let G be a transitive subgroup of $\mathrm{Aut}(M, s, \omega)$ stabilized by the map $g \to s_o g s_o$ for some point $o \in M$. Let J be the momentum map associated to the central extension of G induced by ω. Then the map*

$$\phi : M \times M \to TM : (x, y) \mapsto \left(J_{*x}^{-1} \circ \pi_x \circ J\right)(y)$$

is $\mathrm{Aut}(M, s, \omega)$-equivariant, and is equal to the canonical map, independently of o and G.

This implies that to compute the canonical map, we may restrict ourselves to the transvection group, which is the smallest transitive subgroup of the automorphism group stabilized by σ.

4.3 Morphisms

Proposition 4.3.1. *Let M, M' be two connected and simply connected symplectic symmetric spaces, let ϕ, ϕ' be the corresponding canonical maps, and let $f : M \to M'$ be a morphism of symplectic symmetric spaces. For each $x \in M$, denote $p_x : T_{f(x)}M' \to f_{*_x}(T_xM)$ the projector on $f_{*_x}(T_xM)$ parallel to $f_{*_x}(T_xM)^\perp$, and $\left(f_{*_x}\right)^{-1}$ the left inverse of f_{*_x} defined by $\left(f_{*_x}\right)^{-1}\Big|_{f_{*_x}(T_xM)} \circ p_x$. Then*

$$\phi = f^*\phi',$$

where $(f^*\phi')(x, y) = \left(f_{*_x}\right)^{-1} \circ \phi'(f(x), f(y))$.

Proof. We begin with a few lemmas. Let o be a base point in M, and $o' = f(o)$. Let G (resp. G') be the transvection group of M (resp. M') with involution σ (resp. σ') given the conjugation by s_o (resp. s'_o). Let $\tilde{G} = G \times \mathbb{R}E$ (resp. $\tilde{G}' = G' \times \mathbb{R}E'$) be the central extension associated to the base point o (resp. o'), and J (resp. J') the corresponding momentum map. Let $\tilde{f} : (G, \sigma) \to (G', \sigma')$ be the morphism given by $\tilde{f}(s_x s_y) = s'_{f(x)} s'_{f(y)}$ for all $x, y \in M$. It is such that $f(g \cdot x) = \tilde{f}(g) \cdot f(x)$ for all $g \in G$ and $x \in M$.

Lemma 4.3.2. *We have*

$$J = f^*J',$$

where $f^*J' := (\tilde{f}_{*_e})^* \circ J' \circ f$ *with* $(\tilde{f}_{*_e})^* : \mathfrak{g}'^* \to \mathfrak{g}^* : \xi \mapsto \xi \circ \tilde{f}_{*_e}$.

Proof. Since $f^*\omega' = \omega$, $\tilde{f}$ extends to a homomorphism from $\tilde{G}$ to $\tilde{G}'$ sending E on E'. Defining the map $L = (\tilde{f}_{*_e})^* \circ J' \circ f$, we will check that L is a $\tilde{G}$-equivariant map from M to $\mathfrak{g}^*$ such that $L(o) = E^*$, so that it must be equal to J.

Indeed, as $J' : M' \to \mathfrak{g}'^*$ and $f : M \to M'$, we have $L : M \to \mathfrak{g}^*$. Moreover, since $f(o) = o'$, $L(o) = (\tilde{f}_{*_e})^* \circ J'(o') = (\tilde{f}_{*_e})^* E'^* = E^*$. Finally, for $g \in G$, $x \in M$, using $f(g \cdot x) = \tilde{f}(g) \cdot f(x)$ and the G'-equivariance of J', we have

$$\begin{aligned} L(g \cdot x) &= \left((\tilde{f}_{*_e})^* \circ J' \circ \tilde{f}(g) \circ f\right)(x) \\ &= \left((\tilde{f}_{*_e})^* \circ \mathrm{Ad}^*(\tilde{f}(g)) \circ J' \circ f\right)(x) \\ &= \left(\mathrm{Ad}^*(g) \circ (\tilde{f}_{*_e})^* \circ J' \circ f\right)(x) \\ &= \mathrm{Ad}^*(g) L(x). \end{aligned}$$

□

Consider, as in Section 4.1, the involution on the Lie algebra of the automorphism group of M defined by $\mathrm{Ad}(s_x)$. As the transvection group G is a normal subgroup of the automorphism group, $\mathrm{Ad}(s_x)$ restricts in a map σ_x on $\mathfrak{g}$, whose dual on $\mathfrak{g}^*$ we also call σ_x. Define now π_x as the endomorphism of $\mathfrak{g}^*$ given by $(\mathrm{Id} - \sigma_x)/2$. Define the same maps for M'.

Lemma 4.3.3. *We have for all* $x \in M$, $(\tilde{f}_{*_e})^* \circ \pi'_{f(x)} = \pi_x \circ (\tilde{f}_{*_e})^*$.

Proof. As $\sigma'_{f(x)} = \mathrm{Ad}^*(s'_{f(x)}) = \mathrm{Ad}^*(\tilde{f}(s_x))$, this is clear since then, $(\tilde{f}_{*_e})^* \circ \sigma'_{f(x)} = (\tilde{f}_{*_e})^* \circ \mathrm{Ad}^*(\tilde{f}(s_x)) = \mathrm{Ad}^*(s_x) \circ (\tilde{f}_{*_e})^* = \sigma_x \circ (\tilde{f}_{*_e})^*$. □

Lemma 4.3.4. *We have for all* $x \in M$, $(\tilde{f}_{*_e})^* \circ J'_{*_x} = (\tilde{f}_{*_e})^* \circ J'_{*_x} \circ p_x$.

Proof. Let $X \in \mathfrak{g}'$. Then we have

$$X \cdot f(x) \in \ker\left((\tilde{f}_{*_e})^* \circ J'_{*_x}\right)$$

if and only if for all $Y \in \mathfrak{g}$,

$$\begin{aligned} 0 &= \left\langle (\tilde{f}_{*_e})^* \circ J'_{*_x}(X \cdot f(x)), Y \right\rangle \\ &= \left\langle J'_{*_x}(X \cdot f(x)), \tilde{f}_{*_e} Y \right\rangle \\ &= d\left(\lambda_{\tilde{f}_{*_e}(Y)}\right)(X \cdot f(x)) \\ &= \omega_{f(x)}(f_{*_x}(Y \cdot x), X \cdot f(x)) \end{aligned}$$

if and only if $X \cdot f(x) \in \left(f_{*_x}(T_xM)\right)^\perp = \ker(p_x)$. □

Notice that by definition of p_x and $\left(f_{*_x}\right)^{-1}$, we have

$$f_{*_x} \circ \left(f_{*_x}\right)^{-1} = p_x, \qquad \left(f_{*_x}\right)^{-1} \circ f_{*_x} = \mathrm{Id}_{T_xM}.$$

Lemma 4.3.5. *We have for all $x \in M$, $(\tilde{f}_{*_e})^* \circ J'_{*_{f(x)}} = J_{*_x} \circ \left(f_{*_x}\right)^{-1}$.*

Proof. We have, using successively lemma 4.3.4, the above note, and lemma 4.3.2,

$$\begin{aligned}(\tilde{f}_{*_e})^* \circ J'_{*_{f(x)}} &= (\tilde{f}_{*_e})^* \circ J'_{*_{f(x)}} \circ p_x \\ &= (\tilde{f}_{*_e})^* \circ J'_{*_{f(x)}} \circ f_{*_x} \circ \left(f_{*_x}\right)^{-1} \\ &= J_{*_x} \circ \left(f_{*_x}\right)^{-1}.\end{aligned}$$

□

We may now finish the proof of Proposition 4.3.1. Lemmas 4.3.2, 4.3.3 and 4.3.5 give

$$\begin{aligned}\phi(x,y) &= \left(J_{*_x}\right)^{-1} \circ \pi_x \circ J(y) \\ &= \left(J_{*_x}\right)^{-1} \circ \pi_x \circ (\tilde{f}_{*_e})^* \circ J(f(y)) \\ &= \left(J_{*_x}\right)^{-1} \circ (\tilde{f}_{*_e})^* \circ \pi'_{f(x)} \circ J(f(y)) \\ &= (f_{*_x})^{-1} \circ \left(J'_{*_{f(x)}}\right)^{-1} \circ \pi'_{f(x)} \circ J(f(y)) \\ &= (f_{*_x})^{-1} \circ \phi'(f(x), f(y)) \\ &= (f^*\phi')(x,y).\end{aligned}$$

□

Corollary 4.3.6. *Let M_i, $i = 1, 2$, be two connected and simply connected symplectic symmetric spaces, let ϕ_i be the corresponding canonical maps, and let $f : M_1 \to M_2$ be an isomorphism of symplectic symmetric spaces. Then $\phi_1 = f^*\phi_2$.*

Corollary 4.3.7. *Let M be a connected and simply connected symplectic symmetric space, and let $f : N \to M$ be a symplectic symmetric subspace. Then $\phi^N = f^*\phi^M$.*

Corollary 4.3.8. *Let M_i, $i = 1, 2$, be two connected and simply connected symplectic symmetric spaces, and let ϕ_i be the corresponding canonical maps. Then $\phi_1 \times \phi_2$ is the canonical map of the direct product symplectic symmetric space $M = M_1 \times M_2$.*

Proof. It follows from Proposition 4.3.1, but there is also a direct proof. Consider the automorphism groups $G_i = \mathrm{Aut}(M_i, s_i, \omega_i)$ and some base points $o_i \in M_i$, $i = 1, 2$. Consider the corresponding central extensions $\tilde{G}_i$, and two momentum maps $J_1 : M_1 \to \tilde{\mathfrak{g}}_1^*$, $J_2 : M_2 \to \tilde{\mathfrak{g}}_2^*$. Then obviously $J_1 \times J_2 : M \to \tilde{\mathfrak{g}}_1^* \times \tilde{\mathfrak{g}}_2^*$ is a momentum map for the direct product action of $G = \tilde{G}_1 \times \tilde{G}_2$ on $M_1 \times M_2$. Moreover, G acts transitively by automorphisms of M and is stabilized by the conjugation by $s_{o_1} \times s_{o_2}$. Finally, if $(\pi_x)_i : \tilde{\mathfrak{g}}_i^* \to \tilde{\mathfrak{g}}_i^*$ are the linear projections coming from the symmetries, then $(\pi_{x_1})_1 \times (\pi_{x_2})_2$ is clearly the corresponding projection for $\tilde{\mathfrak{g}}$. Hence, the canonical map for M is given by $(J_1 \times J_2)^{-1}_{*(x_1,x_2)} \circ ((\pi_{x_1})_1 \times (\pi_{x_2})_2) \circ (J_1 \times J_2) = \phi_{x_1} \times \phi_{x_2}$. □

Corollary 4.3.9. *Let (M, s) be a connected and simply connected symmetric space such that there exists a symplectic form ω such that (M, s, ω) is a symplectic symmetric space. For each such ω, denote ϕ^ω the corresponding canonical map. Then, for each $A \in \mathrm{Aut}(M, s)$, we have*

$$A^* \phi^\omega = \phi^{A^*\omega}.$$

These corollaries imply that if the canonical maps give rise to quantizations, then the latter behave nicely with respect to direct products, subspaces, (iso)morphisms, action of the automorphism group of the symmetric pair, etc.

4.4 Reduction

We now study the behaviour of the projection under symplectic reduction, as defined in Section 3.4. What we observe is that the reduction procedure is actually completely encoded in the coadjoint orbit realization of symplectic symmetric spaces, so that reduction is in fact built-in the canonical projection.

Proposition 4.4.1. *Let (M, s, ω, o) be a pointed, connected and simply connected symplectic symmetric space, and let N be a closed connected normal subspace through o. Let G (resp. $H^\perp$) be the central extension by ω_o (resp. by the restriction of ω_o to $N^\perp$) of the transvection group of M (resp. of $N^\perp$), and denote σ the conjugation by s_o. Let H be the minimal normal σ-invariant Lie subgroup of G which is transitive on N. Denote $i : N^\perp \to M$ and $j : \mathfrak{h}^\perp \to \mathfrak{g}$ the inclusions. Then*

1. *the map $J \circ i$ takes values in $(\mathfrak{g}/\mathfrak{h})^* \subset \mathfrak{g}^*$, and*
2. *the map $\tilde{J} = j^* \circ J \circ i$ is $(H \cap H^\perp)$-invariant.*

As a consequence, $\tilde{J}$ passes to the quotient as a momentum map $J' : M' \to (\mathfrak{h}^\perp/\mathfrak{h})^*$ for the action of $H^\perp/(H \cap H^\perp)$ on $M' = N^\perp/(N \cap N^\perp)$.

Proof. Assertion 1. is a direct consequence of the definition of $N^\perp$ as $(J^\mathfrak{h})^{-1}(J(o))$ (see Proposition 3.4.4).

For assertion 2., observe that for $X \in \mathfrak{h}$, $x \in N^\perp$ and $Y \in \mathfrak{h}^\perp$, since $e^{-\operatorname{ad} X} Y \subset Y + \mathfrak{h}$, the first assertion implies

$$\begin{aligned}\left\langle \tilde{J}\left(e^X \cdot x\right), Y\right\rangle &= \left\langle J(x), e^{-\operatorname{ad} X} Y\right\rangle \\ &= \langle J(x), Y\rangle = \left\langle \tilde{J}(x), Y\right\rangle,\end{aligned}$$

so that $\tilde{J}$ is H-invariant.

Hence $\tilde{J}$ yields a $(H^\perp/H)$-equivariant map $J' : M' \to (\mathfrak{h}^\perp/\mathfrak{h})^*$ which is a momentum map since J was. □

Corollary 4.4.2. *Under the assumptions and notation of Proposition 4.4.1, denote C the congruence of $N^\perp$ corresponding to $N^\perp \cap N$. Denote $\phi_o : M \to T_oM$ (resp. $\phi'_{o'} : M' \to T_{o'}M'$) the canonical projection of M at o (resp. of $M' = N^\perp/(N^\perp \cap N)$ at o', the image of o in M'). Denote $\pi_\mathfrak{n} : \mathfrak{m} \to \mathfrak{m}/\mathfrak{n}$ the projection. Then the map $\pi_\mathfrak{n} \circ \phi_o \circ i$*

1. *takes values in $\mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$,*
2. *only depends on the equivalence classes of C, and*
3. *passes to the quotient M' as $\phi'_{o'}$.*

Proof. Assertion 1. of Proposition 4.4.1 readily implies that $\pi \circ J \circ i$ has values in $(\mathfrak{m}/\mathfrak{n})^*$. But J_{*o} restricts in a bijection $X \cdot o \mapsto \operatorname{ad}^*_X o$ between $\mathfrak{n}^\perp$ and $(\mathfrak{m}/\mathfrak{n})^*$, so that $\pi_\mathfrak{n} \circ J_*^{-1}|_{(\mathfrak{m}/\mathfrak{n})^*}$ has values in $\mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n})$ and only depends on the images of the projection $(\mathfrak{m}/\mathfrak{n})^* \to (\mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n}))^*$.

As a consequence, we may replace in the expression

$$\pi_\mathfrak{n} \circ \phi_o \circ i = \pi_\mathfrak{n} \circ J_{*o}^{-1} \circ \pi_o \circ J \circ i,$$

the map $J \circ i$ by $j^* \circ J \circ i$, the map π_o by its equivalent $\pi'_{o'} : (\mathfrak{h}^\perp/(\mathfrak{h}^\perp \cap \mathfrak{h}))^* \to (\mathfrak{n}^\perp/(\mathfrak{n}^\perp \cap \mathfrak{n}))^*$ for $\mathfrak{h}^\perp/(\mathfrak{h}^\perp \cap \mathfrak{h})$, and $\pi_\mathfrak{n} \circ J_{*o}^{-1}$ by $J_{*o'}'^{-1}$, without changing its value.

By point 2. of Proposition 4.4.1, this map then passes to the quotient M' as $J_{*o'}'^{-1} \circ \pi'_{o'} \circ J' = \phi'_{o'}$. □

When considering the map

$$\tilde{\phi}_o = \omega_o^\flat \circ \phi_o : M \to T_o^*M,$$

there is a similar statement, replacing

$$\pi_\mathfrak{n} \circ \phi_o \circ i \qquad \text{by} \qquad (i_{*o})^* \circ \tilde{\phi}_o \circ i.$$

4.5 Diffeomorphism condition

In this section, we investigate the question of when the Ševera projection is a global diffeomorphism. We will prove the following theorem.

Theorem 4.5.1. *Let (M, s, ω) be a connected and simply connected symplectic symmetric space. Then the Ševera projection $\check{\phi} : M \times \overline{M} \to T^*M$ is everywhere a local diffeomorphism if and only if M is exponential. If, moreover, M is semisimple or solvable, then $\check{\phi}$ is a global diffeomorphism.*

We split the proof in two parts: a local result (Theorem 4.5.2) and a global one (Theorem 4.5.12).

Theorem 4.5.2. *Let (M, s, ω) be a connected and simply connected symplectic symmetric space, and consider $\phi_o : M \to T_oM$ its canonical projection at some point $o \in M$. Then ϕ_o is everywhere a local diffeomorphism if and only if M is exponential.*

Proof. Let (G, σ) be a symmetric pair such that $M = G/G^\sigma$. As in the definition of the canonical projection, consider the map $\alpha_o : G \to M : g \mapsto g \cdot o$ and the corresponding left-invariant 2-cocycle $\omega_G = \alpha_o^* \omega$ on G. Denote Ω its value at the identity. Take the central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}Z$ of $\mathfrak{g}$ by Ω and the corresponding Lie group central extension $\tilde{G}$ of G given by Proposition 1.3.2. Define the momentum map $J : M \to \mathfrak{g}^*$ by $J(g \cdot o) = \mathrm{Ad}^*_g Z^*$. Since we are concerned with local properties and since the momentum map is a covering, we may identify M with its associated coadjoint orbit through J. We need three short lemmas.

Lemma 4.5.3. *The canonical projection is everywhere a local diffeomorphism on* $\mathrm{Im}(\mathrm{Exp}_o)$ *if and only if*

$$\det(\cosh(\mathrm{ad}\, X)|_{\mathfrak{m}}) \neq 0, \qquad \forall X \in \mathfrak{m}.$$

Proof. Let $g \in G$. The tangent spaces to M are spanned by the images of left-invariant vector fields on G. At $g \cdot o$, those images read $\mathrm{Ad}^*_g \mathrm{ad}^*_Y o$ for $Y \in \mathfrak{g}$. Since $\mathrm{ad}^*_Y o = 0$ for $X \in \mathfrak{k}$ we may as well restrict ourselves to $X \in \mathfrak{m}$. The condition that the projection be a local diffeomorphism at $g \cdot o$ thus means that, for all $Y \in \mathfrak{m}$, we have $\mathrm{Ad}^*_g \mathrm{ad}^*_Y o \notin \mathfrak{k}^*$. This is equivalent to $\sigma(\mathrm{Ad}^*_g \mathrm{ad}^*_Y o) \neq \mathrm{Ad}^*_g \mathrm{ad}^*_Y o$, that is, $(\mathrm{Ad}^*_g + \mathrm{Ad}^*_{\sigma(g)}) \mathrm{ad}^*_Y o \neq 0$. When we restrict to $g = \exp X$ with $X \in \mathfrak{m}$, this means that $\cosh(\mathrm{ad}^*_X) \mathrm{ad}^*_Y o = (\mathrm{ad}^*_Y o) \circ \cosh(\mathrm{ad}_X) \neq 0$ for all $Y \in \mathfrak{m}$. Since $\mathrm{ad}^*_Y o|_{\mathfrak{k}} = 0$, the condition is finally equivalent to $\det(\cosh(\mathrm{ad}_X)|_{\mathfrak{m}}) \neq 0$ for all $X \in \mathfrak{m}$. □

Lemma 4.5.4. *Let $X \in \mathfrak{m}$. Then the non-zero eigenvalues of $(\mathrm{ad}_X)^2|_{\mathfrak{m}}$ and $(\mathrm{ad}_X)^2|_{\mathfrak{k}}$ are the same. In particular, we have*

$$\det(\cosh(\mathrm{ad}_X)) = \det(\cosh(\mathrm{ad}_X)|_{\mathfrak{m}})^2.$$

Proof. Let $X \in \mathfrak{m}$, and denote $p = \dim \mathfrak{m}$ and $k = \dim \mathfrak{k}$. Then, decomposing $\mathfrak{g}$ as $\mathfrak{m} \oplus \mathfrak{k}$, we may write $\mathrm{ad}_X = \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ where A and B are $p \times k$ and $k \times p$ matrices respectively, and we have $(\mathrm{ad}_X)^2 = \left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right)$. By Sylvester's determinant formula, we get

$$\begin{aligned} \det(AB - \lambda I_p) &= (-\lambda)^p \det(I_p - \frac{1}{\lambda} AB) \\ &= (-\lambda)^p \det(I_k - \frac{1}{\lambda} BA) \\ &= (-\lambda)^{p-k} \det(BA - \lambda I_k), \end{aligned}$$

thus the eigenvalues of AB and BA are identical, up to $|p-k|$ zeros. □

Lemma 4.5.5. *For all $X \in \mathfrak{m}$, the function $\det(\cosh(t\,\mathrm{ad}_X))$ is non-zero for all $t \in \mathbb{R}$ if and only if ad_X has no non-zero purely imaginary eigenvalues.*

Proof. Let $X \in \mathfrak{m}$ and $B^{-1}NB$ a Jordan decomposition of ad_X seen as a linear operator on $\mathfrak{g}_{\mathbb{C}}$, where N is an upper triangular matrix with the complex eigenvalues $\lambda_1, \ldots, \lambda_n$ of ad_X on the diagonal. Taking the matrix exponentials, we then have

$$\det(\cosh(\mathrm{ad}_X)|_{\mathfrak{m}}) = \prod_{i=1}^{n} \cosh(\lambda_i).$$

But for $\lambda = a + bi$, we have

$$\cosh(a+bi) = \cosh(a)\cos(b) + i\sinh(a)\sin(b)$$

and this expression vanishes only when $a = 0$ and $\cos(b) = 0$. Hence the function $\cosh(t(a+bi))$ is non-zero for all $t \in \mathbb{R}$ if and only if $a + bi = 0$ or $a \neq 0$. □

We may now finish the proof of Theorem 4.5.2. By Lemma 4.5.3, the projection is a local diffeomorphism everywhere on $\mathrm{Im}(\mathrm{Exp}_o)$ if and only if we have $\det(\cosh(\mathrm{ad}_X)|_{\mathfrak{m}}) \neq 0$ for all $X \in \mathfrak{m}$. By Lemmas 4.5.5 and 4.5.4 and the fact that M is connected and simply connected, this is equivalent to (G, σ) having property $(\mathfrak{m})$, which by Theorem 2.2.20, is equivalent to M being exponential. But in that case, Exp_o is surjective, so that the projection is everywhere a local diffeomorphism. □

We now turn to the global question. Our strategy is inspired from that used in [MS03] to prove the Dixmier-Saito theorem for exponential Lie groups (Proposition 2.2.8 in the present text).

Lemma 4.5.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be the entire function defined by $f(z) = \frac{\sinh(z)}{z}$ for all $0 \neq z \in \mathbb{C}$ and $f(0) = 0$. Then the derivatives $f^{(n)}$, $n \geq 0$, are of the form*

$$f^{(n)}(z) = \frac{1}{z^{n+1}} \left(E_n(z)\sinh(z) + O_n(z)\cosh(z)\right), \tag{4.1}$$

where

- *E_n is an even polynomial having a non-zero term in degree 0 and of overal degree $2\left\lfloor \frac{n}{2} \right\rfloor$, the largest even number less than or equal to n,*
- *O_n is an odd polynomial having a non-zero term in degree 1 and of overal degree $2\left\lfloor \frac{n+1}{2} \right\rfloor - 1$, the largest odd number less than or equal to n.*

Proof. We compute

$$\begin{aligned} f^{(1)}(z) &= \frac{1}{z^2}(z\cosh(z) - \sinh(z)) \\ f^{(2)}(z) &= \frac{1}{z^3}\left(z^2\sinh(z) - 2z\cosh(z) + 2\sinh(z)\right), \end{aligned}$$

so that it is the case for the the first n's. Assuming the result is true for some n, we have

$$\begin{aligned} f^{(n+1)}(z) = \frac{1}{z^{n+2}}\Big(&(-(n+1)E_n(z) + zE_n'(z) + zO_n(z))\sinh(z) \\ &+ (-(n+1)O_n(z) + zE_n(z) + zO_n'(z))\cosh(z)\Big) \end{aligned}$$

from which we readily check the assertions about E_{n+1} and O_{n+1}. □

Lemma 4.5.7. *Let $c > 0$, put $S_c = \{z = x + iy \mid |y| \leq c|x|\}$, and let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = P(z)\sinh(z) + Q(z)\cosh(z)$ for all $z \in \mathbb{C}$, with P and Q two complex polynomials. If $P \neq \pm Q$ then, in the region S_c, $|f(z)|^2$ grows at least like $e^{|z|/\sqrt{1+c^2}}$ when $z \to \infty$.*

Proof. Let $z \in \mathbb{C}$. Denote $z = a + bi$ with $a, b \in \mathbb{R}$, and $P = P_1 + iP_2$, $Q = Q_1 + iQ_2$ with P_1, P_2, Q_1, Q_2 real polynomials in a and b.

Straightforward computations yield

$$\begin{aligned} f(a+bi) = &\left(P_1(a,b) + iP_2(a,b)\right)(\sinh a\cos b + i\cosh a\sin b) \\ &+ \left(Q_1(a,b) + iQ_2(a,b)\right)(\cosh a\cos b + i\sinh a\sin b) \\ = &\frac{1}{2}e^a\left((Q_1 + P_1)\cos b - (Q_2 + P_2)\sin b\right) \\ &+ \frac{1}{2}e^{-a}\left((Q_1 - P_1)\cos b + (Q_2 - P_2)\sin b\right) \end{aligned}$$

$$+ i\frac{1}{2}e^{a}((Q_2+P_2)\cos b + (Q_1+P_1)\sin b)$$
$$+ i\frac{1}{2}e^{-a}((Q_2-P_2)\cos b - (Q_1-P_1)\sin b)$$

and thus

$$|f(a+bi)|^2 = \frac{e^{2a}}{4}\left((Q_1+P_1)^2+(Q_2+P_2)^2\right)+\frac{e^{-2a}}{4}\left((Q_1-P_1)^2+(Q_2-P_2)^2\right)$$
$$+\cos(2b)\frac{1}{2}\left(|Q|^2-|P|^2\right)+\sin(2b)(P_1Q_2-P_2Q_1)$$

If $P \neq \pm Q$, then for z sufficiently large, the factors of $e^{\pm 2a}$ are both greater than one. Moreover, if z remains in S_c for some $c > 0$, then

$$|a| \leq |z| \leq \sqrt{1+c^2}|a|,$$

from which we deduce that $|f(z)|$ grows like $e^{|z|/\sqrt{1+c^2}}$. □

Let us denote $A_X = \frac{\sinh(\operatorname{ad} X)}{\operatorname{ad} X}$ for all $X \in \mathfrak{m}$.

Lemma 4.5.8. *Let $(\mathfrak{g},\sigma)$ be a symmetric pair, let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ be the corresponding decomposition in $+1$ and -1 eigenspaces, and let $X \in \mathfrak{m}$ be such that the eigenvalues of $\operatorname{ad} X$ all lie in some fixed set S_c with $c > 0$. Then for all $Y \in \mathfrak{m}$, either $A_{tX}Y = Y$ for all $t \in \mathbb{R}$, or $A_{tX}Y$ goes to infinity when $t \to \pm\infty$.*

Proof. For every eigenvalue λ_i of $\operatorname{ad} X$, denote $\mathfrak{g}_i$ the corresponding invariant subspace of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Since $\operatorname{ad} X$ stabilizes $\mathfrak{g}$ in $\mathfrak{g}_{\mathbb{C}}$ and $A_{tX}Y$ stabilizes $\mathfrak{m}$, it is sufficient to show the result for each $\mathfrak{g}_i$ and then restrict to $\mathfrak{m}$.

Let $0 \neq Y \in \mathfrak{g}_i$. In a basis of $\mathfrak{g}_i$ where $\operatorname{ad} X|_{\mathfrak{g}_i}$ has a Jordan canonical form

$$\begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$$

the components of $A_{tX}Y$ are linear combinations of $(t\lambda_i)^n f^{(n)}(t\lambda_i)$ for $n = 1,\ldots,\dim\mathfrak{g}_i$.

Assume first that the eigenvalue λ_i is not zero. By Lemma 4.5.6, the components of $A_{tX}Y$ are of the form

$$\frac{1}{t\lambda_i}(E(t\lambda_i)\sinh(t\lambda_i)+O(t\lambda_i)\cosh(t\lambda_i))$$

where the polynomials E and O are respectively linear combinations of the E_n's and the O_n's of the same lemma. In particular, E is even and O is odd, so

that $E \neq \pm O$ unless $E = O = 0$. But this cannot happen for every component since $Y \neq 0$ and A_{tX} is invertible (because of the assumption on X). Since λ_i is not purely imaginary, $t\lambda_i$ stays, for all $t \in \mathbb{R}$, in a set S_c for some $c > 0$ (see Lemma 4.5.7). Hence by Lemma 4.5.7, all the non-zero components grow like $e^{|t|/\sqrt{1+c^2}}$ as $t \to \pm\infty$. Since $Y \neq 0$, this implies the result on $\mathfrak{g}_i$.

Assume now that $\lambda_i = 0$, so that $\operatorname{ad} X|_{\mathfrak{g}_i}$ is nilpotent. If $\operatorname{ad}(X)Y = 0$, then $A_{tX}Y = Y$ for all $t \in \mathbb{R}$. If $\operatorname{ad}(X)Y \neq 0$, then $A_{tX}Y$ is a non-constant polynomial in t and thus goes to infinity with t. □

Let us state a lemma from [MS03].

Lemma 4.5.9 ([MS03, Lemma 5.7])**.** *Let $\mathfrak{g}$ be an exponential solvable Lie algebra. Then there exists a positive real number c such that for every $X \in \mathfrak{g}$, all the eigenvalues of $\operatorname{ad} X$ lie in S_c.*

The idea of the proof is that Lie's theorem implies that the eigenvalues $\lambda_i(X)$ of $\operatorname{ad} X$ are linear functionals on $\mathfrak{g}$. Taking for each i, the supremum c_i of $\frac{\operatorname{Im}\lambda_i(X)}{\operatorname{Re}\lambda_i(X)}$ over the unit sphere for some random norm, and then taking the maximum of $c_1, \ldots, c_n$, yields the number c.

There is a corresponding statement for semisimple exponential symmetric pairs.

Lemma 4.5.10. *Let $(\mathfrak{g}, \sigma)$ be a semisimple exponential symmetric pair. Then any non-negative real number c is such that for every $X \in \mathfrak{g}$, all the eigenvalues of $\operatorname{ad} X$ lie in S_c.*

Proof. By Theorem 2.2.20, we may assume that σ is a Cartan involution. Then, the $\operatorname{ad} X$ for $X \in \mathfrak{m}$ all have real eigenvalues since they are symmetric operators for the Killing form. Hence any $c \geq 0$ is suitable. □

Lemma 4.5.11. *Let $(\mathfrak{g}, \sigma)$ be a symmetric pair with the property that there exists $c \geq 0$ such that for every $X \in \mathfrak{m}$, all the eigenvalues of $\operatorname{ad} X$ lie in S_c, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the associated decomposition. Let $\|\cdot\|$ be some norm of $\mathfrak{m}$. Then, there exists $C > 0$ such that $\|A_X Y\| \geq C\|Y\|$ for all $X, Y \in \mathfrak{m}$.*

Proof. This is an application of Lemma 4.5.8. The function $f : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R} : (X, Y) \mapsto \|A_X Y\|$ is continuous and nowhere vanishing. Moreover, for each non-zero $Y \in \mathfrak{m}$ and for each $X \in \mathfrak{m}$, the function $t \mapsto f(tX, Y)$ is eventually greater than 1 when $|t|$ goes to infinity, hence its infimum is strictly positive. Denote $C(X, Y)$ this infimum, and observe that it is a continuous function of X and Y. Taking the infimum of $(X, Y) \mapsto C(X, Y)$ over the the set $\{(X, Y) \in \mathfrak{m} \times \mathfrak{m} \mid \|X\| = \|Y\| = 1\}$ yields a positive real number C. This number has the desired property. □

Theorem 4.5.12. *Let (M, s, ω) be an exponential symplectic symmetric space which is either semisimple or solvable. Then the Ševera projection $\check{\phi} : M \times \overline{M} \to T^*M$ is a global diffeomorphism.*

Proof. It is equivalent to show that the canonical projection ϕ_o at some point $o \in M$ is a diffeomorphism. And since we have seen in Theorem 4.5.2 that the canonical map is everywhere a local diffeomorphism, hence is an open map, all that remains to see is that it is proper. Indeed, it would then be closed, and thus be a covering of T_oM which is simply connected, hence a diffeomorphism.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, Z, \Omega$ be as in the beginning of the proof of Theorem 4.5.2. Denote $\xi = Z^* \in \mathfrak{g}^*$. Using the diffeomorphisms $\mathrm{Exp}_o : \mathfrak{m} \cong T_oM \to M$ and $T_o^*M \cong \mathfrak{m}^*$, our goal is to see that the map $p : \mathfrak{m} \to \mathfrak{m}^* : X \mapsto \mathrm{Ad}^*_{e^X} \xi\big|_{\mathfrak{m}}$ is proper. Since $\mathfrak{m}$ and $\mathfrak{m}^*$ are finite dimensional vector spaces and since p is continuous, it is sufficient to see that the preimage of any bounded subset is bounded, or equivalently that $p(X) \to \infty$ if $X \to \infty$.

For $X, Y \in \mathfrak{m}$, we have

$$\begin{aligned}\left|\left\langle \mathrm{Ad}^*_{e^X} \xi, Y \right\rangle\right| &= \left|\left\langle \xi, e^{-\mathrm{ad}\, X} Y \right\rangle\right| \\ &= \left|\langle \xi, \sinh(\mathrm{ad}\, X) Y \rangle\right| \\ &= \left|\left\langle \xi, \left[X, \frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X} Y\right]\right\rangle\right| \\ &= \left|\Omega\left(X, A_X Y\right)\right|.\end{aligned}$$

Let us choose some norm $\|\cdot\|$ on $\mathfrak{m}$, and use the same notation for the corresponding norm on $\mathfrak{m}^*$: $\|\eta\| = \sup_{Y \in \mathfrak{m}, \|Y\|=1} \left|\langle \eta, Y \rangle\right|$. Then

$$\left\|\mathrm{Ad}^*_{e^X} \xi\right\| = \sup_{Y \in \mathfrak{m}, \|Y\|=1} \left|\Omega\left(X, A_X Y\right)\right|.$$

Now, Lemma 4.5.11 gives a constant $C > 0$ such that $\|A_X Y\| \geq C\|Y\|$ for all $X, Y \in \mathfrak{m}$, so that

$$\begin{aligned}\left\|\mathrm{Ad}^*_{e^X} \xi\right\| &= \sup_{Y \in \mathfrak{m}, \|Y\|=1} \|A_X Y\| . \left|\Omega\left(X, \frac{A_X Y}{\|A_X Y\|}\right)\right| \\ &\geq C \sup_{Y \in \mathfrak{m}, \|Y\|=1} \left|\Omega\left(X, \frac{A_X Y}{\|A_X Y\|}\right)\right| \\ &= C \sup_{Y \in \mathfrak{m}, \|Y\|=1} \left|\Omega\left(X, Y\right)\right| \\ &\geq C' \|X\|,\end{aligned}$$

for some $C' > 0$ independent of X, and our map is indeed proper. □

In the solvable case, Theorem 4.5.12 provides a generalization, and a more elementary proof, of Corollary 6.3 of [Bie02] for the elementary solvable symplectic symmetric spaces (the "twisting map" in that paper is in fact the non-trivial part of the canonical projection at the base point).

Notice that the only place where structure theory enters the game is in Lemmas 4.5.9 and 4.5.10. We believe that Theorem 4.5.12 is valid for any exponential symplectic symmetric space, regardless of its kind. This was the reason why we adopted such an approach, since all that remains to be done is to prove an analogue of Lemma 4.5.10 for the general case.

As we show below, this theorem can be proved more directly in the semisimple case, using the ad-invariance of the Killing form, and the fact that an exponential semisimple symplectic symmetric space is Hermitian.

Proposition 4.5.13. *Let (M, s, ω) be an exponential semisimple symplectic symmetric space. Then M is a Hermitian symmetric space of the non-compact type.*

Proof. In Section 2.2, it was proved that an exponential semisimple symmetric space is a Riemannian symmetric space of the non-compact type. Let thus G be a non-compact semisimple Lie group and θ a Cartan involution on G, with fixed subgroup K. Denote B the Killing form on $\mathfrak{g}$ and recall that $B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $B_\theta(X, Y) = B(X, \theta Y)$ is a positive definite bilinear form on $\mathfrak{g}$.

Assume now that G/K has a G-invariant symplectic form and denote Ω the corresponding 2-cocycle on $\mathfrak{g}$. Denote $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ a decomposition in simple ideals. By invariance, for $i \neq j$ we have

$$\begin{aligned} 0 &= \Omega([Z_i, X_i], Y_j) + \Omega(X_i, [Z_i, Y_j]) \\ &= \Omega([Z_i, X_i], Y_j) \end{aligned}$$

for all $Z_i \in \mathfrak{k}_i$, $X_i \in \mathfrak{m}_i$ and $Y_j \in \mathfrak{m}_j$. Hence $\Omega(\mathfrak{m}_i, \mathfrak{m}_j) = \Omega([\mathfrak{k}_i, \mathfrak{m}_i], \mathfrak{m}_j) = 0$, and the decomposition of $\mathfrak{g}$ is a symplectic, Riemannian, and Lie algebra direct sum.

Using the previous paragraph, we may assume that G is simple. By the Whitehead lemmas, there exists a unique $\xi \in \mathfrak{g}^*$ such that $\Omega(X, Y) = \langle \xi, [X, Y] \rangle$ for all $X, Y \in \mathfrak{g}$. Using the scalar product, there exists then a unique $Z_0 \in \mathfrak{g}$ such that $\Omega(X, Y) = B_\theta(Z_0, [X, Y])$. Since $\Omega(\mathfrak{k}, \mathfrak{g}) = 0$ we get $\langle \xi, [\mathfrak{k}, \mathfrak{g}] \rangle = 0$. But $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ implies $[\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$, so that in particular $\langle \xi, \mathfrak{m} \rangle = 0$, that is, $\xi \in \mathfrak{k}^*$. Now for $Z \in \mathfrak{k}$ and $X, Y \in \mathfrak{m}$, we have

$$\begin{aligned} 0 &= \Omega([Z, X], Y) + \Omega(X, [Z, Y]) \\ &= B_\theta(Z_0, [[Z, X], Y] + [X, [Z, Y]]) \\ &= B_\theta(Z_0, [Z, [X, Y]]) \\ &= -B_\theta([Z, Z_0], [X, Y]). \end{aligned}$$

Since $[\mathfrak{m},\mathfrak{m}] = \mathfrak{k}$ and since B_θ is non-degenerate on $\mathfrak{k}$, we get $[\mathfrak{k}, Z_0] = 0$. Hence the center $\mathcal{Z}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. By [Koh65, Lemma I], $\mathcal{Z}(\mathfrak{k})$ is then of dimension one and for every element e of it, $\mathrm{ad}_{\mathfrak{m}} e$ has exactly two eigenvalues $\pm ai$, where a is some non-zero real number. Thus there exists an element Z of $\mathcal{Z}(\mathfrak{k})$ such that $(\mathrm{ad}_{\mathfrak{m}} Z)^2 = -\mathrm{Id}_{\mathfrak{m}}$. By $\mathfrak{k}$-invariance of Ω, this yields $\Omega(JX, JY) = \Omega(X, Y)$ for all $X, Y \in \mathfrak{m}$, and since $Z \in \mathcal{Z}(\mathfrak{k})$ we have $J \circ \mathrm{ad}\, T = \mathrm{ad}\, T \circ J$ for all $T \in \mathfrak{k}$. Hence J yields a Hermitian structure on G/K. □

Proposition 4.5.14. *Let (M, s, ω) be an exponential semisimple symplectic symmetric space. Then the canonical map is a global diffeomorphism.*

Proof. As remarked in the proof of Theorem 4.5.12, all we have to show is that the canonical map at the base point o is proper.

By Proposition 4.5.13, M may be represented by G/K where G is connected semisimple and K is the fixed point subgroup of a Cartan involution θ, and the invariant symplectic form ω is given at $o = eK$ by $\Omega(X, Y) = \langle \xi, [X, Y]\rangle = B_\theta(Z_0, [X, Y])$ for some unique $\xi \in \mathfrak{k}^*$ and $Z_0 \in \mathcal{Z}(\mathfrak{k})$. The momentum map, in this context, is defined by $J(o) = \xi$ and by G-equivariance. Our goal is thus to see that the map $\mathfrak{m} \to \mathfrak{m}^* : X \mapsto \mathrm{Ad}^*_{e^X} \xi$ is proper.

For $X, Y \in \mathfrak{m}$, we have

$$\begin{aligned}
\left|\left\langle \mathrm{Ad}^*_{e^X} \xi, Y \right\rangle\right| &= \left|\left\langle \xi, e^{-\mathrm{ad}\, X} Y \right\rangle\right| \\
&= |\langle \xi, \sinh(\mathrm{ad}\, X) Y\rangle| \\
&= \left|\left\langle \xi, \left[X, \frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X} Y\right]\right\rangle\right| \\
&= \left|B_\theta\left(Z_0, \left[X, \frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X} Y\right]\right)\right| \\
&= \left|B_\theta\left([Z_0, X], \frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X} Y\right)\right|
\end{aligned}$$

so that with $Y = [Z_0, X]$, we get

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} B_\theta\left((\mathrm{ad}\, X)^k [Z_0, X], (\mathrm{ad}\, X)^k [Z_0, X]\right) \\
&\geq B_\theta\left([Z_0, X], [Z_0, X]\right) \\
&= \|[Z_0, X]\|^2 .
\end{aligned}$$

Hence

$$\begin{aligned}\|\mathrm{Ad}^*_{e^X}\,\xi\| &= \sup_{Y\in\mathfrak{m}} \frac{\left|\left\langle \mathrm{Ad}^*_{e^X}\,\xi, Y\right\rangle\right|}{\|Y\|} \\ &\geq \frac{\left|\left\langle \mathrm{Ad}^*_{e^X}\,\xi, [Z_0, X]\right\rangle\right|}{\|[Z_0, X]\|} \\ &\geq \|[Z_0, X]\| \\ &\geq c\|X\|\end{aligned}$$

for some constant $c > 0$ independent of X, since $\mathrm{ad}\, Z_0$ is, on each simple direct summand, a multiple of the complex structure (see the proof of Proposition 4.5.13), which is an isometry. This proves that the canonical map is proper. □

4.6 Area of double triangles

We now turn to the geometric meaning of the Ševera projection. We first recall a theorem of Ševera [Ševera06].

Theorem 4.6.1 ([Ševera06])**.** *Let (M, s, ω) be a connected symplectic symmetric space. The Ševera projection $M \times \overline{M} \to T^*M$ is a symplectomorphism (at least locally around the diagonal).*

The characterization of the Ševera projection in terms of the area of the double triangle will directly follow from results of Qian [Qia97] and from the above theorem of Ševera. We reproduce here those results of Qian, as his thesis is not widely available. To avoid unnecessary technicalities, some of the results stated here are less general than those Qian proved.

Definition 4.6.2 (Qian, *op. cit.*)**.** *Let (Γ, Ω) be a symplectic manifold. Then a diffeomorphism $\iota : \Gamma \to \Gamma$ is called a* symplectic inversion *if*

$$\begin{aligned}\iota^2 &= \mathrm{Id}, \\ \iota^*\Omega &= -\Omega.\end{aligned}$$

When a symplectic inversion ι is defined and when its fixed point set Γ_0 is not empty, a submersion $\gamma : \Gamma \to \Gamma_0$ is called an inversion-invariant polarization *of $(\Gamma, \Omega, \iota, \Gamma_0)$ if for each $x \in \Gamma_0$,*

1. $\gamma(x) = x$,
2. $\gamma \circ \iota = \gamma$,

3. *$\gamma^{-1}(x)$ is a Lagrangian submanifold of (Γ, Ω).*

Definition 4.6.3 (Qian, *op. cit.*)**.** *Let $(\Gamma, \Omega; \iota; \gamma)$ be an exact symplectic manifold with a symplectic inversion ι and an inversion-invariant polarization γ. A 1-form θ on Γ is called the* Liouville 1-form *of γ if it satisfies*

$$\begin{aligned} d\theta &= \Omega, \\ \iota^*\theta &= -\theta, \\ i^*_{\gamma^{-1}(x)}\theta &= 0 \end{aligned}$$

for each $x \in \Gamma_0$, where $i_E : E \to \Gamma$ denotes the inclusion of a submanifold $E \subset \Gamma$.

One can show that the Liouville 1-form of γ is unique.

In what follows, given (M, s, ω, γ) a symplectic symmetric space with midpoint map, we will always consider for (Γ, Ω) the *pair groupoid* $(M \times M, \omega \ominus \omega)$, for which we recall the basic notation.

The *space of units* is $\Gamma_0 = \mathrm{Diag}(M \times M)$. Denote $\alpha : \Gamma \to \Gamma_0 : (x, y) \mapsto (x, x)$ the *source map*, and $\beta : \Gamma \to \Gamma_0 : (x, y) \mapsto (y, y)$ the *target map*. Define $\Gamma^{(2)}$ as the submanifold of *compatible pairs* $\{(a, b) \mid \beta(a) = \alpha(b), a, b \in \Gamma\}$, and denote $m : \Gamma^{(2)} \to \Gamma$ the associative *groupoid multiplication* $m((x, y), (y, z)) = (x, z)$. As above, we will generally denote $x, y, \ldots$ the elements of M, and $a, b, \ldots$ those of Γ.

Then the exchange map $\iota(x, y) = (y, x)$ is a symplectic inversion, and the midpoint map γ is an inversion-invariant polarization, with Γ_0 the diagonal in $M \times M$ —we will sometimes identify $x \in M$ with $(x, x) \in \mathrm{Diag}(M \times M)$. Indeed, the first two conditions on γ mean $\gamma(x, x) = x$ and $\gamma(x, y) = \gamma(y, x)$ for all $x, y \in M$, which is clearly the case for a midpoint map. And the third condition is verified as well since $\gamma^{-1}(x)$ is the graph of s_x which is a symplectic map, and thus has a Lagrangian graph.

Theorem 4.6.4 (Qian, *op. cit.*)**.** *Let (M, s, ω, γ) be a symplectic symmetric space with midpoint map. Within the above notations for Γ, Γ_0, Ω, ι, α, β, m, assume the Liouville 1-form θ of γ exists. Then*

1. *for each $a \in \Gamma_0$, there exists a unique smooth function $\tilde{S}_a$ on some neighbourhood $U_{(a,a)}$ of (a, a) in $m^{-1}(\gamma^{-1}(a))$ such that*

$$\begin{aligned} d\tilde{S}_a &= i^*_{U_{(a,a)}}(\theta, \theta) \\ \tilde{S}_a(a, a) &= 0; \end{aligned}$$

2. *(a, a) is a non-degenerate critical point of $\tilde{S}_a$;*

3. *there exists a unique function $\tilde{S}$ on some open neighbourhood U_{Δ_3} of the triple diagonal $\Delta_3 = \{(a, a, a) \mid a \in \Gamma_0\}$ in the graph manifold of the groupoid*

multiplication m such that

$$\begin{aligned} i^*_{\Delta_3}\tilde{S} &= 0 \\ d\tilde{S} &= i^*_{U_{\Delta_3}}(-\theta,\theta,\theta); \end{aligned}$$

4. *for each* $(b,c) \in U_{(a,a)}$ *with* $(m(b,c),b,c) \in U_{\Delta_3}$ *where* $a \in \Gamma_0$,

$$\tilde{S}(m(b,c),b,c) = \tilde{S}_{\gamma(m(b,c))}(b,c);$$

5. *for each* $(a,b,c) \in U_{\Delta_3}$, *we have*

$$\begin{aligned} \tilde{S}(a,b,c) &= -\tilde{S}(b,a,\iota(c)) \\ &= \tilde{S}(\iota(c),\iota(a),b) \\ &= -\tilde{S}(\iota(a),\iota(c),\iota(b)); \end{aligned}$$

6. *let* $\Gamma^{(3)} = \{(a,b,c) \in \Gamma^3 \mid (a,b) \in \Gamma^{(2)}, (b,c) \in \Gamma^{(2)}\}$. *We define two functions* $\tilde{S}_1$, $\tilde{S}_2$ *on* $\Gamma^{(3)}$ *be setting for each* $(a,b,c) \in \Gamma^{(3)}$

$$\begin{aligned} \tilde{S}_1(a,b,c) &= \tilde{S}(m(m(a,b),c),m(a,b),c) + \tilde{S}(m(a,b),a,b), \\ \tilde{S}_2(a,b,c) &= \tilde{S}(m(a,m(b,c)),a,m(b,c)) + \tilde{S}(m(b,c),b,c). \end{aligned}$$

Then $\tilde{S}_1 = \tilde{S}_2$.

Actually, on a symplectic symmetric space the unique function $\tilde{S}$ of the above theorem may be described explicitly:

Theorem 4.6.5 (Weinstein [Wei94], Qian, *op. cit.*)**.** *Let* (M,s,ω,γ) *be a connected symplectic symmetric space with midpoint map. Suppose the Liouville 1-form* θ *of* γ *exists. Then for each* $A,B,C \in M$ *such that* $((A,C),(A,B),(B,C)) \in U_{\Delta_3}$, *the value of* $\tilde{S}((A,C),(A,B),(B,C))$ *is the symplectic area of the geodesic surface in* M *bounded by the clockwise-oriented triangle whose sides* A,B, B,C *and* C,A *are invariant under the symmetries* s_x, s_y *and* s_z *respectively, where* $x = \gamma(A,B)$, $y = \gamma(B,C)$, *and* $z = \gamma(C,A)$.

Now, in our context, we have the following proposition.

Proposition 4.6.6. *Let* (M,s,ω,γ) *be a symplectic symmetric space with midpoint map. Let* α *denote the canonical 1-form on* T^*M, *and* $\check{\phi} : M \times M \to T^*M$ *the Ševera projection. Then*

$$\check{\theta} = \check{\phi}^*\alpha$$

is a Liouville 1-form of γ *as in Definition 4.6.3.*

Proof. We check the three defining conditions for a Liouville 1-form. By Theorem 4.6.1, $\check{\phi}$ is a symplectomorphism, so that $d\theta = \check{\phi}^* d\alpha = \omega \ominus \omega$. For the relation with ι, note that $\iota_{T^*M} : T^*M \to T^*M$ given by $\iota_{T^*M}(x,p) = (x,-p)$ is a symplectic inversion such that $\iota_{T^*M}^* \alpha = -\alpha$, and that $\check{\phi}$ intertwines ι and ι_{T^*M}. For the third condition, recall from Definition 4.1.6 that $\check{\phi}$ sends the graph of s_x to the fiber T_x^*M, on which α vanishes. □

Definition 4.6.7. *Let (M,s,ω,γ) be a symplectic symmetric space with midpoint map such that γ_3 is a diffeomorphism. Define the* Weinstein phase function *$S : M \times M \times M \to \mathbb{R}$ by*

$$S(x,y,z) = \tilde{S}((A,C),(A,B),(B,C))$$

where $(A,B,C) = \gamma_3^{-1}(x,y,z)$ and $\tilde{S}$ is the function defined in Theorem 4.6.5. In other words, S is the symplectic area of the double triangle of x,y,z.

The Weinstein phase function was introduced in [Wei94] and was constructed in [Qia97] as a solution of the following *eikonal equations* obtained by Weinstein about the phase function of a deformed associative product.

Definition 4.6.8 (Qian, *op. cit.*)**.** *Let (M,γ) be a manifold with weak midpoint map (see Definition 2.1.1). We define the smooth map Ψ_γ from M^4 to M^6 by setting for each $p,q,r,s \in M$,*

$$\Psi_\gamma(p,q,r,s) = (\gamma(r,p), \gamma(q,s), \gamma(s,p), \gamma(p,q), \gamma(q,r), \gamma(r,s)).$$

This map associates to the vertices of a quadrangle on M the midpoints of the two diagonals and of the four edges.

Definition 4.6.9 (Qian, *op. cit.*)**.** *Let (M,γ) be a manifold with weak midpoint map. A smooth function S is* γ-compatible *if it satisfies the* eikonal equations *of γ at the six points $(u,v,w,x,y,z) = \Psi_\gamma(p,q,r,s)$ for each $p,q,r,s \in M$ where the eikonal equations of γ at $(u,v,w,x,y,z) \in M^6$ are defined as the following six linear partial differential equations:*

$$\begin{aligned}
\frac{\partial S}{\partial w}(w,u,z) &= \frac{\partial S}{\partial w}(w,x,v)\\
\frac{\partial S}{\partial x}(u,x,y) &= \frac{\partial S}{\partial x}(w,x,v)\\
\frac{\partial S}{\partial y}(u,x,y) &= \frac{\partial S}{\partial y}(v,y,z)\\
\frac{\partial S}{\partial z}(w,u,z) &= \frac{\partial S}{\partial z}(v,y,z)
\end{aligned}$$

$$\frac{\partial S}{\partial u}(u,x,y) = -\frac{\partial S}{\partial u}(w,u,z)$$
$$\frac{\partial S}{\partial v}(v,y,z) = -\frac{\partial S}{\partial v}(w,x,v)$$

Theorem 4.6.10 (Qian, *op. cit.*). *Suppose that γ is a weak midpoint map on M, and S is a skewsymmetric γ-compatible function on M^3. Then there exists a unique smooth map ϕ_S from $M \times M$ to the cotangent bundle T^*M such that*

1. *$\gamma = \pi \circ \phi_S$ where π is the projection map from T^*M to M;*
2. *$\phi_S \circ \iota_{M\times M} = \iota_{T^*M} \circ \phi_S$ where $\iota_{M\times M} = \iota$ and ι_{T^*M} are the inversions defined above. In particular, $\phi_S(p,p) = (p,0)$;*
3. *for each $p,q,r \in M$,*

$$\phi_S \times \phi_S \times \phi_S((r,p),(p,q),(q,r)) = (u,y,z,dS(u,y,z))$$

where $u = \gamma(r,p)$, $y = \gamma(p,q)$ and $z = \gamma(q,r)$.

Definition 4.6.11 (Qian, *op. cit.*). *Under the assumptions of Theorem 4.6.10, we define a 1-form θ_S on $M \times M$ by setting*

$$\theta_S = \phi_S^* \alpha$$

where α is the Liouville 1-form on T^*M.

Theorem 4.6.12 (Qian, *op. cit.*). *Under the assumptions of Theorem 4.6.10, the 1-form defined in Definition 4.6.11 is the unique 1-form on $M \times M$ which satisfies*

1. *$i^*_{\gamma^{-1}(x)}\theta = 0$ for each $x \in M$;*
2. *$\iota^*_{M\times M}\theta = -\theta$; and*
3. *$d\theta = \phi_S^*\omega$ where $\omega = d\alpha$ is the standard symplectic structure on T^*M.*

Furthermore, θ_S also satisfies

$$i^*_{\mathcal{M}} \circ (\gamma \times \gamma \times \gamma)^*(dS) = i^*_{\mathcal{M}}(-\theta_S \times \theta_S \times \theta_S)$$

where $\mathcal{M} = \{((p,r),(p,q),(q,r)) \mid p,q,r \in M\}$.

Theorem 4.6.13 (Qian, *op. cit.*). *Suppose that γ is a weak midpoint map on M and there exists a 1-form θ on $M \times M$ and a smooth function S on M^3 such that for each $x \in M$*

$$i^*_{\gamma^{-1}(x)}\theta = 0$$

$$i^*_{M\times M}\theta = -\theta$$
$$S(x,x,x) = 0$$
$$i^*_{\mathcal{M}} \circ (\gamma\times\gamma\times\gamma)^*(dS) = i^*_{\mathcal{M}}(-\theta\times\theta\times\theta),$$

where $\mathcal{M}$ is defined in Corollary 4.6.12. Then S is a skewsymmetric γ-compatible function and $\theta = \theta_S$.

Applying now these theorems to a symplectic symmetric space with midpoint map such that γ_3 is a diffeomorphism, we get a characterization of the Ševera projection in terms of the Weinstein phase function.

Theorem 4.6.14. *Let (M,s,γ) be a symmetric space with midpoint map such that γ_3 is a diffeomorphism. Then the Ševera projection $\phi : M\times M \to T^*M$ and the associated Liouville 1-form $\theta = \check{\phi}^*\alpha$ are such that, for all $x,y,z\in M$,*

$$\phi\times\phi\times\phi((A,B),(B,C),(C,A)) = ((x,y,z),dS(x,y,z)) \tag{4.2}$$
$$i^*_{\mathcal{M}}(-\theta\times\theta\times\theta) = i^*_{\mathcal{M}} \circ (\gamma\times\gamma\times\gamma)^*(dS),$$

where $x=\gamma(A,B)$, $y=\gamma(B,C)$ and $z=\gamma(C,A)$, $\mathcal{M}$ is defined in Corollary 4.6.12 and S is the Weinstein phase function. In the first formula above, we identified $(T^*M)^3$ with $T^*(M^3)$.

What is thus added here to Qian's framework is an explicit construction of the unique map ϕ_S associated to a phase function and the proof that it is a global diffeomorphism for a large class of spaces.

4.7 Another definition

To conclude this chapter, we briefly mention a promising alternative viewpoint on the Ševera projection, which should lead to a better insight.

Let J, J' be momentum maps of M, M' respectively, for the same group G. Then their sum $J+J' : M\times M' \to \mathfrak{g}^*$ is also a momentum map (to be precise, the map is $\pi_1^*J + \pi_2^*J$ with $\pi_1 : M\times M' \to M$, $\pi_1 : M\times M' \to M$ the projections on the first and second components). Taking $M' = \overline{M}$ and $J' = -J$, we get a map

$$M\times\overline{M} \to \mathfrak{g}^* : (x,y) \mapsto J(x) - J(y).$$

With G and J as in the beginning of Section 4.1, the image of (x,y) is the tangent vector at the point $J(\gamma(x,y))$ given by the projection.

Recall that G was a central extension of the automorphism group of M. Denoting E a generator of the Lie algebra extension, and $\iota : \mathfrak{g}^* \to \mathbb{R}E^*$ the

corresponding projection, we observe that $\iota(J(x)) = \iota(J(y))$ for all $x, y \in M$, that is, the momentum map $J - J : M \times M \to \mathfrak{g}^*$ does not depend on the central extension.

That leads us to yet another reformulation, this time in terms of the fundamental groupoid $\Pi(M)$ of M. The latter is defined as the set of homotopy classes $[\gamma]$ of paths in M with fixed endpoints, and is a covering of the pair groupoid $M \times \overline{M}$. It was proved in [CDW87, Théorème 3.1] that *any symplectic action of a Lie group on a symplectic manifold lifts to a strongly Hamiltonian action on* $\Pi(M)$. The corresponding momentum map J^Π is given by

$$\left\langle J^\Pi([\gamma]), X \right\rangle = \int_\gamma \iota_{X^*}\omega.$$

Assuming that M is simply connected, a class $[\gamma]$ is really equivalent to a pair (x, y) of points in M. To show that J^Π gives the same map as before, observe that if G has a strongly hamiltonian action on M, then $i_{X^*}\omega = -d\lambda_X$ so that the integral gives $-\lambda_X(y) + \lambda_X(x) = \langle J(x) - J(y), X \rangle$. Hence we indeed have $J^\Pi = J - J$.

This viewpoint on the Ševera projection has the advantage of being more geometric and of not requiring central extensions for its definition. In this respect, it is more satisfactory than Definition 4.1.5 since the canonical aspect of the projection is really explicit in its definition and since it should lead to a better geometric understanding of it, and to more geometric proofs of the results obtained here.

Chapter 5

Quantization

In this chapter, we produce an explicit example of quantization using the Ševera projection, along the method briefly described in the Introduction. The spaces that we quantize are the elementary normal spaces, which have the same underlying symmetric spaces as the split-solvable Ricci-type symplectic symmetric spaces, but may have different symplectic forms.

During our study, we compute the full automorphism group of these spaces, as well as the explicit form of the Weinstein phase function.

We give, for each invariant symplectic form, an explicit integral formula for the star product, we show that the map $\omega \mapsto *^{\omega}$ associating a product to a symplectic form is equivariant under the automorphism group of the underlying symmetric space, and observe that changing the deformation parameter may simply be seen as moving in the set of invariant symplectic forms.

5.1 Symplectic Fourier transform

We begin by describing the Liouville volume form.

Lemma 5.1.1. *Let (V,ω) be a symplectic vector space of dimension $2n$ and let ω_{ij} be the matrix of ω in some basis $e_1,\dots,e_{2n}$ of V (so that we have $\omega = \sum_{i<j} \omega_{ij} e_i^* \wedge e_j^*$). Then in that basis the Liouville volume form reads*

$$\frac{\omega^n}{n!} = \pm\sqrt{\det(\omega_{ij})}\, e_1^* \wedge \cdots \wedge e_{2n}^*. \tag{5.1}$$

In particular, the associated measure on V is equal to $\sqrt{\det(\omega_{ij})}$ times the Lebesgue measure with respect to that basis.

Proof. Remark first that if $e_1,\dots,e_{2n}$ and $f_1,\dots,f_{2n}$ are two bases of V, related by

an invertible matrix A through $f_i = A_i^j e_j$, then on the one hand

$$f_1^* \wedge \cdots \wedge f_{2n}^* = \det(A^{-1}) e_1^* \wedge \cdots \wedge e_{2n}^*,$$

and on the other hand the respective matrices ω_{ij} and ω'_{ij} of ω are such that $\omega'_{ij} = A_i^k \omega_{kl} A_j^l$, so that

$$\sqrt{\det(\omega'_{ij})} = |\det(A)| \sqrt{\det(\omega_{ij})}.$$

As a result, the right hand side of equation (5.1) is independent of the coordinates since

$$\begin{aligned}\sqrt{\det(\omega'_{ij})}\, f_1^* \wedge \cdots \wedge f_{2n}^* &= |\det(A)| \sqrt{\det(\omega_{ij})} \det(A^{-1})\, e_1^* \wedge \cdots \wedge e_{2n}^* \\ &= \pm \sqrt{\det(\omega_{ij})}\, e_1^* \wedge \cdots \wedge e_{2n}^*.\end{aligned}$$

But choosing a basis $e_1, \ldots, e_{2n}$ in which the matrix of ω is $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$, one computes that

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} e_1^* \wedge \cdots \wedge e_{2n}^* = \pm \sqrt{\det\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)} e_1^* \wedge \cdots \wedge e_{2n}^*,$$

which concludes the proof. □

If V is a vector space, $\mathcal{S}(V)$ will denote the space of Schwartz functions on V.

Definition 5.1.2. *Let (V, ω) be a symplectic vector space of dimesion $2n$. The* symplectic Fourier transform *on (V, ω) is the map $\mathcal{F}^\omega : \mathcal{S}(V) \to \mathcal{S}(V)$ defined by*

$$(\mathcal{F}^\omega u)(v) = \frac{1}{(2\pi)^n} \int_V e^{-i\omega(v,w)} u(w) d^\omega w, \tag{5.2}$$

where $d^\omega w$ denotes the Liouville measure associated to ω on V.

Proposition 5.1.3. *In the setting of the previous definition, the symplectic Fourier transform is well-defined, in the sense that $\mathcal{F}^\omega u \in \mathcal{S}(V)$ for all $u \in \mathcal{S}(V)$, and is an involutive isomorphism: $\mathcal{F}^\omega \circ \mathcal{F}^\omega = \mathrm{Id}$.*

Proof. Choose a basis $e_1, \ldots, e_{2n}$ of V, in which the matrix of ω is denoted ω_{ij}, and let $u \in \mathcal{S}(V)$. Then we have

$$(\mathcal{F}^\omega u)(v) = \sqrt{\det(\omega_{ij})} (\mathcal{F} u)(\omega^\flat(v)),$$

where $\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$ is the usual Fourier transform with respect to the Lebesgue measure corresponding to the chosen basis, and where $\omega^\flat$ is the linear isomorphism $V \to V^* : v \mapsto \omega^\flat(v) = \omega(v, \cdot)$. As $\omega^\flat$ lifts to an isomorphism $\mathcal{S}(V^*) \to \mathcal{S}(V)$, this shows the first assertion.

But the change of variable $w = (\omega^\flat)^{-1}(\xi)$ in formula (5.2) also yields

$$(\mathcal{F}^\omega u)(v) = \sqrt{\det(\omega_{ij})^{-1}}\mathcal{F}^{-1}\left(u \circ (\omega^\flat)^{-1}\right)(v),$$

which, combined with the first equation above, gives

$$\begin{aligned}
\mathcal{F}^\omega(\mathcal{F}^\omega u)(v) &= \sqrt{\det(\omega_{ij})^{-1}}\mathcal{F}^{-1}\left((\mathcal{F}^\omega u) \circ (\omega^\flat)^{-1}\right)(v) \\
&= \sqrt{\det(\omega_{ij})^{-1}}\mathcal{F}^{-1}\left(\sqrt{\det(\omega_{ij})}(\mathcal{F}u) \circ \omega^\flat \circ (\omega^\flat)^{-1}\right)(v) \\
&= \mathcal{F}^{-1}(\mathcal{F}u)(v) \\
&= u(v)
\end{aligned}$$

by the Fourier inversion theorem. □

Replacing ω by $\frac{1}{\epsilon}\omega$, we may introduce a parameter ϵ in the symplectic Fourier transform. Since, under this transformation, the Liouville measure is multiplied by $\frac{1}{\epsilon^n}$, denoting $\mathcal{F}_\epsilon^\omega = \mathcal{F}^{\frac{\omega}{\epsilon}}$, we have

$$\mathcal{F}_\epsilon^\omega(f)(x) = \frac{1}{(2\pi\epsilon)^n}\int_V e^{\frac{i}{\epsilon}\omega(x,y)} f(y)dy$$

where $dy = \frac{\omega^n}{n!}$ is the Liouville measure for ω, for $f \in \mathcal{S}(V)$.

Since $\mathcal{F}_\epsilon^\omega$ is merely a symplectic Fourier transform with a modified symplectic form, we still have

$$\mathcal{F}_\epsilon^\omega \circ \mathcal{F}_\epsilon^\omega = \mathrm{Id}.$$

5.2 Invariant quantization

For the two main classes of spaces which have a WKB quantization in the sense of [Bieo2], namely the elementary solvable symplectic symmetric spaces [Bieo2], and the elementary normal symplectic symmetric spaces [Bieo7], the quantization is obtained by an "equivalence" T from the Weyl product on a symplectic vector space.

The construction goes as follows.

Definition 5.2.1. *Let (M, ω) be a symplectic manifold with a strongly hamiltonian action of a connected Lie group G, and denote μ_X the dual moment functions, for*

$X \in \mathfrak{g}$. *A star product* $*_\nu$ *on* $C^\infty(M)[[\nu]]$ *is said to be* G-covariant *if*

$$[\lambda_X, \lambda_Y]_{*_\nu} := \lambda_X *_\nu \lambda_Y - \lambda_Y *_\nu \lambda_X = 2\nu\{\lambda_X, \lambda_Y\}.$$

When a star product is G-covariant on M, we may define a *representation* of $\mathfrak{g}$ on $C^\infty(M)[[\nu]]$ by

$$\rho_\nu(X)u = \frac{1}{2\nu}[\lambda_X, u]_{*_\nu}.$$

Indeed, we then have

$$\begin{aligned}
\rho_\nu([X,Y])u &= \frac{1}{2\nu}[\lambda_{[X,Y]}, u]_{*_\nu} \\
&= \frac{1}{2\nu}[\{\lambda_X, \lambda_Y\}, u]_{*_\nu} \\
&= \frac{1}{2\nu}\left[\frac{1}{2\nu}[\lambda_X, \lambda_Y]_{*_\nu}, u\right]_{*_\nu} \\
&= \frac{1}{2\nu}\left[\lambda_X, \frac{1}{2\nu}[\lambda_Y, u]_{*_\nu}\right]_{*_\nu} - \frac{1}{2\nu}\left[\lambda_Y, \frac{1}{2\nu}[\lambda_X, u]_{*_\nu}\right]_{*_\nu} \\
&= \rho_\nu(X)\rho_\nu(Y)u - \rho_\nu(Y)\rho_\nu(X)u \\
&= [\rho_\nu(X), \rho_\nu(Y)]u.
\end{aligned}$$

The maps $\rho_\nu(X)$ are moreover *derivations* of the algebra $(C^\infty(M)[[\nu]], *_\nu)$. Indeed, we have

$$\begin{aligned}
\rho_\nu(X)u *_\nu v + u *_\nu \rho_\nu(X)v &= (\lambda_X *_\nu u - u *_\nu \lambda_X) *_\nu v + u *_\nu (\lambda_X *_\nu v - v *_\nu \lambda_X) \\
&= \lambda_X *_\nu (u *_\nu v) - (u *_\nu v) *_\nu \lambda_X \\
&= \rho_\nu(X)(u *_\nu v).
\end{aligned}$$

Denote $(gu)(x) = u(g^{-1}x)$ the left regular action of G on functions on M.

Definition 5.2.2. *A star product* $*$ *on* M *is said to be* G*-invariant if*

$$g(u * v) = (gu * gv),$$

for all $g \in G$ *and* $u, v \in C^\infty(M)[[\nu]]$.

Since G is connected, the G-invariance is equivalent to its infinitesimal counterpart, which reads

$$X^*(u * v) = X^*u * v + u * X^*v,$$

where X^* is the fundamental vector field

$$X^*_x u = \frac{d}{dt} u(\exp(-tX)x)\Big|_{t=0}$$

associated to $X \in \mathfrak{g}$.

Consider now a manifold M endowed with a strongly hamiltonian action of a Lie group G, and with a G-covariant star product $*$. The 0-order term of the representation ρ_ν is $\{\lambda_X, \cdot\} = X^*$, so that it is a deformation of the representation by fundamental vector fields. A method to obtain a G-invariant product is thus to look for a transformation T of the set of functions on which the product is defined, such that

$$T^{-1} \circ \rho_\nu(X) \circ T = X^*,$$

since then the product $*'$ defined by

$$u *' v = T^{-1}(Tu * Tv)$$

will be G-invariant, since

$$\begin{aligned} X^*(u *' v) &= X^* T^{-1}(Tu * Tv) \\ &= T^{-1}\rho_\nu(X)(Tu * Tv) \\ &= T^{-1}(\rho_\nu(X)Tu * v + Tu * \rho_\nu(X)Tv) \\ &= T^{-1}(TX^*u * v + Tu * TX^*v) \\ &= X^*u *' v + u *' X^*v. \end{aligned}$$

For the above mentioned explicit cases, the transformation T had to be guessed in [BM01, Bie02] from the looks of the representation ρ_ν associated to the Moyal product

$$u *^M_\nu v = uv + \nu\{u, v\} + \sum_{k=2}^{\infty} \frac{\nu^k}{k!} \sum_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}} \Omega^{i_1 j_1} \dots \Omega^{i_k j_k}\, \partial_{i_1 \dots i_k} u\, \partial_{j_1 \dots j_k} v, \tag{5.3}$$

on $\mathbb{R} \times V \times \mathbb{R}$ with parameter $\nu = \frac{\epsilon}{i}$. As an example, for the elementary normal spaces, in coordinates $(a, v, l) \in \mathbb{R} \times V \times \mathbb{R}$ on M, it looked like

$$T = \mathcal{F} \circ \varphi_\epsilon \circ \mathcal{F}$$

with $\mathcal{F}$ the partial Fourier transform

$$\mathcal{F}u(a, v, \xi) = \frac{1}{\sqrt{2\pi}} \int e^{-il\xi} u(a, v, l) dl,$$

and $\varphi_\epsilon : M \to M$ the diffeomorphism

$$\varphi_\epsilon(a, v, \xi) = \left(a, \frac{v}{\cosh(\epsilon\xi)}, \frac{1}{2\epsilon}\sinh(2\epsilon\xi)\right).$$

Now our observation is that T can be rewritten as

$$T = \mathcal{F}^{\omega/\epsilon} \circ m_{\cosh^{2n}} \circ \phi^* \circ \mathcal{F}^{\omega/\epsilon}$$

where $(m_{\cosh^{2n}} f)(a, v, l) = \cosh(a)^{2n} f(a, v, l)$, $\mathcal{F}^{\omega/\epsilon}$ is the symplectic Fourier transform of Section 5.1, and ϕ is the canonical projection

$$\phi(a, v, l) = \left(\frac{1}{2}\sinh(2a), \cosh(a)v, l\right)$$

(see Section 5.5 for the computation). A computation shows that dropping the multiplication operator $m_{\cosh^{2n}}$ still yields an invariant product. Hence we may consider the formula

$$T = \mathcal{F}^{\omega/\epsilon} \circ \phi^* \circ \mathcal{F}^{\omega/\epsilon}$$

which only depends on the choice of a global Darboux chart, and take this as an ansatz for defining an invariant product on a sympletic symmetric space.

This is what we will do in the rest of this chapter for a class of spaces generalizing the elementary normal spaces, which we describe now.

5.3 Elementary normal spaces

Every group G acts simply transitively on its associated symmetric space $M = (G \times G)/G$, through the identification $G \cong G \times \{0\} \subset G \times G$. But generally $G \times G$ is too large to form the invariance group of a symplectic structure on M. Actually [Bie95, p. 13], if a connected Lie group G is such that there exists on M a symplectic form invariant under $G \times G$, then G must be *abelian*.

However, this does not necessarily prevent M from being a (non-trivial) symplectic symmetric space. Indeed, denoting $\mathfrak{g} \times \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, with $\mathfrak{k} = \{(x, x) \mid x \in \mathfrak{g}\}$ and $\mathfrak{m} = \{(x, -x) \mid x \in g\}$, we see that the transvection algebra is $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m} + \mathfrak{k}'$ with $\mathfrak{k}' = \{(x, x) \mid x \in [\mathfrak{g}, \mathfrak{g}]\}$, so that in general it is different from $\mathfrak{g} \times \mathfrak{g}$.

A particular case of this is the following.

Lemma 5.3.1. *Let $\mathfrak{s}$ be a Lie algebra which is a semi-direct product $\mathfrak{s} = \mathfrak{a} \ltimes_\rho \mathfrak{n}$ of a Lie subalgebra $\mathfrak{a}$ and an ideal $\mathfrak{n}$ with extension morphism $\rho : \mathfrak{a} \to \mathrm{Der}(\mathfrak{n})$. Then*

1. *the semi-direct product $\mathfrak{g} = \mathfrak{a} \ltimes (\mathfrak{n} \oplus \mathfrak{n})$ of $\mathfrak{a}$ and two commuting copies of $\mathfrak{n}$, with*

extension morphism $\rho \oplus (-\rho)$*, is a Lie algebra if and only if* $[\mathfrak{a}, \mathfrak{a}]$ *acts trivially on* $\mathfrak{n}$ *;*

2. *the map* $\sigma : \mathfrak{g} \to \mathfrak{g}$ *defined by* $\sigma|_{\mathfrak{a}} = -\mathrm{Id}$ *and* $\sigma((n, n')) = (n', n)$ *is an involutive automorphism of Lie algebra if and only if* $\mathfrak{a}$ *is abelian.*

Proof. For the first assertion, as the two copies of $\mathfrak{n}$ commute and $\mathfrak{a}$ acts separately on each copy, it must only be checked that $-\rho$ is a Lie algebra morphism. But this will happen if and only if $[\mathfrak{a}, \mathfrak{a}]$ acts trivially.

For the second assertion, just note that an anti-automorphism of a Lie algebra is an automorphism if and only if the Lie algebra is abelian. □

When it is defined, we will call $(\mathfrak{g}, \sigma)$ *the symmetric pair associated to* $\mathfrak{s}$ (although it also depends on $\mathfrak{a}$ and $\mathfrak{n}$). The decomposition of $\mathfrak{g}$ induced by σ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{k} = \{(n, n) \,|\, n \in \mathfrak{n}\}$ and $\mathfrak{m} = \{a + (n, -n) \,|\, a \in \mathfrak{a}, n \in \mathfrak{n}\}$. When $\rho(\mathfrak{a})\mathfrak{n} = \mathfrak{n}$, $\mathfrak{g}$ is the transvection algebra of the symmetric pair $(\mathfrak{s} \times \mathfrak{s}, \sigma_{\mathfrak{s}})$, where $\sigma_{\mathfrak{s}}(x, y) = (y, x)$ for all $x, y \in \mathfrak{s}$.

The main examples we have in mind come from the Hermitian symmetric spaces of the noncompact type, and are contractions thereof.

Example 5.3.2. Let $(\mathfrak{g}, \theta)$ be a noncompact semisimple symmetric pair of Hermitian type (i.e., θ is a Cartan involution and in the associated decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{k}$ has one-dimensional center). Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be the solvable part in an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then $\mathfrak{a}$ is abelian and the image of its adjoint action on $\mathfrak{n}$ is the whole $\mathfrak{n}$, so that $\mathfrak{g}_{\mathfrak{s}} = \mathfrak{a} \ltimes (\mathfrak{n} \oplus \mathfrak{n})$ is the transvection algebra of $(\mathfrak{s} \times \mathfrak{s}, \sigma_{\mathfrak{s}})$. Moreover $\mathfrak{g}_{\mathfrak{s}}$ is a contraction of $\mathfrak{g}$ in the following precise sense.

The commutation relations of a *contracted Lie algebra*, or *contraction*, $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$, are given by the limit:

$$[x, y]' = \lim_{\epsilon \to 0} U_\epsilon^{-1}([U_\epsilon(x), U_\epsilon(y)]),$$

where, for $0 < \epsilon \leq \epsilon_0$, $U_\epsilon \in GL(n, \mathbb{R})$ is a non singular linear transformation of $\mathfrak{g}$, and U_0 is singular (see [IW53, FdMo6] for example).

In the above Hermitian setting, let Δ be the root system corresponding to the chosen maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$, and Δ^+ be the choice of positive roots defining $\mathfrak{n}$. We may thus write

$$\mathfrak{g} = \mathfrak{m}_0 \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$$

avec

$$\mathfrak{m}_0 = \mathcal{Z}_{\mathfrak{k}}(\mathfrak{a}), \quad \mathfrak{g}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}, \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta_0^+} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta_0^+} \mathfrak{g}_{-\alpha}.$$

Choosing a set of simple roots α_i, $i = 1, \ldots, \dim \mathfrak{a}$, we define U_ϵ by

$$U_\epsilon|_{\mathfrak{m}_0} = \epsilon \operatorname{Id}, \quad U_\epsilon|_{\mathfrak{g}_{\pm\alpha_i}} = \epsilon \operatorname{Id}, \quad U_\epsilon|_{\mathfrak{a}} = \operatorname{Id},$$

and by extending it to $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ in such a way that, restricted to these subalgebras, it be a homomorphism. This would yield for example $U_\epsilon|_{\mathfrak{g}_{2\alpha_i+\alpha_j}} = \epsilon^3 \operatorname{Id}$.

We may check that the associated contraction is such that

1. the structures of $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are (separately) preserved,
2. $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are in direct sum in the limit $\epsilon \to 0$,
3. $\mathfrak{m}_0$ becomes abelian and commutes with $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ in the limit $\epsilon \to 0$.

Hence, we have a contraction towards the direct sum

$$\mathfrak{g}' = \mathbb{R}^{\dim \mathfrak{m}_0} \oplus \mathfrak{g}_{\mathfrak{s}}.$$

Since θ remains an automorphism during the whole process, this yields a path in the variety of Lie triple systems connecting a Hermitian Lie triple system of the noncompact type to its solvable contraction. ◁

Although all the symmetric pairs $(\mathfrak{g}, \theta)$ above underlie symplectic symmetric spaces, their contractions do not always bear an invariant symplectic form. In fact, only those with rank one do, as shows the following theorem due to P. Bieliavsky and V. Gayral, which we rephrase in the language of double extensions (see Section 3.2):

Theorem 5.3.3 ([BG09])**.** *Let $\mathfrak{s} = \mathfrak{a} \ltimes_\rho \mathfrak{n}$ be a semi-direct product with $\mathfrak{a}$ abelian, such that the associated symmetric pair $\mathfrak{g}_{\mathfrak{s}} = \mathfrak{a} \ltimes_{\rho\oplus-\rho} (\mathfrak{n} \oplus \mathfrak{n})$ underlies a symplectic symmetric space. Then, for the symplectic form pulled-back through the injection $\mathfrak{s} \to \mathfrak{g}_{\mathfrak{s}}$, $\mathfrak{s}$ is a symplectic Lie algebra, direct product $\mathfrak{s} = \tilde{\mathfrak{a}} \oplus \mathfrak{s}'$ of a symplectic vector space $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ with a symplectic double extension $\mathfrak{s}' = \mathfrak{a}' \ltimes (V \oplus \mathfrak{a}'^*)$ of an abelian symplectic Lie algebra V by an abelian Lie algebra $\mathfrak{a}' \subset \mathfrak{a}$.*

Moreover, if the symplectic form on $\mathfrak{s}$ is exact and the action of $\mathfrak{a}$ on $\mathfrak{n}$ is semisimple, then $\mathfrak{s}$ is isomorphic to a direct product of elementary normal Lie algebras.

We will use the notations $n_{\mathfrak{k}} = \frac{1}{2}(n, n)$ and $n_{\mathfrak{m}} = \frac{1}{2}(n, -n)$ for all $n \in \mathfrak{n}$. It should be noted that in [Bie07], a different normalization was used (without the factor $\frac{1}{2}$), leading to numerous little differences propagating throughout the computations, and to the apparent fact that the symplectic forms on $\mathfrak{s}$ and on $\mathfrak{m}$ give different relative weights to the H, E and the V parts. To clarify the differences, we introduce for the rest of this section a parameter α which will be set to $\frac{1}{2}$ afterwards, and which took the value 1 in [Bie07].

For $v \in V$, we will denote $v_1 = (v, 0)$ and $v_2 = (0, v)$, and similarly $E_1 = (E, 0)$ and $E_2 = (0, E)$. With this parameter, we thus have

$$\begin{aligned} v_{\mathfrak{m}} &= \alpha(v_1 - v_2) & v_{\mathfrak{k}} &= \alpha(v_1 + v_2) \\ E_{\mathfrak{m}} &= \alpha(E_1 - E_2) & E_{\mathfrak{k}} &= \alpha(E_1 + E_2), \end{aligned}$$

the $[\mathfrak{m}, \mathfrak{m}]$ brackets read

$$\begin{aligned} [H, v_{\mathfrak{m}}] &= v_{\mathfrak{k}} \\ [H, E_{\mathfrak{m}}] &= 2E_{\mathfrak{k}} \\ [v_{\mathfrak{m}}, v'_{\mathfrak{m}}] &= \alpha\omega_V(v, v')E_{\mathfrak{k}} \end{aligned}$$

and analogous expressions hold, with the same factors, for the $[\mathfrak{k}, \mathfrak{m}]$ and $[\mathfrak{k}, \mathfrak{k}]$ brackets.

Lemma 5.3.4. *Let $\mathfrak{s}$ be an elementary normal Lie algebra and let $(\mathfrak{g}, \sigma)$ be the symmetric pair associated to $\mathfrak{s}$.*

1. *Any 2-cocycle ω on $\mathfrak{s}$ is a 2-coboundary and is of the form*

$$\omega = \lambda\omega_0 + \delta i_w \omega_V$$

where $\lambda \in \mathbb{R}$, $w \in V$ and $\omega_0 = 2H^* \wedge H^* + \omega_V$ (ω_V is extended to $\mathfrak{s}$ by zero on H and E). It is non-degenerate if and only if $\lambda \neq 0$.

2. *Any 2-cocycle Ω on $\mathfrak{g}$ vanishing on $\mathfrak{k}$ is of the form*

$$\Omega = \lambda\Omega_0 + H^* \wedge i_{w_{\mathfrak{m}}} \Omega_V$$

where $\lambda \in \mathbb{R}$, $w \in V$, $\Omega_0 = 2H^* \wedge E^* + \Omega_V$, and Ω_V is the 2-form on $\mathfrak{g}$ vanishing on $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathbb{R}E_{\mathfrak{m}}$ and such that $\Omega_V(v_{\mathfrak{m}}, v'_{\mathfrak{m}}) = 2\alpha\omega_V(v, v')$. It is non-degenerate on $\mathfrak{m} \times \mathfrak{m}$ if and only if $\lambda \neq 0$.

Proof. The first part was proved in [AP97, Lemma 3.3].

For the second, let Ω be a general 2-form on $\mathfrak{g}$ vanishing on $\mathfrak{k}$: let

$$\Omega(H, v_{\mathfrak{m}}) = \theta(v), \quad \Omega(H, E_{\mathfrak{m}}) = 2\lambda, \quad \Omega(v_{\mathfrak{m}}, v'_{\mathfrak{m}}) = \beta(v, v'), \quad \Omega(v_{\mathfrak{m}}, E_{\mathfrak{m}}) = \mu(v),$$

for some 1-forms θ, μ, some 2-form β and some real number λ. Then Ω will be $\mathfrak{k}$-invariant if and only if the bilinear form $(x, y) \in \mathfrak{m} \times \mathfrak{m} \mapsto \Omega([k, x], y)$ is symmetric for each $k \in \mathfrak{k}$.

We have

$$\Omega([v'_{\mathfrak{k}} + l'E_{\mathfrak{k}}, aH + v_{\mathfrak{m}} + lE_{\mathfrak{m}}], \bar{a}H + \bar{v}_{\mathfrak{m}} + \bar{l}E_{\mathfrak{m}})$$

$$
\begin{aligned}
&= \Omega(-av'_{\mathfrak{m}} - (2al' - \alpha\omega_V(v',v))E_{\mathfrak{m}}, \bar{a}H + \bar{v}_{\mathfrak{m}} + \bar{l}E_{\mathfrak{m}}) \\
&= a\bar{a}\theta(v') + 2\lambda\bar{a}(2al' - \alpha\omega_V(v',v)) \\
&\quad - a\beta(v',\bar{v}) - a\bar{l}\mu(v') + (2al' - \alpha\omega_V(v',v))\mu(\bar{v}).
\end{aligned}
$$

Setting $v' = 0$ and $l' = 1$ yields $4\lambda a\bar{a} + 2a\mu(\bar{v})$, which imposes $\mu = 0$. And with $l' = 0$, we get $a\bar{a}\theta(v') - 2\lambda\bar{a}\omega_V(v',v) - a\beta(v',\bar{v})$, which forces $\beta = 2\lambda\alpha\omega_V$.

Finally, θ may be written as $2\alpha i_w\omega_V$ for some $w \in V$. □

The correspondance between the symplectic structures on $\mathfrak{s}$ and $\mathfrak{g}$ is as follows. If $\Omega = \lambda\Omega_0 + H^* \wedge i_{w_{\mathfrak{m}}}\Omega_V$ as above, then its pullback $\omega = j^*\Omega$ on $\mathfrak{s}$ by $j : \mathfrak{s} \to \mathfrak{g} : aH + v + lE \mapsto aH + v_1 + lE_1$ is given by

$$
\omega(H,v) = i_w\omega_V \qquad \omega(H,E) = \frac{\lambda}{\alpha} \qquad \omega(v,v') = \frac{\lambda}{2\alpha}\omega_V(v,v').
$$

For the rest of this chapter, as promised, we fix $\alpha = \frac{1}{2}$, and $(\mathfrak{g},\sigma)$ will always designate the symmetric pair associated to an elementary normal Lie algebra $\mathfrak{s}$.

5.4 Automorphism group

Let us compute the automorphism group of the symmetric pair $(\mathfrak{g},\sigma)$. Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, and let K be the connected subgroup with Lie algebra $\mathfrak{k}$.

Definition 5.4.1. *If $L : \mathfrak{n} \to \mathfrak{n}$ is a linear map, we will denote $\overline{L} : \mathfrak{g} \to \mathfrak{g}$ the "double map"*

$$
\overline{L}\left(aH + n_{\mathfrak{m}} + n'_{\mathfrak{k}}\right) = aH + (L(n))_{\mathfrak{m}} + (L(n'))_{\mathfrak{k}}.
$$

Lemma 5.4.2. *If g is an automorphism of $\mathfrak{n}$ commuting with $\rho(A)$ for all $A \in \mathfrak{a}$, then $\overline{g}$ is an automorphism of $(\mathfrak{g},\sigma)$.*

Proof. Note that the above definition of $\overline{g}$ is equivalent to $\overline{g}(aH + (n,n')) = aH + (g(n),g(n'))$. Now, as the two copies of $\mathfrak{n}$ commute, the result follows directly from the hypotheses. □

Proposition 5.4.3. *1. The following sets G_1, G_2, G_3, G_4, G_5 form five groups of automorphisms of $(\mathfrak{g},\sigma)$:*

- *(a) Inner automorphisms: $G_1 = \operatorname{Ad} K$;*
- *(b) Symmetry: $G_2 = \{\mathrm{Id}, \sigma\}$;*
- *(c) Inversion: $G_3 = \{\mathrm{Id}, \imath\}$ with*

$$
\imath(v + eE) = \imath_V(v) - eE
$$

where $\iota_V : V \to V$ *is defined by* $\iota_V = \mathrm{Id}_L \oplus (-\mathrm{Id})_{L'}$ *for some fixed pair of transverse Lagrangians* L, L' *of* V*;*

(d) *Dilations:* $G_4 = \left\{ \overline{D_r} \,\middle|\, r > 0 \right\}$ *where*

$$D_r(v + eE) = rv + er^2 E;$$

(e) *Symplectomorphisms of* V*:* $G_5 = \left\{ \overline{S_F} \,\middle|\, F \in Sp(V, \Omega_V) \right\}$ *where*

$$S_F(v + eE) = F(v) + eE.$$

2. *Every automorphism* g *of* $(\mathfrak{g}, \sigma)$ *can be written in a unique way as a product*

$$g = g_1 g_2 g_3 g_4 g_5$$

with $g_i \in G_i$ *for* $i = 1, \ldots, 5$*. In other words, we have*

$$\begin{aligned} \mathrm{Aut}(\mathfrak{g}, \sigma) &= G_2 \times (G_4 \times (G_3 \ltimes G_5)) \ltimes G_1 \\ &= \{\mathrm{Id}, \sigma\} \times \overline{\mathrm{Out}(\mathfrak{n})} \ltimes \mathrm{Ad}\, K \end{aligned} \tag{5.4}$$

where we denoted $\mathrm{Out}(\mathfrak{n})$ *the subgroup of* $\mathrm{Aut}(\mathfrak{n})$ *isomorphic to the outer automorphisms of* $\mathfrak{n}$ *and composed of the dilations, symplectomorphisms of* V *and the inversion.*

Proof. 1. That the G_i's form linear Lie groups of transformations of the vector space $\mathfrak{g}$ is obvious. Now by definition of σ and K, G_1 and G_2 act by automorphisms of $(\mathfrak{g}, \sigma)$. And by lemma 5.4.2, the same is true for G_3, G_4 and G_5 since the dilations, inversion and symplectomorphisms of V are well-known to form automorphisms of the Heisenberg algebra $\mathfrak{n}$ [Fol89].

2. Let $g \in \mathrm{Aut}(\mathfrak{g}, \sigma)$. We have $g(H) = hH + w_{\mathfrak{m}} + 2tE_{\mathfrak{m}}$ for some $h, t \in \mathbb{R}$ and $w \in V$. Composing with $\mathrm{Ad}(e^{(w+tE)_{\mathfrak{k}}})$, we may assume that $g(H) = hH$ for some $h \in \mathbb{R}$. As the derived series of a Lie algebra is always composed of characteristic ideals, we must have $g(E_{\mathfrak{m}}) = eE_{\mathfrak{m}}$ and $g(E_{\mathfrak{k}}) = e'E_{\mathfrak{k}}$ for some $e, e' \in \mathbb{R}_0$. Thus

$$\begin{aligned} 2e'E_{\mathfrak{k}} &= g(2E_{\mathfrak{k}}) = g([H, E_{\mathfrak{m}}]) = [g(H), g(E_{\mathfrak{m}})] = [hH, eE_{\mathfrak{m}}] = 2heE_{\mathfrak{k}} \\ 2eE_{\mathfrak{m}} &= g(2E_{\mathfrak{m}}) = g([H, E_{\mathfrak{k}}]) = [g(H), g(E_{\mathfrak{k}})] = [hH, e'E_{\mathfrak{k}}] = 2he'E_{\mathfrak{m}} \end{aligned}$$

so that $h = \pm 1$ and $e' = he$. Composing with σ we may assume that $h = 1$ and $e = e'$, and composing with the inversion, we may assume that $e > 0$. Composing further by a dilation, me way assume that $e = 1$, so that g fixes H, $E_{\mathfrak{m}}$ and $E_{\mathfrak{k}}$. On $V_{\mathfrak{k}}$ and $V_{\mathfrak{m}}$, we must have $g(v_{\mathfrak{m}}) = A(v)_{\mathfrak{m}} + \alpha(v)E_{\mathfrak{m}}$ and

$g(v_\mathfrak{k}) = A'(v)_\mathfrak{k} + \alpha'(v)E_\mathfrak{k}$ for some linear maps $A, A' : V \to V$ and $\alpha, \alpha' : V \to \mathbb{R}$ with the conditions

$$A(v)_\mathfrak{m} + \alpha(v)E_\mathfrak{m} = g([H, v_\mathfrak{k}]) = [g(H), g(v_\mathfrak{k})] = A'(v)_\mathfrak{m} + 2\alpha'(v)E_\mathfrak{m}$$
$$A'(v)_\mathfrak{k} + \alpha'(v)E_\mathfrak{k} = g([H, v_\mathfrak{m}]) = [g(H), g(v_\mathfrak{m})] = A(v)_\mathfrak{k} + 2\alpha(v)E_\mathfrak{k}$$

which imply that $\alpha = \alpha' = 0$ and $A = A'$, and the condition

$$\frac{1}{2}\Omega_V(v, w)E_\mathfrak{k} = g([v_\mathfrak{m}, w_\mathfrak{m}]) = [g(v_\mathfrak{m}), g(w_\mathfrak{m})] = \frac{1}{2}\Omega_V(A(v), A(w))E_\mathfrak{k}$$

which forces g to be in G_5.

Note to finish the proof, that the elements of G_i, $i = 1, \ldots, 5$, we had to choose were necessarily unique.

□

Lemma 5.4.4. *The action of $Sp(n, \mathbb{R})$ on $\mathbb{R}^{2n}$ has only two orbits, namely $\{0\}$ and $\mathbb{R}^{2n} \setminus \{0\}$.*

Proof. The maximal compact subgroup of $Sp(n, \mathbb{R})$ is $U(n)$ with the inclusion given by $M \in U(n) \mapsto j(M) = \left(\begin{smallmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{smallmatrix}\right) \in O(2n) \subset Sp(n, \mathbb{R})$. Viewing $U(n-1)$ as the subgroup of $U(n)$ stabilizing $(1, 0, \ldots, 0)^t \in \mathbb{C}^n$, one checks that the stabilizer of $e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^{2n}$ in $j(U(n))$ is exactly $j(U(n-1))$. Since $U(n)/U(n-1) \cong S^{2n-1}$, the orbit of e_1 under $Sp(n, \mathbb{R})$ contains the unit sphere.

Now $Sp(n, \mathbb{R})$ also acts by dilations as for each $\lambda \in \mathbb{R}_0$ the matrix $\left(\begin{smallmatrix} D^\lambda & 0 \\ 0 & D^{1/\lambda} \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$, where $D^a = \operatorname{Diag}(a, 1 \ldots, 1)$, sends the vector $e_1 = (1, 0, \ldots, 0)^t$ to λe_1. □

Definition 5.4.5. *We will denote the set of 2-cocycles on $\mathfrak{g}$ vanishing on $\mathfrak{k}$ by $C^{2,\mathfrak{k}} = \{\Omega^{\lambda,w} \mid \lambda \in \mathbb{R}, w \in V\}$ where*

$$\Omega^{\lambda,w} = \lambda\Omega_0 + H^* \wedge i_w\Omega_V.$$

A short calculation yields the following result.

Proposition 5.4.6. *The left regular action of $\operatorname{Aut}(\mathfrak{g}, \sigma)$ on $C^{2,\mathfrak{k}}$, defined by $(g \cdot \Omega)(x, y) = \Omega(g^{-1}x, g^{-1}y)$ for all $g \in \operatorname{Aut}(\mathfrak{g}, \sigma)$, $\Omega \in C^{2,\mathfrak{k}}$ and $x, y \in \mathfrak{g}$, is given by*

$$\operatorname{Ad} k \cdot \Omega^{\lambda,w} = \Omega^{\lambda,w}$$
$$\sigma \cdot \Omega^{\lambda,w} = \Omega^{\lambda,w}$$
$$\bar{\imath} \cdot \Omega^{\lambda,w} = \Omega^{-\lambda,-w}$$
$$\overline{D_r} \cdot \Omega^{\lambda,w} = \Omega^{\lambda/r^2, w/r}$$
$$\overline{S_F} \cdot \Omega^{\lambda,w} = \Omega^{\lambda, F^{-1}w}$$

In particular, there are four orbits of $\mathrm{Aut}(\mathfrak{g},\sigma)$ *in* $C^{2,\mathfrak{k}}$*, namely those of* $\Omega^{0,0}$, $\Omega^{0,w}$, $\Omega^{1,0}$ *and* $\Omega^{1,w}$ *for some non-zero* $w \in V$*. Moreover, denoting*

$$\mathrm{Stab}_{Sp}(w) = \left\{ \overline{S_F} \,\middle|\, F \in Sp(V,\Omega_V), Fw = w \right\},$$

we have

$$\begin{aligned} \mathrm{Aut}\left(\mathfrak{g},\sigma,\Omega^{0,0}\right) &= \mathrm{Aut}\left(\mathfrak{g},\sigma\right) \\ \mathrm{Aut}\left(\mathfrak{g},\sigma,\Omega^{\lambda,w}\right) &= \{\mathrm{Id},\sigma\} \times \mathrm{Stab}_{Sp}(w) \ltimes \mathrm{Ad}\, K \qquad \text{if } \lambda \neq 0 \text{ or } w \neq 0. \end{aligned}$$

As $\Omega^{\lambda,w}$ is non-degenerate if and only if $\lambda \neq 0$, this shows that, up to isomorphism, there are exactly two symplectic symmetric spaces having $(\mathfrak{g},\sigma)$ as symmetric pair. The first one, with $w = 0$, has been studied in [Bieo7]. The purpose of what follows is to study the second one, with $w \neq 0$.

5.5 Ševera projection

In this section, we compute the canonical map (see Definition 4.1.5) associated to a symplectic symmetric triple $(\mathfrak{g},\sigma,\Omega^{\lambda,w})$. We will denote $M = G/K$ the connected and simply connected symmetric space associated to the symmetric pair $(\mathfrak{g},\sigma)$, and $\omega^{\lambda,w}$ the invariant closed 2-form on M associated to the 2-cocycle $\Omega^{\lambda,w}$ on $\mathfrak{g}$.

Proposition 5.5.1. *Let* Ω *be any* $\mathfrak{k}$*-invariant 2-form on* $(\mathfrak{g},\sigma)$ *vanishing on* $\mathfrak{k}$*, and let* ω *be the* G*-invariant closed 2-form on* $M = G/K$ *whose value at* eK *is* $\Omega|_{\mathfrak{m}\times\mathfrak{m}}$*. Then the map*

$$I : (\mathfrak{m},\Omega) \to (M,\omega) : aH + n_{\mathfrak{m}} \mapsto e^{aH} e^{n_{\mathfrak{m}}} K$$

is a global Darboux chart.

Proof. Denoting $\alpha : G\times M \to M : (g,x) \mapsto \alpha_g(x) = \alpha^x(g)$ the action of G on M, and $\tilde{I} : \mathfrak{m} \to G : aH + n_{\mathfrak{m}} \mapsto e^{aH} e^{n_{\mathfrak{m}}}$ —so that $I(r) = \tilde{I}(r)K = \alpha_{\tilde{I}(r)}(eK)$ for all $r \in \mathfrak{m}$—, we have for any $r \in \mathfrak{m}$ and any $A, B \in T_r\mathfrak{m} \cong \mathfrak{m}$,

$$\begin{aligned} \left(I^*\omega\right)_r (A,B) &= \omega_{I(r)}\left(I_{*_r}A, I_{*_r}B\right) \\ &= \omega_{eK}\left(\left(\alpha_{\tilde{I}(r)^{-1}}\right)_* I_{*_r}A, \left(\alpha_{\tilde{I}(r)^{-1}}\right)_* I_{*_r}B\right). \end{aligned}$$

But

$$\left(\alpha_{\tilde{I}(r)^{-1}}\right)_* I_{*_r}A = \left.\frac{d}{dt}\alpha_{\tilde{I}(r)^{-1}}\alpha_{\tilde{I}(r+tA)}(eK)\right|_{t=0} = \left(\alpha^{eK}\right)_{*_e}\left(L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_e}A\right)$$

where L_g is left-translation in G by $g \in G$, so that we have

$$(I^*\omega)_r(A,B) = \Omega\left(L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}A, L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}B\right)$$

since $\left(\left(\alpha^{eK}\right)_{*e}\right)^* \omega_{eK} = \Omega$.

Now we compute, with $r = aH + (v+eE)_{\mathfrak{m}}$,

$$\begin{aligned}
L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}H &= \frac{d}{dt}e^{-(v+eE)_{\mathfrak{m}}}e^{-aH}e^{(a+t)H}e^{(v+eE)_{\mathfrak{m}}}\Big|_{t=0} \\
&= \mathrm{Ad}\left(e^{-(v+eE)_{\mathfrak{m}}}\right)H \\
&= H + (v+2eE)_{\mathfrak{k}} \\
L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}w_{\mathfrak{m}} &= \frac{d}{dt}e^{-(v+eE)_{\mathfrak{m}}}e^{(v+tw+eE)_{\mathfrak{m}}}\Big|_{t=0} \\
&= w_{\mathfrak{m}} + \frac{1}{4}\Omega_V(w,v)E_{\mathfrak{k}} \\
L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}E_{\mathfrak{m}} &= \frac{d}{dt}e^{-(v+eE)_{\mathfrak{m}}}e^{(v+(e+t)E)_{\mathfrak{m}}}\Big|_{t=0} \\
&= E_{\mathfrak{m}}.
\end{aligned}$$

Remembering that Ω vanishes on $\mathfrak{k}$, we may drop all terms in $\mathfrak{k}$ just above and see that the map $L_{\tilde{I}(r)^{-1}}\tilde{I}_{*_r}$ is the identity modulo $\mathfrak{k}$, so that we indeed have

$$(I^*\omega)_r(A,B) = \Omega(A,B).$$

□

From now on, we will denote the elements of $\mathfrak{m}$ as $(a,v,l) = aH + v_{\mathfrak{m}} + lE_{\mathfrak{m}}$, i.e., we will identify $\mathfrak{m}$ with $\mathbb{R}\times V\times\mathbb{R}$.

Lemma 5.5.2. *Denote $L^w : \mathfrak{m} \to \mathfrak{m}$ the linear map given by $L^w(a,v,l) = (a,v,l - \frac{1}{2}\Omega_V(w,v))$. Then for any $\Omega^{\lambda,\lambda w}$, $I^w = I \circ L^w$ is global Darboux chart from $(\mathfrak{m},\Omega^{\lambda,0})$ to $(M,\omega^{\lambda,\lambda w})$.*

Proof. We just have to check that $L^w : (\mathfrak{m},\Omega^{\lambda,0}) \to (\mathfrak{m},\Omega^{\lambda,\lambda w})$ is symplectic:

$$\begin{aligned}
&(L^w)^*\Omega^{\lambda,\lambda w}((a,v,l),(a',v',l')) \\
&\quad= \Omega^{\lambda,\lambda w}((a,v,l-\Omega_V(w,v)),(a',v',l'-\Omega_V(w,v'))) \\
&\quad= \lambda(2al' - a\Omega_V(w,v') - 2a'l + a'\Omega_V(w,v) + \Omega_V(v,v')) \\
&\qquad + a\Omega_V(\lambda w,v') - a'\Omega_V(\lambda w,v) \\
&\quad= \lambda(2al' - 2a'l + \Omega_V(v,v')) \\
&\quad= \Omega^{\lambda,0}((a,v,l),(a',v',l')).
\end{aligned}$$

□

Proposition 5.5.3. *In the above global Darboux chart I with respect to the cocycle $\Omega = \Omega^{\lambda,\lambda w}$, $\lambda \neq 0$, $w \in V$, the dual moments read*

$$\begin{aligned}
\mu_H(a,v,l) &= \lambda(2l + \Omega_V(w,v)) \\
\mu_{v_{\mathfrak{m}}}(a,v,l) &= \lambda\Omega_V(-\cosh(a)v - \sinh(a)w, v) \\
\mu_{E_{\mathfrak{m}}}(a,v,l) &= \lambda(-\sinh(2a)) \\
\mu_{v_{\mathfrak{k}}}(a,v,l) &= \lambda\Omega_V(\sinh(a)v + (\cosh(a)-1)w, v) \\
\mu_{E_{\mathfrak{k}}}(a,v,l) &= \lambda(\cosh(2a)-1) \\
\mu_Z(a,v,l) &= 1
\end{aligned}$$

(with $v \in V$), and the canonical map reads

$$\begin{aligned}
\phi(a,v,l) = \Big(&\frac{1}{2}\sinh(2a), \\
&\cosh(a)v + \Big(\sinh(a) - \frac{1}{2}\sinh(2a)\Big)w, \\
&l - \frac{1}{2}(\cosh(a)-1)\Omega_V(w,v)\Big).
\end{aligned}$$

In the chart I^w the canonical map is

$$\begin{aligned}
\phi(a,v,l) = \Big(&\frac{1}{2}\sinh(2a), \\
&\cosh(a)v + \Big(\sinh(a) - \frac{1}{2}\sinh(2a)\Big)w, \\
&l\Big).
\end{aligned}$$

Proof. As remarked in Proposition 3.4.9, the momentum map J is an equivariant diffeomorphism. Hence $J \circ I$ provides a global Darboux chart on the coadjoint orbit.

In this computation, for all $v \in V$ and $\zeta \in \mathbb{R}$, we will denote $(v,\zeta)_{\mathfrak{m}} = v_{\mathfrak{m}} + \zeta E_{\mathfrak{m}}$ and $(v,\zeta)_{\mathfrak{k}} = v_{\mathfrak{k}} + \zeta E_{\mathfrak{k}}$. We have

$$\begin{aligned}
&\langle (J\circ I)(a,v,l), \alpha H + (v,\zeta)_{\mathfrak{m}} + (v',\zeta')_{\mathfrak{k}} + zZ\rangle \\
&\quad = \langle \exp(v,l)_{\mathfrak{m}} \cdot Z^*, \exp(-aH)(\alpha H + (v,\zeta)_{\mathfrak{m}} + (v',\zeta')_{\mathfrak{k}} + zZ)\rangle \\
&\quad = \langle \exp(v,l)_{\mathfrak{m}} \cdot Z^*, \alpha H \\
&\qquad + (-\sinh(a)v, -\sinh(2a)\zeta)_{\mathfrak{k}} + (\cosh(a)v, \cosh(2a)\zeta)_{\mathfrak{m}} \\
&\qquad + (-\sinh(a)v', -\sinh(2a)\zeta')_{\mathfrak{m}} + (\cosh(a)v', \cosh(2a)\zeta')_{\mathfrak{k}}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda \sinh(a)\Omega_V(w,v)Z + \lambda(\cosh(a)-1)\Omega_V(w,v')Z \\
&\quad - \lambda \sinh(2a)\zeta Z + \lambda(\cosh(2a)-1)\zeta' Z \\
&\quad + zZ\rangle \\
&= \langle Z^*, \alpha H + \alpha(v,2l)_{\mathfrak{k}} + \lambda 2\alpha l Z + \lambda\alpha\Omega(w,v)Z \\
&\quad + (1-\mathrm{ad}(v,l)_{\mathfrak{m}}) \\
&\qquad \Big((-\sinh(a)v + \cosh(a)v', -\sinh(2a)\zeta + \cosh(2a)\zeta')_{\mathfrak{k}} \\
&\qquad + (\cosh(a)v - \sinh(a)v', \cosh(2a)\zeta - \sinh(2a)\zeta')_{\mathfrak{m}}\Big) \\
&\quad - \lambda \sinh(a)\Omega_V(w,v)Z + \lambda(\cosh(a)-1)\Omega_V(w,v')Z \\
&\quad - \lambda \sinh(2a)\zeta Z + \lambda(\cosh(2a)-1)\zeta' Z \\
&\quad + zZ\rangle \\
&= \langle Z^*, \alpha H + \alpha(v,2l)_{\mathfrak{k}} + \lambda 2\alpha l Z + \lambda\alpha\Omega(w,v)Z \\
&\quad (-\sinh(a)v + \cosh(a)v', -\sinh(2a)\zeta + \cosh(2a)\zeta')_{\mathfrak{k}} \\
&\quad + (\cosh(a)v - \sinh(a)v', \cosh(2a)\zeta - \sinh(2a)\zeta')_{\mathfrak{m}} \\
&\quad - \left(0, -\tfrac{1}{2}\sinh(a)\Omega_V(v,v) + \tfrac{1}{2}\cosh(a)\Omega_V(v,v')\right)_{\mathfrak{m}} \\
&\quad - \left(0, -\tfrac{1}{2}\sinh(a)\Omega_V(v,v') + \tfrac{1}{2}\cosh(a)\Omega_V(v,v)\right)_{\mathfrak{k}} \\
&\quad - \lambda\left(-\sinh(a)\Omega_V(v,v') + \cosh(a)\Omega_V(v,v)\right)Z \\
&\quad - \lambda \sinh(a)\Omega_V(w,v)Z + \lambda(\cosh(a)-1)\Omega_V(w,v')Z \\
&\quad - \lambda \sinh(2a)\zeta Z + \lambda(\cosh(2a)-1)\zeta' Z \\
&\quad + zZ\rangle \\
&= z + \lambda\Big(2\alpha l + \alpha\Omega(w,v) - \sinh(2a)\zeta + (\cosh(2a)-1)\zeta' \\
&\qquad + \sinh(a)\Omega_V(v,v') - \cosh(a)\Omega_V(v,v) \\
&\qquad - \sinh(a)\Omega_V(w,v) + (\cosh(a)-1)\Omega_V(w,v')\Big) \\
&= z + \lambda\Big(\left(2l + \Omega_V(w,v)\right)\alpha - \Omega_V\left(\cosh(a)v + \sinh(a)w, v\right) - \sinh(2a)\zeta \\
&\qquad + \Omega_V\left(\sinh(a)v + (\cosh(a)-1)w, v'\right) + (\cosh(2a)-1)\zeta'\Big)
\end{aligned}
$$

from which we deduce the dual moments.

Morevover, differentiating this expression with respect to a, v, l at $(0,0,0)$, we get

$$
\begin{aligned}
&\langle (J\circ I)_{*_{(0,0,0)}}(A,V,L), \alpha H + (v,\zeta)_{\mathfrak{m}}\rangle \\
&\qquad = \lambda\Big(\left(2L + \Omega_V(w,V)\right)\alpha - \Omega_V(V + Aw, v) - 2A\zeta\Big)
\end{aligned}
$$

so that we have

$$(J \circ I)(a, v, l)|_{\mathfrak{m}} = (J \circ I)_{*_{(0,0,0)}}\Big(\tfrac{1}{2}\sinh(2a),\ \cosh(a)v + \big(\sinh(a) - \tfrac{1}{2}\sinh(2a)\big)w,\ l - \tfrac{1}{2}(\cosh(a) - 1)\Omega_V(w, v)\Big)$$

which yields the canonical map ϕ at o. In the chart I^w, the projection is given by $(L^w)^{-1} \circ \phi \circ L^w$. □

5.6 Phases

In this section, we consider two candidates for a phase: the Weinstein phase function, and the Ševera phase function. Given three points x, y, z in our symplectic symmetric space M, the former is the symplectic area of the double triangle of x, y, z, whereas the latter is the flat symplectic area of the triangle in T_xM whose vertices are the images of x, y and z through the canonical map at x.

Quite surprisingly, for all symplectic symmetric spaces that are quantized in [Bie02] and [Bie07], one may check that these two phase candidates actually *coincide*, and are indeed equal to the phase of the products obtained. Hence, with these examples only, it is not clear which candidate should be the "good one" in general.

One of the interests of our new example is that it is the first for which the two candidates differ, singling out the Weinstein phase function as the correct phase for the product (see Section 5.8). Nevertheless, this is not the end of the story, as when considering Hermitian symmetric spaces, we have seen in Theorem 2.3.12 that the Weinstein phase function is not everywhere defined, whereas Theorem 4.5.12 shows that the Ševera phase function is well-defined, and [BDS09] shows that the latter naturally appears in the quantization of the hyperbolic plane.

Area of the double triangle

In this section, we compute the area of a double triangle in our symplectic symmetric space. We first compute explicit formulæ for the geodesics and the vertices of double triangles, and use these to compute the area of a double triangle by integrating a symplectic potential on the geodesic edges.

Proposition 5.6.1. *The symmetries in* $M^{\lambda,\lambda w} := (M, \omega^{\lambda,\lambda w})$ *are expressed, in the*

Darboux chart I^w, by

$$\begin{aligned} s_{(a,v,l)}(a',v',l') = \Big(&2a-a',\\ &2\cosh(a-a')\,v-v',\\ &2\cosh(2(a-a'))\,l-l'\\ &+(\cosh(a-a')-\cosh(2(a-a')))\,\Omega_V(w,v)\\ &+\sinh(a-a')\,\Omega_V(v,v')\Big). \end{aligned}$$

Proof. In the chart

$$I^w(a,v,l) = e^{aH}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K$$

(see Lemma 5.5.2), we compute

$$e^{\tilde{a}H}\cdot I^w(a,v,l) = I^w(a+\tilde{a},v,l)$$

$$\begin{aligned} e^{\tilde{v}_{\mathfrak{m}}}\cdot I^w(a,v,l) &= e^{aH}e^{e^{-a\operatorname{ad}H}\tilde{v}_{\mathfrak{m}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K\\ &= e^{aH}e^{\cosh(a)\tilde{v}_{\mathfrak{m}}-\sinh(a)\tilde{v}_{\mathfrak{k}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K\\ &= e^{aH}e^{\cosh(a)\tilde{v}_{\mathfrak{m}}}e^{-\sinh(a)\tilde{v}_{\mathfrak{k}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K\\ &= e^{aH}e^{\cosh(a)\tilde{v}_{\mathfrak{m}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}e^{\frac{1}{2}\Omega_V(-\sinh(a)\tilde{v},v)E_{\mathfrak{m}}}e^{-\sinh(a)\tilde{v}_{\mathfrak{k}}}K\\ &= e^{aH}e^{(v+\cosh(a)\tilde{v})_{\mathfrak{m}}}\\ &\quad\times e^{\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}+\frac{1}{4}\Omega_V(\cosh(a)\tilde{v},v)E_{\mathfrak{k}}+\frac{1}{2}\Omega_V(-\sinh(a)\tilde{v},v)E_{\mathfrak{m}}}K\\ &= e^{aH}e^{(v+\cosh(a)\tilde{v})_{\mathfrak{m}}}\\ &\quad\times e^{\left[\left(l+\frac{1}{2}\Omega_V(w,\cosh(a)\tilde{v})+\frac{1}{2}\Omega_V(v,\sinh(a)\tilde{v})\right)-\frac{1}{2}\Omega_V(w,v+\cosh(a)\tilde{v})\right]E_{\mathfrak{m}}}K\\ &= I^w\left(a,v+\cosh(a)\tilde{v},l+\frac{1}{2}\Omega_V(\cosh(a)w+\sinh(a)v,\tilde{v})\right) \end{aligned}$$

$$\begin{aligned} e^{\tilde{l}E_{\mathfrak{m}}}\cdot I^w(a,v,l) &= e^{aH}e^{e^{-a\operatorname{ad}H}\tilde{l}E_{\mathfrak{m}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K\\ &= e^{aH}e^{\cosh(2a)\tilde{l}E_{\mathfrak{m}}-\sinh(2a)\tilde{l}E_{\mathfrak{k}}}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}K\\ &= I^w(a,v,l+\cosh(2a)\tilde{l}). \end{aligned}$$

Now since

$$I^w(a,v,l) = e^{aH}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}\cdot I^w(0,0,0)$$

we have

$$\begin{aligned} s_{(0,0,0)}(a',v',l') &= \sigma\left(e^{aH}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}\right)K \\ &= I^w(-a',-v',-l') \end{aligned}$$

and

$$\begin{aligned} s_{(a,v,l)}(a',v',l') &= e^{aH}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}\cdot s_{(0,0,0)} \\ &\quad \cdot e^{-v_{\mathfrak{m}}-\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}e^{-aH}\cdot I^w(a',v',l') \\ &= e^{aH}e^{v_{\mathfrak{m}}+\left(l-\frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}}\cdot s_{(0,0,0)} \\ &\quad \cdot I^w\Big(a'-a, v'-\cosh(a'-a)v, \\ &\qquad l'-\frac{1}{2}\Omega_V(\cosh(a'-a)w+\sinh(a'-a)v',v) \\ &\qquad -\cosh(2(a'-a))\Big(l-\frac{1}{2}\Omega_V(w,v)\Big)\Big) \\ &= I^w\Big(2a-a', \\ &\qquad 2\cosh(a-a')v-v', \\ &\qquad 2\cosh(2(a-a'))l-l' \\ &\qquad +(\cosh(a-a')-\cosh(2(a-a')))\Omega_V(w,v) \\ &\qquad +\sinh(a-a')\Omega_V(v,v')\Big). \end{aligned}$$

□

Proposition 5.6.2. *Let $x_1=(a_1,v_1,l_1)$ and $x_2=(a_2,v_2,l_2)$ be two points in $M^{\lambda,\lambda w}$. The geodesic from x_1 to x_2 is $\gamma_{x_1,x_2}:[0,1]\to M^{\lambda,\lambda w}$ defined by*

$$\begin{aligned} \gamma_{x_1,x_2}(t) = \Bigg(& a_1+t(a_2-a_1), \\ & \frac{\sinh((1-t)(a_2-a_1))v_1+\sinh(t(a_2-a_1))v_2}{\sinh(a_2-a_1)}, \\ & \frac{\sinh(2(1-t)(a_2-a_1))l_1+\sinh(2t(a_2-a_1))l_2}{\sinh(2(a_2-a_1))} \\ & +\frac{1}{2}\Omega_V\left(w,\frac{\sinh((1-t)(a_2-a_1))v_1+\sinh(t(a_2-a_1))v_2}{\sinh(a_2-a_1)}\right) \\ & -\frac{1}{2}\Omega_V\left(w,\frac{\sinh(2(1-t)(a_2-a_1))v_1+\sinh(2t(a_2-a_1))v_2}{\sinh(2(a_2-a_1))}\right) \\ & -\frac{\sinh(t(a_2-a_1))\sinh((1-t)(a_2-a_1))}{\sinh(2(a_2-a_1))}\Omega_V(v_1,v_2)\Bigg) \end{aligned}$$

Proof. Compute the geodesics through o by computing $\exp(tX)\cdot o$ for $X \in \mathfrak{m}$. Then translate to x_1 by $s_{\frac{x_1}{2}}$, and finally solve for X so that the geodesic passes through x_2 at $t = 1$. □

Proposition 5.6.3. *Let $x = (a_1, v_1, l_1)$, $y = (a_2, v_2, l_2)$, $z = (a_3, v_3, l_3)$ be three points in $M^{\lambda,\lambda w}$. The equation $A = s_x s_y s_z A$ has a solution A in $M^{\lambda,\lambda w}$. Moreover, it is unique, and given by the formula*

$$\begin{aligned}
A = \Big(& a_1 - a_2 + a_3, \\
& \cosh(a_1 - a_2)v_3 - \cosh(a_3 - a_1)v_2 + \cosh(a_2 - a_3)v_1, \\
& \cosh(2(a_1 - a_2))l_3 - \cosh(2(a_3 - a_1))l_2 + \cosh(2(a_2 - a_3))l_1 \\
& + \frac{1}{2}(\cosh(a_1 - a_2) - \cosh(2(a_1 - a_2)))\Omega_V(w, v_3) \\
& - \frac{1}{2}(\cosh(a_3 - a_1) - \cosh(2(a_3 - a_1)))\Omega_V(w, v_2) \\
& + \frac{1}{2}(\cosh(a_2 - a_3) - \cosh(2(a_2 - a_3)))\Omega_V(w, v_1) \\
& + \frac{1}{2}\Omega_V(v_1, v_2)\sinh(2a_3 - a_1 - a_2) \\
& + \frac{1}{2}\Omega_V(v_3, v_1)\sinh(a_3 - a_1) \\
& - \frac{1}{2}\Omega_V(v_2, v_3)\sinh(2a_1 - a_2 - a_3) \Big)
\end{aligned}$$

Proof. The existence and uniqueness follow from Theorem 2.3.12 and the fact that $M^{\lambda,\lambda w}$ is solvable and exponential. The explicit solution is obtained by a long but straightforward computation. □

Proposition 5.6.4. *Let $o = (0, 0, 0)$ be the base point, and $x_2 = (a_2, v_2, l_2)$, $x_2 = (a_3, v_3, l_3)$ be any two points in $M^{\lambda,\lambda w}$. Then the symplectic area of the double triangle built on o, x_2 and x_3 is given by*

$$\begin{aligned}
S_W(o, x_2, x_3) = & \, 2\lambda\cosh(a_2)\cosh(a_3)\Omega_V(v_2, v_3) \\
& + \lambda(\sinh(2a_2)l_3 - \sinh(2a_3)l_2) \\
& + 2\lambda\Big(\sinh(a_2) - \frac{1}{2}\sinh(2a_2)\Big)\Omega_V(w, v_3) \\
& - 2\lambda\Big(\sinh(a_3) - \frac{1}{2}\sinh(2a_3)\Big)\Omega_V(w, v_2)
\end{aligned}$$

Proof. The symplectic form being constant in our coordinates, we may choose a global symplectic potential θ and use the Stokes theorem to compute the area of the double triangle by integrating the potential over the edges of the double

triangle. We choose $\theta_{(a,v,l)} = -2\lambda l da + \lambda(\theta_V)_v$ for some potential θ_V of Ω_V such that $(\theta_V)_v(w) - (\theta_V)_w(v) = \Omega_v(v,w)$.

The three vertices of the double triangle are given by

$$\begin{aligned}
A = \Big(& a_3 - a_2, \\
& \cosh(a_2)v_3 - \cosh(a_3)v_2, \\
& \cosh(2a_2)l_3 - \cosh(2a_3)l_2 \\
& + \frac{1}{2}(\cosh(a_2) - \cosh(2a_2))\Omega_V(w, v_3) \\
& - \frac{1}{2}(\cosh(a_3) - \cosh(2a_3))\Omega_V(w, v_2) \\
& + \frac{1}{2}\sinh(a_2 + a_3)\Omega_V(v_2, v_3)\Big) \\
B = & -A \\
C = \Big(& a_2 + a_3, \\
& \cosh(a_3)v_2 + \cosh(a_2)v_3, \\
& + \cosh(2a_2)l_3 + \cosh(2a_3)l_2 \\
& + \frac{1}{2}(\cosh(a_2) - \cosh(2a_2))\Omega_V(w, v_3) \\
& + \frac{1}{2}(\cosh(a_3) - \cosh(2a_3))\Omega_V(w, v_2) \\
& + \frac{1}{2}\sinh(a_2 - a_3)\Omega_V(v_2, v_3)\Big).
\end{aligned}$$

The three geodesics forming the edges are thus given by $\gamma_{A,B}, \gamma_{B,C}, \gamma_{C,A} : [0,1] \to M^{\lambda,\lambda w}$, with

$$\begin{aligned}
\gamma_{A,B}(t) = \Bigg(& (a_2 - a_3)(-1 + 2t), \\
& \frac{\sinh((2t-1)(a_2 - a_3))}{\sinh(a_2 - a_3)}(\cosh(a_3)v_2 - \cosh(a_2)v_3), \\
& \frac{\sinh((2t-1)(a_2 - a_3))}{\sinh(a_2 - a_3)}\Omega_V(w, \cosh(a_3)v_2 - \cosh(a_2)v_3) \\
& + \frac{\sinh((2t-1)2(a_2 - a_3))}{\sinh(2(a_2 - a_3))}\Big(\Omega_V(w, \cosh(2a_2)v_3 - \cosh(2a_3)v_2) \\
& - (\cosh(2a_2)l_3 - \cosh(2a_3)l_2) - \sinh(a_2 + a_3)\Omega_V(v_2, v_3)\Big)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\gamma_{B,C}(t) = \Big(& a_2 + a_3(2t-1), \\
& \cosh((1-2t)a_3)v_2 - \cosh(a_2)\frac{\sinh((1-2t)a_3)}{\sinh(a_3)}v_3, \\
& \frac{1}{2\sinh(2a_3)}\times \\
& \Big(-2\cosh(a_3)\sinh((1-2t)a_3)\sinh(a_2+(1-2t)a_3)\Omega_V(v_2,v_3) \\
& \quad + 2\Big(\sinh(2a_3)\cosh((1-2t)2a_3)l_2 - \cosh(2a_2)\sinh((1-2t)2a_3)l_3\Big) \\
& \quad + \frac{1}{2}\Big(\sinh((1+2t)a_3) + \sinh((1+2(1-t))a_3) \\
& \qquad - \sinh(4ta_3) - \sinh(4(1-t)a_3)\Big)\Omega_V(w,v_2) \\
& \quad + \Big(\cosh(a_2)(\sinh(2ta_3) - \sinh(2(1-t)a_3)) \\
& \qquad + \cosh(2a_2)\sinh((1-2t)2a_3)\Big)\Omega_V(w,v_3)\Big)\Big)
\end{aligned}
$$

$\gamma_{C,A}(t) =$ same expression as $\gamma_{B,C}(t)$ with $x_2 \leftrightarrow x_3$ and $t \to 1-t$.

Now we integrate the symplectic one-form θ on the edges. First, remark that the integral on $\gamma_{A,B}$ vanishes. Indeed, as $\gamma_{A,B}$ is a geodesic passing by o at $t = 1/2$ and $s_o = -\mathrm{Id}$, we have $\gamma_{A,B}(1-t) = -\gamma_{A,B}(t)$ so that $\dot\gamma_{A,B}(1-t) = \dot\gamma_{A,B}(t)$ for all $t \in [0,1]$. Thus

$$
\begin{aligned}
\int_0^1 \theta_{\gamma_{A,B}(t)}(\dot\gamma_{A,B}(t))dt &= \int_0^{1/2} \theta_{\gamma_{A,B}(t)}(\dot\gamma_{A,B}(t))dt + \int_{1/2}^1 \theta_{\gamma_{A,B}(t)}(\dot\gamma_{A,B}(t))dt \\
&= \int_0^{1/2} \theta_{\gamma_{A,B}(t)}(\dot\gamma_{A,B}(t))dt + \int_0^{1/2} \theta_{\gamma_{A,B}(1-s)}(\dot\gamma_{A,B}(1-s))ds \\
&= \int_0^{1/2} \theta_{\gamma_{A,B}(t)}(\dot\gamma_{A,B}(t))dt - \int_0^{1/2} \theta_{\gamma_{A,B}(s)}(\dot\gamma_{A,B}(s))ds \\
&= 0.
\end{aligned}
$$

Next, remark that the integral over $\gamma_{C,A}$ is equal to minus that over $\gamma_{B,C}$, with x_2 and x_3 interchanged, so that we only need to compute the integral over $\gamma_{B,C}$ and care for the terms which are antisymmetric in x_2 and x_3.

We have

$$
\begin{aligned}
&\int_0^1 \theta_{\gamma_{B,C}(t)}(\dot\gamma_{B,C}(t))dt \\
&\quad = \lambda \int_0^1 \Bigg[\frac{-1}{2\sinh(2a_3)}\times
\end{aligned}
$$

$$
\begin{aligned}
&\Big(-2\cosh(a_3)\sinh((1-2t)a_3)\sinh(a_2+(1-2t)a_3)\Omega_V(v_2,v_3) \\
&\quad + 2\sinh(2a_3)\cosh((1-2t)2a_3)l_2 \\
&\quad - 2\cosh(2a_2)\sinh((1-2t)2a_3)l_3 \\
&\quad + \frac{1}{2}\Big(\sinh((1+2t)a_3)+\sinh((1+2(1-t))a_3) \\
&\qquad - \sinh(4ta_3)-\sinh(4(1-t)a_3)\Big)\Omega_V(w,v_2) \\
&\quad + \Big(\cosh(a_2)(\sinh(2ta_3)-\sinh(2(1-t)a_3)) \\
&\qquad + \cosh(2a_2)\sinh((1-2t)2a_3)\Big)\Omega_V(w,v_3)\Big)\times 2a_3 \\
&+ \cosh((1-2t)a_3)(-2a_3)\sinh((1-2t)a_3)(\theta_V)_{v_2}(v_2) \\
&+ \cosh((1-2t)a_3)2a_3\cosh(a_2)\frac{\cosh((1-2t)a_3)}{\sinh(a_3)}(\theta_V)_{v_2}(v_3) \\
&- \cosh(a_2)\frac{\sinh((1-2t)a_3)}{\sinh(a_3)}(-2a_3)\sinh((1-2t)a_3)(\theta_V)_{v_3}(v_2) \\
&- \cosh(a_2)\frac{\sinh((1-2t)a_3)}{\sinh(a_3)}2a_3\cosh(a_2)\frac{\cosh((1-2t)a_3)}{\sinh(a_3)}(\theta_V)_{v_3}(v_3)\Big]dt \\
=& -2\lambda\Big(\sinh(a_3)-\frac{1}{2}\sinh(2a_3)\Big)\Omega_V(w,v_2)-\lambda\sinh(2a_3)l_2 \\
&+ \lambda\Omega_V(v_2,v_3)\cosh(a_2)\cosh(a_3) \\
&+ \lambda\cosh(a_2)\cosh(a_3)\Big((\theta_V)_{v_2}(v_3)+(\theta_V)_{v_3}(v_2)\Big).
\end{aligned}
$$

Putting everything together, we get the result. □

Ševera area

Definition 5.6.5. *Let (M,s,ω) be a symplectic symmetric space whose canonical map ϕ is a global diffeomorphism. The* Ševera phase function *$\check{S}$ is defined by*

$$\check{S}(x,y,z) = 2\omega_x(\phi_x(y),\phi_x(z)).$$

Lemma 5.6.6. *The Ševera phase function is invariant under the diagonal action of* $\mathrm{Aut}(M,s,\omega)$.

Proof. Let $g\in\mathrm{Aut}(M,s,\omega)$ and $x,y,z\in M$. Since ω is invariant and ϕ is equivariant, we get

$$
\begin{aligned}
\check{S}(gx,gy,gz) &= 2\omega_{gx}(\phi_{gx}(gy)\phi_{gx}(gz)) \\
&= 2\omega_x(g^{-1}g\phi_x y, g^{-1}g\phi_x z) \\
&= \check{S}(x,y,z).
\end{aligned}
$$

□

In our case, we compute in the chart I^w:

$$\begin{aligned}
\check{S}(o,x_2,x_3) &= 2\Omega^{\lambda,0}(\phi_o(x_2),\phi_o(x_3)) \\
&= 2\lambda\Big(\frac{1}{2}\sinh(2a_2)\,l_3 - \frac{1}{2}\sinh(2a_3)\,l_2 + \Omega_V(\cosh(a_2)\,v_2,\cosh(a_3)\,v_3) \\
&\quad + \Omega_V\Big(\cosh(a_2)\,v_2,\Big(\sinh(a_3)-\frac{1}{2}\sinh(2a_3)\Big)w\Big) \\
&\quad - \Omega_V\Big(\cosh(a_3)\,v_3,\Big(\sinh(a_2)-\frac{1}{2}\sinh(2a_2)\Big)w\Big)\Big) \\
&= \lambda\Big(\sinh(2a_2)\,l_3 - \sinh(2a_3)\,l_2 + 2\cosh(a_2)\cosh(a_3)\,\Omega_V(v_2,v_3) \\
&\quad - 2\cosh(a_2)\Big(\sinh(a_3)-\frac{1}{2}\sinh(2a_3)\Big)\Omega_V(w,v_2) \\
&\quad + 2\cosh(a_3)\Big(\sinh(a_2)-\frac{1}{2}\sinh(2a_2)\Big)\Omega_V(w,v_3)\Big)
\end{aligned}$$

and we check that this function does not satisfy the relation

$$\check{S}(o,x_2,s_{x_2}x_3) = -\check{S}(o,x_2,x_3)$$

due to the factors $\cosh(a_2)$ and $\cosh(a_3)$ in the last two lines. Indeed, for example

$$\begin{aligned}
\big(\phi^*\Omega^{\lambda,0}\big)((a,0,0),(0,v,0)) &= -\lambda\Big(\sinh(a)-\frac{1}{2}\sinh(2a)\Big)\Omega_V(v,w) \\
\big(\phi^*\Omega^{\lambda,0}\big)\big((a,0,0),s_{(a,0,0)}(0,v,0)\big) &= \big(\phi^*\Omega^{\lambda,0}\big)((a,0,0),(2a,-v,0)) \\
&= -\lambda\cosh(2a)\Big(\sinh(a)-\frac{1}{2}\sinh(2a)\Big)\Omega_V(-v,w)
\end{aligned}$$

In contrast, the Weinstein phase function is equal to the above function, except that it doesn't have these cosh factors. It has been shown in [Bie02] to be admissible (whenever defined) for any symplectic symmetric space.

5.7 Amplitudes

As we mentioned in the general introduction, as a byproduct of his heuristic argument leading to the area of the double triangle as a phase candidate, Weinstein in [Wei94] also conjectured a formula for an amplitude in an invariant

WKB product:

$$A_W(x,y,z) = \frac{1}{\sqrt{\det\left(\mathrm{Id} - (s_x s_y s_z)_{*A}\right)}}.$$

Proposition 5.7.1. *In $M^{\lambda,\lambda w}$, we have $A_W \equiv \frac{1}{2^n}$, where $2n = \dim M^{\lambda,\lambda w}$.*

Proof. From the expression of the symmetries computed in Proposition 5.6.1, we observe that, for all $x, y \in M^{\lambda,\lambda w}$,

$$(s_x)_{*y} = \begin{pmatrix} -1 & 0 & 0 \\ * & -\mathrm{Id} & 0 \\ * & * & -1 \end{pmatrix},$$

hence the same holds for a product of three such matrices, so that the determinant in the expression of A_W is equal to $2^{\dim(M^{\lambda,\lambda w})} = 2^{2n}$. □

We will see in the next section that this is not the amplitude appearing in our product. Moreover, in the case $w = 0$, all possible amplitudes were computed in [Bie07] and none of them is constant. A corresponding result holds in the other cases (with $w \neq 0$) so that A_W is not a good candidate in general.

5.8 Product

As in section 5.1, define the symplectic Fourier transform on the symplectic vector space $(\mathfrak{m}, \Omega = \Omega^{\lambda,0})$ (that is, on the symplectic symmetric space $(M, \omega^{\lambda,\lambda w})$ seen through the global Darboux chart $I^w : (\mathfrak{m}, \Omega^{\lambda,0}) \to (M, \omega^{\lambda,\lambda w})$) by

$$(\mathcal{F}^\Omega u)(v) = \frac{1}{(2\pi)^n} \int_{\mathfrak{m}} e^{-i\Omega(v,w)} u(w) d^\Omega w, \tag{5.5}$$

and define the symplectic Frourier transform with parameter ϵ by

$$\mathcal{F}^\Omega_\epsilon = \mathcal{F}^{\Omega/\epsilon}.$$

Define the operator T_ϵ by

$$T_\epsilon = \mathcal{F}^\Omega_\epsilon \circ \phi^* \circ \mathcal{F}^\Omega_\epsilon,$$

where $\phi : \mathfrak{m} \to \mathfrak{m}$ is the canonical projection expressed in the Darboux coordinates I^w.

Denote $\rho_\nu(X) = \frac{1}{2\nu}[\mu_X, \cdot]_*$, where $\nu = \frac{\epsilon}{2i}$, and $* = *^M_\nu$ is the Moyal product defined in Equation (5.3). Let us first check that the operator $T_{\epsilon/2}$ will map the

action ρ_ν by star commutator, to the action by fundamental vector fields.

By antisymmetry of Ω^{ij}, we have

$$\begin{aligned}\rho_\nu(X)u &= \frac{1}{2\nu}\left(\mu_X *_\nu^M u - u *_\nu^M \mu_X\right)\\ &= \frac{1}{2\nu}\sum_{k=1}^{\infty}\frac{\nu^k}{k!}\sum_{\substack{i_1\dots i_k\\ j_1\dots j_k}}\Omega^{i_1 j_1}\dots\Omega^{i_k j_k}\left(\partial_{i_1\dots i_k}\mu_X\,\partial_{j_1\dots j_k}u - \partial_{i_1\dots i_k}u\,\partial_{j_1\dots j_k}\mu_X\right)\\ &= \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\nu^{2k+1}}{(2k+1)!}\sum_{\substack{i_1\dots i_{2k+1}\\ j_1\dots j_{2k+1}}}\Omega^{i_1 j_1}\dots\Omega^{i_{2k+1}j_{2k+1}}\partial_{i_1\dots i_{2k+1}}\mu_X\,\partial_{j_1\dots j_{2k+1}}u.\end{aligned}$$

Note that the non-zero components (for $\Omega = \Omega^{\lambda,0}$) of $\Omega_{\cdot\cdot}$ and $\Omega^{\cdot\cdot}$ (defined by $\Omega_{ij}\Omega^{kj} = \delta_i^k$) are

$$\begin{aligned}\Omega_{al} &= 2\lambda = -\Omega_{la} & \Omega_{v_i v_j} &= \lambda(\Omega_V)_{ij}\\ \Omega^{al} &= \frac{1}{2\lambda} = -\Omega^{la} & \Omega^{v_i v_j} &= \frac{1}{\lambda}(\Omega_V)^{ij}.\end{aligned}$$

Computation on $\mathbb{R}H$. The only non-zero derivative of λ_H being $\partial_l\mu_H = 2\lambda$, we get

$$\begin{aligned}\rho_\nu(H)u &= \frac{1}{\nu}\nu\Omega^{la}\partial_l\mu_H\,\partial_a u\\ &= -\partial_a u.\end{aligned}$$

Computation on $V_{\mathfrak{m}}$. The non-zero derivatives of $\mu_{\nu_{\mathfrak{m}}}$ being

$$\begin{aligned}\partial_a^{2k+1}\mu_{\nu_{\mathfrak{m}}} &= -\mu\Omega\left(\sinh(a)v + \cosh(a)w, \nu\right)\\ \partial_a^{2k}\mu_{\nu_{\mathfrak{m}}} &= -\lambda\Omega\left(\cosh(a)v + \sinh(a)w, \nu\right)\\ \partial_a^{2k+1}\partial_{v^i}\mu_{\nu_{\mathfrak{m}}} &= -\lambda\sinh(a)(\Omega_V)_{ij}\nu^j\\ \partial_a^{2k}\partial_{v^i}\mu_{\nu_{\mathfrak{m}}} &= -\lambda\cosh(a)(\Omega_V)_{ij}\nu^j.\end{aligned}$$

the representation reads ($2n = \dim V$)

$$\begin{aligned}\rho_\nu(\nu_{\mathfrak{m}})u = \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\nu^{2k+1}}{(2k+1)!}\Bigg[&\left(\Omega^{al}\right)^{2k+1}\partial_a^{2k+1}\mu_{\nu_{\mathfrak{m}}}\,\partial_l^{2k+1}u\\ &+(2k+1)\sum_{i,j=1}^{2n}\left(\Omega^{al}\right)^{2k}(\Omega_V)_{im}\partial_a^{2k}\partial_{v^i}\mu_{\nu_{\mathfrak{m}}}\,\partial_l^{2k}\partial_{v^m}u\Bigg]\end{aligned}$$

$$= \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\nu^{2k+1}}{(2k+1)!}\Big[\lambda\Omega_V(-\sinh(a)v-\cosh(a)w,v)\Big(\frac{1}{2\lambda}\Big)^{2k+1}\,\partial_l^{2k+1}u$$

$$+(2k+1)(-\lambda)\cosh(a)$$

$$\times\sum_{i,j,m=1}^{2n}(\Omega_V)_{ij}\,v^j\Big(\frac{1}{2\lambda}\Big)^{2k}\frac{1}{\lambda}(\Omega_V)^{im}\,\partial_l^{2k}\partial_{v^m}u\Big]$$

$$= -\frac{\lambda}{\nu}\Omega_V(\sinh(a)v+\cosh(a)w,v)\sinh\Big(\frac{\nu}{2\lambda}\partial_l\Big)u$$

$$-\cosh(a)v^i\partial_{v^i}\cosh\Big(\frac{\nu}{2\lambda}\partial_l\Big)u.$$

Computation on $\mathbb{R}E_{\mathfrak{m}}$. Finally, with

$$\partial_a^{2k+1}\mu_{E_{\mathfrak{m}}} = 2^{2k+1}\,(-\cosh(2a))$$

we get

$$\rho_\nu(E)u = \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\nu^{2k+1}}{(2k+1)!}\left(\Omega^{al}\right)^{2k+1}\partial_a^{2k+1}\mu_E\,\partial_l^{2k+1}u$$

$$= \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\nu^{2k+1}}{(2k+1)!}\Big(\frac{1}{2\lambda}\Big)^{2k+1}2^{2k+1}\lambda\,(-\cosh(2a))\,\partial_l^{2k+1}u$$

$$= -\cosh(2a)\frac{\lambda}{\nu}\sinh\Big(\frac{\nu}{\lambda}\partial_l\Big)u.$$

The representation of $\mathfrak{g}$ by fundamental vector fields is the zeroth order in ν in the above expressions, so that we have

$$H^* = -\partial_a$$
$$\nu_{\mathfrak{m}}^* = -\frac{1}{2}\Omega_V(\sinh(a)v+\cosh(a)w,v)\partial_l - \cosh(a)v^k\partial_{v^k}$$
$$E^* = -\cosh(2a)\partial_l.$$

In what follows, we will take $\nu = \frac{\varepsilon}{2i}$ with ε real (in physics, ε stands for $\hbar$), and we will denote $\rho := \rho_{\nu=\frac{\varepsilon}{2i}}$, $\mathcal{F} = \mathcal{F}^{\Omega}_{\epsilon/2}$.

We now compute the symplectic Fourier transform with parameter of these two representations. Note that we have

$$\mathcal{F}\circ\partial_{x^k}\circ\mathcal{F} = m_{-\frac{2i}{\epsilon}\Omega_{kl}x^l}$$
$$\mathcal{F}\circ m_{x^k}\circ\mathcal{F} = -\frac{\epsilon}{2i}\Omega^{kl}\partial_{x^l}.$$

Thus we get

$$\mathcal{F}\circ\rho(H)\circ\mathcal{F}=m_{\frac{4i\lambda}{\epsilon}l}$$

$$\begin{aligned}\mathcal{F}\circ\rho(v_{\mathfrak{m}})\circ\mathcal{F}&=\left(\mathcal{F}\circ m_{-\frac{2i\lambda}{\epsilon}\Omega_V(\sinh(a)v+\cosh(a)w,v)}\circ\mathcal{F}\right)\circ\left(\mathcal{F}\circ\sinh\left(\frac{\epsilon}{4i\lambda}\partial_l\right)\circ\mathcal{F}\right)\\&\quad-\left(\mathcal{F}\circ m_{\cosh(a)}\circ\mathcal{F}\right)\circ\left(\mathcal{F}\circ v^k\partial_{v^k}\circ\mathcal{F}\right)\circ\left(\mathcal{F}\circ\cosh\left(\frac{\epsilon}{4i\lambda}\partial_l\right)\circ\mathcal{F}\right)\\&=-\frac{2i\lambda}{\epsilon}\Bigg[\sinh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)(\Omega_V)_{jk}\left(-\frac{\epsilon}{2i\lambda}(\Omega_V)^{jl}\partial_{v^l}\right)v^k\\&\qquad+\cosh\left(\frac{\epsilon}{4i\lambda}\partial_l\right)\circ m_{\Omega_V(w,v)}\Bigg]\circ m_{\sinh(a)}\\&\quad-\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\circ m_{v^j\left(-\frac{2i\lambda}{\epsilon}(\Omega_V)_{jk}v^k\right)}\circ m_{\cosh(a)}\\&=\left[\sinh\left(-\frac{\epsilon}{2i\lambda}\partial_l\right)v^k\partial_{v^k}-m_{\frac{2i\lambda}{\epsilon}\Omega_V(w,v)}\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\right]\circ m_{\sinh(a)}\\&\quad-\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\circ m_{-\frac{2i\lambda}{\epsilon}\cosh(a)\Omega_V(v,v)}\end{aligned}$$

$$\mathcal{F}\circ\rho(E)\circ\mathcal{F}=-\frac{2i\lambda}{\epsilon}\cosh\left(-\frac{\epsilon}{2i\lambda}\partial_l\right)\circ m_{\sinh(2a)}$$

and on the other hand

$$\mathcal{F}\circ H^*\circ\mathcal{F}=m_{\frac{4i\lambda}{\epsilon}l}$$

$$\begin{aligned}\mathcal{F}\circ v_{\mathfrak{m}}^*\circ\mathcal{F}&=\left[\frac{\epsilon}{4i\lambda}\sinh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)v^k\partial_{v^k}-\frac{1}{2}m_{\Omega_V(w,v)}\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\right]\circ m_{\frac{4i\lambda}{\epsilon}a}\\&\quad-\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\circ m_{-\frac{2i\lambda}{\epsilon}\Omega_V(v,v)}\end{aligned}$$

$$\mathcal{F}\circ E^*\circ\mathcal{F}=\cosh\left(-\frac{\epsilon}{2i\lambda}\partial_l\right)m_{-\frac{4i\lambda}{\epsilon}a}.$$

Now we will check that $\phi^*\circ(\mathcal{F}\circ X^*\circ\mathcal{F})\circ{\phi^{-1}}^*$ is equal to $\mathcal{F}\circ\rho(X)\circ\mathcal{F}$ for all $X\in\mathfrak{m}$.

We have

$${\phi^{-1}}_{*_{\phi(a,v,l)}}=\left(\phi_{*(a,v,l)}\right)^{-1}$$

$$
\begin{aligned}
&= \begin{pmatrix} \cosh(2a) & 0 & 0 \\ \sinh(a)v + (\cosh(a) - \cosh(2a))w & \cosh(a).\,\mathrm{Id} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{\cosh(2a)} & 0 & 0 \\ -\frac{\cosh(a)}{\cosh(2a)}\left(\sinh(a)v + (\cosh(a) - \cosh(2a))w\right) & \frac{1}{\cosh(a)}.\,\mathrm{Id} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

so that

$$
\begin{aligned}
\phi^* \circ \partial_a \circ \phi^{-1^*} &= \frac{1}{\cosh(2a)}\partial_a - \frac{\cosh(a)}{\cosh(2a)}\left(\sinh(a)v^k + (\cosh(a) - \cosh(2a))w^k\right)\partial_{v^k} \\
\phi^* \circ \partial_{v^k} \circ \phi^{-1^*} &= \frac{1}{\cosh(a)}\partial_{v^k} \\
\phi^* \circ \partial_l \circ \phi^{-1^*} &= \partial_l,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\phi^* \circ (\mathcal{F} \circ H^* \circ \mathcal{F}) \circ \phi^{-1^*} &= \phi^* \circ m_{\frac{4i\lambda}{\epsilon} l} \circ \phi^{-1^*} \\
&= m_{\frac{4i\lambda}{\epsilon} l} \\
&= \mathcal{F} \circ \rho(H) \circ \mathcal{F}
\end{aligned}
$$

$$
\begin{aligned}
&\phi^* \circ (\mathcal{F} \circ \nu_{\mathfrak{m}}^* \circ \mathcal{F}) \circ \phi^{-1^*} \\
&= \left(\frac{\epsilon}{4i\lambda}\sinh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)v^k \frac{1}{\cosh(a)}\partial_{v^k} - \frac{1}{2}\Omega_V(w,v)\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\right) \\
&\qquad \circ m_{\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2a)} \\
&\quad - \cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right) \circ m_{-\frac{2i\lambda}{\epsilon}\Omega_V\left(v,\cosh(a)v + \left(\sinh(a) - \frac{1}{2}\sinh(2a)\right)w\right)} \\
&= \sinh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)v^k \frac{1}{\cosh(a)}\sinh(a)\cosh(a)\partial_{v^k} \\
&\quad - \frac{1}{2}\Omega_V(w,v)\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2a)\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right) \\
&\quad + \cosh(a)\frac{2i\lambda}{\epsilon}\Omega_V(v,v)\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right) \\
&\quad - \Omega_V(w,v)\frac{2i\lambda}{\epsilon}\left(\sinh(a) - \frac{1}{2}\sinh(2a)\right)\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right) \\
&= \left(\sinh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)v^k\partial_{v^k} - m_{\frac{2i\lambda}{\epsilon}\Omega_V(w,v)}\cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right)\right) \circ m_{\sinh(a)} \\
&\quad - \cosh\left(-\frac{\epsilon}{4i\lambda}\partial_l\right) \circ m_{-\frac{2i\lambda}{\epsilon}\cosh(a)\Omega_V(v,v)} \\
&= \mathcal{F} \circ \rho(\nu_{\mathfrak{m}}) \circ \mathcal{F}.
\end{aligned}
$$

$$\begin{aligned}\phi^* \circ (\mathcal{F} \circ E^* \circ \mathcal{F}) \circ \phi^{-1*} &= \phi^* \circ \cosh\left(-\frac{\epsilon}{2i\lambda}\partial_l\right) m_{-\frac{4i\lambda}{\epsilon}a} \circ \phi^{-1*} \\ &= \cosh\left(-\frac{\epsilon}{2i\lambda}\partial_l\right) m_{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2a)} \\ &= \mathcal{F} \circ \rho(E) \circ \mathcal{F}.\end{aligned}$$

We have thus shown that $T_{\epsilon/2}$ intertwines the representations $X \mapsto \rho(X)$ and $X \mapsto X^*$:

$$T_{\epsilon/2} \circ X^* = \rho_{\epsilon/2i}(X) \circ T_{\epsilon/2} \quad \text{for all } X \in \mathfrak{m}.$$

Hence applying this intertwiner to the Weyl product should lead to an associative product invariant under the transvection group. This is indeed the case, as shows the following theorem.

Theorem 5.8.1. *The product $*$ on $(M, s, \omega = \omega^{\lambda,\lambda w})$ defined by*

$$f * g = (T_{\epsilon/2})^{-1}\left(T_{\epsilon/2} f *^M_{\epsilon/2} T_{\epsilon/2} g\right)$$

is an $\mathrm{Aut}(M, s, \omega^{\lambda,\lambda w})$-invariant associative WKB product in the sense of [Bie02] given, in the Darboux coordinates $I^w : (\mathfrak{m}, \Omega^{\lambda,0}) \to (M, \omega^{\lambda,\lambda w})$, by

$$f * g(x_0) = \frac{1}{(\pi\epsilon)^{\dim M}} \int_{M\times M} e^{\frac{2i}{\epsilon}S(x_0,x_1,x_2)} A(x_0,x_1,x_2) f(x_1) g(x_2) d^{\Omega}x_1 d^{\Omega}x_2$$

where the phase S and the amplitude A are given by

$$\begin{aligned}S(x_0,x_1,x_2) = \lambda \bigoplus_{0,1,2} \Big(&\Omega_V(\cosh(a_2-a_0)v_1, \cosh(a_0-a_1)v_2) \\ &- \sinh(2(a_1-a_2))\, z_0 \\ &- \Big(\sinh(a_1-a_2) - \frac{1}{2}\sinh(2(a_1-a_2))\Big)\Omega_V(w, v_0)\Big) \\ A(x_0,x_1,x_2) = \cosh(2(a_2-a_1))&\cosh(a_2-a_1)^{2n-2}\end{aligned}$$

Moreover the assignment $\Omega \mapsto *^{\Omega}$ is equivariant with respect to the actions of $\mathrm{Aut}(M, s)$.

Proof. Remember that $\Omega = \Omega^{\lambda,0}$ and $d^{2\Omega/\epsilon}x = \left(\frac{2\lambda}{\epsilon}\right)^n d^{\Omega^{1,0}}x = 2\left(\frac{2\lambda}{\epsilon}\right)^n da\, d^{\Omega_V} v\, dl$. We compute

$$\begin{aligned}Tf(a,v,l) &= \frac{1}{(2\pi)^n}\int e^{-\frac{2i}{\epsilon}\Omega(x,x')}(\mathcal{F}f)(\phi(x'))d^{2\Omega/\epsilon}x' \\ &= \left(\frac{\lambda}{\pi\epsilon}\right)^{2n}\int e^{-\frac{2i}{\epsilon}\Omega(x,x')} e^{-\frac{2i}{\epsilon}\Omega(\phi(x'),x'')} f(x'') d^{\Omega^{1,0}}x'\, d^{\Omega^{1,0}}x''\end{aligned}$$

$$= 4\left(\frac{\lambda}{\pi\epsilon}\right)^{2n} \int e^{-\frac{2i\lambda}{\epsilon}(2(al'-a'l)+\Omega_V(v,v'))}$$
$$\times e^{-\frac{2i\lambda}{\epsilon}2\left(\frac{1}{2}\sinh(2a')l''-a''l'\right)}$$
$$\times e^{-\frac{2i\lambda}{\epsilon}\Omega_V\left(\cosh(a')v'+\left(\sinh(a')-\frac{1}{2}\sinh(2a')\right)w,v''\right)}$$
$$\times f(a'',v'',l'')\,da'\,d^{\Omega_V}v'\,dl'\,da''\,d^{\Omega_V}v''\,dl''$$

$(l' \to \tilde{l}' = \frac{4\lambda}{\epsilon}l',\ v' \to \tilde{v}' = \frac{2\lambda}{\epsilon}v',\ v'' \to \tilde{v}'' = \cosh(a')v'')$

$$= \frac{2\lambda}{\pi\epsilon}\frac{1}{(2\pi)^{2n-1}} \int e^{-i(a-a'')\tilde{l}'} e^{-i\Omega_V(v-\tilde{v}'',\tilde{v}')} e^{-\frac{4i\lambda}{\epsilon}\left(\frac{1}{2}\sinh(2a')l''-a'l\right)}$$
$$\times e^{-\frac{2i\lambda}{\epsilon}\Omega_V\left(\frac{\sinh(a')-\frac{1}{2}\sinh(2a')}{\cosh(a')}w,\tilde{v}''\right)}\frac{1}{\cosh(a')^{2n-2}}$$
$$\times f\left(a'',\frac{\tilde{v}''}{\cosh(a')},l''\right)da'\,d^{\Omega_V}\tilde{v}'\,d\tilde{l}'\,da''\,d^{\Omega_V}\tilde{v}''\,dl''$$
$$= \frac{2\lambda}{\pi\epsilon}\int e^{-\frac{2i\lambda}{\epsilon}\left(2\left(\frac{1}{2}\sinh(2a')l''-a'l\right)+\frac{\sinh(a')-\frac{1}{2}\sinh(2a')}{\cosh(a')}\Omega_V(w,v)\right)}$$
$$\times \frac{1}{\cosh(a')^{2n-2}} f\left(a,\frac{v}{\cosh(a')},l''\right)da'dl''$$

and

$$T^{-1}f(a,v,l) = \left(\mathcal{F}\circ\phi^{-1^*}\circ\mathcal{F}\right)(f)(x)$$
$$= \left(\frac{\lambda}{\pi\epsilon}\right)^{2n}\int e^{-\frac{2i}{\epsilon}\Omega(x,x')}e^{-\frac{2i}{\epsilon}\Omega(\phi^{-1}(x'),x'')}f(x'')d^{\Omega^{1,0}}x'\,d^{\Omega^{1,0}}x''$$
$$= \left(\frac{\lambda}{\pi\epsilon}\right)^{2n}\int e^{-\frac{2i\lambda}{\epsilon}\Omega^{1,0}(x,\phi(x'))}e^{-\frac{2i\lambda}{\epsilon}\Omega^{1,0}(x',x'')}$$
$$\times \mathrm{Jac}(\phi)_{x'}\,f(x'')\;d^{\Omega^{1,0}}x'\;d^{\Omega^{1,0}}x''$$
$$= 4\left(\frac{\lambda}{\pi\epsilon}\right)^{2n}\int e^{-\frac{2i\lambda}{\epsilon}\left(2\left(al'-\frac{1}{2}\sinh(2a')l\right)+\Omega_V\left(v,\cosh(a')v'+\left(\sinh(a')-\frac{1}{2}\sinh(2a')\right)w\right)\right)}$$
$$\times e^{-\frac{2i\lambda}{\epsilon}(2(a'l''-a''l')+\Omega_V(v',v''))}\cosh(2a')\cosh(a')^{2n-2}$$
$$\times f(a,\cosh(a')v,l'')\;da'\;d^{\Omega_V}v'\;dl'\;da''\;d^{\Omega_V}v''\;dl''$$

$$(v' \to \tilde{v}' = \tfrac{2\lambda}{\epsilon} v',\, l' \to \tilde{l}' = \tfrac{4\lambda}{\epsilon} l',\, \int e^{-i\Omega_V(v',v''-\cosh(a')v)} dv',\, \int e^{-il'(a-a'')} dl')$$

$$= \frac{2\lambda}{\pi\epsilon} \int e^{-\frac{2i\lambda}{\epsilon}\left(2\left(a'l''-\frac{1}{2}\sinh(2a')l\right)+\left(\sinh(a')-\frac{1}{2}\sinh(2a')\right)\Omega_V(v,w)\right)}$$
$$\times \cosh(2a') \cosh(a')^{2n-2} f(a, \cosh(a')v, l'')\, da'\, dl''$$

Then

$$(Tf * Tg)(a_0, v_0, l_0)$$
$$= \frac{1}{(\pi\epsilon)^{2n}} \int e^{\frac{2i}{\epsilon}\Omega^{\lambda,0}(x_1-x_0, x_2-x_0)} (Tf)(x_1)(Tg)(x_2)\, d^{\Omega^{\lambda,0}} x_1\; d^{\Omega^{\lambda,0}} x_2$$
$$= \frac{1}{(\pi\epsilon)^{2n}} \int e^{\frac{2i\lambda}{\epsilon}(2((a_1-a_0)(l_2-l_0)-(a_2-a_0)(l_1-l_0))+\Omega_V(v_1-v_0, v_2-v_0))}$$
$$\times \frac{2\lambda}{\pi\epsilon} e^{-\frac{4i\lambda}{\epsilon}\left(\frac{1}{2}\sinh\left(2a_1'\right)-a_1'l_1\right)} e^{-\frac{2i\lambda}{\epsilon}\frac{\sinh\left(a_1'\right)-\frac{1}{2}\sinh\left(2a_1'\right)}{\cosh\left(a_1'\right)}\Omega_V(w, v_1)}$$
$$\times \frac{1}{\cosh\left(a_1'\right)^{2n-2}} f\left(a_1, \frac{v_1}{\cosh\left(a_1'\right)}, l_1''\right) da_1'\; dl_1''$$
$$\times \frac{2\lambda}{\pi\epsilon} e^{-\frac{4i\lambda}{\epsilon}\left(\frac{1}{2}\sinh\left(2a_2'\right)-a_2'l_2\right)} e^{-\frac{2i\lambda}{\epsilon}\frac{\sinh\left(a_2'\right)-\frac{1}{2}\sinh\left(2a_2'\right)}{\cosh\left(a_2'\right)}\Omega_V(w, v_2)}$$
$$\times \frac{1}{\cosh\left(a_2'\right)^{2n-2}} g\left(a_2, \frac{v_2}{\cosh\left(a_2'\right)}, l_2''\right) da_2'\; dl_2''$$
$$\times 2\lambda^n\, da_1\; d^{\Omega_V} v_1\; dl_1\; 2\lambda^n\, da_2\; d^{\Omega_V} v_2\; dl_2$$

$$(l_1 \to \tilde{l}_1 = \tfrac{4\lambda}{\epsilon} l_1,\, l_2 \to \tilde{l}_2 = \tfrac{4\lambda}{\epsilon} l_2,\, \int e^{i\left(a_1'-(a_2-a_0)\right)\tilde{l}_1}\, d\tilde{l}_1\,,\, \int e^{i\left(a_2'+(a_1-a_0)\right)\tilde{l}_2}\, d\tilde{l}_2\,)$$

$$= 4\left(\frac{\lambda}{\pi\epsilon}\right)^{2n} \int e^{\frac{2i\lambda}{\epsilon}(2(a_2-a_1)l_0+\Omega_V(v_1-v_0, v_2-v_0))}$$
$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_2-a_0))l_1''} e^{-\frac{2i\lambda}{\epsilon}\frac{\sinh(a_2-a_0)-\frac{1}{2}\sinh(2(a_2-a_0))}{\cosh(a_2-a_0)}\Omega_V(w, v_1)}$$
$$\times \frac{1}{\cosh(a_2-a_0)^{2n-2}} f\left(a_1, \frac{v_1}{\cosh(a_2-a_0)}, l_1''\right) dl_1''$$
$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_0-a_1))l_2''} e^{-\frac{2i\lambda}{\epsilon}\frac{\sinh(a_0-a_1)-\frac{1}{2}\sinh(2(a_0-a_1))}{\cosh(a_0-a_1)}\Omega_V(w, v_2)}$$
$$\times \frac{1}{\cosh(a_0-a_1)^{2n-2}} g\left(a_2, \frac{v_2}{\cosh(a_0-a_1)}, l_2''\right) dl_2''$$

$$\times\, da_1\ d^{\Omega_V} v_1\ da_2\ d^{\Omega_V} v_2$$

$(v_2 \to \tilde{v}_2 = \frac{v_2}{\cosh(a_0-a_1)},\ v_1 \to \tilde{v}_1 = \frac{v_1}{\cosh(a_2-a_0)})$

$$= 4\left(\frac{\lambda}{\pi\epsilon}\right)^{2n} \int e^{\frac{2i\lambda}{\epsilon}(2(a_2-a_1)l_0+\Omega_V(\cosh(a_2-a_0)v_1-v_0,\cosh(a_0-a_1)v_2-v_0))}$$

$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_2-a_0))l_1''}$$

$$\times e^{-\frac{2i\lambda}{\epsilon}\left(\sinh(a_2-a_0)-\frac{1}{2}\sinh(2(a_2-a_0))\right)\Omega_V(w,v_1)}$$

$$\times f(a_1, v_1, l_1'')\ dl_1''$$

$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_0-a_1))l_2''}$$

$$\times e^{-\frac{2i\lambda}{\epsilon}\left(\sinh(a_0-a_1)-\frac{1}{2}\sinh(2(a_0-a_1))\right)\Omega_V(w,v_2)}$$

$$\times g(a_2, v_2, l_2'')\ dl_2''$$

$$\times da_1\ d^{\Omega_V} v_1\ da_2\ d^{\Omega_V} v_2$$

Then

$$T^{-1}(Tf * Tg)(a_0, v_0, l_0)$$

$$= 2\frac{\lambda}{\pi\epsilon} \int e^{-\frac{2i\lambda}{\epsilon}\left(2\left(a_0' l_0''-\frac{1}{2}\sinh\left(2a_0'\right)l_0\right)+\left(\sinh\left(a_0'\right)-\frac{1}{2}\sinh\left(2a_0'\right)\right)\Omega_V(v_0,w)\right)}$$

$$\times \cosh(2a_0')\cosh(a_0')^{2n-2}(Tf * Tg)(a_0, \cosh(a_0')v_0, l_0'')\, da_0' dl_0''$$

$$= 2\frac{\lambda}{\pi\epsilon} \int e^{-\frac{2i\lambda}{\epsilon}\left(2\left(a_0' l_0''-\frac{1}{2}\sinh\left(2a_0'\right)l_0\right)+\left(\sinh\left(a_0'\right)-\frac{1}{2}\sinh\left(2a_0'\right)\right)\Omega_V(v_0,w)\right)}$$

$$\times \cosh(2a_0')\cosh(a_0')^{2n-2}$$

$$\times 4\left(\frac{\lambda}{\pi\epsilon}\right)^{2n} e^{\frac{2i\lambda}{\epsilon}\left(2(a_2-a_1)l_0''\right)}$$

$$\times e^{\frac{2i\lambda}{\epsilon}\Omega_V(\cosh(a_2-a_0)v_1-\cosh(a_0')v_0,\cosh(a_0-a_1)v_2-\cosh(a_0')v_0)}$$

$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_2-a_0))l_1''} e^{-\frac{2i\lambda}{\epsilon}\left(\sinh(a_2-a_0)-\frac{1}{2}\sinh(2(a_2-a_0))\right)\Omega_V(w,v_1)}$$

$$\times f(a_1, v_1, l_1'')\ dl_1''$$

$$\times e^{-\frac{4i\lambda}{\epsilon}\frac{1}{2}\sinh(2(a_0-a_1))l_2''} e^{-\frac{2i\lambda}{\epsilon}\left(\sinh(a_0-a_1)-\frac{1}{2}\sinh(2(a_0-a_1))\right)\Omega_V(w,v_2)}$$

$$\times g(a_2, v_2, l_2'')\ dl_2''$$

$$\times da_1\ d^{\Omega_V}v_1\ da_2\ d^{\Omega_V}v_2\ da_0' dl_0''$$

$$(l_0'' \to \tilde{l}_0'' = \tfrac{4\lambda}{\epsilon} l_0'', \int e^{i\tilde{l}_0''\left((a_2-a_1)-a_0'\right)}\ d\tilde{l}_0'',\ l_1'' \to l_1,\ l_2'' \to l_2)$$

$$= \left(\frac{1}{\pi\epsilon}\right)^{2n} \int e^{-\frac{2i\lambda}{\epsilon}(\sinh(2(a_1-a_2))l_0+\sinh(2(a_2-a_0))l_1+\sinh(2(a_0-a_1))l_2)}$$

$$\times e^{\frac{2i\lambda}{\epsilon}\Omega_V(\cosh(a_2-a_0)v_1-\cosh(a_2-a_1)v_0,\cosh(a_0-a_1)v_1-\cosh(a_2-a_1)v_0)}$$

$$\times e^{-\frac{2i\lambda}{\epsilon}\left(\left(\sinh(a_1-a_2)-\frac{1}{2}\sinh(2(a_1-a_2))\right)\Omega_V(w,v_0)\right.}$$

$$+\left(\sinh(a_2-a_0)-\frac{1}{2}\sinh(2(a_2-a_0))\right)\Omega_V(w,v_1)$$

$$\left.+\left(\sinh(a_0-a_1)-\frac{1}{2}\sinh(2(a_0-a_1))\right)\Omega_V(w,v_2)\right)$$

$$\times \cosh(2(a_2-a_1))\cosh(a_2-a_1)^{2n-2} f(a_1, v_1, l_1) g(a_2, v_2, l_2)$$

$$\times 2\lambda^n\ da_1\ d^{\Omega_V}v_1\ dl_1\ 2\lambda^n\ da_2\ d^{\Omega_V}v_2\ dl_2$$

which is the anounced formula.

The phase that we obtain is exactly the Weinstein phase function computed in Proposition 5.6.4, and the latter is by construction invariant under the diagonal action of the automorphism group $\mathrm{Aut}(M, s, \omega)$. It is actually equivariant under $\mathrm{Aut}(M, s)$. Indeed, let us denote S^ω the Weinstein phase function associated to the symplectic form ω on M, and $\Delta(x, y, z)$ the geodesic surface bounded by the double triangle on x, y, z. Then

$$S^\omega(gx, gy, gz) = \int_{\Delta(gx,gy,gz)} \omega = \int_{g(\Delta(x,y,z))} \omega = \int_{\Delta(x,y,z)} g^*\omega = S^{g^*\omega}(x, y, z)$$

for all $x, y, z \in M$.

That the amplitude is invariant under the full automorphism group of the symmetric space (M, s) is seen as follows. We computed in Proposition 5.4.3 the automorphism group of (g, σ). Since M is connected, this group is equal to the stabilizer of the base point o in $\mathrm{Aut}(M, s, \omega)$, with the action defined by $h \cdot (e^X \cdot o) = e^{h(X)} \cdot o$, for all $h \in \mathrm{Aut}(\mathfrak{g}, \sigma)$ and $X \in \mathfrak{g}$, and extended as an

automorphism.

Any element $h \in \mathrm{Aut}(M,s)$ may be written as $h = tk$ with t in the transvection group, and k in the stabilizer of the base point o.

Since $\mathrm{Aut}(\mathfrak{g},\sigma)$ vanishes on $\mathbb{R}H$, $\mathrm{Stab}(o)$ acts trivially on the quotient symmetric space $M' = M/N$ of M by the normal subspace $N = \{I^w(0,v,l) \mid v \in V, l \in \mathbb{R}\}$. Moreover, we deduce from the expression of the symmetries given in Proposition 5.6.1 that each t in the transvection group acts on M' by translation by a constant element. Consequently, the same holds for any element $h \in \mathrm{Aut}(M,s)$. Since the amplitude $A(x_1,x_2,x_3)$ only depends on $a_1 - a_2$, it is invariant under the diagonal action of $\mathrm{Aut}(M,s)$.

The Liouville measure is of course also $\mathrm{Aut}(M,s)$-equivariant. Hence, denoting $*^\omega$ the product for some $\omega = \omega^{\lambda,\lambda w}$, the association $\omega \mapsto *^\omega$ is equivariant in the sense that, for all $h \in \mathrm{Aut}(M,s)$ and $x \in M$,

$$h^*(f *^\omega g) = \left(h^* f *^{h^*\omega} h^* g\right).$$

□

Chapter 6

Universal deformation formulæ

In this last chapter, we provide a universal deformation formula for the strongly continuous tempered actions of some Lie groups acting simply transitively on the elementary normal spaces of Chapter 5. This is done by a straightforward adaptation of the arguments in [BG11a].

Beside the fact that the above arguments hold for the non-Kählerian elementary normal spaces, the main point of this chapter is to show that for exponential solvable symplectic symmetric spaces with a simply transitive group of automorphisms, some map entering the definition of a *tempered pair* is always a global diffeomorphism, thanks to the results of Chapters 2 and 4. We briefly indicate how temperedness and admissibility could be approached in general, using the methods of Chapter 3.

6.1 Framework

In this section, we extract almost verbatim from [BG11a] a minimal set of statements needed to use their final result.

We start with the notions related to temperedness.

Lemma 6.1.1. *Let G be a connected real Lie group and $\psi : \mathbb{R}^m \to G$ be a global diffeomorphism. Then the multiplication and inverse operations seen through ψ are tempered functions (in the ordinary sense of $\mathbb{R}^n$) if and only if for every element $A \in \mathcal{U}(\mathfrak{g})$ their derivatives along $\widetilde{A}$ is bounded by a polynomial function within the chart ψ.*

Definition 6.1.2. *A Lie group G is called* tempered *if there exists a global coordinate system $\mathbb{R}^m \to G$ where the multiplication and inverse operations are tempered functions. A smooth function on a tempered group is called a* tempered function *if for any $X \in \mathcal{U}(\mathfrak{g})$, its derivative along $\widetilde{X}$ is bounded by a polynomial in the coordinates of the chart $\mathbb{R}^m \to G$.*

The following is one of the main definitions.

Definition 6.1.3. *The pair (G, S) is called* tempered *if the following two properties are satisfied:*

1. *The map*

$$\varphi : G \to \mathfrak{g}^* : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto (dS)_x\left(\widetilde{X}\right) = \left(\widetilde{X}.S\right)(x)\right], \tag{6.1}$$

   is a global diffeomorphism.

2. *The inverse map $\varphi^{-1} : \mathfrak{g}^* \cong \mathbb{R}^m \to G$ endows G with the structure of a tempered group.*

Given a tempered pair (G, S), we consider a vector space decomposition:

$$\mathfrak{g} = \bigoplus_{n=0}^{N} V_n, \tag{6.2}$$

and for every $n = 0, \ldots, N$, an ordered basis $\left\{e_{j_n}^n\right\}_{j_n=1,\ldots,\dim(V_n)}$ of V_n. We get global coordinates on G:

$$x_n^{j_n} := \left(\widetilde{e_{j_n}^n} S\right)(x), \qquad n = 0, \ldots, N, \quad j_n = 1, \ldots, \dim(V_n). \tag{6.3}$$

We choose a scalar product on each V_n and let $|.|_n$ be the associated Euclidean norm. We will always identify the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the symmetric algebra $\mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$, through the Poincaré-Birkhoff-Witt linear isomorphism.

We now pass to regularity assumptions regarding the function S. This is the second main definition.

Definition 6.1.4. *Set*

$$E := \exp\{iS\}. \tag{6.4}$$

A tempered pair (G, S) is called admissible, *if there exists a decomposition (6.2) with associated coordinate system (6.3), such that for every $n = 0, \ldots, N$, there exists an element $X_n \in \mathcal{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$ whose associated multiplier α_n, defined as*

$$\widetilde{X}_n E =: \alpha_n E, \tag{6.5}$$

satisfies the following properties:

1. *There exist $C_n > 0$ and $\rho_n > 0$ such that:*

$$|\alpha_n| \geq C_n\Big(1 + |x_n|_n^{\rho_n}\Big),$$

   where $x_n := (x_n^{j_n})_{j_n=1,\dots,\dim(V_n)}$.

2. *For all $n = 0,\dots,N$, there exists a tempered function $0 < \mu_n \in C^\infty(G)$ such that:*

   (a) *For every $A \in \prod_{k=0}^n \mathcal{S}(V_k) \subset \mathcal{U}(\mathfrak{g})$ there exists $C_A > 0$ such that:*

$$\left|\widetilde{A}\alpha_n\right| \leq C_A\,|\alpha_n|\,\mu_n. \tag{6.6}$$

   (b) *The function μ_n is independent of the variables $\{x_r^{j_r}\}_{j_r=1,\dots,\dim(V_r)}$, for all $r \leq n$:*

$$\frac{\partial \mu_n}{\partial x_r^{j_r}} = 0, \quad \forall r \leq n, \forall j_r = 1,\dots,\dim(V_r). \tag{6.7}$$

Definition 6.1.5. *A tempered Lie group G, with associated diffeomorphism $\phi : G \to \mathfrak{g}^\star$ is called* tame *(on the left) if there exist a Euclidean norm $|.|$ on $\mathfrak{g}^\star$, a tempered weight μ_ϕ and two positive constants C, ρ such that*

$$C\left(1 + |\phi(x)|^2\right)^{\frac{\rho}{2}} \leq \mu_\phi(x).$$

We then set

$$\nu_\phi(x) := \left(1 + |\phi(x)|^2\right)^{\frac{1}{2}}.$$

We now quickly describe the main result of [BG11a]. It begins with a strongly continuous tempered action on a Fréchet algebra.

Definition 6.1.6. *Let α be an action of a tempered Lie group G on a Fréchet algebra $\mathcal{A}$. Then,*

1. *α is said to be* tempered *if for all $j \in \mathbb{N}$ there exists a tempered weight μ_j^α such that for all $a \in \mathcal{A}$ we have*

$$\|\alpha_g(a)\|_j \leq \mu_j^\alpha(g)\|a\|_j, \quad \forall g \in G,$$

2. *α is said to be* strongly continuous *if for each $a \in \mathcal{A}$, the map $G \to \mathcal{A} : g \mapsto \alpha_g(a)$ is continuous,*

3. *$a \in \mathcal{A}$ is called a* smooth vector *of the action if the map $G \to \mathcal{A} : g \mapsto \alpha_g(a)$ is smooth.*

The set of smooth vectors of the action is denoted $\mathcal{A}^\infty$.

In [BG11a], the authors prove that, for every negatively curved Kählerian Lie group (i.e., for every normal j-group) G, there is a function $S : G \times G \to \mathbb{R}$ such that $(G \times G, S)$ is a tame admissible tempered pair. They then use this fact, together with the datum of an associative WKB product (in the sense of [BG11a]) on G, to build a universal deformation formula for the algebras of smooth vectors for strongly continuous tempered actions of G on Fréchet algebras.

If $\widetilde{K}_\theta$ is the three-point kernel of the WKB quantization, $K_\theta(x,y) = \widetilde{K}_\theta(e,x,y)$ is the corresponding two-point kernel, and if $(\mathcal{A}, \mu)$ is a Fréchet algebra with multiplication μ, one may give an oscillatory integral meaning to the following expression, with $a, b \in \mathcal{A}^\infty$,

$$a *_\theta^\alpha b := \int_{G\times G} K_\theta(x,y)\mu(\alpha_x(a), \alpha_y(b))dxdy, \tag{6.8}$$

in such a way that the following main theorem holds.

Theorem 6.1.7. *Let $(\mathcal{A}, \alpha, G)$ be a Fréchet algebra endowed with a tempered and strongly continuous action α of a normal j-group G. Let also $\theta \in \mathbb{R}^*$. Then, $\left(\mathcal{A}^\infty, *_\theta^\alpha\right)$ is an associative Fréchet algebra with jointly continuous product.*

It turns out that the only ingredient that is needed in their proofs is precisely a tame admissible tempered pair $(G \times G, S)$ endowed with an associative WKB product[1].

In the next sections, we will proceed to prove these properties for the examples of Chapter 5, yielding thus universal deformation formulæ for the (strongly continuous tempered) actions of these groups on Fréchet algebras.

6.2 General remark

Our purpose is to prove the following theorem.

Theorem 6.2.1. *Let (M, s, ω, o) be a pointed exponential solvable symplectic symmetric space, and G a simply transitive Lie group of automorphisms of M acting through $\alpha : G \times M \to M : (g, x) \mapsto \alpha_g(x) = \alpha^x(g)$. Denote $S : G \times G \to \mathbb{R}$ the two-point function defined by*

$$S(x_1, x_2) = S_W(e, x_1, x_2),$$

where S_W is the Weinstein phase function (see Definition 4.6.7) of M pulled back on G through α^o. Then the map $\varphi : G \times G \to \mathfrak{g}^* \times \mathfrak{g}^*$ of Definition 6.1.3 is a global

[1] Actually, a *weakly associative* product, see [BG11a, Definition 2.7].

diffeomorphism.

We first prove two lemmas. The main observation is the following.

Lemma 6.2.2. *In the setting of Theorem 6.2.1, let ϕ_o be the canonical map of M at o, and $\beta : G \times G \to G \times G$ the map*

$$\beta(x_1, x_2) = L_{(x_1^{-1}, x_2^{-1})}(C, A), \tag{6.9}$$

where C and A are defined by $C = s_{x_1} s_o s_{x_2} C = s_{x_1} s_o A$. Let Ω be the symplectic form at $e \in G$ and $\Omega^\flat : \mathfrak{g} \to \mathfrak{g}^*$ the symplectic isomorphism. Then the map φ reads

$$\varphi = 2\left(\Omega^\flat \times \Omega^\flat\right) \circ (\phi_e \times \phi_e) \circ \beta.$$

Proof. Let $x_1, x_2 \in G$ and define $(A, B, C) = \gamma_3^{-1}(e, x_1, x_2)$. By definition, the evaluation $\varphi(x_1, x_2)$ is the linear map

$$(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g} \mapsto (dS_W)_{(e,x_1,x_2)}(0, \widetilde{X}_1, \widetilde{X}_2),$$

where $\widetilde{X}_1, \widetilde{X}_2$ are the left-invariant vector fields associated to X_1, X_2. By Equation (4.2) in Theorem 4.6.14, we thus have

$$\begin{aligned}
\langle \varphi(x_1, x_2), (X_1, X_2) \rangle &= (dS_W)_{(e,x_1,x_2)}\left(0, L_{x_1 *} X_1, L_{x_2 *} X_2\right) \\
&= \left\langle \left(\check{\phi}(B, C), \check{\phi}(C, A)\right), \left(L_{x_1 *} X_1, L_{x_2 *} X_2\right) \right\rangle \\
&= \left\langle \left(\check{\phi}(L_{x_1^{-1}} B, L_{x_1^{-1}} C), \check{\phi}(L_{x_2^{-1}} C, L_{x_2^{-1}} A)\right), (X_1, X_2) \right\rangle \\
&= (\Omega \times \Omega)\left(\left(2\phi_e(L_{x_1^{-1}} C), 2\phi_e(L_{x_2^{-1}} A)\right), (X_1, X_2)\right).
\end{aligned}$$

Above, $\check{\phi}$ and ϕ are respectively the Ševera projection (Definition 4.1.6) and the canonical map (Definition 4.1.5), and we used the facts that G acts on itself by symmetric space automorphisms and that the Ševera projection is equivariant under all such automorphisms. □

Lemma 6.2.3. *In the setting of Theorem 6.2.1, the map β defined in Equation* (6.9) *is a global diffeomorphism.*

Proof. Injectivity. Assume $\beta(x_1, x_2) = \beta(x_1', x_2')$, so that

$$L_{x_1^{-1}} C = L_{x_1'^{-1}} C' \tag{6.10}$$

$$L_{x_2^{-1}} A = L_{x_2'^{-1}} A' \tag{6.11}$$

Since $A = s_{x_2} C$ and $A' = s_{x_2'} C'$, and since $L_{x^{-1}} \circ s_x = s_e \circ L_{x^{-1}}$ for all $x \in G$, Equation (6.11) is equivalent to $C' = L_{x_2' x_2^{-1}} C$. Combining with Equation (6.10), we

get $x_1' x_1^{-1} = x_2' x_2^{-1}$.

In the same way, we get $B' = L_{x_1' x_1^{-1}} B$. Hence

$$\begin{aligned}(e, x_1', x_2') &= \gamma_3(A', B', C') \\ &= L_{x_1' x_1^{-1}} \gamma_3(A, B, C) \\ &= L_{x_1' x_1^{-1}}(e, x_1, x_2)\end{aligned}$$

which yields $x_1' x_1^{-1} = e$, and thus also $x_2' x_2^{-1} = e$. This proves this injectivity.

Surjectivity. We will in fact explicitly compute the inverse, which will prove its smoothness at the same time.

Let $x_1, x_2 \in G$ and put $(y_1, y_2) = \beta(x_1, x_2)$. With $(A, B, C) = \gamma_3^{-1}(e, x_1, x_2)$, we thus have $(C, A) = \left(L_{x_1} y_1, L_{x_2} y_2\right)$ which, remembering the definition of γ_3, gives

$$x_1 = \gamma\left(s_e L_{x_2} y_2, L_{x_1} y_1\right) \tag{6.12}$$

$$x_2 = \gamma\left(L_{x_2} y_2, L_{x_1} y_1\right). \tag{6.13}$$

Using Equation (6.12), we get

$$L_{x_1} s_e y_1 = s_{x_1} L_{x_1} y_1 = s_e L_{x_2} y_2$$

from which we may isolate x_2:

$$x_2 = \left(s_e L_{x_1} s_o y_1\right).y_2^{-1}.$$

Using this expression in Equation (6.13) yields

$$\begin{aligned}L_{x_1} y_1 &= s_{\left(s_e L_{x_1} s_e y_1\right).y_2^{-1}}\left(s_e L_{x_1} s_e y_1\right) \\ &= L_{s_e L_{x_1} s_e y_1} s_{y_2^{-1}} e.\end{aligned}$$

Multiplying by $\left(s_{y_2^{-1}} e\right)^{-1}$ on the right, we get

$$L_{x_1} y_1 \left(s_{y_2^{-1}} e\right)^{-1} = s_e L_{x_1} s_e y_1 = L_{x_1} s_{x_1^{-1}} s_e y_1$$

and finally, simplifying L_{x_1} on the left, using the uniqueness of the midpoints and taking the inverse, we obtain

$$x_1 = \left[\gamma\left(s_e y_1, y_1.\left(s_{y_2^{-1}} e\right)^{-1}\right)\right]^{-1}.$$

This conludes the proof since we have explicit smooth expressions for x_1, x_2 in

terms of y_1, y_2. □

Proof of Theorem 6.2.1. Since M is exponential, the map $\phi_e \times \phi_e$ is a global diffeomorphism, and by Lemma 6.2.3, so is β. Hence by Lemma 6.2.2, $\varphi = 2\left(\Omega^\flat \times \Omega^\flat\right) \circ (\phi_e \times \phi_e) \circ \beta$ is a global diffeomorphism. □

For the next few paragraphs, we indicate some ideas on how the methods developed in the previous chapters could be use to extend the results of [BG11a] about the elementary normal j-algebras to a larger class of spaces. The subsequent sections will then be devoted to the explicit example of Chapter 5.

As a first step, the following class of spaces should be considered.

Definition 6.2.4. *Let $(\mathfrak{m}, \diamond, \Omega), (\mathfrak{m}', \diamond', \Omega')$ be two symplectic combined algebras (see Definition 3.4.11). Then $(\mathfrak{m}, \diamond, \Omega)$ is said to be a* double extension *of $(\mathfrak{m}', \diamond', \Omega')$ if $(\mathfrak{m}, \diamond)$ has an isotropic two-sided ideal $\mathfrak{n}$ such that $\mathfrak{n}^\perp$ is also a two-sided ideal and there exists a supplementary combined subalgebra $\mathfrak{a} \subset \mathfrak{m}$.*

Definition 6.2.5. *A* split combined algebra *is a symplectic combined algebra such that the associated Lie triple system is completely solvable, and which is obtained by a sequence of double extensions by one-dimensional spaces.*

These correspond to completely solvable symplectic symmetric spaces endowed with a simply transitive group of automorphisms, such that both are obtained by a sequence of (compatible) double extensions by one-dimensional systems.

Let $(\mathfrak{m}, \diamond, \Omega)$ be a split combined algebra, and let $H_1, \ldots, H_m$, $E_1, \ldots, E_m$ be a basis of $\mathfrak{m}$ realizing the sequence of double extensions:

$$\mathfrak{m} = \mathbb{R}H_1 \ltimes (\mathbb{R}H_2 \ltimes (\cdots \ltimes (\mathbb{R}H_m \ltimes \mathbb{R}E_m) \cdots + \mathbb{R}E_2) + \mathbb{R}E_1).$$

Let M (resp. G) be the connected and simply connected symplectic symmetric space (resp. locally simply transitive Lie group of automorphisms) associated to $(\mathfrak{m}, \diamond, \Omega)$. Let $\mathfrak{h} = \mathfrak{m} + \mathrm{Der}(\mathfrak{m}, \Omega)$ be the Lie algebra of $\mathrm{Aut}(M, s, \omega)$, and $\mathfrak{g} = \{x^\mathfrak{g} = x + L_x \mid x \in \mathfrak{m}\} \subset \mathfrak{h}$ be the Lie algebra of G. Now the tentative ideas that there are global coordinate systems on M and G given by

$$\mathbb{R}^m \times \mathbb{R}^m \to M : (a, l) \mapsto e^{a_1 H_1} \ldots e^{a_m H_m} e^{l_m E_m} \ldots e^{l_1 E_1} \cdot o$$
$$\mathbb{R}^m \times \mathbb{R}^m \to G : (a, l) \mapsto e^{a_1 H_1^\mathfrak{g}} \ldots e^{a_m H_m^\mathfrak{g}} e^{l_m E_m^\mathfrak{g}} \ldots e^{l_1 E_1^\mathfrak{g}}$$

and that using the special properties of the double extensions (E_1 is central in its symplectic orthogonal, which is an ideal, etc.) and of the Ševera projection should prove temperedness, and maybe admissibility also.

In Section 6.4, the admissibility will follow from the following general lemma.

Lemma 6.2.6. *Let (G,S) be a tempered pair, let $\mathfrak{g} = \sum_{k=0}^{N} V_k$ be a decomposition in vector spaces of dimension d_k, let $\{e_{k,1},\dots,e_{k,d_k}\}$ be a basis of V_k, for each $k = 0,\dots,N$, and endow $\mathfrak{g}$ and each V_k with the Euclidian norm associated to this choice of basis. Denote $x_{k,j} : G \to \mathbb{R} : x \mapsto \varphi(x)(e_{k,j})$ the associated coordinates on G, and denote $\vec{x}_k = (x_{k,1},\dots,x_{k,d_k})$ for each $k = 0,\dots,N$.*

Assume that for each $k = 0,\dots,N$, there exists a tempered function $m_k > 0$ such that

1. *$\frac{\partial m_k}{\partial x_{k',j}} = 0$ for every $j = 1,\dots,d_{k'}$ and $k' \leq k$,*
2. *for every $X \in \mathcal{S}\left(\sum_{k'=0}^{k} V_{k'}\right)$, there exists a constant $C_X > 0$ with*

$$\left|\widetilde{X}\vec{x}_k\right| \leq C_X m_k (1 + |\vec{x}_k|).$$

Then, defining

$$X_k = 1 - \sum_{j=1}^{d_k} \left(e_{k,j}\right)^2 \in \mathcal{U}(\mathfrak{g}),$$

the corresponding multipliers $\alpha_k = e^{-iS}\widetilde{X}_k e^{iS}$ satisfy conditions 1. and 2. of Definition 6.1.4.

Proof. The proof is the same as that of Proposition 3.6 of [BG11a] about elementary normal j-algebras, as it does not rely on its particular setting of normal j-groups. □

6.3 Temperedness

We now focus on the main example that was quantized in Chapter 5, and essentially follow [BG11a] to show that their result holds also in the non-Kahlerian case $w \neq 0$.

Let $(\mathfrak{h}, \sigma, \Omega^{\lambda,\lambda w})$ be an elementary normal symplectic triple, with associated decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}$, let H be the connected and simply connected Lie group with Lie algebra $\mathfrak{h}$, let K be its connected subgroup with Lie algebra $\mathfrak{k}$, and $M = H/K$ the corresponding symplectic symmetric space.

Let G be the connected Lie subgroup of H with Lie algebra $\mathfrak{g} = \mathfrak{a} + (\mathfrak{n} \oplus \{0\})$, denote $\alpha : G \times M \to M$ its action on M, and $\alpha_o : G \to M : g \mapsto \alpha(g,o)$.

Proposition 6.3.1. *The Lie group G acts simply transitively on M, and the map*

$$I_G^w : \mathfrak{g} \to G : aH + v_1 + lE_1 \mapsto e^{aH} e^{v_1 + \left(l - \frac{1}{2}\Omega_V(w,v)\right)E_1}$$

is a global diffeomorphism such that, identifying $\mathfrak{g}$ with $\mathbb{R} \times V \times \mathbb{R}$ by setting $(a, v, l) = aH + v_1 + lE_1$, we have

$$\alpha_o \circ I_G^w = I^w, \tag{6.14}$$

where I^w was defined in Lemma 5.5.2.

In the chart I_G^w, the group law reads

$$I_G^w(a,v,l).I_G^w(a',v',l') = I_G^w\Big(a + A', e^{-a'}v + v', \\ e^{-2a'}l + l' + \frac{1}{2}\left(e^{-a'} - e^{-2a'}\right)\Omega_V(w,v) + \frac{1}{2}e^{-a'}\Omega_V(v,v')\Big)$$

and the inverse is

$$I_G^w(a,v,l)^{-1} = I_G^w\Big(-a, -e^a v, -e^{2a}l - \frac{1}{2}\left(e^a - e^{2a}\right)\Omega(w,v)\Big).$$

Proof. The Lie algebra $\mathfrak{g}$ is a semi-direct product of two exponential Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Moreover, the subgroup G of H is simply connected since any connected subgroup of an exponential Lie group is simply connected. Hence the Lie algebra semi-direct product yields a Lie group semi-direct product, so that any element in G is uniquely written as a product $e^X e^Y$ with $X \in \mathfrak{a}$, $Y \in \mathfrak{n}$. This proves that I_G^w is a global diffeomorphism.

We compute, for all $v \in V$ and $l \in \mathbb{R}$,

$$\begin{aligned} e^{-v_{\mathfrak{m}} - lE_{\mathfrak{m}}} e^{v_1 + lE_1} &= e^{-v_{\mathfrak{m}} - lE_{\mathfrak{m}}} e^{v_{\mathfrak{m}} + lE_{\mathfrak{m}} + v_{\mathfrak{k}} + lE_{\mathfrak{k}}} \\ &= e^{-v_{\mathfrak{m}} - lE_{\mathfrak{m}}} e^{v_{\mathfrak{m}} + lE_{\mathfrak{m}}} e^{v_{\mathfrak{k}} + lE_{\mathfrak{k}}} \\ &= e^{v_{\mathfrak{k}} + lE_{\mathfrak{k}}} \in \mathfrak{k}. \end{aligned}$$

Hence

$$\begin{aligned} \alpha_o \circ I_G^w(a,v,l) &= e^{aH} e^{v_1 + \left(l - \frac{1}{2}\Omega_V(w,v)\right)E_1} K \\ &= e^{aH} e^{v_{\mathfrak{m}} + \left(l - \frac{1}{2}\Omega_V(w,v)\right)E_{\mathfrak{m}}} K \\ &= I^w(a,v,l), \end{aligned}$$

which proves at the same time the anounced formula, and the fact that G acts simply transitively on M. □

Remark 6.3.2. Notice that with our identifications of $\mathfrak{m}$ and $\mathfrak{g}$ with $\mathbb{R}\times V\times\mathbb{R}$, we obtain two *a priori* different symplectic forms on the latter vector space, but that they coincide, and are given by $\lambda(2da\wedge dl+\Omega_V)$. ○

Proposition 6.3.3. *In the I_G^w coordinates, the map $\varphi' : G\times G\to\mathfrak{g}\times\mathfrak{g}$ defined by $\varphi'(x_1,x_2)=(\phi_e\times\phi_e)\circ\beta$, is given by*

$$
\begin{aligned}
&\varphi'((a_1,v_1,l_1),(a_2,v_2,l_2))\\
&\quad=\Big(\Big(\frac{1}{2}\sinh(2a_2),\\
&\qquad\cosh(a_2)(\cosh(a_1)v_2+\sinh(a_2)v_1)+\Big(\sinh(a_2)-\frac{1}{2}\sinh(2a_2)\Big)w,\\
&\qquad\cosh(2a_1)l_2+\sinh(2a_2)l_1+\frac{1}{2}(\sinh(a_2)-\sinh(2a_2))\Omega_V(w,v_1)\\
&\qquad+\frac{1}{2}(\cosh(a_1)-\cosh(2a_1))\Omega_V(w,v_2)-\frac{1}{2}e^{-a_1}\cosh(a_2)\Omega_V(v_1,v_2)\Big),\\
&\qquad\Big(-\frac{1}{2}\sinh(2a_1),\\
&\qquad-\cosh(a_1)(\sinh(a_1)v_2+\cosh(a_2)v_1)-\Big(\sinh(a_1)-\frac{1}{2}\sinh(2a_1)\Big)w,\\
&\qquad-\sinh(2a_1)l_2-\cosh(2a_2)l_1-\frac{1}{2}(\sinh(a_1)-\sinh(2a_1))\Omega_V(w,v_2)\\
&\qquad-\frac{1}{2}(\cosh(a_2)-\cosh(2a_2))\Omega_V(w,v_1)-\frac{1}{2}e^{-a_2}\cosh(a_1)\Omega_V(v_1,v_2)\Big)\Big).
\end{aligned}
$$

Proof. We compute

$$
\begin{aligned}
x_1^{-1}.C&=I_G^w\Big(-a_1,-e^{a_1}v_1,-e^{2a_1}l_1-\frac{1}{2}\Big(e^{a_1}-e^{2a_1}\Big)\Omega_V(w,v_1)\Big)\\
&\quad.I_G^w\Big(a_1+a_2,\cosh(a_2)v_1+\cosh(a_1)v_2,\\
&\qquad\cosh(2a_1)l_2+\cosh(2a_2)l_1+\frac{1}{2}\sinh(a_1-a_2)\Omega_V(v_1,v_2)\\
&\qquad+\frac{1}{2}\Omega_V(w,(\cosh(a_1)-\cosh(2a_1))v_2+(\cosh(a_2)-\cosh(2a_2)v_1)\Big)\\
&=I_G^w\Big(a_2,e^{-(a_1+a_2)}(e^{a_1}v_1)+\cosh(a_1)v_2+\cosh(a_2)v_1,\\
&\qquad e^{-2(a_1+a_2)}\Big(-e^{2a_1}l_1-\frac{1}{2}\Big(e^{a_1}-e^{2a_1}\Big)\Omega_V(w,v_1)\Big)\\
&\qquad+\cosh(2a_1)l_2+\cosh(2a_2)l_1\\
&\qquad+\frac{1}{2}\Omega_V(w,(\cosh(a_1)-\cosh(2a_1))v_2+(\cosh(a_2)-\cosh(2a_2))v_1)\\
&\qquad+\frac{1}{2}\sinh(a_1-a_2)\Omega_V(v_1,v_2)+\frac{1}{2}\Big(e^{-(a_1+a_2)}-e^{-2(a_1+a_2)}\Big)\Omega_V(w,-e^{a_1}v_1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{2} e^{-(a_1+a_2)} \Omega_V \left(-e^{a_1} v_1, \cosh(a_1) v_2 + \cosh(a_2) v_1\right) \Big) \\
&= I_G^w \Big(a_2, \cosh(a_1) v_2 + \sinh(a_2) v_1, \\
&\qquad \cosh(2a_1) l_2 + \sinh(2a_2) l_1 - \frac{1}{2} e^{-a_1} \cosh(a_2) \Omega_V(v_1, v_2) \\
&\qquad + \frac{1}{2} \left(\sinh(a_2) - \sinh(2a_2)\right) \Omega_V(w, v_1) \\
&\qquad + \frac{1}{2} \left(\cosh(a_1) - \cosh(2a_1)\right) \Omega_V(w, v_2) \Big).
\end{aligned}
$$

Applying the canonical map to this, we get

$$
\begin{aligned}
\phi(x_1^{-1}.C) = I_G^w \Big(&\frac{1}{2} \sinh(2a_2), \\
&\cosh(a_2) \left(\cosh(a_1) v_2 + \sinh(a_2) v_1\right) + \Big(\sinh(a_2) - \frac{1}{2} \sinh(2a_2)\Big) w, \\
&\cosh(2a_1) l_2 + \sinh(2a_2) l_1 + \frac{1}{2} \left(\sinh(a_2) - \sinh(2a_2)\right) \Omega_V(w, v_1) \\
&+ \frac{1}{2} \left(\cosh(a_1) - \cosh(2a_1)\right) \Omega_V(w, v_2) - \frac{1}{2} e^{-a_1} \cosh(a_2) \Omega_V(v_1, v_2) \Big)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\phi(x_2^{-1}.A) = I_G^w \Big(&-\frac{1}{2} \sinh(2a_1), \\
&-\cosh(a_1) \left(\sinh(a_1) v_2 + \cosh(a_2) v_1\right) - \Big(\sinh(a_1) - \frac{1}{2} \sinh(2a_1)\Big) w, \\
&-\sinh(2a_1) l_2 - \cosh(2a_2) l_1 - \frac{1}{2} \left(\sinh(a_1) - \sinh(2a_1)\right) \Omega_V(w, v_2) \\
&- \frac{1}{2} \left(\cosh(a_2) - \cosh(2a_2)\right) \Omega_V(w, v_1) - \frac{1}{2} e^{-a_2} \cosh(a_1) \Omega_V(v_1, v_2) \Big).
\end{aligned}
$$

□

In order to prove the admissibility, we now fix a particular basis of $\mathfrak{g} \times \mathfrak{g}$. We take the same as in [BG11a] since only the symplectic form changes. Let $V = L \oplus L'$ be a decomposition of V as a direct sum of two transverse lagrangians, and let $e_1, \ldots, e_n$, and $f_1, \ldots, f_n$ be some mutually symplectically dual bases of L and L' respectively.

Define the basis elements

$$
\begin{aligned}
H_1 &= H \oplus \{0\} & H_2 &= \{0\} \oplus H \\
f_{j1} &= f_j \oplus \{0\} & f_{j2} &= \{0\} \oplus f_j \\
e_{j1} &= e_j \oplus \{0\} & e_{j2} &= \{0\} \oplus e_j
\end{aligned}
$$

$$E_1 = E \oplus \{0\} \qquad\qquad E_2 = \{0\} \oplus E$$

and the vector spaces

$$\begin{array}{lll}
V_{10} = \mathbb{R}H_1 & V_{20} = \mathbb{R}H_2 & V_0 = V_{10} \oplus V_{20} \\
V_{11} = \left\langle f_{j1} \mid j = 1, \ldots, n \right\rangle & V_{21} = \left\langle f_{j2} \mid j = 1, \ldots, n \right\rangle & V_1 = V_{11} \oplus V_{21} \\
V_{12} = \left\langle e_{j1} \mid j = 1, \ldots, n \right\rangle & V_{22} = \left\langle e_{j2} \mid j = 1, \ldots, n \right\rangle & V_2 = V_{12} \oplus V_{22} \\
V_{13} = \mathbb{R}E_1 & V_{23} = \mathbb{R}E_2 & V_3 = V_{13} \oplus V_{23},
\end{array}$$

and take as coordinates

$$\begin{array}{ll}
x_{10} = \varphi(x)(H_1) & x_{20} = \varphi(x)(H_2) \\
x_{11j} = \varphi(x)(f_{j1}) & x_{21j} = \varphi(x)(f_{j2}) \\
x_{12j} = \varphi(x)(e_{j1}) & x_{22j} = \varphi(x)(e_{j2}) \\
x_{13} = \varphi(x)(E_1) & x_{23} = \varphi(x)(E_2).
\end{array}$$

Using the explicit form of the map φ' computed above, and the fact that $\varphi = 2(\Omega^\flat \times \Omega^\flat) \circ \varphi'$, the coordinates read

$$\begin{aligned}
x_{10} = & -\lambda\Big(2\cosh(2a_1)l_2 + 2\sinh(2a_2)l_1 - e^{-a_1}\cosh(a_2)\Omega_V(v_1, v_2) \\
& \qquad (\sinh(a_2) - \sinh(2a_2))\Omega_V(w, v_1) + (\cosh(a_1) - \cosh(2a_1))\Omega_V(w, v_2)\Big) \\
x_{20} = & \lambda\Big(2\cosh(2a_2)l_1 + 2\sinh(2a_1)l_2 + e^{-a_2}\cosh(a_1)\Omega_V(v_1, v_2) \\
& \qquad (\sinh(a_1) - \sinh(2a_1))\Omega_V(w, v_2) + (\cosh(a_2) - \cosh(2a_2))\Omega_V(w, v_1)\Big) \\
x_{11j} = & \lambda\cosh(a_2)\Big(\sinh(a_2)\Omega_V(v_1, f_j) + \cosh(a_1)\Omega_V(v_2, f_j)\Big) \\
& + \lambda\left(\sinh(a_2) - \frac{1}{2}\sinh(2a_2)\right)\Omega_V(w, f_j) \\
x_{21j} = & -\lambda\cosh(a_1)\Big(\cosh(a_2)\Omega_V(v_1, f_j) + \sinh(a_1)\Omega_V(v_2, f_j)\Big) \\
& - \lambda\left(\sinh(a_1) - \frac{1}{2}\sinh(2a_1)\right)\Omega_V(w, f_j) \\
x_{12j} = & \lambda\cosh(a_2)\Big(\sinh(a_2)\Omega_V(v_1, e_j) + \cosh(a_1)\Omega_V(v_2, e_j)\Big) \\
& + \lambda\left(\sinh(a_2) - \frac{1}{2}\sinh(2a_2)\right)\Omega_V(w, e_j) \\
x_{22j} = & -\lambda\cosh(a_1)\Big(\cosh(a_2)\Omega_V(v_1, e_j) + \sinh(a_1)\Omega_V(v_2, e_j)\Big) \\
& - \lambda\left(\sinh(a_1) - \frac{1}{2}\sinh(2a_1)\right)\Omega_V(w, e_j) \\
x_{13} = & \lambda\sinh(2a_2)
\end{aligned}$$

$x_{23} = -\lambda \sinh(2a_1)$.

Incidentally, observing that these coordinates have an at most polynomial growth in the adapted tempered coordinates $\sinh(a_1), \sinh(a_2), v_1, v_2, l_1, l_2$, this proves explicitly that $(G \times G, S_W)$ is indeed a tempered pair.

6.4 Admissibility

Proposition 6.4.1. *In the chart I_G^w, the left-invariant vector fields at a point (a, v, l) read*

$$\begin{aligned}
\tilde{H} &= \partial_a + \partial_v + \left(-2l + \frac{1}{2}\Omega_v(w, v)\right)\partial_l \\
\tilde{u} &= \partial_u + \frac{1}{2}\Omega_V(v, u)\partial_l \\
\tilde{E} &= \partial_l
\end{aligned}$$

where $\partial_v = \sum_{i=1}^n \left(\Omega_V(v, f_i)\partial_{e_i} - \Omega_V(v, e_i)\partial_{f_i}\right)$ for all $v \in V$.

Proof. We compute, for $\tilde{H}$,

$$\begin{aligned}
L_{I_G^w(a,v,l)_*} I_{G*}^w \partial_a &= \frac{d}{dt} I_G^w(a, v, l).I_G^w(t, 0, 0)\Big|_0 \\
&= I_{G*(a,v,l)}^w \left(1, -v, -2l + \frac{1}{2}\Omega_V(w, v)\right).
\end{aligned}$$

The others are similar. □

We want to check the conditions of Lemma 6.2.6.

On V_0

Denote

$$\begin{aligned}
A = A(a_1, a_2) &= \lambda \begin{pmatrix} -2\sinh(2a_2) & -2\cosh(2a_1) \\ 2\cosh(2a_2) & 2\sinh(2a_1) \end{pmatrix} \\
D &= \frac{1}{2} A\left(\frac{a_1}{2}, \frac{a_2}{2}\right) - \frac{1}{2} A(a_1, a_2) \\
\vec{\gamma} &= \lambda \begin{pmatrix} e^{-a_1}\cosh(a_2) \\ e^{-a_2}\cosh(a_1) \end{pmatrix} \\
\vec{x_0} &= \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \qquad \vec{l} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \qquad \vec{u} = \begin{pmatrix} \Omega_V(w, v_1) \\ \Omega_V(w, v_2) \end{pmatrix}
\end{aligned}$$

so that

$$\vec{x}_0 = A.\vec{l} + D.\vec{u} + \Omega_V(v_1, v_2).\vec{\gamma}.$$

The cyclicity of the derivatives of the exponential and hyperbolic functions and the fact that the determinant of A is independant of a_1, a_2 show the following result.

Lemma 6.4.2. *There exist finitely many matrices $M_{(r)}$ and finitely many vectors $\vec{\gamma}_{(s)}$ such that for all $N_1, N_2 \in \mathbb{N}$ and all $x, y, z \in \mathbb{R}$, the matrices $\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.(xA + yA^{-1} + zD)$ (resp. the vectors $\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{\gamma}$) are linear combinations of the $M_{(r)}$'s (resp. the $\vec{\gamma}_{(s)}$'s).*

Lemma 6.4.3. *There exist finitely many matrices $M_{(r)} \in \mathbb{R}^{2\times 2}[e^{\pm a_1}, e^{\pm a_2}]$ and finitely many vectors $v_{(s)} \in \mathbb{R}^2[e^{\pm a_1}, e^{\pm a_2}]$ such that for all $N_1, N_2 \in \mathbb{N}$,*

$$\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_0 = M_{N_1N_2}\vec{x}_0 + M'_{N_1N_2}\vec{u} + \Omega_V(v_1, v_2)\vec{v}_{N_1N_2}, \tag{6.15}$$

where $M_{N_1N_2} \in \operatorname{span}\langle M_{(r)}\rangle$, $M'_{N_1N_2} \in \operatorname{span}\langle M'_{(r')}\rangle$, and $v_{N_1N_2} \in \operatorname{span}\langle v_{(s)}\rangle$.

Proof. We have

$$\begin{aligned}
\widetilde{H}_j.l_k &= \Big(-2l_j + \frac{1}{2}\Omega_V(w, v_j)\Big)\delta_{jk} \\
\widetilde{H}_j.\Omega_V(w, v_k) &= -\Omega_V(w, v_j)\delta_{jk} \\
\widetilde{H}_j.\Omega_V(v_1, v_2) &= -\Omega_V(v_1, v_2)
\end{aligned}$$

and

$$\vec{l} = A^{-1}\left(\vec{x}_0 - D.\vec{u} - \Omega_V(v_1, v_2).\vec{\gamma}\right).$$

Consequently, any derivative $\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_0$ is at most linear in l_1, l_2, and may be written as

$$\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_0 = M_{N_1N_2}\vec{x}_0 + (\text{terms in } v_1, v_2 \text{ and } a_1, a_2)$$

with $M_{N_1N_2}$ a linear combination of products of derivatives of A with respect to a_1, a_2 (only a finite number of these derivatives being linearly independent) by $\pi_j.A^{-1}$, $j = 1, 2$, where $\pi_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\pi_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Similarly, the terms linear (resp. quadratic) in v_1, v_2 are obtained as linear combinations of products of π_j and of derivatives of A and D (resp. and $\vec{\gamma}$), only a finite number of these derivatives being linearly independent. □

Corollary 6.4.4. *There exists a tempered function m_0 independent of the variables*

x_{10}, x_{20} and such that for all $X \in \mathcal{S}(V_0)$, there exists a constant $C_X > 0$ with

$$\left|\widetilde{X}\vec{x}_0\right| \leq C_X m_0(1 + |\vec{x}_0|).$$

Proof. Any $X \in \mathcal{S}(V_0)$ is of the form $\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}$ for some $N_1, N_2 \in \mathbb{N}$. Using Equation (6.15) and the explicit expressions for the $x_\cdot$ variables, we then see that

$$\left|\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_0\right| \leq C'_{N_1,N_2} f(\vec{x}_3)|\vec{x}_0| + C''_{N_1,N_2} g(\vec{x}_1, \vec{x}_2, \vec{x}_3)$$

for some positive functions f, g of polynomial growth, given essentially by the sums of the norms of the finite number of matrices and vectors of Lemma 6.4.3. We may then set $m_0 = 1 + f + g$. □

On V_1 and V_2

Denote

$$B = \lambda \begin{pmatrix} \cosh(a_2) & 0 \\ 0 & \cosh(a_1) \end{pmatrix} \begin{pmatrix} \sinh(a_2) & \cosh(a_1) \\ -\cosh(a_2) & -\sinh(a_1) \end{pmatrix}$$

$$\vec{w}_j = \lambda \Omega_V(w, f_j) \begin{pmatrix} \sinh(a_2) - \frac{1}{2}\sinh(2a_2) \\ -\sinh(a_1) + \frac{1}{2}\sinh(2a_1) \end{pmatrix}$$

$$\vec{x}_{1j} = \begin{pmatrix} x_{11j} \\ x_{21j} \end{pmatrix}, \qquad \vec{v}_j = \begin{pmatrix} \Omega_V(v_1, f_j) \\ \Omega_V(v_2, f_j) \end{pmatrix}$$

so that

$$\vec{x}_{1j} = B.\vec{v}_j + \vec{w}_j.$$

Just as for V_0, we compute that $\widetilde{f_{jm}}.\vec{x}_{1j} = 0$ for $m = 1, 2$, and that all the derivatives of x_{1j} in the H_1, H_2 directions are affine functions of v_j with coefficients being linear combinations of finitely many matrices with polynomial entries in $e^{\pm a_1}, e^{\pm a_2}$. Since the determinant of B is always greater than 1, $\vec{v}_j$ is a tempered function of $\vec{x}_1$ and $\vec{x}_3$.

Replacing in the above expressions, f_j by e_j, we get the same result, except that now $\widetilde{e_{jm}}.\vec{x}_{2k} = 0$ and $\widetilde{f_{jm}}.\vec{x}_{2k} = -\delta_{jk}\delta_{m1}B.\binom{1}{0} - \delta_{jk}\delta_{m2}B.\binom{0}{1}$.

Putting these together, we obtain.

Lemma 6.4.5. *There exist finitely many matrices $M_{(r)} \in \mathbb{R}^{2\times 2}[e^{\pm a_1}, e^{\pm a_2}]$ and finitely many vectors $v_{(s)} \in \mathbb{R}^2[e^{\pm a_1}, e^{\pm a_2}]$ such that for all $m = 1, 2$, $j = 1, \ldots, n$ and all $N_1, N_2 \in \mathbb{N}$,*

$$\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_{mj} = M_{N_1N_2}\vec{x}_{mj} + \vec{v}_{N_1N_2}, \tag{6.16}$$

where $M_{N_1N_2} \in \operatorname{span}\left\langle M_{(r)}\right\rangle$, *and* $v_{N_1N_2} \in \operatorname{span}\left\langle v_{(s)}\right\rangle$.

Corollary 6.4.6. *There exists a tempered function* m_1 *independent of the variables* $\vec{x}_0, \vec{x}_1$ *and such that for all* $X \in \mathcal{S}(V_0 \oplus V_1)$, *there exists a constant* $C_X > 0$ *with*

$$\left|\widetilde{X}\vec{x}_1\right| \leq C_X m_1(1 + |\vec{x}_1|).$$

Corollary 6.4.7. *There exists a tempered function* m_2 *independent of the variables* $\vec{x}_0, \vec{x}_1, \vec{x}_2$ *and such that for all* $X \in \mathcal{S}(V_0 \oplus V_1 \oplus V_2)$, *there exists a constant* $C_X > 0$ *with*

$$\left|\widetilde{X}\vec{x}_2\right| \leq C_X m_2(1 + |\vec{x}_2|).$$

On V_3

We have

$$\widetilde{E}_m.\vec{x}_3 = 0$$
$$\widetilde{e}_{mj}.\vec{x}_3 = 0$$
$$\widetilde{f}_{mj}.\vec{x}_3 = 0$$

and $\widetilde{H}_1^{N_1}\widetilde{H}_2^{N_2}.\vec{x}_3$ is a linear combination of the vectors given by multiplying $\cosh(2a_m)$ and $\sinh(2a_m)$, with $m = 1, 2$, by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

Lemma 6.4.8. *For all* $X \in \mathcal{S}(\mathfrak{g})$, *there exists a constant* $C_X > 0$ *with*

$$\left|\widetilde{X}\vec{x}_3\right| \leq C_X(1 + |\vec{x}_3|).$$

Putting all this together and applying Lemma 6.2.6, we get the admissibility.

Proposition 6.4.9. *The pair* $(G \times G, S)$ *is tempered admissible.*

Using the same kind of arguments, we also have the tameness.

Proposition 6.4.10. *The pair* $(G \times G, S)$ *is tame.*

Proof. Let $\varphi : G \to \mathfrak{g}^*$ be the global coordinates associated to the tempered pair $(G \times G, S)$. We want to show that the function $v_\varphi : G \to \mathbb{R} : x \mapsto (1 + |\varphi(x)|^2)^{\rho/2}$ is tempered, which means that in the coordinates φ, the map $x \mapsto (1 + |x|^2)^{\rho/2}$ is tempered. The result follows directly from these three observations:

1. every polynomial in the variables $e^{\pm a_m}$, v_{mkj} and l_m (we will call such a polynomial a *function of type* (P)) is of at most polynomial growth in the variables x_{m0}, x_{m1j}, x_{m2j} and x_{m3};

2. the left-invariant vector fields of Proposition 6.4.1 send functions of type (P) to functions of type (P);

3. the coordinate functions x_{m0}, x_{m1j}, x_{m2j} and x_{m3}, and thus their square too, are functions of type (P).

□

Using the same method as in [BM01], we can show that the product obtained in Theorem 5.8.1 is closed on some transported Schwartz space and, as in [BG11a], is thus weakly associative. This, together with the tame admissible tempered pair computed above, provides the anounced universal deformation formula.

Conclusion and perspectives

We briefly review our contributions presented in this thesis.

Many formal and non-formal quantization procedures exist in the litterature, and some apply to quite large classes of spaces, but few can pretend to yield a non-formal symbol calculus on functions on the initial symplectic manifold. Following examples, insights and programmes by Weinstein, Qian, Bieliavsky, Landsman, etc., the spaces for which this is most likely possible may heuristically be identified as those symmetric spaces admitting a *midpoint map*. Using the structure theory of involutive Lie algebras, we have completely characterized them as the symmetric spaces such that their exponential map is a global diffeomorphism, and have further given an algebraic criterion on the spectrum of the adjoint representation, generalizing the Dixmier-Saito theorem in the Lie group case.

Building on this characterization, we have studied the symmetric spaces in which the notion of double triangle is globally well-defined, and have identified them as the *exponential solvable symmetric spaces*, providing a precise idea of where the Weinstein conjecture about double triangles and quantization may hold without change.

As an attempt to study these spaces more precisely, we have uncovered a basic structure underlying every symplectic symmetric space, that we have termed a *primitive structure*. The latter plays a role analoguous to the canonical affine structure, or left-symmetric algebra, of a symplectic Lie group. With its help, we have defined a notion of T^**-extension* of symplectic Lie triple systems, allowing us to characterize those systems that have a lagrangian (necessarily abelian) ideal. As a second application of this structure, we introduced a version of *symplectic reduction* adapted to the symmetric structure. We then used the reverse notion of *double extension* to describe all the completely solvable and all the nilpotent symplectic Lie triple systems as sequences of such extensions by one-dimensional spaces. The T^*-extensions and the double extensions encompass all known examples that have a WKB quantization in the sense of [Bie02], and provide the most natural generalizations thereof.

We have then studied the *Ševera projection* providing, for a symplectic symmetric space M, a local symplectomorphism between $M \times \overline{M}$ and T^*M. We have studied its basic properties with respect to automorphisms, direct products, quotients, and have shown that it commutes with symplectic reduction. We have then shown that it is a global diffeomorphism as soon as the space has a midpoint map and is semisimple or solvable, thus providing explicitly computable global symplectomorphisms $M \times \overline{M} \to T^*M$ for the class of spaces we are interested in.

In the last two chapters, we have explicitly quantized a new class of examples stemming from the elementary normal j-groups, but with no Kähler structure, using the previously defined Ševera projection. We then showed that they lead to universal deformation formulæ by using recent work by Bieliavsky and Gayral.

The main achievement of this thesis is to have provided what seems to be the right framework for studying all questions related to solvable symplectic symmetric spaces, and in particular for generalizing the existing WKB products in terms of general properties of these spaces and not of particular computations.

As such, we are left with more questions than answers. Here are a few of these questions:

- We know that there exist symplectic symmetric spaces, such as the sphere, which do not have any simply transitive Lie groups of automorphisms. We also know that there exist symplectic Lie groups which, viewed as symmetric spaces, cannot be endowed with an invariant symplectic form: think of the irreducible normal j-groups of rank greater than one. However [DM96a], every symplectic Lie group either is a semi-direct product of two symplectic subgroups, or is a double extension of a smaller one. In the above case of normal j-groups, these are all semi-direct products of rank one normal j-groups, whose symmetric spaces *do* have an invariant symplectic structure.

  If we name such spaces *group symplectic symmetric spaces*, this raises the following question: *is every symplectic Lie group a sequence of semi-direct products of group symplectic symmetric spaces?* If not, can we extend the notion of group symplectic symmetric space, or restrict the set of symplectic Lie groups, to have such a result? The initial guess would be quite constraining as the Lie groups underlying the group symplectic symmetric spaces must be three step solvable, for example [BG09].

  In the affirmative, this could on the one hand have applications to universal deformation formulæ for actions of such groups, and would on the other hand somewhat clarify the relationship between Lie groups and symmetric spaces in the, here very rigid, symplectic world.

- Authors like Pedersen, Pukanszky, Arnal and Cortet have developed methods for finding global Darboux charts on coadjoint orbits of solvable Lie groups, and for working with "good" (covariant) star products on these spaces. We have been trying for some time to apply these methods to solvable symplectic symmetric spaces in order to compute inductively the Ševera projection and a corresponding star product with the method of Chapter 5. However, we repeatedly failed because of a lack of knowledge of the fine structure of the spaces. Developing the method of double extensions should help in that respect. As a first step, given a decomposition as a sequence of double extensions by one-dimensional spaces, we should try to define an adapted choice of global Darboux coordinates.

- The intertwining operator defined in Chapter 5 in terms of the projection seems to allow us to recover the known WBK quantizations (in the sense of [Bie02]) so far. However, we still have no idea why this works. Successfully carrying out the previous point would certainly provide insight about this, but in the meantime its geometric content is unclear, as is, in particular, the fact that we apply a symplectic Fourier transform to a function on a curved space by the use of a global Darboux chart.

  This, of course, suggests that the intertwiner is not the right object. Modding out one symplectic Fourier transform with one in the Weyl product leads us to consider the twisted convolution instead [Mai86], and to see the remaining operator $\phi^* \circ \mathcal{F}^\omega$ as a single object maybe playing somehow the role of an adapted Fourier transform.

- One of the general quantization methods generalizing the Weyl quantization is that of Stratonovich-Weyl quantizers [GV95, PPT02], and a related method is developed in [BG11b] for negatively curved Kählerian Lie groups. These methods often rely on a very particular polarization: most examples are cotangent bundles of Riemannian manifolds in the first case, and the Lie groups and symmetric spaces have a Lagrangian (abelian) ideal in the second case. The T^*-extensions describe exactly this kind of structure, so that it would be interesting to try to apply these methods to a T^*-extension of a (say, exponential) primitive system. Conversely, we would like to know whether large classes of spaces can be realized as T^*-extensions. As a comparison, recall that we mentioned in Chapter 3 that all nilpotent metrized Lie triple systems are T^*-extensions (in a different sense than ours, adapted to the metrized structure).

- Still another approach, which brings us back closer to our initial goal of realizing Weinstein's programme, is that of geometric quantization, and

particularly the work [dMRT08] of de M. Rios and Tuynman on obtaining integral formulas involving areas of double triangles for symplectic symmetric spaces.

They successfully derive such formulas from geometric quantization, with some restriction on the domain of integration, but unfortunately they do not say anything about the associativity of such products, so that in the present state they are not usable. One of the difficulties they encounter comes from their use of a map from TM to $M \times M$ which is compatible with the symmetric structure, but not with the symplectic one. (Another difficulty lies in the fact that two of their three examples either do not have a midpoint map or do not have all double triangles.) They use the "symmetric exponential" $(x, v) \mapsto (\mathrm{Exp}_x(v), \mathrm{Exp}_x(-v))$. Our study of the Ševera projection suggests using that one instead of the symmetric exponential, since it is also a symplectomorphism and since it acts somehow like a slowed down exponential map. It would be interesting to pursue that direction.

- A new look at the quantization by groupoids programme was presented in the insightful paper [Haw08] of Hawkins. His main point is that when performing geometric quantization on a symplectic groupoid to get a quantization of its base Poisson manifold, the polarization should not be any ordinary polarization, but should be what he calls a *symplectic groupoid polarization* and which is an involutive, multiplicative, Hermitian and Lagrangian distribution of the complexified tangent groupoid. It would be interesting to check whether the canonical polarization on $M \times \overline{M}$ induced by the symplectomorphism $\check{\phi} : M \times \overline{M} \to T^*M$ is a (real) groupoid polarization or not (it should be, given the relation of $\check{\phi}$ with the graph of the groupoid multiplication), and if it is the case, to apply Hawkins' methods to symplectic symmetric spaces.

- There is a theorem of McCrudden (see [ÐH97]) saying that *in a connected real Lie group G, the set of divisible elements is equal to the image of the exponential map*, where an element $g \in G$ is *divisible* if for each $n \in \mathbb{N}$ there exists an $x \in G$ such that $x^n = g$. This implies that a connected real Lie group is divisible if and only if it is weakly exponential (in the sense that the exponential map is surjective). With the notion of *unique* divisibility, we get back the strong exponentiality.

  We could formulate the same problem for pointed symmetric spaces, and for dyadic divisibility. To avoid fixing a base point, we could put the following definition. A subset E of a symmetric space is *uniquely 2-divisible*

if it admits a midpoint map (in the sense of a smooth map $\gamma : E \times E \to E$ such that $s_{\gamma(x,y)}x = y$ for all $x, y \in E$).

Is the following sentence true? A subset of a connected symmetric space is 2-divisible if and only if its closure is a completely geodesic subspace.

This would generalize our theorem about midpoint maps and exponentiality. It would also be related, though more remotely and with no comparison of difficulty, to the beautiful (now largely solved) problem of embeddability of infinitely divisible distributions on linear Lie groups [DM92a, DM07].

- We defined the double extensions of symplectic Lie triple systems in an implicit manner. However, to really be able to work with them, it is best to have an explicit characterization thereof, probably in terms of some cohomological objects. We have made some computations, but have not yet arrived at such an understandable characterization.

- Given the fact that the primitive structures are the analogues for symplectic symmetric spaces of the affine structures for symplectic Lie groups, one may wonder what is the geometric content of a primitive structure. A symplectic Lie group has a locally simply transitive action by affine transformations of its Lie algebra. What about a symplectic symmetric space M? There is the obvious fact, already pointed out in Chapter 3, that $\mathfrak{m}^*$ is a special kind of $\mathfrak{m}$-module. Are there nice, complementary, algebraic and geometric points of view on this, as in the Lie group case?

- What could we retrieve from the reinterpretation of the Ševera projection in terms of the fundamental groupoid, apart from shorter proofs?

- It would be interesting to relate the quantization formulæ (and methods) that we obtain with those that Beltitsa and Beltitsa [BB09a, BB09b, BB11] define for cotangent bundles of nilpotent Lie groups. They consider finite-dimensional coadjoint orbits of some locally convex infinite-dimensional Lie groups obtained as semi-direct products of some function space with a nilpotent Lie algebra. They obtain very explicit formulas for the symbol calculus and for the quantization maps.

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