On the representability of actions of non-associative algebras

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Actions / Split extensions of groups

- Let B, X be groups. An action of B on X is a map

$$\xi\colon B \times X \to X\colon (b,x) \mapsto b \cdot x$$

  such that $(bb')\cdot x = b\cdot(b'\cdot x)$, $1_B \cdot x = x$ and $b\cdot(xx') = (b\cdot x)(b\cdot x')$.
- One can construct the *semi-direct product* $B \ltimes X$ with respect to ξ, that is $B \times X$ with multiplication

$$(b,x)(b',x') = (bb', x(b\cdot x')),$$

  where $1_{B\ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1}\cdot x^{-1})$.
- We can associate with ξ the split extension

$$\begin{tikzcd} 1 \arrow[r] & X \arrow[r, "i_2"] & B \ltimes X \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "i_1"] \arrow[r] & 1. \end{tikzcd}$$

- Conversely, given

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

  then we can define an action

$$B \times X \to X \colon (b, x) \mapsto \beta(b)k(x)\beta(b)^{-1}$$

  and we have a bijection between $\mathrm{Act}(B, X)$ and $\mathrm{SplExt}(B, X)$.
- Split extensions of groups are in bijection with the homomorphisms $B \to \mathrm{Aut}(X)$: given a split extension of groups as above, one can define

$$\varphi \colon B \to \mathrm{Aut}(X) \colon b \mapsto \varphi_b,$$

  where $\varphi_b(x) = \beta(b)k(x)\beta(b)^{-1}$.
- Conversely, any homomorphism $\varphi \colon B \to \mathrm{Aut}(X)$ defines a semi-direct product $B \ltimes X$ with multiplication

$$(b, x)(b', x') = (bb', x\varphi_b(x'))$$

  and thus a split extension of B by X.

The semi-abelian context

- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts).
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

  in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.
- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

  where $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- We have a natural isomorphism $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$.

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp** and **Lie**.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ *and let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathbf{AbAlg}$ or $\mathcal{V} = \mathbf{Lie}$.*

- More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-,X)$ admits a weak representation. This means that there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau\colon \mathsf{SplExt}(-,X) \rightarrowtail \mathsf{Hom}_{\mathcal{C}}(-,T).$$

An object T as above is called *weak actor* of X, the pair (T,τ) is called *weak representation* of $\mathsf{SplExt}(-,X)$ and $(\varphi\colon B \to T) \in \mathsf{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The category **Assoc** is weakly action representable. A weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f*-, -*f) \in \mathrm{End}(X)\times\mathrm{End}(X)^{\mathrm{op}} \mid f*(xy) = (f*x)y,$$
$$(xy)*f = x(y*f), x(f*y) = (x*f)y, \ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y),$$
$$D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}.$$

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a commutative algebra $(X, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in X.$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible.

Remark (A. S. Cigoli and S. Mantovani, 2012)

The category **Jord** of Jordan algebras over a field $\mathbb{F}$ is not action accessible.

Conclusion: **Jord** is not weakly action representable.

W.R.A. of non-associative algebras

- Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$;
- We suppose that $\mathcal{V}$ is *operadic*, i.e. all the identities of $\mathcal{V}$ are *multilinear*. This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible*, or equivalently that $\mathcal{V}$ is an *Orzech category of interest*;
- This is equivalent to saying that the λ/μ *rules* hold, i.e. there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$x(yz) = \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) +$$

$$+\lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx),$$

$$(yz)x = \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) +$$

$$+\mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx)$$

are identities in $\mathcal{V}$.

The commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of commutative algebras over $\mathbb{F}$. The λ/μ rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some $\alpha, \beta \in \mathbb{F}$.

Proposition (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

$\mathcal{V}$ is associative or satisfies the Jacobi identity $(x(yz) + y(zx) + z(xy) = 0)$.

Corollary (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let $\mathrm{char}(\mathbb{F}) \neq 3$. *If $\mathcal{V}$ is quadratic, then $\mathcal{V} = \mathbf{CAssoc}$, $\mathcal{V} = \mathbf{JJord}$ or $\mathcal{V} = \mathbf{Nil}_2(\mathrm{Com})$.*

Jacobi-Jordan algebras

Definition

A commutative algebra X is called a *Jacobi-Jordan* algebra if it satisfies the *Jacobi identity*

$$x(yz) + y(zx) + z(xy) = 0, \quad \forall x, y, z \in X.$$

Definition

An *anti-derivation* is a linear map $d \colon X \to X$ such that

$$d(xy) = -d(x)y - d(y)x, \quad \forall x, y \in X.$$

Example

The (left) multiplications $L_x = x \cdot -$, $x \in X$, are particular anti-derivations, called *inner anti-derivations*.

Split extensions

- Let

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

  be a split extension of Jacobi-Jordan algebras.
- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

  and the Jacobi-Jordan algebra structure is given by

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + b' * x).$$

- The action of B on X is defined by

$$b * x = \beta(b) \cdot_A k(x), \quad \forall b \in B, \, \forall x \in X.$$

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ using the formula above:

Proposition

$(B \oplus X, \cdot)$ is a Jacobi-Jordan algebra if and only if

1. $b * (xx') = -(b * x)x' - (b * x) * x'$;
2. $(bb') * x = -b * (b' * x) - b' * (b' * x)$;

for every $b, b' \in B$ and $x, x' \in X$.

A split extension of B by X (or an action of B on X) is a linear map

$$B \to \mathrm{ADer}(X)\colon\, b \mapsto b * -$$

such that

$$(bb') * - = \{b * -, b' * -\}, \quad \forall b, b' \in B,$$

where $\{f, f'\} = -f \circ f' - f' \circ f$ denotes the *anti-commutator* between two linear endomorphisms of X.

Remark

The space $\mathrm{ADer}(X)$ endowed with the anti-commutator $\{-,-\}$ is not in general a (Jacobi-Jordan) algebra (for instance, if $X = \mathbb{F}$, then $\mathrm{ADer}(X) = \mathrm{End}(X)$ does not satisfy the Jacobi identity).

Definition (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

A *partial algebra* is a vector space X endowed with a *bilinear partial operation*

$$\Omega \subseteq X \times X \to X.$$

A homomorphism of partial algebras is a linear map $f\colon X \to Y$ such that $f(xy) = f(x)f(y)$, whenever xy and $f(x)f(y)$ are defined.

Definition

We denote by **PAlg** the category whose objects are the partial algebras over $\mathbb{F}$ and morphisms are the homomorphisms of partial algebras.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let X be a Jacobi-Jordan algebra. Then

(i) $\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathrm{ADer}(X))$, *where* $U \colon \mathbf{JJord} \to \mathbf{PAlg}$ *denotes the forgetful functor.*

(ii) *if* $\mathrm{ADer}(X)$ *is a Jacobi-Jordan algebra, then it is the actor of* X.

Two-step nilpotent commutative algebras

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

- *The variety* $\mathbf{Nil}_2(\mathrm{Com})$ *is weakly action representable;*
- *A weak actor of an object* X *of* $\mathbf{Nil}_2(\mathrm{Com})$ *is the abelian algebra*

$$[X]_2 = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y = 0, \ \forall x \in X\}.$$

- *An arrow* $B \to [X]_2 \colon b \mapsto b * -$ *is an acting morphism if and only if*

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \ \forall x \in X.$$

The non-commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of non-associative algebras over $\mathbb{F}$ and let

$$\Phi_{k,i}(x_1, \ldots, x_k) = 0, \quad i = 1, \ldots, n,$$

where $k = \deg(\Phi_{k,i})$, be the multilinear identities which define $\mathcal{V}$.
- We look for a vector space $\mathcal{E}(X)$ such that

$$\mathrm{Inn}(X) = \{(L_x, R_x) \mid x \in X\} \leq \mathcal{E}(X) \leq \mathrm{End}(X)^2$$

and we want to endow it with a bilinear partial operation

$$\langle -, - \rangle \colon \Omega \subseteq X \times X \to X,$$

such that we can associate in a natural way a morphism of partial algebras $B \to \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$.

Split extensions

- Let

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

  be a split extension in $\mathcal{V}$.
- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

  with

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b').$$

- The action of B on X is defined by the pair of bilinear maps

$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$

  with $b * x' = l(b, x') = \beta(b) \cdot_A k(x')$, $x * b' = r(x, b') = k(x) \cdot_A \beta(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as above:

Proposition

Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps

$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as before. Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

where at least one of the α_1, ..., α_k is an element of of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$.

- The resulting algebra is the *semi-direct product* $B \ltimes X$.
- We denote $b * x \coloneqq (b, 0) \cdot (0, x)$ and $x * b \coloneqq (0, x) \cdot (b, 0)$.

The partial algebra $\mathcal{E}(X)$

$\mathcal{E}(X)$ is the subspace of all pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j \in \{1, \dots, k\}$ and $fx := f * x$, $xf := x * f$.
We define $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$\begin{aligned} x * h =& \lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &+ \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x =& \mu_1 (x * f) * g + \mu_2 (f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &+ \mu_5 (x * g) * f + \mu_6 (g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Example

If $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) = \mathrm{Bim}(X)$. If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$.

Remark

A split extension of B by X in $\mathcal{V}$ is the same as a linear map $B \to \mathcal{E}(X)$, defined by $b \mapsto (b * -, - * b)$, such that

$$((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$$

and $\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0$, for $i = 1, \dots, n$, where $\alpha_1, \dots, \alpha_k$ are as before.

Remark

The bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$ defines a partial operation $\langle -, - \rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage $\langle -, - \rangle^{-1}(\mathcal{E}(X))$ of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let $\mathcal{V}$ be a variety and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon \operatorname{SplExt}(-, X) \rightarrowtail \operatorname{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X))$$

  where τ_B sends an element of $\operatorname{SplExt}(B, X)$ to $(b * -, - * b)$.

- *The morphism of partial algebras $U(B) \to \mathcal{E}(X)\colon b \mapsto (b * -, - * b)$ belongs to $\operatorname{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$, as before.*
- *If $\mathcal{E}(X)$ is an object of $\mathcal{V}$, then $\mathcal{E}(X)$ is a weak actor of X.*
- *If $xy = \pm yx$ is an identity of $\mathcal{V}$, then $\mathcal{E}(X)$ is isomorphic to*

$$\{f \in \operatorname{End}(X) \mid \Phi_{k,i}(f, x_2, \ldots, x_k) = 0,\ \forall x_2, \ldots, x_k \in X\}$$

  endowed with $\langle f, g\rangle = \alpha(f \circ g) + \beta(g \circ f)$, where α, $\beta \in \mathbb{F}$ are given by the λ/μ rules.

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of $\mathsf{SplExt}(-, X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Example

We may check that, with the *obvious* choices of the λ/μ rules:

1. if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;
2. if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathsf{M}(X)$ is an external actor of X;
3. if $\mathcal{V} = \mathbf{JJord}$, then $\mathcal{E}(X) \cong \mathsf{ADer}(X)$ is an external actor of X;
4. if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathsf{Der}(X)$ is the actor of X;
5. if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAlg})$, then $\mathcal{E}(X) \cong [X]_2$ is a weak actor of X;
6. if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and

$$\mathsf{SplExt}(-, X) \cong \mathsf{Hom}_{\mathbf{Alt}}(-, X)$$

Open problems and future directions

- Does the converse of the implication

  $$\textit{weakly action representable category} \Rightarrow \textit{action accessible category}$$

  hold in the context of varieties of algebras over a field?
- Is it possible to generalize the costruction of the external weak actor to any operadic and action accessible variety of algebras with two not necessarily associative bilinear operations? We already have positive answers for the categories **Pois** and **CPois** of (commutative) Poisson algebras.
- We can use the construction of the external weak actor to study the representability of actions of varieties of unitary algebras, which are *ideally exact categories* in the sense of G. Janelidze. For instance, we have that the category of unitary associative algebras and that of unitary alternative algebras are action representable.

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