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ESSAYS ON SOCIAL AND ECONOMIC NETWORKS

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General Introduction

This dissertation studies the outcome of social and economic interactions when agents have the possibility to establish bilateral or multilateral partnerships. In chapters 1-3, we analyze the formation of networks resulting from the creation of bilateral partnerships among agents while in chapter 4 we develop a new solution concept to determine the outcome of games when any group of players may coordinate their moves.

Networks play an important role in the trade of many goods and services, and are the basis of the provision of mutual insurance in developing countries. They are critical to the scientific collaboration and to the dissemination of information. Starting with Jackson and Wolinsky (1996), the literature on network has typically addressed the question of which networks will eventually form when agents have the discretion to choose their connections, and has focused on whether those networks are socially efficient. A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that agents do not benefit from altering the structure of the network. The game-theoretic approach to network formation uses two different notions of a deviation by a coalition. Pairwise deviations (Jackson and Wolinsky, 1996) are deviations involving a single link at a time. Link addition is bilateral (two players that would be involved in the link must agree to add the link), and link deletion is unilateral (at least one player involved in the link must agree to delete the link). A network is pairwise stable if no player benefits from severing one of her links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Jackson and Wolinsky (1996) have shown that pairwise stable networks are not necessarily those that generate the highest total societal value. Groupwise deviations (Jackson and van den Nouweland, 2005) are deviations involving several links within some group of players at a time. Link addition is bilateral, link deletion is unilateral, and multiple link changes can take place at a time. A network is strongly stable if no group of agents benefit from rearranging their links. Whether a pairwise deviation or a groupwise deviation makes more sense will depend on the setting within which network formation takes place.

Chapters 1, 2 and 3 analyze which networks can be sustained in the long run when agents are farsighted, rather than myopic, in the sense that they are able to

forecast how other agents would react to their choice of partners. We also analyze whether the farsightedness of the agents solves the aforementioned conflict between stability and efficiency. In his survey of models of network formation, Jackson (2005) has mentioned that this is an important consideration in some appropriate context. He has stated that "in large networks it might be that players have very little ability to forecast how the network might change in reaction to the addition or deletion of a link. In such situations the myopic solutions are quite reasonable. However, if players have very good information about how others might react to changes in the network, then these are things that one wants to allow for either in the specification of the game or in the definition of the stability concept". There are two main approaches to tackle the question of farsightedness in network formation. The first approach follows the work of von Neumann and Morgenstern (1944), Harsanyi (1974) and Chwe (1994) and regroups notions which are based on indirect dominance and consistency. Dutta, Ghoshal and Ray (2005) have proposed another approach, closer in spirit to noncooperative games, where network formation is modelled through a dynamic game.

von Neumann and Morgenstern (1944) introduced the abstract stable set, defined as a set of networks which is such that no network in the set directly dominates another network in the set (internal stability) and each network not in the set is directly dominated by a network in the set (external stability). In this sense, a deviation from a stable network leading to a network outside the set is not accounted for since the network reached is itself unstable. The stable set is defined to be consistent. Harsanyi (1974) has suggested that something is missing with the use of the direct dominance relationship. A further deviation from an unstable network needs not invalidate but may actually encourage a deviation. "Suppose that coalition S does get payoff vector x replaced by some unstable payoff vector y , and that the latter in turn is later replaced by a third payoff vector y' . Then, this payoff vector y' itself may very well be a stable imputation lying in (the stable) set V , and may very possibly yield every player i in S a higher payoff $y'_i > x_i$ than the first payoff vector x would have yielded him. But, should this be the case, then every player i in S will have obtained a lasting benefit by getting x replaced with y (so as to permit another coalition S' in turn to get y replaced with the desirable payoff vector y'). Hence, a given payoff vector x may turn out to be unstable, even though it does belong to set V . (Harsanyi, 1974, p. 1474)". To circumvent this issue,

Harsanyi has proposed the notion of indirect dominance for general social environment. In the context of network formation, Herings, Mauleon and Vannetelbosch (2009) have introduced the pairwise farsighted improving path to identify the set of networks that indirectly dominate another when only one link can be changed at a time. A *pairwise farsighted improving path* is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the end network.¹ When players are farsighted and realize that deviations can be followed by further deviations and so on, it may be the case that multiple final outcomes may emerge after a move. All notions of farsighted stability based on consistency and indirect dominance require that the final outcome is itself stable, that is belongs to the set of stable networks. The different notions differ however on the behavioral characteristics of the agents. Some presume that the agents are optimistic when they deviate and consider the best outcome that might arise after a move (notions based on the von-Neuman-Morgenstern stable set such as the path dominance stable set of Page and Wooders (2009) or the optimistic stable standard of behavior of Greenberg (1990)), while other notions assume that agents are pessimistic and consider the worst credible outcome that might emerge after a deviation (Greenberg (1990) pessimistic standard of behavior, Chwe (1994) largest consistent set or Herings, Mauleon and Vannetelbosch (2009) pairwise farsightedly stable set). We work extensively in this dissertation with the pairwise farsightedly stable set of Herings, Mauleon and Vannetelbosch (2009). A pairwise farsightedly stable set is a set of networks that is immune to deviations leading to networks which are not in the set (external deviations), that satisfies external stability and that does not contain a subset of networks being a pairwise farsightedly stable set. A deviation from a network in the set to some network outside the set is deterred if, from the network reached, there is another network in the set that can be obtained by a farsighted

¹Page and Wooders (2009) have considered another notion of dominance, the path dominance. A network g path dominates another network g' if there is a finite sequence of networks, beginning in g and ending in g' where there is a farsighted improving path from each network along the sequence to its successor.

improving path such that at least one agent who has deviated initially is worse off. External stability requires that each network outside the set is indirectly dominated by some network in the set. The pairwise farsightedly stable set differs from the pairwise consistent set (Chwe, 1994) which is required to be immune not only to external deviations but also to internal deviations (deviations from a network in the set to another network in the set). In addition pairwise consistent set are not required to satisfy external stability, but Chwe (1994) has shown that the pairwise consistent set that contains all other pairwise consistent set, the largest pairwise consistent set, satisfies external stability.

Dutta, Ghosal and Ray (2005) have proposed another approach, closer in spirit to noncooperative game theory, where network formation is modelled through a dynamic game. At any date, a pair of agents is selected at random and has the possibility to add a link between them and to delete any existing link with any other player. The strategy of a player is a specification of which action to take at each possible network, for each possible pair of active players. A strategy profile then determines a Markov process on the set of networks, which creates a value for each player that corresponds to the discounted sum of future payoff. An equilibrium process of network formation is a strategy profile with the property that no active pair of players can benefit by departing from the prescribed strategy at every network, for each possible pair of active players. There are two main differences between this approach and those discussed previously. First, all the stream of payoffs that arises when one decides to induce a move matters, and not only the payoff obtained in the final network. Second, players have beliefs about how the game is played in every network reached, which are required to be correct at equilibrium. Thus, when a player deviates, he evaluates his future payoff according to his equilibrium beliefs, and does not have to formulate pessimistic or optimistic beliefs concerning future outcome that might arise.

All the notions of farsighted stability suffer from being very difficult to apply.²

²We have implemented an algorithm to identify the pairwise farsightedly stable sets of networks. Unfortunately, we have only been able to run it successfully for the case of three players. For more than three players, the number of candidates becomes too high to be treated by Matlab. Indeed, among n agents, there are possibly $n(n-1)/2$ links and thus $K = \sum_{i=0}^{n(n-1)/2} C_i^{n(n-1)/2}$ networks. Among those K networks, there are possibly $\sum_{i=1}^K C_i^K$ equilibrium set candidates, a number that explodes with n as it corresponds to 256 candidates when $n = 3$ and $1,844,67E + 19$ candidates when $n = 4$. Notice that the problem arises from the number of candidates, not from the definition

To our knowledge, no existing work characterizes the farsightedly stable networks of some classical games of network formation. We believe that the absence of such work arises from its complexity. This is in fact not surprising as even pairwise stable networks cannot always be fully characterized. Our objective in the three first chapters is to better understand the formation of networks when agents are farsighted. We have characterized the pairwise farsightedly stable set of network of Herings, Mauleon and Vannetelbosch (2009) of some games of network formation or under specific conditions on the primitives of the game, and we have related our results with those obtained under alternative notions of myopic and farsighted stability. We have adopted the concept of Herings, Mauleon and Vannetelbosch (2009) not because we believe that the concept is better at modelling the behavior of farsighted agents, but because it is more tractable. First, Herings, Mauleon and Vannetelbosch (2009) have proposed easy to verify conditions to characterize a pairwise farsightedly stable set (Theorem 3, p.533) and to characterize the unique pairwise farsightedly stable set (Theorem 5, p.533). Second, there are situations where we are not able to fully characterize the set of pairwise farsightedly stable networks, but we are still able to identify some equilibrium candidates. By definition, the largest pairwise consistent set is unique and contains all pairwise consistent set. Thus, if one is not able to find them all, nothing can be said about the largest consistent set and each consistent set is not necessarily externally stable.

The first chapter "Connections among farsighted agents", joint with Ana Mauleon and Vincent Vannetelbosch, identifies necessary and sufficient conditions on the primitives of the game so that the notion of pairwise farsighted stability coincides with the set of efficient networks and with the set of networks that are immune to coalitional deviations when agents are myopic (strongly stable networks). This result is established when the value of each component of a network does not depend on the structure of the other components and is distributed equally among the players within each component. In that case, we show that the pairwise farsightedly stable set refines the notion of pairwise stability by eliminating the inefficient pairwise stable networks if and only if the value function is top-convex, that is if and only if the per-capita value generated by the efficient network is higher than the one generated by networks formed by any subgroup of agents. We then reconsider some classical models of network formation. In the symmetric connections model (Jackson and

of the concept. We would thus not be able to run an algorithm for any other set-valued concept.

Wolinsky, 1996), agents form links in order to exchange information. It is assumed that information that travels a long distance in the network becomes diluted and is less valuable than the one obtained from a closer neighbor, providing incentives to create direct connections. However, each link formed results in a cost for both players. As a result, the pairwise stable networks have a star architecture as long as the cost of link formation is neither too high (no link is created), nor too small (every pair of agents is connected). There are however values of the costs that are such that no link is formed, and thus no value is generated, while the star network would be efficient. This results in a conflict between stability and efficiency which arises because the center of the star supports half of the total costs and is better off by cutting links. We show that replacing myopic by farsighted players in this model does not eliminate this conflict. The concept of pairwise stability is quite robust to the introduction of farsighted players because, for a large range of parameters, we have that pairwise stable networks belong to pairwise farsightedly stable sets. In the model of buyer-seller networks of Kranton and Minehart (2001), each buyer has an object to sell and buyers and sellers form links to potentially exchange an object. The buyers who are connected to a seller participate in a standard second-price auction, which determines the price of each good sold and its allocation to a buyer. We show that while myopic agents always form inefficient networks, some efficient networks can be sustained at equilibrium by farsighted agents. Farsightedness may in this case lead utility maximizing agents behaving in a way that is socially efficient.

In the second chapter "A characterization of farsightedly stable networks", joint with Ana Mauleon and Vincent Vannetelbosch, we first introduce the notion of groupwise farsightedly stable set of networks, which is the counterpart of the pairwise farsightedly stable set of networks when groups of agents may jointly reorganize their links. We show by means of examples that there are no general relationships between pairwise and groupwise farsightedly stable sets of networks. We provide an algorithm that characterizes the unique pairwise farsightedly stable set of networks when the value is allocated equally among the players of a component. The algorithm selects networks as follows. First pick a component that maximizes the per capita value out of those that can be formed among all the agents. Then remove the players needed to form this component and pick a second component that maximizes the per capita value out of those that can be formed among the remaining agents. By iteration, we obtain a network composed of the identified components. We show

that the unique pairwise farsightedly stable set of networks is the set of all networks that can be found through this algorithm. The main result of the chapter is that this set coincides with the unique groupwise myopically and farsightedly stable set of networks but not with the unique pairwise myopically stable set of networks. We conclude that, (i) if groupwise deviations are allowed then whether players are farsighted or myopic does not matter; (ii) if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter.

In the third chapter "Risk-sharing networks and farsighted stability", I study how the formation of risk-sharing networks is affected by the presence of farsighted agents. There are regions in developing countries where economic fluctuations are important but the access to a formal insurance market is limited. A growing empirical literature has shown that individuals rely on informal risk-sharing agreements to help those that are in need (see the synthesis of the literature of Alderman and Paxson, 1994). Bramoullé and Kranton (2007a) have proposed a model where pairs of agents may create bilateral agreements to insure each other against shocks to their income. They have shown that the efficient networks are such that each agent is indirectly connected to the others, involving the maximal level of insurance in the population, while myopic agents form networks which do not imply full income pooling. By introducing farsighted agents in the model of Bramoullé and Kranton (2007a), I find that for intermediate costs of establishing and maintaining a partnership, the farsightedness of the agents leads to a reduction of the tension between stability and efficiency that arises when agents are myopic. Some pairwise farsightedly stable sets are composed of efficient networks only while fully connected networks cannot be sustained at equilibrium when agents are myopic. Two mechanisms explain this result: (i) Farsighted agents belonging to small groups may decide to create new partnerships that are not directly profitable to them, because they realize that other partners will further join this bigger and more attractive group. In other words, the farsightedness of the agents may solve a coordination problem. (ii) Farsighted agents may refrain from deleting costly links if they belong to a big group, as they understand that this may induce others to rearrange their partnerships in a way that deters the myopic incentives to delete the link at first.

In chapter 4 "Strongly rational sets for normal form games", (joint with Ana Mauleon and Vincent Vannetelbosch), we propose a new solution concept, the strong curb set, to analyze stability in non-cooperative games when groups of agents may

coordinate their moves. When groups of agents select a joint action, they select an action profile which ensures that the expected payoff of each member of the group is increased with respect to the payoff he would have obtained by playing individually. Strong curb sets are product sets of pure strategies such that a player's set of recommended strategies contains all the actions that are in some coalitional best responses, for every coalition he might belong to, and every belief each coalition member may have that is consistent with the recommendations to the other players. We show that a strong curb set always exists, as opposed to other coalitional Nash equilibrium concepts. If a Nash equilibrium is immune to coalitional deviations (strong Nash equilibrium), then it is a strong curb set. Our new concept is thus a set-theoretic coarsening of the strong Nash equilibrium concept. We provide a class of dynamic learning processes where groups of agents may coordinate their actions. A game is played at discrete points in time. At the beginning of each period the players are partitioned into coalitions to form a coalition structure. Each coalition structure has a positive probability to occur at each period. Players observe how the game has been played in the recent past, form their beliefs upon these observations, and select an action profile jointly with their coalition partners. We show that at the limit, if memory is long enough, play settles down in a minimal strong curb set.

Outside the equilibrium framework, Bernheim (1984) and Pearce (1984) have introduced the notion of rationalizability. Strategies that survive an iterative procedure which eliminates at each round strategies that are never best responses are rationalizable. Basu and Weibull (1991) have proposed the notion of curb sets, which are product set of strategies containing all best responses of each player to whatever belief they may have that is consistent with the recommendations to the other players. Basu and Weibull (1991) have shown that every strategy contained in a minimal curb set is rationalizable, and that the set of rationalizable strategies coincide with the maximal tight curb set where tight curb sets are curb sets that are identical with their own best response correspondences. Ambrus (2006) has introduced the concept of coalitional rationalizability by proposing an iterative procedure that eliminates at each round strategies that groups of players should not play. At some step of the iterative procedure, members of a coalition may agree to reduce the set of actions they will play. They do so if, for any player in the coalition, any belief to which he has a best response strategy outside the agreement yields a strictly lower expected payoff than the payoff he gets by best responding to any belief that

is consistent with other players in the coalition keeping the agreement, holding the marginal expectation concerning the play of non-members fixed. Strategies that survive this iterative procedure are coalitionally rationalizable. Ambrus (2006) has shown that the set of coalitionally rationalizable strategies is a tight curb set. It is thus a subset of the set of rationalizable strategies. One might expect minimal strong curb strategies to be coalitionally rationalizable. This is however not true. Indeed, minimal strong curb strategies are not necessarily (individually) rationalizable. The absence of relationship between the two notions comes from the different approaches to model group play. Under coalitional rationalizability, players may, by introspection, understand that some group of agents could reach common gain by restricting the set of strategies they use. A restriction has to be self-enforcing: if a player believes that all others play according to the restriction, then it is in his own interest to play according to it as well. On the other hand, under the strong curb notion, we assume that agents may talk to each other and commit to take a joint action, even if this joint action is not self-enforcing in the sense that the outcome reached by the deviating agents need not be stable. To illustrate this point, consider the prisoner's dilemma example. Coalitional rationalizability predicts that the Pareto dominated outcome, which is the unique Nash equilibrium, will be played since individual agents have a dominated strategy that they will never use. However, the Nash equilibrium is not immune to joint deviation, and as such is not a strong Nash equilibrium. The only minimal strong curb set of this game is the product set of all strategies.³ By allowing for every possible deviations, our concept may be weak at determining actual play, but predicts with confidence: if players have beliefs with support in a minimal strong curb set (say because they realize this is an equilibrium or because they are able to observe history of play to gain information on likely choices of others), then we can predict with confidence that players will actually choose an action in the set.

To sum up, chapter 1 identifies necessary and sufficient conditions on the primitives of the game so that farsighted agents form efficient networks. It shows that under those conditions, pairwise farsighted stability refines pairwise stability by

³Note however that the Nash equilibrium of the prisoner's dilemma example is coalition-proof. Coalition-proofness requires deviations to be self enforcing in the sense that if a coalition deviates, every possible subcoalition should be willing to conform to the coalitional move. It is straightforward to modify the definition of strong curb set to consider only self-enforcing deviations.

eliminating the inefficient pairwise stable networks. In chapter 2, we provide an algorithm that characterizes the unique pairwise farsightedly stable set of networks when the value is allocated equally among the players of a component. It is shown that (i) if groupwise deviations are allowed then whether players are farsighted or myopic does not matter; (ii) if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter. Chapter 3 analyzes the formation of risk-sharing networks among farsighted agents in rural areas of developing countries. We provide a theoretical explanation of the observation that risk-sharing takes place among agents having common characteristics (neighborhood, professional or religious affiliation, kinship, etc.). In chapter 4, we propose a new solution concept, the strong curb set, to analyze stability in non-cooperative games when groups of agents may coordinate their moves. We relate our concept with the standard notions of stability. It is shown that there is a class of dynamic learning processes such that at the limit, if memory is long enough, play settles down in a strong curb set.

Chapter 1.

Connections among farsighted agents⁴

1.1. Introduction

The network structure of social interactions influences a variety of behaviors and economic outcomes, including the formation of opinions, decisions on which products to buy, investment in education, access to jobs, social mobility, informal borrowing and lending, favor exchange, and participation in loan programs.⁵ A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that individuals do not benefit from altering the structure of the network. Jackson and Wolinsky (1996) have proposed the notion of pairwise stability. A network is pairwise stable if no individual benefits from severing one of her links and no two individuals benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Pairwise stability is a myopic definition. Individuals are not farsighted in the sense that they do not forecast how others might react to their actions. For instance, the adding or severing of one link might lead to subsequent addition or severing of another link. If individuals have very good information about how others might react to changes in the network, then these are things one wants to allow for in the definition of the stability concept. For instance, a network could be stable because individuals might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original individuals.

Herings, Mauleon and Vannetelbosch (2009) have proposed the notion of pairwise farsightedly stable sets of networks that predicts which networks one might expect to emerge in the long run when individuals are farsighted.⁶ A set of networks G is

⁴This chapter is based on an article co-authored with Ana Mauleon and Vincent Vannetelbosch forthcoming in *Journal of Public Economic Theory*.

⁵See Jackson (2008) or Goyal (2007) for a comprehensive introduction to the theory of social and economic networks.

⁶Other approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon, and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Page, Wooders and Kamat (2005), Dutta, Ghosal, and Ray (2005), and Page and Wooders (2009).

pairwise farsightedly stable (i) if all possible pairwise deviations from any network $g \in G$ to a network outside G are deterred by the threat of ending worse off or equally well off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set,⁷ and (iii) if there is no proper subset of G satisfying Conditions (i) and (ii). A non-empty pairwise farsightedly stable set always exists. Herings, Mauleon and Vannetelbosch (2009) have provided a full characterization of unique pairwise farsightedly stable sets of networks. Contrary to other pairwise concepts, pairwise farsighted stability yields a Pareto dominant network, if it exists, as the unique outcome.⁸

The objective of this chapter is twofold. First, we provide some primitive conditions on value functions and allocation rules so that the set of strongly efficient networks is the unique pairwise farsightedly stable set. We find that, under the componentwise egalitarian allocation rule, the set of strongly efficient networks and the set of pairwise myopically stable networks that are immune to coalitional deviations are the unique pairwise farsightedly stable set if and only if the value function is top convex. A value function is top convex if some strongly efficient network also maximizes the per capita value among individuals.

Second, we investigate in some classical models of social and economic networks whether the pairwise farsightedly stable sets of networks coincide with the set of pairwise myopically stable networks and the set of strongly efficient networks. We reconsider three classical models of network formation in which the aforementioned primitive conditions on value functions and/or allocation rules break down: Jackson and Wolinsky (1996) symmetric connections model, Corominas-Bosch (2004) model of trading networks with bilateral bargaining, and Kranton and Minehart (2001) model of buyer-seller networks. We have chosen to analyze those models because

⁷A farsighted improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network.

⁸Herings, Mauleon and Vannetelbosch (2009) have also studied the relationship between pairwise farsighted stability and other concepts. Any von Neumann-Morgenstern pairwise farsightedly stable set is a pairwise farsightedly stable set. But, von Neumann-Morgenstern pairwise farsightedly stable set may fail to exist. Pairwise farsightedly stable sets have no relationship to either largest pairwise consistent sets or sets of pairwise stable networks.

they have different features. The symmetric connections model is a situation where *homogeneous* individuals obtain payoffs not only from direct but also from *indirect* connections (where links represent social relationships between individuals such as friendships), while the models of buyer-seller networks are situations where *heterogeneous* individuals (sellers and buyers) bargain over prices for trade (where *direct* links are necessary for a transaction to occur). We find that, in the symmetric connections model, farsightedness does not eliminate the conflict between stability and strong efficiency that may occur when costs are intermediate. However, farsightedness helps to reduce the conflict when costs are large enough. In the bargaining model of Corominas-Bosch (2004), myopic or farsighted notions of stability sustain the set of strongly efficient networks when the costs of maintaining links are not too large. In the Kranton and Minehart (2001) model, pairwise farsighted stability may sustain the strongly efficient network while pairwise myopic stability only sustains networks that are strongly inefficient or even Pareto dominated.

The chapter is organized as follows. In Section 2 we introduce some notations and basic properties and definitions for networks. In Section 3 we define the notions of pairwise myopically stable sets and of pairwise farsightedly stable sets. In Section 4 we look at the relationship between farsighted stability and efficiency of networks. In Section 5 we reconsider Jackson and Wolinsky (1996) symmetric connections model, Corominas-Bosch (2004) model of trading networks, and Kranton and Minehart (2001) model of buyer-seller networks. In Section 6 we conclude.

1.2. Networks

Let $N = \{1, \dots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. We write $ij \in g$ to indicate that i and j are linked under the network g . Let g^S be the set of all subsets of $S \subseteq N$ of size 2.⁹ So, g^N is the complete network. The set of all possible networks or graphs on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N . The

⁹Throughout the dissertation we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. Let $g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}$. Thus, $g|_S$ is the network found deleting all links except those that are between players in S . For any network g , let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g . A path in a network $g \in \mathbb{G}$ between i and j is a sequence of players $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A non-empty network $h \subseteq g$ is a component of g , if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j , and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$.

A value function is a function $v : \mathbb{G} \rightarrow \mathbb{R}$ that keeps track of how the total societal value varies across different networks. The set of all possible value functions is denoted by $\mathcal{V}$. An allocation rule is a function $Y : \mathbb{G} \times \mathcal{V} \rightarrow \mathbb{R}^N$ that keeps track of how the value is allocated among the players forming a network. It satisfies $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g .

Jackson and Wolinsky (1996) have proposed a number of basic properties of value functions and allocation rules. A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in \mathbb{G}$. Component additive value functions are the ones for which the value of a network is the sum of the value of its components. Given a permutation of players π and any $g \in \mathbb{G}$, let $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\}$. Thus, g^π is a network that is identical to g up to a permutation of the players. A value function is *anonymous* if for any permutation π and any $g \in \mathbb{G}$, $v(g^\pi) = v(g)$.

For a component additive v and network g , the *componentwise egalitarian allocation rule* Y^{ce} is such that for any $h \in C(g)$ and each $i \in N(h)$, $Y_i^{ce}(g, v) = v(h)/\#N(h)$. For a v that is not component additive, $Y^{ce}(g, v) = v(g)/n$ for all g ; thus, Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive.

In evaluating societal welfare, we may take various perspectives. A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in \mathbb{G}$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with at least one strict inequality. A network $g \in \mathbb{G}$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in \mathbb{G}$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that agents do not benefit from altering

the structure of the network. A weak version of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of her links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Formally, a network g is pairwise stable with respect to value function v and allocation rule Y if (i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and (ii) for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.

1.3. Pairwise farsightedly stable sets of networks

Herings, Mauleon and Vannetelbosch (2009) have proposed the notion of pairwise myopically stable sets of networks which is a generalization of Jackson and Wolinsky (1996) pairwise stability notion. Pairwise stable networks do not always exist. A pairwise myopically stable set of networks is a set such that (i) from any network outside this set, there is a myopic improving path leading to some network in the set, (ii) each deviation outside the set is deterred because the deviating players do not prefer the resulting network, and (iii) there is no proper subset satisfying (i) and (ii). The notion of a myopic improving path was first introduced in Jackson and Watts (2002). A myopic improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the resulting network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the resulting network.

Jackson and Watts (2002) have defined the notion of a closed cycle. A set of networks C is a *cycle* if for any $g \in C$ and $g' \in C \setminus \{g\}$, there exists a myopic improving path connecting g to g' . A cycle C is a *maximal cycle* if it is not a proper subset of a cycle. A cycle C is a *closed cycle* if no network in C lies on a myopic improving path leading to a network that is not in C . A closed cycle is necessarily a maximal cycle. Herings, Mauleon and Vannetelbosch (2009) have shown that the set of networks consisting of all networks that belong to a closed cycle is the unique pairwise myopically stable set.

A *farsighted improving path* is a sequence of networks that can emerge when

players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network. Formally, a farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of graphs $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ or $Y_j(g_K, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ and $Y_j(g_K, v) \geq Y_j(g_k, v)$. For a given network g , let $F(g)$ be the set of networks that can be reached by a farsighted improving path from g . Notice that $F(g)$ may contain many networks and that a network $g' \in F(g)$ might be the endpoint of several farsighted improving paths starting in g .

We now introduce a solution concept due to Herings, Mauleon and Vannetelbosch (2009), the pairwise farsightedly stable set.

Definition 1.1. A set of networks $G \subseteq \mathbb{G}$ is pairwise farsightedly stable with respect v and Y if

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $\exists g' \in F(g + ij) \cap G$ such that $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ or $Y_i(g', v) < Y_i(g, v)$ or $Y_j(g', v) < Y_j(g, v)$,
- (ib) $\forall ij \in g$ such that $g - ij \notin G$, $\exists g', g'' \in F(g - ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$.
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Condition (i) in Definition 1.1 requires the deterrence of external deviations. Condition (ia) captures that adding a link ij to a network $g \in G$ that leads to a network outside of G , is deterred by the threat of ending in g' . Here g' is such that there is a farsighted improving path from $g + ij$ to g' . Moreover, g' belongs to G , which makes g' a credible threat. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) in Definition 1.1 requires external

stability and implies that the networks within the set are robust to perturbations. From any network outside of G there is a farsighted improving path leading to some network in G . Condition (ii) implies that if a set of networks is pairwise farsightedly stable, it is non-empty. Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 1.1. This motivates the requirement of a minimality condition, namely Condition (iii). Herings, Mauleon and Vannetelbosch (2009) have shown that a pairwise farsightedly stable set of networks always exists.

A network g strictly Pareto dominates all other networks if g is such that for all $g' \in \mathbb{G} \setminus \{g\}$ it holds that, for all i , $Y_i(g, v) > Y_i(g', v)$. Although the network that strictly Pareto dominates all others is pairwise stable, there might be many more pairwise stable networks. Herings, Mauleon and Vannetelbosch (2009) have shown that, if there is a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique pairwise farsightedly stable set. Thus, pairwise farsighted stability singles out the Pareto dominating network as the unique pairwise farsightedly stable set.

1.4. Farsighted stability and efficiency

We now provide some alternative primitive conditions on value functions and allocation rules so that the set of strongly efficient networks is the unique pairwise farsightedly stable set. It will turn out that under the conditions we will impose the notion of pairwise farsighted stability refines the notion of pairwise stability by eliminating the inefficient pairwise stable networks.

A value function v is *top convex* if some strongly efficient network also maximizes the per capita value among players. Let $\rho(v, S) = \max_{g \in g^S} v(g)/\#S$. The value function v is *top convex* if $\rho(v, N) \geq \rho(v, S)$ for all $S \subseteq N$.

Proposition 1.1. *Consider any anonymous and component additive value function v . The set of strongly efficient networks $E(v)$ is the unique pairwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce} if and only if v is top convex.*

The proof of all propositions can be found in the appendix. The intuition behind the proof of this proposition is as follows. Take any anonymous and component additive value function v . First, if the value function v is top convex then all components of a strongly efficient network must lead to the same per-capita value.

In addition, under the componentwise egalitarian allocation rule Y^{ce} , any strongly efficient network Pareto dominates all inefficient networks. Then, it is immediate that there are no farsighted improving paths emanating from a strongly efficient network, and that from any inefficient network there are improving paths to each strongly efficient network. Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \emptyset\}$ and for every $g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$, we have that the set of strongly efficient networks, $E(v)$, is the unique pairwise farsightedly stable set.

Second, if $E(v)$ is the unique pairwise farsightedly stable set, then we know from Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) that there are no improving paths emanating from any strongly efficient network. It follows that all players receive the same allocation in any strongly efficient network under the componentwise egalitarian allocation rule and that all players receive more in any strongly efficient network than in any inefficient network. Otherwise, there would exist a farsighted improving path from the strongly efficient network $g \in E(v)$. Thus, we have that v is top convex.

Jackson and van den Nouweland (2005) have shown that the set of strongly efficient networks coincides with the set of strongly stable networks under the componentwise egalitarian allocation rule if and only if v is top convex.¹⁰ Hence, the set of strongly stable networks is the unique pairwise farsightedly stable set under the componentwise egalitarian allocation rule if and only if the value function is top convex. So, pairwise farsighted stability selects under Y^{ce} the pairwise stable networks that are immune to coalitional deviations if and only if v is top convex.

Notice that top convexity is a condition that is satisfied in some natural situations. For instance, the value function of Jackson and Wolinsky (1996) symmetric connections model is top convex for all values of $\delta \in [0, 1)$ and $c \geq 0$, so that all strongly efficient networks with respect to v form the unique pairwise farsightedly stable set with respect to Y^{ce} and v .¹¹

¹⁰Jackson and van den Nouweland (2005) have proposed a refinement of pairwise stability where coalitionwise deviations are allowed: the strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component.

¹¹Provided that n is even, the value function of Jackson and Wolinsky (1996) co-author model

1.5. Two models of social and economic networks

We now investigate in some classical models of social and economic networks whether the pairwise farsightedly stable sets of networks coincide with the set of pairwise (myopically) stable networks and the set of strongly efficient networks. We reconsider three classical models of network formation when the aforementioned primitive conditions on value functions and/or allocation rules break down. In Jackson and Wolinsky (1996) symmetric connections model and in Corominas-Bosch (2004) model of trading networks with bilateral bargaining, the value function is top convex but the allocation rule is not componentwise egalitarian. In Kranton and Minehart (2001) model of buyer-seller networks, the value function violates top convexity and the allocation rule is not componentwise egalitarian.

1.5.1. The symmetric connections model

In Jackson and Wolinsky (1996) symmetric connections model, players form links with each other in order to exchange information. If player i is connected to player j by a path of t links, then player i receives a payoff of δ^t from her indirect connection with player j . It is assumed that $0 < \delta < 1$, and so the payoff δ^t decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost c to both i and j . This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Player i 's payoff from a network g is given by

$$Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j: ij \in g} c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). Here the value of network g equals $v(g) = \sum_{i \in N} Y_i(g)$ and is top convex. Let g^* denote a star network encompassing everyone and $g^\emptyset$ be the empty network (no links).

is top convex as the strongly efficient network always involves pairs of players who are linked to each other. The value function of Herings, Mauleon and Vannetelbosch (2009) criminal networks model is top convex too. Finally, the value function of Bramoullé and Kranton (2007) risk sharing networks model is top convex when the utility function is quadratic.

Jackson and Wolinsky (1996) have shown that the unique strongly efficient network is (i) the complete network g^N if $c < \delta(1 - \delta)$, (ii) a star encompassing everyone if $\delta(1 - \delta) < c < \delta + ((n - 2)/2)\delta^2$, and (iii) the empty network if $\delta + ((n - 2)/2)\delta^2 < c$. For $c < \delta(1 - \delta)$, the unique pairwise stable network is the complete network g^N . For $\delta(1 - \delta) < c < \delta$, a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable network. For $\delta < c$, any pairwise stable network which is non-empty is such that each player has at least two links and thus is inefficient. Thus, there is a conflict between efficiency and pairwise stability for a large range of the parameters. Indeed, only for $c < \delta(1 - \delta)$, there is no conflict between the efficient and the pairwise stable networks. When $\delta(1 - \delta) < c < \delta$, the efficient network is pairwise stable, but there are other pairwise stable networks that are not efficient. For $\delta < c < \delta + ((n - 2)/2)\delta^2$, the efficient network is never pairwise stable. And, finally, for $\delta + ((n - 2)/2)\delta^2 < c$, the efficient network is pairwise stable, but there could be other pairwise stable networks that are not efficient.¹² Let us define $\bar{c}(n)$ as the highest cost of links formation such that the payoff of all players is nonnegative in at least one network other than the empty network. Formally $\bar{c}(n) = \max\{c \in \mathbb{R} \mid \max_{g \in \mathcal{G} \setminus \{g^\emptyset\}} (\min_{i \in \mathbb{N}} (Y_i(g))) = 0\}$. We show in the appendix that $\delta < \bar{c}(n) < \delta + ((n - 2)/2)\delta^2$ if $n \geq 4$.

Proposition 1.2. Take the symmetric connections model.

- (i) For $c < \delta(1 - \delta)$, a set consisting of the complete network, $\{g^N\}$, is the unique pairwise farsightedly stable set.
- (ii) For $\delta(1 - \delta) < c < \delta$, every set consisting of a star network encompassing all players, $\{g^*\}$, is a pairwise farsightedly stable set of networks, but they are not necessarily the unique pairwise farsightedly stable sets.
- (iii) For $c > \delta$, a set consisting of the empty network, $\{g^\emptyset\}$, is the unique pairwise farsightedly stable set if $c > \bar{c}(n)$, while $\{g^\emptyset\}$ is not a pairwise farsightedly stable set if $c \leq \bar{c}(n)$.

Proposition 1.2 shows that replacing myopic by farsighted players in the symmetric connections model does not eliminate the conflict between strong efficiency

¹²If $\delta > c > n(\delta - \delta^{n-1})$ then the myopically stable set consists only of pairwise stable networks. Similarly, if c is very small or very large then the symmetric connections model has no cycles. See Jackson and Watts (2001).

and stability but, sometimes, it may help to reduce it. For instance, when $\delta + ((n - 2)/2)\delta^2 < c$, a set consisting of the unique strongly efficient network is the unique pairwise farsightedly stable set while other networks may be pairwise stable.¹³ Thus, pairwise farsighted stability may single out the strongly efficient network even though the allocation rule is not componentwise egalitarian. In fact, each player receives the same payoff in the empty network $g^\varnothing$ (so no player would like to take the role of another player in $g^\varnothing$) as if the componentwise egalitarian allocation rule was used. It follows that in any other network g there is some player with a negative payoff that prefers $g^\varnothing$ and hence there is no farsighted improving from $g^\varnothing$ to g but there is a farsighted improving path from g to $g^\varnothing$ (players with negative payoffs in g have incentives to delete their links looking forward to $g^\varnothing$). That is, $g \notin F(g^\varnothing)$, $F(g^\varnothing) = \varnothing$ and $g^\varnothing \in F(g)$ for all $g \neq g^\varnothing$. Regarding the relationship between pairwise stability and pairwise farsighted stability, we observe that the concept of pairwise stability is quite robust to the introduction of farsighted players because, for a large range of parameters, we have that pairwise stable networks belong to pairwise farsightedly stable sets.

Watts (2001) has analyzed the process of network formation in a dynamic framework where pairs of myopic players meet and decide whether or not to form or sever links with each other based on the improvement the resulting network offers relative to the current network. If the benefit from maintaining an indirect link is greater than the net benefit from maintaining a direct link (case (ii) of Proposition 1.2), then it is difficult for the strongly efficient network (which is the star network) to form. In fact, starting at the empty network, the strongly efficient network only forms if the order in which the players meet takes a particular pattern. Moreover, as the number of players increases it becomes less likely that the strongly efficient network forms. These results contrast with ours, for such parameter values, since every set consisting of a star network is a pairwise farsightedly stable set whatever the number of farsighted players. Thus, it is not unlikely that forward looking players will increase the chances of the star forming.

¹³For instance, Jackson and Wolinsky (1996) have shown that a "tetrahedron" involving 16 players is pairwise stable but is not strongly efficient.

1.5.2. Buyer-seller networks

A model of trading networks with bilateral bargaining

Corominas-Bosch (2004) has developed a simple model of trading networks with bilateral bargaining. The market consists of m sellers $1, 2, \dots, m$ and m buyers $m + 1, m + 2, \dots, 2m$. We denote the set of buyers as B and the set of sellers as S . Each seller owns a single object to sell that has no value to the seller. Buyers have a valuation of 1 for an object and do not care from whom they purchase the good. If a seller and a buyer trade at price p , the seller receives a payoff of p and the buyer a payoff of $1 - p$. Agents are embedded in a network that links sellers and buyers, and trade is only possible among linked agents. That is, a link in the network represents the opportunity for a buyer and a seller to bargain and potentially exchange an object.¹⁴ Let $\mathbb{G}(S, B) = \{g \in \mathbb{G} \mid ij \in g \Leftrightarrow i \in S \text{ and } j \in B\}$ be the set of feasible buyer-seller networks. Agents incur a cost of maintaining each link equal to c_s for sellers and to c_b for buyers. So the payoff to an agent is her payoff from any trade on the network, less the cost of maintaining any links that she is involved with.

In the first period sellers simultaneously call out prices. A buyer can only select from the prices that she has heard called out by the sellers to whom she is linked. Buyers simultaneously respond by either choosing to accept a single price offer received or rejecting all price offers received.¹⁵ At the end of the period, trades are made and buyers and sellers who have traded are cleared from the market. In the next period the situation reverses and buyers call out prices. These are then either accepted or rejected by the sellers connected to them. Each period the role of proposer and responder alternates and this process repeats itself until all remaining buyers and sellers are not linked to each other. Buyers and sellers are impatient so that a transaction at price p in period t is worth $\delta^t p$ to a seller and $\delta^t(1 - p)$ to a buyer with $0 < \delta < 1$ being the common discount factor. In a subgame perfect equilibrium with very patient agents (δ close to 1), there are effectively three possible outcomes for any given agent (ignoring the costs of maintaining links): either she

¹⁴A link is necessary between a buyer and a seller for a transaction to occur, but if an agent has several links, then there are several possible trading patterns. The network structure essentially determines the bargaining power of buyers and sellers.

¹⁵If there are several sellers who have called out the same price and/or several buyers who have accepted the same price, and there is any discretion under the given network connections as to which trades should occur, then there is a careful protocol for determining which trades occur. The protocol is essentially designed to maximize the number of transactions.

gets all the available gains from trade (1), or half of the gains from trade ($1/2$), or none of the available gains from trade (0). Corominas-Bosch (2004) has provided an algorithm that subdivides any network into three types of subnetworks: those in which a set of sellers are collectively linked to a larger set of buyers (sellers obtain 1 as payoffs, and buyers receive 0); those in which the collective set of sellers is linked to the same-sized collective set of buyers (each receives $1/2$); and those in which sellers outnumber buyers (sellers receive 0, and buyers get 1).¹⁶ In Figure 1 we give the limit payoffs of Corominas-Bosch model for some networks. The value function is simply the sum of the payoffs (including the cost of maintaining links) and is top convex.

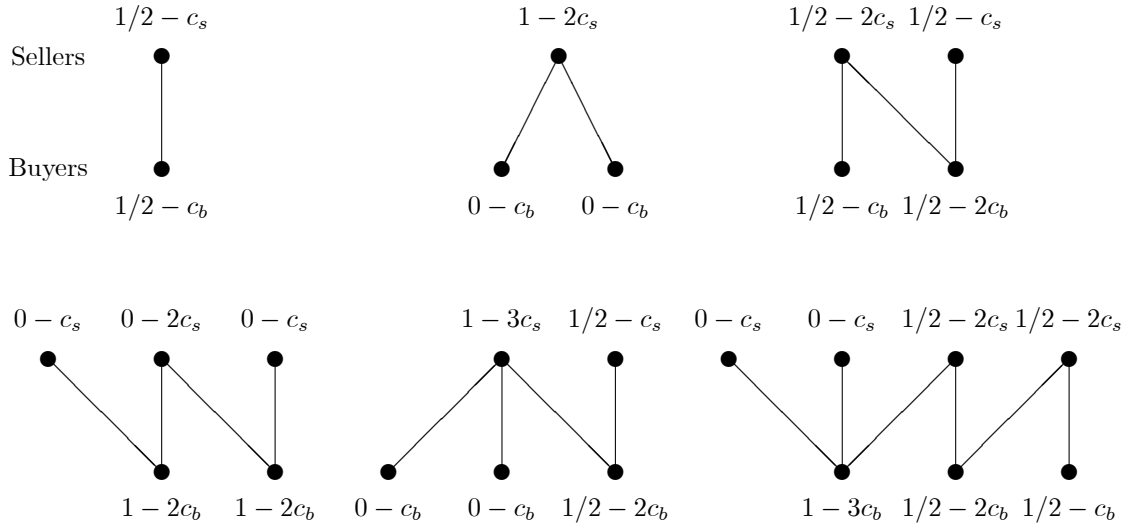


Figure 1 : Limit payoffs in the Corominas-Bosch (2004) model for some networks.

Let G_2 be the set of all buyer-seller networks consisting of pairs and so that the maximum number of potential pairs must form. That is, $G_2 = \{g \in \mathbb{G}(S, B) \mid l_i(g) = 1 \ \forall i \in S \cup B\}$ where $l_i(g)$ is the number of links player i has in g . Jackson

¹⁶The algorithm works as follows. Step 1a: Identify groups of two or more sellers who are all linked only to the same buyer. Regardless of that buyer's other connections, eliminate that set of sellers and buyer (with the buyer obtaining 1 and the sellers receiving 0). Step 1b: On the remaining network, repeat step 1a but with the role of buyers and sellers reversed. Step k : Proceed inductively in k , each time identifying subsets of at least k sellers who are collectively linked to some set of fewer-than- k buyers, or some collection of at least k buyers who are collectively linked to some set of fewer-than- k sellers. End: When all such subgraphs are removed, the buyers and sellers in the remaining network are such that every subset of sellers is linked to at least as many buyers and vice versa, and the buyers and sellers in that subnetwork get $1/2$.

(2003) has shown that, in the Corominas-Bosch model with $1/2 > c_s > 0$ and $1/2 > c_b > 0$, the set of pairwise stable networks is G_2 which is exactly the set of strongly efficient networks.¹⁷ The intuition for this result is straightforward. An agent having a payoff of 0 cannot have any links since by deleting a link she could save the link cost and not lose any benefit. So, all agents who have links must obtain payoffs of $1/2$ (ignoring the costs of maintaining links). Then, we can show that if there are extra links in such a network relative to the strongly efficient network which consists of a maximal number of disjoint linked pairs, some links could be deleted without changing the payoffs from trade but saving link costs. Thus, a pairwise stable network must consist of linked pairs, and the maximum number of potential pairs must form. Notice that if $1/2 < c_s$ and/or $1/2 < c_b$ then the empty network is the unique pairwise stable network. The empty network is strongly efficient only if $c_s + c_b \geq 1$.

Proposition 1.3. In the Corominas-Bosch model with $1/2 > c_s > 0$ and $1/2 > c_b > 0$, the set G_2 is the unique pairwise farsightedly stable set of networks.

In Corominas-Bosch (2004) model of trading networks with bilateral bargaining, the value function is top convex and pairwise farsighted stability singles out the strongly efficient network even though the allocation rule is again not componentwise egalitarian. Myopic or farsighted notions of stability sustain the set of strongly efficient networks G_2 when the costs of maintaining links are not too large.¹⁸ The intuition is that each buyer obtains the same payoff $(1/2 - c_b)$ in any efficient network, each seller obtains the same payoff $(1/2 - c_s)$ in any efficient network, and there is at least one seller or one buyer who has links and obtains a payoff strictly less than $1/2 - c_s$ or $1/2 - c_b$ in any other network g . Hence, there is no farsighted improving from any efficient network to such network g but there is a farsighted improving path from g to some efficient network. That is, $F(g') = \emptyset$ for all $g' \in G_2$, and for each $g \notin G_2$ there is some $g' \in G_2$ such that $g' \in F(g)$.

Notice that if $1/2 < c_s$ and/or $1/2 < c_b$ then a set consisting of the empty network is obviously the unique pairwise farsightedly stable set. In that case, on at

¹⁷ G_2 is also the myopically stable set.

¹⁸Myopic and farsighted notions of stability still sustain the set of strongly efficient networks when $\#S \neq \#B$. But then, each set of all networks consisting of pairs among $\min\{\#S, \#B\}$ sellers and $\min\{\#S, \#B\}$ buyers and so that the maximum number of potential pairs form is a pairwise farsightedly stable set.

least one side of the market (buyers or sellers) agents who have some link in any network receive a payoff strictly less than 0 and thus are willing to delete their links looking forward to the empty network. It also implies that there are no farsighted improving path emanating from the empty network.

A model of buyer-seller networks

The Kranton and Minehart (2001) model of buyer-seller networks is similar to the Corominas-Bosch model except that the valuations of the buyers for an object are random and the determination of prices is made through an auction rather than alternating-offer bargaining. Consider a version of the model with one seller ($\#S = 1$) and some potential buyers ($\#B \geq 1$). So, there is one seller who has an indivisible object for sale and b potential buyers who have utilities for the object, denoted u_i , which are uniformly and independently distributed on $[0, 1]$. The object to sell has no value to the seller. Each buyer knows her own valuation, but only the distribution over the buyers' valuations. The seller also knows only the distribution of buyers' valuations. The object is sold by means of a standard second-price auction. Only the buyers who are linked to the seller participate to the auction. The number of buyers linked to the seller is given by $l(g)$. For a cost per link of c_s to the seller and c_b to the buyer, the allocation rule for any network g with $l(g) \geq 1$ links between the buyers and the seller is

$$Y_i(g) = \begin{cases} 1/[l(g)(l(g) + 1)] - c_b & \text{if } i \text{ is a linked buyer} \\ (l(g) - 1)/(l(g) + 1) - l(g)c_s & \text{if } i \text{ is the seller} \\ 0 & \text{if } i \text{ is a buyer without any links.} \end{cases},$$

and is not componentwise egalitarian. The value function is $v(g) = l(g)/(l(g) + 1) - l(g)(c_s + c_b)$, which is simply the expected value of the object to the highest valued buyer less the cost of links, and violates top convexity. Let l_s^* be the number of links l such that $2/[l(l + 1)] \geq c_s$ and $2/[(l + 1)(l + 2)] < c_s$, which is the number of links that maximizes the seller's payoff. Let l_b^* be the number of links l such that $1/[l(l + 1)] \geq c_b$ and $1/[(l + 1)(l + 2)] < c_b$, which is the maximal number of links up to which buyers make positive payoffs. A network g such that $l(g) = \min\{l_s^*, l_b^*\}$ is pairwise stable. Notice that if $2/[l_s^*(l_s^* + 1)] = c_s$, $1/[l_b^*(l_b^* + 1)] = c_b$ and $l_s^* = l_b^*$ then $g - ij$ such that $l(g) = \min\{l_s^*, l_b^*\}$ is pairwise stable too. Strongly efficient networks are not necessarily pairwise stable.¹⁹ If $c_s = 0$ then the pairwise stable

¹⁹For instance, if $c_s = c_b = 1/100$ then the pairwise stable networks have 10 links, while networks

networks are exactly the efficient ones.²⁰

Proposition 1.4. In the Kranton and Minehart model with one seller,

- (i) If $2/[l_s^*(l_s^* + 1)] > c_s$ and/or $1/[l_b^*(l_b^* + 1)] > c_b$ and/or $l_s^* \neq l_b^*$ then $\{g\}$ with $g \in G_1 = \{g \in \mathbb{G}(\{s\}, B) \mid l(g) = \min\{l_s^*, l_b^*\}\}$ are the unique pairwise farsightedly stable sets.
- (ii) If $2/[l_s^*(l_s^* + 1)] = c_s$, $1/[l_b^*(l_b^* + 1)] = c_b$ and $l_s^* = l_b^*$ then $G_1 \cup G_{-1}$ with $G_{-1} = \{g \in \mathbb{G}(\{s\}, B) \mid l(g) = l_s^* - 1\}$ is the unique pairwise farsightedly stable set.

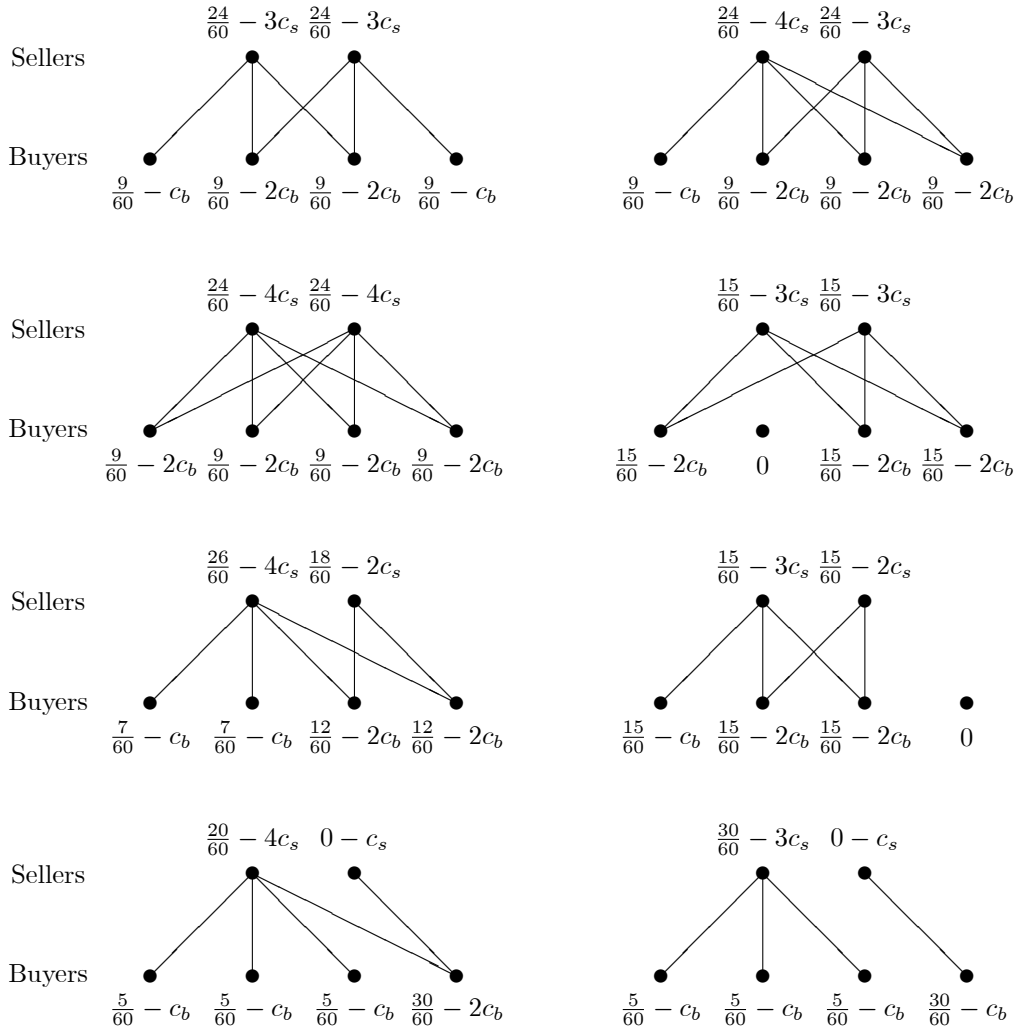


Figure 2 : Payoffs in the Kranton and Minehart (2001) model for selected networks.

with only 6 links are the strongly efficient ones.

²⁰There are no cycles. The myopically stable set consists only of pairwise stable networks.

Pairwise myopically or farsightedly stable networks may not be strongly efficient, however, they are Pareto efficient. When there are more sellers it is possible for non-trivial pairwise myopically stable networks to be Pareto inefficient. Consider a population with two sellers and four buyers. Let agents 1 and 2 be the sellers and 3, 4, 5 and 6 be the buyers. Some straightforward but tedious calculations lead to the payoffs which are given in Figure 2 and Figure 3 for selected networks.

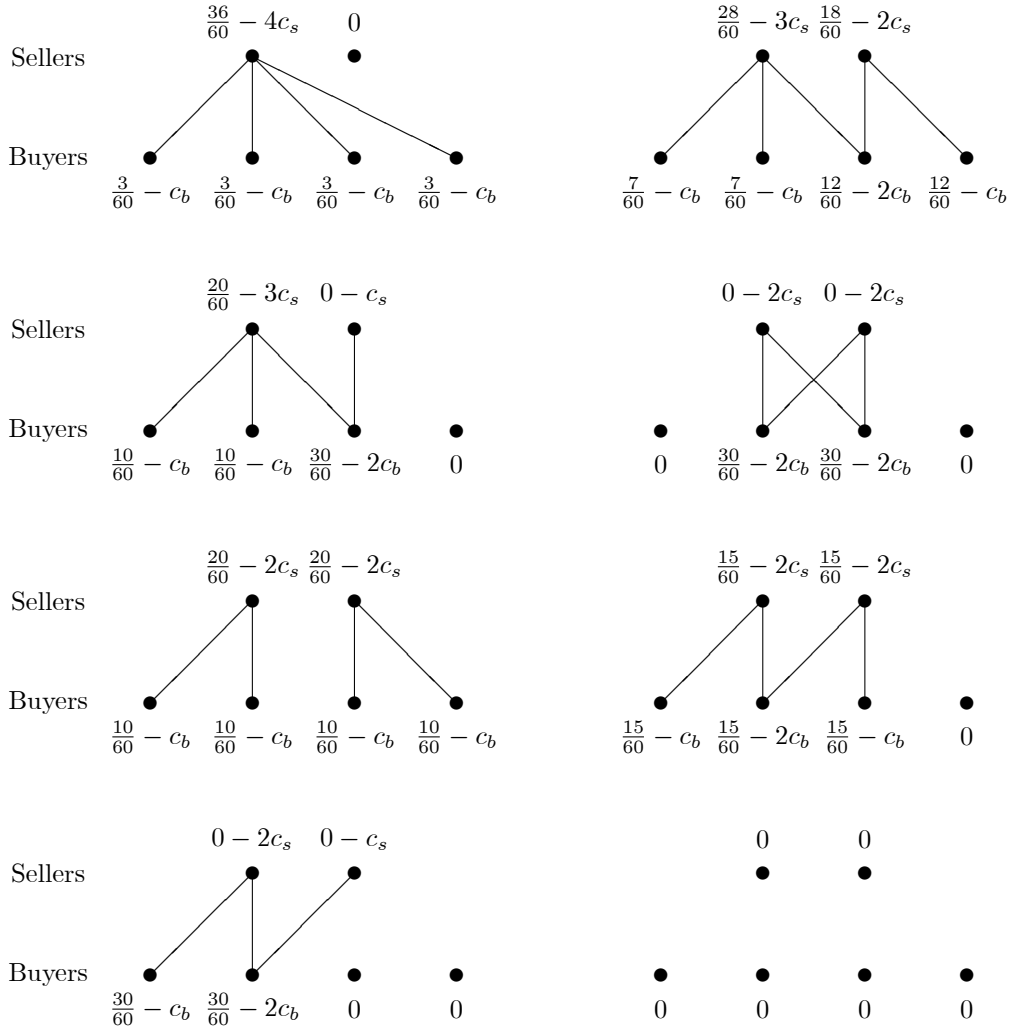


Figure 3 : Payoffs in the Kranton and Minehart (2001) model for selected networks (continued).

For instance, when $c_s = 5/60$ and $c_b = 1/60$, there are three types of pairwise stable networks: the empty network, networks that look like $\{13, 14, 15, 16\}$, and networks that look like $\{13, 14, 15, 24, 25, 26\}$. Both the empty network and $\{13, 14, 15, 24, 25, 26\}$ are not Pareto efficient, while $\{13, 14, 15, 16\}$ is. The empty network and the network $\{13, 14, 15, 24, 25, 26\}$ are Pareto dominated by the network

$\{13, 14, 25, 26\}$. In addition, the network $\{13, 14, 15, 16\}$ is not strongly efficient. The network $\{13, 14, 25, 26\}$ is strongly efficient but is not pairwise stable since agents 1 and 5 have incentives to add a link. However, the network $\{13, 14, 25, 26\}$ is pairwise farsightedly stable. Indeed, we have that $G' = \{g \mid d_1(g) = d_2(g) = 2 \text{ and } d_3(g) = d_4(g) = d_5(g) = d_6(g) = 1\}$ is a pairwise farsightedly stable set since for every $g' \notin G'$ we have $F(g') \cap G' \neq \emptyset$ and for every $g \in G'$, $F(g) \cap G' = \emptyset$. Thus, contrary to pairwise stability, pairwise farsighted stability may sustain strongly efficient networks when there are more than one seller. So, in Kranton and Minehart buyer (2001) model of buyer-seller networks, the allocation rule is not componentwise egalitarian and the value function violates top convexity. Then, no general conclusions hold, and pairwise myopically or farsightedly stable networks may not be strongly efficient depending on the number of sellers and buyers. One open question is whether Pareto inefficient networks could belong to some pairwise farsightedly stable set with many sellers and buyers.

1.6. Conclusion

We have studied the stability of social and economic networks when players are farsighted. In particular, we have first shown that under the componentwise egalitarian allocation rule, the set of strongly efficient networks and the set of pairwise (myopically) stable networks that are immune to coalitional deviations are the unique pairwise farsightedly stable set if and only if the value function is top convex. We have then examined whether the networks formed by farsighted players are different from those formed by myopic players in Jackson and Wolinsky (1996) symmetric connections model, in Corominas-Bosch (2004) model of trading networks with bilateral bargaining, and in Kranton and Minehart (2001) model of buyer-seller networks.

Appendix 1.A. Proofs.

Proof of Proposition 1.1.

Take any anonymous and component additive value function v .

($\Leftarrow$)

Top convexity implies that all components of a strongly efficient network must lead to the same per-capita value (if some component led to a lower per capita value than the average, then another component would have to lead to a higher per capita value than the average which would contradict top convexity). Top convexity also implies that under the componentwise egalitarian allocation rule any $g \in E(v)$ Pareto dominates all $g' \notin E(v)$. Then, it is immediate that $g \in F(g')$ for all $g' \in \mathbb{G} \setminus E(v)$ and that $F(g) = \emptyset$. Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \emptyset\}$ and for every $g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$, we have that $E(v)$ is the unique pairwise farsightedly stable set.

($\Rightarrow$)

Since $E(v)$ is the unique pairwise farsightedly stable set, we have $F(g) = \emptyset$ for all $g \in E(v)$. We will show that this implies that under the componentwise egalitarian allocation rule (i) $Y_i^{ce}(g, v) = Y_j^{ce}(g, v) = Y_i^{ce}(g', v) = Y_j^{ce}(g', v)$ for all $i, j \in N$ and for all $g, g' \in E(v)$; (ii) $Y_i^{ce}(g, v) \geq Y_i^{ce}(g', v)$ for all $i \in N$, for all $g \in E(v)$, for all $g' \notin E(v)$. Thus, v is top convex.

(i.a) Suppose first that $Y_i^{ce}(g, v) \neq Y_j^{ce}(g, v)$ for some $g \in E(v)$ and some pair of players i, j . Without loss of generality, assume that $Y_i^{ce}(g, v) > Y_j^{ce}(g, v)$. Notice that $i \in N(g)$, since if it was not the case, player j could delete successively links from the network g to reach a network g' satisfying $Y_j^{ce}(g', v) \geq Y_i^{ce}(g, v) > Y_j^{ce}(g, v)$. This would then contradict the fact that $F(g) = \emptyset$. Let $h \subseteq g$ be such that $i \in N(h)$. Let $N_i(g) = \{l \in N \mid il \in g\}$. To see that our assumption leads to a contradiction, let us construct a path from the network g to the network $\tilde{g} = g - \{jl \mid jl \in g\} + \{jl \mid l \in N_i(g)\} - \{il \mid l \in N_i(g)\}$ as follows. From the network g , player j successively cuts all her links to reach the network $g' = g - \{jl \mid jl \in g\}$. In g' , players having a link with player i in the network g successively add a link with player j to reach $g'' = g - \{jl \mid jl \in g\} + \{jl \mid l \in N_i(g)\}$. Then from g'' , the players having a link with i successively delete this link and we reach the network $\tilde{g}$. In each network g_k in the sequence going from g to g' we have that $Y_j^{ce}(g_k, v) \leq Y_j^{ce}(g, v)$ since otherwise we would have $g_k \in F(g) \neq \emptyset$. Each network $\bar{g}$ in the path from $g' + jl$, where $l \in N_i(g)$,

to $\tilde{g}$ is such that $Y_k(\bar{g}) = Y_m(\bar{g})$ for all $k, m \in N_i(g) \cup \{j\}$, since they belong to the same component. In addition, we have $Y_k(\bar{g}) < Y_k(g)$ for $k \in N_i(g)$ as otherwise, we would have $\bar{g} \in F(g)$. This contradicts anonymity of the value function since in the network $\tilde{g}$, the players from $N_i(g)$ are in a component identical to h but where player j has replaced player i . This establishes that $Y_i^{ce}(g, v) = Y_j^{ce}(g, v)$ for all $i, j \in N$ and for all $g \in E(v)$.

(i.b) To see that $Y_i^{ce}(g, v) = Y_i^{ce}(g', v)$ for all $g, g' \in E(v)$, suppose on the contrary that $Y_i^{ce}(g, v) \neq Y_i^{ce}(g', v)$, say $Y_i^{ce}(g, v) > Y_i^{ce}(g', v)$. Using the result (i.a), we then have $Y_j^{ce}(g, v) > Y_j^{ce}(g', v)$ for all $j \in N$, contradicting that g' is strongly efficient.

(ii) By contradiction, suppose that $Y_l^{ce}(g, v) < Y_l^{ce}(g', v)$ for some player l , and for some pair of networks $g \in E(v)$, $g' \notin E(v)$. Notice that $l \in N(g')$, since if it was not the case, player l could delete successively links from the network g to reach a network g'' satisfying $Y_l^{ce}(g'', v) \geq Y_l^{ce}(g', v) > Y_l^{ce}(g, v)$. This would then contradict the fact that $F(g) = \emptyset$. Let $h' \subseteq g'$ such that $l \in N(h')$ be the component to which l belongs in the network g' . We have that $Y_i^{ce}(g, v) = Y_l^{ce}(g, v)$ for all $i \in N$ since $g \in E(v)$ (see part (i.a)). In addition, $Y_l^{ce}(g', v) = Y_i^{ce}(g', v)$ for all $i \in N(h')$ by definition of the componentwise allocation rule. Thus $Y_i^{ce}(g, v) < Y_i^{ce}(g', v)$ for all $i \in N(h')$. These relations imply that $F(g) \neq \emptyset$, a contradiction. To see this, let us construct a path from the network g to the network $\tilde{g} = h' + g|_{N \setminus N(h')}$ as follows. From the network g , some player $k \in N(h')$ adds successively a link with all the other players from $N(h')$ to reach the network $g_1 = g + \{kj \mid j \in N(h')\}$. In g_1 , let the players from $N(h')$ add the links that belong to h' but do not belong to g , leading to the formation of the network $g_2 = g_1 + \{ij \mid ij \in h' \setminus g\}$. Then let the players from $N(h')$ delete successively the links that belong to g_2 but do not belong to h' to reach $\tilde{g} = h' + g|_{N \setminus N(h')}$. Let $\bar{g}$ be a network in the path described from g to $\tilde{g}$. Let $S(\bar{g}) \subseteq N(h')$ be the set of players who have added or deleted a link in the path from the network g to the network $\bar{g}$. We have $Y_i^{ce}(\bar{g}, v) = Y_j^{ce}(\bar{g}, v)$ for all $i, j \in S(\bar{g})$ since any pair of players in $S(\bar{g})$ is in the same component in the network $\bar{g}$. Since $F(g) = \emptyset$, we have $Y_i^{ce}(\bar{g}, v) \leq Y_i^{ce}(g, v)$ for all $i \in S(\bar{g})$. This must be true for all network $\bar{g}$ in the path from g to $\tilde{g}$. However, we have supposed that $Y_l^{ce}(g, v) < Y_l^{ce}(g', v) = Y_l^{ce}(\tilde{g}, v)$ for all $l \in N(h')$, a contradiction. ■

Proof of Proposition 1.2.

(i) Suppose that $c < \delta(1 - \delta)$. Since $\delta < 1$, we have that $(\delta - c) > \delta^2 > \delta^3 > \dots > \delta^{n-1}$. Thus, any two agents who are not directly connected benefit from forming a link. In this case, the complete network g^N strictly Pareto dominates all other networks. That is, for every $g \in \mathbb{G} \setminus \{g^N\}$ we have $Y_i(g^N) > Y_i(g)$ for all $i \in N$. Theorem 7 in Herings, Mauleon and Vannetelbosch (2009) states that if there is a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique pairwise farsightedly stable set. Hence, we have that $\{g^N\}$ is the unique pairwise farsightedly stable set.

(ii) Suppose that $\delta(1 - \delta) < c < \delta$. Since $\delta^2 > (\delta - c)$, and $\delta^2 > \delta^3 > \dots > \delta^{n-1}$, each agent prefers an indirect link at a distance of two to any direct link and to any indirect link at a distance greater than two. In a star network encompassing all players g^s there is $n - 1$ links connecting one given player i (the hub player) to each other player $j \in N$, $j \neq i$ (spoke players). Denote $i(g^s)$ the hub player at the star g^s . Then, the payoff of the hub player $i(g^s)$ is $Y_i(g^s) = (n - 1)(\delta - c)$ and the payoff of any spoke player j , $j \neq i(g^s)$, is $Y_j(g^s) = (\delta - c) + (n - 2)\delta^2$. Notice that the payoff of the spoke players is the maximum payoff a player can get in any network $g \in \mathbb{G}$. From Theorem 4 in Herings, Mauleon and Vannetelbosch (2009) a singleton set $\{g\}$ is a pairwise farsightedly stable set if and only if for every $g' \in \mathbb{G} \setminus \{g\}$ we have $g \in F(g')$. We will prove that every star network encompassing all players $\{g^s\}$ is a pairwise farsightedly stable set since $g^s \in F(g)$ for all $g \neq g^s$.

(ii.1) Consider first any network g containing at most $n - 1$ links. If g is another star ($g \neq g^s$) encompassing all players, let the hub player at g , $i(g)$, delete a link. Otherwise, let any linked player $j \neq i(g^s)$ delete one link. In the next steps, any linked player different than $i(g^s)$ cuts one link until the empty network $g^\emptyset$ is reached. From $g^\emptyset$, add successively the links between player $i(g^s)$ and the rest of the players until g^s is formed. We have that $g^s \in F(g)$ since each deviating player prefers g^s to the network she was facing before deviating in order to go to g^s .

(ii.2) Consider next any network g containing more than $n - 1$ links. In such network g , there is always at least a player $j \neq i(g^s)$ with more than one direct link who would like to move to g^s . From g , let one of such players delete one of her links. If the resulting network has still more than $n - 1$ links, choose again a player $l \neq i(g^s)$ with more than one direct link and let her delete one link. The process continue until we reach at some point a network g' with at most $n - 1$ links. If $g' = g^s$, we

stop here. Otherwise, we know from (ii.1) that there is a farsighted improving path from g' leading to g^s . Thus, $g^s \in F(g)$ and Theorem 4 in Herings, Mauleon and Vannetelbosch (2009) applies.

(iii) Suppose that $c > \delta$. Let $\bar{c}(n) = \max\{c \in \mathbb{R} \mid \max_{g \in \mathbb{G} \setminus \{g^\emptyset\}} (\min_{i \in N} (Y_i(g))) = 0\}$. For $c > \bar{c}(n)$, $Y_i(g) < 0$ for all $g \in \mathbb{G} \setminus \{g^\emptyset\}$, for some $i \in N$ and for $c \leq \bar{c}(n)$, $Y_i(g) \geq 0$ for all $i \in N$ for some $g \in \mathbb{G} \setminus \{g^\emptyset\}$. Thus, $\delta < \bar{c}(n) < \delta + ((n-2)/2)\delta^2$ if $n \geq 4$. Indeed, the payoff of a player in the circle g^c of n agents (a circle of n agents is a network where each agent is indirectly connected to the others and has two links) is $Y_i(g^c) = 2\delta + 2\delta^2 + \dots + \delta^{n/2} - 2c$ if n is even and $Y_i(g) = 2\delta + 2\delta^2 + \dots + 2\delta^{(n-1)/2} - 2c$ if n is odd. We have $Y_i(g^c) > 0$ for all $i \in N$ if $c = \delta$ as long as $n \geq 4$, implying that $\bar{c}(n) > \delta$. In addition, if $c = \delta + ((n-2)/2)\delta^2$, the star networks and the empty network are strongly efficient and generate a value of 0. In the star network, the hub player has a negative payoff. Inefficient networks generate a negative value, implying that some agent has a negative payoff. There is no network different than the empty network which ensures a nonnegative payoff to all agents when $c = \delta + ((n-2)/2)\delta^2$. Thus $\bar{c}(n) < \delta + ((n-2)/2)\delta^2$.²¹

(iii.1) Suppose first that $c > \bar{c}(n)$. In order to show that a set consisting of the empty network (with a payoff of 0 for all players) is the unique pairwise farsightedly stable set of networks, we need to show that Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) applies. That is, we need to show that $g^\emptyset \in F(g)$ for all $g \neq g^\emptyset$ and that $F(g^\emptyset) = \emptyset$. Since $c \geq c(\bar{n})$, in any other network g , there is some player with a negative payoff who prefers the empty network and hence, we have that $g \notin F(g^\emptyset)$. Now, from g , let one of the players with a negative payoff delete one of her links. Since in any resulting network g' there is some player preferring the empty network, by letting one of such players deleting one of her links at each step, we finally end up at the empty network $g^\emptyset$, and $g^\emptyset \in F(g)$. Thus, $g^\emptyset \in F(g)$ for all $g \neq g^\emptyset$ and Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) applies.

(iii.2) If $\delta < c \leq \bar{c}(n)$, then $\min_i Y_i(g) \geq 0$ for all i for some $g \neq g^\emptyset$. We have that $g^\emptyset \notin F(g)$ and then $\{g^\emptyset\}$ is not a pairwise farsightedly stable set. ■

²¹When $n = 4$, the network that maximizes the allocation of the agent with the smaller payoff is the circle. We thus have $\bar{c}(4) = \delta + \delta^2/2$. In general the network that maximizes the allocation of the agent with the smaller payoff is such that each agent is indirectly connected and possesses the same number of links.

Proof of Proposition 1.3.

First, we show that for every $g' \notin G_2$ there is $g \in G_2$ such that $g \in F(g')$. For every $g \in G_2$, each seller i receives $Y_i(g) = 1/2 - c_s > 0$, and each buyer j receives $Y_j(g) = 1/2 - c_b > 0$. Start with g' and build a sequence of networks as follows. At each step some agent l who receives a payoff smaller than $1/2 - c_l$ deletes a link looking forward to g until we reach a network g'' consisting only of linked pairs of agents and/or agents having no links. Then, agents successively add the missing links that belong to some $g \in G_2$ such that $g \supseteq g''$.

Second, for every $g \in G_2$ we have that $F(g) \cap G_2 = \emptyset$ since $Y_i(g) = 1/2 - c_s > 0$ and $Y_j(g) = 1/2 - c_b > 0$ for every $g \in G_2$. In addition, for each $g \in G_2$ we have $F(g) \cap (\mathbb{G} \setminus G_2) = \emptyset$ since the only networks $g' \notin G_2$ that some forward looking agents may prefer to $g \in G_2$ are such that the agents deviating from g obtain a payoff of $1 - kc$ in g' for some $k \geq 2$. To obtain 1 the deviating agents will have to form links along the sequence with agents that will obtain $-c$ in g' . But, before forming these additional links with the original deviating agents, these agents have a payoff of $1/2 - c$, and thus, they have incentives to block the formation of any additional costly link. Hence, we have shown that $F(g') \cap G_2 \neq \emptyset$ for all $g' \notin G_2$ and $F(g) = \emptyset$ for all $g \in G_2$. Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) states that the set G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \emptyset\}$ and for every $g' \in \mathbb{G} \setminus G_2$, $F(g') \cap G_2 \neq \emptyset$. Thus, G_2 is the unique pairwise farsightedly stable set. ■

Proof of Proposition 1.4.

(i) Suppose $2/[l_s^*(l_s^* + 1)] > c_s$ and/or $1/[l_b^*(l_b^* + 1)] > c_b$ and/or $l_s^* \neq l_b^*$; and let $G_1 = \{g \in \mathbb{G}(\{s\}, B) \mid l(g) = \min\{l_s^*, l_b^*\}\}$. It is quite straightforward that (a) $g' \notin F(g)$ for all $g' \notin G_1$ and $g \in G_1$; (b) $g' \in F(g)$ for all $g, g' \in G_1$; (c) $g \in F(g')$ for all $g \in G_1$, $g' \notin G_1$. Then, it follows that $\{g\}$ with $g \in G_1$ are the unique pairwise farsightedly stable sets.

(ii) Suppose $2/[l_s^*(l_s^* + 1)] = c_s$, $1/[l_b^*(l_b^* + 1)] = c_b$ and $l_s^* = l_b^*$. Let $G_{-1} = \{g \in \mathbb{G}(\{s\}, B) \mid l(g) = l_s^* - 1\}$. We have $Y_s(g) = Y_s(g')$ for all $g, g' \in G_1 \cup G_{-1}$; $Y_i(g) = 0$ for all $g \in G_1$, $i \in B$; $Y_i(g) = 0$ for all $g \in G_{-1}$, $i \in B$ with $l_i(g) = 0$. It follows that (a) $g' \notin F(g)$ for all $g, g' \in G_1 \cup G_{-1}$; (b) for all $g' \notin G_1 \cup G_{-1}$ there is $g \in F(g')$ such that $g \in G_1 \cup G_{-1}$; (c) $g' \notin F(g)$ for all $g' \notin G_1 \cup G_{-1}$ and $g \in G_1 \cup G_{-1}$. (a) and (b) imply that $G_1 \cup G_{-1}$ is a pairwise farsightedly stable set while (c) implies that $G_1 \cup G_{-1}$ are the unique pairwise farsightedly stable sets. ■

Chapter 2.

A characterization of farsightedly stable networks²²

2.1. Introduction

The organization of agents into networks and groups or coalitions plays an important role in the determination of the outcome of many social and economic interactions.²³ A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that players do not benefit from altering the structure of the network. An example of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of her links and no two players benefit from adding a link between them. Pairwise stability is a myopic definition. Players are not farsighted in the sense that they do not forecast how others might react to their actions. For instance, the adding or severing of one link might lead to subsequent addition or severing of another link. If players have very good information about how others might react to changes in the network, then these are things one wants to allow for in the definition of the stability concept. For instance, a network could be stable because players might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original players.

In this chapter we address the question of which networks one might expect to emerge in the long run when players are either farsighted or myopic. Herings, Mauleon and Vannetelbosch (2009) have first extended the Jackson and Wolinsky pairwise stability notion to a new set-valued solution concept, called the pairwise myopically stable set. A set of networks G is pairwise myopically stable (i) if all possible myopic pairwise deviations from any network $g \in G$ to a network outside the set are deterred by the threat of ending worse off or equally well off, (ii) if there exists

²²This chapter is based on an article co-authored with Ana Mauleon and Vincent Vannetelbosch published in *Games*, Vol. 1, pages 226-241.

²³See Jackson (2003, 2005, 2008), or Goyal (2007) for a comprehensive introduction to the theory of social and economic networks.

a myopic improving path from any network outside the set leading to some network in the set, and (iii) if there is no proper subset of G satisfying Conditions (i) and (ii). The pairwise myopically stable set is non-empty, unique and contains all pairwise stable networks. They have then introduced the pairwise farsightedly stable set, to predict which networks may be formed among farsighted players.²⁴ The definition corresponds to the one of a pairwise myopically stable set with myopic deviations and myopic improving paths replaced by farsighted deviations and farsighted improving paths. A farsighted improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the end network. Similarly, it is straightforward to define the notions of groupwise myopically stable sets and of groupwise farsightedly stable sets for situations in which players can deviate in group. Herings, Mauleon and Vannetelbosch (2009) have shown that a non-empty pairwise (groupwise) farsightedly stable set always exists. In addition, they have provided necessary and sufficient conditions for a set to be a unique pairwise (groupwise) farsightedly stable set of networks.

We first provide an algorithm that characterizes the unique pairwise and groupwise farsightedly stable set of networks under the componentwise egalitarian allocation rule. We then show that this set coincides with the unique groupwise myopically stable set of networks but not with the unique pairwise myopically stable set of networks. We conclude that, if groupwise deviations are allowed then whether players are farsighted or myopic does not matter; if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter. In addition, we show that alternative notions of farsighted stability also single out the same set as the unique farsighted stable set.

The chapter is organized as follows. In Section 2 we introduce some notations and basic properties. In Section 3 we define the notions of myopically stable sets and

²⁴Other approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Dutta, Ghosal and Ray (2005), Page, Wooders and Kamat (2005), and Page and Wooders (2009).

of farsightedly stable sets. In Section 4 we characterize the unique farsightedly stable set of networks under the componentwise egalitarian allocation rule. In Section 5 we consider other concepts of farsighted stability. In Section 6 we conclude.

2.2. Networks

Let $N = \{1, \dots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. We write $ij \in g$ to indicate that i and j are linked under the network g . Let g^S be the set of all subsets of $S \subseteq N$ of size 2. So, g^N is the complete network. The set of all possible networks or graphs on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N . The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. Let

$$g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}.$$

Thus, $g|_S$ is the network found deleting all links except those that are between players in S . For any network g , let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g . A path in a network $g \in \mathbb{G}$ between i and j is a sequence of players $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j . A non-empty network $h \subseteq g$ is a component of g , if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j , and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$. Knowing the components of a network, we can partition the players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by the network g .

A value function is a function $v : \mathbb{G} \rightarrow \mathbb{R}$ that keeps track of how the total societal value varies across different networks. The set of all possible value functions is denoted by $\mathcal{V}$. An allocation rule is a function $Y : \mathbb{G} \times \mathcal{V} \rightarrow \mathbb{R}^N$ that keeps track of how the value is allocated among the players forming a network. It satisfies $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g .

Jackson and Wolinsky (1996) have proposed a number of basic properties of value functions and allocation rules. A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in \mathbb{G}$. Component additive value functions are the ones for which the value of a network is the sum of the value of its components. For a component additive v and network g , the *componentwise egalitarian allocation rule* Y^{ce} is such that for any $h \in C(g)$ and each $i \in N(h)$, $Y_i^{ce}(g, v) = v(h)/\#N(h)$. For a v that is not component additive, $Y^{ce}(g, v) = v(g)/n$ for all g ; thus, Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive.

Which networks are likely to emerge in the long run? The game-theoretic approach to network formation uses two different notions of a deviation by a coalition. *Pairwise deviations* (Jackson and Wolinsky 1996) are deviations involving a single link at a time. That is, link addition is bilateral (two players that would be involved in the link must agree to adding the link), link deletion is unilateral (at least one player involved in the link must agree to deleting the link), and network changes take place one link at a time. *Groupwise deviations* (Jackson and van den Nouweland 2005) are deviations involving several links within some group of players at a time. Link addition is bilateral, link deletion is unilateral, and multiple link changes can take place at a time. Whether a pairwise deviation or a groupwise deviation makes more sense will depend on the setting within which network formation takes place. The definitions of stability we consider allow for a deviation by a coalition to be valid only if all members of the coalition are *strictly* better off, and in doing so we deviate from the original definitions by Herings, Mauleon and Vannetelbosch (2009) where it is sufficient that at least one coalition member is strictly better off while all other members are at least as well off.

2.3. Definitions of Stable Sets of Networks

2.3.1. Myopic Definitions

We first introduce the notion of pairwise myopically stable sets of networks due to Herings, Mauleon and Vannetelbosch (2009) which is a generalization of Jackson and Wolinsky (1996) pairwise stability notion.²⁵ Pairwise stable networks do not

²⁵A network $g \in \mathbb{G}$ is pairwise stable with respect v and Y if no player benefits from severing one of their links and no two players benefit from adding a link between them. The original definition of Jackson and Wolinsky (1996) allows for a pairwise deviation to be valid if one deviating player

always exist. A pairwise myopically stable set of networks is a set such that from any network outside this set, there is a myopic improving path leading to some network in the set, and each deviation outside the set is deterred because the deviating players do not prefer the resulting network. The notion of a myopic improving path was first introduced in Jackson and Watts (2002). A myopic improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the resulting network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the resulting network.

Formally, a *pairwise myopic improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ or $Y_j(g_{k+1}, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ and $Y_j(g_{k+1}, v) > Y_j(g_k, v)$. For a given network g , let $m(g)$ be the set of networks that can be reached by a pairwise myopic improving path from g .

Definition 2.1. A set of networks $G \subseteq \mathbb{G}$ is pairwise myopically stable with respect to v and Y if

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $Y_i(g + ij, v) \leq Y_i(g, v)$ or $Y_j(g + ij, v) \leq Y_j(g, v)$,
- (ib) $\forall ij \in g$ such that $g - ij \notin G$, $Y_i(g - ij, v) \leq Y_i(g, v)$ and $Y_j(g - ij, v) \leq Y_j(g, v)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G$, $m(g') \cap G \neq \emptyset$,
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Conditions (ia) and (ib) in Definition 2.1 capture deterrence of external deviations. In Condition (ia) the addition of a link ij to a network $g \in G$ that leads to a network outside G is deterred because the two players involved do not prefer the resulting network to network g . Condition (ib) is a similar requirement, but

is better off and the other one is at least as well off.

then for the case where a link is severed. Condition (ii) requires external stability. External stability asks for the existence of a pairwise myopic improving path from any network outside G leading to some network in G . Condition (ii) implies that if a set of networks is pairwise myopically stable, it is non-empty. Condition (iii) is the minimality condition.

Jackson and Watts (2002) have defined the notion of a closed cycle. A set of networks C is a *cycle* if for any $g \in C$ and $g' \in C \setminus \{g\}$, there exists a pairwise myopic improving path connecting g to g' . A cycle C is a maximal cycle if it is not a proper subset of a cycle. A cycle C is a *closed cycle* if no network in C lies on a pairwise myopic improving path leading to a network that is not in C . A closed cycle is necessarily a maximal cycle. Herings, Mauleon and Vannetelbosch (2009) have shown that the set of networks consisting of all networks that belong to a closed cycle is the unique pairwise myopically stable set.

The notion of pairwise myopically stable set only considers deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise myopic stability. A network $g' \in G$ is *obtainable* from $g \in G$ via deviations by group $S \subseteq N$ if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \emptyset$.

A *groupwise myopic improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$: g_{k+1} is obtainable from g_k via deviations by $S_k \subseteq N$ and $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ for all $i \in S_k$. For a given network g , let $M(g)$ be the set of networks that can be reached by a groupwise myopic improving path from g .

Definition 2.2. A set of networks $G \subseteq \mathbb{G}$ is groupwise myopically stable with respect v and Y if

- (i) $\forall g \in G, S \subseteq N, g' \notin G$ that is obtainable from g via deviations by S , there exists $i \in S$ such that $Y_i(g', v) \leq Y_i(g, v)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G, M(g') \cap G \neq \emptyset$,
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Replacing the notion of pairwise improving path by the notion of groupwise improving path in the definition of a closed cycle, we have that the set of networks

consisting of all networks that belong to a closed cycle is the unique groupwise myopically stable set. The notion of groupwise myopically stable set is a generalization of Dutta and Mutuswami (1997) strong stability notion.²⁶ In Figure 1 we have depicted an example where the unique pairwise myopically stable set is $\{g_0, g_7\}$ while the unique groupwise myopically stable set is $\{g_7\}$. The networks g_0 and g_7 are pairwise stable but only g_7 is strongly stable, and there are no closed cycles of networks consisting of more than one network when players can modify their links either in pairs or in groups. There is no network such that there is a pairwise myopic improving path from any other network leading to it: $m(g_0) = \emptyset$, $m(g_1) = \{g_0, g_4, g_5, g_7\}$, $m(g_2) = \{g_0, g_4, g_6, g_7\}$, $m(g_3) = \{g_0, g_5, g_6, g_7\}$, $m(g_4) = \{g_7\}$, $m(g_5) = \{g_7\}$, $m(g_6) = \{g_7\}$, and $m(g_7) = \emptyset$. Hence, a set formed by the empty network g_0 and the complete network g_7 is a pairwise myopically stable set. However, the groupwise myopically stable set consists only of the complete network since $g_7 \in M(g) \forall g \neq g_7$ and $M(g_7) = \emptyset$. Indeed, we have $M(g_0) = \{g_4, g_5, g_6, g_7\}$, $M(g_1) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_2) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_3) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_4) = \{g_7\}$, $M(g_5) = \{g_7\}$, $M(g_6) = \{g_7\}$, and $M(g_7) = \emptyset$.

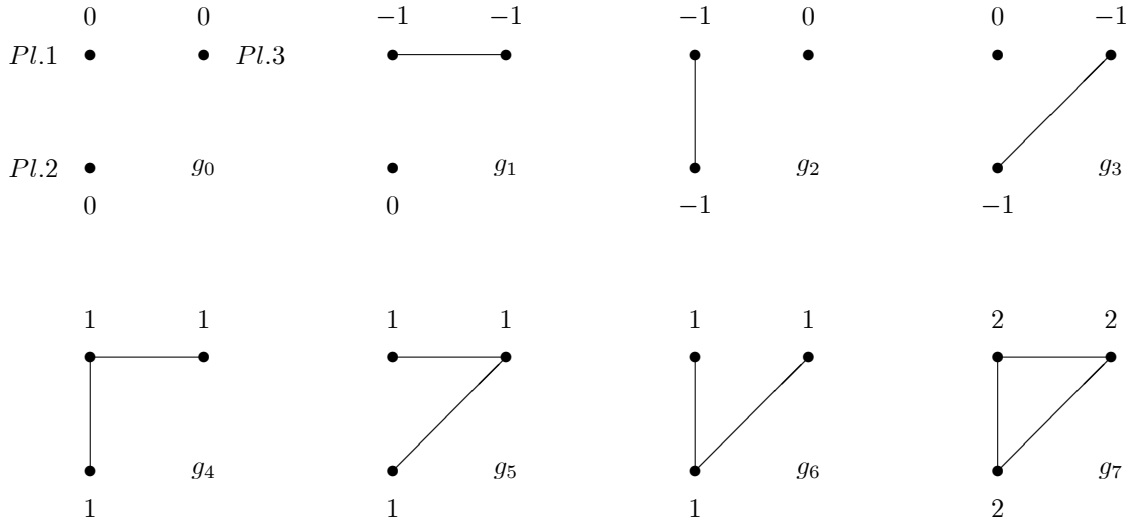


Figure 1: An example with three players.

²⁶A set g is strongly stable with respect v and Y if $\forall S \subseteq N$, g' that is obtainable from g via deviations by S , there exists $i \in S$ such that $Y_i(g', v) \leq Y_i(g, v)$. Jackson and van den Nouweland (2005) have introduced a slightly stronger definition where a deviation is valid if some members are better off and others are at least as well off. For many value functions and allocation rules these definitions coincide.

In Figure 2 we have depicted Jackson and Wolinsky co-author model with three players. It is easy to verify that the complete network g_7 is the unique pairwise stable network and that the pairwise myopically stable set is $\{g_7\}$. However, there is no strongly stable network. The groupwise myopically stable set is $\{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ and consists only of networks that belong to a cycle. Indeed, we have $M(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_1) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_2) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_3) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_4) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_5) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_6) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, and $M(g_7) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$.

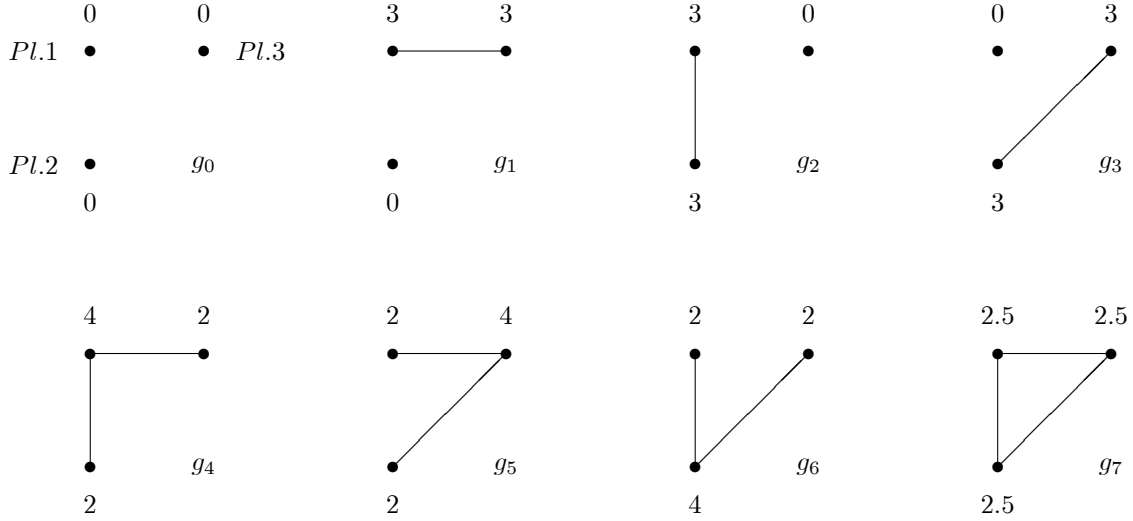


Figure 2: The co-author model with three players.

2.3.2. Farsighted Definitions

A *pairwise farsighted improving path* is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both strictly prefer the end network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the end network. Formally, a pairwise farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that

$Y_i(g_K, v) > Y_i(g_k, v)$ or $Y_j(g_K, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ and $Y_j(g_K, v) > Y_j(g_k, v)$. For a given network g , let $f(g)$ be the set of networks that can be reached by a pairwise farsighted improving path from g .

We now give the definition of a pairwise farsightedly stable set due to Herings, Mauleon and Vannetelbosch (2009).²⁷

Definition 2.3. A set of networks $G \subseteq \mathbb{G}$ is a pairwise farsightedly stable set with respect v and Y if

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $\exists g' \in f(g + ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ or $Y_j(g', v) \leq Y_j(g, v)$,
- (ib) $\forall ij \in g$ such that $g - ij \notin G$, $\exists g', g'' \in f(g - ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \emptyset$.
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Condition (i) in Definition 2.3 requires the deterrence of external deviations. Condition (ia) captures that adding a link ij to a network $g \in G$ that leads to a network outside of G , is deterred by the threat of ending in g' . Here g' is such that there is a pairwise farsighted improving path from $g + ij$ to g' . Moreover, g' belongs to G , which makes g' a credible threat. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) in Definition 2.3 requires external stability and implies that the networks within the set are robust to perturbations. From any network outside of G there is a farsighted improving path leading to some network in G . Condition (ii) implies that if a set of networks is pairwise farsightedly stable, it is non-empty. Condition (iii) is the minimality condition. Herings, Mauleon and Vannetelbosch (2009) have shown that a pairwise farsightedly stable set of networks always exists.

In Figure 3 we have depicted another example with three players. We have that $f(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $f(g_1) = \{g_2, g_4, g_5, g_6, g_7\}$, $f(g_2) = \{g_3, g_4, g_5, g_6, g_7\}$,

²⁷In the original definition of Herings, Mauleon and Vannetelbosch (2009), pairwise deviations are valid if one player is strictly better off and the other is at least as well off in the end network.

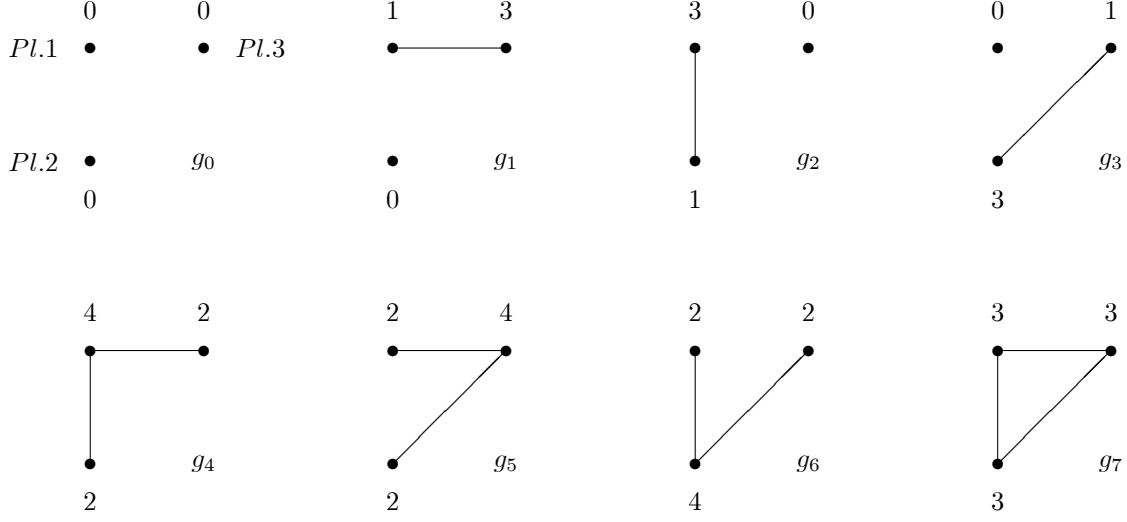


Figure 3: Another example with three players.

$f(g_3) = \{g_1, g_4, g_5, g_6, g_7\}$, $f(g_4) = \{g_5, g_6, g_7\}$, $f(g_5) = \{g_4, g_6, g_7\}$, $f(g_6) = \{g_4, g_5, g_7\}$, and $f(g_7) = \{g_4, g_5, g_6\}$. Theorem 4 in Herings, Mauleon and Vannetelbosch (2009) states that a set $\{g\}$ is a pairwise farsightedly stable set if and only if for every $g' \in \mathbb{G} \setminus \{g\}$ we have $g \in f(g')$. It follows that $\{g_4\}$, $\{g_5\}$, $\{g_6\}$ and $\{g_7\}$ are pairwise farsightedly stable sets. The network g_4 is not pairwise stable but it is farsightedly stable because the profitable deviation to g_7 is deterred by the threat of ending back to g_4 . Indeed, $g_4 \in f(g_7)$. For instance, $(g_7, g_6, g_3, g_0, g_1, g_4)$ is a farsighted improving path starting in g_7 and ending in g_4 .

A *groupwise farsighted improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$: g_{k+1} is obtainable from g_k via deviations by $S_k \subseteq N$ and $Y_i(g_K, v) > Y_i(g_k, v)$ for all $i \in S_k$. For a given network g , let $F(g)$ be the set of networks that can be reached by a groupwise farsighted improving path from g .

Definition 2.4. A set of networks $G \subseteq \mathbb{G}$ is groupwise farsightedly stable with respect v and Y if

- (i) $\forall g \in G, S \subseteq N, g' \notin G$ that is obtainable from g via deviations by S , there exists $g'' \in F(g') \cap G$ such that $Y_i(g'', v) \leq Y_i(g, v)$ for some $i \in S$,
- (ii) $\forall g' \in \mathbb{G} \setminus G, F(g') \cap G \neq \emptyset$,
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Let us reconsider the co-author model with three players depicted in Figure 2. No singleton set is pairwise farsightedly stable. Indeed, there is no network such that there is a farsighted improving path from any other network leading to it. More precisely, $f(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $f(g_1) = \{g_4, g_5\}$, $f(g_2) = \{g_4, g_6\}$, $f(g_3) = \{g_5, g_6\}$, $f(g_4) = \{g_7\}$, $f(g_5) = \{g_7\}$, $f(g_6) = \{g_7\}$, and $f(g_7) = \emptyset$. Theorem 3 in Herings, Mauleon and Vannetelbosch (2009) states that if for every $g' \in \mathbb{G} \setminus G$ we have $f(g') \cap G \neq \emptyset$ and for every $g \in G$, $f(g) \cap G = \emptyset$, then G is a pairwise farsightedly stable set. Hence, $\{g_1, g_2, g_3, g_7\}$ is a pairwise farsightedly stable set. However, a set formed by the complete and two star networks is also a pairwise farsightedly stable set of networks. Indeed, $\{g_4, g_5, g_7\}$, $\{g_4, g_6, g_7\}$, and $\{g_5, g_6, g_7\}$ are pairwise farsightedly stable sets in the co-author model with three players. Suppose that we allow now for groupwise deviations. Then, we have $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $F(g_1) = \{g_4, g_5\}$, $F(g_2) = \{g_4, g_6\}$, $F(g_3) = \{g_5, g_6\}$, $F(g_4) = \{g_3, g_7\}$, $F(g_5) = \{g_2, g_7\}$, $F(g_6) = \{g_1, g_7\}$, and $F(g_7) = \{g_1, g_2, g_3\}$. Hence, $\{g_1, g_2, g_3\}$ becomes a groupwise farsightedly stable set. But, this is not the unique groupwise farsightedly stable set. The others are $\{g_1, g_2, g_4, g_5, g_6\}$, $\{g_1, g_3, g_4, g_5, g_6\}$, $\{g_2, g_3, g_4, g_5, g_6\}$, $\{g_4, g_5, g_6, g_7\}$.

2.4. Farsighted Stability under the Componentwise Egalitarian Allocation Rule

We now investigate whether the pairwise or groupwise farsighted stability coincide with the pairwise or groupwise myopically stability under the componentwise egalitarian allocation. Let

$$g(v, S) = \left\{ g \subseteq g^S \mid \frac{v(g)}{\#N(g)} \geq \frac{v(g')}{\#N(g')} \forall g' \subseteq g^S, g' \neq \emptyset \right\}$$

be the set of networks with the highest per capita value out of those that can be formed by players in $S \subseteq N$. Given a component additive value function v , find a network g^v through the following algorithm due to Banerjee (1999). Pick some $h_1 \in g(v, N)$. Next, pick some $h_2 \in g(v, N \setminus N(h_1))$. At stage k pick some $h_k \in g(v, N \setminus \cup_{i \leq k-1} N(h_i))$. Since N is finite this process stops after a finite number K of stages. The union of the components picked in this way defines a network g^v . We denote by G^v the set of all networks that can be found through this algorithm.

More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

Lemma 2.1 tells us that there is no pairwise or groupwise farsighted improving path emanating from each $g \in G^v$.

Lemma 2.1. Consider any component additive value function v . For all $g \in G^v$ we have $f(g) = \emptyset$ and $F(g) = \emptyset$ under the componentwise egalitarian allocation rule Y^{ce} .

Proof. Take any $g \in G^v$ where $g = \bigcup_{k=1}^K h_k$ with $h_k \in g(v, N \setminus \bigcup_{i \leq k-1} N(h_i))$. Players belonging to $N(h_1)$ in g who are looking forward will never engage in a move since they can never be strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce} . Players belonging to $N(h_2)$ in g who are forward looking will only engage in a move if they can end up in some h such that $v(h)/\#N(h) > v(h_2)/\#N(h_2)$. Suppose there exists some h such that $v(h)/\#N(h) > v(h_2)/\#N(h_2)$. Since $h_2 \in g(v, N \setminus N(h_1))$ it follows that $N(h) \cap N(h_1) \neq \emptyset$. Given that players in $N(h_1)$ will never engage in a move, players belonging to $N(h_2)$ can never end up strictly better off than in g under the componentwise egalitarian allocation rule Y^{ce} . So, players belonging to $N(h_2)$ in g will never engage in a move. Players belonging to $N(h_k)$ in g who are forward looking will only engage in a move if they can end up in some h such that $v(h)/\#N(h) > v(h_k)/\#N(h_k)$. Suppose there exists some h such that $v(h)/\#N(h) > v(h_k)/\#N(h_k)$. Since $h_k \in g(v, N \setminus \bigcup_{i \leq k-1} N(h_i))$ it follows that $N(h) \cap \{\bigcup_{i \leq k-1} N(h_i)\} \neq \emptyset$. Given that players in $\bigcup_{i \leq k-1} N(h_i)$ will never engage in a move, players belonging to $N(h_k)$ can never end up strictly better off than in g under the componentwise egalitarian allocation rule Y^{ce} . So, players belonging to $N(h_k)$ in g will never engage in a move; and so on. Thus, $f(g) = \emptyset$ and $F(g) = \emptyset$. ■

From Lemma 2.1 it follows that there is no pairwise or groupwise myopic improving path emanating from each $g \in G^v$.

Corollary 2.1. Consider any component additive value function v . For all $g \in G^v$ we have $m(g) = \emptyset$ and $M(g) = \emptyset$ under the componentwise egalitarian allocation rule Y^{ce} .

Lemma 2.2 tells us that from any $g' \notin G^v$ there is some pairwise farsighted improving path going to some $g \in G^v$. Hence, from any $g' \notin G^v$ there is some groupwise farsighted improving path going to some $g \in G^v$.

Lemma 2.2. Consider any component additive value function v . For all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce} .

Proof. We show in a constructive way that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce} . Take any $g' \notin G^v$.

Step 1: If there exists some $h_1 \in g(v, N)$ such that $h_1 \in C(g')$ then go to Step 2 with $g_1 = g'$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g')$ such that $h_1 \subsetneq h$ for some $h_1 \in g(v, N)$. Then, take $h_1 \in g(v, N)$ such that there does not exist $h'_1 \in g(v, N)$ with $h_1 \subsetneq h'_1 \subsetneq h$. From g' , let the players who belong to $N(h_1)$ and who look forward to $g \in G^v$ delete successively their links that are not in h_1 to reach $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\}$. Along the sequence from g' to g_1 all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g')$ such that $h_1 \subsetneq h$ with $h_1 \in g(v, N)$. Pick $h_1 \in g(v, N)$ such that there does not exist $h'_1 \in g(v, N)$ with $h'_1 \subsetneq h_1$. From g' , let the players who belong to $N(h_1)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$ first delete successively their links not in h_1 and then build successively the links in h_1 that are not in g' leading to $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1), ij \in h_1 \text{ and } ij \notin g'\}$. Along the sequence from g' to g_1 all players who are moving always prefer the end network g to the current network. Once g_1 and h_1 are formed, we move to Step 2.

Step 2: If there exists some $h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g_1)$ then go to Step 3 with $g_2 = g_1$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g')$ such that $h_2 \subsetneq h$ for some $h_2 \in g(v, N \setminus N(h_1))$. Then, take $h_2 \in g(v, N \setminus N(h_1))$ such that there does not exist $h'_2 \in g(v, N \setminus N(h_1))$ with $h_2 \subsetneq h'_2 \subsetneq h$. From g_1 let the players who belong to $N(h_2)$ and who look forward to $g \in G^v$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ delete successively all their links that are not in h_2 to reach $g_2 = g_1 - \{ij \mid i \in N(h_2) \text{ and } ij \notin h_2\}$. Along the sequence from g_1 to g_2 all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g')$ such that $h_2 \subsetneq h$ with $h_2 \in g(v, N \setminus N(h_1))$. Pick $h_2 \in g(v, N \setminus N(h_1))$ such that there does not exist $h'_2 \in g(v, N \setminus N(h_1))$ with $h'_2 \subsetneq h_2$. From g_1

let the players who belong to $N(h_2)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ first delete successively their links not in h_2 and then build successively the links in h_2 that are not in g_1 leading to $g_2 = g_1 - \{ij \mid i \in N(h_2) \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\}$. Along the sequence from g_1 to g_2 all players who are moving always prefer the end network g to the current network. Once g_2 and h_2 are formed, we move to Step 3.

Step k : If there exists some $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ such that $h_k \in C(g_{k-1})$ then go to Step $k+1$ with $g_k = g_{k-1}$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g')$ such that $h_k \subsetneq h$ for some $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$. Then, take $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ such that there does not exist $h'_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ with $h_k \subsetneq h'_k \subsetneq h$. From g_{k-1} let the players who belong to $N(h_k)$ and who look forward to $g \in G^v$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ..., $h_k \in C(g)$ delete successively their links not in h_k to reach $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\}$. Along the sequence from g_{k-1} to g_k all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g')$ such that $h_k \subsetneq h$ with $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$. Pick $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ such that there does not exist $h'_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ with $h'_k \subsetneq h_k$. From g_{k-1} let the players who belong to $N(h_k)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ..., $h_k \in C(g)$ first delete successively their links not in h_k and then build successively the links in h_k that are not in g_{k-1} leading to $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\} + \{ij \mid i \in N(h_k), ij \in h_k \text{ and } ij \notin g_{k-1}\}$. Along the sequence from g_{k-1} to g_k all players who are moving always prefer the end network g to the current network. Once g_k and h_k are formed, we move to Step $k+1$; and so on until we reach the network $g = \bigcup_{k=1}^K h_k$ with $h_k \in g(v, N \setminus \bigcup_{i \leq k-1} N(h_i))$. Thus, we have build a pairwise farsightedly improving path from g' to g ; $g \in f(g')$. Since $f(g') \subseteq F(g')$, we also have that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in F(g')$ under the componentwise egalitarian allocation rule Y^{ce} . ■

The next proposition tells us that once players are farsighted it does not matter whether groupwise or only pairwise deviations are feasible. Both pairwise farsighted

stability and groupwise farsighted stability single out the same unique set.

Proposition 2.1. Consider any component additive value function v . The set G^v is both the unique pairwise farsightedly stable set and the unique groupwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce} .

Proof. Consider any anonymous and component additive value function v . From Lemma 2.1 we know that $f(g) = \emptyset$ and $F(g) = \emptyset$ for all $g \in G^v$ under the componentwise egalitarian allocation rule Y^{ce} . From Lemma 2.2 we have that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce} . Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid f(g) = \emptyset\}$ and for every $g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \emptyset$, we have that G^v is the unique pairwise farsightedly stable set. In case of groupwise deviations, Theorem 5 says that G is the unique groupwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \emptyset\}$ and for every $g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$. Since $f(g) \subseteq F(g) \forall g \in \mathbb{G}$, we have that G^v is also the unique groupwise farsightedly stable set. ■

Lemma 2.3 tells us that from any $g' \notin G^v$ there is some groupwise myopic improving path going to some $g \in G^v$.

Lemma 2.3. Consider any component additive value function v . For all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce} .

Proof. We show in a constructive way that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce} . Take any $g' \notin G^v$.

Step 1: If there exists some $h_1 \in g(v, N)$ such that $h_1 \in C(g')$ then go to Step 2 with $g_1 = g'$. Otherwise, pick some $h_1 \in g(v, N)$. In g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce} . Then, we have that all members of $N(h_1)$ have incentives to deviate from g' to $g_1 = g'|_{N \setminus N(h_1)} \cup h_1$. Indeed, g_1 is obtainable from g' via deviations by $N(h_1) \subseteq N$ and $Y_i(g_1, v) > Y_i(g', v)$ for all $i \in N(h_1)$. In words, players who belong to $N(h_1)$ delete their links in g' with players not in $N(h_1)$ and build the missing links of h_1 . Once g_1 and h_1 are formed, we move to Step 2.

Step 2: If there exists some $h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g_1)$ then go to Step 3 with $g_2 = g_1$. Otherwise, pick some $h_2 \in g(v, N \setminus N(h_1))$. In g_1 all the remaining players who are belonging to $N \setminus N(h_1)$ are strictly worse off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce} . Then, we have that all members of $N(h_2)$ have incentives to deviate from g_1 to $g_2 = g_1|_{N \setminus N(h_2)} \cup h_2$. Indeed, g_2 is obtainable from g_1 via deviations by $N(h_2) \subseteq N$ and $Y_i(g_2, v) > Y_i(g_1, v)$ for all $i \in N(h_2)$. Once g_2 and h_2 are formed, we move to Step 3.

Step k : If there exists some $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$ such that $h_k \in C(g_{k-1})$ then go to Step $k+1$ with $g_k = g_{k-1}$. Otherwise, pick some $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(k-1)\})$. In g_{k-1} all the remaining players who are belonging to $N \setminus \{N(h_1) \cup \dots \cup N(k-1)\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce} . Then, we have that all members of $N(h_k)$ have incentives to deviate from g_{k-1} to $g_k = g_{k-1}|_{N \setminus N(h_k)} \cup h_k$. Indeed, g_k is obtainable from g_{k-1} via deviations by $N(h_k) \subseteq N$ and $Y_i(g_k, v) > Y_i(g_{k-1}, v)$ for all $i \in N(h_k)$. Once g_k and h_k are formed, we move to Step $k+1$; and so on until we reach the network $g = \bigcup_{k=1}^K h_k$ with $h_k \in g(v, N \setminus \bigcup_{i \leq k-1} N(h_i))$. Thus, we have build a groupwise myopically improving path from g' to g ; $g \in M(g')$. ■

The next proposition tells us that groupwise myopic stability singles out the same unique set as pairwise and groupwise farsighted stability do.

Proposition 2.2. Consider any component additive value function v . The set G^v is the unique groupwise myopically stable set under the componentwise egalitarian allocation rule Y^{ce} .

Proof. Since the set of networks consisting of all networks that belong to a closed cycle is the unique groupwise myopically stable set, we have to show that the set of all networks that belong to a closed cycle is G^v . From Lemma 2.3 we know that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce} . By Corollary 2.1 we have that $M(g) = \emptyset$ for all $g \in G^v$. Thus, it follows that each $g \in G^v$ is a closed cycle, all closed cycles belong to G^v , and G^v is the unique groupwise myopically stable set. ■

Notice that all networks belonging to G^v are pairwise stable networks in a strict sense. However, the pairwise myopically stable set may include networks that do

not belong to G^v . Thus, if players are myopic it matters whether groupwise or only pairwise deviations are feasible. So, pairwise farsighted stability, groupwise farsighted stability and groupwise myopic stability refines the notion of pairwise stability under Y^{ce} when deviations are valid only if all deviating players are strictly better off.

In the example of Figure 1, the value function is component additive and the allocation rule is the componentwise egalitarian one. Using the algorithm we obtain that $G^v = \{g_7\}$. Hence, $\{g_7\}$ is the unique pairwise (groupwise) farsightedly stable set and the unique groupwise myopically stable set.

2.5. Other Notions of Farsighted Stability

In this section we show that other notions of farsighted stability also single out the set G^v .²⁸ The largest consistent set is a concept that has been defined in Chwe (1994) for general social environments. By considering a network as a social environment, we obtain the definition of the largest consistent set.

Definition 2.5. G is a consistent set if $\forall g \in G, S \subseteq N, g' \in \mathbb{G}$ that is obtainable from g via deviations by S , there exists $g'' \in G$, where $g'' = g'$ or $g'' \in F(g') \cap G$ such that $Y_i(g'', v) \leq Y_i(g, v)$ for some $i \in S$. The largest consistent set is the consistent set that contains any consistent set.

Proposition 2.3. Consider any component additive value function v . The set G^v is the largest consistent set under the componentwise egalitarian allocation rule Y^{ce} .

Proof. First, we show in a constructive way that any $g' \notin G^v$ cannot belong to a consistent because there always exists a deviation which is not deterred. Take any $g' \notin G^v$.

Suppose $\nexists h_1 \in g(v, N)$ such that $h_1 \in C(g')$. Then, in g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce} . We have that the deviation by all members of $N(h_1)$ from g' to $g'' = g'|_{N \setminus N(h_1)} \cup h_1$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_1) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_1)$. In words, players

²⁸Herings, Mauleon and Vannetelbosch (2009) have studied the relationship between pairwise farsighted stability and other farsighted concepts. In general, pairwise (groupwise) farsightedly stable sets have no relationships with the largest consistent set.

who belong to $N(h_1)$ delete their links in g' with players not in $N(h_1)$ and build the missing links of h_1 . In addition, for any $g^* \neq g''$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_1)$. So, for any $g''' \in F(g'')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_1)$. Thus, g' cannot belong to a consistent set.

Suppose that $\exists h_1 \in g(v, N)$ such that $h_1 \in C(g')$ but $\nexists h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g')$. Then, in g' all players who belong to $N \setminus N(h_1)$ are strictly worse off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce} . Then, we have that the deviation by all members of $N(h_2)$ from g' to $g'' = g'|_{N \setminus N(h_2)} \cup h_2$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_2) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_2)$. In addition, for any $g^* \neq g''$, $g^* \subseteq g^{N \setminus N(h_1)}$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_2)$. So, for any $g''' \in F(g'')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_2)$. Thus, g' cannot belong to a consistent set.

Suppose that $\exists h_1, h_2, h_3, \dots, h_{k-1}$ with $h_l \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(h_{l-1})\})$, $l = 2, \dots, k-1$, and $h_l \in C(g')$ but $\nexists h_k \in C(g')$ such that $h_k \in g(v, N \setminus \{N(h_1) \cup \dots \cup N(h_{k-1})\})$. Then, in g' all players who are belonging to $N \setminus \{N(h_1) \cup \dots \cup N(h_{k-1})\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce} . Then, we have that the deviation by all members of $N(h_k)$ from g' to $g'' = g'|_{N \setminus N(h_k)} \cup h_k$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_k) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_k)$. In addition, for any $g^* \neq g''$, $g^* \subseteq g^{N \setminus \{N(h_1) \cup \dots \cup N(h_{k-1})\}}$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_k)$. So, for any $g''' \in F(g'')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_k)$. Thus, g' cannot belong to a consistent set. And so forth.

Second, we have from Lemma 2.1 that $F(g) = \emptyset \ \forall g \in G^v$. Hence, each $\{g\}$ with $g \in G^v$ is a consistent set. Thus, G^v is the largest consistent set under the componentwise egalitarian allocation rule Y^{ce} . ■

The von Neumann-Morgenstern stable set (von Neumann and Morgenstern 1944) imposes internal and external stability. Incorporating the notion of farsighted improving paths into the original definition of the von Neumann-Morgenstern stable set, we obtain the von Neumann-Morgenstern farsightedly stable set. von Neumann-Morgenstern farsightedly stable sets do not always exist. Corollary 5 in Herings, Mauleon and Vannetelbosch (2009) tells us that if G is the unique pairwise (groupwise) farsightedly stable set, then G is the unique von Neumann-Morgenstern pairwise (groupwise) farsightedly stable set. Hence, the set G^v is both the unique

von Neumann-Morgenstern pairwise farsightedly stable set and the unique von Neumann-Morgenstern groupwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce} .²⁹

The definitions of stability we have considered allow for a deviation by a coalition to be valid only if all members of the coalition are strictly better off. On the contrary, if we require that at least one coalition member is strictly better off while all other members are at least as well off, then it is not excluded that there are pairwise (groupwise) farsighted improving paths emanating from networks belonging to G^v going to other networks belonging to G^v . Then, it follows that G^v is no more a von Neumann-Morgenstern pairwise farsightedly stable set since internal stability is violated, and from Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) we have that G^v is no more the unique pairwise and groupwise farsightedly stable set.

2.6. Conclusion

We have studied the stability of social and economic networks when players are farsighted. We have provided an algorithm that characterizes the unique pairwise and groupwise farsightedly stable set of networks under the componentwise egalitarian allocation rule. We have then shown that this set coincides with the unique groupwise myopically stable set of networks but not with the pairwise myopically stable set. Thus, if group deviations are allowed then whether players are farsighted or myopic does not matter, or if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter.

²⁹Page and Wooders (2009) have introduced the notion of path dominance core. In general, the pairwise (groupwise) path dominance core is contained in each pairwise (groupwise) farsightedly stable set of networks. However, a path dominance core may fail to exist while a pairwise (groupwise) farsightedly stable set always exists. The set G^v is both the pairwise and groupwise path dominance core under the componentwise egalitarian allocation rule.

Chapter 3.

Risk-sharing networks and farsighted stability

3.1. Introduction

In this chapter, we study the formation of risk-sharing networks. There are regions in developing countries where the access to a formal insurance market is limited. Some villages lack for instance institutions that can enforce contracts or repayments of loans. Economic fluctuations, due to climate shocks, crop pests, illness or funeral expenditures are important in those low income areas. Informal risk-sharing appears to be one of the prominent strategy used to cope with these shocks (see the survey of Alderman and Paxson, 1994). That is, households in need receive help from others, in the form of free loans or transfers. A growing empirical literature (see Fafchamps, 1992; Grimmard, 1997; Fafchamps and Lund, 2003; De Weerdt and Dercon, 2006) has shown that a fully efficient risk-pooling equilibrium is not reached: risk-sharing does not take place within exogenous group such as the village, but rather within networks involving agents having common characteristics (neighborhood, professional or religious affiliation, kinship, etc).

Most of the theoretical papers on informal risk-sharing in developing countries assume that no binding agreement can be enforced. In this context, if the risk occurs only once, the fortunate agent has no incentive to transfer money ex-post. However, this effect disappears in a dynamic setting where multiple shocks are expected to occur since agents who transfer money today may expect to be reciprocated at a future date. This literature offers a theoretical argument to explain the observed lack of complete income pooling at the village level by analyzing the transfer schemes which are such that each agent is willing to conform to the agreement once the uncertain income shock occurs.³⁰

³⁰Kimball (1988) has shown that if individuals are sufficiently patient, then some first-best allocation, i.e. allocations that would be efficient if agents had the possibility to commit to a transfer scheme, can be implemented as a subgame perfect equilibrium. Coate and Revallion (1993) have studied the symmetric two-player model by restricting their analysis to stationary transfers. They have improved upon Kimball (1988) in that they have endogenized the amount of

Platteau (2002) has argued that risk-sharing usually occurs among relatives and even if the institutional context does not provide the tools to enforce contracts, agents involved in a risk-sharing relationship may be committed to the agreement because the social norm imposes it. Bramoullé and Kranton (2007a) have developed a model, where agents establish their informal insurance relationships endogenously, assuming that linked pairs can commit to share equally their income. They have considered agents who are *ex ante* homogeneous, but differ through their position in the network.³¹ They have shown that the efficient risk-sharing networks are such that each agent is indirectly connected to the others, involving the maximal level of insurance in the population, and that networks formed by myopic agents connect fewer individuals than the efficient ones. They have thus provided another theoretical explanation of the observation that informal insurance does not occur at the village level.

Empirical studies support the idea that mutual risk-sharing agreements are formed endogenously. For instance, Rosenweig and Stark (1989) have observed that marriages between households from different regions in India occur to diversify geographically the risks.³² Dekker (2004) has studied the endogenous formation transfers while previous work had considered only the extreme cases of complete income pooling or no transfer at all. Kocherlakota (1996) has further analysed the case of impatient agents and has shown that the Pareto-undominated subgame perfect allocations imply a positive correlation between individual consumption and current and lagged income. Ligon, Thomas and Worrall (2002) have shown that allowing transfers to depend upon history of transfers is payoff improving for the agents. Genicot and Ray (2003) have further studied this problem by adding the possibility that groups of agents jointly deviate from the prescribed transfer scheme. Finally, Bloch, Genicot and Ray (2005) have adapted previous work for situations where transfers occur through a network rather than through a group.

³¹There have been other attempts to model the formation of informal network in the spirit of Jackson and Wolinsky (1996), where direct links involve benefits and costs, while indirect links affect positively or negatively the agents depending on the nature of the network externalities. Bramoullé and Kranton (2007b) have studied the formation of risk-sharing networks among agents living in two different villages, assuming that the shock to the income of households is village specific. Comola (2008) has proposed a model of network formation, where benefits and costs to links formation are heterogeneous. Krishnan and Sciubba (2008) have extended the co-author model of Jackson and Wolinsky (1996) to study bilateral labor exchange agreements among heterogeneous agents.

³²Grimmard (1997) has found evidence of transfers and migrations in Côte d'Ivoire supporting this idea.

of risk-sharing networks in four resettlement villages in rural Zimbabwe. She has found that in a social environment where blood relatives are scarce, resettled households have strongly invested in activities to establish links with surrogate relatives. Comola (2008) has observed that the structure of the network, that is the social position of an agent with respect to the others, is critical to understand the choice of risk-sharing partners. Based on data on the village Nyakatoke in Tanzania, she has found that "not only the characteristics of direct friends, but also the characteristics of indirect contacts are taken into account when a link is created".

This chapter analyzes which pattern of insurance relationships emerges in the long run when agents are farsighted, rather than myopic, in the sense that they are able to forecast how other agents would react to their choice of partners. In his survey of models of network formation, Jackson (2005) has mentioned that farsightedness is an important consideration in some appropriate context. He has stated that "in large networks it might be that players have very little ability to forecast how the network might change in reaction to the addition or deletion of a link. In such situations the myopic solutions are quite reasonable. However, if players have very good information about how others might react to changes in the network, then these are things that one wants to allow for either in the specification of the game or in the definition of the stability concept". To our knowledge, no existing work has attempted to establish whether agents are farsighted or not when creating their network in rural areas of developing countries. However, we believe that the key ingredients mentioned by Jackson (2005) for farsightedness to matter are present in this framework: our focus is on small communities, where agents have good information about each other. Agents in the model of Bramoullé and Kranton (2007a) are strategic: they establish links with other agents, anticipating that these connections might be profitable in the future if they face negative income shocks. In this chapter, we assume that agents are a bit more strategic: in addition to forming connections in anticipation of likely future negative shocks, they also realize that their choice of partners may determine others' choices of partners. Such anticipation is consistent with Comola (2008)'s observation that the full architecture of bilateral agreements determines the incentives for a pair of agents to establish a partnership. We adopt the notion of pairwise farsightedly stable set due to Herings, Mauleon and Vannetelbosch (2009) to determine which networks are formed by farsighted

agents.³³ We find that for small costs of establishing and maintaining a partnership, farsighted agents may form efficient networks that involve full income pooling while myopic agents form networks connecting fewer individuals. Two mechanisms explain this result: (i) Farsighted agents belonging to small groups may decide to create new partnerships that are not directly profitable to them, because they realize that other partners will further join this bigger and more attractive group. In other words, the farsightedness of the agents may solve a coordination problem. (ii) Farsighted agents may refrain from deleting costly links if they belong to a big group, as they understand that this may induce others to rearrange their partnerships in a way that deters the myopic incentives to delete the link at first. We have already mentioned that empirical studies have revealed that risk-sharing occurs among agents having common characteristics. Farsightedness may be a factor rationalizing this observation.

The chapter is organized as follows. In Section 2 we introduce some notations and definitions for networks, and we present the model of risk-sharing networks of Bramoullé and Kranton (2007a). In Section 3, we investigate the formation of risk-sharing networks when agents are myopic. Section 4 provides a characterization of the pairwise farsightedly stable set of risk-sharing networks. In Section 5, we analyze more in detail the formation of risk-sharing networks when agents have a quadratic utility function. In Section 6, we conclude.

3.2. Model and notation

Networks

A network (N, g) is defined by a set of agents $N = \{1, \dots, n\}$ and a list g of which pairs of individuals among the agent set N are linked to each other. For sake of notation we simply use the set of links g to refer to the network when the player set N is fixed. The network relationships are reciprocal and the network is thus modelled as a non-directed graph. Individuals are the nodes in the network and links indicate bilateral relationships between individuals. We write $ij \in g$ to indicate that i and j are linked under the network g . Let g^N be the collection of all subsets of N with

³³Other approaches to farsightedness in network formation are suggested by the work of Xue (1998), Herings, Mauleon, and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Page, Wooders and Kamat (2005), Dutta, Ghosal, and Ray (2005), and Page and Wooders (2009).

cardinality 2, so g^N is the complete network. The set of all possible networks on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N . The network obtained by adding the link ij to an existing network g is denoted $g + ij$ and the network that results from deleting the link ij from an existing network g is denoted $g - ij$. For any network g , let $N(g) = \{i \in N \mid \exists j \text{ such that } ij \in g\}$ be the set of agents who have at least one link in the network g . The degree of agent i in a network g is the number of links that involve that agent: $d_i(g) = \#\{j \in N \mid ij \in g\}$. The total number of links of a network g is given by $d(g) = \sum_{i \in N} d_i(g)/2$. A path in a network $g \in \mathbb{G}$ between i and j is a sequence of agents $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j . A component of a network (N, g) , is a nonempty subnetwork (N', g') such that $\emptyset \neq N' \subseteq N$, $g' \subseteq g$ satisfying (i) (N', g') is connected, and (ii) if $i \in N'$ and $ij \in g$, then $j \in N'$ and $ij \in g'$.³⁴ The set of components of g is denoted by $C(g)$. A component of a network is minimally connected if the path between any two agents in that component is unique. The set of networks composed of minimally connected components is denoted G^m . Formally, $G^m = \{g \in \mathbb{G} \mid \#C(g) < \#C(g - ij) \text{ for each } ij \in g\} \cup \{g^\emptyset\}$, where $g^\emptyset$ is the empty network. A network is minimally connected if all the agents are in the same minimally connected component. The set of minimally connected networks is $G^M = \{g \in G^m \mid \#C(g) = 1\}$. We use the measure of betweenness centrality of Freeman (1977). Letting $P_i(kj)$ denote the number of shortest paths between k and j that i lies on, and $P(kj) = \sum_{i \notin \{k, j\}} P_i(kj)$, the betweenness centrality of an agent i is given by $Ce_i^B(g) = (2/((n-1)(n-2))) \sum_{k \neq j: i \notin \{k, j\}} P_i(kj)/P(kj)$. This measure will be used to determine the central agents of a connected line. A connected line is a minimally connected network (N, g) such that no agent in N has more than two links. The set of connected lines is $G^L = \{g \in G^M \mid d_i(g) \leq 2 \text{ for all } i \in N\}$. The central elements of a line $g \in G^L$ are the agents with the highest measure of betweenness centrality. Formally, for $g \in G^L$, $Ce(g) = \{i \in N \mid Ce_i^B(g) \geq Ce_j^B(g) \text{ for all } j \in N\}$.

³⁴This definition of components is proposed by Jackson (2008) and implies that an agent without links in a network is considered as a component.

Model

We further investigate the model of Bramoullé and Kranton (2007a) where n ex-ante identical individuals are risk averse and face shocks to their income. Each individual's income, y_i , is a random variable which is independently and identically distributed with mean $\bar{y}$ and variance σ^2 . Agents have identical preferences, represented by the utility function v , which is increasing and strictly concave in monetary holdings. Individuals may create links with each other. By doing so, they commit to pool their income with the other agents in their component and to share it equally.³⁵ It follows that risk-sharing benefits only depend on the number of individuals in the component. If agents $1, 2, \dots, s$ belong to a component of size s , then the monetary holdings of each agent in this component are $(y_1 + y_2 + \dots + y_s)/s$ and their expected utility are given by $u(s) = Ev((y_1 + y_2 + \dots + y_s)/s)$, where E denotes the expectation over the realization of incomes. The expected monetary holdings of an agent are independent of the network, but the variance of her expected monetary holdings is decreasing with the size of the component to which she belongs. Since agents are risk-averse, the expected utility function $u(s)$ is increasing in the size of the component, that is $u(s + 1) > u(s)$ for all integer s . In addition, we assume that it increases at a decreasing rate, i.e. $u(s + 2) - u(s + 1) < u(s + 1) - u(s)$ for all s . That is, the bigger is the set of agents with whom an agent shares her risk, the smaller is her benefit to have a new insurance partner. Each direct link ij results in a cost c to both i and j . This cost should be interpreted as an amount of resources needed to ensure that the transfers are realized ex-post, once the shock is realized. In other words, it is assumed that a richer agent will share her revenue with a poorer agent to whom she is linked, because those agents have developed a relationship of trust among themselves, which was costly to establish. We assume that these costs are non-monetary and as such, they cannot be shared with other members of the component. Bramoullé and Kranton (2007a) have motivated this assumption by saying that "some costs, such as the time incurred to build a relation are not easy to compensate or transfer". The payoff of agent i in the network g is

³⁵In reality, full income pooling is not observed. Ligon (1998) finds that information asymmetry is the main factor explaining incomplete income pooling in rural India. Lack of commitment (Coates and Revallion, 1993) is another explanation of this observation. In our model, we assume full information and that agents have the ability to commit to a future contingent transfer. As such, the choice of the equal sharing rule seems appropriate as it is the optimal one.

given by

$$U_i(g) = u(s_i) - d_i(g)c,$$

where $d_i(g)$ indicates the number of links agent i has and s_i denotes the size of the component to which she belongs, $s_i = \#S$, where $i \in S$ and $(S, h) \in C(g)$.

Efficiency

A network $g \in \mathbb{G}$ is *efficient* if it maximizes the total societal value, that is if $\sum_{i \in N} U_i(g) \geq \sum_{i \in N} U_i(g')$ for all $g' \in \mathbb{G}$. Efficient networks are composed of minimally connected components since otherwise, productive resources would be wasted. The total utility of the agents in a network g composed of k minimally connected components is given by $\sum_{i \in N} U_i(g) = \sum_{j=1}^k s_j u(s_j) - 2c(n - k)$, where s_j is the size of the component j . The assumption on the expected utility function ensures that the total value of a component is increasing with the size of this component, namely $(s + 1)u(s + 1) > su(s)$. Bramoullé and Kranton (2007a) have shown that this total value could increase at an increasing or decreasing rate, *i.e.* $(s + 2)u(s + 2) - (s + 1)u(s + 1)$ could be bigger or smaller than $(s + 1)u(s + 1) - su(s)$.³⁶ We assume in this chapter that the total value of a component is increasing with the size of this component at a nondecreasing rate. Efficient networks can then only be of two types: either nobody is linked or everybody is indirectly connected. Let us note by $c^* = (n[u(n) - u(1)])/(2(n - 1))$ the critical cost of link formation such that the empty network generates the same total utility than a minimally connected network. When $(s + 2)u(s + 2) - (s + 1)u(s + 1) > (s + 1)u(s + 1) - su(s)$ for all $s \in \{1, 2, \dots, n - 2\}$, the empty network is efficient if $c > c^*$, while an efficient network is composed of one component connecting minimally the n agents if $c < c^*$.

³⁶No general properties of v and y determine the curvature of $su(s)$. Bramoullé and Kranton (2007a) have shown that when the primitive utility function is CARA: $v(y) = v_0 - e^{-\mu y}$, where $\mu > 0$ denotes the level of absolute risk-aversion, and if income is normally distributed, then $su(s)$ is increasing with s at a decreasing rate, while if we consider the quadratic utility function: $v(y) = y - \lambda y^2$, where λ is a positive parameter, then $su(s)$ is increasing linearly with s .

3.3. Stable risk-sharing networks when agents are myopic

In this section, we investigate the formation of stable risk-sharing networks when agents are myopic. We adopt the notion of pairwise myopically stable sets due to Herings, Mauleon and Vannetelbosch (2009) which is a generalization of Jackson and Wolinsky (1996) pairwise stability notion. A pairwise myopically stable set is such that from any network outside this set, there is a myopic improving path leading to some network in the set, and each deviation outside the set is deterred because the deviating agents do not prefer the resulting network. The notion of myopic improving path is due to Jackson and Watts (2002) and is defined as a sequence of networks that might be observed when agents are adding or deleting links, one at a time, in order to improve their current payoff. Formally, a myopic improving path from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $U_i(g_{k+1}) > U_i(g_k)$ or $U_j(g_{k+1}) > U_j(g_k)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $U_i(g_{k+1}) > U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$. For a given network g , we denote by $M(g)$ the set of networks that can be reached through a myopic improving path from g .

Definition 3.1. A set of networks $G \subseteq \mathbb{G}$ is pairwise myopically stable if

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $(U_i(g + ij), U_j(g + ij)) = (U_i(g), U_j(g))$ or $U_i(g + ij) < U_i(g)$ or $U_j(g + ij) < U_j(g)$,
- (ib) $\forall ij \in g$ such that $g - ij \notin G$, $U_i(g - ij) \leq U_i(g)$ and $U_j(g - ij) \leq U_j(g)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G$, $M(g') \cap G \neq \emptyset$,
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii)..

Conditions (ia) and (ib) in Definition 3.1 capture deterrence of external deviations. In Condition (ia) the addition of a link ij to a network $g \in G$ that leads to a network outside G is deterred because the two agents involved do not prefer the resulting network to network g . Condition (ib) is a similar requirement, but then

for the case where a link is severed. Condition (ii) requires external stability. External stability asks for the existence of a myopic improving path from any network outside G leading to some network in G . Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 3.1. This motivates Condition (iii), the minimality condition. Jackson and Watts (2002) have defined the notion of a closed cycle. A closed cycle is a set of networks C such that for any pair of networks $g, g' \in C$, where $g \neq g'$, there exists a myopic improving path from g to g' , and each myopic improving path emanating from a network in the set C does not reach a network outside C . Each pairwise stable network is a closed cycle. Herings, Mauleon and Vannetelbosch (2009) have proved that the pairwise myopically stable set coincides with the set of networks that belong to a closed cycle.

Bramoullé and Kranton (2007a) have shown that the set of closed cycles consists only of pairwise stable risk-sharing networks if some exist. In addition, they have analyzed the architecture of pairwise stable networks and the conditions on the parameters that guarantee their existence. To summarize their results, let s^* be the critical size of a component such that the benefit of adding another member to the component is less than the cost of doing so:

$$s^* = \max \{s \in \mathbb{N} \mid u(s) - u(s-1) \geq c\}.$$

Since the expected utility function u is increasing at a decreasing rate, an agent belonging to a component of size $s < s^*$ is willing to add a link with a singleton while an agent belonging to a component of size $s > s^*$ has incentives to cut a link if she reaches a component of size $s-1$. Another threshold s^{**} defines the maximal size of a component of a pairwise stable network if another component in the network has size s^* . The threshold s^{**} is defined as follows:

$$s^{**} = \max \{s \leq s^* \mid u(s^* + s) - u(s^*) \leq c, \text{ with the inequality being strict if } s < s^*\}.$$

An agent in a component of size s^* is not willing to add a link with an agent in a component of size smaller than or equal to s^{**} . Each component of a pairwise stable network is minimally connected and at least one of these components has size $s = \min\{s^*, n\}$. If $s^* = s^{**}$, pairwise stable networks always exist. They are composed of a maximal number of components of size s^* and of one component of smaller size. If $s^* > s^{**}$, pairwise stable networks exist if and only if $s^* + s^{**} \geq n$. They are then composed of one component size s^* and of another of size $n - s^*$. Let

G^* be the set of networks composed of minimally connected components of size s^* and of one minimally connected component of size $s = n - s^* \text{int}(n/s^*)$ if $s \neq 0$.³⁷ Formally, $G^* = \{g \in G^m \mid \text{if } (S, h) \in C(g) \text{ and } \#S \neq s^*, \text{ then (i) } \#S < s^* \text{ and (ii) for all } (S', h') \in C(g) \text{ with } S' \neq S, \text{ we have } \#S' = s^*\}$. Bramoullé and Kranton (2007a) have shown that the pairwise myopically stable set G is a superset of G^* . Furthermore, the two sets G and G^* coincide if and only if $s^* + s^{**} \geq n$, or $s^* = s^{**}$.

Proposition 3.1. (Bramoullé and Kranton, 2007) The pairwise myopically stable set G is such that $G^* \subseteq G$. In addition, G^* is the unique pairwise myopically stable set if and only if either $s^* = s^{**}$, or $n \leq s^{**} + s^*$.

All proofs are presented in the appendix. Let us introduce another threshold, $\bar{s}$, which is the maximal integer such that two agents in different components of size $\bar{s}/2$ are willing to add a link between them. Formally, it is defined as follows:

$$\bar{s} = \max \{s \in \mathbb{N} \mid \text{(i) } s \text{ is even and (ii) } u(s) - u(s/2) > c\}.$$

Notice that the addition of a link is not profitable for at least one of the agents involved if one of them belongs to a component of size bigger than $\bar{s}/2$.³⁸ The following proposition states that each network in the pairwise myopically stable set is composed of minimally connected components of size smaller than or equal to $\bar{s}$, and contains at least $s^* - 1$ links, but no more than $n - 1 - \text{int}((n - 1)/\bar{s})$ links.

Proposition 3.2. Each network g in the pairwise myopically stable set G is such that (i) $g \in G^m$, (ii) $\#S \leq \bar{s}$ for all $(S, h) \in g$ and (iii) $s^* - 1 \leq d(g) \leq n - 1 - \text{int}((n - 1)/\bar{s})$.

Intuitively, there exists a myopic improving path from every network composed of components which are not minimally connected to some network composed of minimally connected components if the agents delete unnecessary links. However, the converse does not hold. Once a network composed of minimally connected components is reached, every myopic improving path leads to other networks in G^m since the addition of a useless link is costly. In addition, from networks composed of big-sized components, agents are willing to cut links with peripheral agents (agents having only one link) as long as the size of their component is bigger than s^* . On

³⁷The operator $\text{int}(x)$ gives the integer part of the real x .

³⁸This result is established in the appendix (see Lemma 3.A.2).

the other hand, from networks composed of small-sized components, no myopic improving path is leading to a network having a component of size bigger than $\bar{s}$ since, if it was the case, an agent should add a link at some point in the path when she is currently a member of a component of size $s > \bar{s}/2$, but the addition of that link is not profitable. Each network of the pairwise myopically stable set has more than $s^* - 1$ links since no agent is willing to cut a link if there are s^* agents or fewer in her component. Finally, each network in G has less than $n - 1 - \text{int}((n - 1)/\bar{s})$ links as this number of links is obtained when the agents form a maximal number of components of size $\bar{s}$.

3.4. Stable risk-sharing networks when agents are farsighted

Myopic agents are assessing the profitability of their decision to create new mutual insurance agreements or to remove old ones by considering that their choice has no impact on others' decisions. In this section, we analyze the formation risk-sharing networks when agents are farsighted, rather than myopic, in the sense that they are able to anticipate how other agents would react to their choice of partners.

Herings, Mauleon and Vannetelbosch (2009) have proposed a solution concept to address the question of stability when agents are farsighted: the pairwise farsightedly stable set. Before defining the concept, let us introduce the notion of a farsighted improving path, which is the counterpart of the myopic improving path described in the previous section. A *farsighted improving path* is a sequence of networks that can emerge when agents form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two agents involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two agents involved in the link strictly prefers the end network. Formally, it is defined as follows. A farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \dots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K - 1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $U_i(g_K) > U_i(g_k)$ or $U_j(g_K) > U_j(g_k)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $U_i(g_K) > U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$. For a given network g , let $F(g) = \{g' \in G \mid$

there is a farsighted improving path from g to g' }.

We now introduce the concept of pairwise farsightedly stable set. It is a set of networks such that (i) the deletion or addition of any link from a network in the set leading to a network outside the set is deterred by a credible threat of ending worse off, once other agents further react to the initial deviation, (ii) from any network outside the set, there is a farsighted improving path leading to some network in the set, and (iii) no proper subset of this set satisfies the two first conditions. Formally, pairwise farsightedly stable sets are defined as follows.

Definition 3.2. A set of networks $G \subseteq \mathbb{G}$ is pairwise farsightedly stable with respect v and Y if

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $\exists g' \in F(g + ij) \cap G$ such that $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ or $Y_i(g', v) < Y_i(g, v)$ or $Y_j(g', v) < Y_j(g, v)$,
- (ib) $\forall ij \in g$ such that $g - ij \notin G$, $\exists g', g'' \in F(g - ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,
- (ii) $\forall g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$.
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Condition (i) in Definition 3.2 requires the deterrence of external deviations. Condition (ia) captures that adding a link ij to a network $g \in G$ that leads to a network outside of G , is deterred by the threat of ending in g' . Here g' is such that there is a farsighted improving path from $g + ij$ to g' . Moreover, g' belongs to G , which makes g' a credible threat. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) in Definition 3.2 requires external stability and implies that the networks within the set are robust to perturbations. From any network outside of G there is a farsighted improving path leading to some network in G . Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 3.2. This motivates the requirement of a minimality condition, namely Condition (iii).

We now provide a partial characterization of the pairwise farsightedly stable sets of risk-sharing networks. We analyze the case of very small costs of link formation, the case of small costs of link formation and the case of high costs of link formation.

Very small costs of link formation

Proposition 3.3 characterizes partially the pairwise farsightedly stable sets when the costs of link formation satisfy $c < u(n) - u(n-1)$. For such costs, we find that (a) each pairwise farsightedly stable set contains at least one connected network, (b) each set G composed of a minimally connected network $\tilde{g} \in G^M$ and all other networks $g \in G^M$ such that $U_i(g) = U_i(\tilde{g})$ for all $i \in N$ is a pairwise farsightedly stable set, and (c) each set G composed of a minimally connected network $g_1 \in G^M$ and another minimally connected network $g_2 \in G^M$ such that $U_i(g_1) \neq U_i(g_2)$ for some agent $i \in N$ and g_1 and g_2 are not star networks (i.e. they are not such that one agent connects directly all the others) is a pairwise farsightedly stable set.

Proposition 3.3. If $0 < c \leq u(n) - u(n-1)$, then

- (a) If G is a pairwise farsightedly stable set, then $\#C(g) = 1$ for some $g \in G$.
- (b) For each $\tilde{g} \in G^M$, the set $G(\tilde{g}) = \{g \in G^M \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N\}$ is a pairwise farsightedly stable set.
- (c) The set $G = \{g_1, g_2\} \subseteq G^M$ such that $U_i(g_1) \neq U_i(g_2)$ for some agent $i \in N$ and both g_1 and g_2 are not star networks is a pairwise farsightedly stable set.

A connected line Pareto dominates each network composed of multiple components for such costs. There are thus no farsighted improving paths from a connected line to a network composed of multiple components. It then follows that a set of networks that does not include at least one network that connects indirectly all the agents is not externally stable. To prove part (b) and (c), we first show that there is a farsighted improving path from any network outside $G(\tilde{g}) = \{g \in G^M \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N, \text{ for some } \tilde{g} \in G^M\}$ leading to each network in $G(\tilde{g})$. Intuitively, this holds since from any network g outside $G(\tilde{g})$, there always exists an agent willing to cut a link from g or a pair of agents willing to add a link from g , looking forward to the formation of a network in $G(\tilde{g})$. Thus, the sets proposed in part (b) and (c) of the proposition are such that there are farsighted improving paths from each network outside the set to some network in the set. Notice in addition that each pairwise deviation from a network in one of those sets is deterred by the threat of coming back at the same network in one step.

A star network as a singleton is a pairwise farsightedly stable set according to part (b) of Proposition 3.3. It is thus required that g_1 and g_2 are not star networks in

part (c) of Proposition 3.3 as otherwise, the set of networks g_1 and g_2 would fail to be minimal. In the last section of the chapter, we analyze the quadratic utility function case and we show that when there are four agents, some pairwise farsightedly stable sets of networks are exclusively composed of networks connecting all the population but not at the minimal cost. Whether this result holds for general utility function and for any number of agents remains an open question.

Small costs of link formation

In the next proposition, we show that each set composed of connected line $\tilde{g}$ and of all other lines where the payoff of the agents is equal to their payoff in $\tilde{g}$ constitutes a pairwise farsightedly stable set if the cost of link formation satisfies $c < \min\{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$.

Proposition 3.4. If $c < \min\{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$, we have that

- (a) For each $\tilde{g} \in G^L$, the set $G(\tilde{g}) = \{g \in G^L \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N\}$ is a pairwise farsightedly stable set.
- (b) The set $\{g_1, g_2\}$ where $g_1, g_2 \in G^L$ and $U_i(g_1) \neq U_i(g_2)$ for some $i \in N$ is a pairwise farsightedly stable set.

In the proof of this proposition, it is first established that there is a farsighted improving path from any network outside $G(\tilde{g}) = \{g \in G^L \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N, \text{ for some } \tilde{g} \in G^L\}$ leading to each network in $G(\tilde{g})$. From a network outside $G(\tilde{g})$, farsighted agents who have more links than in the connected line $\tilde{g}$ or who have the same number of links but are indirectly connected to less than n agents, cut their links until the empty network is reached, looking forward to the formation of the network $\tilde{g}$. From the empty network, the agents add links in order to build $\tilde{g}$ such that the last link to be added is the central link of the line. This last move is profitable for the two agents involved in that link since $c < u(n) - u(\text{int}((n+1)/2))$. The addition of each other link from the empty network to $\tilde{g}$ is profitable for farsighted agents having already one link as this allows them to move from a component of size smaller than $\text{int}((n+1)/2)$ to the connected network $\tilde{g}$, and it is profitable for isolated agents since $u(n) - 2c > u(1)$. Notice in addition that each pairwise deviation from a network in the set is deterred by the threat of coming back at the same network.

The pairwise farsightedly stable sets described in Proposition 3.4 are not necessarily unique. We will see in Section 5 that inefficient networks may also belong to some pairwise farsightedly stable sets.

High costs of link formation

In Proposition 3.5, we show that when the cost of link formation is sufficiently high, (a) the set of all networks in which each agent belongs to a component of size 2 is the only pairwise farsightedly stable set if the number of agent is even, (b) the set of all networks in which the same agent is not connected while the remaining agents belong to a component of size 2 is a pairwise farsightedly stable set if the number of agent is odd, and (c) the set composed of one network where a maximal number of linked pairs forms among all the agents but agent k and of one network where a maximal number of linked pairs forms among all the agents but agent $l \neq k$ is a pairwise farsightedly stable set if the number of agent is odd.

Proposition 3.5. If $u(n) - u(2) < c < u(2) - u(1)$, then

- (a) The set $G = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N\}$ is the unique pairwise farsightedly stable set if n is even.
- (b) For $k \in N$, the set $G_k = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N \setminus \{k\}, \text{ and } d_k(g) = 0\}$ is a pairwise farsightedly stable set if $(u(3) - u(1))/2 < c$ and n is odd,
- (c) For $k, l \in N$ with $k \neq l$, the set $G = \{g_k, g_l\}$, where for $m \in \{k, l\}$, $G_m = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N \setminus \{m\}, \text{ and } d_m(g) = 0\}$ is a pairwise farsightedly stable set if $(u(3) - u(1))/2 < c$, n is odd and $n \geq 5$.

When the costs of link formation satisfy $u(n) - u(2) < c < u(2) - u(1)$, each agent prefers to be in a network in which she is a member of a component of size two rather than in any network in which her number of links is different than one. From any network that does not belong to the set of networks composed of a maximal number of linked pairs, agents having more than one link are willing to cut a link, and agents having no links are willing to add a link, looking forward to a network composed of a maximal number of linked pairs. Part (a) of the proposition is then derived from the fact that there are no farsighted improving paths emanating from a network composed of a maximal number of linked pairs if n is even. When the number of agents is odd, an agent, say k , is not connected in a network composed of a maximal

number of linked pairs $g \in G_k$. Then, part (b) and part (c) of the proposition follow from the fact that from each network outside G_k , there is a farsighted improving path to each network in G_k . Also, each deviation from G_k is deterred by the threat of coming back at the same network in one step.

The characterization of pairwise farsightedly stable sets when the cost of link formation is intermediate, i.e. when $\min\{(u(n) - u(1))/2, u(n) - u(n/2)\} \leq c \leq u(n) - u(2)$ for n even, or when $\min\{(u(n) - u(1))/2, u(n) - u((n+1)/2)\} \leq c \leq \max\{(u(3) - u(1))/2, u(n) - u(2)\}$ for n odd, remains an open question.

Let us summarize the results obtained concerning the structure of risk-sharing networks formed by farsighted agents and compare them with the networks formed by myopic agents and the efficient ones. For very small costs of link formation ($c \leq u(n) - u(n-1)$) or high ones ($u(n) - u(2) < c$ when the population size is even, or $\max\{u(n) - u(2), (u(3) - u(1))/2\} < c$ if the population size is odd), farsighted and myopic agents form the same networks. For very small costs of link formation, the pairwise myopically stable set is the set of efficient networks (each efficient network is pairwise stable since $s^* \geq n$ for such costs), and each efficient network belongs to some pairwise farsightedly stable set. For high costs of link formation, the pairwise myopically stable set contains all networks composed of a maximal number of linked pairs ($s^* = 2 = s^{**}$ when $u(n) - u(2) < c < u(2) - u(1)$, implying that each network in the pairwise myopically stable set is pairwise stable). The union of the pairwise farsightedly stable sets coincides with the pairwise myopically stable set when n is even, and contains the pairwise myopically stable set when n is odd. Whether a network not contained in the pairwise myopically stable set belongs to some pairwise farsightedly stable set when n is odd remains an open question. The networks formed differ however for small costs of link formation. When $u(n) - u(n-1) < c < \min\{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$, each line connecting all the agents and other lines in which the degree of the agents is similar constitutes a pairwise farsightedly stable set. Farsighted agents may form efficient networks, while myopic agents cannot sustain those networks at equilibrium since from an efficient network, the agents have myopic incentives to cut a link with a peripheral agent as long as $u(n) - u(n-1) < c$. Farsighted agents are not willing to cut those links as they fear that the others will in turn modify sequentially their choice of insurance partners so that the network that will form in the end will be the same line connecting all the agents.

3.5. Application: the quadratic utility function

In this section, we analyze more in detail the formation of risk-sharing networks when agents have a quadratic utility function. We illustrate through this example the implications of the theorems presented in the previous sections about the formation of risk-sharing networks by farsighted and myopic agents. In particular, we investigate three questions. First, we analyze what is the impact of the risk-aversion of the agents, of their initial wealth, and of the variance of the income shock on the formation of risk-sharing networks. Second, we study to which extent the range of costs for which farsightedly stable networks can be identified shrinks as the size of the population increases. Third, we investigate whether additional results can be obtained when the characterization we have established is incomplete, that is when the cost of link formation is very small, small or intermediate.

Let v be a quadratic utility function $v(y) = y - \lambda y^2$ where λ represents the level of risk-aversion of an individual. As shown in Bramoullé and Kranton (2007a), the expected utility function is then $u(s) = v(\bar{y}) - (\lambda\sigma^2)/s$ where $\bar{y}$ and σ^2 are respectively the mean and the variance of the income distribution. This expected utility function is increasing at a nondecreasing rate, and the total utility of the members of a component ($su(s)$) is increasing at a constant rate, so that quadratic utility functions verify our assumptions.

The stability of sets of networks is determined by comparing the expected utility of an agent when she belongs to components of different sizes. For quadratic utility functions, the variation of expected utility of an agent if she moves from a component of size k to a component of size l is $u(l) - u(k) = \lambda\sigma^2(l - k)/lk$. As far as stability is concerned, the mean of income ($\bar{y}$) thus does not matter. The relevant parameters of the utility functions are the variance of the shock (σ^2) and the parameter that represents the risk aversion (λ). In addition, only their product matters so that uncertainty and risk aversion play the same role.

We have depicted in Figure 1 the evolution of the thresholds of link cost that are relevant for the theorems as a function of the number of agents.³⁹ The uncertainty (σ^2) and the risk aversion (λ) change the scale of Figure 1 but not its shape since a modification of one of those parameters affects the various thresholds in the same way. We assume in Figure 1 and in the rest of this section that $\lambda\sigma^2 = 9$. When the

³⁹We have not represented the threshold $(u(n) - u(1))/2$ since $\min\{(u(n) - u(1))/2; u(n) - u(\text{int}((n+1)/2))\} = u(n) - u(\text{int}((n+1)/2))$ when the utility function is quadratic.

number of agents increases, the range of costs for which Propositions 3.3, 3.4 and 3.5 applies shrinks while the range of intermediate costs (i.e. for which we have not characterized the pairwise farsightedly stable sets) increases. For high costs of link formation ($u(n) - u(2) < c < u(2) - u(1)$ when n is even and $\max\{u(n) - u(2), (u(3) - u(1))/2\} < c < u(2) - u(1)$ when n is odd), the lower bound of the interval is nondecreasing with n while the upper bound is fixed. For very small costs of link formation ($0 < c \leq u(n) - u(n-1)$), the upper bound of the interval is decreasing with n since by assumption, the bigger is the set of agents with whom an agent shares her risk, the smaller is her benefit to have a new insurance partner. When n is even, the range of costs for which Proposition 3.4 applies ($u(n) - u(n-1) < c < u(n) - u(\text{int}(n+1)/2)$) decreases with n as long as $n \geq 4$. Similarly, this range decreases with n when n is odd for $n \geq 5$. The range of intermediate costs is determined by the condition $u(n) - u(n/2) \leq c \leq u(n) - u(2)$ when n is even and by the condition $u(n) - u(\text{int}((n+1)/2)) \leq c \leq \max\{u(n) - u(2), (u(3) - u(1))/2\}$ when n is odd. It is increasing with n since the lower bound of the interval ($u(n) - u(\text{int}((n+1)/2))$) is decreasing in n while the upper bound is nondecreasing in n .

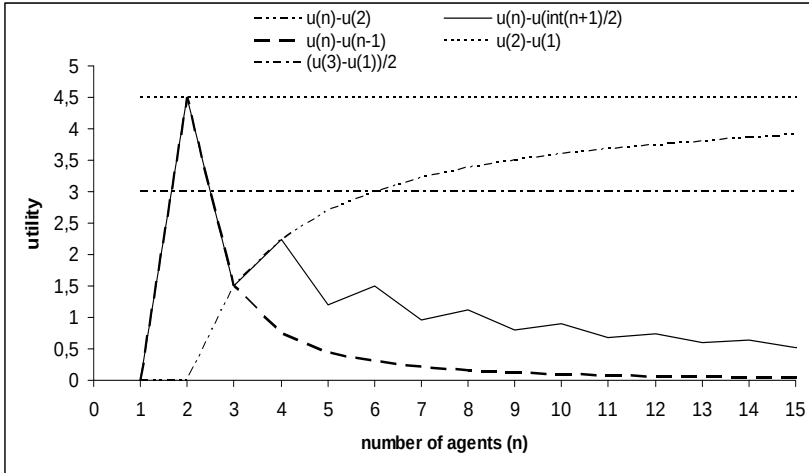


Figure 1. Cost thresholds and population size

The characterizations proposed in the theorems are not complete. The theorems identify some equilibrium candidates but there may exist other equilibria. We have developed an algorithm that aims at identifying all the pairwise farsightedly stable sets to investigate whether additional results can be obtained when the characterization we have established is incomplete.⁴⁰ Among n agents, there are possibly

⁴⁰The algorithm is available upon request from the author. We explain in Appendix 3.B the main steps for its construction.

$K = \sum_{i=0}^{n(n-1)/2} C_i^{n(n-1)/2}$ networks and $\sum_{i=1}^K C_i^K$ equilibrium set candidates.⁴¹ It is thus increasingly complex to fully characterize the pairwise farsightedly stable sets since the number of candidates explodes as n increases. When $n = 3$, there are 8 different networks that can form 256 different equilibrium set candidates and for $n = 4$, there are 64 different possible networks and $1,8447 E + 19$ candidates.⁴² Having associated to each network a number between 1 and K , the output of the algorithm is (i) a square matrix F of dimension $K \times K$, where K is the total number of networks among n agents, such that $F(i, j) = 1$ if there is a farsighted improving path from the network number $i \in \{1, K\}$ leading to the network $j \in \{1, K\}$ and (ii) a matrix $PFFS$ of dimension $L \times K$, where L is the total number of pairwise farsightedly stable sets such that a set composed of networks associated with non-zero elements of a line is a pairwise farsightedly stable set of networks.

We have used the algorithm to determine the farsightedly stable sets of networks formed among 3 agents. For more than 3 agents, we are not able to build a matrix whose number of lines corresponds to the number of candidates. We then have considered subsets of the full set of equilibrium candidates.

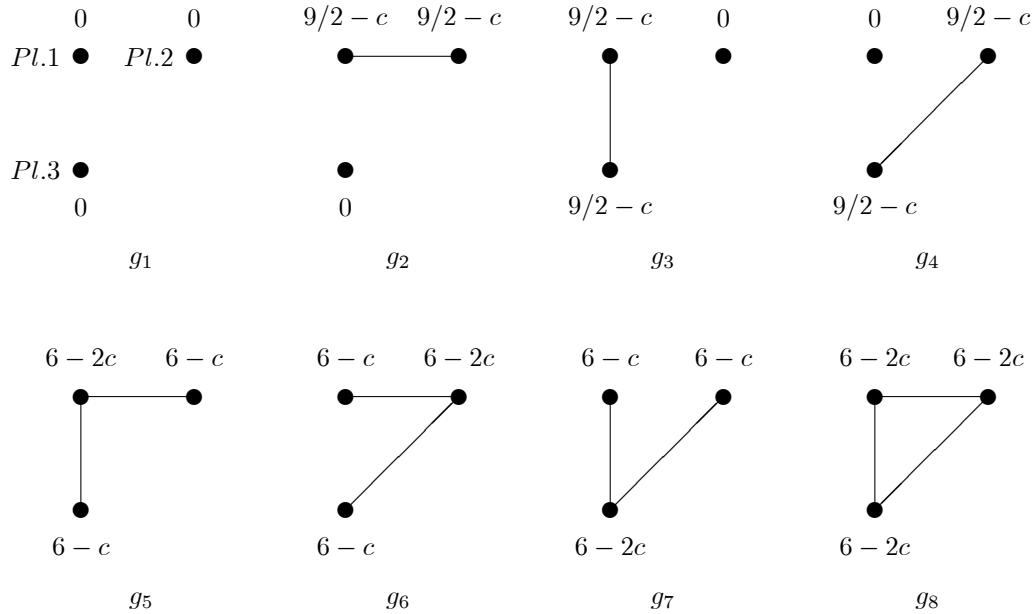


Figure 2. Risk-sharing networks among 3 agents.

⁴¹ C_k^n gives the combination of n things taken k at a time without repetition and is equal to $n!/(k!(n-k)!)$.

⁴²When $n = 5$, there are 1024 different networks and an infinite number of candidates.

In Figure 2, we depict the risk-sharing networks that could be formed among three agents. We assume that $\lambda\sigma^2 = 9 = v(\bar{y})$ so that the payoff of the agents is normalized to 0 in the empty network. Proposition 3.1 completely characterizes the pairwise myopically stable set of networks which coincides with the set of pairwise stable networks. The simulations reveal that when $c \leq 1,5$ and $c > 3$ there are no other pairwise farsightedly stable sets than those described in Theorems 3.4 and 3.5. When $1,5 < c \leq 3$, a set composed of two networks connecting 2 agents is a pairwise farsightedly stable set, and a set composed of one network connecting 2 agents and a star network, where the hub in the star is not connected in the linked pair network is a pairwise farsightedly stable set of networks.⁴³ Farsighted agents thus form efficient networks while myopic agents do not. Table 1 summarizes these results.

Cost	Pairwise farsightedly stable set	Pairwise myopically stable set
$0 < c \leq 1,5$	$\{g_5\}, \{g_6\}, \{g_7\}$	$\{g_5, g_6, g_7\}$
$1,5 < c \leq 3$	$\{g_2, g_3\}, \{g_2, g_4\}, \{g_3, g_4\},$ $\{g_2, g_7\}, \{g_3, g_6\}, \{g_4, g_5\}$	$\{g_2, g_3, g_4\}$
$3 < c < 4,5$	$\{g_2\}, \{g_3\}, \{g_4\}$	$\{g_2, g_3, g_4\}$
$c = 4,5$	$\{g_1, g_2, g_3, g_4\}$	$\{g_1, g_2, g_3, g_4\}$
$c > 4,5$	$\{g_1\}$	$\{g_1\}$

Table 1. Farsightedly and myopically stable sets when $n=3$ and $\lambda\sigma^2=9$.

In Figure 3, we have depicted all the networks that could be formed among 4 agents. Table 2 summarizes the results obtained with the simulations. To simplify the presentation, we have not written down all the equilibrium candidates, but rather all the classes of equilibrium candidates. The candidates that are symmetric to those identified in Table 2 are also farsightedly stable. By Proposition 3.5, the only pairwise farsightedly stable set is the set of all linked pairs networks when $2,25 < c < 4,5$. When $c < 2,25$, each set composed of two lines of 4 agents is a pairwise farsightedly stable set (Proposition 3.4). Also, when $c \leq 0,75$, each star network as a singleton is a pairwise farsightedly stable set (Proposition 3.3). These propositions cover all the costs but $c = 2,25$, and as long as $c < 2,25$, the characterization may be incomplete. The simulations we have realized do not allow us to provide a complete identification of the pairwise farsightedly stable sets. We

⁴³The hub in a star is the agent who is directly connected to all the other agents.

have however considered all candidates of at most six networks for all costs of link formation. We have also considered sets of more than six networks, by focusing our attention on specific candidates.⁴⁴

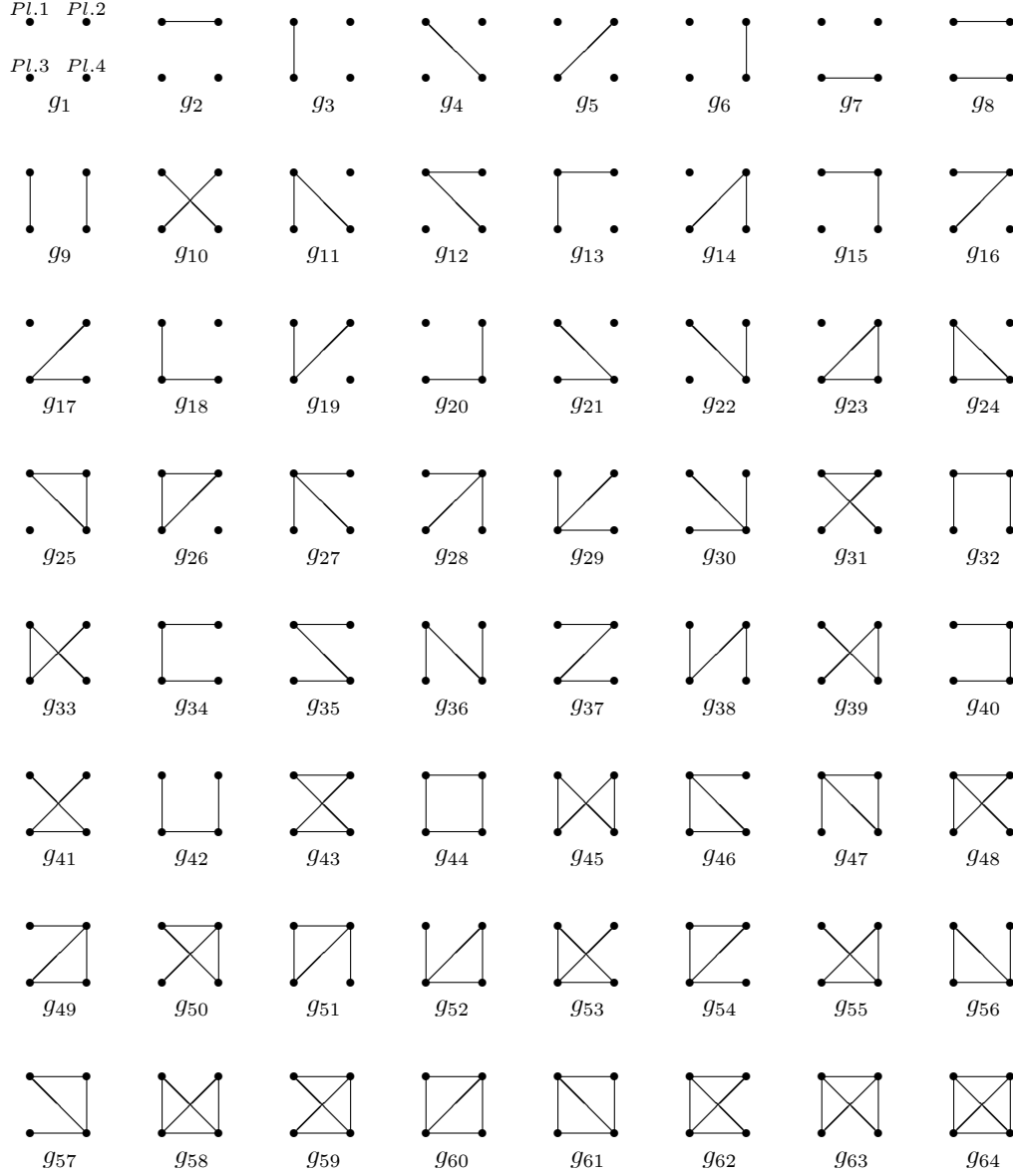


Figure 3: Risk-sharing networks among four agents

When $c = 2, 25$, there are no farsighted improving paths from the networks composed of a maximal number of linked pairs, implying that they belong to each

⁴⁴For instance, we have considered sets of ten networks by eliminating the candidates involving networks that are not minimally connected. By doing so, we have identified pairwise farsightedly stable sets of more than six networks when $1, 5 < c \leq 2, 25$.

pairwise farsightedly stable sets of networks. The set G of all networks composed of a maximal number of linked pairs and of the lines of 4 agents is a pairwise farsightedly stable set. Indeed, there is a farsighted improving path from each network other than the lines of 4 agents or the circles⁴⁵ of 4 agents leading to each network composed of a maximal number of linked pairs. The set G thus satisfies external stability. The deviations from the networks composed of a maximal number of linked pairs leading to a network outside the set necessarily involve the deletion of a link. These deviations are deterred by the threat of coming back at the same network in one step (see Lemma 3.A.3). Deviations from lines of 4 agents involving the addition of a link are deterred by application of Lemma 3.A.3, while those involving the deletion of a link are deterred by the threat of ending in a network composed of a maximal number of linked pairs. Minimality is satisfied since any subset of G would violate external stability. A set G composed of the three networks involving a maximal number of linked pairs, two networks of one link such that each agent has a total of one link in those two networks, and three lines of 3 agents (g_a, g_b, g_c) such that each agent has a total of three links in those three networks, is also a pairwise farsightedly stable set. We have $g \in F(g')$ for some $g \in \{g_a, g_b, g_c\}$, for all $g' \in G^L \cup G^c$, where G^c is the set of circles of 4 agents. External stability is thus guaranteed since there is a path from any other network not in the set leading to a network composed of a maximal number of linked pairs. The addition of a link from a network in the set leading to a network outside the set is never profitable by application of Lemma 3.A.3. For the same reason, a deviation involving the deletion of one link from a component of size 2 is deterred. Also, the hub of a line of 3 agents has no incentives to delete one of her links because she may fear to end up without connections in a network of one link. Minimality holds since external stability would be violated for any $G' \subset G$ such that less than three lines of 3 agents belong to G' while deterrence of external deviations would be violated for any $G' \subset G$ such that less than two lines of 2 agents belong to G' . The class of candidates we have just identified remains farsightedly stable when $1, 5 < c < 2, 25$. However, the set composed of all the lines of 4 agents and all the networks composed of a maximal number of linked pairs is not since it fails to satisfy minimality.⁴⁶ For $0, 75 < c \leq 1, 5$, a set composed of two lines

⁴⁵ A circle is a network where each agent has two links.

⁴⁶ Any set composed of two lines of 4 agents is a pairwise farsightedly stable set by Proposition 3.4.

of 3 agents such that the isolated agent is not the same agent in the two lines and the set of peripheral agents is different in the two networks is a pairwise farsightedly stable set. This candidate trivially satisfies external stability and minimality, and external deviations are deterred by application of Lemma 3.A.3. If the two lines of 3 agents shared the same set of peripheral agents, external stability would be violated since there are no farsighted improving paths from a line of 4 agents leading to a line of 3 agents such that the peripheral agents are identical in those two networks. External stability would also be violated if the isolated agent was the same agent in the two lines of 3 agents because there are no farsighted improving paths from a star leading to a line of 3 agents such that the hub in the star is not connected in the line. It follows that a set composed of three networks, two lines of 3 agents such that the isolated agent is the same agent in those lines and a star where the hub in the star is not connected in the lines, is also a pairwise farsightedly stable set. When $c > 1,5$, those candidates fail to satisfy external stability because there are no farsighted improving paths from a network composed of components of size 2 or less leading to a line of 3 agents. The simulations reveal that another class of candidates is pairwise farsightedly stable when $0,75 < c \leq 1,5$. A set composed of one line of 4 agents and another line of 3 agents such that the set of peripheral agents in the two lines is not identical is a pairwise farsightedly stable set. External stability holds by application of Lemma 3.A.5, which establishes that there is a farsighted improving path from every network leading to a line of 4 agents provided the payoff of the agents is different in the initial and final networks. The deletion of a link from a line of 4 agents is deterred by the threat of coming back to the same network (see Lemma 3.A.5) while every other deviation is deterred by application of Lemma 3.A.3. When $c > 1,5$, such candidates are no longer pairwise farsightedly stable because the hub in the line of 3 agents cannot be deterred from cutting one link. Indeed, the only stable outcome she might reach by doing so is a line of four agents. When $c < 0,75$, a set composed of six networks that are not minimally connected, three networks where one agent has three links while 2 other agents are connected and three networks where another agent has three links while 2 other agents are connected, is a pairwise farsightedly stable set. To see that such candidates satisfy external stability, notice that from a line of 4 agents, there is always an agent i with two links in the line who may cut successively her two links, looking forward to the following succession of moves: one agent deletes the remaining link, then the

3 agents other than agent i add a link between them, starting first with the links involving the agent j who has three links in the final network.⁴⁷ Finally, agents i and j form a link. Deviations from a network in the set are deterred: an agent who cuts a link may always fear to come back in the set with at least the same number of links, and a pair of agents who add a link between them are worse off by doing so, and could come back to the same network in one step by cutting the link they have just added.

Cost	Pairwise farsightedly stable sets	Pairwise myopically stable sets
$0 < c < 0,75$	$\{g_{27}\}, \{g_{28}\}, \{g_{29}\}, \{g_{30}\};$ $\{g_{31}, g_{32}\}, \{g_{31}, g_{33}\}, \dots, \{g_{31}, g_{42}\};$ $\{g_{46}, g_{47}, g_{48}, g_{49}, g_{50}, g_{51}\}$	$\{g_{27}, g_{28}, \dots, g_{42}\}$
$c = 0,75$	$\{g_{27}\}, \{g_{28}\}, \{g_{29}\}, \{g_{30}\};$ $\{g_{31}, g_{32}\}, \{g_{31}, g_{33}\}, \dots, \{g_{31}, g_{42}\}$	$\{g_{27}, g_{28}, \dots, g_{42}\}$
$0,75 < c \leq 1,5$	$\{g_{11}, g_{12}\}, \{g_{11}, g_{13}\}, \{g_{11}, g_{15}\},$ $\{g_{11}, g_{16}\}, \{g_{11}, g_{17}\}, \{g_{11}, g_{19}\},$ $\{g_{11}, g_{20}\}, \{g_{11}, g_{22}\};$ $\{g_{11}, g_{33}\}, \{g_{11}, g_{34}\}, \dots, \{g_{11}, g_{42}\};$ $\{g_{31}, g_{32}\}, \{g_{31}, g_{33}\}, \dots, \{g_{31}, g_{42}\};$ $\{g_{11}, g_{18}, g_{28}\}, \{g_{11}, g_{21}, g_{28}\}$	$\{g_{11}, g_{12}, \dots, g_{22}\}$
$1,5 < c < 2,25$	$\{g_{31}, g_{32}\}, \{g_{31}, g_{33}\}, \dots, \{g_{31}, g_{42}\};$ $\{g_2, g_7, g_8, g_9, g_{10}, g_{12}, g_{14}, g_{18}\}$	$\{g_2, g_3, \dots, g_{22}, g_{31}, g_{32}, \dots, g_{42}\}$
$c = 2,25$	$\{g_8, g_9, g_{10}, g_{31}, g_{32}, \dots, g_{42}\};$ $\{g_2, g_7, g_8, g_9, g_{10}, g_{12}, g_{14}, g_{18}\}$	$\{g_8, g_9, g_{10}\}$
$2,25 < c < 4,5$	$\{g_8, g_9, g_{10}\}$	$\{g_8, g_9, g_{10}\}$

Table 2. Farsightedly and myopically stable sets when $n=4$ and $\lambda\sigma^2=9$.

It is not easy to determine whether farsightedness helps reducing a conflict between myopic stability and efficiency, mainly because it implies a comparison of the efficiency of sets of networks. In addition, the pairwise myopically stable set is unique by definition while pairwise farsightedly stable sets are not. In what

⁴⁷The two agents already connected to agent j add a link between them in this step. This move is profitable for them as long as $c < 0,75$. When $c \geq 0,75$, there are no farsighted improving paths from a network composed of minimally connected components leading to a network composed of components that are not minimally connected.

follows, we summarize the new insights obtained with the simulations when four agents form a risk-sharing network. To compare the same object when discussing the issue of stability versus efficiency, we contrast the set of networks that belong to some pairwise farsightedly stable set with the pairwise myopically stable set. (i) For very small costs of link formation ($c \leq 0,75$), myopic agents always form efficient networks. Each efficient network also belongs to some pairwise farsightedly stable but it is not excluded that an equilibrium candidate is composed of networks that are not minimally connected when agents are farsighted, because farsighted agents can move from an efficient network looking forward to an inefficient network where her situation has improved. (ii) When the costs of link formation are small ($0,75 < c < 2,25$), we should further distinguish two cost thresholds. (ii.1) When $c \leq 1,5$, the pairwise myopically stable set consists of all lines of three agents, which are pairwise stable. The set of networks included in some farsightedly stable sets are all the lines of three and four agents, and the star networks. This set is thus the union of the pairwise myopically stable set and the set of efficient networks. (ii.2) When $c > 1,5$, the set of networks included in some pairwise farsightedly stable set and those included in the pairwise myopically stable set consist in all the networks composed of one link, two links, and the lines of four agents. (iii) For intermediate costs of link formation ($c = 2,25$), the pairwise stable networks are composed of a maximal number of linked pairs. Those network also belong to some pairwise farsightedly stable set in addition to the networks composed of lines involving two, three, and four agents (iv) for high costs of link formation ($c > 2,25$), Proposition 3.5 characterizes completely the farsightedly stable set of networks. It is the set of all the networks composed of a maximal number of linked pairs, and coincides with the pairwise myopically stable set.

In some cases, the set of networks belonging to some pairwise farsightedly stable set coincides with the pairwise myopically stable set. In others, the two sets are different. When the two sets are different, the pairwise myopically stable set is included in the set of networks belonging to some pairwise farsightedly stable set, and each pair of networks in the pairwise myopically stable set generates the same value. The networks that are farsightedly stable but not myopically stable do not necessarily generate more value than the networks that are pairwise stable. This is indeed the case when the costs of link formation are very small or intermediate. When the costs are small on the other hand, each network that is farsightedly stable

but not myopically stable is efficient.

3.6. Conclusion

In this chapter, we have analyzed the formation of risk-sharing networks. A growing empirical literature (see Fafchamps, 1992; Grimmard, 1997; Fafchamps and Lund, 2003; De Weerd and Dercon, 2006) has shown that a fully efficient risk-pooling equilibrium is not reached: risk-sharing does not take place within exogenous group such as the village, but rather within networks involving agents having common characteristics (neighborhood, professional or religious affiliation, kinship, etc). Most of the theoretical papers on informal risk-sharing in developing countries assume that no binding agreement can be enforced (see Kimball (1988); Coate and Ravallion (1993); Kocherlakota (1996); Ligon, Thomas and Worrall (2002); Genicot and Ray (2003); Bloch, Genicot and Ray (2005)). These papers model the risk-sharing process as a dynamic game where agents have incentives to transfer money today because they expect to be reciprocated at a future date. The self-enforcing transfer schemes identified involve incomplete income pooling, providing a theoretical argument to explain the observed pattern of informal insurance relationships.

Platteau (2000) has argued that kinship groups, membership of a clan or of a religious group are factors that help to imposing norms on members, enhancing trust and that increase the ability to punish deviant behaviors, thereby making risk-sharing easier. Thus, the lack of formal institutions allowing agents to commit to future transfers may be relevant at the village level, but not within the aforementioned communities.

Bramoullé and Kranton (2007a) have developed a model of insurance network formation, where agents invest in costly bilateral relationships in order to become members of a group of agents insuring each other against income or expenditure shocks. Their model is a decentralized model of coalition formation, where a coalition is a set of agents that are directly or indirectly connected to each other. Each member of a coalition commits to share her income with her insurance partners. They have shown that the efficient network is such that each agent is indirectly connected to each other, leading to the maximal level of insurance in the population, while strategic agents form networks involving income pooling in smaller groups, because the gain for an agent from adding new insurance partners to the group is

decreasing with the size of the group while its cost is constant. They have thus provided another theoretical explanation of the observation that full income pooling is not achieved in rural areas of developing countries, but they fail to explain why risk-sharing occurs within networks involving agents having common characteristics.

This chapter analyzes which pattern of insurance relationships emerges in the long run when agents are farsighted, rather than myopic, in the sense that they are able to forecast how other agents would react to their choice of partners. In his survey of models of network formation, Jackson (2005) provides support to this behavioral assumption by mentioning that farsightedness is important when agents have good information about each other, which we suspect is the case in the aforementioned communities.

We find that for small costs of establishing and maintaining a partnership, farsighted agents may form efficient networks that involve full income pooling while myopic agents form networks connecting fewer individuals. Two mechanisms explain this result: (i) Farsighted agents belonging to small groups may decide to create new partnerships that are not directly profitable to them, because they realize that other partners will further join this bigger and more attractive group. In other words, the farsightedness of the agents may solve a coordination problem. (ii) Farsighted agents may refrain from deleting costly links if they belong to a big group, as they understand that this may induce others to rearrange their partnerships in a way that deters the myopic incentives to delete the link at first. Farsightedness may thus reconcile the theory with the data observed in small communities. This conclusion does not hold for all cost values. In particular, for very small cost of link formation, myopic agents form efficient networks only while farsighted agents may form inefficient networks. These inefficient networks nonetheless involve full risk-sharing, but not at the minimal cost.

To our knowledge, no existing work has attempted to establish whether agents are farsighted or not when creating their network in rural areas of developing countries. This chapter offers a characterization of farsightedly and myopically stable networks that could be used in future work to estimate the degree of farsightedness of agents by comparing the observed networks with the predicted ones under the two different behavioral assumptions.

Appendix 3.A. Proofs.

The following lemma is useful in establishing Proposition 3.1.

Lemma 3.A.1. (Bramoullé and Kranton, 2007a) For all $g' \in \mathbb{G} \setminus G^*$, we have $g \in M(g')$ for some $g \in G^*$.

Proof. Take a network $g' \in \mathbb{G} \setminus G^*$. Start with g' and let agents successively delete unnecessary links (links connecting agents who are indirectly connected) until a minimally connected network g'' is reached. Then, let some pair of agents belonging to different components of size smaller than s^* add a link between them. Repeating this operation leads to a network g''' in which less than two components have a size smaller than s^* . From g''' , take successively a component of size strictly bigger than s^* and let an agent from this component delete a link with an agent who has exactly one link in that component. When all these links are deleted, we end up at network g'''' such that if $(S, h) \in C(g''')$, then either (i) $\#S = 1$ or (ii) $\#S = s^*$, or (iii) $1 < \#S < s^*$ and no other component $(S', h') \in C(g''')$ satisfies $1 < \#S' < s^*$. From g'''' , build a sequence of networks where at each step k , a link is added between a singleton and an agent belonging to the biggest component of size strictly smaller than s^* in the network of that step g^k . When all these links have been added, we end up in a network $g \in G^*$. Each move in the sequence of networks going from g' to g is profitable, establishing the result. ■

Proof of Proposition 3.1.

Let G be the pairwise myopically stable set. Suppose that $s^* = s^{**}$ or $s^* > s^{**}$ and $n \leq s^{**} + s^*$, so that G^* consists only of pairwise stable networks. By Lemma 3.A.1, there is a myopic improving path from every network outside G^* to some pairwise stable network in G^* . However, the converse does not hold since each network in G^* is by itself a closed cycle. This establishes that there are no other closed cycle than the pairwise stable networks, that is $G = G^*$. If on the other hand $s^* > s^{**}$ and $n > s^* + s^{**}$, then no pairwise stable network exists. By Lemma 3.A.1, we have that $G \cap G^* \neq \emptyset$. Let $g^* \in G \cap G^*$. Starting from the network g^* , we can reach any network $g \in G^*$ by adding links between members of different components of size s and s' where $s, s' \in \{s^{**} + 1, s^*\}$, by letting agents delete links with peripheral agents from components connecting more than s^* agents, and by adding links between a singleton and a member of a component connecting less than s^* agents. Since each of the suggested move is profitable for the agents involved, we

have $g \in M(g^*)$ for all $g \in G^*$. We then conclude that $G^* \subseteq G$. ■

The following lemma is used in the proof of Proposition 3.2.

Lemma 3.A.2. Let a network g be such that $(S, h) \in C(g)$, $\#S > \bar{s}/2$ and $i \in S$. If $U_i(g + ij) > U_i(g)$, then $U_j(g + ij) < U_j(g)$ for all $j \in N$.

Proof. Let $g \in \mathbb{G}$ be such that $(S, h), (S', h') \in C(g)$ and $\#S > \bar{s}/2$. Take agent $i \in S$ and agent $j \in S'$. (i) If $\#S = \#S' > \bar{s}/2$, then we have that $U_i(g + ij) < U_i(g)$ and $U_j(g + ij) < U_j(g)$ by definition of $\bar{s}$. (ii) If $\#S' > \#S > \bar{s}/2$, then at least agent $j \in S'$ is not willing to add the link ij . Indeed, by definition of $\bar{s}$, she would not be willing to add a link with an agent of another component of size $\#S'$. She is thus not willing to add a link with agent i who belongs to a component of size smaller than $\#S'$. (iii) Similarly, if $\#S > \#S'$, then at least agent $i \in S$ is not willing to add the link ij . ■

Proof of Proposition 3.2.

Let G be the pairwise myopically stable set.

- (i) By contradiction, suppose that a network $g' \in G$ is such that $g' \notin G^m$. Take a network g composed of minimally connected components obtained from g' by deleting unnecessary links. Formally, g is such that $g \in G^m$, and for all $(S', h') \in C(g')$, we have $(S', h) \in C(g)$ for some $h \subseteq h'$. Starting from g' and letting agents delete successively a link that belong to g' but not to g , we find that $g \in M(g')$. Thus the network g belongs to the same closed cycle C as the network g' . However, $g' \notin M(g)$, contradicting the fact that C is a closed cycle. Thus $g' \notin G$.
- (ii) By contradiction, suppose that $g' \in G$ and $(S, h) \in C(g')$ with $\#S > \bar{s}$. Since $\#S > \bar{s} \geq s^*$, there is an agent $i \in S$ willing to cut a link ij where $d_j(g') = 1$ (such link exists since $g' \in G^m$ by part (i)). Thus, $g' - ij \in M(g')$, implying that g' and $g' - ij$ are in the same closed cycle C . However $g' \notin M(g' - ij)$ since every path going from $g' - ij$ to g' implies, at some step, that an agent belonging to a component of size bigger than $\bar{s}/2$ should add a link. By Lemma 3.A.2, this move is not profitable. This in turn contradicts the fact that C is a closed cycle. It follows that $g' \notin G$.
- (iii.a) By contradiction, suppose that $g' \in G$ such that $d(g') = s^* - 1 - t \geq 0$, where $t \in \mathbb{N}_0^+$. Notice that each network whose total number of links is strictly

smaller than $s^* - 1$ is composed of components of size strictly smaller than s^* . By definition of s^* , it follows that every pair of agents can profitably add a link from the network g' , i.e. $g' + ij \in M(g') \forall ij \notin g'$. Thus g' and $g' + ij$ are in the same closed cycle C . However $g' \notin M(g' + ij)$ since every path going from $g' + ij$ to g' involves, at some step, the deletion of a link kl from a network g'' such that $d(g'') = d(g') + 1$. Since the network g'' is composed of components of size smaller than or equal to s^* , the deletion of the link kl is neither profitable for agent k , nor for agent l . This contradicts the fact that g' and $g' + ij$ are in the same closed cycle. Thus $g' \notin G$.

- (iii.b) Take a network $g \in G$. We show that $d(g) \leq n - 1 - \text{int}((n - 1)/\bar{s})$. By part (i) of this proposition, we have that $d(g) = n - \#C(g)$. By part (ii) of this proposition, there are no networks in G with fewer components than a network composed of a maximal number of components of size $\bar{s}$ and of one component with the remaining agents. Let g' be such a network. We have $\#C(g') = 1 + \text{int}((n - 1)/\bar{s})$, implying that $d(g) \leq d(g') = n - 1 - \text{int}((n - 1)/\bar{s})$.

■

The following lemma provides sufficient conditions to have a farsighted improving path from one network to an adjacent one. It implies that the addition of a link from a network such that at least one deviating agent is strictly worse off in the resulting network, or the deletion of a link from a network such that one of the two agents involved in the link is strictly worse off in the resulting network while the other agent is not strictly better off, are deviations that are deterred.

Lemma 3.A.3. Let $g \in \mathbb{G}$. If $U_i(g + ij) < U_i(g)$, then $g \in F(g + ij)$. If $U_i(g - ij) < U_i(g)$ and $U_j(g - ij) \leq U_j(g)$, then $g \in F(g - ij)$.

Proof. Trivial.

Proof of Proposition 3.3.

Let $0 < c \leq u(n) - u(n - 1)$. Notice that for such costs, two agents i, j who are not indirectly connected at a network g are both better off at the network $g + ij$ than at the network g .

- (a) Each farsighted improving path emanating from a connected line $g \in G^L$ reaches a connected network, since g Pareto dominates g' for each $g' \in G^m \setminus$

G^M . Thus, a set of networks G' that does not contain a connected network does not satisfy condition (ii) of Definition 3.2.

- (b) Take a network $\tilde{g} \in G^M$ and let $G = \{g \in \mathbb{G} \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N\}$. In order to prove that G is a pairwise farsightedly stable set, we will show that (b.i) $\tilde{g} \in F(g')$ for every $g' \in \mathbb{G} \setminus G$ and (b.ii) $F(g) \cap G = \emptyset$ for every $g \in G$ and hence, Theorem 3 in Herings, Mauleon and Vannetelbosch (2009) applies.
- (b.i) Take $g' \in \mathbb{G} \setminus G$. (b.i.1) If for all $g \in G$, we have $g \not\subseteq g'$, then start from g' and build a sequence of networks where at each step, an agent who has more links than at the network $\tilde{g}$ cuts a link. When all these links have been deleted, we end up at the network g'' such that $d_i(g'') \leq d_i(\tilde{g})$ for all $i \in N$. Notice that there are at most $n - 1$ links in g'' and that $U_i(g'') < U_i(\tilde{g})$ for some $i \in N$, since $\tilde{g}$ is efficient and $g'' \notin G$. Agent i cuts one link in g'' leading to the network g''' , which is composed of multiple components. In g''' and in the successive networks, an agent who has l links in a component of size s cuts a link, looking forward to the formation of the network $\tilde{g}$, if she has $l + x$ links or less in $\tilde{g}$ and $n \geq s + x$. The network reached through this path is $g^\emptyset$. Once in $g^\emptyset$, agents successively add links to form $\tilde{g}$. Notice that $u(n) - u(n - 1) \geq c$ implies $u(s) - lc \leq u(s + x) - (l + x)c$, if $s + x \leq n$, since the expected utility function is increasing at a decreasing rate. Each agent i cutting a link in a network g in the path where she has l links in a component of size s is willing to do so since her payoff in $\tilde{g}$ is $U_i(\tilde{g}) \geq u(n) - (l + x)c > U_i(g)$. Agents adding links from $g^\emptyset$ to $\tilde{g}$, looking forward to the formation of $\tilde{g}$, are better off in the end network. (b.i.2) If $g \subset g'$ for some $g \in G$, then $d_i(g') \geq d_i(\tilde{g})$ for all $i \in N$, and $d_j(g') > d_j(\tilde{g})$ for some $j \in N$. From g' , let a pair of agents add a link such that at least one of the two agents adding the link has strictly more links at the current network than at $\tilde{g}$. By repeating this step, agents reach the complete network g^N . Once there, they successively delete the links that are not in $\tilde{g}$. Each move of the path is profitable for the deviating agents who are looking forward to the formation of $\tilde{g}$. We thus conclude that $\tilde{g} \in F(g')$ for every $g' \in \mathbb{G} \setminus G$.
- (b.ii) For every $g \in G$, $F(g) \cap G = \emptyset$ since $U_i(g) = U_i(\tilde{g})$ for all $g \in G$.
- (c) Let $G = \{g_1, g_2\}$, where $g_1, g_2 \in G^M$ such that $U_i(g_1) \neq U_i(g_2)$ for some agent

$i \in N$, and g_1 and g_2 are not star networks. In order to prove that G is a pairwise farsightedly stable set, we will show that it satisfies conditions (i), (ii) and (iii) of Definition 3.2.

- (c.i) Every deviation from a network in the set is deterred by application of Lemma 3.A.3.
- (c.ii) We have shown in part (b.i) that $g_1 \in F(g')$ for every $g' \in \mathbb{G} \setminus G^1$, where $G^1 = \{g \in \mathbb{G} \mid U_i(g) = U_i(g_1) \text{ for all } i \in N\}$. By part (b.i), we have that $g_2 \in F(g')$ for every $g' \in \mathbb{G} \setminus G^2$, where $G^2 = \{g \in \mathbb{G} \mid U_i(g) = U_i(g_2) \text{ for all } i \in N\}$. Since $G^1 \cap G^2 = \emptyset$, we have $F(g') \cap G \neq \emptyset$ for all $g' \notin G$.
- (c.iii) Suppose that some subset $G' \subset G$ is a pairwise farsightedly stable set. Without loss of generality, suppose that $G' = \{g_1\}$. Then, G' does not satisfy the condition (ii) of Definition 3.2 since $g_1 \notin F(g')$ for $g' \in G^1 \setminus \{g_1\}$, a contradiction. ■

The following two lemmas are central to the proof of Proposition 3.4.

Lemma 3.A.4. $G^L \subseteq F(g^\emptyset)$ if $c < \min \{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$.

Proof. For such costs, two agents i and j in different components of size $\text{int}((n+1)/2)$ and $\text{int}(n/2)$ at a network g' both prefer the network $g' + ij$ to the network g' . Take a network $g \in G^L$. Let $g' = g - ij$ where $ij \in g$, $i \in Ce(g)$ and $j \in Ce(g)$ if n is even such that the network g' is composed of one component of size $\text{int}((n+1)/2)$ and of another of size $\text{int}(n/2)$. The following path is a farsighted improving path from $g^\emptyset$ to g . From $g^\emptyset$, add successively each link $kl \in g'$ until the network g' is formed. Agents i and j then add the link ij . Let g'' be a network of the path going from $g^\emptyset$ to g in which agent $k \in N$ adds a link. If $d_k(g'') = 0$, then since $c < (u(n) - u(1))/2$, agent k prefers to add a link, looking forward to the formation of the network g . If $d_k(g'') = 1$, it is profitable for agent k to add a link looking forward to the network g since she belongs to a component of size $s \leq \text{int}((n+1)/2)$ in g'' , implying that $U_k(g'') = u(s) - c \leq u(\text{int}((n+1)/2)) - c < u(n) - 2c = U_k(g)$. Since g was chosen arbitrarily, we have that $g \in F(g^\emptyset)$ for all $g \in G^L$. ■

Lemma 3.A.5. For $G = \{g \in G^L \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N \text{ for some } \tilde{g} \in G^L\}$, we have that $G \subseteq F(g')$ for all $g' \notin G$ if $c < \min \{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$.

Proof. Take a network $\tilde{g} \in G^L$ and let $G = \{g \in G^L \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N\}$. Suppose that $c < \min \{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$. Take $g' \in \mathbb{G} \setminus G$. (i.1) Suppose that $d_i(g') \geq d_i(\tilde{g})$ for all $i \in N$. Since $g' \notin G$, we have that $d_j(g') > d_j(\tilde{g})$ for some $j \in N$. From g' , let agent j successively add a link with the agents she is not directly connected to. Then, each pair of agents who are not directly connected adds a link between them to form the complete network. From the complete network, the agents cut the links that do not belong to the network $\tilde{g}$ to reach it. One can see that this sequence of actions describes a farsighted improving path from the network g' leading to the network $\tilde{g}$. (i.2) Suppose that $d_i(g') < d_i(\tilde{g})$ for some $i \in N$. Then, start from g' and build a sequence of networks where at each step, an agent who has more links than at the end network $\tilde{g}$ cuts a link. When all these links have been deleted, we end up at the network g'' such that $d_i(g'') \leq d_i(\tilde{g})$ for all $i \in N$ and $d_j(g'') < d_j(\tilde{g})$ for some $j \in N$. It follows that the network g'' is composed of multiple components. In g'' and in the successive networks, the agents having 2 links cut one link. They are willing to do so since they are looking forward to $\tilde{g}$ where they belong to a bigger component and pay at worse the same cost. When all these links have been deleted, we are in a network composed of components of size 2 and of singletons. Agents having a link successively cut this link to reach $g^\emptyset$ looking forward to $\tilde{g}$. From $g^\emptyset$, there is a farsighted improving path leading to $\tilde{g}$ (Lemma 3.A.4). We conclude that $\tilde{g} \in F(g')$. This does not depend on the choice of the network $\tilde{g} \in G$, thus $G \subseteq F(g')$. ■

Proof of Proposition 3.4.

Suppose that $c < \min \{u(n) - u(\text{int}((n+1)/2)), (u(n) - u(1))/2\}$.

- (a) Take a network $\tilde{g} \in G^L$ and let $G(\tilde{g}) = \{g \in G^L \mid U_i(g) = U_i(\tilde{g}) \text{ for all } i \in N\}$. From Lemma 3.A.5, we have that $G(\tilde{g}) \subseteq F(g')$ for every $g' \in \mathbb{G} \setminus G(\tilde{g})$. In addition, $F(g) \cap G(\tilde{g}) = \emptyset$ for every $g \in G(\tilde{g})$, since $U_i(g) = U_i(g')$ for all $g, g' \in G(\tilde{g})$. Hence, Theorem 3 in Herings, Mauleon and Vannetelbosch (2009) applies.
- (b) Take the set $\{g_1, g_2\}$ where $g_1, g_2 \in G^L$ and $U_i(g_1) \neq U_i(g_2)$ for some $i \in N$. To show that $\{g_1, g_2\}$ is a pairwise farsightedly stable set, we show that it satisfies the three conditions of definition 3.2. (ii.1) Deterrence of external

deviations is satisfied since the network g' reached by adding or deleting a link from g_1 (or g_2) is not a minimally connected line. Thus, from Lemma 3.A.5, $g_1 \in F(g')$ which deters the incentives to deviate from g_1 . (ii.2) External stability is ensured by Lemma 3.A.5 since $g_1 \in F(g')$ for the networks g' such that $U_i(g') \neq U_i(g_1)$ for some agent i , and $g_2 \in F(g')$ for the networks g' such that $U_i(g') = U_i(g_1)$ for all $i \in N$. (i.3) Minimality is ensured since external stability would be violated if the set was smaller. ■

The following lemma is used in the proof of Proposition 3.5.

Lemma 3.A.6. If n is odd, then for all $g' \in \mathbb{G} \setminus G_k$, where $G_k = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N \setminus \{k\}, d_k(g) = 0\}$, we have that $G_k \subseteq F(g')$ if $\max\{(u(3)-u(1))/2, u(n)-u(2)\} < c < u(2) - u(1)$.

Proof. Let n be odd and let $\max\{(u(3) - u(1))/2, u(n) - u(2)\} < c < u(2) - u(1)$. Take $k \in N$ and let $G_k = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N \setminus \{k\}, d_k(g) = 0\}$. Take $g \in G_k$ and $g' \in \mathbb{G} \setminus G_k$. Notice that an agent prefers a network in which she has one link to any network in which she has two or more links when $u(n) - u(2) < c < u(2) - u(1)$.

- (i) If $d_i(g') \leq 1 \forall i \in N$, then start with g' and build a sequence of networks where at each step, either a singleton adds a link that belongs to the network g , or an agent who has two links deletes a link that does not belong to the network g until the network g is reached. Step 1a: A singleton in g' other than agent k adds a link that belongs to the network g . Since at least one agent, say i , has no link at g' then $U_i(g') = u(1) < U_i(g) = u(2) - c$, thus agent i is willing to add the link looking forward to g . The other agent, say j , has either no link at g' or she has one link in a component of size 2. In both cases, she agrees to add the link ij looking forward to g . Step 1b: In the remaining network, if an agent has two links, she deletes a link that does not belong to g . This agent is willing to delete a link looking forward to g where she has one link. Step l : Proceed inductively in l , if an agent other than agent k is a singleton, she adds a link that belongs to g ; then, on the remaining network, if an agent has two links, she deletes a link that does not belong to g . Step L : When all these links are added or removed, we end up at the network g . We conclude

that $g \in F(g')$. Since the choice of $g \in G_k$ does not matter, we conclude that $G_k \subseteq F(g')$.

- (ii) If $d_i(g') > 1$ for some agent $i \in N$, then start with g' and build a sequence of networks where at each step, some agent other than agent k who has more than one link deletes a link. When all these links have been deleted, if agent k has more than one link, she successively deletes all her links but one so that the network g''' is reached with $d_i(g''') \leq 1 \forall i \in N$. From g''' , there is a farsighted improving path going to g (see part (i) of the proof of this lemma).
- Step 1a: An agent, say i , who has more than one link deletes a link ii' such that $i' \neq k$ and agent i' has exactly one link at the current network. Repeat this step until a network is reached in which the agents having more than one link are connected to agents having also more than one link or to agent k .
- Step 1b: In the remaining network, let an agent, say j , who has more than one link, delete a link different than the link jk . An agent deleting a link in one of those steps is willing to do so as she has at least 2 links at the network where she deletes a link and she is looking forward to g in which she has one link.
- Step l : Proceed inductively in l , each time an agent, say i , with two links or more is connected to an agent other than agent k that has exactly one link, agent i deletes that link. Then, on the remaining network, an agent who has two links or more, say agent j , deletes a link other than the link jk .
- Step L : When all these links have been deleted, a network g'' is reached such that $d_i(g'') \leq 1, \forall i \in N \setminus \{k\}$.
- Step $L + 1$: If agent k has 2 links or more, she successively deletes all her links but one. Agent k has s links in a component of size $s + 1$ for some $s \geq 2$ at a network, say g_1 , where she deletes a link. She is willing to delete a link in g_1 looking forward to g since, for $s \geq 2$, we have that $U_k(g_1) = u(s + 1) - sc \leq u(3) - 2c < U_k(g) = u(1)$ when $u(3) - u(1) < 2c$.
- Step $L + 2$: We are in a network g''' such that $d_i(g''') \leq 1$ for all $i \in N$ and $g''' \notin G_k$ by construction. In part (i) of this proposition we have shown that $G_k \in F(g''')$. We conclude that $G_k \in F(g')$. ■

Proof of Proposition 3.5.

Let $u(n) - u(2) < c < u(2) - u(1)$.

- (a) Suppose that n is even. Let $G = \{g \in \mathbb{G} \mid d_i(g) = 1 \text{ for all } i \in N\}$. In order to prove that G is the unique pairwise farsightedly stable set, we will

show that (a.i) for every $g' \in \mathbb{G} \setminus G$, we have that $F(g') \cap G \neq \emptyset$ and (a.ii) for every $g \in G$, $F(g) = \emptyset$ and hence, Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) applies.

- (a.i) Take $g' \in \mathbb{G} \setminus G$. Start with g' and build a sequence of networks where at each step, either an agent with more than one link deletes a link, or two unconnected agents add a link between them. This path leads to the formation of a network $g'' \in G$ and each deviating agent is better off in g'' than in the networks in which she adds/cuts a link. Thus $F(g') \cap G \neq \emptyset$.
- (a.ii) By contradiction, suppose that $F(g) \neq \emptyset$, say $g' \in F(g)$, for some network $g \in G$. Then, at least an agent, say i , is willing to create or delete a link from g looking forward to g' , that is, $U_i(g) < U_i(g')$. Having $U_i(g) < U_i(g')$ implies that agent i has exactly one link in g and belongs to a component of size strictly bigger than 2. Then, at least an agent, say j , has 2 links or more in g . However, every path going from g to g' is such that the payoff of agent j is smaller in g' than in the network in which she adds a second link. This contradicts the fact that $g' \in F(g)$. Thus $F(g) = \emptyset$.
- (b) Suppose that n is odd and that $(u(3) - u(1))/2 < c$. We have that (b.i) for every $g' \in \mathbb{G} \setminus G_k$, $F(g') \cap G_k \neq \emptyset$ (see Lemma 3.A.6), and (b.ii) for every $g \in G_k$, $F(g) \cap G_k = \emptyset$, since $U_i(g) = U_i(g')$ for all $i \in N$, for all $g, g' \in G$. Hence, Theorem 3 in Herings, Mauleon and Vannetelbosch (2009) applies, G_k is a pairwise farsightedly stable set.
- (c) Suppose that n is odd, $n \geq 5$ and $(u(3) - u(1))/2 < c$. Take $g_k \in G_k$ and $g_l \in G_l$ such that $k \neq l$. In order to prove that $G = \{g_k, g_l\}$ is a pairwise farsightedly stable set, we show that G satisfies the three conditions of Definition 3.2.
 - (c.i) Every deviation from a network in the set is deterred by application of Lemma 3.A.3.
 - (c.ii) We have shown in Lemma 3.A.6 that $g_k \in F(g')$ for every $g' \in \mathbb{G} \setminus G_k$ and $g_l \in F(g')$ for every $g' \in \mathbb{G} \setminus G_l$. Since $G_l \cap G_k = \emptyset$, we thus have $F(g') \cap G \neq \emptyset$ for all $g' \notin G$.
 - (c.iii) Suppose by contradiction that some subset $G' \subset G$ is a pairwise farsightedly stable set. Without loss of generality, suppose that $G' = \{g_k\}$. We then

have that condition (ii) of Definition 3.2 is not satisfied since $g_k \notin F(g')$ for $g' \in G^k \setminus \{g_k\}$, and $G^k \setminus \{g_k\} \neq \{\emptyset\}$ when $n \geq 5$, contradicting the fact that G' is a pairwise farsightedly stable set. Thus, G satisfies the minimality condition. ■

Appendix 3.B. Description of the algorithm

Having associated to each network a number between 1 and K , the output of the algorithm is (i) a square matrix F of dimension $K \times K$, where K is the total number of networks among n agents, such that $F(i, j) = 1$ if there is a farsighted improving path from network number $i \in \{1, K\}$ leading to network $j \in \{1, K\}$ and (ii) a matrix $PFFS$ of dimension $L \times K$, where L is the total number of pairwise farsightedly stable sets of networks such that a set composed of networks associated with non-zero elements of a line is a pairwise farsightedly stable set of networks.

To determine whether there is some farsighted improving path from some initial network $a \in \{1, K\}$ to some final one $b \in \{1, K\}$, the algorithm proceeds by steps.

Step 1: The algorithm creates at the first step a vector $G_{ab}(1)$ containing all the possible networks that are adjacent to the initial network a such that either one link is added and both agents are better off in the final network b compared to the initial one, or one link is deleted and at least one agent involved in that link is better off at the end network.

Step 2: At the second step, the algorithm creates a vector $G_{ab}(2)$ which consists of the elements of $G_{ab}(1)$ and all the networks that are adjacent to a network c in $G_{ab}(1)$ but not yet included in $G_{ab}(1)$ and such that either one link is added and both agents are better off in the final network b compared to the current network c , or one link is deleted and at least one agent involved in that link is better off at the end network.

Step p : At the p^{th} step, the algorithm creates a vector $G_{ab}(p)$ which consists of the elements of $G_{ab}(p-1)$ and all the networks that are adjacent to a network c in $G_{ab}(p-1)$ but not yet included in $G_{ab}(p-1)$ and such that either one link is added and both agents are better off in the final network b compared to the current network c , or one link is deleted and at least one agent involved in that link is better off at the end network.

The algorithm stops at step P where P is the smallest integer such that either $b \in G_{ab}(P)$, or $G_{ab}(P) = G_{ab}(P-1)$. The finiteness of the number of networks

implies that the algorithm ends in a finite number of steps. There is a farsighted improving path from the initial network to the final network if the final network b belongs to $G_{ab}(P)$. By running this algorithm for each possible pair of initial network and final network, we obtain a square matrix F of dimension $K \times K$, where $K = \sum_{i=0}^{n(n-1)/2} C_i^{n(n-1)/2}$ is the total number of networks among n agents, such that $F(a, b) = 1$ if there is a farsighted improving path from the network a leading to the network b , and $F(a, b) = 0$ otherwise.

In the second part of the algorithm, we build a matrix $PFSS$ of dimension $L \times K$, where L is the total number of pairwise farsightedly stable sets of networks such that a set composed of networks associated with non-zero elements of a line is a pairwise farsightedly stable set of networks. To find this matrix, we start from a matrix H which contains all the possible equilibrium candidates -each line of the matrix being associated with one candidate- and we successively delete the lines that do not satisfy external stability, deterrence of external deviations, and minimality.

Step 1: First build a matrix H of size $\sum_{i=1}^K C_i^K \times K$, such that each line of H corresponds to a different equilibrium candidate. When $n = 3$, we have 8 different networks that can form 256 different equilibrium set candidates. A number between 1 and 8 is attributed to each network. For candidates of less than 8 networks, we use 0 to fill in the matrix. For example, the first line of H is composed of the number 1 in the first cell and of zeros in the remaining ones, and corresponds to the candidate where the empty network (who is associated with the number 1) as a singleton is a candidate. For the sake of notation, let $l_M(k)$ be the set of the non-zero elements of line k of the matrix M . We thus have $l_H(1) = \{1\}$.

Step 2: Build the matrix H' obtained from H by deleting the lines that do not satisfy the external stability requirement. To do so, look for each line k of the matrix H whether $F(a, b) = 1$ for all $a \in \{1, K\} \setminus l_H(k)$, for some $b \in l_H(k)$.

Step 3: Build the matrix H'' obtained from H' by deleting the lines that do not satisfy deterrence of external deviations. To do so, look for each line k of the matrix H' whether for all network $a \notin l_{H'}(k)$ obtained from a network $b \in l_{H'}(k)$ by adding a link ij , we have $F(a, c) = 1$ for some $c \in l_{H'}(k)$ such that either $Y_i(c) < Y_i(b)$, or $Y_j(c) < Y_j(b)$, or $Y_i(c) = Y_i(b)$ and $Y_j(c) = Y_j(b)$. Then repeat the operation for deviations involving the deletion of a link.

Step 4: Build the matrix $PFSS$ from H'' by removing the lines that do not satisfy the minimality requirement.

Chapter 4.

Strongly rational sets for normal-form games⁴⁸

4.1. Introduction

The notion of Nash equilibrium does not incorporate the possibility that groups of players might coordinate their actions to reach an outcome that is better for all of them. Aumann (1959) was first to introduce this consideration into the theory of noncooperative games by proposing the notion of strong Nash equilibrium. A strategy profile is a strong Nash equilibrium if it is immune not only to individual deviations, but also to coalitional deviations. Later on, Bernheim, Peleg and Whinston (1987) have proposed the notion of coalition-proof Nash equilibrium. A strategy profile is a coalition-proof Nash equilibrium if it is immune to coalitional deviations which are themselves immune to further deviations by subcoalitions. The main weakness of strong Nash equilibrium and coalition-proof Nash equilibrium is that existence is not guaranteed in a natural class of games, as opposed to the Nash equilibrium concept.

Basu and Weibull (1991) have proposed a set-theoretic coarsening of the notion of strict Nash equilibrium: minimal curb (closed under rational behavior) sets.⁴⁹ This set-valued solution concept combines a standard rationality condition, stating that the set of recommended strategies of each player must contain all best responses to whatever belief he may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies as small as possible.

In this chapter we introduce the concept of minimal strong curb sets which is a set-theoretic coarsenings of the notion of (strict) strong Nash equilibrium. We require the sets to be immune not only against individual deviations, but also against

⁴⁸This work is joint with Ana Mauleon and Vincent Vannetelbosch.

⁴⁹Many games of interest lack strict Nash equilibria. A strategy profile is a strict Nash equilibrium if each player's equilibrium strategy is better than all her other strategies, given the other players' strategies. In any non-strict Nash equilibrium, at least one player is indifferent between her equilibrium strategy and some other strategy, given the other players' strategies. Such indifference can make the Nash equilibrium evolutionary unstable. See Weibull (1995).

group deviations. Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies must contain all coalitional best-responses of each coalition to whatever belief each coalition member may have that is consistent with the recommendations to the other players. A strong curb set is minimal if it does not properly contain another strong curb set. Think of the set of recommendations to a player in a minimal strong curb set as a well-packed bag for a sports weekend: you may want to be prepared for different kinds of sports since you may like playing tennis with player 2 or playing golf with player 3 or playing bridge with players 2, 3 and 4 or going alone for a jog. Minimal strong curb sets are shown to exist in general and are compared with other well known solution concepts: strong Nash equilibrium, coalition-proof Nash equilibrium, and coalitional rationalizability.

Finally, we provide a dynamic motivation for the concept of minimal strong curb sets. Hurkens (1995) has proposed a dynamic learning process where players have bounded memory and play best-responses against beliefs, formed on the basis of strategies used in the recent past. This learning process leads the players to playing strategies from a minimal curb set.⁵⁰ We propose a similar learning process except that now groups of players may play coalitional best-responses. A game is played at discrete point in time. For each role in the game there is a pool of players. At the beginning of each period one player is drawn from each pool to play the game in that period. These players are partitioned into coalitions to form a coalition structure. Each coalition structure has a positive probability to occur at each period. Players observe how the game has been played in the recent past, form their beliefs upon these observations, and select an action profile jointly with their coalition partners. We show that, if the memory is long enough, play settle down in a minimal strong curb set.

The chapter is organized as follows. We recall notations and definitions in Section 2. We formally define the concept of minimal strong curb sets in Section 3. We compare minimal strong curb sets with strong Nash equilibria, coalition-proof Nash equilibria and coalitionally rationalizable strategy profiles in Section 4. We provide a dynamic learning process leading the players to playing strategies from a minimal strong curb set in Section 5. We conclude in Section 6.

⁵⁰See also Young (1998).

4.2. Preliminaries

A **normal-form game** is a tuple $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where $N = \{1, 2, \dots, n\}$ is a finite set of players, each player $i \in N$ has a nonempty, finite set of pure strategies (or actions) A_i and a von Neumann-Morgenstern utility function $u_i : A \rightarrow \mathbb{R}$, where $A = \times_{j \in N} A_j$. The set of all games is denoted by Γ . For every $X \subseteq A$, let $X_{-i} = \times_{j \in N \setminus \{i\}} X_j$, $\forall i \in N$. The **subgame** obtained from G by restricting the action set of each player $i \in N$ to a subset $X_i \subseteq A_i$ is denoted – with a minor abuse of notation from restricting the domain of the utility functions u_i to $\times_{j \in N} X_j$ – by $G_X = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$. The set of mixed strategies of player $i \in N$ with support in $X_i \subseteq A_i$ is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. Beliefs are profiles of mixed strategies: correlation is not allowed. The profile of strategies where player $i \in N$ plays $a_i \in A_i$ and her opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ is denoted (a_i, α_{-i}) . For $i \in N$ and $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$,

$$\text{BR}^i(\alpha_{-i}) = \{a_i \in A_i \mid u_i(a_i, \alpha_{-i}) \geq u_i(a'_i, \alpha_{-i}) \text{ for each } a'_i \in A_i\}$$

is the set of pure best responses of player i against her belief α_{-i} .

Basu and Weibull (1991) have introduced the concept of strategy subset closed under rational behavior (curb), which is a set-theoretic coarsening of the notion of strict Nash equilibrium. Formally, **curb sets** are defined as follows.

Definition 4.1. A curb set is a product set $X = \times_{i \in N} X_i$ where

- (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;
- (b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i} , the set X_i contains all best responses of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j), \text{BR}^i(\alpha_{-i}) \subseteq X_i.$$

Curb sets are product sets of pure strategies such that each player's set of recommended strategies must contain all best-replies to whatever belief he may have that is consistent with the recommendations to the other players. Since the full strategy space is always curb, particular attention is devoted to minimal curb sets. A curb set X is minimal if no curb set is a proper subset of X . Basu and Weibull (1991) have

shown that every game G possesses at least one minimal curb set. The set-valued solution concept that assigns to each game its collection of minimal curb sets is denoted by min-curb . Hence, $\text{min-curb}(G) = \{X \subseteq A \mid X \text{ is a minimal curb set of } G\}$. Similarly, $\text{curb}(G) = \{X \subseteq A \mid X \text{ is a curb set of } G\}$.⁵¹

The notion of **strong Nash equilibrium** is due to Aumann (1959). A strong Nash equilibrium is a strategy profile such that no subset of players has a joint deviation that benefits all of them. **Coalitions** are nonempty subsets of players (J such that $J \subseteq N$ and $J \neq \emptyset$). For every $X \subseteq A$, let $X_{-J} = \times_{j \in N \setminus J} X_j$, $\forall J \subseteq N$. The profile of strategies where players belonging to coalition J play according to the strategy profile $a_J \in \times_{i \in J} A_i$ and the remaining players play according to the mixed strategy profile $\alpha_{-J} = (\alpha_j)_{j \in N \setminus J} \in \times_{j \in N \setminus J} \Delta(A_j)$ is denoted (a_J, α_{-J}) . For every $J \subseteq N$, $i \in J$, $X \subseteq A$ and $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Delta(X_j)$, we denote by α_{-i}^{-J} the marginal distribution of α_{-i} over X_{-J} . Formally, the notion of strong Nash equilibrium is defined as follows. The strategy profile $a^* \in \times_{i \in N} A_i$ is a strong Nash equilibrium if and only if, $\forall J \subseteq N$, $\forall a_J \in \times_{j \in J} A_j$ ($a_J \neq a_J^*$), $\exists i \in J$ such that $u_i(a^*) \geq u_i(a_J, a_{-J}^*)$. A strong Nash equilibrium is strict if the last inequality holds strictly.

4.3. Strong curb sets

While the concept of curb sets is a set-theoretic coarsening of the notion of strict Nash equilibrium, we now introduce the concept of strong curb sets which is a set-theoretic coarsening of the notion of strict strong Nash equilibrium. That is, we require the set to be immune not only against individual deviations (as for curb sets), but also against coalitional deviations. Let us generalize the concept of best

⁵¹Voorneveld (2004) has proposed the notion of prep sets which are product sets of pure strategies such that each player's set of recommended strategies must contain at least one best-response to whatever belief she may have that is consistent with the recommendations to the other players. A formal definition is provided in the appendix. Every curb set is a prep set and every curb set contains a minimal prep set. But, minimal prep sets may contain a proper subset of the strategies contained in the minimal curb sets. Kalai and Samet (1984) have introduced the notion of persistent retracts which require the recommendations to each player to contain at least one best-response to beliefs in a small neighborhood of the beliefs restricted to the recommendations to the other players. Voorneveld (2005) has shown that, in generic games, persistent retracts, minimal prep sets and minimal curb sets coincide.

response to coalitions of players.

Definition 4.2. For each vector of beliefs $\alpha = (\alpha_{-i})_{i \in N}$ with $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$, the set of **coalitional best-responses** of coalition $J \subseteq N$ is

$$\begin{aligned} \text{CBR}^J(\alpha) = \{a_J \in \times_{i \in J} A_i \mid & \text{(i) } \forall i \in J, u_i(a_i, \alpha_{-i}) \leq u_i(a_J, \alpha_{-i}^{-J}), \forall a_i \in A_i \text{ and} \\ & \text{(ii) } \nexists a'_J \in \times_{i \in J} A_i \text{ such that } \forall i \in J, u_i(a_J, \alpha_{-i}^{-J}) < u_i(a'_J, \alpha_{-i}^{-J})\}. \end{aligned}$$

Given a vector of beliefs α , a profile of strategies a_J for coalition J is a coalitional best-response if (i) each member $i \in J$ prefers to join coalition J and playing a_J rather than playing her individually best-response against her belief α_{-i} , (ii) there is no other profile $a'_J \neq a_J$ such that all members of J strictly prefer a'_J to a_J . Conditions (i) and (ii) captures some rudimentary form of coalitional rationality. First, a sensible concept of coalitional rationality should prescribe coordination on strategy profiles so that all coalition members have incentives to join the group. Second, it should be conceivable that members of coalition J will never coordinate their play on strategy profiles that are Pareto dominated. Of course, $\text{CBR}^{\{i\}}(\alpha)$ coincides with $\text{BR}^i(\alpha_{-i}) \forall i \in N$.

Example 4.1.

Consider the normal-form games G_1 and G_2 .

	L	R		L	R
U	4, 5	0, 0	U	2, 0	0, 0
D	0, 0	3, 2	D	0, 0	0, 2
	G_1			G_2	

Take the normal-form game G_1 and let $J = \{1, 2\}$. Condition (i) makes that (U, R) and (D, L) are never coalitional best-responses for J whatever α . Condition (ii) makes that (D, R) is not a coalitional best-response for J whatever α . However, the strategy profile (U, L) satisfies both conditions whatever α . Thus, $\text{CBR}^{\{1,2\}}(\alpha) = \{(U, L)\}$. Notice that the set of coalitional best-responses, $\text{CBR}^J(\alpha)$, may be empty if $|J| \geq 2$. Take the normal-form game G_2 and consider the beliefs $\alpha = (\alpha_{-1}, \alpha_{-2})$ with $\alpha_{-1}(L) = 1$ and $\alpha_{-2}(D) = 1$. Then, $\text{BR}^1(\alpha_{-1}) = \{U\}$ and $\text{BR}^2(\alpha_{-2}) = \{R\}$ and the expected payoffs are $u_1(U, \alpha_{-1}) = 2$ and $u_2(R, \alpha_{-2}) = 2$. Thus, we have that $\text{CBR}^{\{1,2\}}(\alpha) = \emptyset$. $\square$

A set X is a strong curb set if the belief that only strategies in X are played implies that players and coalitions have no incentives to use other strategies than those belonging to X . Formally, **strong curb sets** are defined as follows.

Definition 4.3. A strong curb set is a product set $X = \times_{i \in N} X_i$ where

- (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;
- (b) for each $J \subseteq N$ and each vector of beliefs $\alpha = (\alpha_{-1}, \dots, \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i} , the product set $X_J = \times_{j \in J} X_j$ contains all coalitional best-responses of coalition J against the beliefs of its members:

$$\begin{aligned} \forall J \subseteq N, \forall \alpha = (\alpha_{-1}, \dots, \alpha_{-n}) \text{ with } \alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l), i \in N, \\ \text{CBR}^J(\alpha) \subseteq \times_{j \in J} X_j. \end{aligned}$$

Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies must contain all coalitional best-responses of each coalition to whatever belief each coalition member may have that is consistent with the recommendations to the other players.⁵² A set $X \subseteq A$ is not a strong curb set if there exists a coalition having a deviation outside the set of recommended strategies such that each coalition member is at least as well off by deviating for at least one possible belief concerning the play of others in the set of recommended strategies. A deviation is blocked if we can find one player who is strictly better off by blocking the deviation. Notice that each coalition member is allowed to have a different belief concerning the play of others in the set of recommended strategies to assess the profitability of the deviation. In other words, the coalition members may disagree on where the deviation leads to.⁵³ Each strong curb set is a curb set and hence contains the support of at least one Nash equilibrium in mixed strategies.⁵⁴

⁵²We assume that players choose pure strategies. However, the notion of strong curb set can be easily extended to mixed strategies simply by accommodating the definition of CBR. Then, strong curb sets would still be product sets of pure strategies but such that each player's set of recommended strategies contains now the support all coalitional best-responses of each coalition to whatever belief each coalition member may have that is consistent with the recommendations to the other players.

⁵³We are implicitly assuming that players do not update their beliefs by trying to understand why some coalitional action is a best-response for the other players of the coalition.

⁵⁴Similarly to strong curb sets, we can define the notion of strong prep sets. Strong prep sets are product sets of pure strategies such that each player's set of recommended strategies must contain

A strong curb set X is **minimal** if no strong curb set is a proper subset of X . The set-valued solution concept that assigns to each game its collection of minimal strong curb sets is denoted by min-strong-curb . Hence, for a game G , $\text{min-strong-curb}(G) = \{X \subseteq A \mid X \text{ is a minimal strong curb set of } G\}$ and $\text{strong-curb}(G) = \{X \subseteq A \mid X \text{ is a strong curb set of } G\}$. Every normal-form game has a minimal strong curb set.

Proposition 4.1. Every normal-form game G has a minimal strong curb set.

Establishing existence of minimal strong curb sets in finite games is simple. The entire pure-strategy space A is a strong curb set. Hence the collection of strong curb sets is nonempty, finite (since A is finite) and partially ordered by set inclusion. Consequently, a minimal strong curb set exists. In the appendix we show that the existence result holds for every game $G \in \mathcal{G}$, where $\mathcal{G}$ is the class of normal-form games $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ where for each player $i \in N = \{1, 2, \dots, n\}$, A_i is a compact subset of a metric space and $u_i : A \rightarrow \mathbb{R}$ is a continuous von Neumann-Morgenstern utility function.

If X is a minimal strong curb set of $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, then it is a minimal strong curb set of the subgame $G_X = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$. The intuition behind the proof of this result is the following. In the game G , for every possible belief profile with support in X , there is no profitable deviation outside X (since $X \in \text{min-strong-curb}(G)$). Then, there is no deviation from some subset $Y \subseteq X$ outside X for beliefs with support in Y . Since $Y \notin \text{min-strong-curb}(G)$ (as it would contradict that $X \in \text{min-strong-curb}(G)$), there should exist a deviation from Y to $X \setminus Y$. Then, $Y \notin \text{min-strong-curb}(G_X)$.

Proposition 4.2. If $X \in \text{min-strong-curb}(G)$ then $X \in \text{min-strong-curb}(G_X)$.

Proof. Let $X \in \text{min-strong-curb}(G)$. X is a trivial strong curb set of the subgame G_X : $X \in \text{strong-curb}(G_X)$. We will show that there is no $Y \subsetneq X$ such that $Y \in \text{strong-curb}(G_X)$. Suppose, on the contrary, that there exists $Y \subsetneq X$ such that $Y \in \text{strong-curb}(G_X)$. Since Y is not a minimal strong curb set of G , there exists a vector of beliefs concentrated on Y and a coalition $J \subseteq N$ such that each member of the coalition prefers to play a strategy profile outside the set Y rather

at least one coalitional best-response of each coalition to whatever belief each coalition member may have that is consistent with the recommendations to the other players. We provide a formal definition of strong prep sets in the appendix.

than playing a best-response in Y to his belief. Formally, since $Y \notin \text{min-strong-curb}(G)$, there exists $J \subseteq N$, $a_J \in \times_{j \in J} A_j \setminus Y_j$ and $\alpha = (\alpha_{-1}, \dots, \alpha_{-N})$ with $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Y_j)$, $i \in N$, such that $u_j(a_J, \alpha_{-j}^-) \geq u_j(a_j, \alpha_{-j})$ for all $j \in J$, for all $a_j \in Y_j$. Since $Y \in \text{strong-curb}(G_X)$, the aforementioned deviation of coalition J does not belong to $\times_{j \in J} X_j \setminus Y_j$, we have $a_J \in \times_{j \in J} A_j \setminus X_j$. Since $X \in \text{strong-curb}(G)$ and $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) \forall i \in N$ (since $\times_{j \in N \setminus \{i\}} \Delta(Y_j) \subsetneq \times_{j \in N \setminus \{i\}} \Delta(X_j)$), at least one member $j^* \in J$ prefers to play a best-response in X against the belief α_{-j^*} than playing according to a_J . Thus, we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) > u_{j^*}(a_J, \alpha_{-j^*}^-)$ for some $b_{j^*} \in X_{j^*}$. Since $u_{j^*}(a_J, \alpha_{-j^*}^-) \geq u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for all $a_{j^*} \in Y_{j^*}$ ($Y \notin \text{strong-curb}(G)$), we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) > u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for some $b_{j^*} \in X_{j^*}$, for all $a_{j^*} \in Y_{j^*}$. This contradicts the fact that $Y \in \text{strong-curb}(G_X)$ since we have identified a belief α which is such that $\text{BR}^{j^*}(\alpha_{-j^*}) \not\subseteq Y$. ■

4.4. Relationships with other solution concepts

In this section we relate the concept of minimal strong curb set to the concepts of strong Nash equilibrium, coalition-proof Nash equilibrium and coalitional rationalizability. The product set of actions chosen in every strict strong Nash equilibrium is a minimal strong curb set. Conversely, for every minimal strong curb set composed of one action per player, the strategy profile in which each player selects this action is a strict strong Nash equilibrium. The main weakness of the strong Nash equilibrium concept is that it fails to exist in a natural class of games. However, the existence of minimal strong curb sets is guaranteed in general. The question we now address is whether minimal strong curb sets allow us to make reasonable predictions in games in which a strong Nash equilibrium does not exist. We provide below a game in which a strong Nash equilibrium does not exist but the unique minimal strong curb set is a proper subset of the full strategy space.

Example 4.2.

Consider the normal-form game G_3 .

	L	C	R
U	4, 4	0, 5	0, 0
M	0, 3	2, 2	0, 0
D	0, 0	0, 0	a , 1

For $a < 4$ the game G_3 has no strong Nash equilibrium while the minimal strong curb set is unique: $\text{min-strong-curb}(G_3) = \{\{U, M\} \times \{L, C\}\}$. Indeed, when each player believes that the other player plays in the set, each player's individual best-responses lie in the set. In addition, any coalitional deviations outside the set is blocked by player 2. $\square$

The collection of minimal strong curb sets may be composed of more elements than the product set of actions chosen in every strong Nash equilibria even when strong Nash equilibria exist. Consider again the game G_3 for $a > 4$. The strategy profile (D, R) is the unique strong Nash equilibrium of the game. The set composed of those actions is thus a minimal strong curb set. But, $\{U, M\} \times \{L, C\}$ is another minimal strong curb set. As a consequence, the unique strong Nash equilibrium may not be the only reasonable prediction in this game.

We now establish that if $X \subseteq A$ is a strong curb set and $a \in \times_{i \in N} X_i$ is a strict strong Nash equilibrium of the subgame restricted to X , then a is a strict strong Nash equilibrium of the original game.

Proposition 4.3. For every game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, if $X \subseteq A$ is a strong curb set of G and $a \in \times_{i \in N} X_i$ is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, then a is a strict strong Nash equilibrium of the original game G .

Proof. Consider a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. By contradiction, suppose $X \subseteq A$ is a strong curb set of G , $a \in \times_{i \in N} X_i$ is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ but a is not a strict strong Nash equilibrium of the original game G . Since a is not a strict strong Nash equilibrium of the original game G , there exists a coalition $J \subseteq N$ and a strategy profile $a'_J \in \times_{j \in J} A_j$ which satisfies $u_i(a'_J, a_{-J}) \geq u_i(a) \forall i \in J$. Since X is a strong curb set of the original game, $a'_J \in \times_{j \in J} X_j$ ($a'_J \notin \times_{j \in J} (A_j \setminus X_j)$). It contradicts the fact that a is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$. $\blacksquare$

When a coalition-proof Nash equilibrium exists, its support is not necessarily contained in a minimal strong curb set, as the following example shows.

Example 4.3. Consider the normal-form game G_4 .

	L	C	R		L	C	R
U	2, 1, 0	0, 0, 0	-9, -9, -9	U	1, 2, 0	0, 2, 1	-9, -9, -9
M	2, 0, 1	1, 0, 2	-9, -9, -9	M	0, 0, 0	0, 1, 2	-9, -9, -9
D	-9, -9, -9	-9, -9, -9	-9, -9, -9	D	-9, -9, -9	-9, -9, -9	-9, -9, -9
	l				c		
	L	C	R				
U	-9, -9, -9	-9, -9, -9	-9, -9, -9				
M	-9, -9, -9	-9, -9, -9	-9, -9, -9				
D	-9, -9, -9	-9, -9, -9	-8, -8, -8				
	r						

The unique coalition-proof Nash equilibrium of G_4 is (D, R, r) , while the unique minimal strong curb set is $\min\text{-strong-curb}(G_3) = \{\{U, M\} \times \{L, C\} \times \{l, c\}\}$. The predictions obtained under the minimal strong curb set seem more reasonable than the one given by the coalition-proof Nash equilibrium. $\square$

Outside the equilibrium framework Bernheim (1984) and Pearce (1984) have proposed the concept of rationalizability which consists of an iterative procedure that eliminates at each round strategies that are never best-response. Strategies that survive this iterative procedure are said to be rationalizable. Basu and Weibull (1991) have shown that every strategy contained in a minimal curb set is rationalizable.⁵⁵ However, contrary to curb sets, strong curb sets may include strategies that are strictly dominated or even not rationalizable.⁵⁶

Example 4.4. Consider the prisoners dilemma G_5 .

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

We have that the action U (L) is strictly dominated for player 1 (2) but belongs to the unique minimal strong curb set of G_5 . Indeed, $\min\text{-strong-curb}(G_5) =$

⁵⁵See Bernheim (1984), Pearce (1984), Herings and Vannetelbosch (1999, 2000) for the definitions of rationalizability for normal-form games and of its refinements. The set of rationalizable strategies coincide with the maximal tight curb set where tight curb sets are curb sets which are identical with their own best responses.

⁵⁶Hofbauer and Weibull (1996) have provided a class of evolutionary selection dynamics under which strictly dominated strategies do survive for some games.

$\{\{U, D\} \times \{L, R\}\}$. $\square$

Ambrus (2006) has proposed the concept of coalitional rationalizability using an iterative procedure.⁵⁷ The construction is similar to the original definition of rationalizability provided by Bernheim (1984) and Pearce (1984), except that not only never best-response strategies of individual players are deleted by the procedure, but also strategies of group of players. Strategies of group of players are deleted if it is in their mutual interest to restrict their play to the remaining set of strategies. The set of coalitionally rationalizable strategies is the set of strategies that survive the iterative procedure of restrictions.⁵⁸

Coalitional rationalizability may have more cutting power than minimal strong curb sets, as the following example shows.

Example 4.5. Consider the normal-form game G_6 .

	L	R		L	R
U	2, 2, 2	0, 0, 0	U	0, 0, 0	0, 0, 0
D	0, 0, 0	3, 3, 0	D	0, 0, 0	1, 1, 1
	l			r	

The game G_6 has a unique coalitionally rationalizable strategy profile which is (D, R, r) . Intuitively, player 1 and player 2 both recognize that they have a dominant strategy profile (D, R) . Anticipating this choice, player 3 selects r . On the other hand, $\{D\} \times \{R\} \times \{r\}$ is not a strong curb set since the deviation of the three players from (D, R, r) to (U, L, l) is Pareto improving. The unique strong curb set of G_6 is the full strategy space. $\square$

However, the converse may also be true. Minimal strong curb sets may have more cutting power than coalitional rationalizability.

Example 4.6. Consider the normal-form game G_7 .

	L	C	R
U	-2, 1	-1, 0	1, -2
M	0, -1	0, 0	0, -1
D	1, -2	-1, 0	-2, 1

⁵⁷Ambrus (2009) has provided an alternative concept of best-response to coalitions of players and he has offered epistemic definitions of coalitional rationalizability in normal-form games.

⁵⁸Another approach is Herings, Mauleon and Vannetelbosch (2004) who have introduced the notion of social rationalizability

In G_7 the strategy profile (M, C) is a strict strong Nash equilibrium and min-strong-curb(G_7) = $\{\{M\} \times \{C\}\}$. But, any strategy profile is coalitionally rationalizable. $\square$

4.5. Learning to play min-strong-curb strategies

We now provide a class of dynamic learning processes in which groups of players may coordinate their actions. In line with Hurkens (1995),⁵⁹ players observe actions played recently, form their beliefs upon these observations, and play best-responses to those beliefs. The new feature of the processes we propose is that players are allowed to play coalitional best-responses. That is, players are allowed to select a joint action if by doing so, the expected payoff of each member of the group is increased with respect to the payoff she would have obtained by playing individually. We will show that the learning processes we propose lead the players to play only strategies from a minimal strong curb set, and thus provide a dynamic motivation for the concept of minimal strong curb set.

A game $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is played once every period. Suppose the players are partitioned into classes $C_1, C_2, \dots, C_n$ such that $u_i = u_j$ and $A_i = A_j$ if $i, j \in C_k$. In each period, one player is drawn at random from each of n disjoint classes $C_1, C_2, \dots, C_n$, to play the game G in that period. These players are partitioned into coalitions to form a coalition structure. A coalition structure $\mathbf{J} = (J_1, J_2, \dots, J_M)$ is a partition of the player set $N = \{1, 2, \dots, n\}$ such that $J_k \cap J_l = \emptyset$ for $k \neq l$ and $\cup_{k=1}^M J_k = N$. Let $\mathbb{J}$ be the finite set of coalition structures. Each coalition structure $\mathbf{J} \in \mathbb{J}$ has a positive probability to occur at each period. Players have information about how the game has been played in the recent past. In a given period t , a particular history $h^t = (a^{t-K}, \dots, a^{t-1})$ is a description of how the game has been played in the K previous periods, where $a^{t-k} \in A$ is the action profile chosen by the n players in period $t - k$ for $k \in \{1, \dots, K\}$. We define the state space $H = A^K$ to consist of all histories $h = (a^{-K}, \dots, a^{-1})$ of length K . The choices of the players are time-independent, the learning process can thus be described by a stationary Markov chain on the state space $H = A^K$. Call $\hat{h} \in H$ a successor

⁵⁹See also Fudenberg and Levine (1998) or Young (1998). Kets and Voorneveld (2008) have provided an alternative dynamic learning process in which players display a bias towards recent choices and choose best-responses to beliefs supported by observed play in the recent past. The limit behavior of this learning process is shown to eventually settle down in minimal prep sets.

of $h \in H$ if $\hat{h}$ is obtained from h by deleting the leftmost element and by adding some element $a \in A$ to the right. Let $r(\hat{h})$ denote the rightmost element of $\hat{h} \in H$. For $h = (a^{-K}, \dots, a^{-1}) \in H$, let $\pi_i(h, k) = \{a_i^{-k}, \dots, a_i^{-1}\}$ denote the set of strategies played by player i in the k last periods, for $k \leq K$. Let $P : H \times H \rightarrow [0, 1]$ be a transition matrix, where $P(h, \hat{h})$ is the probability of moving from state $h \in H$ to state $\hat{h} \in H$ in one period and $\sum_{\hat{h} \in H} P(h, \hat{h}) = 1$ for all $h \in H$. A learning process is described by a transition matrix $P \in \mathcal{P}$, where $\mathcal{P}$ is defined as follows.

Definition 4.4. Let $\mathcal{P}$ be the set of transition matrices P that satisfy for all histories $h, \hat{h} \in H$, $P(h, \hat{h}) > 0$ if and only if (i) $\hat{h}$ is a successor of h , (ii) there exists some $\mathbf{J} \in \mathbb{J}$ and $\boldsymbol{\alpha} = (\alpha_{-1}, \dots, \alpha_{-n})$ with $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\pi_j(h, K))$ such that $r(\hat{h}) = (a_J)_{J \in \mathbf{J}}$ with $a_J \in \text{CBR}^J(\boldsymbol{\alpha})$ if $\text{CBR}^J(\boldsymbol{\alpha}) \neq \emptyset$ and $a_J \in \times_{i \in J} \text{BR}^i(\alpha_{-i})$ otherwise.

At each period every player chooses an action. This action can be chosen individually or in group, and is chosen after having observed the recent past play. When a group of players coordinate their actions, they choose a Pareto undominated action profile such that each member of the group benefits from playing jointly. In state h , if coalition $J \subseteq N$ has a coalitional best-response $a_J \in \text{CBR}^J(\boldsymbol{\alpha})$ given a profile of beliefs with support in the set of strategies played in the recent past, then the process moves with positive probability from state h to some state $\hat{h}$ in which coalition J plays a_J . To determine the outcome of such learning processes, what matters is to identify, for each state h , the set of states that can be reached from h in one period with positive probability and those that cannot be reached. Since the exact probability does not matter, we do not have to specify a particular process of belief formation nor a protocol of coalition formation. We only require that every such belief with support in the set of actions played recently and every partition of the players occur with positive probability.

For each $k \in \mathbb{N}$, $P^k : H \times H \rightarrow [0, 1]$ denotes the k -step transition probabilities of the Markov process with transition matrix $P \in \mathcal{P}$: $P^1 = P$ and $P^k = P \circ P^{k-1}$ for $k > 1$. We will write $h \rightsquigarrow \hat{h}$ if there exists $k \in \mathbb{N}$ satisfying $P^k(h, \hat{h}) > 0$. Now $\rightsquigarrow$ defines a weak order on H . We can define an equivalence relation on H : $h \sim \hat{h} \Leftrightarrow h \rightsquigarrow \hat{h}$ and $\hat{h} \rightsquigarrow h$. Let $[h]$ denote the equivalence class that contains h and let $Q = \{[h] \mid h \in H\}$ denote the set of equivalence classes. A partial order $\preceq$ on Q is given by: $[h] \preceq [\hat{h}] \Leftrightarrow \hat{h} \rightsquigarrow h$. The minimal elements with respect to the order $\preceq$ are called ergodic sets. The other elements are called transient sets.

If the process leaves a transient set it can never return to that set. If the process is in an ergodic set it can never leave that set. The elements belonging to ergodic and transient sets are called ergodic and transient states, respectively. In any finite Markov chain, no matter where the process starts, the probability that the process is in an ergodic state after k steps tends to 1 as k tends to infinity (see Kemeny and Snell, 1976). Proposition 4.4 states that if memory is long enough (K high enough), the probability that the players are playing a minimal strong curb strategy profile after k steps of the learning process tends to 1 as k tends to infinity. To prove this result we show that each ergodic set Z of every Markov chain with transition matrix $P \in \mathcal{P}$ satisfies $Z \subseteq X^K$ for some $X \in \text{min-strong-curb}(G)$.

Proposition 4.4 There exists $\underline{K} \in \mathbb{N}$ such that for all finite $K \geq \underline{K}$ and every Markov chain with transition matrix $P \in \mathcal{P}$, if $Z \subseteq H$ is an ergodic set then $Z \subseteq X^K$ for some minimal strong curb set X .

The following lemma will be useful to prove Proposition 4.4.

Lemma 4.1. Let $h^t = (x^{K-t}, \dots, x^1, a^1, \dots, a^t)$ be a particular history. (a) If the players draw their beliefs from $\times_{i \in N} \pi_i(h^t, t)$, the process moves with positive probability to an history h^{t+1} such that $\times_{i \in N} \pi_i(h^t, t) \subsetneq \times_{i \in N} \pi_i(h^{t+1}, t+1)$ if $\times_{i \in N} \pi_i(h^t, t)$ is not a strong curb set. (b) If the players draw their beliefs from $\times_{i \in N} \pi_i(h^t, t)$, the process moves with probability 1 to an history h^{t+1} such that $\times_{i \in N} \pi_i(h^t, t) = \times_{i \in N} \pi_i(h^{t+1}, t+1)$ if $\times_{i \in N} \pi_i(h^t, t)$ is a strong curb set.

Proof. Easy and therefore omitted.

Proof of Proposition 4.4. Let $L = \sum_{i=1}^n |A_i| - (n-1)$. Let $M = \sum_{i=1}^n |A_i| - n$. Take $\underline{K} = L + M$ and let $K \geq \underline{K}$ be finite. Let $P \in \mathcal{P}$. To prove Proposition 4.4 we will show that (i) from any history $h^1 \in H$, the process moves with positive probability in $L-1$ steps to a state $h^L \in H$ such that $\times_{i \in N} \pi_i(h^L, L)$ is a strong curb set, (ii) from state h^L , the process moves with positive probability in M steps to a state $h^{L+M} \in H$ such that $\times_{i \in N} \pi_i(h^{L+M}, M)$ is a minimal strong curb set, and (iii) steps (i) and (ii) imply that if $Z \subseteq H$ is an ergodic set then $Z \subseteq X^K$ for some minimal strong curb set X . (i) Let $a^1, \dots, a^T \in A$ be such that $a^{t+1} \notin \times_{i \in N} \pi_i(h^t, t)$ for all $t = 1, \dots, T-1$. By definition of L , we have $T \leq L$ since $\times_{i \in N} \pi_i(h^1, 1)$ contains n actions, $\times_{i \in N} \pi_i(h^{t+1}, t+1)$ contains at least one additional action than $\times_{i \in N} \pi_i(h^t, t)$ and the action space A , which is the largest strong curb set, contains $\sum_{i=1}^n |A_i|$ of them. Thus, there exists a $\tau \leq L$ such that, starting from h^1 and

applying τ times part (a) of Lemma 4.1, we have $h^1 \rightsquigarrow h^\tau = (x^{K-\tau}, \dots, x^1, a^1, \dots, a^\tau)$ and $\times_{i \in N} \pi_i(h^\tau, \tau)$ is a strong curb set. From part (b) of Lemma 4.1, we have $h^\tau \rightsquigarrow h^L = (x^{K-L}, \dots, x^1, a^1, \dots, a^\tau, \dots, a^L)$ such that $\times_{i \in N} \pi_i(h^\tau, \tau)$ is a strong curb set and $\times_{i \in N} \pi_i(h^L, L) = \times_{i \in N} \pi_i(h^\tau, \tau)$.

(ii) Let $X \subseteq \times_{i \in N} \pi_i(h^L, L)$ be a minimal strong curb set. Since $K \geq L + M$ and since every strategy in a minimal strong curb set is an element of a coalitional best-response to some belief concentrated on the set, there exists a set $\{b^1, \dots, b^M\}$ that spans X and such that $h^L \rightsquigarrow h^{L+M} = (\dots, a^1, \dots, a^L, b^1, \dots, b^M)$. That is, from h^L there is a positive probability that each player $i \in N$ draws specific beliefs from $\times_{j \in N \setminus \{i\}} \Delta(\pi_j(h^L, L))$ and is assigned to specific coalitions during M periods in a row such that each coalition chooses a coalitional best-response in each period and the process reaches $h^{L+M} = (\dots, a^1, \dots, a^L, b^1, \dots, b^M)$.⁶⁰ Once in h^{L+M} , each player draws with positive probability her beliefs from the minimal strong curb set $\times_{i \in N} \Delta(\pi_i(h^{L+M}, M))$ during $K - M$ periods in a row. Then, the process reaches history $h^{L+K} = (b^1, \dots, b^M, c^1, \dots, c^{K-M})$ such that $\times_{i \in N} \pi_i(h^{L+K}, K - M) \subseteq X$. By definition of $\mathcal{P}$, when the process reaches state h^{L+K} , each player draws her beliefs from X with probability 1 and plays coalitional best-responses to her beliefs by selecting with probability one actions from X . So, $\times_{i \in N} \pi_i(h^{L+K+1}, K) \subseteq X$. Repeating the previous argument, we have that $\times_{i \in N} \pi_i(h^{L+K+k}, K) \subseteq X$ for all $k \in \mathbb{N}$ and for all h^{L+K+k} such that $h^{L+K} \rightsquigarrow h^{L+K+k}$. Once in h^{L+K} , each player plays with probability one actions from the minimal strong curb set X in all future periods. The set X^K thus contains an ergodic set.

(iii) By contradiction, suppose there exists an ergodic set Z such that $Z \not\subseteq X^K$ for any minimal strong curb set X . Thus Z contains an ergodic state $h \in H$ such that $h \notin X^K$ for all minimal strong curb set X . Applying (i) and (ii), we have $h \rightsquigarrow h'$ such that $h' \in Y^K$ for some minimal strong curb set Y and h' is an ergodic state of some ergodic set $W \subseteq Y^K$. Notice that $h \notin W$ since $h \notin Y^K$, which implies that we do not have $h' \rightsquigarrow h$. This contradicts the fact that h is an ergodic state and thus that Z is an ergodic set. $\square$

Remark 4.1. Take any game $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$. Suppose we are in state h^L and $X \subseteq \times_{i \in N} \pi_i(h^L, L)$ is a minimal strong curb set of G . Let $k = \max(|X_1|, \dots, |X_n|)$ and let $l = \sum_{i=1}^n |X_i| - n$. (a) If $X = \times_{i \in N} X_i$ is such that every action $a_i \in X_i$ of each player $i \in N$ is an individual best-response to some

⁶⁰See Remark 4.1 for an explanation of the value of M .

belief in the set, the process can converge in exactly k periods from h^L to $h^{L+k} = (\dots, a^1, \dots, a^L, b^1, \dots, b^k)$ with the property that $\{b^1, \dots, b^k\}$ spans X . (b) If $X = \times_{i \in N} X_i$ is such that some action $a_i \in X_i$ of player i only belongs to some coalitional best-response, the time of convergence of the process can be longer even more if the same player is involved in different coalitional moves. It can take at most l periods to move from h^L to $h^{L+l} = (\dots, a^1, \dots, a^L, b^1, \dots, b^l)$ with the property that $\{b^1, \dots, b^l\}$ spans X . This is illustrated through the following example.

Example 4.7. Consider the normal form game G_8

	L	R		L	R
U	2, 2, 1	0, 3, 3	U	0, -1, 0	2, 0, 2
D	3, 0, 1	1, 1, 1	D	1, -1, 0	3, 0, 0
	l			r	

We have that $\text{min-strong-curb}(G_8) = \{\{U, D\} \times \{L, R\} \times \{l, r\}\}$ and $M = \sum_{i=1}^n |A_i| - n = 3$. Suppose the process is in state h^M where $\times_{i \in N} \pi_i(h^M, M) = A$. Let k be the smallest integer such that $h^M \rightsquigarrow h^{M+k}$ with the property that $\times_{i \in N} \pi_i(h^{M+k}, k) = A$. We have $k = 3$ since player 2 selects her strategy L only when coalition $\{1, 2\}$ plays (U, L) . Player 3 selects her strategy r only when coalition $\{1, 3\}$ plays (U, r) . A third period is needed for player 1 to play D . $\square$

4.6. Conclusion

Basu and Weibull (1991) have introduced the notion of curb sets which are product sets of pure strategies containing all individual best-responses against beliefs restricted to the recommendations to the remaining players. The concept of minimal curb sets is a set-theoretic coarsening of the notion of strict Nash equilibrium. In this chapter we have introduced the concept of minimal strong curb sets which is a set-theoretic coarsening of the notion of strong Nash equilibrium. Strong curb sets require sets to be immune not only against individual deviations, but also against group deviations. Strong curb sets are product sets of pure strategies such that each player's set of recommended strategies must contain all coalitional best-responses of each coalition to whatever belief each coalition member may have that is consistent with the recommendations to the other players. We have shown that minimal strong curb sets exist in general. We have also compared minimal strong curb sets with

other well known solution concepts. Finally, we have provided a dynamic learning process leading the players to playing strategies from a minimal strong curb set only.

Appendix 4.A. Existence of strong curb sets

We now show that the existence of minimal strong curb sets holds in general. Let $\mathcal{G}$ be the class of normal-form games $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ where for each player $i \in N = \{1, 2, \dots, n\}$, A_i is a compact subset of a metric space and $u_i : A \rightarrow \mathbb{R}$ is a continuous von Neumann-Morgenstern utility function. Payoffs are extended to mixed strategies in the usual way. Let $\Delta(A_i)$ be the set of Borel probability measures over A_i . If $B_i \subseteq A_i$ is a Borel set, then $\Delta(B_i)$ denotes the set of Borel probability measures with support in B_i : $\Delta(B_i) = \{\alpha_i \in \Delta(A_i) \mid \alpha_i(B) = 1\}$. If $G \in \mathcal{G}$, that is, payoff functions are continuous and strategy sets compact, then each set $\text{BR}^i(\alpha_{-i}) \subseteq A_i$ is nonempty and compact. A strong curb set is a product set $X = \times_{i \in N} X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies; (b) $\forall J \subseteq N, \forall \alpha = (\alpha_{-1}, \dots, \alpha_{-n})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l)$, $i \in N$, $\text{CBR}^J(\alpha) \subseteq \times_{j \in J} X_j$.

Proposition 4.A.1. Every game $G \in \mathcal{G}$ has a minimal strong curb set

Proof. Let $Q = \text{strong-curb}(G)$ denote the collection of all strong curb sets of G . A is a strong curb set of G since for every $J \subseteq N$ and $\alpha = (\alpha_{-1}, \dots, \alpha_{-N})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(A_l)$, $i \in N$, we have $\text{CBR}^J(\alpha) \subseteq \times_{j \in J} A_j$. So Q is nonempty and partially ordered via set inclusion. According to the Hausdorff Maximality Principle, Q contains a maximal nested subset R . For each $i \in N$, let $X_i = \cap_{Y \in R} Y_i$ be the intersection of player i 's strategies in the nested set R . The set X_i is nonempty since the conditions of the Cantor intersection principle⁶¹ are satisfied, i.e. (i) the collection $\{Y_i \mid Y \in R\}$ is nested and thus satisfies the finite intersection property and (ii) each Y_i is nonempty and compact. It remains to prove that $X = \times_{i \in N} X_i$ is a minimal strong curb set. Take $\alpha = (\alpha_{-1}, \dots, \alpha_{-N})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l)$, $i \in N$. We have that $\text{CBR}^J(\alpha) \cap \times_{j \in J} (A_j \setminus X_j) = \emptyset$ for $J \subseteq N$ since $\text{CBR}^J(\alpha) \cap \times_{j \in J} (A_j \setminus X_j) = \text{CBR}^J(\alpha) \cap (\times_{j \in J} \cup_{Y \in R} (A_j \setminus Y_j)) \subseteq \text{CBR}^J(\alpha) \cap (\cup_{Y \in R} \times_{j \in J} (A_j \setminus Y_j)) = \cup_{Y \in R} (\text{CBR}^J(\alpha) \cap \times_{j \in J} (A_j \setminus Y_j))$ and $\text{CBR}^J(\alpha) \cap \times_{j \in J} (A_j \setminus Y_j) = \emptyset$ for all $Y \in R$ (Y is a strong curb set). This establishes that X is

⁶¹In words, the Cantor intersection principle tells us that to show that the intersection of an infinite number of elements of a set Z is nonempty and compact, we just need to show that the intersection is nonempty and compact for every subset of Z composed of finite elements.

a strong curb set. The fact that it is minimal follows directly from the fact that R is a maximal nested subset of Q . ■

Appendix 4.B. Strong prep sets

Voorneveld (2004) has proposed another set-valued solution concept, prep sets, which are formally defined as follows.

Definition 4.B.1 A prep set is a product set $X = \times_{i \in N} X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies; (b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i} , the set X_i contains at least one best response of player i against his belief : $\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j), BR^i(\alpha_{-i}) \cap X_i \neq \emptyset$.

A prep set X is minimal if no prep set is a proper subset of X . Voorneveld (2004) has shown that every game G possesses at least one minimal prep set. The set-valued solution concept that assigns to each game its collection of minimal prep sets is denoted by min-prep.⁶² Similarly to strong curb sets, we can define the notion of strong prep sets as follows.

Definition 4.B.2 A strong prep set is a product set $X = \times_{i \in N} X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies; (b) for each $J \subseteq N$ and each vector of beliefs $\alpha = (\alpha_{-1}, \dots, \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i} , the product set $X_J = \times_{j \in J} X_j$ contains at least one coalitional best response of coalition J against the beliefs of its members: $\forall J \subseteq N, \forall \alpha = (\alpha_{-1}, \dots, \alpha_{-n})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l), i \in N, CBR^J(\alpha) \cap \times_{j \in J} X_j \neq \emptyset$.

A strong prep set X is minimal if no strong prep set is a proper subset of X . Every strong curb set is a strong prep set, so if a strong curb set is contained in a minimal strong prep set, the two sets are necessarily equal. Similarly to Proposition 4.2 we have that, if X is a minimal strong prep set of G , then it is a minimal strong prep set of the subgame G_X .

⁶²Voorneveld, Kets and Norde (2005) have provided axiomatizations of minimal prep sets and minimal curb sets.

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