

On the minimal and maximal smooth convex interpolating functions

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1 Introduction

Given a list of points with associated function values and gradients, we consider the problem of finding an interpolating smooth convex function, where smooth is defined as
Definition 1. Given a constant $L > 0$, a differentiable function F is said to be L -smooth if and only if it satisfies :

$$\|\nabla F(x) - \nabla F(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbf{E}.$$

Formally, the input of the interpolation problem we consider is a set of triplets $\{(point, gradient, function\ value)\}$:

Definition 2. A set of triplets $\{(x_i, g_i, f_i)\}_{i \in I} \in \mathbf{E} \times \mathbf{E}^* \times \mathbf{R}$ is said to be L -smooth convex interpolable if and only if there exists an L -smooth convex function F such as $F(x_i) = f_i$ and $\nabla F(x_i) = g_i \quad \forall i \in I$.

Smooth convex interpolation was characterized in [1]:

Theorem 1. A set of triplets $\{(x_i, g_i, f_i)\}_{i \in I} \in \mathbf{E} \times \mathbf{E}^* \times \mathbf{R}$; $i \in I$ is L -smooth interpolable if and only if

$$\forall i, j \in I, f_i \geq f_j + \langle g_j, x_i - x_j \rangle + \frac{1}{2L} \|g_i - g_j\|^2.$$

In this work, we search for extremal interpolating functions, i.e. we identify the pointwise maximal and minimal smooth convex interpolating functions. Our main result is as follows.

Theorem 2. Consider a set of triplets $\{(x_i, g_i, f_i)\}_{i \in I}$. If this set is L -smooth convex interpolable, there exist two L -smooth convex interpolating functions F_l and F_h such as any L -smooth convex function F interpolating the set $\{(x_i, g_i, f_i)\}_{i \in I}$ satisfies:

$$F_l(x) \leq F(x) \leq F_h(x) \quad \forall x \in \mathbf{E}.$$

2 Extremal smooth interpolating functions

Let us consider a L -smooth convex interpolable set of triplets $\{(x_i, g_i, f_i)\}$ and an interpolating function F . It follows [1, Theorem 2] that the convex conjugate F^* of F is a $\frac{1}{L}$ -strongly convex functions that interpolates the new set of triplets $\{(g_i, x_i, \langle g_i, x_i \rangle - f_i)\}$. Defining now the function $h(y) = F^*(y) - \frac{1}{2L} \|y\|^2 \quad \forall y \in \mathbf{E}^*$ it follows that h is convex.

Remark 1. h is a convex interpolating function for the set $\{(g_i, x_i - \frac{1}{L}g_i, \langle g_i, x_i \rangle - f_i - \frac{1}{2L} \|g_i\|^2)\}_{i \in I}$.

From this we reduce our initial problem to that of identifying extremal convex interpolating functions (without the smoothness requirement) [1, Theorem 4], which is given by the following

Theorem 3. If the set $\{(x_i, g_i, f_i)\}$ is interpolable by a convex function, then any convex interpolable function f satisfies $f_l(x) \leq f(x) \leq f_h(x)$ for all x where f_l and f_h are the lowest and highest interpolating functions defined by

$$f_l(x) = \max_{i \in I} f_i + \langle g_i, x - x_i \rangle.$$

$$f_h(x) = \min_{\lambda_i \geq 0} \sum_{i \in I} \lambda_i f_i \text{ s.t. } \sum_{i \in I} \lambda_i = 1 \text{ and } x = \sum_{i \in I} \lambda_i x_i.$$

Using Remark 1 and Theorem 3, we can find two convex functions h_l and h_h interpolating the set $\{(g_i, x_i - \frac{1}{L}g_i, \langle g_i, x_i \rangle - f_i - \frac{1}{2L} \|g_i\|^2)\}_{i \in I}$ and such that $h_l(y) \leq h(y) \leq h_h(y)$ for all $y \in \mathbf{E}^*$. Observe now that F can be recovered by h with $F = (h + \frac{1}{2L} \|y\|^2)^*$. Since this is based on two order-preserving/reversing transformations we can construct from h_l and h_h two extremal L -smooth convex functions F_l, F_h that are guaranteed to bound F :

$$F_h(x) = \sup_{y \in \mathbf{E}^*} \langle y, x \rangle - h_l(y) - \frac{1}{2L} \|y\|^2,$$

$$F_l(x) = \sup_{y \in \mathbf{E}^*} \langle y, x \rangle - h_h(y) - \frac{1}{2L} \|y\|^2.$$

3 Special case : One-dimensional functions

Let us now consider the special case of one-dimensional functions $\mathbf{E} = \mathbf{E}^* = \mathbb{R}$. Assume without loss of generality that points $x_i \quad i \in I$ are ordered such that $x_i \leq x_{i+1}$ for all $i \in I$. Using this fact, we find simplified expressions for the bounding functions of Theorem 3 and derive from that closed-form expressions for F_h and F_l .

4 Conclusion and Perspectives

We prove the existence and give explicit expressions for the pointwise maximal and minimal L -smooth convex functions interpolating a given set of triplets. Future work will be devoted to the search for simplified expressions in higher dimensions.

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References

- [1] A.B. Taylor, J.M. Hendrickx, F. Glineur *Smooth strongly convex interpolation and exact worst-case performance of first-order methods*. Mathematical Programming , 2017.